Chapter 6
Decomposing Dynamical Systems

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1 Introduction

A dynamical system is a kind of mathematical model that is intended to capture the intuitive notion of an arbitrary deterministic system, either reversible or irreversible, with discrete or continuous time or state space [1, 2, 4, 5]. Giunti and Mazzola [3] generalize the standard notion of a dynamical system by simply taking the state space M to be a non-empty set, and by requiring for the time set T a quite simple algebraic structure—namely, a monoid $L = (T, +)$, which they call the time model. They thus define a dynamical system DS_L on a monoid L, which they claim to be the minimal mathematical model that still captures the intuitive idea of a deterministic dynamics; such a model consists of a state space M together with a family, indexed by T, of functions from M to M, which satisfy an identity and a composition condition.

Cellular automata are well known examples of dynamical systems with discrete time set (the non-negative integers Z^+) and discrete state space. Wolfram [6] showed that the state space of a particular finite cellular automaton (*rule* 90_{10}, see Fig. 6.1) can be exhaustively decomposed into seventy mutually disconnected constituents. Each constituent subspace is internally connected, and it turns out to be the state space of a subsystem of the cellular automaton; the complete dynamics of the cellular automaton can thus be obtained as the sum of the dynamics of its seventy constituent systems (see Fig. 6.2).

Wolfram's example is not peculiar to the quite special case he analyzed. In fact, I will show in this paper that the state space of any dynamical system DS_L on a monoid L can be exhaustively decomposed into a set of mutually disconnected constituents (Decomposition Theorem, Theorem 1), where each constituent is internally

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Twelve cells arranged in a circle.  Each cell
may take value 0 or 1.  Thus, this CA has 2^12 =
4096 possible states.
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Rule	111	110	101	100	011	010	001	000
	0	1	0	1	1	0	1	0

Rule number (base 2) $01011010_2 = 90_{10}$ (base 10)

Time 0	0	1	0	0	1	0	1	0	1	1	1	1
Time 1	0	0	1	1	0	0	0	0	1	0	0	1

Fig. 6.1: A finite cellular automaton with 12 cells—*rule* 90_{10}

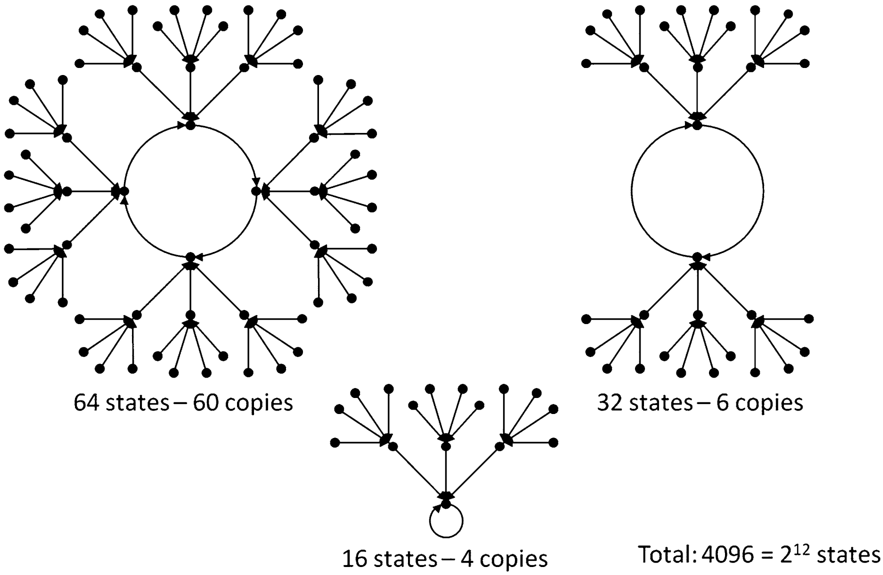

Fig. 6.2: The seventy constituent systems of finite CA *rule* 90_{10}

connected and is the state space of a subsystem of DS_L (called a constituent system of DS_L). In addition, constituent systems are themselves indecomposable (Proposition 6), even though they may very well be complex. Finally, I will show that any dynamical system DS_L is in fact identical to the sum of all its constituent systems (Composition Theorem, Theorem 2). Constituent systems can thus be thought as the indecomposable, but possibly complex, building blocks to which the dynamics of an arbitrary complex system fully reduces. However, no further reduction of the constituents is possible, even if they are themselves complex.

2 Dynamical Systems on Monoids

As mentioned above, a dynamical system is a kind of mathematical model that purports to formally capture the intuitive notion of an arbitrary deterministic system, either reversible or irreversible, with discrete or continuous time or state space. Let $T = Z^+$ (non-negative integers), Z (integers), R^+ (non-negative reals), or R (reals). Below is a standard definition of a dynamical system [1, 2, 4, 5].

