On conjugacy classes in a reductive group

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Dedicated to David Vogan
on the occasion of his 60th birthday

Abstract Let G be a connected reductive group over an algebraically closed field. We define a decomposition of G into finitely many strata such that each stratum is a union of conjugacy classes of fixed dimension; the strata are indexed purely in terms of the Weyl group and the indexing set is independent of the characteristic.

Key words: Conjugacy class, Springer correspondence, reductive group, Weyl group

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Introduction

0.1 Let $\mathbf{k}$ be an algebraically closed field of characteristic $p \geq 0$ and let G be a connected reductive algebraic group over $\mathbf{k}$. Let W be the Weyl group of G. Let $\mathrm{cl}(W)$ be the set of conjugacy classes of W.

In [St] Steinberg defined the notion of regular element in G (an element whose conjugacy class has dimension as large as possible, that is $\dim(G) - \mathrm{rk}(G)$) and showed that the set of regular elements in G form an open dense subset G_{reg}. The goal of this paper is to define a partition of G into finitely many strata, one of which is G_{reg}. Each stratum of G is a union of conjugacy classes of G of the same dimension. The set of strata is naturally indexed by a set which depends only on W

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as a Coxeter group, not on the underlying root system and not on the ground field **k**. We give two descriptions of the indexing set above:

(i) one in terms of a class of irreducible representations of W which we call 2-special representations (they are obtained by truncated induction from special representations of certain reflection subgroups of W);
(ii) one in terms of $\mathrm{cl}(W)$ (modulo a certain equivalence relation).

In the case where W is irreducible we give a third description of the indexing set above:

(iii) in terms of the sets of unipotent classes in the various versions of G over $\overline{\mathbf{F}}_r$ for a variable prime number r, glued together according to the set of unipotent classes in the version of G over $\mathbf{C}$.

The definition of strata in the form (i) and (iii) are based on Springer's correspondence (see [Spr] when $p = 0$ or $p \gg 0$ and [L3] for any p) connecting irreducible representations of W with unipotent classes; when W is irreducible, the definition of strata in the form (iii) is related to that in the form (ii) by the results of [L8, L10] connecting $\mathrm{cl}(W)$ with unipotent classes in G.

Since (i),(ii) are two incarnations of our indexing set, they are in canonical bijection with each other. In particular we obtain a canonical map from $\mathrm{cl}(W)$ to the set of irreducible representations of W whose image consists of the 2-special representations (when G is $GL_n(\mathbf{k})$ this is a bijection). We also show that the dimension of a conjugacy class in a stratum of G is independent of the ground field. (This statement makes sense since the parametrization of the strata is independent of the ground field.) In particular, we see that if $n \geq 1$, then the following three conditions on an integer k are equivalent:

- there exists a conjugacy class of dimension k in $\mathrm{SO}_{2n+1}(\mathbf{C})$;
- there exists a conjugacy class of dimension k in $\mathrm{Sp}_{2n}(\mathbf{C})$;
- there exists a conjugacy class of dimension k in $\mathrm{Sp}_{2n}(\overline{\mathbf{F}}_2)$.

The proof shows that the following fourth condition is equivalent to the three conditions above: there exists a unipotent conjugacy class of dimension k in $\mathrm{Sp}_{2n}(\overline{\mathbf{F}}_2)$.

In Section 5 we sketch an alternative approach to the definition of strata which is based on an extension of the ideas in [L8], and Springer's correspondence does not appear in it.

In Section 6 we dicuss extensions of our results to the Lie algebra of G and to the case where G is replaced by a disconnected reductive group. We also define a partition of the set of compact regular semisimple elements in a loop group into strata analogous to the partition of G into strata. Moreover, we give a conjectural description of the strata of G (assuming that $\mathbf{k} = \mathbf{C}$) which is based on an extension of a construction in [KL].

0.2 Notation. For an algebraic group H over **k**, we denote by H^0 the identity component of H. For a subgroup T of H we denote by $N_H T$ the normalizer of T in H. Let $\mathfrak{g}$ be the Lie algebra of G. For $g \in G$ we denote by $Z_G(g)$ the centralizer of g in G and by g_s (resp. g_u) the semisimple (resp. unipotent) part of g. Let $\mathcal{B}$ be the variety of Borel subgroups of G. Let $\mathcal{B}_g = \{B \in \mathcal{B}; g \in B\}$. Let l be a prime number $\neq p$. For an algebraic variety X over $\mathbf{k}$ we denote by $H^i(X)$ the l-adic cohomology of X in degree i; if X is projective let $H_i(X) = \mathrm{Hom}(H^i(X), \mathbf{Q}_l)$.

For any (finite) Weyl group Γ, we denote by Irr Γ a set of representatives for the isomorphism classes of irreducible representations of Γ over $\mathbf{Q}$. For any $\tau \in \mathrm{Irr} W$ let n_τ be the smallest integer $i \geq 0$ such that τ appears with > 0 multiplicity in the i-th symmetric power of the reflection representation of W; if this multiplicity is 1, we say that τ is *good*.

A *bipartition* is a sequence $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ in $\mathbf{N}$ such that $\lambda_m = 0$ for $m \gg 0$ and $\lambda_1 \geq \lambda_3 \geq \lambda_5 \geq \dots, \lambda_2 \geq \lambda_4 \geq \lambda_6 \geq \dots$. We write $|\lambda| = \lambda_1 + \lambda_2 + \lambda_3 + \dots$. We say that λ is a bipartition of n if $|\lambda| = n$. Let BP^n be the set of bipartitions of n. Let $e, e' \in \mathbf{N}$. We say that a bipartition $(\lambda_1, \lambda_2, \lambda_3, \dots)$ has excess (e, e') if $\lambda_i + e \geq \lambda_{i+1}$ for $i = 1, 3, 5, \dots$ and $\lambda_i + e' \geq \lambda_{i+1}$ for $i = 2, 4, 6, \dots$. Let $BP^n_{e,e'}$ be the set of bipartitions of n which have excess (e, e').

A *partition* is a sequence $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ in $\mathbf{N}$ such that $\lambda_m = 0$ for $m \gg 0$ and $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$. Thus a partition is the same as a bipartition of excess $(0, 0)$. On the other hand, a bipartition is the same as an ordered pair of partitions $((\lambda_1, \lambda_3, \lambda_5, \dots), (\lambda_2, \lambda_4, \lambda_6, \dots))$.

Let $\mathcal{P} = \{2, 3, 5, \dots\}$ be the set of prime numbers.

1 The 2-special representations of a Weyl group

1.1 Let V, V^* be finite-dimensional $\mathbf{Q}$-vector spaces with a given perfect bilinear pairing $\langle , \rangle : V \times V^* \to \mathbf{Q}$. Let R (resp. $\check{R}$) be a finite subset of $V - \{0\}$ (resp. $V^* - \{0\}$) with a given bijection $\alpha \leftrightarrow \check{\alpha}$, $R \leftrightarrow \check{R}$, such that $\langle \alpha, \check{\alpha} \rangle = 2$ for any $\alpha \in R$ and $\langle \alpha, \check{\beta} \rangle \in \mathbf{Z}$ for any $\alpha, \beta \in R$; it is assumed that $\beta - \langle \beta, \check{\alpha} \rangle \alpha \in R$, $\check{\beta} - \langle \alpha, \beta \rangle \check{\alpha} \in \check{R}$ for any $\alpha, \beta \in R$ and that $\alpha \in R \implies \alpha/2 \notin R$. Thus, $(V, V^*, R, \check{R})$ is a reduced root system. Let V_0 (resp. V_0^*) be the $\mathbf{Q}$-subspace of V (resp. V^*) spanned by R (resp. $\check{R}$). Let $\mathrm{rk}(R) = \dim V_0 = \dim V_0^*$. Let W be the (finite) subgroup of $GL(V)$ generated by the reflections $s_\alpha : x \mapsto x - \langle x, \check{\alpha} \rangle \alpha$ in V for various $a \in R$; it may be identified with the subgroup of $GL(V^*)$ generated by the reflections ${}^t s_a : x' \mapsto x' - \langle \alpha, x' \rangle \check{\alpha}$ in V^* for various $\alpha \in R$. For any $e \in V$ let $R_e = \{\alpha \in R; \langle e, \check{\alpha} \rangle \in \mathbf{Z}\}$, $\check{R}_e = \{\check{\alpha}; \alpha \in R_e\}$; note that $(V, V^*, R_e, \check{R}_e)$ is a root system with Weyl group $W_e = \{w \in W; w(e) - e \in \sum_{\alpha \in R} \mathbf{Z}\alpha\}$. Similarly, for any $e' \in V^*$ let $R_{e'} = \{\alpha \in R; \langle \alpha, e' \rangle \in \mathbf{Z}\}$, $\check{R}_{e'} = \{\check{\alpha}; \alpha \in R_{e'}\}$; note that $(V, V^*, R_{e'}, \check{R}_{e'})$ is a root system with Weyl group $W_{e'} = \{w \in W; w(e') - e' \in \sum_{\alpha \in R} \mathbf{Z}\check{\alpha}\}$. For any $(e, e') \in V \times V^*$ let $R_{e,e'} = R_e \cap R_{e'}$, $\check{R}_{e,e'} = \check{R}_e \cap \check{R}_{e'}$. Then $(V, V^*, R_{e,e'}, \check{R}_{e,e'})$ is a root system; let $W_{e,e'}$ be its Weyl group (a subgroup of $W_e \cap W_{e'}$). Note that $W_{0,e'} = W_{e'}$, $W_{e,0} = W_e$, $W_{0,0} = W$. For $E \in \mathrm{Irr}(W_{e,e'})$ let n_E be as in 0.2.

Let $(e_1, e'_1) \in V \times V^*$, $(e_2, e'_2) \in V \times V^*$ be such that $R_{e_1,e'_1} \subset R_{e_2,e'_2}$ (so that $W_{e_1,e'_1} \subset W_{e_2,e'_2}$). In this case, if $E \in \mathrm{Irr}(W_{e_1,e'_1})$ is good, there is a unique

$E_0 \in \mathrm{Irr}(W_{e_2,e_2'})$ such that E_0 appears in $\mathrm{Ind}_{W_{e_1,e_1'}}^{W_{e_2,e_2'}}(E)$ and $n_{E_0} = n_E$, see [LS1, 3.2]; moreover, E_0 is good. We set $E_0 = j_{W_{e_1,e_1'}}^{W_{e_2,e_2'}}(E)$. Note that if we have also $R_{e_2,e_2'} \subset R_{e_3,e_3'}$ where $(e_3, e_3') \in V \times V^*$, then we have the transitivity property:

(a) $j_{W_{e_1,e_1'}}^{W_{e_3,e_3'}}(E) = j_{W_{e_2,e_2'}}^{W_{e_3,e_3'}}(j_{W_{e_1,e_1'}}^{W_{e_2,e_2'}}(E))$.

Let $\mathcal{S}(W_{e,e'}) \subset \mathrm{Irr}(W_{e,e'})$ be the set of *special* representations of $W_{e,e'}$, see [L1]; note that any $E \in \mathcal{S}(W_{e,e'})$ is good. Hence $j_{W_{e,e'}}^W(E) \in \mathrm{Irr}(W)$ is defined. We say that $E_0 \in \mathrm{Irr}(W)$ is 2-*special* if $E_0 = j_{W_{e,e'}}^W(E)$ for some $(e, e') \in V \times V^*$ and some $E \in \mathcal{S}(W_{e,e'})$. Let $\mathcal{S}_2(W)$ be the set of all 2-special representations of W (up to isomorphism). From the definition we see that

(b) *$\mathcal{S}_2(W)$ is unchanged when $(V, V^*, R, \check{R})$ is replaced by $(V^*, V, \check{R}, R)$.*

Let $\mathcal{S}_1(W)$ (resp. $'\mathcal{S}_1(W)$) be the set of all $E_0 \in \mathrm{Irr}(W)$ such that $E_0 = j_{W_e}^W(E)$ (resp. $E_0 = j_{W_{e'}}^W(E)$) for some $e \in V$, $E \in \mathcal{S}(W_e)$ (resp. $e' \in V^*$, $E \in \mathcal{S}(W_{e'})$). The analogue of (b) with $\mathcal{S}_2(W)$ replaced by $\mathcal{S}_1(W)$ is not true in general; instead, if $(V, V^*, R, \check{R})$ is replaced by $(V^*, V, \check{R}, R)$, then $\mathcal{S}_1(W)$ becomes $'\mathcal{S}_1(W)$ and $'\mathcal{S}_1(W)$ becomes $\mathcal{S}_1(W)$.

Now, for any $e' \in V^*$ the subset $\mathcal{S}_1(W_{e'}) \subset \mathrm{Irr}(W_{e'})$ is defined; it consists of all $E' \in \mathrm{Irr}(W_{e'})$ such that $E' = j_{W_{e,e'}}^{We'}(E)$ for some $e \in V$ and some $E \in \mathcal{S}(W_{e,e'})$. Note that any $E' \in \mathcal{S}_1(W_{e'})$ is good. From (a) we see that

(c) *$\mathcal{S}_2(W)$ consists of all $E_0 \in \mathrm{Irr}(W)$ such that $E_0 = j_{W_{e'}}^W(E')$ for some $e' \in V^*$ and some $E' \in \mathcal{S}_1(W_{e'})$.*

We say that $e' \in V^*$ (resp. $(e, e') \in V \times V^*$) is *isolated* if $\mathrm{rk}(R_{e'}) = \mathrm{rk}(R)$ (resp. $\mathrm{rk}(R_{e,e'}) = \mathrm{rk}(R)$). We show:

(d) *$\mathcal{S}_2(W)$ consists of all $E_0 \in \mathrm{Irr}(W)$ such that $E_0 = j_{W_{e,e'}}^W(E)$ for some isolated $(e, e') \in V \times V^*$ and some $E \in \mathcal{S}(W_{e,e'})$.*

Let $E_0 \in \mathcal{S}_2(W)$. By definition, we can find $(e, e') \in V \times V^*$ and $E \in \mathcal{S}(W_{e,e'})$ such that $E_0 = j_{W_{e,e'}}^W(E)$. We can find an isolated $e_1' \in V^*$ such that $R_{e'}$ is rationally closed in $R_{e_1'}$ that is, $R_{e_1'} \cap \sum_{\alpha \in R_{e'}} \mathbf{Q}\alpha = R_{e'}$. Applying the analogous statement to $(V^*, V, \check{R}_{e_1'}, R_{e_1'})$, e, instead of $(V, V^*, R, \check{R})$, e', we can find $e_1 \in V$ such that $\mathrm{rk}(R_{e_1} \cap R_{e_1'}) = \mathrm{rk}(R_{e_1'})$ and $R_e \cap R_{e_1'}$ is rationally closed in $R_{e_1} \cap R_{e_1'}$. It follows that (e_1, e_1') is isolated and $R_e \cap R_{e'}$ is rationally closed in $R_{e_1} \cap R_{e_1'}$; hence $E_1 := j_{W_{e,e'}}^{W_{e_1,e_1'}}(E)$ is in $\mathcal{S}(W_{e_1,e_1'})$, see [L1]. By (a), we have $E_0 = j_{W_{e_1,e_1'}}^W(E_1)$. This proves (d).

