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Representations of Reductive Groups

In Honor of the 60th Birthday of David A. Vogan, Jr.

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To David Vogan, with admiration

Contents

Preface

This volume is an outgrowth of the conference *Representations of Reductive Groups: A conference dedicated to David Vogan on his 60th birthday*, which took place at MIT from May 19–23, 2014. This celebratory conference showcased developments in the representation theory of reductive Lie groups and algebraic groups over finite and local fields, as well as connections of this theory with other subjects, such as number theory, automorphic forms, algebraic geometry and combinatorics. It was an occasion to mark the 60th birthday of David Vogan, who has inspired and shaped the development of this field for almost 40 years. A list of conference speakers appears on page xiii.

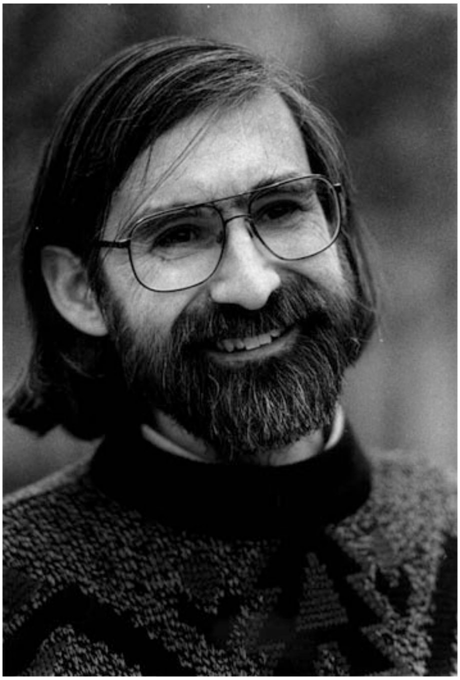
David Vogan has proven himself a leader in the field in many ways. The most evident of these are his vast and influential mathematical contributions, an overview of which appears in the chapter *The Mathematical Work of David A. Vogan, Jr.* in this volume. Also, of significant importance to so many — and to us personally — has been David's mentorship, insight and guidance. We include a subset of those who have benefitted from working closely with David — namely, his PhD students to date, as well as some of his mathematical descendants who were able to attend the conference — on pages xvi to xviii.

AMS Centennial Fellow (1977), Plenary Lecturer of the International Congress of Mathematicians (1986), Herman Weyl Lecturer (IAS, 1986), Member of the American Academy of Arts and Sciences (1996), Robert E. Collins Distinguished Scholar, MIT, (2007–2012), winner of the Levi L. Conant Prize (2011) for his paper "The character table of E_8", Member of the National Academy of Science (2013), and current Norbert Weiner Chair at MIT (2014–2019), David Vogan has been often been distinguished by his peers and by the mathematical community. Significantly, he has also given back through substantial work in academic service, including as Head of the MIT Department of Mathematics (1999–2004) and President of the American Mathematical Society (2013–2015).

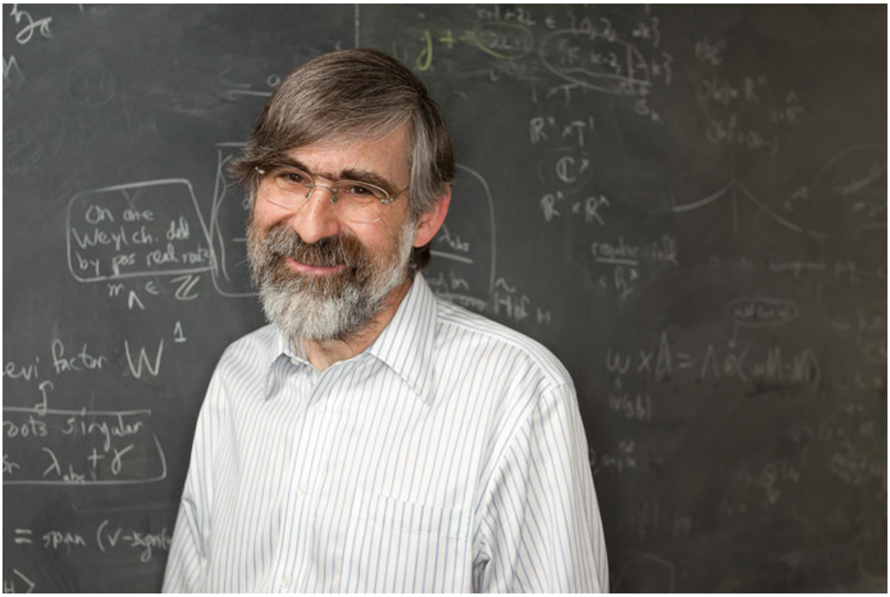
This volume represents a step towards expressing our thanks.

Acknowledgements It is a pleasure to thank our conference co-organizers, Roman Bezrukavnikov, Pavel Etingof, and George Lusztig, as well as members of the MIT administrative staff — particularly Shirley Entzminger, Jonathan Harmon, and Dennis Porche, whose assistance and support were indispensable. We are especially grateful for the editorial assistance of Ann Kostant in the preparation of this work.

Ottawa, Canada — Monica Nevins
Salt Lake City, USA — Peter E. Trapa
May 2015



David A. Vogan, Jr., 1999
Photo courtesy of Donna Coveney, MIT.



David A. Vogan, Jr., 2011
Photo courtesy of Bryce Vickmark.

Representations of reductive groups
A conference dedicated to David Vogan on his 60th birthday
M.I.T., May 19 – May 23, 2014

Conference Speakers

Pramod Achar	Equivariant coherent sheaves on the nilpotent cone of a reductive algebraic group
Jeffrey Adams	Galois and theta cohomology of real groups
James Arthur	On Langlands' automorphic Galois group and Weil's explicit formulas
Joseph Bernstein	Stacks in Representation Theory: What is a representation of an algebraic group?
Dan Ciubotaru	On formal degrees of unipotent discrete series representations of semisimple p-adic groups
Stephen DeBacker	Nilpotent orbits revisited
Michel Duflo	On Frobenius Lie subalgebras of simple Lie algebras
Meinolf Geck	Computing with left cells of type E_8
Benedict Gross	Newforms for odd orthogonal groups
Xuhua He	Cocenters and representations of affine Hecke algebras
Jing-Song Huang	Elliptic representations, Dirac cohomology and endoscopy
Toshiyuki Kobayashi	Branching problems of representations of real reductive Lie groups
Bert Kostant	Generalized Amitsur-Levitski Theorem and equations for sheets in a reductive complex Lie algebra
George Lusztig	Conjugacy classes in a reductive group

W. Monty McGovern	Upper semicontinuity of KLV polynomials for certain blocks of Harish-Chandra modules
Diana Shelstad	Transfer results for real groups
Wilfried Schmid	On the $\mathfrak{n}$-cohomology of the limits of the discrete series
Wolfgang Soergel	Graded versions of categories of representations and true motives
Peter Trapa	Relationships between unitary representations of real and p-adic groups
Akshay Venkatesh	Analytic number theory and harmonic analysis on semisimple Lie groups
Jean-Loup Waldspurger	Stabilization of the twisted trace formula: the local theorem
Nolan Wallach	Gleason's theorem and unentangled orthonormal bases
Geordie Williamson	Global and local Hodge theory of Soergel bimodules

Conference Participants, MIT, May 2014.
Photo reprinted courtesy of Dennis Porche.

