Mechanical Proofs of Properties of the Tribonacci Word

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Abstract. We implement a decision procedure for answering questions about a class of infinite words that might be called (for lack of a better name) "Tribonacci-automatic". This class includes, for example, the famous Tribonacci word $\mathbf{T} = 0102010010201\cdots$, the fixed point of the morphism $0 \to 01$, $1 \to 02$, $2 \to 0$. We use our decision procedure to reprove some old results about the Tribonacci word from the literature, such as assertions about the occurrences in $\mathbf{T}$ of squares, cubes, palindromes, and so forth. We also obtain some new results, including on enumeration.

1 Introduction

In several previous papers (e.g., [1,21–24,32] we have explored the ramifications of a decision procedure for the logical theory $\mathrm{Th}(\mathbb{N}, +, a(n))$, where $(a(n))_{n\geq 0}$ is an infinite sequence specified by a finite-state machine M. Furthermore, in many cases we can explicitly enumerate various aspects of such sequences, such as subword complexity [8]. Roughly speaking, given a predicate P of one or more variables in the logical theory, the method transforms M to a new automaton M' that accepts the representations of those variables making the predicate true. The ideas are based on extensions of the logical theory $\mathrm{Th}(\mathbb{N}, +)$, sometimes called *Presburger arithmetic* [28,29]. See, for example, [6].

A critical point is what sort of representations are allowed. According to the results in [5], it suffices that the representations be based on a Pisot number. The critical point is that there must be an automaton that can perform addition of two numbers in the appropriate representation [15,16].

In the papers mentioned above, we applied our method to the so-called k-automatic sequences, which correspond to automata that work with the ordinary base-k expansions of numbers. More recently, we also proved a number of new results using Fibonacci (or "Zeckendorf") representation [13], which is based on writing integers as a sum of Fibonacci numbers.

It is our contention that the power of this approach has not been widely appreciated, and that many results, previously proved using long and involved ad hoc techniques, can be proved with much less effort by phrasing them as logical predicates and employing a decision procedure. Furthermore, many enumeration questions can be solved with a similar approach. Although the worst-case

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running time of the procedure is enormously large (of the form $2^{2^{\cdot^{\cdot^{\cdot^{2^{p(n)}}}}}}$ where the number of 2's is the number of quantifier alternations), n is the size of the polynomial, and p is a polynomial), in practice the procedure often terminates in a reasonable time.

In this paper we discuss our implementation of an analogous algorithm for Tribonacci representation. We use it to reprove some old results from the literature purely mechanically, as well as obtain some new results. The implementation of the decision procedure was created by the first author. It is called `Walnut`, and is available for free download at https://www.cs.uwaterloo.ca/~shallit/papers.html.

We have not rigorously proved the correctness of this implementation, but it has been tested in a large number of different ways (including some results verified with independently-written programs). In this, we are well in the tradition of many other results in combinatorics on words that have been verified with machine computations — despite a lack of formal verification of the code. Even if the code were formally verified, one could reasonably ask for a proof of the correctness of the verification code! We believe that publication of our code, allowing checking by any interested reader, serves as an adequate check.

We view our work as part of a modern trend in mathematics. For other works on using computerized formal methods to prove theorems see, for example, [25,27].

2 Tribonacci Representation

Let the Tribonacci numbers be defined, as usual, by the linear recurrence $T_n = T_{n-1} + T_{n-2} + T_{n-3}$ for $n \geq 3$ with initial values $T_0 = 0$, $T_1 = 1$, $T_2 = 1$. (We caution the reader that some authors use a different indexing for these numbers.) Here are the first few values of this sequence.

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
T_n	0	1	1	2	4	7	13	24	44	81	149	274	504	927	1705	3136	5768

From the theory of linear recurrences we know that

$$T_n = c_1\alpha^n + c_2\beta^n + c_3\gamma^n$$

where α, β, γ are the zeros of the polynomial $x^3 - x^2 - x - 1$. The only real zero is $\alpha \doteq 1.83928675521416113$; the other two zeros are complex and are of magnitude $< 3/4$. Solving for the constants, we find that $c_1 \doteq 0.33622811699494109 4225$, the real zero of the polynomial $44x^3 - 2x - 1 = 0$. It follows that $T_n = c_1\alpha^n + O(.75^n)$. In particular $T_n/T_{n-1} = \alpha + O(.41^n)$.

It is well-known that every non-negative integer can be represented, in an essentially unique way, as a sum of Tribonacci numbers $(T_i)_{i\geq 2}$, subject to the

constraint that no three consecutive Tribonacci numbers are used [7]. For example, $43 = T_7 + T_6 + T_4 + T_3$.

Such a representation can be written as a binary word $a_1 a_2 \cdots a_n$ representing the integer $\sum_{1 \le i \le n} a_i T_{n+2-i}$. For example, the binary word 110110 is the Tribonacci representation of 43.

Let $\Sigma_2 = \{0, 1\}$. For $w = a_1 a_2 \cdots a_n \in \Sigma_2^*$, we define $[a_1 a_2 \cdots a_n]_T := \sum_{1 \le i \le n} a_i T_{n+2-i}$, even if $a_1 a_2 \cdots a_n$ has leading zeros or occurrences of the word 111.

By $(n)_T$ we mean the *canonical* Tribonacci representation for the integer n, having no leading zeros or occurrences of 111. Note that $(0)_T = \epsilon$, the empty word. The language of all canonical representations of elements of $\mathbb{N}$ is $\epsilon + (1 + 11)(0 + 01 + 011)^*$.

Just as Tribonacci representation is an analogue of base-k representation, we can define the notion of *Tribonacci-automatic sequence* as the analogue of the more familiar notation of k-automatic sequence [2,11]. We say that an infinite word $\mathbf{a} = (a_n)_{n \ge 0}$ is Tribonacci-automatic if there exists an automaton with output $M = (Q, \Sigma_2, q_0, \delta, \kappa, \Delta)$ for a coding κ such that $a_n = \kappa(\delta(q_0, (n)_T))$ for all $n \ge 0$. An example of a Tribonacci-automatic sequence is the infinite Tribonacci word,

$$\mathbf{T} = T_0 T_1 T_2 \cdots = 0102010010201 \cdots$$

which is generated by the following 3-state automaton (Fig. 1):

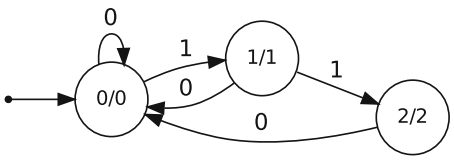

Fig. 1. Automaton generating the Tribonacci sequence

To compute T_i, we express i in canonical Tribonacci representation, and feed it into the automaton. Then T_i is the output associated with the last state reached (denoted by the symbol after the slash).

A basic fact about Tribonacci representation is that addition can be performed by a finite automaton. To make this precise, we need to generalize our notion of Tribonacci representation to r-tuples of integers for $r \ge 1$. A representation for $(x_1, x_2, \ldots, x_r)$ consists of a string of symbols z over the alphabet Σ_2^r, such that the projection $\pi_i(z)$ over the i'th coordinate gives a Tribonacci representation of x_i. Notice that since the canonical Tribonacci representations of the individual x_i may have different lengths, padding with leading zeros will often be necessary. A representation for $(x_1, x_2, \ldots, x_r)$ is called canonical if it has no leading $[0, 0, \ldots 0]$ symbols and the projections into individual coordinates have no occurrences of 111. We write the canonical representation as

$(x_1, x_2, \ldots, x_r)_T$. Thus, for example, the canonical representation for $(9, 16)$ is $[0, 1][1, 0][0, 0][1, 1][0, 1]$.