We have the following variant of (d):

(e) $\mathcal{S}_2(W)$ *consists of all* $E_0 \in \mathrm{Irr}(W)$ *such that* $E_0 = j_{W_{e'}}^W(\widetilde{E})$ *for some isolated* $e' \in V^*$ *and some* $\widetilde{E} \in \mathcal{S}_1(W_{e'})$.

Let $E_0 \in \mathcal{S}_2(W)$. Let E, e, e' be as in (d). We have $E = j_{W_{e'}}^W(\widetilde{E})$ where $\widetilde{E} = j_{W_{e,e'}}^{W_{e'}}(E) \in \mathcal{S}_1(W_{e'})$ and $\mathrm{rk}(R_{e'}) = \mathrm{rk}(R)$. Conversely, if $e' \in V^*$ and $\widetilde{E} \in \mathcal{S}_1(W_{e'})$, then, by (c), $j_{W_{e'}}^W(\widetilde{E}) \in \mathcal{S}_2(W)$ (even without the assumption that $\mathrm{rk}(R_{e'}) = \mathrm{rk}(R)$). This proves (e).

Let $R' \subset R$ be such that (if $\check{R}'$ is the image of R' under $R \leftrightarrow \check{R}$), $(V, V^*, R', \check{R}')$ is a root system (with Weyl group W') and R' is rationally closed in R. Note that $R' = R_e$ for some $e \in V$ and $R' = R_{e'}$ for some $e' \in V^*$. We show:

(f) If $E \in \mathcal{S}_1(W')$, then $j_{W'}^W(E) \in \mathcal{S}_1(W)$.
(g) If $E \in \mathcal{S}_2(W')$, then $j_{W'}^W(E) \in \mathcal{S}_2(W)$.

We prove (f). Let $e' \in V^*$ be such that $R' = R_{e'}$. We have $E = j_{W_{e,e'}}^{W_{e'}}(E')$ for some $e \in V$ and some $E' \in \mathcal{S}(W_{e,e'})$. Hence $j_{W'}^W(E) = j_{W_{e,e'}}^W(E') = j_{W_e}^W(E'')$ where $E'' = j_{W_{e,e'}}^{W_e}(E')$. Now $R_{e,e'}$ is rationally closed in R_e, hence $E'' \in \mathcal{S}(W_e)$, see [L1]. We see that $j_{W'}^W(E) \in \mathcal{S}_1(W)$.

We prove (g). Let $e \in V$ be such that $R' = R_e$. We have $E = j_{W_{e,e'}}^{W_e}(E')$ for some $e' \in V^*$ and some $E' \in \mathcal{S}_1(W_{e,e'})$. Hence $j_{W'}^W(E) = j_{W_{e,e'}}^W(E') = j_{W_{e'}}^W(E'')$ where $E'' = j_{W_{e,e'}}^{W_{e'}}(E')$. Now $R_{e,e'}$ is rationally closed in $R_{e'}$, hence $E'' \in \mathcal{S}(W_{e'})$, see (f). We see that $j_{W'}^W(E) \in \mathcal{S}_2(W)$.

1.2 There are unique direct sum decompositions $V_0 = \oplus_{i\in I} V_i$, $V_0^* = \oplus_{i\in I} V_i^*$ such that $R = \sqcup_{i\in I}(R \cap V_i)$, $\check{R} = \sqcup_{i\in I}(\check{R} \cap V_i)$ and for any $i \in I$, $(V_i, V_i^*, R \cap V_i, \check{R} \cap V_i)$ is an irreducible root system for (with Weyl group W_i); the bijection $R \cap V_i \leftrightarrow \check{R} \cap V_i$ is induced by $R \leftrightarrow \check{R}$). We have canonically $W = \prod_{I\in I} W_i$ and $\mathcal{S}_2(W) = \prod_{i\in I} \mathcal{S}_2(W_i)$ (via external tensor product).

1.3 In this subsection we assume that $(V, V^*, R, \check{R})$ is irreducible. Now W acts naturally on the set of subgroups W' of W of form $W_{e'}$ for various isolated $e' \in V^*$. The types of various W' which appear in this way are well known and are described below in each case.

(a) R of type $A_n, n \geq 0$: W' of type A_n.
(b) R of type $B_n, n \geq 2$: W' of type $B_a \times D_b$ where $a \in \mathbf{N}, b \in \mathbf{N}-\{1\}, a+b = n$.
(c) R of type $C_n, n \geq 2$: W' of type $C_a \times C_b$ where $a, b \in \mathbf{N}$, $a + b = n$.
(d) R of type $D_n, n \geq 4$: W' of type $D_a \times D_b$ where $a, b \in \mathbf{N} - \{1\}$, $a + b = n$.
(e) R of type E_6: W' of type E_6, A_5A_1, $A_2A_2A_2$.
(f) R of type E_7: W' of type E_7, D_6A_1, A_7, A_5A_2, $A_3A_3A_1$.
(g) R of type E_8: W' of type E_8, E_7A_1, E_6A_2, D_5A_3, A_4A_4, $A_5A_2A_1$, A_7A_1, A_8, D_8.

(h) R of type F_4: W' of type F_4, B_3A_1, A_2A_2, A_3A_1, B_4.
(i) R of type G_2: W' of type G_2, A_2, A_1A_1.

(We use the convention that a Weyl group of type B_n or D_n with $n = 0$ is $\{1\}$.)

1.4 In this subsection we assume that $(V, V^*, R, \check{R})$ is irreducible. Now W acts naturally on the set of subgroups W' of W of form $W_{e,e'}$ for various isolated $(e, e') \in V \times V^*$. The types of various W' which appear in this way are described below in each case. (For type F_4 and G_2 we denote by τ a non-inner involution of W).

(a) R of type A_n: W' of type A_n.
(b) R of type B_n or C_n: W' of type $B_a \times B_b \times D_c \times D_d$ where $a, b \in \mathbf{N}$, $c, d \in \mathbf{N} - \{1\}$, $a + b + c + d = n$.
(c) R of type D_n: W' of type $D_a \times D_b \times D_c \times D_d$ where $a, b, c, d \in \mathbf{N} - \{1\}$, $a + b + c + d = n$.
(d) R of type E_6: W' as in 1.3(e).
(e) R of type E_7: W' as in 1.3(f) and also W' of type $D_4A_1A_1A_1$.
(f) R of type E_8: W' as in 1.3(g) and also W' of type D_6D_2, D_4D_4, $A_3A_3A_1A_1$, $A_2A_2A_2A_2$.
(g) R of type F_4: W' as in 1.3(h), the images under τ of the subgroups W' of type A_3A_1, B_4 in 1.3(h) and also W' of type B_2B_2.
(h) R of type G_2: W' as in 1.3(i) and the image under τ of the subgroup W' of type A_2 in 1.3(i).

1.5 If $R' \subset R$, $\check{R}' \subset \check{R}$ are such that $(V, V^*, R', \check{R}')$ is a root system (with the bijection $R' \leftrightarrow \check{R}'$ being induced by $R \leftrightarrow \check{R}$) then, setting $\overline{R}' = R \cap \sum_{\alpha \in R'} \mathbf{Q}\alpha$, $\overline{\check{R}}' = \check{R} \cap \sum_{\alpha \in R'} \mathbf{Q}\check{\alpha}$, we obtain a root system $(V, V^*, \overline{R}', \overline{\check{R}}')$. We set

$$N_{R'} = \sharp\Big(\sum_{\alpha \in \overline{R}'} \mathbf{Z}\alpha / \sum_{\alpha \in R'} \mathbf{Z}\alpha\Big) \in \mathbf{Z}_{\geq 1}.$$

For any $e' \in V^*$ we set $N_{e'} = N_{R_{e'}}$.

Now let $r \in \mathcal{P}$. Let $\mathcal{S}_2^r(W)$ be the set of all $E_0 \in \mathrm{Irr}(W)$ such that for some isolated $e' \in V^*$ with $N_{e'} = r^k$ for some $k \in \mathbf{N}$ and for some $E \in \mathcal{S}_1(W_{e'})$ we have $E_0 = j_{W_{e'}}^W(E)$. Note that $\mathcal{S}^1(W) \subset \mathcal{S}_2^r(W) \subset \mathcal{S}_2(W)$.

Now assume that $(V, V^*, R, \check{R})$ is irreducible. We show:

(a) If R is of type A_n, $n \geq 0$, then $\mathcal{S}_2^r(W) = \mathcal{S}_2(W) = \mathcal{S}_1(W) = \mathcal{S}(W)$.
(b) If R is of type B_n or C_n, $n \geq 2$, then $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$ if $r \neq 2$ and $\mathcal{S}_2^2(W) = \mathcal{S}_2(W)$.
(c) If R is of type D_n, $n \geq 4$, then $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$ if $r \neq 2$ and $\mathcal{S}_2^2(W) = \mathcal{S}_2(W)$.
(d) If R is of type E_6, then $\mathcal{S}_2^r(W) = \mathcal{S}_2(W) = \mathcal{S}_1(W)$.
(e) If R is of type E_7, then $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$ if $r \neq 2$ and $\mathcal{S}_2^2(W) = \mathcal{S}_2(W)$.
(f) If R is of type E_8, then $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$ if $r \notin \{2, 3\}$ and $\mathcal{S}_2^2(W) \cup \mathcal{S}_2^3(W) = \mathcal{S}_2(W)$.

(g) If R is of type F_4, then $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$ if $r \neq 2$ and $\mathcal{S}_2^2(W) = \mathcal{S}_2(W)$.
(h) If R is of type G_2, then $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$ if $r \neq 3$ and $\mathcal{S}_2^3(W) = \mathcal{S}_2(W)$.

We prove (a). In this case for any isolated $e' \in V^*$ we have $N_{e'} = 1$ and the result follows from 1.1(d),(e), 1.3.

We prove (b), (c). In these cases for any isolated $e' \in V^*$, $N_{e'}$ is a power of 2 (see 1.3) and the equality $\mathcal{S}_2^2(W) = \mathcal{S}_2(W)$ follows from 1.1(e). Moreover, if e' is isolated and $N_{e'}$ is not divisible by 2, then $W_{e'} = W$ so that for $r \neq 2$ we have $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$.

In cases (d), (e), (f) we shall use the fact that for any $e' \in V^*$:

(i) we can find $e \in V$ such that $W_{e'} = W_e$, so that if $E \in \mathcal{S}(W_{e'})$, then $j_{W_{e'}}^W(E) \in \mathcal{S}_1(W)$.

(This property does not always hold in cases (g),(h).)

We prove (d). If $e' \in V^*$ is isolated and $W_{e'} \neq W$, then from 1.3 we see that $W_{e'}$ is of type $A_2A_2A_2$ or A_5A_1 so that $\mathcal{S}_1(W_{e'}) = \mathcal{S}(W_{e'})$; using this and 1.1(e) we see that $\mathcal{S}_2(W) = \mathcal{S}_2^r = \mathcal{S}_1(W)$. (We have used (i).)

We prove (e). If $e' \in V^*$ is isolated and $W_{e'}$ is not of type E_7 (with $N_{e'} = 1$) or D_6A_1 (with $N_{e'} = 2$), then from 1.3 we see that $W_{e'}$ is of type A_7 or A_5A_2 or $A_3A_3A_1$ so that $\mathcal{S}_1(W_{e'}) = \mathcal{S}(W_{e'})$. We see that $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$ if $r \neq 2$ and $\mathcal{S}_2^2(W) = \mathcal{S}_2(W)$. (We have used (i).)

We prove (f). If $e' \in V^*$ is isolated and $W_{e'}$ is not of type E_8 (with $N_{e'} = 1$) or E_7A_1 (with $N_{e'} = 2$) or E_6A_2 (with $N_{e'} = 3$) or D_5A_3 (with $N_{e'} = 4$) or D_8 (with $N_{e'} = 2$), then from 1.3 we see that $W_{e'}$ is of type A_4A_4 or $A_5A_2A_1$ or A_7A_1 or A_8, so that $\mathcal{S}_1(W_{e'}) = \mathcal{S}(W_{e'})$; we see that $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$ if $r \notin \{2,3\}$ and $\mathcal{S}_2^2(W) \cup \mathcal{S}_2^3(W) = \mathcal{S}_2(W)$. (We have used (i).)

We prove (g). If $e' \in V^*$ is isolated and $W_{e'}$ is not of type F_4 (when $N_{e'} = 1$) or B_3A_1 (with $N_{e'}$ a power of 2) or B_4 (with $N_{e'}$ a power of 2), then from 1.3 we see that $W_{e'}$ is of type A_2A_2 (with $N_{e'} = 3$) or A_3A_1 (with $N_{e'}$ a power of 2) so that $\mathcal{S}_1(W_{e'}) = \mathcal{S}(W_{e'})$. Moreover, if $e' \in V^*$ is isolated and $W_{e'}$ is of type A_2A_2, then (i) holds for this e'. We see that $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$ if $r \neq 2$ and $\mathcal{S}_2^2(W) = \mathcal{S}_2(W)$.

We prove (h). If $e' \in V^*$ is isolated and $W_{e'}$ is not of type G_2 (with $N_{e'} = 1$), then from 1.3 we see that $W_{e'}$ is of type A_2 (with $N_{e'} = 3$) or A_1A_1 (when $N_{e'} = 2$) so that $\mathcal{S}_1(W_{e'}) = \mathcal{S}(W_{e'})$. Moreover, if $e' \in V^*$ is isolated and $W_{e'}$ is of type A_1A_1, then (i) holds for this e'. We see that $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$ if $r \neq 3$ and $\mathcal{S}_2^3(W) = \mathcal{S}_2(W)$.

This proves (a)–(h). From (a)–(h) we deduce:

(j) We have $\mathcal{S}_2(W) = \mathcal{S}_2^2(W) \cup \mathcal{S}_2^3(W)$. If $r \in \mathcal{P} - \{2,3\}$, then $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$.

The following result can be verified by computation.

(k) If R is of type E_7, then $\mathcal{S}_2^2(W) - \mathcal{S}_1(W) = \{84_{15}\}$. If R is of type E_8, then $\mathcal{S}_2^2(W) - \mathcal{S}_1(W) = \{1050_{10}, 840_{14}, 168_{24}, 972_{32}\}$ and $\mathcal{S}_2^3(W) - \mathcal{S}_1(W) = \{175_{12}\}$. If R is of type F_4, then $\mathcal{S}_2^2(W) - \mathcal{S}_1(W) = \{9_6, 4_7, 4_8, 2_{16}\}$. If R is of type G_2, then $\mathcal{S}_2^3(W) - \mathcal{S}_1(W) = \{1_3\}$.

(In each case we specify a representation E by a symbol d_n where d is the degree of E and $n = n_E$. For type F_4 and G_2 the specified representations are uniquely determined by the additional condition that they are not in $\mathcal{S}_1(W)$.)

(l) $\mathcal{S}_2^2(W) \cap \mathcal{S}_2^3(W) = \mathcal{S}_1(W)$.

The inclusion $\mathcal{S}_1(W) \subset \mathcal{S}_2^2(W) \cap \mathcal{S}_2^3(W)$ is obvious. The reverse inclusion for R of type $\neq E_8$ follows from the fact that for such R we have either $\mathcal{S}_2^2(W) = \mathcal{S}_1(W)$ or $\mathcal{S}_2^3(W) = \mathcal{S}_1(W)$, see (a)–(h). Thus we can assume that R is of type E_8. In this case the result follows from (k).