Definition 1 (DYNAMICAL SYSTEM). *DS is a dynamical system* := *DS* is a pair $(M, (g^t)_{t \in T})$ such that

1. T is either Z^+, Z, R^+, or R;
   - any $t \in T$ is called *a duration*, and T *the time set* of *DS*;
2. M is a non-empty set;
   - any $x \in M$ is called *a state*, and M *the state space* of *DS*;
3. $(g^t)_{t \in T}$ is a family, indexed by T, of functions from M to M;
   - for any $t \in T$, g^t is called *the (state) transition of duration t* or, briefly, *the t-transition* or *the t-advance* of *DS*;
4. for any $v, t \in T$, for any $x \in M$,

   a. $g^0(x) = x$;
   b. $g^{v+t}(x) = g^v(g^t(x))$.

Example 1. The following are all examples of dynamical systems.

1. Discrete time set $(T = Z^+)$ and discrete state space: Finite state machines, Turing machines, cellular automata restricted to finite configurations.[1]
2. Discrete time set $(T = Z^+)$ and continuous state space: Many systems specified by difference equations, iterated mappings on R, unrestricted cellular automata.
3. Continuous time set $(T = R)$ and continuous state space: Systems specified by ordinary differential equations, many neural networks.

Giunti and Mazzola [3] point out that the standard definition of a dynamical system (Definition 1) is not fully explicit, for it does not make clear exactly which

[1] The state space of a cellular automaton is discrete (i.e. finite or countably infinite) if the cellular automaton state space only includes finite configurations, that is to say, configurations where all but a finite number of cells are in the quiescent state. If this condition is not satisfied, the state space has the power of the continuum.

structure on the time set T is needed in order to support appropriate dynamics. By condition 1 of Definition 1, T is either Z^+, Z, R^+, or R. With respect to the addition operation, these four models share the structure of a linearly ordered commutative monoid; but it is by no means obvious that all this structure on T is needed for a general definition of a dynamical system.

Giunti and Mazzola [3] maintain that the minimal structure on the time set that makes for a materially adequate definition of a dynamical system is just that of a *monoid*. Accordingly, they generalize Definition 1 as follows.

Definition 2 (Dynamical System on a Monoid). *DS_L is a dynamical system on L* := DS_L is a pair $(M, (g^t)_{t \in T})$ and L is a pair $(T, +)$ such that

1. L is a monoid;
   – any $t \in T$ is called *a duration*, T *the time set*, and L *the time model* of DS;
2. M is a non-empty set;
   – any $x \in M$ is called *a state*, and M *the state space* of DS_L;
3. $(g^t)_{t \in T}$ is a family, indexed by T, of functions from M to M;
   – for any $t \in T$, g^t is called *the (state) transition of duration t* or, briefly, *the t-transition* or *the t-advance* of DS_L;
4. for any $v, t \in T$, for any $x \in M$,
   a. $g^0(x) = x$, where 0 is the unity of L;
   b. $g^{v+t}(x) = g^v(g^t(x))$.

It is interesting to realize that any vector space VS_F over a field F (and, even more generally, any unital left module US_R over a ring with unity R) turns out to be a dynamical system DS_L on a monoid L, where appropriate further structure has been added to both DS_L and L. Thus, the theory of dynamical systems on monoids is a natural generalization of (1) the theory of vector spaces over fields, and (2) the theory of unital left modules over rings with unity.[2]

3 Subspaces and Subsystems

If $DS_L = (M, (g^t)_{t \in T})$ is a dynamical system on a monoid $L = (T, +)$, a subspace of the state space M is any non-empty subset of M which is closed under all t-transitions g^t. More formally,

Definition 3. Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$.

X is a subspace of M := $X \subseteq M, X \neq \emptyset$ and, $\forall x \in X, \forall t \in T, g^t(x) \in X$.

[2] My thanks to Tomasz Kowalski for pointing out to me the relation between vector spaces over fields and dynamical systems on monoids.

Proposition 1 *Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$; let $X \subseteq M$, and g^t/X be the restriction of g^t to X.*

$(X, (g^t/X)_{t \in T})$ is a dynamical system on L iff X is a subspace of M.

Proof. Suppose that $(X, (g^t/X)_{t \in T})$ is a dynamical system on $L = (T, +)$; then, by 2 of Definition 2, $X \neq \emptyset$ and, by 3 of Definition 2, $\forall x \in X, \forall t \in T, g^t/X(x) \in X$. As g^t/X is the restriction of g^t to X, $\forall x \in X, \forall t \in T, g^t(x) = g^t/X(x) \in X$. Hence, as $X \subseteq M$, by Definition 3, X is a subspace of M.