Thus, our claim about addition in Tribonacci representation is that there exists a deterministic finite automaton (DFA) M_{add} that takes input words of the form $[0, 0, 0]^*(x, y, z)_T$, and accepts if and only if $x+y = z$. Thus, for example, M_{add} accepts $[1, 0, 1][0, 1, 1][0, 0, 0]$ since the three words obtained by projection are 100, 010, and 110, which represent, respectively, 4, 2, and 6 in Tribonacci representation.

Since this automaton does not appear to have been given explicitly in the literature and it is essential to our implementation, we give it below in Table 1. This automaton actually works even for non-canonical expansions having three consecutive 1's. The initial state is state 1. The state 0 is a "dead state" that can safely be ignored.

We briefly sketch a proof of the correctness of this automaton. States can be identified with certain sequences, as follows: if x, y, z are the identical-length words arising from projection of a word that takes M_{add} from the initial state 1 to the state t, then t is identified with the integer sequence $([x0^n]_T + [y0^n]_T - [z0^n]_T)_{n\geq 0}$. State 0 corresponds to sequences that can never lead to 0, as they are too positive or too negative.

When we intersect this automaton with the appropriate regular language (ruling out input triples containing 111 in any coordinate), we get an automaton with 149 states accepting $0^*(x, y, z)_T$ such that $x + y = z$.

Another basic fact about Tribonacci representation is that, for canonical representations containing no three consecutive 1's or leading zeros, the radix order on representations is the same as the ordinary ordering on $\mathbb{N}$. It follows that a very simple automaton can, on input $(x, y)_T$, decide whether $x < y$.

Putting this all together, we get the following decision procedure:

Procedure 1 (Decision Procedure for Tribonacci-Automatic Words)
Input:

- $m, n \in \mathbb{N}$;
- *m DFA's generating the Tribonacci-automatic words* $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$;
- *a first-order proposition with n free variables* $\varphi(v_1, v_2, \ldots, v_n)$ *using constants and relations definable in* $Th(\mathbb{N}, 0, 1, +)$ *and indexing into* $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_m$.

Output: *DFA with input alphabet* Σ_2^n *accepting* $\{(k_1, k_2, \ldots, k_n)_T : \varphi(k_1, k_2, \ldots, k_n)$ *holds*$\}$.

3 Mechanical Proofs of Properties of the Infinite Tribonacci Word

Recall that a word x, whether finite or infinite, is said to have period p if $x[i] = x[i + p]$ for all i for which this equality is meaningful. Thus, for example, the English word `alfalfa` has period 3. The *exponent* of a finite word x, written $\exp(x)$, is $|x|/P$, where P is the smallest period of x. Thus $\exp(\texttt{alfalfa}) = 7/3$.

Table 1. Transition table for M_{add} for Tribonacci addition

q	[0,0,0]	[0,0,1]	[0,1,0]	[0,1,1]	[1,0,0]	[1,0,1]	[1,1,0]	[1,1,1]	acc/rej
0	0	0	0	0	0	0	0	0	0
1	1	2	3	1	3	1	0	3	1
2	4	0	5	4	5	4	6	5	0
3	0	7	0	0	0	0	0	0	0
4	0	0	0	0	0	0	8	0	0
5	9	0	10	9	10	9	11	10	0
6	12	13	0	12	0	12	0	0	1
7	0	14	0	0	0	0	0	0	0
8	0	0	9	0	9	0	10	9	0
9	0	0	4	0	4	0	5	4	0
10	2	15	1	2	1	2	3	1	0
11	7	16	0	7	0	7	0	0	1
12	14	17	0	14	0	14	0	0	1
13	18	19	20	18	20	18	21	20	0
14	3	1	0	3	0	3	0	0	0
15	0	0	0	0	0	0	22	0	0
16	20	18	21	20	21	20	0	21	1
17	5	4	6	5	6	5	23	6	1
18	0	0	8	0	8	0	24	8	0
19	0	0	0	0	0	0	25	0	0
20	10	9	11	10	11	10	0	11	1
21	0	12	0	0	0	0	0	0	0
22	0	0	26	0	26	0	27	26	0
23	0	28	0	0	0	0	0	0	0
24	13	29	12	13	12	13	0	12	0
25	0	0	0	0	0	0	26	0	0
26	0	0	0	0	0	0	4	0	0
27	15	0	2	15	2	15	1	2	0
28	0	30	0	0	0	0	0	0	0
29	0	0	31	0	31	0	32	31	0
30	0	3	0	0	0	0	0	0	0
31	0	0	0	0	0	0	33	0	0
32	26	0	27	26	27	26	34	27	0
33	0	0	0	0	0	0	9	0	0
34	16	35	7	16	7	16	0	7	0
35	31	0	32	31	32	31	36	32	0
36	37	38	39	37	39	37	0	39	1
37	17	40	14	17	14	17	0	14	0
38	19	0	18	19	18	19	20	18	0
39	0	41	0	0	0	0	0	0	1
40	0	0	22	0	22	0	42	22	0
41	21	20	0	21	0	21	0	0	0
42	38	43	37	38	37	38	39	37	0
43	0	0	0	0	0	0	31	0	0

If $\mathbf{x}$ is an infinite word with a finite period, we say it is *ultimately periodic*. An infinite word $\mathbf{x}$ is ultimately periodic if and only if there are finite words u, v such that $x = uv^\omega$, where $v^\omega = vvv\cdots$.

A nonempty word of the form xx is called a *square*, and a nonempty word of the form xxx is called a *cube*. More generally, a nonempty word of the form x^n is called an n'th power. By the *order* of a square xx, cube xxx, or n'th power x^n, we mean the length $|x|$.

The infinite Tribonacci word $\mathbf{T} = 0102010\cdots = T_0T_1T_2\cdots$ can be described in many different ways. In addition to our definition in terms of automata, it is also the fixed point of the morphism $\varphi(0) = 01$, $\varphi(1) = 02$, and $\varphi(1) = 0$. This word has been studied extensively in the literature; see, for example, [3,9,14,17,30,31,33,34].

It can also be described as the limit of the finite Tribonacci words $(Y_n)_{n\geq 0}$, defined as follows:

$$Y_0 = \epsilon \qquad Y_1 = 2 \qquad Y_2 = 0$$
$$Y_3 = 01 \qquad Y_n = Y_{n-1}Y_{n-2}Y_{n-3} \text{ for } n \geq 4.$$

Note that Y_n, for $n \geq 2$, is the prefix of length T_n of $\mathbf{T}$.

In the next subsection, we use our implementation to prove a variety of results about repetitions in $\mathbf{T}$.

3.1 Repetitions

It is known that all strict epistandard words (or Arnoux-Rauzy words), are not ultimately periodic (see, for example, [19]). Since $\mathbf{T}$ is in this class, we have the following known result which we can reprove using our method.

Theorem 2. *The word* $\mathbf{T}$ *is not ultimately periodic.*

Proof. We construct a predicate asserting that the integer $p \geq 1$ is a period of some suffix of $\mathbf{T}$:

$$(p \geq 1) \ \wedge \ \exists n \ \forall i \geq n \ \mathbf{T}[i] = \mathbf{T}[i+p].$$

(Note: unless otherwise indicated, whenever we refer to a variable in a predicate, the range of the variable is assumed to be $\mathbb{N} = \{0, 1, 2, \ldots\}$.) From this predicate, using our program, we constructed an automaton accepting the language

$$L = 0^* \ \{(p)_T \ : \ (p \geq 1) \ \wedge \ \exists n \ \forall i \geq n \ \mathbf{T}[i] = \mathbf{T}[i+p]\}.$$

This automaton accepts the empty language, and so it follows that $\mathbf{T}$ is not ultimately periodic.