1.6 Let $r \in \mathcal{P}$. Let $V_r^* = \{e' \in V^*; N_{e'}/r \notin \mathbf{Z}\}$. Let $\widetilde{\mathcal{S}}_2^r(W)$ be the set of all $E_0 \in \mathrm{Irr}(W)$ such that for some $e' \in V_r^*$ and some $E \in \mathcal{S}_2^r(W_{e'})$ we have $E_0 = j_{W_{e'}}^W(E)$. (Note that any $E \in \mathcal{S}_2^r(W_{e'})$ is good.) Note that $\mathcal{S}_2^r(W) \subset \widetilde{\mathcal{S}}_2^r(W)$ (take $e' = 0$ in the definition of $\widetilde{\mathcal{S}}_2^r(W)$). We show:

(a) $\mathcal{S}_2(W) \subset \widetilde{\mathcal{S}}_2^r(W)$.

We can assume that $(V, V^*, R, \check{R})$ is irreducible. Let $E_0 \in \mathcal{S}_2(W)$. We must show that $E_0 \in \widetilde{\mathcal{S}}_2^r(W)$. By 1.1(e) we can find an isolated $e' \notin V^*$ and $\widetilde{E} \in \mathcal{S}_1(W_{e'})$ such that $E_0 = j_{W_{e'}}^W(\widetilde{E})$. If $N_{e'}/r \notin \mathbf{Z}$ then we have $E_0 \in \widetilde{\mathcal{S}}_2^r(W)$ since $\mathcal{S}_1(W_{e'}) \subset \mathcal{S}_2^r(W_{e'})$. If $N_{e'}$ is a power of r, then from definitions we have $E_0 \in \mathcal{S}_2^r(W)$, hence $E_0 \in \widetilde{\mathcal{S}}_2^r(W)$. Thus we may assume that $N_{e'}$ is not a power of r and is $N_{e'}/r \in \mathbf{Z}$. This forces R to be of type E_8 and $W_{e'}$ to be of type $A_5A_2A_1$ (see 1.3); we then have $N_{e'} = 6$ and $r \in \{2, 3\}$. In particular we must have $\widetilde{E} \in \mathcal{S}(W_{e'})$. If $\widetilde{E}$ is not the sign representation of $W_{e'}$, then we have $\widetilde{E} = j_{W_{e'_1}}^{W_{e'}}(\text{sign})$ for some $e'_1 \in V^*$ such that $W_{e'_1}$ is a proper parabolic subgroup of $W_{e'}$. Replacing $W_{e'_1}$ by a W-conjugate we can assume that $W_{e'_1}$ is a proper parabolic subgroup of W so that $j_{W_{e'}}^W(\text{sign}) \in \mathcal{S}(W)$ and in particular, $E_0 \in \widetilde{\mathcal{S}}_2^r(W)$. Thus we can assume that $\widetilde{E}$ is the sign representation of $W_{e'}$. We have $W_{e'} \subset W_{e'_2}$ where $W_{e'_2}$ is of type E_7A_1 and by the definition of $\mathcal{S}_1(W_{e'_2})$ we have

$$\widetilde{E}_2 := j_{W_{e'}}^{W_{e'_2}}(\text{sign}) \in \mathcal{S}_1(W_{e'_2}).$$

If $r = 3$, we have $e'_2 \in V_r^*$ hence $E_0 = j_{W_{e'_2}}^W(\widetilde{E}_2) \in \widetilde{\mathcal{S}}_2^r(W)$. We have $W_{e'} \subset W_{e'_3}$ where $W_{e'_3}$ is of type E_6A_2 and by the definition of $\mathcal{S}_1(W_{e'_3})$, we have $\widetilde{E}_3 := j_{W_{e'}}^{W_{e'_3}}(\text{sign}) \in \mathcal{S}_1(W_{e'_3})$. If $r = 2$, we have $e'_3 \in V_r^*$ hence $E_0 = j_{W_{e'_3}}^W(\widetilde{E}_3) \in \widetilde{\mathcal{S}}_2^r(W)$. This completes the proof of (a).

We show:

(b) $\widetilde{\mathcal{S}}_2^r(W) \subset \mathcal{S}_2(W)$.

We can assume that $(V, V^*, R, \check{R})$ is irreducible. Let $E_0 \in \widetilde{\mathcal{S}}_2^r(W)$. We must show that $E_0 \in \mathcal{S}_2(W)$. Assume first that $r \notin \{2, 3\}$. Then by results in 1.5 we have

$E \in \mathcal{S}_1(W_{e'})$, hence by 1.1(c) we have $E_0 \in \mathcal{S}_2(W)$. Next we assume that $r = 3$. If $W_{e'} \neq W$, then by results in 1.5 we have $E \in \mathcal{S}_1(W_{e'})$ hence by 1.1(c) we have $E_0 \in \mathcal{S}_2(W)$. Thus we can assume that $W_{e'} = W$ so that $E_0 = E \in \mathcal{S}_2^r(W)$. Since $\mathcal{S}_2^r(W) \subset \mathcal{S}_2(W)$ we see that $E_0 \in \mathcal{S}_2(W)$.

We now assume that $r = 2$. We can find $e' \in V_r^*$ and $E \in \mathcal{S}_2^r(W_{e'})$ such that $E_0 = j_{W_{e'}}^W(E)$. We can find an isolated $e'_1 \in V^*$ such that $N_{e'_1}$ is odd, $R_{e'} \subset R_{e'_1}$ and $R_{e'}$ is rationally closed in $R_{e'_1}$. Let $E' = j_{W_{e'}}^{W_{e'_1}}(E)$. Since $E \in \mathcal{S}_2(W_{e'})$ we have $E' \in \mathcal{S}_2(W_{e'_1})$, see 1.1(g) and $E_0 = j_{W_{e'_1}}^W(E')$. It is then enough to prove the following statement:

(c) If $e' \in V_r^*$ is isolated ($r = 2$) and $E \in \mathcal{S}_2(W_{e'})$, then $E_0 = j_{W_{e'}}^W(E) \in \mathcal{S}_2(W)$.

If $W_{e'} = W$, then $E_0 = E \in \mathcal{S}_2(W)$, as required. If R is of type A_n, B_n, C_n, D_n, then in (c) we have automatically $W_{e'} = W$ hence (c) holds in these cases. Thus we can assume in (c) that R is of exceptional type and $W_{e'} \neq W$. Then $W_{e'}$ is of the following type: $A_2A_2A_2$ (if R is of type E_6); A_5A_2 (if R is of type E_7); A_4A_4 or A_8 or E_6A_2 (if R is of type E_8); A_2A_2, as in 1.3(h) (if R is of type F_4); A_2, as in 1.3(i) (if R is of type G_2). In each case we have $\mathcal{S}_2(W_{e'}) = \mathcal{S}_1(W_{e'})$, see 1.5. Thus $E \in \mathcal{S}_1(W_{e'})$. Using 1.1(e) we see that $E_0 \in \mathcal{S}_2(W)$. This proves (c) hence (b).

Combining (a), (b) we obtain

(d) $\widetilde{\mathcal{S}}_2^r(W) = \mathcal{S}_2(W)$.

In the case where $r = 0$, we set $V_0^* = V^*$, $\mathcal{S}_2^0(W) = \mathcal{S}_1(W)$, $\widetilde{\mathcal{S}}_2^0(W) = \mathcal{S}_2(W)$.

2 The strata of G

2.1 We return to the setup of the introduction. Thus G is a connected reductive algebraic group over $\mathbf{k}$. Let $\mathcal{T}$ be "the" maximal torus of G; let $X = \mathrm{Hom}(\mathcal{T}, \mathbf{k}^*)$, $Y = \mathrm{Hom}(\mathbf{k}^*, \mathcal{T})$, $V = \mathbf{Q} \otimes X$, $V^* = \mathbf{Q} \otimes Y$. We have an obvious perfect bilinear pairing $\langle, \rangle : V \times V^* \to \mathbf{Q}$. Let $R \subset V$ be the set of roots and let $\check{R} \subset V^*$ be the set of corrots. Then $(V, V^*, R, \check{R})$ is as in 1.1. The associated Weyl group W (as in 1.1) that is, the Weyl group of G, can be viewed as an indexing set for the orbits of G acting diagonally on $\mathcal{B} \times \mathcal{B}$; we denote by $\mathcal{O}_w$ the orbit corresponding to $w \in W$. Note that W is naturally a Coxeter group.

Let $g \in G$. Let W_g be the Weyl group of the connected reductive group $H := Z_G(g_s)^0$. We can view W_g as a subgroup of W as follows. Let β be a Borel subgroup of H and let T be a maximal torus of β. We define an isomorphism $b_{T,\beta} : N_H T/T \xrightarrow{\sim} W_g$ by $n'T \mapsto H$-orbit of $(\beta, n'\beta n'^{-1})$. Similarly for any $B \in \mathcal{B}$ such that $T \subset B$ we define an isomorphism $a_{T,B} : N_G T/T \xrightarrow{\sim} W$ by $n'T \mapsto G$-orbit of $(B, n'Bn'^{-1})$. Now assume that $B \in \mathcal{B}$ is such that $B \cap H = \beta$.

We define an embedding $c_{T,\beta,B} : W_g \to W$ as the composition $W_g \xrightarrow{b_{T,\beta}^{-1}} N_H T/T \to N_G T/T \xrightarrow{a_{T,B}} W$ where the middle map is the obvious embedding. If $B' \in \mathcal{B}$ also satisfies $B' \cap H = \beta$, then we have $B' = nBn^{-1}$ for some $n \in N_G T$ and from the definitions we have $c_{T,\beta,B'}(w) = a_{T,B}(nT)c_{T,\beta,B}(w)a_{B,T}(nT)^{-1}$ for any $w \in W_g$. Thus $c_{T,\beta,B}$ depends (up to composition with an inner automorphism of W) only on T, β and we can denote it by $c_{T,\beta}$. Since the set of pairs T, β as above form a homogeneous space for the connected group H, we see that $c_{T,\beta}$ is independent of T, β (up to composition with an inner automorphism of W) hence it does not depend on any choice. We see that there is a well-defined collection $\mathcal{C}$ of embeddings $W_g \to W$ so that any two of them differ only by composition by an inner automorphism of W.

Define $\rho \in \mathrm{Irr}(W_g)$ by the condition that under the Springer correspondence for H, ρ corresponds to the H-conjugacy class of g_u and the trivial local system on it. We choose $f \in \mathcal{C}$; then we can view ρ as an irreducible representation of $f(W_g)$, a subgroup of W such that $f(W_g) = W_{e'}$ for some $e' \in V_p^*$, see 1.6. By [L5, 1.4] we have $\rho \in \mathcal{S}_2^p(f(W_g))$, see 1.5, 1.6. Hence $\widetilde{\rho} := j_{f(W_g)}^W(\rho) \in \widetilde{\mathcal{S}}_2^p(W)$ is well defined. Since $\widetilde{\mathcal{S}}_2^p(W) = \mathcal{S}_2(W)$, see 1.6, we have $\widetilde{\rho} \in \mathcal{S}_2(W)$. This is independent of the choice of f since f is well defined up to composition by an inner automorphism of W.

2.2 Let $g \in G$. Let $d = d_g = \dim \mathcal{B}_g$. The embedding $h_g : \mathcal{B}_g \to \mathcal{B}$ induces a linear map $h_{g*} : H_{2d}(\mathcal{B}_g) \to H_{2d}(\mathcal{B})$. Now $H^{2d}(\mathcal{B}_g), H^{2d}(\mathcal{B})$ carry natural W-actions, see [L3], and this induces natural W-actions on $H_{2d}(\mathcal{B}_g), H_{2d}(\mathcal{B})$ which are compatible with h_{g*}. Hence W acts naturally on the subspace $h_{g*}(H_{2d}(\mathcal{B}_g))$ of $H_{2d}(\mathcal{B})$.

The following result gives an alternative description of the map $g \mapsto \widetilde{\rho}$ (in 2.1) from G to $\mathrm{Irr}W$.

(a) *The W-submodule $h_{g*}(H_{2d}(\mathcal{B}_g))$ of $H_{2d}(\mathcal{B})$ is isomorphic to the W-module $\overline{\mathbf{Q}}_l \otimes \widetilde{\rho}$ where $\rho, \widetilde{\rho}$ are associated to g as in 2.1.*

First, we note that $h_{g*}(H_{2d}(\mathcal{B}_g)) \neq 0$; indeed it is clear that for any irreducible component D of $\mathcal{B}_g$ (necessarily of dimension d), the image of the fundamental class of D under h_{g*} is nonzero (we ignore Tate twists). Let $\mathcal{B}'$ be the variety of Borel subgroups of $Z_G(g_s)^0$. Let $\mathcal{B}'_{g_u} = \{\beta \in \mathcal{B}'; g_u \in \beta\}$. Then $\dim \mathcal{B}' = d$ and W_g (see 2.1) acts naturally on $H_{2d}(\mathcal{B}'_{g_u})$; from the definitions, the W-module $H_{2d}(\mathcal{B}_g)$ is isomorphic to $\mathrm{Ind}_{W_g}^W H_{2d}(\mathcal{B}'_{g_u})$. From the definitions we have $n_\rho = d$ and the W_g-module $H_{2d}(\mathcal{B}'_{g_u})$ is of the form $\oplus_{i \in [1,s]}(\overline{\mathbf{Q}}_l \otimes E_i)^{\oplus c_i}$ where $E_i \in \mathrm{Irr}(W_g)$, $c_i \in \mathbf{N}$ satisfy $E_1 = \rho$, $c_1 = 1$ and $n_{E_i} > d$ for $i > 1$. It follows that the W-module $H_{2d}(\mathcal{B}_g)$ is of the form $\oplus_{i \in [1,s]}(\mathrm{Ind}_{W_g}^W(\overline{\mathbf{Q}}_l \otimes E_i))^{\oplus c_i}$. Now $\mathrm{Ind}_{W_g}^W(\overline{\mathbf{Q}}_l \otimes E_1)$ contains $\overline{\mathbf{Q}}_l \otimes \widetilde{\rho}$ with multiplicity 1 and all its other irreducible constituents are of the form $\overline{\mathbf{Q}}_l \otimes E$ with $n_E > d$; moreover, for $i > 1$, any irreducible constituent E of $\mathrm{Ind}_{W_g}^W(\overline{\mathbf{Q}}_l \otimes E_i)$ satisfies $n_E > d$. Thus the W-module $H_{2d}(\mathcal{B}_g)$ contains $\overline{\mathbf{Q}}_l \otimes \widetilde{\rho}$ with multiplicity 1 and all its other irreducible constituents are of the form $\overline{\mathbf{Q}}_l \otimes E$ with $n_E > d$; these other irreducible constituents are necessarily mapped to 0 by h_{g*} and the irreducible constituent isomorphic to $\overline{\mathbf{Q}}_l \otimes \widetilde{\rho}$ is mapped injectively by h_{g*} since $h_{g*} \neq 0$. It follows that the image of h_{g*} is isomorphic to $\overline{\mathbf{Q}}_l \otimes \widetilde{\rho}$ as a W-module. This proves (a).