Ph.D. students of David A. Vogan, Jr.

Luis Casian, 1983	A Global Jacquet Functor for Harish-Chandra Modules
Joseph Johnson, 1983	Lie Algebra Cohomology and Representation Theory
Susana Salamanca Riba, 1986	On Unitary Representations with Regular Infinitesimal Character
Jesper Bang-Jensen, 1987	The Multiplicities of Certain K-Types in Irreducible Spherical Representations of Semisimple Lie Groups
W. Montgomery McGovern, 1987	Primitive Ideals and Nilpotent Orbits in Complex Semisimple Lie Algebras
James Schwartz, 1987	The Determination of the Admissible Nilpotent Orbits in Real Classical Groups
Hisayosi Matumoto, 1988	Whittaker Vectors and Whittaker Functions for Real Semisimple Lie Groups
Jing-Song Huang, 1989	The Unitary Dual of the Universal Covering Group of $GL(n, R)$
Iwan Pranata, 1989	Structures of Dixmier Algebras
Eugenio Garnica-Vigil, 1992	On the Classification of Tempered Representations for a Group in the Harish-Chandra Class
William Graham, 1992	Regular Functions on the Universal Cover of the Principal Nilpotent Orbit
Kian Boon Tay, 1994	Nilpotent Orbits and Multiplicity-Free Representations
Diko Mihov, 1996	Quantization of Nilpotent Coadjoint Orbits
Hongyu He, 1998	Howe's Rank and Dual Pair Correspondence in Semistable Range
Monica Nevins, 1998	Admissible Nilpotent Coadjoint Orbits of p-adic Reductive Lie Groups

continued on page xviii

David A. Vogan, Jr., with some of his students and mathematical descendants, at MIT, May 2014.
First row (left to right) : Monica Nevins, Jing-Song Huang, Susana Salamanca, David A. Vogan, Jr., Peter Trapa
Second row (left to right) : Chunying Fang, Pramod Achar, Laura Rider, Myron Minn-Thu-Aye, Amber Russell, Benjamin Harris
Back row (left to right) : Kei Yuen Chan, William Graham, Thomas Pietraho, Haian He, Monty McGovern,
Kehei Yahiro, Jacob Matherne, Takuma Hayashi, Jun Yu,
Photo reprinted courtesy of Dennis Porche.

Ph.D. students of David A. Vogan, Jr.
(continued)

Peter Trapa, 1998	Unitary Representations of $\mathrm{U}(p, q)$
Adam Lucas, 1999	Small Unitary Representations of the Double Cover of $\mathrm{SL}(m)$
Wentang Kuo, 2000	Principal Nilpotent Orbits and Reducible Principal Series
Dana Pascovici, 2000	Regular Functions on Principal Nilpotent Orbits and R-Groups; Representation Theory of Real Lie Groups
Pramod Achar, 2001	Equivariant Coherent Sheaves on the Nilpotent Cone for Complex Reductive Lie Groups
Thomas Pietraho, 2001	Orbital Varieties and Unipotent Representations of Classical Semisimple Lie Groups
Alessandra Pantano, 2004	Weyl Group Representations and Signatures of Intertwining Operators
Wai Ling Yee, 2004	On the Signature of the Shapovalov Form
Christopher Malon, 2005	The p-adic Local Langlands Conjecture
Chuying Fang, 2007	Ad-nilpotent ideals of complex and real reductive groups
Jerin Gu, 2008	Single-petaled K-types and Weyl group representations for classical groups
Markéta Havlíčková, 2008	Boundaries of K-types in discrete series
Benjamin Harris, 2011	Fourier Transforms of Nilpotent Orbits, Limit Formulas for Reductive Lie Groups, and Wave Front Cycles of Tempered Representations
Peter Speh, 2012	A Classification of Real and Complex Nilpotent Orbits of Reductive Groups in Terms of Complex Even Nilpotent Orbits
Eric Marberg, 2013	Coxeter Systems, Multiplicity Free Representations, and Twisted Kazhdan–Lusztig Theory

First row (left to right): Allison Corman-Vogan, Lois Corman, Aparna Kumar
Back row (left to right): David Vogan, Jonathan Vogan
On the occasion of Jonathan and Aparna's engagement, 2009.

Four generations: David A. Vogan, Sr. (left), Jonathan Vogan, newborn Theodore Kumar Vogan and David A. Vogan, Jr., 2014.

The Mathematical Work of David A. Vogan, Jr.

William M. McGovern and Peter E. Trapa

To David, with gratitude and respect

Abstract Over four decades David Vogan's groundbreaking work in representation theory has changed the face of the subject. We give a brief summary here.

Key words: unitary representations, semisimple Lie groups

MSC (2010): 22E46

It is difficult to give a complete overview in a few short pages of the impact of the work of David Vogan, but it is easy to identify the starting point: his 1976 MIT Ph.D. thesis [V76], completed at the age of 21 under the direction of Bertram Kostant, was a striking advance in the subject. It paved the way for an algebraic classification of irreducible (not necessarily unitary) representations for a reductive Lie group G at a time when the existing approaches to such classification problems (in the work of Harish-Chandra, Langlands, Knapp–Zuckerman, and others) were heavily analytic. David's classification (published as an announcement in [V77] and partially in [V79d]) was later streamlined and extended with Zuckerman using Zuckerman's new technology of cohomological induction, which complemented the Lie algebra cohomology techniques developed in David's thesis. A full exposition, including an influential list of problems, appeared in [V81a], completed in 1980.

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Harish-Chandra developed the theory of non-unitary representations in large part to study unitary representations, and indeed unitary representation theory emerged as the central thread in virtually all of David's work. Along the way, David naturally proved many results about non-unitary representations, but rarely without consideration of their relevance for unitarity. This is beautifully laid out in [V83a]; in [V87], an exposition of David's Weyl Lectures at the Institute in 1986; and in [V87], the notes from David's 1986 Plenary ICM address.

