Two Frameworks for Mathematical Reasoning at Preschool Level

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Abstract In this chapter, young children's mathematical reasoning is explored using two different frameworks. Two cases of reasoning are analysed and discussed in order to illustrate how the mathematical foundation is used in young children's arguments and choices that they make when solving mathematical problems. The first framework focuses on arguments and warrants and is used to analyse individual reasoning. The second identifies strategy choices and categorises different types of reasoning that are developed in groups. In both frameworks, the mathematical foundation is central.

Introduction

I'm sitting on the train. My seat is one of four sharing a table. The other three seats are occupied by a mum and two small children, a boy and a girl. The boy, who is the oldest of the two, turns to me and says: 'I'm four!'. I smile and ask the little girl how old she is. 'I'm four!', she replies. The boy laughs and says: 'No, she is two!' and shows me two fingers to illustrate. 'Ok. So you are four and your sister is two. How much older are you than your sister?', I ask the boy. He looks at me a bit puzzled. Then he holds up four fingers on his left hand and two fingers on his right and places the hands opposite each other, so he can compare the number of fingers. I can see him nodding when he is counting the fingers on his left hand which do not match a finger on the right hand. One nod. One more nod. 'Two!', he says with a smile. The mother looks at me and says, 'I have never seen him doing that before'. The boy turns to me again: 'You are a big girl, aren't you?'

Research has shown that young children are more capable of developing mathematical concepts and processes than previously thought (Clements and Sarama 2007; Mulligan and Vergnaud 2006). This is further emphasised by studies focusing on general mathematical processes such as problem solving, argumentation and justification (Perry and Dockett 2007), early algebraic reasoning (Papic et al. 2011), and

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modelling and statistical reasoning (English 2012). Recent Swedish research shows how young children can use different mathematical competencies, alternatively labelled processes (NCTM 2000), in their mathematical reasoning (Säfström 2013). A mathematical competence is defined as:

> the ability to understand, judge, do, and use mathematics in a variety of intra- and extra-mathematical contexts and situations in which mathematics plays or could play a role. (Niss 2003, p. 7)

In Säfström's (2013) research the children questioned other children's arguments and justified their own. One of the conclusions from this research is that mathematical reasoning is something that children can use from an early age; other skills do not have to be developed before this competence can be used. We also know that mathematical reasoning predicts mathematical achievement later in school (Nunes et al. 2012).

Mathematical reasoning is also a social activity. The negotiation carried out by children described by Säfström (2013) was a central part of the interaction structure when creating collective mathematical reasoning (Voigt 1994). As most preschools in Sweden focus on social skills such as learning to cooperate and share, mathematics and mathematical problem solving could provide opportunities for learning mathematical reasoning as well as learning constructive social learning.

Although the development of process and sense making of mathematical concepts can take place without explicit guidance (McMullen et al. 2013), there seems also to be evidence that children do not develop these competencies without support (Bobis et al. 2005, 2008). Children need to be part of situations which provide opportunities to learn (Hiebert 2003). Guidance from an adult is more likely to support children to gain more extensive and explicitly investigated mathematical ideas (Björklund 2008; Lee and Ginsburg 2009; van Oers 1996). For example, the boy on the train faced a mathematical problem but was supported in solving it by the questions asked by a guide (a 'big girl'). Nevertheless, the solution strategy was a product of his own creativity.

Although there is a growing body of research about preschool children's mathematical reasoning, few studies incorporate theories about mathematical reasoning and theoretical concepts are seldom discussed explicitly. In this chapter, I explore young children's mathematical reasoning with two theoretical frameworks. The two frameworks highlight different aspects of reasoning in relationship to individual and collective reasoning.

Mathematical Reasoning

Mueller (2009) suggested 'mathematical understanding and thus mathematical knowledge depend upon reasoning' (p. 138). Therefore, it is not surprising to find mathematical reasoning included in several frameworks that describe teaching/learning pathways, such as curricula (e.g. NCTM 2000; Niss 2003) including the

Swedish curriculum from preschool up to upper secondary school level (National Agency for Education 2011a–c). One of the goals that Swedish preschools should aim for is that children 'develop their mathematical skill in putting forward and following reasoning' (National Agency for Education 2011a, p. 10). This appears to be a challenging goal, given that Swedish children struggle with mathematical reasoning and problem solving later, as documented in international tests such as TIMSS (National Agency for Education 2012).