2.3 By 2.1, 2.2 we have a well-defined map $\phi : G \to \mathcal{S}_2(W)$, $g \mapsto \widetilde{\rho}$ where $\mathbf{Q}_l \otimes \widetilde{\rho} = h_{g*}((H_{2d_g}(\mathcal{B}_g))$ (notation of 2.1, 2.2). The fibres $G_E = \phi^{-1}(E)$ of ϕ ($E \in \mathcal{S}_2(W)$) are called the *strata* of G. They are clearly unions of conjugacy classes of G. Note the strata of G are indexed by the finite set $\mathcal{S}_2(W)$ which depends only on the Weyl group W and not on the underlying root system (see 1.1(b)) or on the characteristic of $\mathbf{k}$.

One can show that any stratum of G is a union of pieces in the partition of G defined in [L3, 3.1]; in particular, it is a constructible subset of G.

2.4 We have the following result.

(a) *Any stratum G_E ($E \in \mathcal{S}_2(W)$) of G is a (non-empty) union of G-conjugacy classes of fixed dimension, namely* $2\dim\mathcal{B} - 2n$ *where $n = n_E$, see 0.2. At most one G-conjugacy class in G_E is unipotent.*

Since $\mathcal{S}_2(W) = \widetilde{\mathcal{S}}_2^p(W)$, see 1.6, we have $E \in \widetilde{\mathcal{S}}_2^p(W)$. Hence there exists $e' \in V_p^*$ and $\rho \in \mathcal{S}_2^p(W_{e'})$ such that $E = j_{W_{e'}}^W(\rho)$. We can find a semisimple element of finite order $s \in G$ such that W_s (viewed as a subgroup of W as in 2.1) is equal to $W_{e'}$. By [L5, 1.4] we can find a unipotent element u in $Z_G(s)^0$ such that ρ is the Springer representation of W_s defined by u and the trivial local system on its $Z_G(s)^0$-conjugacy class. Then $E = \phi(su)$ so that $G_E \neq \emptyset$. Let γ be a G-conjugacy class in G_E. Let $g \in \gamma$. Let ρ (resp. $\widetilde{\rho}$) be the irreducible representation of W_g (resp. W) defined by g_u as in 2.1. Let $n_\rho, n_{\widetilde{\rho}}$ be as in 0.2. By the definition of $\widetilde{\rho}$ we have $n_\rho = n_{\widetilde{\rho}}$. By assumption we have $\widetilde{\rho} = E$, hence $n_{\widetilde{\rho}} = n$ and $n_\rho = n$. By a known property of Springer's representations, n_ρ is equal to the dimension of the variety of Borel subgroups of $Z_G(g_s)^0$ that contain g_u; hence by a result of Steinberg (for $p = 0$) and Spaltenstein [Spa, 10.15] (for any p), n_ρ is equal to

$$(\dim(Z_{Z_G(g_s)^0}(g_u)^0 - \mathrm{rk}(Z_G(g_s)^0))/2 = (\dim(Z_G(g)^0) - \mathrm{rk}(G))/2.$$

It follows that $(\dim(Z_G(g)^0) - \mathrm{rk}(G))/2 = n$ and the desired formula for $\dim\gamma$ follows. Now assume that γ, γ' are two unipotent G-conjugacy classes contained in G_E. Then the Springer representation of W associated to γ is the same as that associated to γ', namely E. By properties of Springer representations, it follows that $\gamma = \gamma'$. This proves (a).

2.5 In this and the next subsection we assume that W is irreducibble. Let $r \in \mathcal{P} \cup \{0\}$. Let G^r be a connected reductive group of the same type as G over an algebraically closed field of characteristic r, whose Weyl group is identified with W. Let $\mathcal{U}^r$ be the set of unipotent classes of G^r. By [L5, 1.4] we have a canonical bijection

$$\psi^r : \mathcal{U}^r \xrightarrow{\sim} \mathcal{S}_2^r(W)$$

which, to a unipotent class γ, associates the Springer representation of W corresponding to γ and the constant local system on γ. We define an embedding $h^r : \mathcal{U}^0 \to \mathcal{U}^r$ as the composition

$$\mathcal{U}^0 \xrightarrow{\psi^0} \mathcal{S}_2^0(W) = \mathcal{S}_1(W) \to \mathcal{S}_2^r(W) \xrightarrow{(\psi^r)^{-1}} \mathcal{U}^r$$

where the unnamed map is the inclusion.

Consider the relation $\cong$ on $\sqcup_{r \in \mathcal{P}} \mathcal{U}^r$ for which $x \in \mathcal{U}^r$, $y \in \mathcal{U}^{r'}$ (where $r, r' \in \mathcal{P}$) satisfy $x \cong y$ if either $r = r'$ and $x = y$ or $r \neq r'$ and $x = h^r(z)$, $y = h^{r'}(z)$ for some $z \in \mathcal{U}^0$. We show that $\cong$ is an equivalence relation. It is enough to show that if $x \in \mathcal{U}^r$, $y \in \mathcal{U}^{r'}$, $u \in \mathcal{U}^{r''}$ are such that $r \neq r'$, $r' \neq r''$ and $x = h^r(z)$, $y = h^{r'}(z)$, $y = h^{r'}(\tilde{z})$, $u = h^{r''}(\tilde{z})$ for some $z \in \mathcal{U}^0, \tilde{z} \in \mathcal{U}^0$, then $x \cong u$. From $h^{r'}(z) = h^{r'}(\tilde{z})$ and the injectivity of $h^{r'}$ we have $z = \tilde{z}$. Thus, if $r \neq r''$, we have $x \cong u$, while if $r = r''$, we have $x = u$. Thus, $\cong$ is indeed an equivalence relation.

Let $\mathcal{U}^*$ be $\sqcup_{r \in \mathcal{P}} \mathcal{U}^r$ modulo the equivalence relation $\cong$. Let $\sqcup_{r \in \mathcal{P}} \mathcal{U}^r \to \mathcal{S}_2(W)$ be the map whose restriction to $\mathcal{U}^r$ is ψ^r followed by the inclusion $\mathcal{S}_2^r(W) \subset \mathcal{S}_2(W)$ (for any r). We show:

(a) *This map induces a bijection* $\psi^* : \mathcal{U}^* \xrightarrow{\sim} \mathcal{S}_2(W)$.

To show that ψ^* is a well-defined map it is enough to verify that if $z \in \mathcal{U}^0$, then for any $r, r' \in \mathcal{P}$, we have $\psi^r h^r(z) = \psi^{r'} h^{r'}(z)$ in $\mathcal{S}_2(W)$; but both sides of the equality to be verified are equal to $\psi^0(z)$. Let $E \in \mathcal{S}_2(W)$. By 1.5(j) there exists $r \in \mathcal{P}$ such that $E \in \mathcal{S}_2^r(W)$, hence $E = \psi^r(x)$ for some $x \in \mathcal{U}^r$. It follows that ψ^* is surjective. We show that ψ^* is injective. It is enough to show that

(b) if $x \in \mathcal{U}^r$, $y \in \mathcal{U}^{r'}$ ($r, r' \in \mathcal{P}$ distinct) satisfy $\psi^r(x) = \psi^{r'}(y)$, then there exists $z \in \mathcal{U}^0$ such that $x = h^r(z)$, $y = h^{r'}(z)$.

If $r \neq \{2, 3\}$, then $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$, hence $\psi^r(x) = \psi^0(z)$ for some $z \in \mathcal{U}^0$. We then have $\psi^{r'}(y) = \psi^0(z)$. It follows that $h^r(z) = x$, $h^{r'}(z) = y$, as required. Similarly, if $r' \neq \{2, 3\}$, then the conclusion of (b) holds. Thus we can assume that $r \in \{2, 3\}$, $r' \in \{2, 3\}$. Since $r \neq r'$ we have $\{r, r'\} = \{2, 3\}$. Hence $\psi^r(x) = \psi^{r'}(y) \in \mathcal{S}_2^2(W) \cap \mathcal{S}_2^3(W) = \mathcal{S}_1(W)$; the last equality follows from 1.5(l). Thus we have $\psi^r(x) = \psi^{r'}(y) = \psi^0(z)$ for some $z \in \mathcal{U}^0$. It follows that $h^r(z) = x$, $h^{r'}(z) = y$, as required.

From (a) we deduce the following:

(c) *The strata of G are naturally indexed by the set $\mathcal{U}^*$.*

The proof of (a) shows also that $\mathcal{U}^*$ is equal to $\mathcal{U}^2 \sqcup \mathcal{U}^3$ with the identification of $h^2(z), h^3(z)$ for any $z \in \mathcal{U}^0$.

We can now state the following result.

(d) *Let $E \in \mathcal{S}_2(W)$. Then for some $r \in \mathcal{P}$, the stratum G_E^r contains a unipotent class. In fact, r can be assumed to be* 2 *or* 3.

Under (a), E corresponds to an element of $\mathcal{U}^*$ which is the equivalence class of some element $\gamma \in \mathcal{U}^r$ with $r \in \{2, 3\}$. Let $g \in G^r$ be an element in the unipotent conjugacy class γ. From the definitions we see that $g \in G^r_E$. This proves (d).

2.6 We show that the set $\mathcal{U}^*$ has a natural partial order. If $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$ (type A and E_6), we have $\mathcal{U}^* = \mathcal{U}^0$ which has a natural partial order defined by the closure relation of unipotent classes in G^0. If $\mathcal{S}_1(W) \neq \mathcal{S}_2^r(W)$ for a unique $r \in \mathcal{P}$ (type $\neq A, E_6, E_8$), we have $\mathcal{U}^* = \mathcal{U}^r$ which has a natural partial order defined by the closure relation of unipotent classes in G^r. Assume now that G is of type E_8. Then we can identify $\mathcal{U}^2, \mathcal{U}^3$ with subsets of $\mathcal{U}^*$ whose union is $\mathcal{U}^*$ and whose intersection is $\mathcal{U}^0$. Both subsets $\mathcal{U}^2, \mathcal{U}^3$ have natural partial orders defined by the closure relation of unipotent classes in G^2 and G^3. If $\gamma, \gamma' \in \mathcal{U}^*$, we say that $\gamma \leq \gamma'$ if there exists a sequence $\gamma = \gamma_0, \gamma_1, \dots, \gamma_s = \gamma'$ in $\mathcal{U}^*$ such that for any $i \in [1, s]$ there exists $r \in \{2, 3\}$ such that

(a) $\gamma_{i-1} \in \mathcal{U}^r, \gamma_i \in \mathcal{U}^r, \gamma_{i-1} \leq \gamma_i$ in the partial order of unipotent classes in G^r;

note that if for some i, (a) holds for both $r = 2$ and $r = 3$, then we have $\gamma_{i-1} \in \mathcal{U}^0$, $\gamma_i \in \mathcal{U}^0$, $\gamma_{i-1} \leq \gamma_i$ in the partial order of unipotent classes in G^0. One can show that this partial order on $\mathcal{U}^*$ induces the usual partial orders on the subsets $\mathcal{U}^2$, $\mathcal{U}^3, \mathcal{U}^0$.

2.7 Let W_a be the semidirect product of W with the subgroup of V generated by R (an affine Weyl group); let $'W_a$ be the semidirect product of W with the subgroup of V^* generated by $\check{R}$ (another affine Weyl group). We consider four triples:

(a) $(\mathcal{S}(W), X_0, Z_0)$
(b) $(\mathcal{S}_1(W), X_1, Z_1)$
(c) $('\mathcal{S}_1(W), 'X_1, 'Z_1)$
(d) $(\mathcal{S}_2(W), X_2, Z_2)$

where $X_0, X_1, 'X_1$ is the set of two-sided cells in $W, W_a, 'W_a$ respectively, Z_0 is the set of special unipotent classes in G with $p = 0$, Z_1 is the set of unipotent classes in G with $p = 0$, $'Z_1$ is the set of unipotent classes in the Langlands dual G^* of G with $p = 0$, Z_2 is the set of strata of G with $p = 0$ and X_2 remains to be defined. The three sets in each of these four triples are in canonical bijection with each other (assuming that X_2 has been defined). Moreover, each set in (a) is naturally contained in the corresponding set in (b) and (replacing G by G^*) in the corresponding set in (c); each set in (b) is contained in the corresponding set in (d) and (replacing G by G^*) each set in (c) is contained in the corresponding set in (d).

It remains to define X_2. It seems plausible that the (trigonometric) double affine Hecke algebra **H** associated by Cherednik to W has a natural filtration by two-sided ideals whose successive subquotients can be called two-sided cells and form the desired set X_2. The inclusion of the Hecke algebra of W_a and that of $'W_a$ into **H** should induce the embeddings $X_1 \subset X_2$, $'X_1 \subset X_2$ and X_2 should be in natural bijection with $\mathcal{S}_2(W)$ and with the set of strata of G.

3 Examples

3.1 We write the adjoint group of G as a product $\prod_i G_i$ where each G_i is simple with Weyl group W_i so that $W = \prod_i W_i$. Let $E \in \mathcal{S}_2(W)$. We have $E = \boxtimes_i E_i$ where $E_i \in \mathcal{S}_2(W_i)$. Now G_E is the inverse image of $\prod_i (G_i)_{E_i}$ under the obvious map $G \to \prod_i G_i$.

When E is the sign representation of W, then G_E is the centre of G; when E is the unit representation of W, G_E is the set of elements of G which are regular in the sense of Steinberg [St].

By 2.5(a) and 2.6 applied to G_i, the set $\mathcal{S}_2(W_i)$ has a natural partial order. Since $\mathcal{S}_2(W)$ can be identified as above with $\prod_i \mathcal{S}_2(W_i)$, $\mathcal{S}_2(W)$ is naturally a partially ordered set (a product of partially ordered sets). Hence by 2.3 the set of strata of G is naturally a partially ordered set.

3.2 Assume that $G = GL(V)$ where V is a $\mathbf{k}$-vector space of dimension $n \geq 1$. Let $g \in G$. For any $x \in \mathbf{k}^*$ let V_x be the generalized x-eigenspace of $g : V \to V$ and let $\lambda_1^x \geq \lambda_2^x \geq \lambda_3^x \geq \dots$ be the sequence in $\mathbf{N}$ whose nonzero terms are the sizes of the Jordan blocks of $x^{-1}g : V_x \to V_x$. Let ${}^g\lambda$ be the sequence ${}^g\lambda_1 \geq {}^g\lambda_2 \geq {}^g\lambda_3 \geq \dots$ given by ${}^g\lambda_j = \sum_{x \in \mathbf{k}^*} \lambda_j^x$. Now $g \mapsto {}^g\lambda$ defines a map from G onto the set of partitions of n. From the definitions we see that the fibres of this map are exactly the strata of G. If $g \in G$ and ${}^g\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$, then

$$\dim(\mathcal{B}_g) = \sum_{k \geq 1} (n - (\lambda_1 + \lambda_2 + \dots + \lambda_k)).$$

3.3 Repeating the definition of sheets in a semisimple Lie algebra over $\mathbf{C}$ (see [Bo]), one can define the sheets of G as the maximal irreducible subsets of G which are unions of conjugacy classes of fixed dimension. One can show that if G is as in 3.2, the sheets of G are the same as the strata of G, as described in 3.2. (In this case, the sheets of G, or rather their Lie algebra analogue, are described in [Pe]. They are smooth varieties.) This is not true for a general G (the sheets of G do not usually form a partition of G; the strata of G are not always irreducible). In [Ca] it is shown that if p is 0 or a good prime for G, then any stratum is a union of sheets and that the closure of a stratum is not necessarily a union of strata, even if G is of type A.