As he was completing his algebraic classification, David tackled a description of irreducible characters, which he solved in [V83] and [LV83] with Lusztig before the completed manuscript [V81a] went to press. David's approach to computing irreducible characters involved expressing them as explicit linear combinations of characters of certain standard parabolically induced representations. Since the latter, in principle, can be computed from Harish-Chandra's results on the discrete series and the tractable effect of parabolic induction on characters, this gives the irreducible characters. The problem is therefore parallel to expressing an irreducible highest weight module as a virtual sum of Verma modules, the subject of the Kazhdan–Lusztig conjectures, and so the irreducible character problem for Harish-Chandra modules became informally known as the "Kazhdan–Lusztig algorithm for real groups". David constructed this algorithm, modulo a judicious technical conjecture about the semisimplicity of certain modules arising from wall-crossing translation functors, in [V79c].

A substantial part of [V79a, V79c] (and its exposition in [V81a]) relied on understanding how irreducible characters behave under coherent continuation. This was also the starting point for the earlier work with Speh [SV80] which addressed fundamental problems of the reducibility of standard modules, clearly of importance to both the classification and irreducible character problem (and to unitarity questions). The intricate arguments in [SV80] were simplified in [V79a] which further made a key connection with dimensions of certain Ext groups in the category of Harish-Chandra modules. This latter connection became an extremely powerful tool in the computation of irreducible characters, culminating in the Kazhdan–Lusztig algorithm for real groups in [V79c]. (When applied to the setting of highest weight modules, the algorithm of [V79c] in fact reduced to the original algorithm of Kazhdan–Lusztig.) The technical conjecture mentioned above, though entirely algebraic in its formulation, was ultimately only surmounted by geometric methods in positive characteristic [LV83] and partly based on the new localization theory of Beilinson–Bernstein and its connection to the cohomological methods in David's classification [V83]. (Particular cases of the Lusztig–Vogan geometry and related settings are considered in the contributions of Graham–Li and McGovern in this volume.) David gave a very accessible "roadmap" to the Kazhdan–Lusztig conjectures for real groups in [V83b], including complete recursion formulas for the polynomials of [LV83] which came to be known as Kazhdan–Lusztig–Vogan, or KLV, polynomials.

Thus, in the span of just of a few spectacularly productive years, the irreducible character problem was solved by a vast array of new techniques. There was much more to be done with these powerful new ideas.

Because one is often interested in extracting less complete (but more accessible) information about irreducible representations than their characters, David was led to consider various invariants of Harish-Chandra modules. His influential paper [V78a] (see also [V80b]) lays the foundation of Gelfand–Kirillov dimension for Harish-Chandra modules and classifies those representations which are generic in the sense that they admit Whittaker models. The contribution of Wallach to this volume discusses GK dimension for smaller discrete series.

Connections to Kazhdan–Lusztig theory and Joseph's theory of primitive ideal cells, as well as weaker formulations of quantization, naturally led to fundamental questions in the theory of primitive ideals in enveloping algebras of complex semisimple Lie algebras initiated by Dixmier, Duflo, Joseph, and others. In [V79b], David studied primitive ideals directly by understanding their behavior under coherent continuation restricted to rank-two root subsystems, generalizing the Borho–Jantzen τ-invariant which considered rank-one subsystems. (The paper of Bonnafé and Geck in this volume takes up many of these ideas.) Later, working with Garfinkle, he proved analogous results for restricting to the root system of type D_4 [GaV92], necessary for any systematic analysis of branched Dynkin diagrams. These turned out to be very powerful computational tools which were exploited to great effect in the work of Garfinkle and others. In [V80a], David related the ordering of primitive ideals to a preorder arising in the original paper of Kazhdan–Lusztig. He and Barbasch carried out the classification of primitive ideals in complex semisimple Lie algebras [BV82, BV83a]. Along the way, they showed that representations of the Weyl group that arise in Joseph's Goldie rank construction are exactly the special ones in the sense of Lusztig, and related them to Fourier inversion of certain unipotent orbital integrals.

As he was developing his algebraic theory, David was naturally led to understand its relation with Langlands' original classification and the larger context of the Local Langlands Conjectures. The dual group makes a fundamental appearance in the technical tour de force [V82] where David uncovered an intricate symmetry in his Kazhdan–Lusztig algorithm for real groups: he proved that computing irreducible characters of real forms G of a complex connected reductive group $G_{\mathbb{C}}$ is dual, in a precise sense, to computing irreducible characters of real forms of the dual group $G_{\mathbb{C}}^{\vee}$. (When applied to the case of complex groups, it can be interpreted as the equality (up to sign) of the Kazhdan–Lusztig polynomials $P_{x,y}$ and $P_{w_0 y, w_0 x}$ proved by Kazhdan–Lusztig in their original paper; here w_0 is the long element of the Weyl group.) The full significance of this deep and beautiful symmetry, now known as Vogan duality, was only fully realized later in [ABV92]. In order to get a perfectly symmetric statement, one must consider multiple real forms at the same time. This immediately leads to the question of when two collections of representations of multiple real forms should be considered equivalent, and eventually to the definition of strong real forms [ABV92], differing in subtle and interesting ways from the classical notion of real forms. On the dual side, it led Adams, Barbasch, and Vogan (building on earlier ideas of Adams and Vogan [AV92a, AV92b]) to reformulate the space of Langlands parameters.

Like the space of classical Langlands parameters, the reformulated space of ABV parameters is a complex algebraic variety on which the complex dual groups acts. Unlike the space of Langlands parameters in the real case, the orbits of the dual group on the ABV space have closures with nontrivial singularities. The main result of [ABV92] is a refinement of the representation-theoretic part of the Local Langlands Conjecture for real groups where K-groups of representations of strong real forms are dual to K-groups of appropriate categories of equivariant perverse sheaves on the space of ABV parameters. This incredibly intricate correspondence is ultimately deduced from [V82]. Using it, [ABV92] makes precise and establishes a series of conjectures of Arthur (for real groups), at the same time providing a different perspective on the Langlands–Shelstad theory of endoscopy. In the ABV theory, Arthur packets of representations are defined in terms of characteristic cycles of perverse sheaves on the space of ABV parameters. Such cycles are still mysterious and poorly understood. (Some related complications are on display in the new examples of Williamson in this volume.) For classical groups, Arthur has recently defined Arthur packets and established his conjectures from a different point of view; comparing his results in the real case to those of [ABV92] is still to be done. Meanwhile, Soergel (generalizing Beilinson–Ginzburg–Soergel) formulated a still-open conjecture extending the main result of [ABV92] (established on the level of K-groups) to a categorical statement. Using the real case as a model, David proposed refinements of the Local Langlands Conjectures in the much more difficult p-adic case as well ([V93a]). This has played an increasingly important role in recent years.

The paper [V84] is most often remembered for its long sought-after proof that cohomological induction preserves unitarity under fairly general hypotheses. As a consequence, certain representations (the so-called $A_\mathfrak{q}(\lambda)$ modules) constructed by Zuckerman from unitary characters are indeed unitary. Earlier, Vogan and Zuckerman [VZ84] had classified all unitary representations with nonzero relative Lie algebra cohomology as $A_\mathfrak{q}$ modules, modulo the conjecture that the modules they had classified were indeed unitary. For the applications to the cohomology of locally symmetric spaces discussed in [VZ84] (and [V97b]), this was not important, but for unitary representation theory it was a central question at the time.

