RBF Neural Networks and Radial Fuzzy Systems

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Abstract. RBF neural networks are an efficient tool for acquisition and representation of functional relations reflected in empirical data. The interpretation of acquired knowledge is, however, generally difficult because the knowledge is encoded into values of the parameters of the network. Contrary to neural networks, fuzzy systems allow a more convenient interpretation of the stored knowledge in the form of IF-THEN rules. This paper contributes to the fusion of these two concepts. Namely, we show that a RBF neural network can be interpreted as the radial fuzzy system. The proposed approach is based on the study of conjunctive and implicative representations of the rule base in radial fuzzy systems. We present conditions under which both representations are computationally close and, as the consequence, a reasonable syntactic interpretation of RBF neural networks can be introduced.

Keywords: RBF neural networks · Radial fuzzy systems · Conjunctive and implicative rule bases

1 Introduction

In the world of nature-inspired models of computation, neural and fuzzy computing possess a solid position. It is well known that neural networks primarily focus on acquisition of functional relations from empirical data in the form of a regression model. In contrast, fuzzy modeling aims on a syntactic description of functional relations. This description is provided in terms of IF-THEN rules that have the form of fuzzy logic formulas and incorporate fuzzy sets to model linguistic terms.

Both computational paradigms are backed by the well-developed theories and their own techniques for grasping information provided by a real-world environment. For a long time there have been attempts for creative combination of both (or even more) modeling approaches. This movement has become known under the term of soft-computing [10]. Our contribution follows the soft-computing line.

The neural networks are extremely good devices for processing data in terms of non-linear regression. They develop (learn) the regression function that for given inputs computes such the outputs that minimize the selected distance

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to the desired outputs (a supervised learning task on a set of training data). In RBF neural networks, the regression function is implemented in the form of a weighted sum of certain selected points from the range of the regression function. The weights correspond to activation levels of radial computational units that represent the neurons of the network.

Fuzzy systems deal with a linguistic description of functional relations and mathematize it. Mathematization is provided by means of translating linguistic terms into fuzzy sets and combining them into IF-THEN rules. A group of IF-THEN rules then determines the rule base of the fuzzy system.

The rule base is the carrier of the knowledge stored in the fuzzy system. There are two basic mathematical representations of the rule base - the *conjunctive* and the *implicative* one. Under the conjunctive representation, the knowledge representation is data-driven. In this case, the rule base can be seen as the list of prototypical examples taken from the relation the fuzzy system accommodates. In contrast, under the implicative representation the rule base is seen as the set of conditions expressed syntactically in terms of fuzzy logic formulas.

Conjunctively represented radial fuzzy systems can be shown to be computationally equivalent to RBF neural networks and vice versa. Hence both devices can be transformed to each other. A conjunctive rule base can be formally represented in the implicative way. Under certain assumptions, it can be shown that this representation is computationally close to the original conjunctive one. Hence one can propose to interpreted the given RBF neural network as the conjunctive radial fuzzy system and then interpret its rule base implicatively. As a result, one gets the syntactic representation of the original RBF neural network in terms of the implicative radial fuzzy system.

The goal of this paper is to mathematically develop the above transformation idea and to study conditions under which conjunctive and implicative representation are computationally equivalent so that the proposed formal translation is reasonable.

The rest of the paper is organized as follows. The next section is the review section. The third section deals with the computational aspects of conjunctive and implicative representations of the rule base in the radial fuzzy systems. The fourth section concludes the paper.

2 RBF Neural Networks and Radial Fuzzy Systems

In this section we review the very basics of RBF neural networks, radial fuzzy systems and fuzzy logic in order to the paper be self-contained. The review is based on the classical textbooks [5], [7], [9], [4] and [6].