Here is the log of our program:

```
p >= 1 with 5 states, in 426ms
i >= n with 13 states, in 3ms
i + p with 150 states, in 31ms
```

```
TR[i] = TR[i + p] with 102 states, in 225ms
i >= n => TR[i] = TR[i + p] with 518 states, in 121ms
Ai i >= n => TR[i] = TR[i + p] with 4 states, in 1098ms
En Ai i >= n => TR[i] = TR[i + p] with 2 states, in 0ms
p >= 1 & En Ai i >= n => TR[i] = TR[i + p] with 2 states, in 1ms
overall time: 1905ms
```

The largest intermediate automaton during the computation had 5999 states.

A few words of explanation are in order: here "T" refers to the sequence **T**, and "E" is our abbreviation for ∃ and "A" is our abbreviation for ∀. The symbol "=>" is logical implication, and "&" is logical and. □

From now on, whenever we discuss the language accepted by an automaton, we will omit the 0^* at the beginning.

We now turn to repetitions. As a particular case of [17, Theorem. 6.31 and Example 7.6, p. 130] and [18, Example 6.21] we have the following result, which we can reprove using our method.

Theorem 3. **T** *contains no fourth powers.*

Proof. A natural predicate for the orders of all fourth powers occurring in **T**:

$$(n > 0) \ \wedge \ \exists i \ \forall t < 3n \ \mathbf{T}[i+t] = \mathbf{T}[i+n+t].$$

However, this predicate could not be run on our prover. It runs out of space while trying to determinize an NFA with 24904 states.

Instead, we make the substitution $j = i + t$, obtaining the new predicate

$$(n > 0) \ \wedge \ \exists i \ \forall j \ ((j \geq i) \wedge (j < i + 3n)) \implies \mathbf{T}[j] = \mathbf{T}[j+n].$$

The resulting automaton accepts nothing, so there are no fourth powers.

Here is the log.

```
n > 0 with 5 states, in 59ms
i <= j with 13 states, in 15ms
3 * n with 147 states, in 423ms
i + 3 * n with 799 states, in 4397ms
j < i + 3 * n with 1103 states, in 4003ms
i <= j & j < i + 3 * n with 1115 states, in 111ms
j + n with 150 states, in 18ms
TR[j] = TR[j + n] with 102 states, in 76ms
i <= j & j < i + 3 * n => TR[j] = TR[j + n] with 6550 states, in 1742ms
Aj i <= j & j < i + 3 * n => TR[j] = TR[j + n] with 4 states, in 69057ms
Ei Aj i <= j & j < i + 3 * n => TR[j] = TR[j + n] with 2 states, in 0ms
n > 0 & Ei Aj i <= j & j < i + 3 * n => TR[j] = TR[j + n] with 2 states, in 0ms
overall time: 79901ms
```

The largest intermediate automaton in the computation had 86711 states. □

Next, we move on to a description of the orders of squares occurring in **T**. We reprove a result of Glen [17, Sect. 6.3.5].

Theorem 4. *All squares in* $\mathbf{T}$ *are of order* T_n *or* $T_n + T_{n-1}$ *for some* $n \geq 2$. *Furthermore, for all* $n \geq 2$, *there exists a square of order* T_n *and* $T_n + T_{n-1}$ *in* $\mathbf{T}$.

Proof. A natural predicate for the lengths of squares is

$$(n > 0) \ \wedge \ \exists i \ \forall t < n \ \mathbf{T}[i + t] = \mathbf{T}[i + n + t].$$

but when we run our solver on this predicate, we get an intermediate NFA of 4612 states that our solver could not determinize in the allotted space. The problem appears to arise from the three different variables indexing T. To get around this problem, we rephrase the predicate, introducing a new variable j that represents $i + t$. This gives the predicate

$$(n > 0) \ \wedge \ \exists i \ \forall j \ ((i \leq j) \wedge (j < i + n)) \implies \mathbf{T}[j] = \mathbf{T}[j + n].$$

and the following log

```
n > 0 with 5 states, in 59ms
i <= j with 13 states, in 15ms
3 * n with 147 states, in 423ms
i + 3 * n with 799 states, in 4397ms
j < i + 3 * n with 1103 states, in 4003ms
i <= j & j < i + 3 * n with 1115 states, in 111ms
j + n with 150 states, in 18ms
TR[j] = TR[j + n] with 102 states, in 76ms
i <= j & j < i + 3 * n => TR[j] = TR[j + n] with 6550 states, in 1742ms
Aj i <= j & j < i + 3 * n => TR[j] = TR[j + n] with 4 states, in 69057ms
Ei Aj i <= j & j < i + 3 * n => TR[j] = TR[j + n] with 2 states, in 0ms
n > 0 & Ei Aj i <= j & j < i + 3 * n => TR[j] = TR[j + n] with 2 states, in 0ms
overall time: 79901ms
```

The resulting automaton accepts exactly the language $10^* + 110^*$. The largest intermediate automaton had 26949 states. □

We can easily get more information about the square occurrences in $\mathbf{T}$. By modifying our previous predicate, we get

$$(n > 0) \ \wedge \ \forall j \ ((i \leq j) \wedge (j < i + n)) \implies \mathbf{T}[j] = \mathbf{T}[j + n]$$

which encodes those (i, n) pairs such that there is a square of order n beginning at position i of $\mathbf{T}$.

This automaton has only 10 states and efficiently encodes the orders and starting positions of each square in $\mathbf{T}$. During the computation, the largest intermediate automaton had 26949 states. Thus we have proved the following new result:

Theorem 5. *The language*

$$\{(i, n)_T \ : \ \textit{there is a square of order } n \textit{ beginning at position } i \textit{ in } \mathbf{T}\}$$

is accepted by the automaton in Fig. 2.

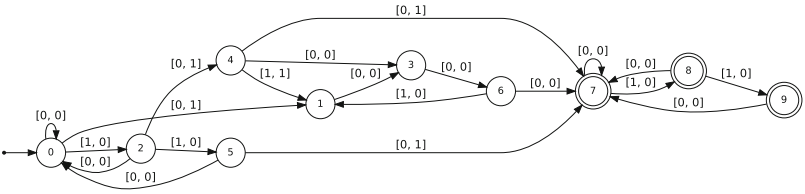

Fig. 2. Automaton accepting orders and positions of all squares in **T**

Next, we examine the cubes in **T**. Evidently Theorem 4 implies that any cube in **T** must be of order T_n or $T_n + T_{n-1}$ for some n. However, not every order occurs. We thus recover the following result of Glen [17, Sect. 6.3.7].