Despite this central role, few theoretical frameworks characterise reasoning in detail (Lithner 2008; Yackel and Hanna 2003). For instance, Skemp (1978) described two different kinds of understandings, instrumental and relational understanding, that is, the base for student's reasoning, but gave no further specification. When Wyndhamn and Säljö (1997) studied children's mathematical reasoning when solving mathematical problems, they focused on the content and rules in students' reasoning. However, no definition of reasoning was provided. At the very least, in a framework about mathematical reasoning, it could be expected that the mathematical content that is the basis for the reasoning should be explicit.

One of the few frameworks providing a definition of reasoning was that of Ball and Bass (2003). They described mathematical understanding as founded on mathematical reasoning. Reasoning is comprised of a 'set of practises and norms that are collective not merely individual or idiosyncratic, and rooted in the discipline' (Ball and Bass 2003, p. 29). Such a framework is helpful when distinguishing between the body of public knowledge and language. However, Ball and Bass (2003) also seem to imply that mathematical reasoning is rooted in logic.

In order to study reasoning based on subjective, rather than mathematical, knowledge including arguments such as 'I do this because my teacher says so', a different approach would be needed. Therefore, the question arises, if mathematical reasoning is thought of as logical thinking, should preschool children be expected to produce such thinking? Yet Ball and Bass (2003) concluded that 'mathematical reasoning is no more than a basic skill' (p. 28). This implies that mathematical reasoning could be found at all levels of mathematical understanding including preschool. To study mathematical reasoning, there is a need for appropriate tools and theories.

For young children, mathematical reasoning is often related to oral language skills (Charlesworth 2005), and therefore one way of studying reasoning is to use understandings of argumentation. By studying individuals' argumentation and the choices that they make when solving tasks (e.g. Lithner 2008; Sumpter 2013), it is possible to identify different types of reasoning.

Säfström (2013) analysed different definitions of mathematical reasoning in order to define this competence as:

> Explicitly justifying choices and conclusions by mathematical arguments. Select, use and create informal and formal arguments. Interpreting and evaluating one's own and others' reasoning. Reflecting on the role of reasoning. Knowing what a proof is. (p. 36)

This definition as with much of the research mentioned earlier was about the reasoning of individuals. However, it is plausible to expect other forms of reasoning than just individual reasoning when preschool children are trying to solve mathematical tasks and exercises, mainly because of social contexts where activities are taking place. Reasoning as a social activity occurs when 'learners participate as they interact with one another' (Yackel and Hanna 2003, p. 228). Therefore, in this chapter I discuss examples of individual and collective reasoning from the perspective of two different frameworks. Two frameworks are used because differences appear when one person makes all the central decisions, in contrast to the situation where several participants contribute to the development of the reasoning.

Individual Reasoning

In Lithner's (2008) framework, an individual's reasoning is defined as the line of thought adopted to produce assertions and reach conclusions in tasks. This line of thought does not have to be based on formal logic; it could even be incorrect. It is produced from starting with a task and ending with some sort of answer. To structure the data, task solving is seen as occurring in four steps:

1. A problematic situation (PS) is met where it is not obvious for the individual as to how to proceed.
2. A strategy choice (SC) is made, a choice that can be supported by a predictive argument.
3. The strategy is implemented (SI) and the implementation can be supported by verifying arguments.
4. A conclusion (C) is obtained.

This is not necessarily a linear structure; an individual can jump between the steps in his or her reasoning. Argumentation is the part of reasoning which aims to convince the individual or others that the reasoning is appropriate. A strategic choice may not result from a conscious decision, but can include actions that are more subconscious.