2.1 RBF Networks

The concept of the radial basis function neural network is very well known [1], [5]. In this paper, we consider the RBF network in the standard three-layered MISO (multiple-input single-output) configuration. That is, the

RBF network represents a function from $\mathbb{R}^n$ to $\mathbb{R}$ space, $n \in \mathbb{N}$. Let the hidden layer consist of $m \in \mathbb{N}$ processing units - neurons. The neurons are mathematically represented by radial functions of the form

$$\phi_j(\boldsymbol{x}) = act(||\boldsymbol{x} - \boldsymbol{a}_j||), \quad j = 1, \ldots, m. \tag{1}$$

In the formula, $act : \mathbb{R} \to \mathbb{R}$ is the so-called activation function that is considered to be be continuous, non-increasing with $act(0) = 1$ and $\lim_{z\to\infty} act(z) = 0$. The point $\boldsymbol{a}_j \in \mathbb{R}^n$ is the center of the radial function ϕ_j. Clearly, ϕ_j corresponds to a non-increasing function of the distance of the argument $\boldsymbol{x}$ from the central point $\boldsymbol{a}_j$. The distance is measured by a norm $||\cdot||$ in the $\mathbb{R}^n$ space.

The function RBF : $\mathbb{R}^n \to \mathbb{R}$ implemented by the RBF network has the form

$$\mathrm{RBF}(\boldsymbol{x}) = \sum_{j=1}^{m} w_j \cdot \phi_j(\boldsymbol{x}) = \sum_{j=1}^{m} w_j \cdot act(||\boldsymbol{x} - \boldsymbol{a}_j||) \tag{2}$$

where w_j, $j = 1, \ldots, m$ is the set of network's weights.

Concerning setting of parameters of the RBF network, namely, the number m of neurons, their central points $\boldsymbol{a}_j$ and weights w_j, $j = 1, \ldots, m$, this task is split into the structure and parameter learning subtasks. The first one is typically carried out by using a sort of a clustering algorithm. The second one corresponds to an optimization task. In our case, the latter is the task of linear regression. In a multi-layered version, the celebrated back propagation algorithm is usually employed here.

2.2 Basics of Fuzzy Systems

A fuzzy system in the MISO configuration computes a function from some input space $X \subseteq \mathbb{R}^n$ to some output space $Y \subseteq \mathbb{R}$. The special about fuzzy systems is how this function is implemented. Canonically, this is made by the combination of the four building blocks called the *fuzzifier*, the *rule base*, the *inference engine* and the *defuziffier* [7], [9].

The fuzzifier transforms points of the input space into fuzzy sets specified on this space. In this paper, we will consider the singleton fuzzifier that is the most common in applications. The singleton fuzzifier associates a given input $\boldsymbol{x}^* \in X$ with the fuzzy set A_{fuzz} such that $A_{fuzz}(\boldsymbol{x}) = 1$ for $\boldsymbol{x} = \boldsymbol{x}^*$, and $A_{fuzz}(\boldsymbol{x}) = 0$ otherwise.

The rule base of the fuzzy system is given by the set of $m \in \mathbb{N}$ IF-THEN rules. The j-th rule represents the fuzzy relation $R_j(\boldsymbol{x}, y)$ on $X \times Y$ space specified as

$$R_j(\boldsymbol{x}, y) = A_{j1}(x_1) \star \cdots \star A_{jn}(x_n) \rhd B_j(y) = A_j(\boldsymbol{x}) \rhd B_j(y). \tag{3}$$

In the formula, A_j and B_j fuzzy sets correspond to the antecedent and consequent fuzzy sets of the rule, respectively. The antecedent A_j is composed from the individual fuzzy sets A_{ji}, $i = 1, \ldots, n$ using a fuzzy conjunction which is implemented by the t-norm $\star$, i.e., we have $A_j(\boldsymbol{x}) = A_{j1}(x_1) \star \cdots \star A_{jn}(x_n)$ for the antecedent. Clearly, $\boldsymbol{x} = (x_1, \ldots, x_n)$. The interpretation of the $\rhd$ symbol depends on the representation of the whole rule base.