Theorem 6. *The cubes in* **T** *are of order* T_n *for* $n \geq 5$*, and a cube of each such order occurs.*

Proof. We use the predicate

$$(n > 0) \ \wedge \ \exists i \ \forall j \ ((i \leq j) \wedge (j < i + 2n)) \implies \mathbf{T}[j] = \mathbf{T}[j + n].$$

When we run our program, we obtain an automaton accepting exactly the language $(1000)0^*$, which corresponds to T_n for $n \geq 5$. The largest intermediate automaton had 60743 states. □

4 Enumeration

Mimicking the base-k ideas in [8], we can also mechanically enumerate many aspects of Tribonacci-automatic sequences. We do this by encoding the factors having the property in terms of paths of an automaton. This gives the concept of *Tribonacci-regular sequence*. Roughly speaking, a sequence $(a(n))_{n \geq 0}$ taking values in $\mathbb{N}$ is Tribonacci-regular if the set of sequences

$$\{(a([xw]_T)_{w \in \Sigma_2^*} \ : \ x \in \Sigma_2^*\}$$

is finitely generated. Here we assume that $a([xw]_T)$ evaluates to 0 if xw contains the word 111. Every Tribonacci-regular sequence $(a(n))_{n \geq 0}$ has a *linear representation* of the form (u, μ, v) where u and v are row and column vectors, respectively, and $\mu : \Sigma_2 \to \mathbb{N}^{d \times d}$ is a matrix-valued morphism, where $\mu(0) = M_0$ and $\mu(1) = M_1$ are $d \times d$ matrices for some $d \geq 1$, such that

$$a(n) = u \cdot \mu(x) \cdot v$$

whenever $[x]_T = n$. The *rank* of the representation is the integer d.

Recall that if $\mathbf{x}$ is an infinite word, then the subword complexity function $\rho_{\mathbf{x}}(n)$ counts the number of distinct factors of length n. Then, in analogy with [8, Theorem. 27], we have

Theorem 7. *If* $\mathbf{x}$ *is Tribonacci-automatic, then the subword complexity function of* $\mathbf{x}$ *is Tribonacci-regular.*

Using our implementation, we can obtain a linear representation of the subword complexity function for $\mathbf{T}$. An obvious choice is to use the language

$$\{(n,i)_T \ : \ \forall j < i \ \mathbf{T}[i..i+n-1] \neq \mathbf{T}[j..j+n-1]\},$$

based on a predicate that expresses the assertion that the factor of length n beginning at position i has never appeared before. Then, for each n, the number of corresponding i gives $\rho_{\mathbf{T}}(n)$.

However, this does not run to completion in our implementation in the allotted time and space. Instead, let us substitute $u = j + t$ and $k = i - j$ to get the predicate

$$\forall k \ (((k > 0) \wedge (k \leq i)) \implies (\exists u \ ((u \geq j) \wedge (u < n + j) \wedge (\mathbf{T}[u] \neq \mathbf{T}[u + k])))).$$

This predicate is close to the upper limit of what we can compute using our program. The largest intermediate automaton had 1230379 states and the program took 12323.82 s, giving us a linear representation (u, μ, v) rank 22. When we minimize this using the algorithm in [4] we get the rank-12 linear representation

$$u = [1\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0]$$

$$M_0 = \begin{bmatrix}
1 & 0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\
0 & 0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\
0 & 0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0 \\
-1 & 0\,1\,0\,1\,0\,0\,0\,0\,0\,0\,0 \\
0 & 0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0 \\
-1 & 0\,1\,0\,0\,0\,1\,0\,0\,0\,0\,0 \\
-2 & 0\,1\,0\,1\,0\,1\,0\,0\,0\,0\,0 \\
-3 & 0\,2\,0\,1\,0\,1\,0\,0\,0\,0\,0 \\
-4 & 0\,2\,0\,2\,0\,1\,0\,0\,0\,0\,0 \\
-5 & 0\,2\,0\,2\,0\,2\,0\,0\,0\,0\,0 \\
-6 & 0\,2\,0\,3\,0\,2\,0\,0\,0\,0\,0 \\
-10 & 0\,3\,0\,4\,0\,4\,0\,0\,0\,0\,0
\end{bmatrix} \qquad M_1 = \begin{bmatrix}
0\,1\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\
0\,0\,0\,1\,0\,0\,0\,0\,0\,0\,0 \\
0\,0\,0\,0\,0\,1\,0\,0\,0\,0\,0 \\
0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\
0\,0\,0\,0\,0\,0\,0\,1\,0\,0\,0 \\
0\,0\,0\,0\,0\,0\,0\,0\,1\,0\,0 \\
0\,0\,0\,0\,0\,0\,0\,0\,0\,1\,0 \\
0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\
0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,1 \\
0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0 \\
0\,0\,0\,0\,0\,0\,0\,0\,0\,0\,0
\end{bmatrix}$$

$$v' = [1\ 3\ 5\ 7\ 9\ 11\ 15\ 17\ 21\ 29\ 33\ 55]^R.$$

Comparing this to an independently-derived linear representation of the function $2n+1$, we see they are the same. From this we get a well-known result (see, e.g., [12, Theorem 7]):

Theorem 8. *The subword complexity function of* $\mathbf{T}$ *is* $2n + 1$.

We now turn to computing the exact number of square occurrences in the finite Tribonacci words Y_n.

To solve this using our approach, we first generalize the problem to consider any length-n prefix of Y_n, and not simply the prefixes of length T_n.

The predicate represents the number of distinct squares in $\mathbf{T}[0..n-1]$:

$$L_{\rm ds} := \{(n,i,j)_T \ : \ (j \geq 1) \text{ and } (i+2j \leq n) \text{ and } \mathbf{T}[i..i+j-1] = \mathbf{T}[i+j..i+2j-1] \text{ and } \forall i' < i \ \mathbf{T}[i'..i'+2j-1] \neq \mathbf{T}[i..i+2j-1]\}.$$

This predicate asserts that $\mathbf{T}[i..i+2j-1]$ is a square occurring in $\mathbf{T}[0..n-1]$ and that furthermore it is the first occurrence of this particular word in $\mathbf{T}[0..n-1]$.

This represents the total number of occurrences of squares in $\mathbf{T}[0..n-1]$:

$$L_{\rm dos} := \{(n,i,j)_T \ : \ (j \geq 1) \text{ and } (i+2j \leq n) \text{ and } \mathbf{T}[i..i+j-1] = \mathbf{T}[i+j..i+2j-1]\}.$$

This predicate asserts that $\mathbf{T}[i..i+2j-1]$ is a square occurring in $\mathbf{T}[0..n-1]$.

Unfortunately, applying our enumeration method to this suffers from the same problem as before, so we rewrite it as

$$(j \geq 1) \wedge \ (i+2j \leq n) \ \wedge \forall u \ ((u \geq i) \wedge (u < i+j)) \implies \mathbf{T}[u] = \mathbf{T}[u+j]$$

When we compute the linear representation of the function counting the number of such i and j, we get a linear representation of rank 63. Now we compute the minimal polynomial of M_0 which is $(x-1)^2(x^2+x+1)^2(x^3-x^2-x-1)^2$. Solving a linear system in terms of the roots (or, more accurately, in terms of the sequences 1, n, T_n, T_{n-1}, T_{n-2}, nT_n, nT_{n-1}, nT_{n-2}) gives

Theorem 9. *The total number of occurrences of squares in the Tribonacci word Y_n is*

$$c(n) = \frac{n}{22}(9T_n - T_{n-1} - 5T_{n-2}) + \frac{1}{44}(-117T_n + 30T_{n-1} + 33T_{n-2}) + n - \frac{7}{4}$$

for $n \geq 5$.