In later work, David clarified the role of the $A_\mathfrak{q}(\lambda)$ in the discrete spectrum of symmetric spaces [V88b], as well as how they appear as isolated representations [V07a]. Basic questions about explicitly constructing the unitary inner product geometrically on minimal globalizations are still open (as explained in [V08]). He returned to the Dirac operator methods of [VZ84] in an influential series of lectures [V97] that spawned an entire new area of investigation, still developing today (for example in the contribution of Huang to this volume).

Striking examples of complete classifications of unitary representations include David's description of the unitary dual of $\mathrm{GL}(n,\mathbb{R})$ (obtained in 1984 and appearing in [V86a]), and later the unitary dual of G_2 [V94] (dedicated to Borel). The description of the unitary duals given in these cases was organized in terms of systematic procedures (like cohomological induction and construction of complementary series) applied to certain building blocks. The systematic role of cohomological induction was clarified later in his deep work with Salamanca-Riba [SaV98], [SaV01]; see also the exposition [V00b]. The groundwork for systematizing the construction of complementary series is [BV84]. But abstracting a general definition of the mysterious building blocks, which came to be known (somewhat imprecisely) as unipotent representations, proved to be more difficult and is a major theme in David's work. The overviews [V93b], [V97a], [V00c] contain many ideas. At the heart of the notion of unipotent is the connection to nilpotent coadjoint orbits in semisimple Lie algebras and modern approaches to geometric quantization pioneered by Kirillov and Kostant. The paper with Graham [GrV98] makes significant progress in the setting of complex groups.

Early on, David proved that the annihilator of an irreducible unitary Harish-Chandra module for a complex group was completely prime. He then formulated a conjectural kind of Nullstellensatz for such ideals in [V86b] in which finite algebra extensions of primitive quotients of the enveloping algebra play a crucial role. Such algebra extensions, sometimes called Dixmier algebras, were further studied in [V90] and the notion of induced ideal was extended to them. Many beautiful facets of David's conception of the orbit method are explained in [V88a].

One of the fundamental ways in which nilpotent orbits appear in the representation theory of real groups is through the asymptotics of the character expansion discovered in [BV80]. In this construction, each Harish-Chandra module gives rise to a real linear combination of real nilpotent coadjoint orbits. Nilpotent orbits also arise through David's construction of the associated cycle of a Harish-Chandra module [V91], a positive integral combination of complex nilpotent coadjoint orbits for symmetric pairs (the setting originally investigated by Kostant–Rallis). The Barbasch–Vogan conjecture, proved by Schmid and Vilonen, asserted that the two kinds of linear combinations coincide perfectly under the Kostant–Sekiguchi bijection. David used these invariants to define conditions on a class of unipotent representation in [V91]. A beautiful example of the explicit desiderata in a case of great interest is given in [AHV98].

A weaker version of the quantization of nilpotent orbits instead focuses on constructing Harish-Chandra modules with prescribed annihilator. Barbasch and Vogan long ago identified a set of interesting infinitesimal characters and sought to understand Harish-Chandra modules annihilated by maximal primitive ideals with those infinitesimal characters, conjecturing that such Harish-Chandra modules were unitary. (This generalizes the study of minimal representations [V81b], where the maximal primitive ideals are the Joseph ideals.) In [BV85], Barbasch and Vogan discovered that many of their interesting infinitesimal characters — conjecturally those arising as the annihilators of unitary representations with automorphic applications — fit perfectly into the framework of the ideas proposed by Arthur. The paper [BV85] gives strikingly simple character formulas for these so-called special unipotent representations in the setting of complex groups (and the ideas make sense for real groups too). An aspect of the proof of the character formulas relied on ways to count special unipotent representations using the decomposition of the coherent continuation representation into cells. The theory of Kazhdan–Lusztig cells for complex groups was extended to real groups with Barbasch in [BV83b].

The theory of [BV85] was generalized in the important closing chapter of [ABV92], where it can be understood entirely in terms of the principle of Langlands functoriality. A central theme in David's work is the extent to which functoriality extends to organize all unitary representations, not just automorphic ones. (A very accessible introduction to this set of ideas is contained in [V01]). The Shimura-type lifting in [ABPTV97] can also be understood in these terms as part of an aim to extend functoriality to certain nonalgebraic groups.

The interaction between the philosophy of the orbit method and the construction of associated varieties was developed further in [V00a], culminating in a general (still unrealized) approach toward proving the unitarity of the special unipotent representations defined in [BV85] and [ABV92]. One facet of this involved relating the $K_{\mathbb{C}}$ equivariant K-theory of the nilpotent cone in the symmetric space setting to the tempered dual of G. David conjectured a precise relationship (independently and earlier conjectured by Lusztig for complex groups) that was later proved in special cases by Achar and by Bezrukavnikov. The article of Achar in this volume provides an up-to-date look at work in this direction.

The unitarity of $A_{\mathfrak{q}}(\lambda)$ in [V84] is a deep and important result, but the theory of signature characters of Harish-Chandra modules that David developed to obtain it has proved to be even more influential. It immediately led Wallach to a shorter proof of the unitarity of the $A_{\mathfrak{q}}(\lambda)$ modules, for example, and was adapted to unramified representations of split p-adic groups by Barbasch and Moy. But for David it was part of an approach to determining the entire unitary dual of a reductive group. The paper [V84] proposes an algorithm (heavily rooted in his Kazhdan–Lusztig theory for real groups and the theory of the Jantzen conjecture) to determine if an irreducible representation specified in the Langlands classification is in fact unitary. The algorithm was predicated on determining certain signs that, at the time, were inaccessible. Determining the signs in the algorithm of [V84] was finally surmounted in [ALTV12], giving a finite effective algorithm to locate the unitary dual of a reductive Lie group in the Langlands classification.

The paper [ALTV12] relies on relating classical invariant Hermitian forms on irreducible Harish-Chandra modules (the object of study in unitary representation theory) to forms with a different, more canonical invariance property. Once this latter invariance property was uncovered, its importance was immediately recognized in other settings (for example in the geometric setting of Schmid and Vilonen explained in their contribution to this volume and the analogous p-adic setting in the contribution of Barbasch and Ciubotaru). Translating between the two kinds of forms in [ALTV12] immediately leads one to certain extended groups which are not the real points of a connected reductive algebraic group, and which are outside the class of groups for which [V83] established a Kazhdan–Lusztig algorithm. In recent work, Lusztig and Vogan [LV14] (generalizing their earlier work

[LV83, LV12]) provide the geometric foundations of Kazhdan–Lusztig theory for such extended groups. In particular, they define a Hecke algebra action on an appropriate Grothendieck group. This action characterizes the "twisted" Kazhdan–Lusztig polynomials in this setting. [LV14] gives explicit formulas for individual Hecke operators, but they depended on certain choices. The effect of these choices is completely understood in the paper by Adams and Vogan in this volume. Meanwhile, Lusztig and Vogan in this volume provide an extension of the results of [LV12] to the setting of arbitrary Coxeter groups using the new theory of Elias and Williamson.