Under the conjunctive representation, $\rhd$ corresponds to the fuzzy conjunction and the relation that implements the j-th rule reads as $R_j(\boldsymbol{x}, y) = A_j(\boldsymbol{x}) \star B_j(y)$. The whole rule base is then determined by a disjunctive combination of individual fuzzy relations

$$RB_{conj}(\boldsymbol{x}, y) = \bigvee_{j=1}^{m} A_j(\boldsymbol{x}) \star B_j(y). \tag{4}$$

In (4), maximum is the typical choice for implementing the fuzzy disjunction $\bigvee$. In the conjunctive case, the rule base RB_{conj} can be seen as the list of prototypic points from the relation on the input-output space that the fuzzy system implements.

Under the implicative representation, $\rhd$ corresponds to the residuated fuzzy implication $\rightarrow$ devised from the t-norm $\star$ by the process of residuation [4], [6]: $u \rightarrow v = \sup_z \{z \,|\, z \star u \le v\,; u, v, z \in [0,1]\}$. The j-th rule is then implemented as the fuzzy relation $R_j(\boldsymbol{x}, y) = A_j(\boldsymbol{x}) \rightarrow B_j(y)$; and the whole rule base is combined conjunctively

$$RB_{impl}(\boldsymbol{x}, y) = \bigwedge_{j=1}^{m} A_j(\boldsymbol{x}) \rightarrow B_j(y). \tag{5}$$

In (5), the minimum operation is typically used to represent the fuzzy conjunction $\bigwedge$. In the implicative case, the rule base can be seen as the set of the conditions that are simultaneously imposed on the points of the relation the fuzzy system implements.

2.3 Radial Fuzzy Systems

Radial fuzzy systems (RFSs) employ radial functions for representing membership functions of fuzzy sets in their rules [2], [3]. Moreover, the RFSs exhibit the so-called *radial property*. This is the shape preservation property in antecedents of IF-THEN rules. The property makes the computational model of radial systems more tractable and the antecedents of rules correspond to multivariate radial functions. Let us be more specific.

Definition 1. *A fuzzy system is called radial if*:

(i) *There exists a continuous function* $act : [0, +\infty) \rightarrow [0, 1]$, $act(0) = 1$ *such that:* (a) *either there exists* $z_0 \in (0, +\infty)$ *such that act is strictly decreasing on* $[0, z_0]$ *and* $act(z) = 0$ *for* $z \in [z_0, +\infty)$ *or* (b) *act is strictly decreasing on* $[0, +\infty)$ *and* $\lim_{z \to +\infty} act(z) = 0$.

(ii) *Fuzzy sets in the antecedent and consequent of the j-th rule are specified as*

$$A_{ji}(x_i) = act\left(\left|\frac{x_i - a_{ji}}{b_i}\right|\right), \quad B_j(y) = act\left(\frac{\max\{0, |y - c_j| - s\}}{d}\right) \tag{6}$$

where $n, m \in \mathbb{N}$; $i = 1, \ldots, n$; $j = 1, \ldots, m$; $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x} = (x_1, \ldots, x_n)$; $\boldsymbol{a}_j \in \mathbb{R}^n$, $\boldsymbol{a}_j = (a_{j1}, \ldots, a_{jn})$; $\boldsymbol{b} = (b_1, \ldots, b_n)$, $b_i > 0$, $c_j \in \mathbb{R}$; $d_j > 0$, $s > 0$.
(iii) *For each* $\boldsymbol{x} \in \mathbb{R}^n$ *the radial property holds, i.e.,*

$$A_j(\boldsymbol{x}) = A_{j1}(x_1) \star \cdots \star A_{jn}(x_n) = act(\, ||\boldsymbol{x} - \boldsymbol{a}_j||_{p,\boldsymbol{b}}), \tag{7}$$

where $||\cdot||_{\boldsymbol{b}}$ *is the scaled* ℓ_p *norm for some* $p \geq 1$, *i.e.,* $||\boldsymbol{u}||_{p,\boldsymbol{b}} = (\sum_i |u_i/b_i|^p)^{1/p}$.

Let us shortly comment on the definition. The first two conditions refer to the specification of the individual fuzzy sets. In fact, they require that one-dimensional fuzzy sets are specified as radial functions with the central points a_{ji}, c_j and scaling parameters b_i and d. The parameter s is the shifting parameter that makes the shape of the consequent fuzzy sets trapezoid-like. In Fig. 1 there are presented examples of radial fuzzy sets.