An important part of this framework is the content of the argument. Lithner (2008) suggested that the argument is anchored in the reasoning by the individual referring to relevant mathematical properties of the components. These components are objects, transformations, and concepts. Objects are fundamental entities, "the 'thing' that one is doing something with" (Lithner 2008, p. 261), e.g. numbers, variables, and functions. A transformation is the process done to an object, with a sequence of these transformations being a procedure, e.g. finding a polynomial maxima or a division algorithm. Concepts are central mathematical ideas built on a set of objects, transformations, and their properties, for example, the infinity concept. However, in the same way objects can be transformed, a transformation can be transformed into an object. As Lithner (2008) discussed, the 'status of a component depends on the situation' (p. 261). Also, some properties are more relevant than others, and the division of surface and intrinsic properties indicates what is relevant, depending on the context:

> In deciding if 9/15 or 2/3 is largest, the size of the numbers (9, 15, 2, 3) is a *surface* property that is insufficient to resolve the problem, while the quotient captures the *intrinsic* property. (Lithner 2008, p. 261)

When the boy on the train solved the problem of the difference between 4 and 2, he did a comparison which is a transformation of the objects 'cardinal number 4' and 'cardinal number 2'. He could also have used subtraction in the meaning of 'take away', for instance, counting down from 4 to 2. This would have been another transformation to the same objects.

In Lithner's (2008) framework, creative and imitative mathematical reasoning are separated. Reasoning is defined as Creative Mathematically Founded Reasoning (CMR) if it fulfils the following conditions (Lithner 2008):

1. Novelty
2. Plausibility
3. Mathematical foundation

Novelty means that a new reasoning sequence is created or re-created. To do this the arguments supporting the strategy choice and/or the implementation of the strategy need to be true or plausible. The mathematical foundation is created when the arguments are anchored in intrinsic mathematical properties. For example, the strategy choice could be about constructing or reconstructing an algorithm where the construction, or more specifically the arguments for the construction, is based on mathematical properties. Global decisions about the strategy choice could be based on CMR, but in the process of solving the problem, a specific local step could involve imitative rather than CMR. Alternatively, the global reasoning about strategy choice could be imitative reasoning with local steps that are CMR (Bergqvist 2006).

It is important to stress that creative mathematical thinking is not restricted to people with an exceptional ability in mathematics, but it can be hard to perform without appropriate interconnected competencies. The competencies are knowledge (the mathematical foundation), heuristics, beliefs, and control (Schoenfeld 1985). They are both cognitive (e.g. the mathematical knowledge) and affective (beliefs). Therefore, students might not even try to produce a CMR (Lithner 2008) even in situations when they easily could have made progress (Bergqvist et al. 2007; Sumpter 2013).

Imitative reasoning is a family of different types of reasoning: *memorised reasoning* (MR) where the strategy choice is founded on recalling an answer and the strategy implementation consists of writing this answer down with no other consideration and *algorithmic reasoning* (AR) where the strategy choice is recalling a certain algorithm (set of rules) that will probably solve the problematic situation. Algorithmic reasoning has three subcategories: familiar AR, delimiting AR, and guided AR. In this chapter, the focus is on the two main categories, CMR and imitative reasoning. (For a longer discussion and further explanations, see Lithner 2008.)

Collective Reasoning

Reasoning, especially when it is a result of social interaction, can also be seen as a collective process. The decisions and arguments are created between a group of people, not just by one person. Mueller (2009) and Mueller et al. (2012) highlighted the importance for collaboration when constructing mathematical arguments. Cobb et al. (1992) argued that it is through participation in the practice of collective argumentation that students learn mathematics. Therefore, this process is social and 'comprises a set of practices and norms that are collective' (Ball and Bass 2003, p. 29). Through collective reasoning, it is possible to study the dynamics of mathematical reasoning such as negotiations of mathematical meaning (Voigt 1994).

Krummheuer (2007) studied students learning mathematics through participation in processes of collective argumentation. Based on Toulmin's (2003) description of argumentation, Krummheuer (2007) described mathematical arguments as consisting of four main components: data, conclusion, warrant, and backing. Depending upon which components are present, it is possible to see how arguments are used, for instance, how they are directed. Previous research has shown that during free outdoor play, Swedish preschool children use a variety of products and procedures in their argumentation when they challenge, support, and take the reasoning forward (Sumpter and Hedefalk 2015). When needed, they use concrete materials to strengthen their arguments and also as an aid for reaching a conclusion but also included abstract social constructs such as jokes as part of their reasoning.