Conversely, suppose that X is a subspace of M. Then, by Definition 3, condition 2 of Definition 2 is satisfied. As g^t/X is the restriction of g^t to X, and by Definition 3, condition 3 of Definition 2 is satisfied as well. Finally, as DS_L is a dynamical system on L and g^t/X is the restriction of g^t to X, conditions 4a and 4b of Definition 2 also hold. Therefore, $(X, (g^t/X)_{t \in T})$ is a dynamical system on $L = (T, +)$. □

A subsystem of a dynamical system $DS1_L = (M, (g^t)_{t \in T})$ on a monoid $L = (T, +)$ is a dynamical system $DS2_L = (N, (h^t)_{t \in T})$ on the same monoid L, whose state space N is a subset of M and whose t-transitions are the restrictions to N of the t-transitions of $DS1_L$. That is to say,

Definition 4. Let $DS1_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$.

$DS2_L$ is a subsystem of $DS1_L$:= $DS2_L = (N, (h^t)_{t \in T})$ is a dynamical system on L, $N \subseteq M$ and, $\forall t \in T$, $h^t = g^t/N$.

Obviously, by Definition 4, any dynamical system DS_L on a monoid L is a subsystem of itself. Furthermore, the following proposition holds.

Proposition 2 *Let $DS1_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$.*

$DS2_L$ is a subsystem of $DS1_L$ iff $DS2_L = (N, (h^t)_{t \in T})$, N is a subspace of M, and $\forall t \in T$, $h^t = g^t/N$.

Proof. The thesis easily follows from Definitions 4, 3 and Proposition 1. □

The concept of a subsystem allows us to introduce a quite general notion of a simple system. In this acceptation, a dynamical system is simple if it does not possess any proper subsystem; otherwise, it is complex.

Definition 5. Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$.

DS_L is simple := DS_L does not have any subsystem but itself.

Example 2. Let $DS_L = (\{x\}, (id^t)_{t \in Z^+})$, where x is an arbitrary object and, $\forall t \in Z^+$, id^t is the t-th iteration of the identity function on $\{x\}$. Then, DS_L is a dynamical system on $L = (Z^+, +)$ and DS_L is simple.

Definition 6. Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$.

DS_L is complex := DS_L is not simple.

4 Past and Future

Let $DS_L = (M, (g^v)_{v \in T})$ be a dynamical system on a monoid $L = (T, +)$. The family of state transitions $(g^v)_{v \in T}$ allows us to introduce purely dynamical concepts of *past* and *future* as follows. Let 0 be the unity of L, $t \in T - \{0\}$, and $x \in M$.

Definition 7. *The t-past of x* $:= P^t(x) := \{y : y \in M$ and $g^t(y) = x\}$.

Definition 8. *The t-future of x* $:= F^t(x) := \{y : y \in M$ and $g^t(x) = y\}$.

Definition 9. *The past of x* $:= P(x) := \bigcup_{t \in T - \{0\}} P^t(x)$.

Definition 10. *The future of x* $:= F(x) := \bigcup_{t \in T - \{0\}} F^t(x)$.

Analogous definitions can be given for a set of states $X \subseteq M$. Let 0 be the unity of the time model $L = (T, +)$, and $t \in T - \{0\}$.

Definition 11. *The t-past of X* $:= P^t(X) := \{y : y \in M$ and $\exists x \in X$ such that $g^t(y) = x\}$.

Definition 12. *The t-future of X* $:= F^t(X) := \{y : y \in M$ and $\exists x \in X$ such that $g^t(x) = y\}$.

Definition 13. *The past of X* $:= P(X) := \bigcup_{t \in T - \{0\}} P^t(X)$.

Definition 14. *The future of X* $:= F(X) := \bigcup_{t \in T - \{0\}} F^t(X)$.

5 Constituent Subspaces and the Decomposition Theorem

By Definitions 14 and 3, any subspace X of the state space M of a DS_L on L contains its future $F(X)$. However, X may not contain its past $P(X)$, as the following example shows.

Example 3. Let $DS_L = (Z, (s^t)_{t \in Z^+})$, where Z are the integers, Z^+ the non-negative integers and, for any $t \in Z^+$, s^t is the t-th iteration of the successor function on Z. Then, in the first place, DS_L is a dynamical system on $L = (Z^+, +)$. Furthermore, Z^+ is a subspace of Z; however, $P(Z^+) \not\subseteq Z^+$.

Whenever X also contains $P(X)$, X is called temporally complete.

Definition 15. Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$.

X is a temporally complete subspace of M $:= X$ is a subspace of M and $P(X) \subseteq X$.

Definition 16. Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$, $x_1, x_2 \in M$, and $n \in Z^+ - \{0\}$.

For any $n \in Z^+ - \{0\}$, the ternary relation x_1 *is temporally n-connected with* x_2 *in M* is recursively defined below.

1. x_1 is temporally 1-connected with x_2 in M iff $x_1 \neq x_2$ and $\exists t \in T$ such that $g^t(x_1) = x_2$ or $g^t(x_2) = x_1$;
2. x_1 is temporally $(n+1)$-connected with x_2 in M iff $x_1 \neq x_2$ and $\exists x_3 \in M$ such that $x_3 \neq x_1$, $x_3 \neq x_2$, x_1 is temporally n-connected with x_3 in M and x_3 is temporally 1-connected with x_2 in M.