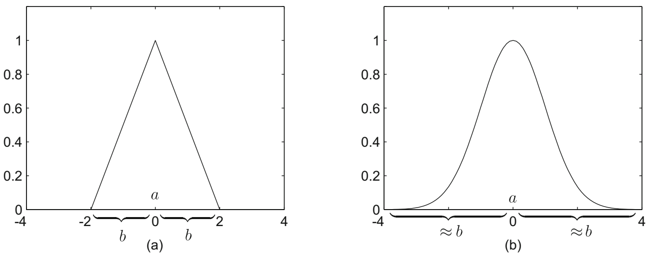

Fig. 1. Examples of radial fuzzy sets; (a) triangular $A(x) = \max\{0, 1 - |x - a|/b\}$ and (b) Gaussian $A(x) = \exp[-((x - a)/b)^2]$ fuzzy set.

The third condition is the radial property. The property requires the shape of individual fuzzy sets, which is determined by the *act* function, to be retained under the combination by the t-norm $\star$. From (7) we see that individual fuzzy sets are transformed into the multi-dimensional one, and this fuzzy set has the same shape as the individual fuzzy sets.

Table 1. Examples of t-norms and corresponding *act* functions.

t-norm	$act(z)$	$A_j(\boldsymbol{x})$
$x \star y = \max\{0, x + y - 1\}$	$\max\{0, 1 - z\}$	$\max(0, 1 - ||\boldsymbol{x} - \boldsymbol{a}_j||_{1,\boldsymbol{b}})$
$x \star y = x \cdot y$	$\exp(-z^2)$	$\exp(-||\boldsymbol{x} - \boldsymbol{a}_j||^2_{2,\boldsymbol{b}})$
$x \star y = \exp\left(-[(-\ln(x))^{1/2} + (-\ln(y))^{1/2})]\right)^2$	$\exp(-z^2)$	$\exp(-||\boldsymbol{x} - \boldsymbol{a}_j||^2_{1,\boldsymbol{b}})$

Remark that the radial property is not trivial. That is, the t-norm and the shape has to be matched somehow in order to the radial property holds. In [2], there is shown that in the case of continuous Archimedean t-norms the matching is provided by setting $act(z) = t^{(-1)}(qz^p)$ where $t^{(-1)}$ is the pseudo-inverse of the additive generator of the t-norm and $q > 0, p > 1$ are parameters. Table 1 presents some allowed pairs connected with the Łukasiewicz, product and Aczél-Alsina t-norm with $\lambda = 1/2$, respectively [6].

2.4 Computational Model of Radial Fuzzy Systems

The specification of the computational model of a fuzzy system depends on the implementation of the inference engine and the chosen deffuzification method.

The inference engine performs a sort of projection of the fuzzy set yielded by the fuzzifier through the fuzzy relation represented by the rule base. The projection can be seen as the composition of the fuzzified input with the knowledge stored in the fuzzy system. The most standard inference engine employed in fuzzy computing is the CRI inference engine referring to the compositional rule of inference [7], [9]. Mathematically, when the singleton fuzzifier is employed, it yields for an input $\boldsymbol{x}^*$ the fuzzy set B specified as

$$B(y) = \sup_{\boldsymbol{x}}\{\boldsymbol{x} \in X | A_{fuzz}(\boldsymbol{x}) \star RB(\boldsymbol{x}, y)\} = RB(\boldsymbol{x}^*, y). \tag{8}$$

The defuzzier transforms the fuzzy set B into the point y^* from the output space Y. There are plenty of methods of defuzzification present [7], [9]; and their use depends on the representation of the rule base.

In the case of the conjunctive representation when $B(y) = \bigvee_j A(\boldsymbol{x}^*) \star B_j(y)$, the method of centroids is popular: $y^* = \left(\sum_{j=1}^m A_j(\boldsymbol{x}^*) \cdot y^*_{B_j}\right) / \left(\sum_{j=1}^m A_j(\boldsymbol{x}^*)\right)$. In radial fuzzy systems the centroids $y^*_{B_j}$ of B_j sets correspond to their central points, i.e., $y^*_{B_j} = c_j$, $j = 1 \ldots, m$.