Over the last fifteen years, David has been deeply involved in the atlas project, the goal of which is to translate much of the mathematics described above into the computer software package `atlas` in the generality of the real points of *any* complex connected reductive algebraic group. This has involved his close collaboration with many people, but especially with Adams, du Cloux, and van Leeuwen. David's Conant Prize winning article [V07b] gives an overview of the first step: the implementation of the computation of irreducible characters and, in particular, the computation of the KLV polynomials for the split real form of E_8. His paper [V07c] is devoted to algorithms at the heart of computing the K-spectrum of any irreducible Harish-Chandra module. At present the software is able to test the unitarity of any irreducible Harish-Chandra module specified in the Langlands classification. The results of [V84] imply that testing a finite number of such representations suffices to determine the entire unitary dual. The implementation of this will almost certainly be complete in the next year or two, a remarkable achievement that no one could have predicted was possible even just a decade ago. In many ways, it is the culmination of David's seminal contributions to unitary representation theory.

The above captures a sliver of the mathematics developed in David's papers. It says little of his influential expositions that have, by now, educated generations. It also says nothing of the immense number of mathematical ideas David gave freely to others, nor of his selfless devotion to the profession of mathematics. But, we hope, it points to the breadth of his influence to date, as well as some of the exciting work left to be done.

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On exotic and perverse-coherent sheaves

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Dedicated to David Vogan on his 60th birthday

Abstract Exotic sheaves are certain complexes of coherent sheaves on the cotangent bundle of the flag variety of a reductive group. They are closely related to perverse-coherent sheaves on the nilpotent cone. This expository article includes the definitions of these two categories, applications, and some structure theory, as well as detailed calculations for SL_2.

Key words: t-structures; coherent sheaves; nilpotent cone; Springer resolution

MSC (2010): Primary 20G05; Secondary 14F05

1 Introduction

Let G be a reductive algebraic group. Let $\mathcal{N}$ be its nilpotent cone, and let $\widetilde{\mathcal{N}}$ be the Springer resolution. This article is concerned with two closely related categories of complexes of coherent sheaves: *exotic sheaves* on $\widetilde{\mathcal{N}}$, and *perverse-coherent sheaves* on $\mathcal{N}$. These categories play key roles in Bezrukavnikov's proof of the Lusztig–Vogan bijection [18] and his study of quantum group cohomology [19]; in the Bezrukavnikov–Mirković proof of Lusztig's conjectures on modular representations of Lie algebras [21]; in the proof of the Mirković–Vilonen conjecture on torsion on the affine Grassmannian [7]; and in various derived equivalences related to local geometric Langlands duality, both in characteristic zero [11, 20] and, more recently, in positive characteristic [8, 36].

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In this expository article, we will introduce these categories via a relatively elementary approach. But we will also see that they each admit numerous alternative descriptions, some of which play crucial roles in various applications. The paper also touches on selected aspects of their structure theory, and it concludes with detailed calculations for $G = \mathrm{SL}_2$. Some proofs are sketched, but many are omitted. There are no new results in this paper.

My interest in these notions began when David Vogan pointed out to me, in 2001, the connection between the papers [17, 18] and my own Ph.D. thesis [1]. It is a pleasure to thank David for suggesting a topic that has kept me interested and busy for the past decade and a half!

2 Definitions and preliminaries

2.1 General notation and conventions

Let $\Bbbk$ be an algebraically closed field, and let G be a connected reductive group over $\Bbbk$. We assume throughout that the following conditions hold:

- The characteristic of $\Bbbk$ is zero or a JMW prime for G.
- The derived subgroup of G is simply connected.
- The Lie algebra $\mathfrak{g}$ of G admits a nondegenerate G-invariant bilinear form.

Here, a *JMW prime* is a prime number that is good for G and such that the main result of [31] holds for G. That result, which is concerned with tilting G-modules under the geometric Satake equivalence, holds for quasisimple G at least when p satisfies the following bounds[1]:

$A_n, B_n, D_n, E_6, F_4, G_2$	C_n	E_7	E_8
p good for G	$p > n$	$p > 19$	$p > 31$

The condition that $\mathfrak{g}$ admit a nondegenerate G-invariant bilinear form is satisfied for $\mathrm{GL}(n)$ in all characteristics, and for quasisimple, simply connected groups not of type A in all good characteristics.

Fix a Borel subgroup $B \subset G$ and a maximal torus $T \subset B$. Let $\mathbf{X}$ be the character lattice of T, and let $\mathbf{X}^+ \subset \mathbf{X}$ be the set of dominant weights corresponding to the positive system *opposite* to the roots of B. (In other words, we think of B as a "negative" Borel subgroup.) Let $\mathfrak{u} \subset \mathfrak{g}$ be the Lie algebra of the unipotent radical of B. Let W be the Weyl group, and let $w_0 \in W$ be the longest element. For $\lambda \in \mathbf{X}$, let

$$\delta_\lambda = \min\{\ell(w) \mid w\lambda \in \mathbf{X}^+\} \qquad \text{and} \qquad \delta^*_\lambda = \delta_{w_0\lambda}.$$

[1] These bounds have recently been improved by Mautner–Riche; see Remark 2.1.

For any weight $\lambda \in \mathbf{X}$, let $\mathsf{dom}(\lambda)$ be the unique dominant weight in its Weyl group orbit. Two partial orders on $\mathbf{X}$ will appear in this paper. For $\lambda, \mu \in \mathbf{X}$, we write

$\lambda \preceq \mu$ if $\mu - \lambda$ is a sum of positive roots;
$\lambda \leq \mu$ if $\mathsf{dom}(\lambda) \prec \mathsf{dom}(\mu)$, or else if $\mathsf{dom}(\lambda) = \mathsf{dom}(\mu)$ and $\lambda \preceq \mu$.

Let $\mathbf{X}_{\mathrm{min}} \subset \mathbf{X}^+$ be the set of minuscule weights. Elements of $-\mathbf{X}_{\mathrm{min}}$ are called *antiminuscule* weights. For any $\lambda \in \mathbf{X}$, there is a unique minuscule, resp. antiminuscule, weight that differs from λ by an element of the root lattice, denoted

$$\overset{+}{\mathsf{m}}(\lambda), \qquad \text{resp.} \qquad \bar{\mathsf{m}}(\lambda).$$

Note that $\bar{\mathsf{m}}(\lambda) = w_0\overset{+}{\mathsf{m}}(\lambda)$.