Lemma 1 (Attraction Lemma) *Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$, and $x^* \in M$.*

If X is a temporally complete subspace of M, $x \in X$, and x is temporally n-connected with x^* in M, then $x^* \in X$.

Proof. By induction on $n \in Z^+ - \{0\}$. The base of the induction follows from 1 of Definition 16 and from Definitions 15, 14, 13, 12, and 11. The step of the induction is a straightforward consequence of 2 of Definition 16. □

Proposition 3 *Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$, and $x^* \in M$. Let X be a temporally complete subspace of M, and $DS_{X_L} = (X, (g^t/X)_{t \in T})$.*

If $x \in X$, and x is temporally n-connected with x^* in M, then x is temporally n-connected with x^* in X.

Proof. By induction on $n \in Z^+ - \{0\}$. The base of the induction follows from the definition of temporal n-connectedness (1 of Definition 16), Proposition 1, and the Attraction Lemma (Lemma 1). The step of the induction is a direct consequence of the definition of temporal n-connectedness (2 of Definition 16). □

Proposition 4 *Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$, $x_1, x_2 \in M$, and $n \in Z^+ - \{0\}$.*

If x_1 is temporally n-connected with x_2 in M, then x_2 is temporally n-connected with x_1 in M.

Proof. By induction on $n \in Z^+ - \{0\}$. The base of the induction immediately follows from 1 of Definition 16. The step of the induction easily follows from 2 of Definition 16. □

Definition 17. Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$, and $x_1, x_2 \in M$.

x_1 is temporally connected with x_2 in M $:= x_1 \neq x_2$ and $\exists n \in Z^+ - \{0\}$ such that x_1 is temporally n-connected with x_2 in M.

Definition 18. Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$.

X is temporally connected in M $:= X \subseteq M$ and, $\forall x_1, x_2 \in X$, if $x_1 \neq x_2$, then x_1 is temporally connected with x_2 in M.

Definition 19. Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$.

X is a constituent subspace of M $:= X$ is a temporally complete subspace of M and X is temporally connected in M.

Definition 20. Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$.

The decomposition of M into its constituent subspaces $:= \mathbf{C}_M := \{X : X$ is a constituent subspace of $M\}$.

In order to prove the *Decomposition Theorem* (Theorem 1 below), we need three more definitions and the *x-Constituent Lemma* (Lemma 2 below).

Definition 21. Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$, $x \in M$, and $n \in Z^+$.

For any $n \in Z^+$, the set $X_x^n \subseteq M$ is recursively defined below.

1. $X_x^0 = \{x\}$;
2. $X_x^{n+1} = P(X_x^n) \cup X_x^n \cup F(X_x^n)$.

Definition 22. Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$, and $x \in M$.

$X_x := \bigcup_{n \in Z^+} X_x^n$; X_x is called the *x-constituent* of M.

Note that, by Definition 22, $\forall z \in X_x, \exists n \in Z^+$ such that $z \in X_x^n$. We then define:

Definition 23. Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$, $x \in M$, and $z \in X_x$.

The level of $z := lev(z) :=$ the minimum n such that $z \in X_x^n$.

Note that, by Definitions 23, 22, and by 2 of Definition 21, $\forall z \in X_x$, $\forall m \in Z^+$, if $m \geq lev(z)$, then $z \in X_x^m$.

Lemma 2 (*x*-Constituent Lemma) *Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$, and $x \in M$.*

X_x is a constituent subspace of M, that is to say,

1. *X_x is a subspace of M;*
2. *X_x is a temporally complete subspace of M;*
3. *X_x is temporally connected in M.*

Proof. We prove the three theses below.

1. **Thesis 1:** X_x is a subspace of M.
2. By the definitions of subspace (Definition 3) and $F(z)$ (Definition 10), it suffice to prove: $\forall z \in X_x, F(z) \subseteq X_x$;
3. suppose: $lev(z) = n \in Z^+$;
4. by 3 and the definition of $lev(z)$ (Definition 23): $z \in X_x^n$;
5. by 4 and by the definitions of $F(z)$ and $F(X_x^n)$ (Definition 10 and 14): $F(z) \subseteq F(X_x^n)$;
6. by 5 and by the definitions of X_x^{n+1} and X_x (2 of Definition 21 and Definition 22): $F(z) \subseteq F(X_x^n) \subseteq X_x^{n+1} \subseteq X_x$.