The problem with this method is that the output is not defined for denominator being 0 nor is generally continuous when we set $y^* = 0$ for the zero denominator. That is why the formula of centroids method is further simplified into the form (in fact, this corresponds to a variant of the Takagi-Sugeno fuzzy system [8])

$$y^* = \sum_{j=1}^{m} A_j(\boldsymbol{x}^*) \cdot c_j. \tag{9}$$

In the case of the implicative representation, the fuzzy set B yielded by the CRI inference engine reads as $B(y) = \bigwedge_j A_j(\boldsymbol{x}^*) \to B_j(y)$. Since for any residuated implication one has $u \to v = 1$ iff $u \leq v$, we see that $A_j(\boldsymbol{x}^*) \to B_j(y)$ are modified B_j sets such that $B'_j(y) = A_j(\boldsymbol{x}^*) \to B_j(y) = 1$ iff $A_j(\boldsymbol{x}^*) \leq B_j(y)$. That is, B'_j sets are normal with the kernels $I_j(\boldsymbol{x}^*) = \{y \,|\, A_j(\boldsymbol{x}^*) \leq B_j(y)\}$.

Any fuzzy conjunction of B'_j sets then yields a fuzzy set that is also normal with the kernel $I(\boldsymbol{x}^*) = \bigcap_j I_j(\boldsymbol{x}^*)$, under the assumption that $\bigcap_j I_j(\boldsymbol{x}^*) \neq \emptyset$. If this is true for any input $\boldsymbol{x}^* \in X$, then the implicative fuzzy system is called *coherent*. If the system under study is coherent, then it is natural to take as the defuzzified point the middle point of $I(\boldsymbol{x}^*)$. In the case of the radial systems it can be shown that $I(\boldsymbol{x}^*)$ corresponds to a closed interval. Hence

$$y^* = \frac{L(I(\boldsymbol{x}^*)) + R(I(\boldsymbol{x}^*))}{2} \tag{10}$$

where $L(I(\boldsymbol{x}^*)$ and $R(\boldsymbol{x}^*))$ are the left and right limit points of $I(\boldsymbol{x}^*)$, respectively.

3 Combination of RBF Networks and Radial Fuzzy Systems

In this section we propose an approach how to interpret the RBF neural network in terms of the implicative radial fuzzy system.

We start with the computational equivalence of RBF networks and radial conjunctive fuzzy systems. The equivalence is based on the comparison of computational models (2) and (9). These models coincide if the number of neurons equals the number of IF-THEN rules and if $\phi_j = A_j$ and $w_j = c_j$ for all $j = 1, \ldots, m$.

Concerning the first equality $\phi_j(\boldsymbol{x}) = A_j(\boldsymbol{x})$, $\boldsymbol{x} \in X = \mathbb{R}^n$ it rests on the radial property of the radial fuzzy systems. It is clear that the representation of neurons and the employed t-norm in the fuzzy system must be matched in order to the radial property holds. In the case of the second equality $w_j = c_j$, we see that it requires the central points of consequents to coincide with the network's weights. The width and shift parameters d and s, respectively, do not affect the computational equivalence between (2) and (9). Hence they can be set freely, however, in the case of the d parameter we will require that the fuzzy system is *strictly coherent*.

Definition 2. *A radial fuzzy system is said to be strictly coherent if for any pair of rules $j, k \in \{1, \ldots, m\}$ the following holds*

$$d \cdot ||\boldsymbol{a}_j - \boldsymbol{a}_k||_{p,\boldsymbol{b}} \geq |c_j - c_k|. \tag{11}$$

Based on the above definition, we set up d in such a way that the built conjunctive fuzzy system is strictly coherent. Hence we search for such the minimal $d = d_{coh}$ that (11) holds. Formally, we have

$$d_{coh} = \min_d \{d \cdot ||\boldsymbol{a}_j - \boldsymbol{a}_k||_{p,\boldsymbol{b}} \geq |c_j - c_k|, \;\; \forall j, k \in \{1, \ldots, m\}\}. \tag{12}$$

This is a straightforward optimization task. Solving this problem we set $d = d_{coh}$ which finalizes the specification of the parameters in the conjunctive radial fuzzy system. The specification of d does not impact the computational equivalence between (2) and (9), however, this specification is important from the implicative representation point of view, because if the implicative system is strictly coherent then it is also coherent in the sense of Section 2.4.