3.4 In the next few subsections we will describe explicitly the strata of G when G is a symplectic or special orthogonal group.

Given a partition $\nu = (\nu_1 \geq \nu_2 \geq \dots)$, a *string* of ν is a maximal subsequence $\nu_i, \nu_{i+1}, \dots, \nu_j$ of ν consisting of equal > 0 numbers; the string is said to have an odd origin if i is odd and an even origin if i is even.

For an even $N \in \mathbf{N}$, let Z_N^1 be the set of partitions $\nu = (\nu_1 \geq \nu_2 \geq \dots)$ of N such that any odd number appears an even number of times in ν. We show:

(a) *There is a canonical bijection $Z_N^1 \leftrightarrow BP_{1,1}^{N/2}$ (notation of 0.2).*

To $\nu \in Z_N^1$ we associate $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ as follows: each string $2a, 2a, \dots, 2a$ in ν is replaced by $a, a, \dots, a$ of the same length; each string $2a + 1, 2a + 1, \dots, 2a + 1$ (necessarily of even length) in ν is replaced by $a, a + 1, a, a + 1, \dots, a, a + 1$ of the same length. The resulting entries form a bipartition $\lambda \in BP_{1,1}^{N/2}$. Now $\nu \mapsto \lambda$ establishes the bijection (a).

For an even $N \in \mathbf{N}$, let Z_N^2 be the set of partitions $\nu = (\nu_1 \geq \nu_2 \geq \dots)$ of N such that any odd number appears an even number of times in ν and any even > 0 number which appears an even > 0 number of times in ν has an associated label 0 or 1. We show:

(b) *There is a canonical bijection $Z_N^2 \leftrightarrow BP_{2,2}^{N/2}$ (notation of 0.2).*

To $\nu \in Z_N^2$ we associate $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ as follows: each string $2a, 2a, \dots, 2a$ of odd length or of even length and label 1 in ν is replaced by $a, a, \dots, a$ of the same length; each string $2a, 2a, \dots, 2a$ of even length and label 0 in ν is replaced by $a - 1, a + 1, a - 1, a + 1, \dots, a - 1, a + 1$ of the same length; each string $2a + 1, 2a + 1, \dots, 2a + 1$ (necessarily of even length) in ν is replaced by $a, a + 1, a, a + 1, \dots, a, a + 1$ of the same length. The resulting entries form a bipartition $\lambda \in BP_{2,2}^{N/2}$. Now $\nu \mapsto \lambda$ establishes the bijection (b).

Assume for example that $N = 6$. The bijection (b) is:

$$
\begin{aligned}
(6\dots) &\leftrightarrow (3\dots)\\
(42\dots) &\leftrightarrow (21\dots)\\
(411\dots) &\leftrightarrow (201\dots)\\
(33\dots) &\leftrightarrow (12\dots)\\
(222\dots) &\leftrightarrow (111\dots)\\
((22)_1 11\dots) &\leftrightarrow (1101\dots)\\
((22)_0 110\dots) &\leftrightarrow (0201\dots)\\
(21111\dots) &\leftrightarrow (10101\dots)\\
(111111\dots) &\leftrightarrow (010101\dots).
\end{aligned}
$$

Here we write $\dots$ instead of $000\dots$. (Compare [LS2, 6.1].)

3.5 Assume that $G = \mathrm{Sp}(V)$ where V is a $\mathbf{k}$-vector space of dimension N with a fixed nondegenerate symplectic form.

Let $g \in G$. For any $x \in \mathbf{k}^*$ let V_x be the generalized x-eigenspace of $g : V \to V$. Let $d_x = \dim V_x$. For any $x \in \mathbf{k}^*$ such that $x^2 \neq 1$ let $\lambda_1^x \geq \lambda_2^x \geq \lambda_3^x \geq \dots$ be the partition of d_x whose nonzero terms are the sizes of the Jordan blocks of $x^{-1}g : V_x \to V_x$.

For $x \in \mathbf{k}^*$ such that $x^2 = 1$, let $\nu^x \in Z_{d_x}^1$ (if $p \neq 2$) and $\nu^x \in Z_{d_x}^2$ (if $p = 2$) be again the partition of d_x whose nonzero terms are the sizes of the Jordan blocks of the unipotent element $x^{-1}g \in \mathrm{Sp}(V_x)$. (When $p = 2$, ν^x should also include a labelling with 0 and 1 associated to $x^{-1}g \in \mathrm{Sp}(V_x)$ as in [L10, 1.4].) Let $\lambda^x = (\lambda^x_1, \lambda^x_2, \lambda^x_3, \dots)$ be the bipartition of $d_x/2$ associated to ν^x by 3.4(a),(b). Thus $\lambda^x \in BP^{d_x/2}_{1,1}$ (if $p \neq 2$), $\lambda^x \in BP^{d_x/2}_{2,2}$ (if $p = 2$). Note that λ^x is the bipartition such that the Springer representation attached to the unipotent element $x^{-1}g \in \mathrm{Sp}(V_x)$ (an irreducible representation of the Weyl group of type $B_{d_x/2}$) is indexed in the standard way by λ^x. Define ${}^g\lambda = ({}^g\lambda_1, {}^g\lambda_2, {}^g\lambda_3, \dots)$ by ${}^g\lambda_j = \sum_x \lambda^x_j$ where x runs over a set of representatives for the orbits of the involution $a \mapsto a^{-1}$ of $\mathbf{k}^*$. Note that ${}^g\lambda \in BP^{N/2}_{2,2}$. Thus we have defined a (surjective) map $g \mapsto {}^g\lambda$, $G \to BP^{N/2}_{2,2}$. From the definitions we see that the fibres of this map are exactly the strata of G.

If $g \in G$ and ${}^g\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$, then

(a) $\dim(\mathcal{B}_g) = \sum_{k \geq 1}((N/2) - (\lambda_1 + \lambda_2 + \dots + \lambda_k))$.

We now consider the case where $N = 4$. In this case we have $\mathcal{S}_2(W) = \mathrm{Irr}(W)$; hence there are five strata. One stratum is the union of all conjugacy classes of dimension 8 (it corresponds to the unit representation); one stratum is the union of all conjugacy classes of dimension 6 (it corresponds to the reflection representation of W). There are two strata which are unions of conjugacy classes of dimension 4 (they correspond to the two one-dimensional representations of W other than unit and sign); if $p = 2$, both these strata are single unipotent classes; if $p \neq 2$, one of these strata is a semisimple class and the other is a unipotent class times the centre of G. The centre of G is a stratum (it corresponds to the sign representation of W).

The results in this subsection show that under the standard identification $\mathrm{Irr}(W) = BP^{N/2}$, we have

(b) $\mathcal{S}_2(W) = BP^{N/2}_{2,2}$.

Under this identification the map $g \mapsto {}^g\lambda$, $G \to BP^{N/2}_{2,2}$ becomes the map $g \mapsto E$ where $g \in G_E$.

3.6 For $N \in \mathbf{N}$, let ${}'Z^1_N$ be the set of partitions $\nu = (\nu_1 \geq \nu_2 \geq \dots)$ such that any even > 0 number appears an even number of times in ν and $\nu_1 + \nu_2 + \dots = N$.

(a) *If N is odd, then there is a canonical bijection* ${}'Z^1_N \leftrightarrow BP^{(N-1)/2}_{2,0}$.

To $\nu \in {}'Z^1_N$ we associate $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ as follows: each string $2a, 2a, \dots, 2a$ of ν (necessarily of even length) is replaced by $a-1, a+1, a-1, a+1, \dots, a-1, a+1$ of the same length (if the string has odd origin) or by $a, a, \dots, a$ of the same length (if the string has even origin); each string $2a+1, 2a+1, \dots, 2a+1$ of ν is replaced by $a, a+1, a, a+1, \dots$ of the same length (if the string has odd origin) or by $a+1, a, a+1, a, \dots$ of the same length (if the string has even origin). The resulting entries form a bipartition $\lambda \in BP^{(N-1)/2}_{2,0}$. Now $\nu \mapsto \lambda$ establishes the bijection (a).

(b) *If N is even, then there is a canonical bijection* $'Z^1_N \leftrightarrow BP^{N/2}_{0,2}$.

To $\nu \in {}'Z^1_N$ we associate $\lambda=(\lambda_1,\lambda_2,\lambda_3,\dots)$ as follows: each string $2a,2a,\dots,2a$ of ν (necessarily of even length) is replaced by $a-1,a+1,a-1,a+1,\dots,a-1,a+1$ of the same length (if the string has even origin) or by $a,a,\dots,a$ of the same length (if the string has odd origin); each string $2a+1,2a+1,\dots,2a+1$ of ν is replaced by $a,a+1,a,a+1,\dots$ of the same length (if the string has even origin) or by $a+1,a,a+1,a,\dots$ of the same length (if the string has odd origin). The resulting entries form a bipartition $\lambda \in BP^{N/2}_{0,2}$. Now $\nu \mapsto \lambda$ establishes the bijection (b).

3.7 Assume that $p \neq 2$ and that $G = \mathrm{SO}(V)$ where V is a $\mathbf{k}$-vector space of odd dimension $N \geq 1$ with a fixed nondegenerate quadratic form.

Let $g \in G$. For any $x \in \mathbf{k}^*$, let V_x be the generalized x-eigenspace of $g: V \to V$. Let $d_x = \dim V_x$. For any $x \in \mathbf{k}^*$ such that $x^2 \neq 1$ let $\lambda^x_1 \geq \lambda^x_2 \geq \lambda^x_3 \geq \dots$ be the partition of d_x whose nonzero terms are the sizes of the Jordan blocks of $x^{-1}g : V_x \to V_x$.

For $x \in \mathbf{k}^*$ such that $x^2 = 1$ let $\nu^x \in {}'Z^1_{d_x}$ again be the partition of d_x whose nonzero terms are the sizes of the Jordan blocks of the unipotent element $x^{-1}g \in \mathrm{SO}(V_x)$. Let $\lambda^x = (\lambda^x_1,\lambda^x_2,\lambda^x_3,\dots)$ be the bipartition of $d_x/2$ associated to ν^x by 3.6(a) if $x = 1$ and by 3.6(b) if $x = -1$. Thus $\lambda^x \in BP^{(d_x-1)/2}_{2,0}$ if $x = 1$, $\lambda^x \in BP^{d_x/2}_{0,2}$ if $x = -1$. Note that λ^x is the bipartition such that the Springer representation attached to the unipotent element $x^{-1}g \in \mathrm{SO}(V_x)$ (an irreducible representation of the Weyl group of type $B_{(d_x-1)/2}$, if $x = 1$, or of type $D_{d_x/2}$, if $x = -1$) is indexed by λ^x. Define ${}^g\lambda = ({}^g\lambda_1, {}^g\lambda_2, {}^g\lambda_3,\dots)$ by ${}^g\lambda_j = \sum_x \lambda^x_j$ where x runs over a set of representatives for the orbits of the involution $a \mapsto a^{-1}$ of $\mathbf{k}^*$. Note that ${}^g\lambda \in BP^{(N-1)/2}_{2,2}$. Thus we have defined a (surjective) map $g \mapsto {}^g\lambda$, $G \to BP^{(N-1)/2}_{2,2}$. From the definitions we see that the fibres of this map are exactly the strata of G. Under the identification $\mathcal{S}_2(W) = BP^{(N-1)/2}_{2,2}$, see 3.5(b), the map $g \mapsto {}^g\lambda$, $G \to BP^{(N-1)/2}_{2,2}$ becomes the map $g \mapsto E$ where $g \in G_E$.

If $g \in G$ and ${}^g\lambda = (\lambda_1,\lambda_2,\lambda_3,\dots)$, then

$$\dim(\mathcal{B}_g) = \sum_{k\geq 1}((N-1)/2 - (\lambda_1+\lambda_2+\dots+\lambda_k)).$$

3.8 Assume that $p = 2$ and that $G = \mathrm{SO}(V)$ where V is a $\mathbf{k}$-vector space of odd dimension $N \geq 1$ with a given quadratic form, such that the associated symplectic form has radical $\mathbf{r}$ of dimension 1 and the restriction of the quadratic form to $\mathbf{r}$ is nonzero. In this case there is an obvious morphism from G to the symplectic group G' of $V/\mathbf{r}$ which is an isomorphism of abstract groups. From the definitions we see that this morphism maps each stratum of G bijectively onto a stratum of G' (which has been described in 3.5).

3.9 For an even $N \in \mathbf{N}$, let $'Z_N^2$ be the set of partitions with labels $\nu = (\nu_1 \geq \nu_2 \geq \dots)$ in Z_N^2 (see 3.4) such that the number of nonzero entries of ν is even.

(a) *If N is even, then there is a canonical bijection $'Z_N^2 \leftrightarrow BP_{0,4}^{N/2}$.*

To $\nu \in {}'Z_N^2$ we associate $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ as follows: each string $2a, 2a, 2a, \dots$ of ν of odd length or of even length and label 1 is replaced by $a-1, a+1, a-1, a+1, \dots$ of the same length (if the string has even origin) or $a+1, a-1, a+1, a-1, \dots$ of the same length (if the string has odd origin); each string $2a, 2a, 2a, \dots$ of ν of even length and label 0 is replaced by $a-2, a+2, a-2, a+2, \dots$ of the same length (if the string has even origin) or $a, a, a, a, \dots$ of the same length (if the string has odd origin); each string $2a+1, 2a+1, 2a+1, \dots$ of ν (necessarily of even length) is replaced by $a-1, a+2, a-1, a+2, \dots$ of the same length (if the string has even origin) or $a+1, a, a+1, a, \dots$ of the same length (if the string has odd origin). The resulting entries form a bipartition $\lambda \in BP_{0,4}^{N/2}$. Now $\nu \mapsto \lambda$ establishes the bijection (a).

Assume for example that $N = 8$. The bijection (a) is:

$$
\begin{aligned}
(62\dots) &\leftrightarrow (40\dots)\\
((44)_1\dots) &\leftrightarrow (31\dots)\\
((44)_0\dots) &\leftrightarrow (22\dots)\\
(4211\dots) &\leftrightarrow (3010\dots)\\
(3311\dots) &\leftrightarrow (2110\dots)\\
((2222)_1\dots) &\leftrightarrow (2020\dots)\\
((2222)_0\dots) &\leftrightarrow (1111\dots)\\
((22)_1 1111\dots) &\leftrightarrow (201010\dots)\\
((22)_0 1111\dots) &\leftrightarrow (111010\dots)\\
(11111111\dots) &\leftrightarrow (10101010\dots).
\end{aligned}
$$

Here we write $\dots$ instead of $000\dots$. (Compare [LS2, 6.2].)