1. **Thesis 2:** X_x is a temporally complete subspace of M.
2. By the definitions of temporally complete subspace (Definition 15), by thesis 1 and by the definitions of $P(X_x)$ and $P(z)$ (Definitions 13 and 9), it suffice to prove: $\forall z \in X_x, P(z) \subseteq X_x$;
3. suppose: $lev(z) = n \in Z^+$;
4. by 3 and the definition of $lev(z)$ (Definition 23): $z \in X_x^n$;
5. by 4 and by the definitions of $P(z)$ and $P(X_x^n)$ (Definitions 9 and 13): $P(z) \subseteq P(X_x^n)$;
6. by 5 and by the definitions of X_x^{n+1} and X_x (2 of Definition 21 and Definition 22): $P(z) \subseteq P(X_x^n) \subseteq X_x^{n+1} \subseteq X_x$.

1. **Thesis 3:** X_x is temporally connected in M.
2. By the definition of temporally connected subset (Definition 18) and by the definition of the relationship of temporal connectedness (Definition 17), theses 3 is equivalent to:
   for any $x_1, x_2 \in X_x$, if $x_1 \neq x_2$, then $\exists n \in Z^+ - \{0\}$ such that x_1 is temporally n-connected with x_2 in M;
3. we prove 2 by double induction, first on the level of x_2, and second on the level of x_1;

   a. **Base of the induction.** Suppose: $lev(x_1) = 0$ and $lev(x_2) = 0$;
   b. by 3a and by the definitions of level and X_x^0 (Definitions 23 and 1 of Definition 21): $x_1, x_2 \in X_x^0 = \{x\}$;
   c. by 3b: $x_1 = x_2$;
   d. thus, by 3c, 2 is vacuously satisfied.

   a. **Step of the induction on the level of** x_2**.** Suppose: $lev(x_1) = 0$, and 2 holds for any x_2 such that $lev(x_2) \leq m \in Z^+$;
   b. by 3a: $x_1 = x$;
   c. by 3b and 3a, it suffice to prove: for any $z_2 \in X_x$, if $lev(z_2) = m+1$ and $x \neq z_2$, then $\exists k \in Z^+ - \{0\}$ such that x is temporally k-connected with z_2 in M;
   d. suppose: $lev(z_2) = m+1$ and $x \neq z_2$;
   e. by 3d: $z_2 \in X_x^{m+1} = P(X_x^m) \cup X_x^m \cup F(X_x^m)$;
   f. by 3e and 3d: $z_2 \in P(X_x^m)$ or $z_2 \in F(X_x^m)$, but $z_2 \notin X_x^m$;
   g. by 3f: $\exists x_2^* \in X_x^m, \exists t \in T - \{0\}$ such that $g^t(z_2) = x_2^*$ or $g^t(x_2^*) = z_2$;
   h. by 3g and by the definition of level (Definition 23): $lev(x_2^*) \leq m$;
   i. by 3h, 3a, and 3b: if $x \neq x_2^*$, $\exists n \in Z^+ - \{0\}$ such that x is temporally n-connected with x_2^* in M;
   j. case: $x = x_2^*$;
      i. by 3j and 3g: $\exists t \in T - \{0\}$ such that $g^t(z_2) = x$ or $g^t(x) = z_2$;
      ii. by 3(j)i, 3d, and by the definition of temporal n-connectedness (1 of Definition 16): x is temporally 1-connected with z_2 in M; [3c is thus proved for case 3j]
   k. case: $x \neq x_2^*$;
      i. by 3f and 3g: $z_2 \neq x_2^*$;

   ii. by 3k and 3i: $\exists n \in Z^{+} - \{0\}$ such that x is temporally n-connected with x_2^* in M;
   iii. by 3(k)i, 3g, and by the definition of temporal n-connectedness (1 of Definition 16): x_2^* is temporally 1-connected with z_2 in M;
   iv. by 3(k)iii, 3(k)ii, 3d, and by the definition of temporal n-connectedness (2 of Definition 16): x is temporally $(n+1)$-connected with z_2 in M. [3c is thus proved]