Let us consider conjunctive radial system with the set of its parameters. Let $\rightarrow$ be the residuated implication derived from the t-norm $\star$ which is employed in the conjunctive fuzzy system. We ask what will happen if we interpret the rule base implicatively with the values of the parameters retained.

Clearly, if the conjunctive and implicative representations are computationally close then the interpretation chain: the RBF network $\Rightarrow$ the conjunctive RFS $\Rightarrow$ the implicative RFS makes sense. In the following two lemmas we will show that both representations are computationally close to each other under certain assumptions.

Lemma 1. *Let $\star$ be a t-norm and $\rightarrow$ its residuated implication. Let the rule base of a radial fuzzy system consists of $m \in \mathbb{N}$ IF-THEN rules built up on the basis of antecedents A_j and consequents B_j, $j = 1, \ldots, m$. Let the radial fuzzy system be strictly coherent and the norm of the radial property be the scaled ℓ_1 norm, i.e., $p = 1$, then for the conjunctive and implicative representations of the rule base one has*

$$RB_{conj}(\boldsymbol{x}, y) \leq RB_{impl}(\boldsymbol{x}, y) \tag{13}$$

for any $\boldsymbol{x} \in \mathbb{R}^n$ and $y \in \mathbb{R}$.

Proof. We start by proving the inequality

$$A_j(\boldsymbol{x}) \star A_k(\boldsymbol{x}) \star B_j(y) \leq B_k(y) \tag{14}$$

for $j, k \in \{1, \ldots, m\}$ and $\boldsymbol{x} \in \mathbb{R}^n$, $y \in \mathbb{R}$. Indeed, by the triangle inequality, non-increasing character of the *act* function and the radial property for the ℓ_1 scaled norm $||\cdot||_{1,\boldsymbol{b}}$, we have the following chain:

$$\begin{aligned}
||\boldsymbol{x} - \boldsymbol{a}_j||_{1,\boldsymbol{b}} + ||\boldsymbol{x} - \boldsymbol{a}_k||_{1,\boldsymbol{b}} &\geq ||\boldsymbol{a}_j - \boldsymbol{a}_k||_{1,\boldsymbol{b}}, \\
act(||\boldsymbol{x} - \boldsymbol{a}_j||_{1,\boldsymbol{b}} + ||\boldsymbol{x} - \boldsymbol{a}_k||_{1,\boldsymbol{b}}) &\leq act(||\boldsymbol{a}_j - \boldsymbol{a}_k||_{1,\boldsymbol{b}}), \\
act(||\boldsymbol{x} - \boldsymbol{a}_j||_{1,\boldsymbol{b}}) \star act(||\boldsymbol{x} - \boldsymbol{a}_k||_{1,\boldsymbol{b}}) &\leq act(||\boldsymbol{a}_j - \boldsymbol{a}_k||_{1,\boldsymbol{b}}), \\
A_j(\boldsymbol{x}) \star A_k(\boldsymbol{x}) &\leq act(||\boldsymbol{a}_j - \boldsymbol{a}_k||_{1,\boldsymbol{b}}).
\end{aligned} \tag{15}$$