3.10 Assume that $G = SO(V)$ where V is a $\mathbf{k}$-vector space of even dimension N with a fixed nondegenerate quadratic form. Let $g \in G$. For any $x \in \mathbf{k}^*$ let V_x be the generalized x-eigenspace of $g : V \to V$. Let $d_x = \dim V_x$. For any $x \in \mathbf{k}^*$ such that $x^2 \neq 1$ let $\lambda_1^x \geq \lambda_2^x \geq \lambda_3^x \geq \dots$ be the partition whose nonzero terms are the sizes of the Jordan blocks of $x^{-1}g : V_x \to V_x$. For $x \in \mathbf{k}^*$ such that $x^2 = 1$ let $\nu^x \in {}'Z_{d_x}^1$ (if $p \neq 2$) and $\nu^x \in {}'Z_{d_x}^2$ (if $p = 2$) be again the partition of d_x whose nonzero terms are the sizes of the Jordan blocks of the unipotent element $x^{-1}g \in SO(V_x)$. (When $p = 2$, ν^x should also include a labelling with 0 and 1 associated to $x^{-1}g$ viewed as an element of $Sp(V_x)$ as in [L10, 1.4].) Let $\lambda^x = (\lambda_1^x, \lambda_2^x, \lambda_3^x, \dots)$ be the bipartition of $d_x/2$ associated to ν^x by 3.6(b), 3.9(a). Thus $\lambda^x \in BP_{0,2}^{d_x/2}$ (if $p \neq 2$), $\lambda^x \in BP_{0,4}^{d_x/2}$ (if $p = 2$). Note that λ^x is the bipartition such that the Springer representation attached to the unipotent element $x^{-1}g \in \mathrm{SO}(V_x)$ (an irreducible representation of the Weyl group of type $D_{d_x/2}$) is indexed by λ^x. Define ${}^g\lambda = ({}^g\lambda_1, {}^g\lambda_2, {}^g\lambda_3, \dots)$ by ${}^g\lambda_j = \sum_x \lambda^x_j$ where x runs over a set of representatives for the orbits of the involution $a \mapsto a^{-1}$ of $\mathbf{k}^*$. Note that ${}^g\lambda \in BP^{N/2}_{0,4}$ and that $g \mapsto {}^g\lambda$ defines a (surjective) map $G \to BP^{N/2}_{0,4}$. From the definitions we see that the fibres of this map are exactly the strata of G (except for the fibre over a bipartition $(\lambda_1, \lambda_2, \lambda_3, \dots)$ with $\lambda_1 = \lambda_2, \lambda_3 = \lambda_4, \dots$ in which case the fibre is a union of two strata). If $g \in G$ and ${}^g\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$, then

(a) $\dim(\mathcal{B}_g) = \sum_{k \geq 1} ((N/2) - (\lambda_1 + \lambda_2 + \cdots + \lambda_k)).$

Viewing W as a subgroup of index 2 of a Weyl group W' of type B_n, we can associate to any $\lambda \in BP^{N/2}$ one or two irreducible representations of W which appear in the restriction to W of the irreducible representation of W' indexed by λ; the representation(s) of W associated to $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \dots)$ are the same as those associated to $\iota(\lambda) := (\lambda_2, \lambda_1, \lambda_4, \lambda_3, \dots)$; here $\iota : BP^{N/2} \to BP^{N/2}$ is an involution and the set of orbits is denoted by $BP^{N/2}/\iota$. This gives a surjective map $f : \mathrm{Irr}(W) \to BP^{N/2}/\iota$ whose fibre at the orbit of λ has one element if $\lambda \neq \iota(\lambda)$ and two elements if $\lambda = \iota(\lambda)$. Let $\iota' : \mathrm{Irr}(W) \to \mathrm{Irr}(W)$ be the involution whose orbits are the fibres of f and let $\mathcal{S}_2(W)/\iota'$ be the set of orbits of the restriction of ι' to $\mathcal{S}_2(W)$. The results in this subsection show that f induces a bijection

(b) $\mathcal{S}_2(W)/\iota' \xrightarrow{\sim} BP^{N/2}_{0,4}$.

We have used the fact that the intersection of $BP^{N/2}_{0,4}$ with an orbit of $\iota : BP^{N/2} \to BP^{N/2}$ has at most one element; more precisely,

$$\{\lambda \in BP^{N/2}; \lambda \in BP^{N/2}_{0,4} \text{ and } \iota(\lambda) \in BP^{N/2}_{0,4}\} = \{\lambda \in BP^{N/2}; \lambda = \iota(\lambda)\}.$$

Under the identification (b), the map $g \mapsto {}^g\lambda$, $G \to BP^{N/2}_{0,4}$ becomes the map $g \mapsto E$ (up to the action of ι') where $g \in G_E$.

3.11 Assume that $p \neq 2$ and $n \geq 3$. If $G = \mathrm{SO}_{2n+1,\mathbf{k}}$ then the stratum of minimal dimension > 0 consists of a semisimple class of dimension $2n$; if $G = \mathrm{Sp}_{2n,\mathbf{k}}/\pm 1$ then the stratum of minimal dimension > 0 consists of a unipotent class of dimension $2n$ (that of transvections). The corresponding $E \in \mathrm{Irr}(W)$ is one-dimensional.

3.12 Assume that G is simple of type E_8. In this case G has exactly 75 strata. If $p \neq 2, 3$ then exactly 70 strata contain unipotent elements. If $p = 2$ (resp. $p = 3$) then exactly 74 (resp. 71) strata contain unipotent elements. The unipotent class of dimension 58 is a stratum. If $p \neq 2$, there is a stratum which is a union of a semisimple class and a unipotent class (both of dimension 128); in particular this stratum is disconnected.

4 A map from conjugacy classes in W to 2-special representations of W

4.1 In this subsection we shall define a canonical surjective map

(a) $'\Phi : \mathrm{cl}(W) \to \mathcal{S}_2(W)$.

We preserve the setup of 2.5. We will first define the map (a) assuming that G is simple. In [L8] we have defined for any $r \in \mathcal{P}$ a surjective map $\mathrm{cl}(W) \to \mathcal{U}^r$; we denote this map by Φ^r. Let $C \in \mathrm{cl}(W)$. We define an element $\Phi(C) \in \mathcal{U}^*$ as follows. If $\Phi^r(C) \in h^r(z_r)$ (with $z_r \in \mathcal{U}^0$) for all $r \in \mathcal{P}$, then $z_r = z$ is independent of r (see [L10, 0.4]) and we define $\Phi(C)$ to be the equivalence class of $h^r(z)$ for any $r \in \mathcal{P}$. If $\Phi^r(C) \notin h^r(\mathcal{U}^0)$ for some $r \in \mathcal{P}$, then r is unique. (The only case where r can be possibly not unique is in type E_8 in which case we use the tables in [L10, 2.6].) We then define $\Phi(C)$ to be the equivalence class of $\Phi^r(C)$. Thus we have defined a surjective map $\Phi : \mathrm{cl}(W) \to \mathcal{U}^*$. By composing Φ^r with $\psi^r : \mathcal{U}^r \xrightarrow{\sim} \mathcal{S}_2^r(W)$, see 2.5, and with the inclusion $\mathcal{S}_2^r(W) \subset \mathcal{S}_2(W)$, we obtain a map $'\Phi^r : \mathrm{cl}(W) \to \mathcal{S}_2(W)$. Similarly, by composing Φ with $\psi^* : \mathcal{U}^* \xrightarrow{\sim} \mathcal{S}_2(W)$, see 2.5(a), we obtain a surjective map $'\Phi : \mathrm{cl}(W) \to \mathcal{S}_2(W)$. Note that for $C \in \mathrm{cl}(W)$, $'\Phi(C)$ can be described as follows. If $'\Phi^r(C) \in \mathcal{S}_1(W)$ for all $r \in \mathcal{P}$, then $'\Phi^r(C)$ is independent of r, and we have $'\Phi(C) = {}'\Phi^r(C)$ for any r. If $'\Phi^r(C) \notin \mathcal{S}_1(W)$ for some $r \in \mathcal{P}$, then such r is unique and we have $'\Phi(C) = {}'\Phi^r(C)$.

We return to the general case. We write the adjoint group of G as a product $\prod_i G_i$ where each G_i is simple with Weyl group W_i. We can identify $W = \prod_i W_i$, $\mathrm{cl}(W) = \prod_i \mathrm{cl}(W_i)$, $\mathcal{S}_2(W) = \prod_u \mathcal{S}_2(W_i)$ (via external tensor product). Then $'\Phi_i : \mathrm{cl}(W_i) \to \mathcal{S}_2(W_i)$ is defined as above for each i. We set $'\Phi = \prod_i {}'\Phi_i : \mathrm{cl}(W) \to \mathcal{S}_2(W)$.

For C, C' in $\mathrm{cl}(W)$ we write $C \sim C'$ if $'\Phi(C) = {}'\Phi(C')$. This is an equivalence relation on $\mathrm{cl}(W)$. Let $\underline{\mathrm{cl}}(W)$ be the set of equivalence classes. Note that:

(b) *$'\Phi$ induces a bijection $\underline{\mathrm{cl}}(W) \to \mathcal{S}_2(W)$.*

We see that, via (b),

(c) *the strata of G are naturally indexed by the set $\underline{\mathrm{cl}}(W)$.*

4.2 We preserve the setup of 2.5. Now $'\Phi$ in 4.1(a) is a map between two sets which depend only on W, not on the underlying root system, see 1.1(b). We show that

(a) *$'\Phi$ itself depends only on W, not on the underlying root system.*

We can assume that G is adjoint, simple. We can also assume that G is not of simply laced type. In this case there is a unique $r \in \mathcal{P}$ such that $\mathcal{S}_2(W) = \mathcal{S}_2^r(W)$ so that we have simply $'\Phi = {}'\Phi^r : \mathrm{cl}(W) \to \mathcal{S}_2(W)$. Thus $'\Phi$ is the composition

(b) $\mathrm{cl}(W) \xrightarrow{\Phi^r} \mathcal{U}^r \xrightarrow{\psi^r} \mathcal{S}_2(W)$.

We now use the fact the maps in (b) are compatible with the exceptional isogeny between groups G^2 of type B_n and C_n or of type F_4 and F_4 (resp. between groups G^3 of type G_2 and G_2). This implies (a).

4.3 Assume that G is simple. The map ${}'\Phi$ in 4.1 is defined in terms of ${}'\Phi^r$ which is the composition of $\Phi^r : \mathrm{cl}(W) \to \mathcal{U}^r$ (which is described explicitly in each case in [L10]) and $\psi^r : \mathcal{U}^r \leftarrow \mathcal{S}_2^r(W)$ which is given by the Springer correspondence. Therefore ${}'\Phi$ is explicitly computable. In this subsection we describe this map in the case where W is of classical type.

If W is of type $A_n, n \geq 1$, then $\mathrm{cl}(W)$ can be identified with the set of partitions of n: to a conjugacy class of a permutation of n objects we associate the partition whose nonzero terms are the sizes of the disjoint cycles of which the permutation is a product. We identify $\mathcal{S}_2(W) = \mathrm{Irr}(W)$ with the set of partitions in the standard way (the unit representation corresponds to the partition $(n, 0, 0 \dots)$). With these identifications the map ${}'\Phi$ is the identity map.

Assume now that W is a Weyl group of type B_n or C_n, $n \geq 2$. Let X be a set with $2n$ elements with a given fixed point free involution τ. We identify W with the group of permutations of X which commute with τ. To any $w \in W$, we can associate an element $\nu \in Z^2_{2n}$ (see 3.4) as follows. The nonzero terms of the partition ν are the sizes of the disjoint cycles of which w is a product. To each string $c, c, \dots, c$ of ν of even length with $c > 0$ even we attach the label 1 if at least one of its terms represents a cycle which commutes with τ; otherwise we attach to it the label 0. This defines a (surjective) map $\mathrm{cl}(W) \to Z^2_{2n}$ which by results of [L10] can be identified with the map $\Phi^2 : \mathrm{cl}(W) \to \mathcal{U}^2$. Composing this with the bijection 3.4(b) we obtain a surjective map $\mathrm{cl}(W) \to BP^n_{2,2}$ or equivalently (see 3.5(b)) $\mathrm{cl}(W) \to \mathcal{S}_2(W)$. This is the same as ${}'\Phi$.

Next we assume that W is a Weyl group of type D_n, $n \geq 4$. We can identify W with the group of *even* permutations of X (as above) which commute with τ (as above). To any $w \in W$ we associate an element $\nu \in Z^2_{2n}$ as for type B_n above. This element is actually contained in ${}'Z^2_{2n}$ (see 3.9) since w is an even permutation. This defines a (surjective) map $\mathrm{cl}(W) \to {}'Z^2_{2n}$ which by results of [L10] can be identified with the composition of $\Phi^2 : \mathrm{cl}(W) \to \mathcal{U}^2$ with the obvious map from $\mathcal{U}^2$ to the set of orbits of the conjugation action of the full orthogonal group on $\mathcal{U}^2$. Composing this with the bijection 3.9(a) we obtain a surjective map $\mathrm{cl}(W) \to BP^n_{0,4}$ or equivalently (see 3.10(b)) a surjective map $\mathrm{cl}(W) \to \mathcal{S}_2(W)/\iota'$ (notation of 3.10). This is the same as the composition of ${}'\Phi$ with the obvious map $\mathcal{S}_2(W) \to \mathcal{S}_2(W)/\iota'$.

4.4 In this and the next five subsections we describe the map ${}'\Phi : \mathrm{cl}(W) \to \mathcal{S}_2(W)$ in the case where W is of exceptional type. The results will be expressed as diagrams $[a, b, \dots] \mapsto d_n$ where $a, b, \dots$ is the list of conjugacy classes in W (with notation of [C]) which are mapped by ${}'\Phi$ to an irreducible representation E denoted d_n (here d denotes the degree of E and the index $n = n_E$ as in 0.2). We also mark by $*_r$ those E which are in $\mathcal{S}_2(W) - \mathcal{S}_1(W)$; here r is the unique prime such that $E \in \mathcal{S}^r_2(W)$. Note that the notation d_n does not determine E for types G_2 and F_4; for these types it may happen that there are two E's with same d_n.