However, argumentation is not the same as reasoning, and so there is a need to establish the relationship between them. As stated previously, arguments are considered to have four components: conclusion, data, warrant, and backing. Data are the facts or the things that are being reasoned about. Warrants can be defined as the statements that legitimise the reasoning. Backings are about what are permitted, representing 'unquestionable basic convictions' (Krummheuer 2007, p. 65). Together, arguments can be linked to each other creating a chain or reasoning sequence which leads to an accepted conclusion that can act as data for a new argument. A chain of arguments is in alignment with Lithner's (2008) definition of mathematical reasoning as the line of thought. A chain of arguments does not necessarily need to be based on logic and may even be incorrect. Based on this, mathematical reasoning is not restricted to deductive logic. Toulmin's diagram shows the relation between the four objects (Krummheuer 2007) (see Fig. 1).

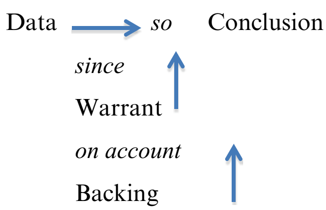

Fig. 1 Toulmin's diagram of argumentation; the implications of *arrows* are given in italics

Table 1 Structure of data

	Data	Argument	Conclusion
Person A			

In this chapter, I analyse children's arguments from a mathematical point of view. Therefore, I use Krummheuer's (2007) structure and notions as a starting point for studying the conclusions, warrants, and backing developed by a collective but add Lithner's (2008) concept of anchoring arguments in mathematical properties. When analysing collective reasoning, a revised version of Krummheuer's (2007) analysis of argumentation (AA) will be used (see Table 1).

In Table 1, the implication arrows 'since' and 'on account' are presented in the column 'Argument'. These arguments are analysed using the notions of objects, transformations, and concepts. Conclusions are analysed in the same way, with references back to arguments when appropriate, e.g. 'Since X, therefore Y'. In this way, data, analysis, and conclusions are presented in the same table. This alternative structure and coding scheme was first tested and used in Sumpter and Hedefalk (2015).

Tom and Jim

This interaction comes from a set of observations made when the author spent 5 days (different times over a period of 2 years; three outdoors sessions and two indoors sessions) in a preschool. As the data were collected by note taking, some details would be missed. However, in this case the interaction was quite short making it more likely that the important details were captured. There was no specific aim with the observations more than the focus on mathematical activities and child–child interactions. This episode comes from an occasion when the preschool took the 5-year-olds, the oldest children of the group, to play in the woods. Tom and Jim (both 5 years) are playing with sticks. The sticks are in their game laser swords and the question, brought up by the boys themselves, is: Which stick is the longest? (Table 2).

In this episode, the boys are allowed to explore, negotiate, and support their own and each other's argument without an adult. Tom and Jim use several mathematical properties concerning measurement when taking their reasoning forward. They compare magnitudes and order, they may have used conservation, and there is a possible use of transitivity when working with three objects. These function as a mathematical foundation; fulfilling the role of warrants, they are the grounds for the arguments. The accepted conclusion $B > A$ is later used as data for a new argument: when they establish $C > B$ and $C > A$. The arguments are directed and there are elements of challenge ('No, it is not!') and support ('It is the longest'.). In this way Tom and Jim arrive at a conclusion they both agree on.

Table 2 Tom and Jim solving 'which stick is the longest?'

	Data	Mathematical properties of the argument	Conclusion
Tom	Look! A laser sword! [picks up a stick from the ground]		
Jim	I got one, too. [picks up a stick from the ground]		
[Tom and Jim are playing with the sticks for a few minutes]			
Tom	My sword is long	The length of Tom's stick is large	
Jim	My sword is longer	The length of Jim's stick is longer than the length of Tom's stick	B > A
Tom	No, it is not!	Objection to B > A	
Jim	Look! [holds up his stick next to Tom's stick]	Comparing magnitudes, here lengths. The length of Jim's stick is longer than the length of Tom's stick	Since my stick is longer than yours, B > A
Tom	Ok. But mine is thicker. Boom boom! [Pretend shooting]	(Identification of another property of the stick, although not relevant to the question, 'which stick is the longest?')	B > A
Jim [looking around for other sticks]	Look at this one then! [Drags out a large branch]		
Tom	That one is longer than mine! That one is a laser cannon!	The branch is longer than Tom's stick. This is concluded without a direct measure. He may be using an understanding of conservation of length	C > A
Jim	It is the longest. Ha, ha! It is a cannon!	Possible use of transitivity: C > B > A	C > B, C > A