The triangle inequality of the second kind reads $|c_j - c_k| \geq \big|\,|y - c_j| - |y - c_k|\,\big|$. Therefore due to the strict coherence we have

$$\begin{aligned}
d \cdot ||\boldsymbol{a}_j - \boldsymbol{a}_k||_{1,\boldsymbol{b}} &\geq \big|\,|y - c_j| - |y - c_k|\,\big|, \\
d \cdot ||\boldsymbol{a}_j - \boldsymbol{a}_k||_{1,\boldsymbol{b}} + |y - c_j| - s &\geq |y - c_k| - s, \\
d \cdot ||\boldsymbol{a}_j - \boldsymbol{a}_k||_{1,\boldsymbol{b}} + \max\{0, |y - c_j| - s\} &\geq \max\{0, |y - c_k| - s\}, \\
act(||\boldsymbol{a}_j - \boldsymbol{a}_k||_{1,\boldsymbol{b}} + \max\{0, |y - c_j| - s\}/d) &\leq act(\max\{0, |y - c_k| - s\}/d).
\end{aligned}$$

Due to the radial property for the the ℓ_1 scaled norm and (15), the left-hand side of the last inequality is greater than

$$act(||\boldsymbol{x} - \boldsymbol{a}_j||_{1,\boldsymbol{b}}) \star act(||\boldsymbol{x} - \boldsymbol{a}_k||_{1,\boldsymbol{b}}) \star act(\max\{0, |y - c_j| - s\}/d)$$

and implies the inequality (14).

For any t-norm $\star$ and its residuated implication $\rightarrow$ one has $u \star v \rightarrow w = u \rightarrow (v \rightarrow w)$ and $u \rightarrow v = 1$ iff $u \leq v$ for $u, v, w \in [0, 1]$, see [4]. Hence we can update (14) as follows

$$\begin{aligned}
A_j(\boldsymbol{x}) \star A_k(\boldsymbol{x}) \star B_j(y) &\leq B_k(y), \\
A_j(\boldsymbol{x}) \star B_j(y) &\leq A_k(\boldsymbol{x}) \rightarrow B_k(y), \\
\max_{j=1}^m\{A_j(\boldsymbol{x}) \star B_j(y)\} &\leq A_k(\boldsymbol{x}) \rightarrow B_k(y), \\
\max_{j=1}^m\{A_j(\boldsymbol{x}) \star B_j(y)\} &\leq \min_{k=1}^m\{A_k(\boldsymbol{x}) \rightarrow B_k(y)\}, \\
RB_{conj}(\boldsymbol{x}, y) &\leq RB_{impl}(\boldsymbol{x}, y).
\end{aligned}$$

This concludes the proof. □

There is the natural question of how computationally close are both representations. In terms of the fuzzy logic, the closeness relation is interpreted as the fuzzy equivalence relation [4]. The fuzzy equivalence is introduced by the formula $x \equiv y = (x \to y) \star (y \to x)$ for $x, y \in [0,1]$. For any fuzzy equivalence one has $x \equiv y = 1$ if $x = y$. If $x \neq y$, then the measure of difference is given either as $x \to y$, if $x \geq y$ or by $y \to x$ if $x \leq y$. For example for Łukasiewicz t-norm it reads as $x \equiv y = 1 - |x - y|$. The next lemma is due to Hájek [4].

Lemma 2. *Let $\star$ be a t-norm and $\to$ its residuated implication. Let the rule base of a radial fuzzy system consists of $m \in \mathbb{N}$ IF-THEN rules built up on the basis of antecedents A_j and consequents B_j, $j = 1, \ldots, m$. Then for the conjunctive and implicative representations of the rule base it holds*

$$\min_{j=1}^{m}\{A_j(\boldsymbol{x}) \star A_j(\boldsymbol{x})\} \to (RB_{impl}(\boldsymbol{x}, y) \leq RB_{conj}(\boldsymbol{x}, y))$$

for any $\boldsymbol{x} \in \mathbb{R}$ and $y \in \mathbb{R}$.