a. **Step of the induction on the level of** x_1**.** Suppose: $lev(x_2) = r \in Z^{+}$, and 2 holds for any x_1 such that $lev(x_1) \leq m \in Z^{+}$;
b. by 3a, it suffice to prove: for any $z_1 \in X_x$, if $lev(z_1) = m+1$ and $z_1 \neq x_2$, then $\exists k \in Z^{+} - \{0\}$ such that z_1 is temporally k-connected with x_2 in M;
c. suppose: $lev(z_1) = m+1$ and $z_1 \neq x_2$;
d. by 3c: $z_1 \in X_x^{m+1} = P(X_x^m) \cup X_x^m \cup F(X_x^m)$;
e. by 3d and 3c: $z_1 \in P(X_x^m)$ or $z_1 \in F(X_x^m)$, but $z_1 \notin X_x^m$;
f. by 3e: $\exists x_1^* \in X_x^m, \exists t \in T - \{0\}$ such that $g^t(z_1) = x_1^*$ or $g^t(x_1^*) = z_1$;
g. by 3f and by the definition of level (Definition 23): $lev(x_1^*) \leq m$;
h. by 3g and 3a: if $x_1^* \neq x_2$, $\exists n \in Z^{+} - \{0\}$ such that x_1^* is temporally n-connected with x_2 in M;
i. case: $x_1^* = x_2$;
   i. by 3i and 3f: $\exists t \in T - \{0\}$ such that $g^t(z_1) = x_2$ or $g^t(x_2) = z_1$;
   ii. by 3(i)i, 3c, and by the definition of temporal n-connectedness (1 of Definition 16): z_1 is temporally 1-connected with x_2 in M; [3b is thus proved for case 3i]
j. case: $x_1^* \neq x_2$;
   i. by 3e and 3f: $z_1 \neq x_1^*$;
   ii. by 3j and 3h: $\exists n \in Z^{+} - \{0\}$ such that x_1^* is temporally n-connected with x_2 in M;
   iii. by 3(j)i, 3f, and by the definition of temporal n-connectedness (1 of Definition 16): x_1^* is temporally 1-connected with z_1 in M;
   iv. by 3(j)iii, 3(j)ii, 3c, by the definition of temporal n-connectedness (2 of Definition 16), and by commutativity of temporal n-connectedness (Proposition 4, applied twice): z_1 is temporally $(n+1)$-connected with x_2 in M. [3b is thus proved]

□

Theorem 1 (Decomposition Theorem) *Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$, and $\mathbf{C}_M$ be the decomposition of M into its constituent subspaces.*

$\mathbf{C}_M$ is a partition of M, that is to say,

1. *for any $X, Z \in \mathbf{C}_M$, if $X \neq Z$, then $X \cap Z = \emptyset$;*
2. *$\bigcup_{X \in \mathbf{C}_M} X = M$.*

Proof. We prove the two theses below.

1. **Thesis 1:** for any $X, Z \in \mathbf{C}_M$, if $X \neq Z$, then $X \cap Z = \emptyset$.
2. Suppose for *reductio*: $\exists X, Z \in \mathbf{C}_M : X \neq Z$ and $X \cap Z \neq \emptyset$;
3. we show below:
   a. $Z \subseteq X$;
   b. $X \subseteq Z$;
      i. **Thesis 3a:** $Z \subseteq X$.
      ii. Suppose for *reductio*: $Z \nsubseteq X$;
      iii. by 3(b)ii: $\exists z : z \in Z$ and $z \notin X$;
      iv. as, by 2, $X \cap Z \neq \emptyset$, suppose: $z^* \in X \cap Z$;
      v. by 2: Z is temporally connected in M;
      vi. by 3(b)iv and 3(b)iii: $z \neq z^*$;
      vii. by 3(b)vi, 3(b)v, 3(b)iv, 3(b)iii and by the definitions of temporally connected subset (Definition 18), temporal connectedness (Definition 17) and temporal n-connectedness (Definition 16): there is $n \in Z^+ - 0$ such that z^* is temporally n-connected with z in M;
      viii. by 2: X is a temporally complete subspace of M;
      ix. by 3(b)viii, 3(b)vii, 3(b)iv, and by the Attraction Lemma (Lemma 1): $z \in X$, contrary to 3(b)iii.
      i. **Thesis 3b:** $X \subseteq Z$.
      ii. the proof of thesis 3b is completely analogous to the proof of thesis 3a.
4. by 3a and 3b: $X = Z$, contrary to 2.

1. **Thesis 2:** $\bigcup_{X \in \mathbf{C}_M} X = M$.
2. by the definitions of $\mathbf{C}_M$ (Definition 20), constituent subspace of M (Definition 19), temporally complete subspace of M (Definition 15), and subspace of M (Definition 3): $\bigcup_{X \in \mathbf{C}_M} X \subseteq M$;
3. by the definition of X_x (Definition 22), and by the x-Constituent Lemma (Lemma 2): $\forall x \in M, x \in X_x \in \mathbf{C}_M$;
4. by 3: $M \subseteq \bigcup_{X \in \mathbf{C}_M} X$;
5. by 4 and 2: $\bigcup_{X \in \mathbf{C}_M} X = M$.

□

Given a set X and a property Φ such that X has Φ, X *is minimal with respect to* Φ := there is no Y such that $Y \subset X$ and Y has Φ; X *is maximal with respect to* Φ := there is no Y such that $Y \supset X$ and Y has Φ. It is then immediate to show:

Corollary 1 *Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$, X be a constituent subspace of M, and Φ be the property of being a constituent subspace of M.*

X is both minimal and maximal with respect to Φ.

Proof. The thesis is an immediate consequence of the definitions of minimality, maximality, Definition 20, and thesis 1 of Theorem 1. □

6 Constituent Subsystems

Definition 24. Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$.