Heidi

This episode comes from a set of video recordings. The author asked parents of children from three preschools if their children could participate in a problem-solving session. The first ones to agree were Heidi's parents. This recording was made in their home to simplify the permissions for recording. Since the focus is on

Heidi's reasoning and not on the social context, the assumption is that this episode provides an illustration of a preschool child's mathematical reasoning when solving a mathematical task. It is not considered a play session, although Heidi might look at it as play. The task was provided by the interviewer as a stimuli to generate strategy choice, strategy implementation, and conclusion. Heidi is at the time 3 years and 5 months old. According to her parents, she has not been engaged in any specific mathematical activities at home nor at preschool.

As an introduction to the problem-solving session, she solved three tasks with the help of blocks: one addition task (4 + 3), one subtraction task (5 − 2), and one division task (4 ÷ 2). In the division task, the interviewer tried to give all the blocks to one teddy. Heidi objected to this and then gave each teddy equal amounts.

Heidi then worked on how nine blocks should be divided by three toy animals (Teddy Bluebear, Rabbit-y, and George the Dog). How many blocks do they get each? There were nine blocks, consisting of three of each colour, green, blue, and yellow, but are mixed up.

Interviewer	Shall we see if George the Dog can count the blocks?
Heidi	Yes. [Counts when the interviewer points to the blocks one by one with George the Dog's paw] One, two, three, four, five, six, seven, eight, nine. Nine!
Interviewer	Nine blocks! Look, George the Dog is really happy!
Heidi	[laughs] Why is he shaking?
Interviewer	He is happy! That is what George the Dog is doing when he is happy. Shall we divide the blocks? Shall we divide so they get a few blocks each? Do you want to do that?
Heidi	Yes.
Interviewer	Shall we do it together? Who should have this one? [Points at a yellow brick that is closest to the interviewer.]
Heidi	Rabbit-y!
Interviewer	Then…?
Heidi	[Points at another yellow block]
Interviewer	Who should have this one?
Heidi	Bluebear. [points at the remaining yellow block]
Interviewer	Who should have this yellow block?
Heidi	The dog.
Interviewer	What should we do now?
Heidi	The green and the blue ones.

Heidi distributes first the green blocks and then the blue blocks to the toy animals.

Reasoning Structure

The data are organised using the reasoning structure suggested by Lithner (2008): problematic situation (PS), strategy choice (SC), strategy implementation (SI), and conclusion (C). Choice should here be interpreted in a wide sense (choose, guess, etc.) and could also include subconscious preferences.

PS: Nine blocks should be divided by three teddies.

SC: Identify property of the blocks: three colours. Group the blocks after colour: $9 = 3 + 3 + 3$. Then perform division: $9/3 = (3 + 3 + 3)/3 = 3/3 + 3/3 + 3/3$.

SI: Straightforward. First yellow, second green, last blue.
C: Each teddy gets 1 blue, 1 green, and 1 yellow resulting in 3 blocks.

In this sequence, although the interviewer asked questions, Heidi was the one making all of the central decisions. She decided which blocks are going to be shared out (except for the first one), in which order they should be shared, and how many at a time. As a strategy choice, Heidi recognised the colour of the blocks and used this property when grouping the blocks into smaller subsets. Then she performed division for each of these subsets, one colour at the time, without any observed hesitation. The task was considered a new problem for Heidi in that sense that she did not have memorised knowledge (e.g. $9 \div 3 = 3$) or used a familiar algorithm based on surface arguments ('this is the algorithm we normally use'). In that way her reasoning was novel. She could have divided objects before in other activities and maybe even solved 9 divided by 3 previously. However, in solving this task, it seemed more likely that the algorithm was reconstructed than Heidi used imitative reasoning. Also, the grouping of the objects was an added transformation rather than 'just sharing'. Her choice to group the blocks, a local step in the reasoning, is plausible and has a mathematical foundation. If she, for instance, used quotient division instead of partitioning, it would have been a different strategy choice from a mathematical point of view. This reasoning is categorised as CMR.