In a similar way, we can count the occurrences of cubes in the finite Tribonacci word Y_n. Here we get a linear representation of rank 46. The minimal polynomial for M_0 is $x^4(x^3-x^2-x-1)^2(x^2+x+1)^2(x-1)^2$. Using analysis exactly like the square case, we easily find

Theorem 10. *Let $C(n)$ denote the number of cube occurrences in the Tribonacci word Y_n. Then for $n \geq 3$ we have*

$$C(n) = \frac{1}{44}(T_n + 2T_{n-1} - 33T_{n-2}) + \frac{n}{22}(-6T_n + 8T_{n-1} + 7T_{n-2}) + \frac{n}{6} - \frac{1}{4}[n \equiv 0 \pmod 3] + \frac{1}{12}[n \equiv 1 \pmod 3] - \frac{7}{12}[n \equiv 2 \pmod 3].$$

Here $[P]$ is Iverson notation, and equals 1 if P holds and 0 otherwise.

5 Additional Results

Next, we encode the orders and positions of all cubes. We build a DFA accepting the language

$$\{(i,n)_T \ : \ (n>0) \ \wedge \ \forall j \ ((i \le j) \wedge (j < i+2n)) \implies \mathbf{T}[j] = \mathbf{T}[j+n]\}.$$

Theorem 11. *The language*

$$\{(n,i)_T \ : \ \textit{there is a cube of order } n \textit{ beginning at position } i \textit{ in } \mathbf{T}\}$$

is accepted by the automaton in Fig. 3.

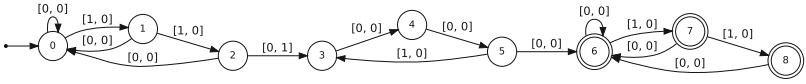

Fig. 3. Automaton accepting orders and positions of all cubes in $\mathbf{T}$

We also computed an automaton accepting those pairs (p,n) such that there is a factor of $\mathbf{T}$ having length n and period p, and n is the largest such length corresponding to the period p. However, this automaton has 266 states, so we do not give it here.

5.1 Palindromes

We now turn to a characterization of the palindromes in $\mathbf{T}$. Once again it turns out that the predicate we previously used in [13], namely,

$$\exists i \ \forall j < n \ \mathbf{T}[i+j] = \mathbf{T}[i+n-1-j],$$

resulted in an intermediate NFA of 5711 states that we could not successfully determinize.

Instead, we used two equivalent predicates. The first accepts n if there is an even-length palindrome, of length $2n$, centered at position i:

$$\exists i \ge n \ \forall j < n \ \mathbf{T}[i+j] = \mathbf{T}[i-j-1].$$

The second accepts n if there is an odd-length palindrome, of length $2n+1$, centered at position i:

$$\exists i \ge n \ \forall j \ (1 \le j \le n) \implies \mathbf{T}[i+j] = \mathbf{T}[i-j].$$

Theorem 12. *There exist palindromes of every length* ≥ 0 *in* $\mathbf{T}$.

Proof. For the first predicate, our program outputs the automaton below. It clearly accepts the Tribonacci representations for all n (Fig. 4).

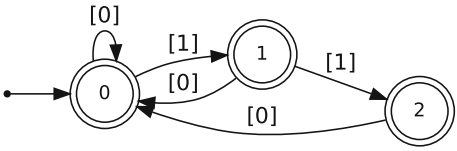

Fig. 4. Automaton accepting lengths of palindromes in **T**

The log of our program follows.

```
i >= n with 13 states, in 34ms
j < n with 13 states, in 8ms
i + j with 150 states, in 53ms
i - 1 with 7 states, in 155ms
i - 1 - j with 150 states, in 166ms
TR[i + j] = TR[i - 1 - j] with 664 states, in 723ms
j < n => TR[i + j] = TR[i - 1 - j] with 3312 states, in 669ms
Aj j < n => TR[i + j] = TR[i - 1 - j] with 24 states, in 5782274ms
i >= n & Aj j < n => TR[i + j] = TR[i - 1 - j] with 24 states, in 0ms
Ei i >= n & Aj j < n => TR[i + j] = TR[i - 1 - j] with 4 states, in 6ms
overall time: 5784088ms
```

The largest intermediate automaton had 918871 states. This was a fairly significant computation, taking about two hours' CPU time on a laptop.

We omit the details of the computation for the odd-length palindromes, which are quite similar. □

Remark 1. A. Glen has pointed out to us that this follows from the fact that **T** is episturmian and hence rich, so a new palindrome is introduced at each new position in T.

We could also characterize the positions of all nonempty palindromes. To illustrate the idea, we generated an automaton accepting (i, n) such that $\mathbf{T}[i-n..i+n-1]$ is an (even-length) palindrome (Fig. 5).

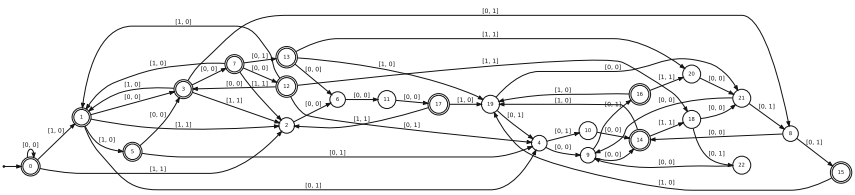

Fig. 5. Automaton accepting orders and positions of all nonempty even-length palindromes in **T**

The prefixes are factors of particular interest. Let us determine which prefixes are palindromes:

Theorem 13. *The prefix* $\mathbf{T}[0..n-1]$ *of length* n *is a palindrome if and only if* $n = 0$ *or* $(n)_T \in 1 + 11 + 10(010)^*(00 + 001 + 0011)$.

Proof. We use the predicate

$$\forall i < n \ \mathbf{T}[i] = \mathbf{T}[n-1-i].$$

The automaton generated is given below (Fig. 6). □

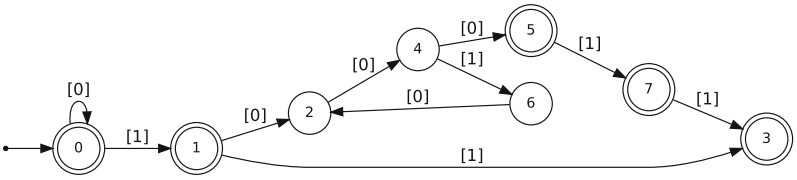

Fig. 6. Automaton accepting lengths of palindromes in $\mathbf{T}$

Remark 2. A. Glen points out to us that the palindromic prefixes of $\mathbf{T}$ are precisely those of the form $\mathrm{Pal}(w)$, where w is a finite prefix of the infinite word $(012)^\omega$ and Pal denotes the "iterated palindromic closure"; see, for example, [19, Example 2.6]. She also points out that these lengths are precisely the integers $(T_i + T_{i+2} - 3)/2$ for $i \geq 1$.

5.2 Quasiperiods

We now turn to quasiperiods. An infinite word $\mathbf{a}$ is said to be *quasiperiodic* if there is some finite nonempty word x such that $\mathbf{a}$ can be completely "covered" with translates of x. Here we study the stronger version of quasiperiodicity where the first copy of x used must be aligned with the left edge of $\mathbf{w}$ and is not allowed to "hang over"; these are called *aligned covers* in [10]. More precisely, for us $\mathbf{a} = a_0a_1a_2\cdots$ is quasiperiodic if there exists x such that for all $i \geq 0$ there exists $j \geq 0$ with $i - n < j \leq i$ such that $a_ja_{j+1}\cdots a_{j+n-1} = x$, where $n = |x|$. Such an x is called a *quasiperiod*. Note that the condition $j \geq 0$ implies that, in this interpretation, any quasiperiod must actually be a prefix of $\mathbf{a}$.