Proof. Remind once again that $u \star v \to w = u \to (v \to w)$. Further, it can be shown that $(u \to v) \to [(u \star w) \to (v \star w)] = 1$ for any $u, v, w \in [0,1]$, see [4]. Hence one has the following chain:

$$\begin{aligned}
&(A_j(\boldsymbol{x}) \to B_j(y)) \to [(A_j(\boldsymbol{x}) \star A_j(\boldsymbol{x})) \to (A_j(\boldsymbol{x}) \star B_j(y))],\\
&(A_j(\boldsymbol{x}) \star A_j(\boldsymbol{x})) \to [(A_j(\boldsymbol{x}) \to B_j(y)) \to (A_j(\boldsymbol{x}) \star B_j(y))],\\
&(A_j(\boldsymbol{x}) \star A_j(\boldsymbol{x})) \to [\min_{j=1}^{m}\{A_j(\boldsymbol{x}) \to B_j(y)\} \to (A_j(\boldsymbol{x}) \star B_j(y))],\\
&(A_j(\boldsymbol{x}) \star A_j(\boldsymbol{x})) \to [\min_{j=1}^{m}\{A_j(\boldsymbol{x}) \to B_j(y)\} \to \max_{j=1}^{m}\{A_j(\boldsymbol{x}) \to B_j(y)\}],\\
&\min_j\{A_j(\boldsymbol{x}) \star A_j(\boldsymbol{x})\} \to [RB_{impl}(\boldsymbol{x}, y) \to RB_{conj}(\boldsymbol{x}, y)].
\end{aligned}$$

This concludes the proof. □

The combination of both lemmas gives us the final answer on how computationally close are both representation of the rule base in the radial fuzzy systems. In order to state the theorem, we will denote as the *degree of equivalence* the number

$$\mathrm{DOE} = \inf_{\boldsymbol{x} \in X}\{\min_j\{A_j(\boldsymbol{x}) \star A_j(\boldsymbol{x})\}\}.$$

Theorem 1. *Let $\star$ be a t-norm and $\to$ its residuated implication. Let the rule base of a radial fuzzy system consists of $m \in \mathbb{N}$ IF-THEN rules built up on the basis of antecedents A_j and consequents B_j, $j = 1, \ldots, m$. Let the radial fuzzy system be strictly coherent and the norm of the radial property be the scaled ℓ_1 norm, i.e., $p = 1$, then for the conjunctive and implicative representations of the rule base one has*

$$\mathrm{DOE} \leq (RB_{impl}(\boldsymbol{x}, y) \equiv RB_{conj}(\boldsymbol{x}, y)).$$

Proof. By the above lemmas and the specification of $\equiv$ relation we get

$$\min_j\{A_j(\boldsymbol{x}) \star A_j(\boldsymbol{x})\} \to [RB_{impl}(\boldsymbol{x}, y) \equiv RB_{conj}(\boldsymbol{x}, y)].$$

Therefore also $\mathrm{DOE} \leq [RB_{impl}(\boldsymbol{x}, y) \equiv RB_{conj}(\boldsymbol{x}, y)]$ because $u \to v = 1$ iff $u \leq v$, $u, v \in [0,1]$. □

Theorem 1 tells us that if the scaled ℓ_p norm occurring in the radial property is the scaled ℓ_1 norm and d is set in such a way that the fuzzy system is strictly coherent, then DOE $\leq RB_{conj}(\boldsymbol{x}, y) \equiv RB_{impl}(\boldsymbol{x}, y)$ for any $\boldsymbol{x} \in \mathbb{R}^n$ and $y \in \mathbb{R}$. Therefore for any input $\boldsymbol{x}^*$ one has also DOE $\leq RB_{conj}(\boldsymbol{x}^*, y) \equiv RB_{impl}(\boldsymbol{x}^*, y)$.

4 Conclusions

The main result presented in the paper is the characterization of the mutual relation between conjunctive and implicative representations of the rule base in the radial fuzzy systems. We have shown that we can measure or control the closeness of computational equivalence by means of the degree of equivalence.

The proposed use of the result follows the line of fusion of concepts of the RBF neural networks and radial fuzzy systems. In fact, we have shown how the RBF neural network can be interpreted in syntactic way in terms of the radial implicative fuzzy system. The process of transforming the RBF neural networks can be also understood as the method for establishing the radial fuzzy systems on the basis of empirical data. In this case, the well-developed machinery for learning the RBF neural networks can be exploited.

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