Type G_2

$[G_2] \mapsto 1_0$ $\quad [A_1+\widetilde{A}_1] \mapsto 2_2$ $\quad [A_1] \mapsto 1_3$

$[A_2] \mapsto 2_1$ $\quad [\widetilde{A}_1] \mapsto 1_3, *_3$ $\quad [A_0] \mapsto 1_6$

4.5 Type F_4.

$[F_4] \mapsto 1_0$ $\quad [A_2+\widetilde{A}_1] \mapsto 4_7$

$[B_4] \mapsto 4_1$ $\quad [\widetilde{A}_2+A_1] \mapsto 4_7, *_2$

$[F_4(a_1)] \mapsto 9_2$ $\quad [B_2] \mapsto 4_8, *_2$

$[D_4, B_3] \mapsto 8_3$ $\quad [\widetilde{A}_2] \mapsto 8_9$

$[C_3+A_1, C_3] \mapsto 8_3$ $\quad [A_2] \mapsto 8_9$

$[D_4(a_1)] \mapsto 12_4$ $\quad [4A_1, 3A_1, 2A_1+\widetilde{A}_1, A_1+\widetilde{A}_1] \mapsto 9_{10}$

$[A_3+\widetilde{A}_1] \mapsto 16_5$ $\quad [2A_1] \mapsto 4_{13}$

$[A_3] \mapsto 9_6$ $\quad [A_1] \mapsto 2_{16}$

$[B_2+A_1] \mapsto 9_6, *_2$ $\quad [\widetilde{A}_1] \mapsto 2_{16}, *_2$

$[\widetilde{A}_2+\widetilde{A}_2] \mapsto 6_6$ $\quad [A_0] \mapsto 1_{24}$

4.6 Type E_6.

$[E_6] \mapsto 1_0$ $\quad [D_4] \mapsto 24_6$ $\quad [2A_2] \mapsto 24_{12}$

$[E_6(a_1)] \mapsto 6_1$ $\quad [A_4] \mapsto 81_6$ $\quad [A_2+A_1] \mapsto 64_{13}$

$[D_5] \mapsto 20_2$ $\quad [D_4(a_1)] \mapsto 80_7$ $\quad [A_2] \mapsto 30_{15}$

$[E_6(a_2)] \mapsto 30_3$ $\quad [A_3+2A_1, A_3+A_1] \mapsto 60_8$ $\quad [4A_1, 3A_1] \mapsto 15_{16}$

$[A_5+A_1, A_5] \mapsto 15_4$ $\quad [3A_2, 2A_2+A_1] \mapsto 10_9$ $\quad [2A_1] \mapsto 20_{20}$

$[D_5(a_1)] \mapsto 64_4$ $\quad [A_3] \mapsto 81_{10}$ $\quad [A_1] \mapsto 6_{25}$

$[A_4+A_1] \mapsto 60_5$ $\quad [A_2+2A_1] \mapsto 60_{11}$ $\quad [A_0] \mapsto 1_{36}$

4.7 Type E_7.

$[E_7] \mapsto 1_0$ $\quad [E_7(a_4)] \mapsto 315_7$

$[E_7(a_1)] \mapsto 7_1$ $\quad [D_5] \mapsto 189_7$

$[E_7(a_2)] \mapsto 27_2$ $\quad [E_6(a_2)] \mapsto 405_8$

$[E_7(a_3)] \mapsto 56_3$ $\quad [D_6(a_2)+A_1, D_6(a_2)] \mapsto 280_8$

$[E_6] \mapsto 21_3$ $\quad [A_5+A_2, (A_5+A_1)'] \mapsto 70_9$

$[E_6(a_1)] \mapsto 120_4$ $\quad [(A_5+A_1)'', A_5''] \mapsto 216_9$

$[D_6+A_1, D_6] \mapsto 35_4$ $\quad [D_5(a_1)+A_1] \mapsto 378_9$

$[A_7] \mapsto 189_5$ $\quad [D_5(a_1)] \mapsto 420_{10}$

$[A_6] \mapsto 105_6$ $\quad [A_4+A_2] \mapsto 210_{10}$

$$[D_6(a_1)] \mapsto 210_6 \qquad [A_4 + A_1] \mapsto 512_{11}$$
$$[D_5 + A_1] \mapsto 168_6 \qquad [A_5'] \mapsto 105_{12}$$

$$[D_4 + 3A_1, D_4 + 2A_1, D_4 + A_1] \mapsto 84_{12}$$
$$[A_4] \mapsto 420_{13}$$
$$[2A_3 + A_1, A_3 + A_2 + A_1] \mapsto 210_{13}$$
$$[A_3 + A_2] \mapsto 378_{14}$$
$$[D_4] \mapsto 105_{15}$$
$$[D_4(a_1) + A_1] \mapsto 405_{15}$$
$$[A_3 + A_2] \mapsto 84_{15}, *_2$$
$$[A_3 + 3A_1, (A_3 + 2A_1)'] \mapsto 216_{16}$$
$$[D_4(a_1)] \mapsto 315_{16}$$
$$[(A_3 + 2A_1)'', (A_3 + A_1)''] \mapsto 280_{17}$$
$$[3A_2, 2A_2 + A_1] \mapsto 70_{18}$$
$$[(A_3 + A_1)'] \mapsto 189_{20}$$

$$[A_3] \mapsto 210_{21} \qquad [A_2] \mapsto 56_{30}$$
$$[2A_2] \mapsto 168_{21} \qquad [(4A_1)'', (3A_1)''] \mapsto 35_{31}$$
$$[A_2 + 3A_1] \mapsto 105_{21} \qquad [(3A_1)'] \mapsto 21_{36}$$
$$[A_2 + 2A_1] \mapsto 189_{22} \qquad [2A_1] \mapsto 27_{37}$$
$$[A_2 + A_1] \mapsto 120_{25} \qquad [A_1] \mapsto 7_{46}$$
$$[7A_1, 6A_1, 5A_1, (4A_1)'] \mapsto 15_{28} \qquad [A_0] \mapsto 1_{63}$$

4.8 Type E_8

$$[E_8] \mapsto 1_0 \qquad [E_7(a_3)] \mapsto 2268_{10}$$
$$[E_8(a_1)] \mapsto 8_1 \qquad [E_6(a_1) + A_1] \mapsto 4096_{11}$$
$$[E_8(a_2)] \mapsto 35_2 \qquad [D_8(a_3)] \mapsto 1400_{11}$$
$$[E_8(a_4)] \mapsto 112_3 \qquad [E_6] \mapsto 525_{12}$$
$$[E_7 + A_1, E_7] \mapsto 84_4 \qquad [D_7(a_2)] \mapsto 4200_{12}$$
$$[E_8(a_5)] \mapsto 210_4 \qquad [D_6 + 2A_1, D_6 + A_1, D_6] \mapsto 972_{12}$$
$$[D_8] \mapsto 560_5 \qquad [E_6(a_1)] \mapsto 2800_{13}$$
$$[E_7(a_1)] \mapsto 567_6 \qquad [A_7 + A_1] \mapsto 4536_{13}$$
$$[E_8(a_3)] \mapsto 700_6 \qquad [A_7'] \mapsto 6075_{14}$$
$$[D_8(a_1), D_7] \mapsto 400_7 \qquad [A_6 + A_1] \mapsto 2835_{14}$$
$$[E_8(a_7)] \mapsto 1400_7 \qquad [D_5 + A_2] \mapsto 840_{14}, *_2$$
$$[E_8(a_6)] \mapsto 1400_8 \qquad [A_6] \mapsto 4200_{15}$$

$[E_7(a_2)+A_1, E_7(a_2)] \mapsto 1344_8$
$[E_6+A_2, E_6+A_1] \mapsto 448_9$
$[D_8(a_2)] \mapsto 3240_9$
$[D_7(a_1)] \mapsto 1050_{10}, *_2$
$[A_7''] \mapsto 175_{12}, *_3$
$[A_8] \mapsto 2240_{10}$
$[D_6(a_1)] \mapsto 5600_{15}$
$[E_8(a_8)] \mapsto 4480_{16}$
$[D_5+2A_1, D_5+A_1] \mapsto 3200_{16}$
$[E_7(a_4)+A_1, E_7(a_4)] \mapsto 7168_{17}$
$[2D_4, D_6(a_2)+A_1, D_6(a_2)] \mapsto 4200_{18}$
$[E_6(a_2)+A_2, E_6(a_2)+A_1] \mapsto 3150_{18}$

$[A_5+A_2+A_1, A_5+A_2, A_5+2A_1, (A_5+A_1)''] \mapsto 2016_{19}$
$[D_5(a_1)+A_3, D_5(a_1)+A_2] \mapsto 1344_{19}$

$[D_5] \mapsto 2100_{20}$
$[2A_4, A_4+A_3] \mapsto 420_{20}$
$[E_6(a_2)] \mapsto 5600_{21}$
$[D_4+A_3] \mapsto 4200_{21}$
$[(A_5+A_1)'] \mapsto 3200_{22}$
$[D_5(a_1)+A_1] \mapsto 6075_{22}$
$[A_4+A_2+A_1] \mapsto 2835_{22}$
$[A_4+A_2] \mapsto 4536_{23}$
$[A_4+2A_1] \mapsto 4200_{24}$
$[D_4+A_2] \mapsto 168_{24}, *_2$
$[D_5(a_1)] \mapsto 2800_{25}$
$[A_4+A_1] \mapsto 4096_{26}$

$[2D_4(a_1), D_4(a_1)+A_3, (2A_3)''] \mapsto 840_{26}$
$[D_4+4A_1, D_4+3A_1, D_4+2A_1, D_4+A_1] \mapsto 700_{28}$
$[D_4(a_1)+A_2] \mapsto 2240_{28}$
$[2A_3+2A_1, A_3+A_2+2A_1, 2A_3+A_1, A_3+A_2+A_1] \mapsto 1400_{29}$
$[A_4] \mapsto 2268_{30}$
$[(2A_3)'] \mapsto 3240_{31}$
$[D_4(a_1)+A_1] \mapsto 1400_{32}$
$[A_3+A_2] \mapsto 972_{32}, *_2$
$[A_3+4A_1, A_3+3A_1, (A_3+2A_1)''] \mapsto 1050_{34}$

$[D_4] \mapsto 525_{36}$
$[4A_2, 3A_2+A_1, 2A_2+2A_1] \mapsto 175_{36}$
$[D_4(a_1)] \mapsto 1400_{37}$
$[(A_3+2A_1)', A_3+A_1] \mapsto 1344_{38}$
$[3A_2, 2A_2+A_1] \mapsto 448_{39}$
$[2A_2] \mapsto 700_{42}$
$[A_2+4A_1, A_2+3A_1] \mapsto 400_{43}$
$[A_3] \mapsto 567_{46}$
$[A_2+2A_1] \mapsto 560_{47}$
$[A_2+A_1] \mapsto 210_{52}$
$[8A_1, 7A_1, 6A_1, 5A_1, (4A_1)''] \mapsto 50_{56}$
$[A_2] \mapsto 112_{63}$
$[(4A_1)', 3A_1] \mapsto 84_{64}$
$[2A_1] \mapsto 35_{74}$
$[A_1] \mapsto 8_{91}$
$[A_0] \mapsto 1_{120}$

4.9 In the tables in 4.4–4.8 the E which are not marked with $*_r$ are in $\mathcal{S}_1(W)$; they are expressed explicitly in the form $j_{W_{e'}}^W(E')$ with $e' \in V^*$, $E' \in \mathcal{S}(W_{e'})$ in the tables of [L6].

We now consider the E in the tables 4.4–4.8 which are marked with $*_r$.

Type G_2:

$1_3 = j_{W'}^W(\text{sign})$ where W' is of type A_2 but not of form $W_{e'}, e' \in V^*$.

Type F_4:

$9_6 = j_{W'}^W(\text{E'})$ where W' is of type B_4 but not of form $W_{e'}, e' \in V^*$ and dim $E' = 6, n_{E'} = 6$;
$4_7 = j_{W'}^W(\text{sign})$ where W' is of type A_3A_1 but not of form $W_{e'}, e' \in V^*$;
$4_8 = j_{W'}^W(\text{sign})$ where W' is of type B_2B_2;
$2_{12} = j_{W'}^W(\text{sign})$ where W' is of type B_4 but not of form $W_{e'}, e' \in V^*$.

Type E_7:

$84_{15} = j_{W'}^W(\text{sign})$ where W' is of type $D_4A_1A_1A_1$.

Type E_8:

$1050_{10} = j_{W'}^W(\text{E'})$ where W' is of type $D_6A_1A_1$ and dim $E' = 30, n_{E'} = 10$;
$175_{12} = j_{W'}^W(\text{sign})$ where W' is of type $A_2A_2A_2A_2$.
$840_{14} = j_{W'}^W(\text{sign})$ where W' is of type $A_3A_3A_1A_1$.
$168_{24} = j_{W'}^W(\text{sign})$ where W' is of type D_4D_4.
$972_{32} = j_{W'}^W(\text{sign})$ where W' is of type $D_6A_1A_1$.

4.10 For any $C \in \text{cl}(W)$ let m_C be the dimension of the 1-eigenspace of an element in C in the reflection representation of W. We have the following result.

(a) *For any $E \in \mathcal{S}_2(W)$, the restriction of $C \mapsto m_C$ to $'\Phi^{-1}(E) \subset \text{cl}(W)$ reaches its minimum at a unique element of $'\Phi^{-1}(E)$, denoted by C_E.*

We can assume that G is simple. When G is of exceptional type, (a) follows from the tables 4.4-4.8. When G is of classical type, (a) follows from [L10, 0.2].

Note that $E \mapsto C_E$ is a cross section of the surjective map $'\Phi : \text{cl}(W) \to \mathcal{S}_2(W)$. It defines a bijection of $\mathcal{S}_2(W)$ with a subset $\text{cl}_0(W)$ of $\text{cl}(W)$.

5 A second approach

5.1 In this section we sketch another approach to defining the strata of G in which Springer representations do not appear. Let $\mathrm{cl}(G)$ be the set of conjugacy classes in G. Let $\underline{l} : W \to \mathbf{N}$ be the length function of the Coxeter group W. For $w \in W$ let

$$G_w = \{g \in G; (B, gBg^{-1}) \in \mathcal{O}_w \text{ for some } B \in \mathcal{B}\}.$$

For $C \in \mathrm{cl}(W)$ let

$$C_{\min} = \{w \in C; \underline{l} : C \to \mathbf{N} \text{ reaches minimum at } w\}$$

and let $G_C = G_w$ where $w \in C_{\min}$.

As pointed out in [L8, 0.2], from [L8, 1.2(a)] and [GP, 8.2.6(b)] it follows that G_C is independent of the choice of w in $C_{\min}$. From [L8] it is known that G_C contains unipotent elements; in particular, $G_C \neq \emptyset$. Clearly, G_C is a union of conjugacy classes. Let

$$\delta_C = \min_{\gamma \in \mathrm{cl}(G); \gamma \subset G_C} \dim \gamma,$$

$$\boxed{G_C} = \bigcup_{\substack{\gamma \in \mathrm{cl}(G); \\ \gamma \subset G_C, \dim \gamma = \delta_C}} \gamma.$$

Then $\boxed{G_C}$ is $\neq \emptyset$, a union of conjugacy classes of fixed dimension, δ_C. We have the following result.

5.2 Theorem Let $C \in \mathrm{cl}(W)$, $E \in \mathcal{S}_2(W)$ be such that $'\Phi(C) = E$, see 4.1. We have $\boxed{G_C} = G_E$.

We can assume that G is almost simple and that $\mathbf{k}$ is an algebraic closure of a finite field. The proof in the case of exceptional groups is reduced in 5.3 to a computer calculation. The proof for classical groups, which is based on combining the techniques of [L8], [L9] and [L12], will be given elsewhere.

5.3 In this subsection we assume that $\mathbf{k}$ is an algebraic closure of a finite field $\mathbf{F}_q$ and that G is simply connected, defined and split over $\mathbf{F}_q$ with Frobenius map $F : G \to G$.

Let γ be an F-stable conjugacy class of G. Let $\gamma' = \{g_s; g \in \gamma\}$, an F-stable semisimple conjugacy class in G. For every $s \in \gamma'$ let $\gamma(s) = \{u \in Z_G(s); u \text{ unipotent}, us \in \gamma\}$, a unipotent conjugacy class of $Z_G(s)$. We fix $s_0 \in \gamma'^F$ and we set $H = Z_G(s_0)$, $\gamma_0 = \gamma(s_0)$. Let W_H be the Weyl group of H. As in 2.1, we can regard W_H as a subgroup of W (the embedding of W_H into W is canonical up to composition with an inner automorphism of W).