Glen, Levé, and Richomme characterized the quasiperiods of a large class of words, including the Tribonacci word [20, Theorem 4.19]. However, their characterization did not explicitly give the lengths of the quasiperiods. We do that in the following new result.

Theorem 14. *A nonempty length-*n *prefix of* $\mathbf{T}$ *is a quasiperiod of* $\mathbf{T}$ *if and only if* n *is accepted by the following automaton (Fig. 7):*

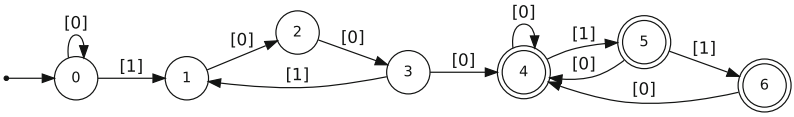

Fig. 7. Automaton accepting lengths of quasiperiods of the Tribonacci sequence

Proof. We write a predicate for the assertion that the length-n prefix is a quasiperiod:

$$\forall i \geq 0 \ \exists j \text{ with } i - n < j \leq i \text{ such that } \forall t < n \ \mathbf{T}[t] = \mathbf{T}[j + t].$$

When we do this, we get the automaton above. These numbers are those i for which $T_n \leq i \leq U_n$ for $n \geq 5$, where $U_2 = 0$, $U_3 = 1$, $U_4 = 3$, and $U_n = U_{n-1} + U_{n-2} + U_{n-3} + 3$ for $n \geq 5$. □

5.3 Unbordered Factors

Next we look at unbordered factors. A word y is said to be a *border* of x if y is both a nonempty proper prefix and suffix of x. A word x is *bordered* if it has at least one border. It is easy to see that if a word y is bordered iff it has a border of length ℓ with $0 < \ell \leq |y|/2$.

Theorem 15. *There is an unbordered factor of length n of $\mathbf{T}$ if and only if $(n)_T$ is accepted by the automaton given below (Fig. 8).*

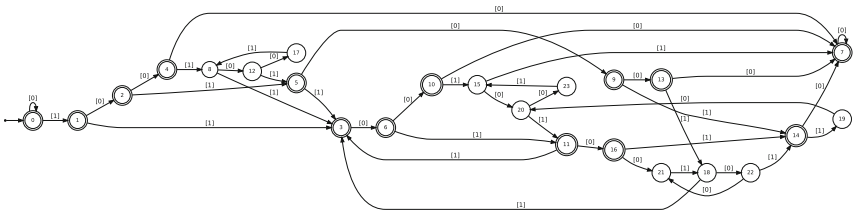

Fig. 8. Automaton accepting lengths of unbordered factors of the Tribonacci sequence

Proof. As in a previous paper [13] we can express the property of having an unbordered factor of length n as follows

$$\exists i \ \forall j, 1 \leq j \leq n/2, \ \exists t < j \ \mathbf{T}[i + t] \neq \mathbf{T}[i + n - j + t].$$

However, this does not run to completion within the available space on our prover. Instead, make the substitutions $t' = n - j$ and $u = i + t$. This gives the predicate

$$\exists i \ \forall t', \ n/2 \leq t' < n, \ \exists u, \ (i \leq u < i + n - t') \ \mathbf{T}[u] \neq \mathbf{T}[u + t'].$$

Here is the log:

```
2 * t with 61 states, in 276ms
n <= 2 * t with 79 states, in 216ms
t < n with 13 states, in 3ms
n <= 2 * t & t < n with 83 states, in 9ms
u >= i with 13 states, in 7ms
i + n with 150 states, in 27ms
i + n - t with 1088 states, in 7365ms
u < i + n - t with 1486 states, in 6041ms
u >= i & u < i + n - t with 1540 states, in 275ms
u + t with 150 states, in 5ms
TR[u] != TR[u + t] with 102 states, in 22ms
u >= i & u < i + n - t & TR[u] != TR[u + t] with 7489 states, in 3364ms
Eu u >= i & u < i + n - t & TR[u] != TR[u + t] with 552 states, in 5246873ms
n <= 2 * t & t < n => Eu u >= i & u < i + n - t & TR[u] != TR[u + t] with 944 states, in 38ms
At n <= 2 * t & t < n => Eu u >= i & u < i + n - t & TR[u] != TR[u + t] with 47 states, in 1184ms
Ei At n <= 2 * t & t < n => Eu u >= i & u < i + n - t & TR[u] != TR[u + t] with 25 states, in 2ms
overall time: 5265707ms
```

□

5.4 Lyndon Words

Next, we turn to some results about Lyndon words. Recall that a nonempty word x is a *Lyndon word* if it is lexicographically less than all of its nonempty proper prefixes.[1]

Theorem 16. *There is a factor of length n of $\mathbf{T}$ that is Lyndon if and only if n is accepted by the automaton given below (Fig. 9).*

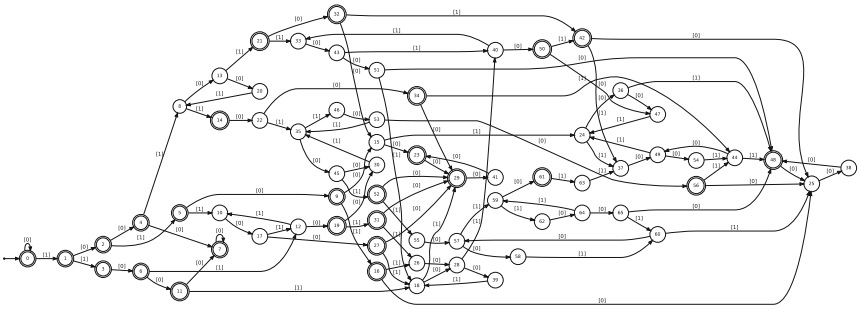

Fig. 9. Automaton accepting lengths of Lyndon factors of the Tribonacci sequence

Proof. Here is a predicate specifying that there is a factor of length n that is Lyndon:

$$\exists i\, \forall j, 1 \le j < n,\ \exists t < n-j\ (\forall u < t\ \mathbf{T}[i+u] = \mathbf{T}[i+j+u]) \wedge \mathbf{T}[i+t] < \mathbf{T}[i+j+t].$$

Unfortunately this predicate did not run to completion, so we substituted $u' := i + u$ to get

$$\exists i\, \forall j, 1 \le j < n,\ \exists t < n-j\ (\forall u', i \le u' < i+t\ \mathbf{T}[u'] = \mathbf{T}[u'+j]) \wedge \mathbf{T}[i+t] < \mathbf{T}[i+j+t].$$ □

[1] There is also a version where "prefixes" is replaced by "suffixes".

5.5 Critical Exponent

Recall from Sect. 3 that $\exp(w) = |w|/P$, where P is the smallest period of w. The *critical exponent* of an infinite word $\mathbf{x}$ is the supremum, over all factors w of $\mathbf{x}$, of $\exp(w)$.

Then Tan and Wen [33] proved that

Theorem 17. *The critical exponent of* $\mathbf{T}$ *is* $\rho \doteq 3.191487883953118747 06$*, the real zero of the polynomial* $2x^3 - 12x^2 + 22x - 13$.

A. Glen points out that this result can also be deduced from [26, Theorem 5.2].

Proof. Let x be any factor of exponent ≥ 3 in $\mathbf{T}$. From Theorem 11 we know that such x exist. Let $n = |x|$ and p be the period, so that $n/p \geq 3$. Then by considering the first $3p$ symbols of x, which form a cube, we have by Theorem 11 that $p = T_n$. So it suffices to determine the largest n corresponding to every p of the form T_n. We did this using the predicate (Fig. 10).