Let $\mathrm{Rep}(G)$ and $\mathrm{Rep}(B)$ denote the categories of finite-dimensional algebraic G- and B-representations, respectively. Let $\mathrm{ind}_B^G : \mathrm{Rep}(B) \to \mathrm{Rep}(G)$ and $\mathrm{res}_B^G : \mathrm{Rep}(G) \to \mathrm{Rep}(B)$ denote the induction and restriction functors. For any $\lambda \in \mathbf{X}$, let $\Bbbk_\lambda \in \mathrm{Rep}(B)$ be the 1-dimensional B-module of weight λ, and let

$$H^i(\lambda) = R^i\,\mathrm{ind}_B^G\,\Bbbk_\lambda.$$

If $\lambda \in \mathbf{X}^+$, then $H^0(\lambda)$ is the dual Weyl module of highest weight λ, and the other $H^i(\lambda)$ vanish. On the other hand, we put

$$V(\lambda) = H^{\dim G/B}(w_0\lambda - 2\rho) \cong H^0(-w_0\lambda)^*.$$

This is the Weyl module of highest weight λ.

Let $\mathcal{N} \subset \mathfrak{g}$ be the nilpotent cone of G, and let $\widetilde{\mathcal{N}} = G \times^B \mathfrak{u}$. Any weight $\lambda \in \mathbf{X}$ gives rise to a line bundle $\mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda)$ on $\widetilde{\mathcal{N}}$. The Springer resolution is denoted by

$$\pi : \widetilde{\mathcal{N}} \to \mathcal{N}.$$

Let the multiplicative group $\mathbb{G}_{\mathrm{m}}$ act on $\mathfrak{g}$ by $z \cdot x = z^{-2}x$, where $z \in \mathbb{G}_{\mathrm{m}}$ and $x \in \mathfrak{g}$. This action commutes with the adjoint action of G. It restricts to an action on $\mathcal{N}$ or on $\mathfrak{u}$; the latter gives rise to an induced action on $\widetilde{\mathcal{N}}$. We write $\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$ for the category of $(G \times \mathbb{G}_{\mathrm{m}})$-equivariant coherent sheaves on $\mathcal{N}$, and likewise for the other varieties. Recall that there is an "induction equivalence"

$$\mathrm{Coh}^{B\times\mathbb{G}_{\mathrm{m}}}(\mathfrak{u}) \cong \mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}}). \tag{1}$$

The $\mathbb{G}_{\mathrm{m}}$-action on $\mathfrak{g}$ corresponds to equipping the coordinate ring $\Bbbk[\mathfrak{g}]$ with a grading in which the space of linear functions $\mathfrak{g}^* \subset \Bbbk[\mathfrak{g}]$ is precisely the space of homogeneous elements of degree 2. Thus, a $(G \times \mathbb{G}_{\mathrm{m}})$-equivariant coherent sheaf on $\mathfrak{g}$ is the same as a finitely-generated graded $\Bbbk[\mathfrak{g}]$-module equipped with a compatible G-action. Similar remarks apply to $\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$ and $\mathrm{Coh}^{B\times\mathbb{G}_{\mathrm{m}}}(\mathfrak{u})$. If $V = \bigoplus_{n\in\mathbb{Z}} V_n$ is a such a graded module (or more generally, a graded vector space), we define $V\langle n\rangle$ to be the graded module given by $(V\langle m\rangle)_n = V_{m+n}$. Given two graded vector modules V, W, we define a new graded vector space

$$\underline{\mathrm{Hom}}(V,W) = \bigoplus_{n\in\mathbb{Z}} \underline{\mathrm{Hom}}(V,W)_n \quad \text{where} \quad \underline{\mathrm{Hom}}(V,W)_n = \mathrm{Hom}(V, W\langle n\rangle).$$

We extend the notation $\mathcal{F} \mapsto \mathcal{F}\langle n\rangle$ to $\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})$ via the equivalence (1).

Remark 2.1. In [35, 36], Mautner and Riche prove that all good primes are JMW primes. Those papers also give new proofs of the foundational results in Sections 2.2 and 2.4 below, placing them in the context of the affine braid group action on $D^\mathrm{b}\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})$ introduced in [24]. The main result of [36] appears in this paper as Theorem 4.3.

2.2 Exotic sheaves

This subsection and the following one introduce the two t-structures we will study. On a first reading, the descriptions given here are, unfortunately, rather opaque. We will see some other approaches in Section 4; for explicit examples, see Section 6.

For $\lambda \in \mathbf{X}$, let $D^\mathrm{b}\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})_{\le\lambda}$ be the full triangulated subcategory of $D^\mathrm{b}\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})$ generated by line bundles $\mathcal{O}_{\widetilde{\mathcal{N}}}(\mu)\langle m\rangle$ with $\mu \le \lambda$ and $m \in \mathbb{Z}$. The category $D^\mathrm{b}\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})_{<\lambda}$ is defined similarly.

Proposition 2.2. *For any $\lambda \in \mathbf{X}$, there are objects $\widehat{\Delta}_\lambda, \widehat{\nabla}_\lambda \in D^\mathrm{b}\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})$ that are uniquely determined by the following properties: there are canonical distinguished triangles*

$$\widehat{\Delta}_\lambda \to \mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda)\langle\delta_\lambda\rangle \to \mathcal{K}_\lambda \to \qquad \textit{and} \qquad \mathcal{K}'_\lambda \to \mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda)\langle\delta_\lambda\rangle \to \widehat{\nabla}_\lambda \to$$

such that $\mathcal{K}_\lambda \in D^\mathrm{b}\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})_{<\lambda}$, $\mathcal{K}'_\lambda \in D^\mathrm{b}\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})_{<w_0\mathsf{dom}(\lambda)}$, and

$$\underline{\mathrm{Hom}}^\bullet(\widehat{\Delta}_\lambda, \mathcal{F}) = \underline{\mathrm{Hom}}^\bullet(\mathcal{F}, \widehat{\nabla}_\lambda) = 0 \qquad \textit{for all } \mathcal{F} \in D^\mathrm{b}\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})_{<\lambda}.$$

Proof (sketch). This is mostly a consequence of the machinery of "mutation of exceptional sets," discussed in [19, §2.1 and §2.3]. The general results of [19, §2.1.4] might at first suggest that $\mathcal{K}_\lambda$ and $\mathcal{K}'_\lambda$ both lie just in $D^\mathrm{b}\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})_{<\lambda}$. To obtain the stronger constraint on $\mathcal{K}'_\lambda$, one first shows that line bundles already have a strong Hom-vanishing property with respect to $\preceq$:

$$\underline{\mathrm{Hom}}^\bullet(\mathcal{O}_{\widetilde{\mathcal{N}}}(\mu), \mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda)) = 0 \qquad \text{if } \mu \not\preceq \lambda. \tag{2}$$