CS_{X_L} is a constituent subsystem of DS_L $:= X \in \mathbf{C}_M$ and $CS_{X_L} = (X, (g^t/X)_{t \in T})$.

Proposition 5 *Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$.*

If CS_{X_L} is a constituent subsystem of DS_L, then CS_{X_L} is a subsystem of DS_L.

Proof. Suppose CS_{X_L} is a constituent subsystem of DS_L. Thus, by Definitions 24, 20, 19, and 15, X is a subspace of M. Hence, by Definition 24 and Proposition 2, CS_{X_L} is a subsystem of DS_L. □

Definition 25. Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$.

The decomposition of DS_L into its constituent subsystems $:= (CS_{X_L})_{X \in \mathbf{C}_M}$.

Note that a constituent subsystem of a dynamical system DS_L on a monoid L may very well be complex, as the following example shows.

Example 4. Let us consider the dynamical system of Example 3. Recall that $DS_L = (Z, (s^t)_{t \in Z^+})$ is a dynamical system on $(Z^+, +)$, where s^t is the t-th iteration of the successor function on Z. In fact, the only constituent subsystem of DS_L is DS_L itself; however, $(Z^+, (s^t/Z^+)_{t \in Z^+}) \neq DS_L$ is a subsystem of DS_L. Thus, DS_L is not simple, that is to say, DS_L is complex.

Definition 26. Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$.

DS_L is indecomposable $:= DS_L$ does not have any constituent subsystem but itself.

Definition 27. Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$.

DS_L is decomposable $:= DS_L$ is not indecomposable.

Example 5. The first example below is a complex indecomposable system, while the second one is decomposable.

1. Let $DS1_L = (Z, (s^t)_{t \in Z^+})$, where s^t is the t-th iteration of the successor function on Z. $DS1_L$ is a dynamical system on $L = (Z^+, +)$, and $DS1_L$ is complex and indecomposable (see Example 4).
2. Let $DS2_L = (Z, (id^t)_{t \in Z^+})$, where id^t is the t-th iteration of the identity function on Z. $DS2_L$ is a dynamical system on $L = (Z^+, +)$, and $DS2_L$ is decomposable. (Hence, $DS2_L$ is complex as well.)

Proposition 6 *Let $DS_L = (M, (g^t)_{t \in T})$ be a dynamical system on a monoid $L = (T, +)$.*

If CS_{X_L} is a constituent subsystem of DS_L, then CS_{X_L} is indecomposable.

Proof. Suppose CS_{X_L} is a constituent subsystem of DS_L and, for *reductio*, that CS_{X_L} is decomposable. Let M_1 and M_2 be the state spaces of two distinct constituent subsystems of CS_{X_L}. Then, in the first place, M_1 is a temporally complete subspace of X. Let $x_1 \in M_1$ and $x_2 \in M_2$. As M_1 is a temporally complete subspace of X, by the Attraction Lemma (Lemma 1), thesis 1 of the Decomposition Theorem (Theorem 1), and by the definitions of temporal n-connectedness (Definition 17) and temporal connectedness (Definition 17), x_1 is not temporally connected with x_2 in X. On the other hand, as CS_{X_L} is a constituent subsystem of DS_L, X is temporally connected in M, and thus x_1 is temporally connected with x_2 in M. Furthermore, X is a temporally complete subspace of M; hence, by Proposition 3, x_1 is temporally connected with x_2 in X. Contradiction. □

7 Composition of Disjoint Dynamical Systems

Definition 28. Let $DS_{1_L} = (M_1, (g_1^t)_{t\in T})$ and $DS_{2_L} = (M_2, (g_2^t)_{t\in T})$ be two dynamical systems on the same monoid $L = (T, +)$.

DS_{1_L} and DS_{2_L} are disjoint $:= M_1 \cap M_2 = \emptyset$.

Proposition 7 *Let $DS_L = (M, (g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T, +)$.*

If CS_{X_L} and CS_{Z_L} are distinct constituent subsystems of DS_L, then CS_{X_L} and CS_{Z_L} are disjoint.

Proof. By the definition of constituent subsystem (Definition 24), both X and Z are constituent subspaces of M. Therefore, by Theorem 1, $X \cap Z = \emptyset$; thus, by the definition of disjoint subsystems (Definition 28), CS_{X_L} and CS_{Z_L} are disjoint. □

Any two disjoint dynamical systems on the same monoid L can always be composed into a single dynamical system on L by means of the composition operation $\oplus$, which is defined below.

Definition 29. Let $DS_{1_L} = (M_1, (g_1^t)_{t\in T})$ and $DS_{2_L} = (M_2, (g_2^t)_{t\in T})$ be two disjoint dynamical systems on the same monoid $L = (T, +)$.