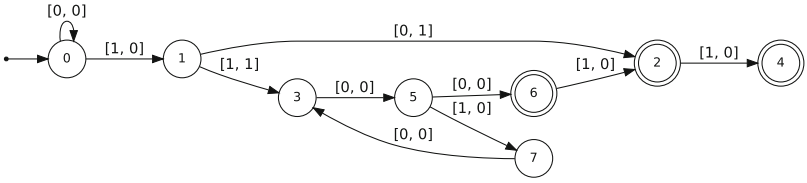

Fig. 10. Length n of longest factors having period $p = T_n$ of Tribonacci sequence

From inspection of the automaton, we see that the maximum length of a factor $n = U_j$ having period $p = T_j$, $j \geq 2$, is given by

$$U_j = \begin{cases} 2, & \text{if } j = 2; \\ 5, & \text{if } j = 3; \\ [110(100)^{i-1}0]_T, & \text{if } j = 3i+1 \geq 4; \\ [110(100)^{i-1}01]_T, & \text{if } j = 3i+2 \geq 5; \\ [110(100)^{i-1}011]_T, & \text{if } j = 3i+3 \geq 6. \end{cases}$$

A tedious induction shows that U_j satisfies the linear recurrence $U_j = U_{j-1} + U_{j-2} + U_{j-3} + 3$ for $j \geq 5$. Hence we can write U_j as a linear combination of Tribonacci sequences and the constant sequence 1, and solving for the constants we get

$$U_j = \frac{5}{2}T_j + T_{j-1} + \frac{1}{2}T_{j-2} - \frac{3}{2}$$

for $j \geq 2$.

The critical exponent of T is then $\sup_{j \geq 1} U_j/T_j$. Now

$$U_j/T_j = \frac{5}{2} + \frac{T_{j-1}}{T_j} + \frac{T_{j-2}}{2T_j} - \frac{3}{2T_j} = \frac{5}{2} + \alpha^{-1} + \frac{1}{2}\alpha^{-2} + O(1.8^{-j}).$$

Hence U_j/T_j tends to $5/2 + \alpha^{-1} + \frac{1}{2}\alpha^{-2} = \rho$. □

We can also ask the same sort of questions about the *initial critical exponent* of a word $\mathbf{w}$, which is the supremum over the exponents of all prefixes of $\mathbf{w}$.

Theorem 18. *The initial critical exponent of* $\mathbf{T}$ *is* $\rho - 1$.

Proof. We create an automaton M_{ice} accepting the language

$$L = \{(n,p)_T \ : \ \mathbf{T}[0..n-1] \text{ has least period } p\}.$$

It is depicted in Fig. 11 below. An analysis similar to that we gave above for the critical exponent gives the result. □

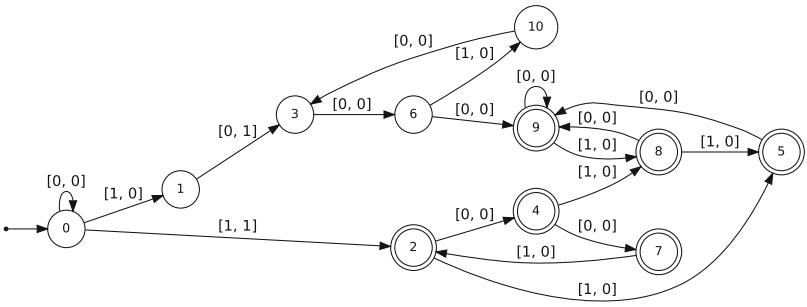

Fig. 11. Automaton accepting least periods of prefixes of length n

Theorem 19. *The only prefixes of the Tribonacci word that are powers are those of length* $2T_n$ *for* $n \geq 5$.

Proof. The predicate

$$\exists d < n \ (\forall j < n - d \ \mathbf{T}[j] = \mathbf{T}[d + j]) \ \wedge \ (\forall k < d \ \mathbf{T}[k] = \mathbf{T}[n - d + k])$$

asserts that the prefix $\mathbf{T}[0..n-1]$ is a power. When we run this through our program, the resulting automaton accepts 100010^*, which corresponds to $F_{n+1} + F_{n-3} = 2T_n$ for $n \geq 5$. □

6 Abelian Properties

We can derive some results about the abelian properties of the Tribonacci word $\mathbf{T}$ by proving the analogue of Theorem 63 of [13]:

Theorem 20. *Let* n *be a non-negative integer and let* $e_1 e_2 \cdots e_j$ *be a Tribonacci representation of* n*, possibly with leading zeros, with* $j \geq 3$. *Then*

(a) $|\mathbf{T}[0..n-1]|_0 = [e_1 e_2 \cdots e_{j-1}]_T + e_j$.
(b) $|\mathbf{T}[0..n-1]|_1 = [e_1 e_2 \cdots e_{j-2}]_T + e_{j-1}$.
(c) $|\mathbf{T}[0..n-1]|_2 = [e_1 e_2 \cdots e_{j-3}]_T + e_{j-2}$.

Proof. By induction, in analogy with the proof of [13, Theorem 63]. □

Recall that the Parikh vector $\psi(x)$ of a word x over an ordered alphabet $\Sigma = \{a_1, a_2, \ldots, a_k\}$ is defined to be $(|x|_{a_1}, \ldots, |x|_{a_k})$, the number of occurrences of each letter in x. Recall that the abelian complexity function $\rho^{\mathrm{ab}}_{\mathbf{w}}(n)$ counts the number of distinct Parikh vectors of the length-n factors of an infinite word $\mathbf{w}$.

Using Theorem 20 we get another proof of a recent result of Turek [34].

Corollary 1. *The abelian complexity function of* $\mathbf{T}$ *is Tribonacci-regular.*

Proof. First, from Theorem 20 there exists an automaton TAB such that $(n, i, j, k)_T$ is accepted iff $(i, j, k) = \psi(\mathbf{T}[0..n-1])$. In fact, such an automaton has 32 states.

Using this automaton, we can create a predicate $P(n, i)$ such that the number of i for which $P(n, i)$ is true equals $\rho^{\mathrm{ab}}_{\mathbf{T}}(n)$. For this we assert that i is the least index at which we find an occurrence of the Parikh vector of $\mathbf{T}[i..i+n-1]$:

$$\forall i' < i \ \exists a_0, a_1, a_2, b_0, b_1, b_2, c_0, c_1, c_2, d_0, d_1, d_2$$
$$\mathrm{TAB}(i{+}n, a_0, a_1, a_2) \wedge \mathrm{TAB}(i, b_0, b_1, b_2) \wedge \mathrm{TAB}(i'{+}n, c_0, c_1, c_2) \wedge \mathrm{TAB}(i', d_0, d_1, d_2) \wedge$$
$$((a_0{-}b_0 \neq c_0{-}d_0) \vee (a_1{-}b_1 \neq c_1{-}d_1) \vee (a_2{-}b_2 \neq c_2{-}d_2)).$$ □

Remark 3. Note that exactly the same proof would work for any word and numeration system where the Parikh vector of prefixes of length n is "synchronized" with n.

Remark 4. In principle we could mechanically compute the Tribonacci-regular representation of the abelian complexity function using this technique, but with our current implementation this is not computationally feasible.