The result is obtained by combining this with a study of the $\widehat{\nabla}_\lambda$ based on Proposition 2.5 below. □

The preceding proposition says, in part, that the $\widehat{\nabla}_\lambda$ constitute a "graded exceptional set" in the sense of [19, §2.3]. In the extreme cases of dominant or antidominant weights, equation (2) actually implies that

$$\widehat{\nabla}_\lambda \cong \mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda) \qquad \text{and} \qquad \widehat{\Delta}_{w_0\lambda} \cong \mathcal{O}_{\widetilde{\mathcal{N}}}(w_0\lambda)\langle\delta_{w_0\lambda}\rangle \qquad \text{for } \lambda \in \mathbf{X}^+. \tag{3}$$

The proposition also implies that the composition $\widehat{\Delta}_\lambda \to \mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda)\langle\delta_\lambda\rangle \to \widehat{\nabla}_\lambda$ is nonzero for all $\lambda \in \mathbf{X}$. This is the morphism $\widehat{\Delta}_\lambda \to \widehat{\nabla}_\lambda$ appearing in the following statement.

Theorem 2.3. *There is a unique t-structure on* $D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})$ *whose heart*

$$\mathrm{ExCoh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}}) \subset D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})$$

is stable under $\langle 1\rangle$ *and contains* $\widehat{\Delta}_\lambda$ *and* $\widehat{\nabla}_\lambda$ *for all* $\lambda \in \mathbf{X}$. *In this category, every object has finite length. The objects*

$$\mathfrak{E}_\lambda\langle n\rangle := \mathrm{im}(\widehat{\Delta}_\lambda\langle n\rangle \to \widehat{\nabla}_\lambda\langle n\rangle)$$

are simple and pairwise nonisomorphic, and every simple object is isomorphic to one of these.

This t-structure is called the *exotic t-structure* on $D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})$. The $\widehat{\Delta}_\lambda$ and $\widehat{\nabla}_\lambda$ are called *standard* and *costandard* objects, respectively. Further properties of these objects will be discussed in Section 3.1.

Proof (sketch). The existence of the t-structure follows from [19, Proposition 4], which describes a general mechanism for constructing t-structures from exceptional sets. However, that general mechanism does not guarantee that the $\widehat{\Delta}_\lambda$ and $\widehat{\nabla}_\lambda$ lie in the heart. One approach to showing that they do lie in the heart (see [8]) is to deduce it from the derived equivalences that we will see in Sections 4.1 (for $\Bbbk = \mathbb{C}$) or 4.2 (in general). □

2.3 Perverse-coherent sheaves

For any $\lambda \in \mathbf{X}$, let

$$A_\lambda = \pi_*\mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda).$$

For $\lambda \in \mathbf{X}^+$, we define $D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\mathcal{N})_{\le\lambda}$ to be the full triangulated subcategory of $D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\mathcal{N})$ generated by the objects $A_\mu\langle m\rangle$ with $\mu \in \mathbf{X}^+$, $\mu \le \lambda$, and $m \in \mathbb{Z}$. The category $D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\mathcal{N})_{<\lambda}$ is defined similarly. Note that these categories are defined only for dominant weights, unlike in the exotic case.

One basic property of the A_λ, often called *Andersen–Jantzen sheaves*, is that if $s_\alpha \in W$ is a simple reflection such that $s_\alpha\lambda \prec \lambda$, then there is a natural map

$$A_{s_\alpha\lambda} \to A_\lambda\langle -2\rangle \tag{4}$$

whose cone lies in $D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\mathcal{N})_{<\mathsf{dom}(\lambda)}$ [3, Lemma 5.3].

For a dominant weight $\lambda \in \mathbf{X}^+$, we introduce the following additional notation:

$$\overline{\Delta}_\lambda = A_{w_0\lambda}\langle\delta^*_\lambda\rangle, \qquad \overline{\nabla}_\lambda = A_\lambda\langle-\delta^*_\lambda\rangle. \tag{5}$$

It follows from (4) that there is a natural nonzero map

$$\overline{\Delta}_\lambda \to \overline{\nabla}_\lambda$$

whose cone lies in $D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})_{<\lambda}$. This morphism appears in the following statement.

Theorem 2.4. *There is a unique t-structure on $D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$ whose heart*

$$\mathrm{PCoh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N}) \subset D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$$

is stable under $\langle 1\rangle$ and contains $\overline{\Delta}_\lambda$ and $\overline{\nabla}_\lambda$ for all $\lambda \in \mathbf{X}^+$. In this category, every object has finite length. The objects

$$\mathfrak{IC}_\lambda\langle n\rangle := \mathrm{im}(\overline{\Delta}_\lambda\langle n\rangle \to \overline{\nabla}_\lambda\langle n\rangle)$$

are simple and pairwise nonisomorphic, and every simple object is isomorphic to one of these.

This t-structure is called the *perverse-coherent t-structure* on $D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$. The justification for this terminology will be discussed in Section 4.3.

Proof (sketch). The first step is to show that the $\overline{\Delta}_\lambda$ and $\overline{\nabla}_\lambda$ satisfy Ext-vanishing properties similar to (but somewhat weaker than) those in Proposition 2.2. (To be precise, the $\overline{\nabla}_\lambda$ constitute a "graded quasiexceptional set," but not an exceptional set. See [18, §2.2 and Remark 8].) The existence of the t-structure follows from a general mechanism, as in Theorem 2.3. In this case, the fact that the $\overline{\Delta}_\lambda$ and the $\overline{\nabla}_\lambda$ lie in the heart can be checked by direct computation; see [18, Lemma 9] or [3, Lemma 5.2]. □

2.4 The relationship between exotic and perverse-coherent sheaves

In this subsection, we briefly outline a proof of the t-exactness of π_*, following [19, §§2.3–2.4]. Let α be a simple root, and let $P_\alpha \supset B$ be the corresponding parabolic subgroup. Let $\mathfrak{u}_\alpha \subset \mathfrak{u}$ be the Lie algebra of the unipotent radical of P_α. The cotangent bundle $T^*(G/P_\alpha)$ can be identified with $G \times^{P_\alpha} \mathfrak{u}_\alpha$. On the other hand, let $\widetilde{\mathcal{N}}_\alpha = G \times^B \mathfrak{u}_\alpha$. There are natural maps

$$T^*(G/P_\alpha) \xleftarrow{\pi_\alpha} \widetilde{\mathcal{N}}_\alpha \xrightarrow{i_\alpha} \widetilde{\mathcal{N}},$$

where i_α is an inclusion of a smooth subvariety of codimension 1, and π_α is a smooth, proper map whose fibers are isomorphic to $P_\alpha/B \cong \mathbb{P}^1$.