By replacing if necessary F by a power of F, we can assume that H contains a maximal torus which is defined and split over $\mathbf{F}_q$. For any F-stable maximal torus

T of G, R_T^1 is the virtual representation of G^F defined as in [DL, 1.20] (with $\theta = 1$ and with B omitted from notation). Replacing T, G by T', H where T' is an F-stable maximal torus of H, we obtain a virtual representation $R^1_{T',H}$ of H^F.

For any $z \in W$ we denote by R_z^1 the virtual representation R_T^1 of G^F where T is an F-stable maximal torus of G of type given by the conjugacy class of z in W. For any $z' \in W_H$ we denote by $R^1_{z',H}$ the virtual representation $R^1_{T',H}$ of H^F where T' is an F-stable maximal torus of H of type given by the conjugacy class of z' in W_H. For $E' \in \mathrm{Irr}W$ we set $R_{E'} = |W|^{-1}\sum_{y\in W}\mathrm{tr}(y, E')R_y^1$. Then for any $z \in W$, we have $R_z^1 = \sum_{E'\in\mathrm{Irr}W}\mathrm{tr}(z, E')R_{E'}$.

Let $w \in W$. We show the following:

(a)
$$\begin{aligned}
&|\{(g, B) \in \gamma^F \times \mathcal{B}^F; (B, gBg^{-1}) \in \mathcal{O}_w\}| \\
&\quad = |G^F||H^F|^{-1} \sum_{\substack{E\in\mathrm{Irr}W, E'\in\mathrm{Irr}W,\\ E''\in\mathrm{Irr}W_H, y}} \mathrm{tr}(T_w, E_q)(\rho_E, R_{E'}) \\
&\quad \times (E'|_{W_H} : E'')|Z_{W_H}(y)|^{-1}\mathrm{tr}(y, E'') \sum_{u\in\gamma_0^F}\mathrm{tr}(u, R^1_{y,H}),
\end{aligned}$$

where y runs over a set of representatives for the conjugacy classes in W_H and T_w, E_q, ρ_E are as in [L8, 1.2]. Let N be the left-hand side of (a). As in [L8, 1.2(c)] we see that

$$N = \sum_{E\in\mathrm{Irr}W}\mathrm{tr}(T_w, E_q)A_E$$

with

$$A_E = |G^F|^{-1}\sum_{g\in\gamma^F}\sum_T |T^F|(\rho_E, R_T^1)\mathrm{tr}(g, R_T^1),$$

where T runs over all maximal tori of G defined over $\mathbf{F}_q$. We have

$$\begin{aligned}
A_E &= |G^F|^{-1}\sum_{s\in\gamma'^F, u\in\gamma(s)^F}\sum_T |T^F|(\rho_E, R_T^1)\mathrm{tr}(su, R_T^1) \\
&= |H^F|^{-1}\sum_{u\in\gamma_0^F}\sum_T |T^F|(\rho_E, R_T^1)\mathrm{tr}(s_0u, R_T^1).
\end{aligned}$$

By [DL, 4.2] we have

$$\mathrm{tr}(s_0u, R_T^1) = |H^F|^{-1}\sum_{x\in G^F; x^{-1}Tx\subset H}\mathrm{tr}(u, R^1_{x^{-1}Tx,H}),$$

hence

$$A_E = |H^F|^{-2} \sum_{u \in \gamma_0^F} \sum_T |T^F|(\rho_E, R_T^1) \sum_{x \in G^F; x^{-1}Tx \subset H} \mathrm{tr}(u, R^1_{x^{-1}Tx,H})$$
$$= |G^F||H^F|^{-2} \sum_{T' \subset H} |T'^F|(\rho_E, R^1_{T'}) \sum_{u \in \gamma_0^F} \mathrm{tr}(u, R^1_{T',H}),$$

where T' runs over the maximal tori of H defined over $\mathbf{F}_q$. Using the classification of maximal tori of H defined over $\mathbf{F}_q$, we obtain

$$A_E = |G^F||H^F|^{-1}|W_H|^{-1} \sum_{z \in W_H} (\rho_E, R_z^1) \sum_{u \in \gamma_0^F} \mathrm{tr}(u, R^1_{z,H})$$
$$= |G^F||H^F|^{-1}|W_H|^{-1} \sum_{z \in W_H} \sum_{E' \in \mathrm{Irr} W} \mathrm{tr}(z, E')(\rho_E, R_{E'}) \sum_{u \in \gamma_0^F} \mathrm{tr}(u, R^1_{z,H}).$$

This clearly implies (a).

Now assume that G is almost simple of exceptional type and that w has minimal length in its conjugacy class in W. We can also assume that $q-1$ is sufficiently divisible. Then the right-hand side of (a) can be explicitly determined using a computer. Indeed, it is an entry of the product of several large matrices whose entries are explicitly known. In particular the quantities $\mathrm{tr}(T_w, E_q)$ (known from the works of Geck and Geck–Michel, see [GP, 11.5.11]) are available through the CHEVIE package [GH]. The quantities $(\rho_E, R_{E'})$ are coefficients of the nonabelian Fourier transform in [L2, 4.15]. The quantities $(E'|_{W_H} : E'')$ are available from the induction tables in the CHEVIE package. The quantities $\mathrm{tr}(y, E'')$ are available through the CHEVIE package. The quantities $\mathrm{tr}(u, R^1_{y,H})$ are Green functions; I thank Frank Lübeck for providing me with the tables of Green functions for groups of rank ≤ 8 in GAP format. I also thank Gongqin Li for her help with programming in GAP to perform the actual computation using these data.

Thus the number $|\{(g, B) \in \gamma^F \times \mathcal{B}^F; (B, gBg^{-1}) \in \mathcal{O}_w\}|$ is explicitly computable. It turns out that it is a polynomial in q. Note that the set $\{(g, B) \in \gamma \times \mathcal{B}; (B, gBg^{-1}) \in \mathcal{O}_w\}$ is nonempty if and only if this polynomial is non zero. Thus the condition that $\gamma \subset G_w$ can be tested. This can be used to check that Theorem 5.2 holds for exceptional groups.

5.4 If C is the conjugacy class containing the Coxeter elements of W, then $G_C = \boxed{G_C}$ is the union of all conjugacy classes of dimension $\dim G - \mathrm{rk}(G)$, see [St].

6 Variants

6.1 The results in this subsection will be proved elsewhere. In this subsection we assume that G is simple and that G' is a disconnected reductive algebraic group G over $\mathbf{k}$ with identity component G, such that G'/G is cyclic of order r and such that the homomorphism $\epsilon : G'/G \to \mathrm{Aut}(W)$ (the automorphism group of W as a Coxeter group) induced by the conjugation action of G'/G on G is injective. Note that (G,r) must be of type $(A_n, 2)$ $(n \geq 2)$ or $(D_n, 2)$ $(n \geq 4)$ or $(D_4, 3)$ or $(E_6, 2)$. Let D be a connected component of G' other than G. We will give a definition of the strata of D, extending the definition of strata of G. Let $\epsilon_D : W \to W$ be the image of D under ϵ. Let $\mathrm{cl}_D W$ be the set of conjugacy classes in W twisted by ϵ_D (as in [L12, 0.1]). Let $\mathrm{cl}(D)$ be the set of G-conjugacy classes in D. For $w \in W$ let

$$D_w = \{g \in D; (B, gBg^{-1}) \in \mathcal{O}_w \text{ for some } B \in \mathcal{B}\}.$$

For $C \in \mathrm{cl}_D(W)$ let

$$C_{\min} = \{w \in C; \underline{l} : C \to \mathbf{N} \text{ reaches minimum at } w\}.$$

and let $D_C = D_w$ where $w \in C_{\min}$. This is independent of the choice of w in $C_{\min}$. One can show that $D_C \neq \emptyset$. Clearly, D_C is a union of G-conjugacy classes in D. Let

$$\delta_C = \min_{\gamma \in \mathrm{cl}(D); \gamma \subset D_C} \dim \gamma,$$

$$\boxed{D_C} = \bigcup_{\substack{\gamma \in \mathrm{cl}(D);\\ \gamma \subset D_C, \dim \gamma = \delta_C}} \gamma.$$

Then $\boxed{D_C}$ is $\neq \emptyset$, a union of G-conjugacy classes of fixed dimension, δ_C. One can show that $\cup_{C \in \mathrm{cl}_D(W)} \boxed{D_C} = D$; moreover, one can show that if $C, C' \in \mathrm{cl}_D(W)$, then $\boxed{D_C}, \boxed{D_{C'}}$ are either equal or disjoint. (Some partial results in this direction are contained in [L12].) Let $\sim$ be the equivalence relation on $\mathrm{cl}_D(W)$ given by $C \sim C'$ if $\boxed{D_C} = \boxed{D_{C'}}$ and let $\underline{\mathrm{cl}}_D(W)$ be the set of equivalence classes. We see that there is a unique partition of D into pieces (called *strata*) indexed by $\underline{\mathrm{cl}}_D(W)$ such that each stratum is of the form $\boxed{D_C}$ for some $C \in \mathrm{cl}_D(W)$. One can show that the equivalence relation $\sim$ on $\mathrm{cl}_D(W)$ and the function $C \mapsto d_C$ on $\mathrm{cl}_D(W)$ depend only on W and its automorphism ϵ_D; in particular they do not depend on $\mathbf{k}$. When $p = r$, each stratum of D contains a unique unipotent G-conjugacy class in D; this gives a bijection $\underline{\mathrm{cl}}_D(W) \leftrightarrow \mathcal{U}_D^r$ where $\mathcal{U}_D^r$ is the set of unipotent G-conjugacy classes in D (with $p = r$). This bijection coincides with the bijection $\underline{\mathrm{cl}}_D(W) \leftrightarrow \mathcal{U}_D^r$ described explicitly in [L11]. Thus the strata of D can also be indexed by $\mathcal{U}_D^r$. We can also index them by a certain set of irreducible representations of W^{ϵ_D} (the fixed point set of $\epsilon_D : W \to W$) using the bijection [L4, II] between $\mathcal{U}_D^r$ and a set of irreducible representations of W^{ϵ_D} (an extension of the Springer correspondence).

6.2 Assume that G is adjoint. We identify $\mathcal{B}$ with the variety of Borel subalgebras of $\mathfrak{g}$. For any $\xi \in \mathfrak{g}$ let $\mathcal{B}_\xi = \{\mathfrak{b} \in \mathcal{B}; \xi \in \mathfrak{b}\}$ and let $d = \dim \mathcal{B}_\xi$. The subspace of $H_{2d}(\mathcal{B})$ spanned by the images of the fundamental classes of the irreducible components of $\mathcal{B}_\xi$ is an irreducible W-module denoted by τ_ξ. We also denote by τ_ξ

the corresponding W-module over $\mathbf{Q}$. Thus we have a well-defined map $\mathfrak{g} \to \mathrm{Irr}W$, $\xi \mapsto \tau_\xi$. The nonempty fibres of this map are called the *strata* of $\mathfrak{g}$. Each stratum of $\mathfrak{g}$ is a union of adjoint orbits of fixed dimension; exactly one of these orbits is nilpotent. The image of the map $\xi \mapsto \tau_\xi$ is the subset of $\mathrm{Irr}(W)$ denoted by $\mathcal{T}_W^p$ in [L7]; when $p = 0$ this is $\mathcal{S}_1(W)$.

6.3 In this subsection we assume that G is semisimple simply connected. Let K be the field of formal power series $\mathbf{k}((\epsilon))$ and let $\hat{G} = G(K)$. Let $\hat{\mathcal{B}}$ be the set of Iwahori subgroups of $\hat{G}$ viewed as an increasing union of projective algebraic varieties over $\mathbf{k}$. Let $\hat{W}$ be the affine Weyl group associated to $\hat{G}$ viewed as an infinite Coxeter group. Let $G(K)_{rsc}$ be the set of all $g \in G(K)$ that are compact (that is such that $\hat{\mathcal{B}}_g = \{B \in \hat{\mathcal{B}}; g \in B\}$ is nonempty) and regular semisimple. If $g \in G(K)_{rsc}$, then $\hat{\mathcal{B}}_g$ is a union of projective algebraic varieties of fixed dimension $d = d_g$ (see [KL] for a closely related result) hence the homology space $H_{2d}(\hat{\mathcal{B}}_g)$ is well defined and it carries a natural $\hat{W}$-action (see [L13]). Similarly the homology space $H_{2d}(\hat{\mathcal{B}})$ is well-defined and it carries a natural $\hat{W}$-action. The embedding $h_g : \hat{\mathcal{B}}_g \to \hat{\mathcal{B}}$ induces a linear map $h_{g*} : H_{2d}(\hat{\mathcal{B}}_g) \to H_{2d}(\hat{\mathcal{B}})$ which is compatible with the $\hat{W}$-actions. Hence $\hat{W}$ acts naturally on the (finite-dimensional) subspace $E_g := h_{g*}(H_{2d}(\hat{\mathcal{B}}_g))$ of $H_{2d}(\hat{\mathcal{B}})$, but this action is not irreducible in general. Note that E_g is the subspace of $H_{2d}(\hat{\mathcal{B}})$ spanned by the images of the fundamental classes of the irreducible components of $\hat{\mathcal{B}}_g, \overline{\mathbf{Q}}_l$ (we ignore Tate twists), hence is $\neq 0$. For $g, g' \in G(K)_{rsc}$ we say that $g \sim g'$ if $d_g = d_{g'}$ and $E_g = E_{g'}$. This is an equivalence relation on $G(K)_{rsc}$. The equivalence classes for $\sim$ are called the *strata* of $G(K)_{rsc}$. Note that $G(K)_{rsc}$ is a union of countably many strata and each stratum is a union of conjugacy classes of $G(K)$ contained in $G(K)_{rsc}$.

6.4 In this subsection we state a conjectural definition of the strata of G in the case where $\mathbf{k} = \mathbf{C}$ based on an extension of a construction in [KL]. Let K be as in 6.3. Let $g \in G$. Let $\mathfrak{z} \subset \mathfrak{g}$ be the Lie algebra of $Z_G(g_s)$ and let $\xi = \log(g_u) \in \mathfrak{z}$. Let $\mathfrak{p}$ be a parahoric subalgebra of $\mathfrak{g}_K := K \otimes \mathfrak{g}$ with pro-nilradical $\mathfrak{p}_n$ such that $\mathfrak{p} = \mathfrak{z} \oplus \mathfrak{p}_n$ as $\mathbf{C}$-vector spaces. By the last corollary in [KL, §6], there exists a non-empty subset $\mathfrak{U}$ of $\xi + \mathfrak{p}_n$ (open in the power series topology) and $\sigma \in \mathrm{cl}(W)$ such that for any $x \in \mathfrak{U}$, x is regular semisimple in a Cartan subalgebra of $\mathfrak{g}_K$ of type σ (see [KL, §1,§6]). Note that σ does not depend on the choice of $\mathfrak{U}$. We expect that it does not depend on the choice of $\mathfrak{p}$ and that $g \mapsto \sigma$ is a map $G \to \mathrm{cl}(W)$ whose fibres are exactly the strata of G. By the identification 4.1(c) this induces an injective map $\underline{\mathrm{cl}}(W) \to \mathrm{cl}(W)$ whose image is expected to be the subset $\mathrm{cl}_0(W)$ in 4.10 and whose composition with the obvious map $\mathrm{cl}(W) \to \underline{\mathrm{cl}}(W)$ is expected to be the identity map of $\underline{\mathrm{cl}}(W)$.

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