The composition of DS_{1_L} and DS_{2_L} $:= DS_{1_L} \oplus DS_{2_L} := (M_1 \cup M_2, (g_1^t \oplus g_2^t)_{t\in T})$ where, $\forall t \in T$, $\forall x \in M_1 \cup M_2$, $g_1^t \oplus g_2^t(x) := g_1^t(x)$, if $x \in M_1$; $g_1^t \oplus g_2^t(x) := g_2^t(x)$, if $x \in M_2$.

Proposition 8 *Let $DS_{1_L} = (M_1, (g_1^t)_{t\in T})$ and $DS_{2_L} = (M_2, (g_2^t)_{t\in T})$ be two disjoint dynamical systems on the same monoid $L = (T, +)$.*

1. *$DS_{1_L} \oplus DS_{2_L}$ is a dynamical system on L;*
2. *$\oplus$ is commutative.*

Proof. Thesis 1 easily follows from the definitions of composition (Definition 29) and dynamical system on a monoid (Definition 2). As for thesis 2, it is an immediate consequence of Definition 29. □

As shown below, the composition operation can be generalized to any family of mutually disjoint dynamical systems on a given monoid L.

Definition 30. Let $L = (T,+)$ be a monoid.

$(DS_{X_L})_{X\in\mathbf{D}}$ is a family indexed by $\mathbf{D}$ *of mutually disjoint dynamical systems on* $L := (DS_{X_L})_{X\in\mathbf{D}} = ((X,(g_X^t)_{t\in T}))_{X\in\mathbf{D}}$ is a family, indexed by $\mathbf{D}$, of dynamical systems on L, and $\mathbf{D}$ is a set of mutually disjoint sets (i.e., for any $X, Z \in \mathbf{D}$, if $X \neq Z$, then $X \cap Z = \emptyset$).

Let $(DS_{X_L})_{X\in\mathbf{D}} = ((X,(g_X^t)_{t\in T}))_{X\in\mathbf{D}}$ be a family, indexed by $\mathbf{D}$, of mutually disjoint dynamical systems on a given monoid $L = (T,+)$. As $\mathbf{D}$ is a set of mutually disjoint sets, $\mathbf{D}$ is a partition of $\bigcup_{X\in\mathbf{D}} X$. Consequently, $\forall x \in \bigcup_{X\in\mathbf{D}} X$, there is exactly one $X \in \mathbf{D}$ such that $x \in X$. Thus, let us define the function χ as follows:

Definition 31. $\chi : \bigcup_{X\in\mathbf{D}} X \to \mathbf{D}$, $\forall x \in \bigcup_{X\in\mathbf{D}} X$, $\chi(x) =$ the $X \in \mathbf{D}$ such that $x \in X$.

We can now generalize the composition operation as follows.

Definition 32. Let $(DS_{X_L})_{X\in\mathbf{D}} = ((X,(g_X^t)_{t\in T}))_{X\in\mathbf{D}}$ be a family, indexed by $\mathbf{D}$, of mutually disjoint dynamical systems on a given monoid $L = (T,+)$.

$$\sum_{X\in\mathbf{D}} DS_{X_L} = (\bigcup_{X\in\mathbf{D}} X, (\sum_{X\in\mathbf{D}} g_X^t)_{t\in T}), \textit{where}$$

$$\forall t \in T, \forall x \in \bigcup_{X\in\mathbf{D}} X, \sum_{X\in\mathbf{D}} g_X^t(x) = g_{\chi(x)}^t(x).$$

Proposition 9 *Let* $(DS_{X_L})_{X\in\mathbf{D}} = ((X,(g_X^t)_{t\in T}))_{X\in\mathbf{D}}$ *be a family, indexed by* $\mathbf{D}$, *of mutually disjoint dynamical systems on a given monoid* $L = (T,+)$.

$\sum_{X\in\mathbf{D}} DS_{X_L}$ *is a dynamical system on* L.

Proof. The thesis follows from the definitions of generalized composition (Definition 32), χ function (Definition 31), and dynamical system on a monoid (Definition 2). □

8 The Composition Theorem

Let $DS_L = (M,(g^t)_{t\in T})$ be a dynamical system on a monoid $L = (T,+)$, and $(CS_{X_L})_{X\in\mathbf{C}_M} = ((X,(g^t/X)_{t\in T}))_{X\in\mathbf{C}_M}$ be the decomposition of DS_L into its constituent subsystems (see Definitions 25 and 24). By Definition 30, $(CS_{X_L})_{X\in\mathbf{C}_M}$ is a family, indexed by $\mathbf{C}_M$, of mutually disjoint dynamical systems on L. Thus, the generalized composition operation applies to $(CS_{X_L})_{X\in\mathbf{C}_M}$, and it holds:

Theorem 2 (Composition Theorem)

$$(\sum_{X\in\mathbf{C}_M} CS_{X_L}) = DS_L$$

Proof. The thesis is a straightforward consequence of the definition of generalized composition (Definition 32), the Decomposition Theorem (Theorem 1), and Proposition 9. □

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