Theorem 21. *Any morphic image of the Tribonacci word is Tribonacci-automatic.*

Proof. In analogy with Corollary 69 of [13]. □

7 Things We Could Not Do Yet

There are a number of things we have not succeeded in computing with our prover because it ran out of space. These include

- Mirror invariance of $\mathbf{T}$ (that is, if x is a finite factor then so is x^R);
- Counting the number of special factors of length n (although it can be deduced from the subword complexity function);

– Statistics about, e.g., lengths of squares, cubes, etc., in the "flipped" Tribonacci sequence [31], the fixed point of $0 \to 01$, $1 \to 20$, $2 \to 0$;
– Recurrence properties of the Tribonacci word;
– Counting the number of distinct squares (not occurrences) in the finite Tribonacci word Y_n.
– Abelian complexity of the Tribonacci word.

In the future, an improved implementation may succeed in resolving these in a mechanical fashion.

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References

1. Allouche, J.P., Rampersad, N., Shallit, J.: Periodicity, repetitions, and orbits of an automatic sequence. Theoret. Comput. Sci. **410**, 2795–2803 (2009)
2. Allouche, J.P., Shallit, J.: Automatic Sequences: Theory, Applications, Generalizations. Cambridge University Press, Cambridge (2003)
3. Barcucci, E., Bélanger, L., Brlek, S.: On Tribonacci sequences. Fibonacci Quart. **42**, 314–319 (2004)
4. Berstel, J., Reutenauer, C.: Noncommutative Rational Series with Applications, Encylopedia of Mathematics and Its Applications, vol. 137. Cambridge University Press, Cambridge (2011)
5. Bruyère, V., Hansel, G.: Bertrand numeration systems and recognizability. Theoret. Comput. Sci. **181**, 17–43 (1997)
6. Bruyère, V., Hansel, G., Michaux, C.,Villemaire, R.: Logic and p-recognizable sets of integers. Bull. Belgian Math. Soc. 1, 191–238 (1994), corrigendum. Bull. Belg. Math. Soc. **1**, 577 (1994)
7. Carlitz, L., Scoville, R., Hoggatt, Jr., V.E.: Fibonacci representations of higher order. Fibonacci Quart. **10**, 43–69, 94 (1972)
8. Charlier, E., Rampersad, N., Shallit, J.: Enumeration and decidable properties of automatic sequences. Int. J. Found. Comp. Sci. **23**, 1035–1066 (2012)
9. Chekhova, N., Hubert, P., Messaoudi, A.: Propriétés combinatoires, ergodiques et arithmétiques de la substitution de Tribonacci. J. Théorie Nombres Bordeaux **13**, 371–394 (2001)
10. Christou, M., Crochemore, M., Iliopoulos, C.S.: Quasiperiodicities in Fibonacci strings (2012), to appear in Ars Combinatoria. Preprint available at http://arxiv.org/abs/1201.6162
11. Cobham, A.: Uniform tag sequences. Math. Syst. Theory **6**, 164–192 (1972)
12. Droubay, X., Justin, J., Pirillo, G.: Episturmian words and some constructions of de Luca and Rauzy. Theoret. Comput. Sci. **255**, 539–553 (2001)
13. Du, C.F., Mousavi, H., Schaeffer, L., Shallit, J.: Decision algorithms for Fibonacci-automatic words, with applications to pattern avoidance (2014). http://arxiv.org/abs/1406.0670
14. Duchêne, E., Rigo, M.: A morphic approach to combinatorial games: theTribonacci case. RAIRO Inform. Théor. App. **42**, 375–393 (2008)
15. Frougny, C.: Representations of numbers and finite automata. Math. Systems Theory **25**, 37–60 (1992)

16. Frougny, C., Solomyak, B.: On representation of integers in linear numeration systems. In: Pollicott, M., Schmidt, K. (eds.) Ergodic Theory of $\mathbb{Z}^d$ Actions (Warwick, 1993–1994). London Mathematical Society Lecture Note Series, vol. 228, pp. 345–368. Cambridge University Press, Cambridge (1996)
17. Glen, A.: On sturmian and episturmian words, and related topics. Ph.D. thesis, University of Adelaide (2006)
18. Glen, A.: Powers in a class of a-strict episturmian words. Theoret. Comput. Sci. **380**, 330–354 (2007)
19. Glen, A., Justin, J.: Episturmian words: a survey. RAIRO Inform. Théor. App. **43**, 402–433 (2009)
20. Glen, A., Levé, F., Richomme, G.: Quasiperiodic and Lyndon episturmian words. Theoret. Comput. Sci. **409**, 578–600 (2008)
21. Goč, D., Henshall, D., Shallit, J.: Automatic theorem-proving in combinatorics on words. In: Moreira, N., Reis, R. (eds.) CIAA 2012. LNCS, vol. 7381, pp. 180–191. Springer, Heidelberg (2012)
22. Goč, D., Mousavi, H., Shallit, J.: On the number of unbordered factors. In: Dediu, A.-H., Martín-Vide, C., Truthe, B. (eds.) LATA 2013. LNCS, vol. 7810, pp. 299–310. Springer, Heidelberg (2013)
23. Goč, D., Saari, K., Shallit, J.: Primitive words and Lyndon words in automatic and linearly recurrent sequences. In: Dediu, A.-H., Martín-Vide, C., Truthe, B. (eds.) LATA 2013. LNCS, vol. 7810, pp. 311–322. Springer, Heidelberg (2013)
24. Goč, D., Schaeffer, L., Shallit, J.: Subword complexity and k-synchronization. In: Béal, M.-P., Carton, O. (eds.) DLT 2013. LNCS, vol. 7907, pp. 252–263. Springer, Heidelberg (2013)
25. Hales, T.C.: Formal proof. Notices Am. Math. Soc. **55**(11), 1370–1380 (2008)
26. Justin, J., Pirillo, G.: Episturmian words and episturmian morphisms. Theoret. Comput. Sci. **276**, 281–313 (2002)
27. Konev, B., Lisitsa, A.: A SAT attack on the Erdős discrepancy problem (2014). Preprint available at http://arxiv.org/abs/1402.2184
28. Presburger, M.: Über die Volständigkeit eines gewissen Systems derArithmetik ganzer Zahlen, in welchem die Addition als einzigeOperation hervortritt. In: Sparawozdanie z I Kongresu matematykówkrajów slowianskich, Warsaw, pp. 92–101, 395 (1929)
29. Presburger, M.: On the completeness of a certain system of arithmetic of whole numbers in which addition occurs as the only operation. Hist. Phil. Logic **12**, 225–233 (1991)
30. Richomme, G., Saari, K., Zamboni, L.Q.: Balance and Abelian complexity of the Tribonacci word. Adv. Appl. Math. **45**, 212–231 (2010)
31. Rosema, S.W., Tijdeman, R.: The Tribonacci substitution. INTEGERS 5 (3), Paper #A13 (2005). Available at http://www.integers-ejcnt.org/vol5-3.html
32. Shallit, J.: Decidability and enumeration for automatic sequences: a survey. In: Bulatov, A.A., Shur, A.M. (eds.) CSR 2013. LNCS, vol. 7913, pp. 49–63. Springer, Heidelberg (2013)
33. Tan, B., Wen, Z.Y.: Some properties of the Tribonacci sequence. Eur. J. Comb. **28**, 1703–1719 (2007)
34. Turek, O.: Abelian complexity function of the Tribonacci word. J. Integer Sequences **18**, Article 15.3.4 (2015). Available at https://cs.uwaterloo.ca/journals/JIS/VOL18/Turek/turek3.html