Discussion

The starting point of this chapter was that although there is a growing body of research studying preschool children's mathematical reasoning, few studies use and anchor their analysis in theories and frameworks about mathematical reasoning. Mathematical reasoning is often considered to represent a high quality of thinking (Lithner 2008), and with such a definition, it is hard to see how Swedish preschools should work in order to help children 'develop their mathematical skill in putting forward and following reasoning' (National Agency for Education 2011a, p. 10). Therefore, there is a need to understand the different types of reasoning that children produce and/or how arguments are used and directed and what they are based on.

In this chapter, two cases of reasoning are described, individual and collective reasoning, and an example of each kind is analysed using two different types of frameworks. In the case of Tom and Jim, analysing their reasoning as a collective process highlighted the content and direction of their arguments. Tom and Jim used a mathematical foundation, the properties of measurement, to reach a shared conclusion. Similar behaviour has been observed in previous studies (Sumpter and Hedefalk 2015). Just as in Säfström (2013), the children showed that they could use different mathematical competencies and the ability to challenge and justify arguments.

In the case of Heidi, individual reasoning was the focus. Even though she interacted with the interviewer (mainly through a teddy), she made all the central decisions, making her reasoning individual. Her strategy choice and conclusion allow her reasoning to be categorised as CMR. It seemed that this type of reasoning was helpful for solving a mathematical problem when a specific solution method both at a global and local level was not known.

It would have been surprising if Heidi performed an imitative reasoning considering that she has not yet been involved in formal mathematical training. Imitative reasoning is more likely to be something produced when an individual has had access to and learnt a lot of mathematical procedures which were linked to specific tasks, such as occurs when working alone with a textbook (Lithner 2008). Heidi has not experienced this yet. Most likely, she does not have a belief that a certain task should be solved with a specific algorithm. Her reasoning, at the moment, is limited to her mathematical knowledge, her creativity, and the milieu in which she operated on a daily basis. Similar to collective reasoning, as part of a social process with practices and norms (Ball and Bass 2003), her 'mathematical world view' (Schoenfeld 1985) would be a factor determining what could be seen as possible and not. The same conclusion can be drawn about Tom and Jim's reasoning. They were not restrained by the idea of trying to find the right algorithm, a behaviour observed, for instance, in upper secondary school students' reasoning (Bergqvist et al. 2007; Sumpter 2013). The implication for preschool mathematics is that there is a need to support children to explore and produce their own reasoning, both individually and collectively, instead of telling what is 'the correct one' or just establishing 'the correct answer'.

The two frameworks highlighted different aspects of reasoning, each of them suitable to the different types of data, individual and collective. In the case of collective reasoning, the direction of arguments can be helpful in order to understand the social processes, such as participation and contributions. If mathematical objects, transformations, and concepts are present, the warrants and backings can be analysed from the point of view of mathematical content. However, it is not possible to state anything about different types of reasoning, CMR or imitative reasoning. Neither does this framework stress different types of strategy choices. In the case of individual problem solving, focusing on strategy choice and the conclusion using the structure of data made it possible to categorise different types of reasoning. It could highlight different strategy choices for different problematic situations. Although it seems suitable for doing an analysis of the solving of task, it does not seem suitable for reflecting on the reasoning used in interactions (Säfström 2013). However, the results are still interesting if it could help us explain and predict behaviour.

The use of one framework in doing an analysis does not exclude also using the other because their foci are on different things. The results from the two analyses will highlight different aspects of the reasoning.

In the curriculum, preschool teachers are responsible for ensuring that children in their preschools 'are stimulated and challenged in their mathematical development' (National Agency for Education 2011a, p. 11). Moreover, research has indicated that students learn when they have the opportunity to learn (Bobis et al. 2005, 2008; Hiebert 2003) and that young children that are guided can expand their mathematical thinking further than without a guide (Björklund 2008; Lee and Ginsburg 2009; van Oers 1996). Given this, it would be interesting to see how Tom's, Jim's, and Heidi's reasoning could be developed through being stimulated, so that they were supported to do what it says in the curriculum 'to develop their mathematical skill in putting forward and following reasoning' (National Agency for Education 2011a, p. 10). What reasoning could they perform, now and later, especially since mathematical reasoning predicts mathematical achievement later in school (Nunes et al. 2012)? Tom, Jim, and Heidi show creativity and skills for putting forward arguments based on mathematical properties. These are good qualities. It is possible to recognise their competencies in this area: the question is, what is being done with it?

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