Let $\rho = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$, where Φ^+ is the set of positive roots. In general, ρ need not lie in $\mathbf{X}$, but because G has a simply-connected derived group, there exists a weight $\widehat{\rho} \in \mathbf{X}$ such that $(\beta^\vee, \widehat{\rho}) = 1$ for all simple coroots $\beta^\vee$.

Let $\Psi_\alpha : D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}}) \to D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$ be the functor

$$\Psi_\alpha(\mathcal{F}) = i_{\alpha*}\pi_\alpha^*\pi_{\alpha*}i_\alpha^*(\mathcal{F}\otimes\mathcal{O}_{\widetilde{\mathcal{N}}}(\widehat{\rho}-\alpha))\otimes\mathcal{O}_{\widetilde{\mathcal{N}}}(-\widehat{\rho})\langle 1\rangle.$$

This coincides with the functor denoted $F_\alpha\langle -1\rangle[1]$ or $F'_\alpha\langle 1\rangle[-1]$ in [19, §2.3]. (Note that the statement of [19, Lemma 6(b)] contains a misprint.)

Proposition 2.5. *1. The functor Ψ_α is self-adjoint.*
2. If $s_\alpha\lambda = \lambda$, then $\Psi_\alpha(\widehat{\Delta}_\lambda) = \Psi_\alpha(\widehat{\nabla}_\lambda) = 0$.
3. If $s_\alpha\lambda \prec \lambda$, then

$$\Psi_\alpha(\widehat{\Delta}_{s_\alpha\lambda}) \cong \Psi_\alpha(\widehat{\Delta}_\lambda)\langle -1\rangle[1] \qquad \textit{and} \qquad \Psi_\alpha(\widehat{\nabla}_{s_\alpha\lambda}) \cong \Psi_\alpha(\widehat{\nabla}_\lambda)\langle 1\rangle[-1].$$

4. If $s_\alpha\lambda \preceq \lambda$, then there are natural distinguished triangles

$$\widehat{\Delta}_{s_\alpha\lambda} \to \widehat{\Delta}_\lambda\langle 1\rangle \to \Psi_\alpha(\widehat{\Delta}_\lambda)[1] \to, \quad \Psi_\alpha(\widehat{\nabla}_\lambda)[-1] \to \widehat{\nabla}_\lambda\langle -1\rangle \to \widehat{\nabla}_{s_\alpha\lambda} \to .$$

Proof (sketch). For costandard objects, parts (1), (3), and (4) appear in [19, Lemma 6 and Proposition 7]. In fact, the proofs of those statements also establish part (2). Similar arguments apply in the case of standard objects. □

It is likely that the distinguished triangles in part (4) above are actually short exact sequences in $\mathrm{ExCoh}^{G\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$; see Section 5.3.

Proposition 2.6. *The functor $\pi_* : D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}}) \to D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$ restricts to an exact functor $\pi_* : \mathrm{ExCoh}^{G\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}}) \to \mathrm{PCoh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$. It satisfies*

$$\pi_*\widehat{\Delta}_\lambda \cong \overline{\Delta}_{\mathsf{dom}(\lambda)}\langle -\delta_\lambda^*\rangle, \qquad \pi_*\widehat{\nabla}_\lambda \cong \overline{\nabla}_{\mathsf{dom}(\lambda)}\langle \delta_\lambda^*\rangle, \qquad \pi_*\mathfrak{E}_\lambda \cong \begin{cases} \mathfrak{IC}_{w_0\lambda} & \textit{if } \lambda \in -\mathbf{X}^+, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof (sketch). One first shows that $\pi_*\circ\Psi_\alpha = 0$ for all simple roots α. The formula for $\pi_*\widehat{\Delta}_\lambda$ (resp. $\pi_*\widehat{\nabla}_\lambda$) is clear when λ is dominant (resp. antidominant), so it follows for general λ from Proposition 2.5. The t-exactness of π_* follows immediately from its behavior on standard and costandard objects.

Since $\mathfrak{E}_\lambda$ is the image of a nonzero map $h : \widehat{\Delta}_\lambda \to \widehat{\nabla}_\lambda$, $\pi_*\mathfrak{E}_\lambda$ is the image of $\pi_*h : \overline{\Delta}_{\mathsf{dom}(\lambda)}\langle -\delta_\lambda^*\rangle \to \overline{\nabla}_{\mathsf{dom}(\lambda)}\langle \delta_\lambda^*\rangle$. But it follows from the definition of a graded quasiexceptional set (see Theorem 2.4) that $\mathrm{Hom}(\overline{\Delta}_{\mathsf{dom}(\lambda)}\langle n\rangle, \overline{\nabla}_{\mathsf{dom}(\lambda)}\langle m\rangle) = 0$ unless $n = m$. Thus, $\pi_*h = 0$ unless $\delta_\lambda^* = 0$. In other words, $\pi_*\mathfrak{E}_\lambda = 0$ if $\lambda \notin -\mathbf{X}^+$.

Assume now that $\lambda \in -\mathbf{X}^+$. To show that $\pi_*\mathfrak{E}_\lambda \cong \mathfrak{IC}_{\mathsf{dom}(\lambda)}$, it suffices to show that π_*h is nonzero. Recall that $\widehat{\Delta}_\lambda \cong \mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda)\langle\delta_\lambda\rangle$. We may assume that h is the map appearing in the second distinguished triangle from Proposition 2.2: $\mathcal{K}'_\lambda \to \widehat{\Delta}_\lambda \xrightarrow{h} \widehat{\nabla}_\lambda \to$. Since $\mathcal{K}'_\lambda$ lies in $D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})_{<\lambda}$, our computation of π_* on standard and costandard objects implies that

$$\pi_* \mathcal{K}'_\lambda \in D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})_{<\mathsf{dom}(\lambda)}. \tag{6}$$

If $\pi_* h = 0$, then $\pi_* \mathcal{K}'_\lambda \cong \overline{\Delta}_{\mathsf{dom}(\lambda)} \oplus \overline{\nabla}_{\mathsf{dom}(\lambda)}[-1]$, contradicting (6). Thus, $\pi_* h \neq 0$, as desired. □

2.5 Remarks on grading choices

Equation (5) involves a choice of normalization in the grading shifts. The choice we have made here (which is consistent with [3, 37, 7]) has the desirable property that it behaves well under Serre–Grothendieck duality (see Section 4.3).

Proposition 2.2 also involves such a choice. Our choice agrees with [8] but differs from [19]. The choice we have made here has two advantages: (i) it is compatible with the choice in (5), in the sense that the formula for $\pi_* \mathfrak{E}_\lambda$ involves no grading shift; and (ii) it behaves well under Verdier duality via the derived equivalence of Section 4.2. The reader should bear this in mind when comparing statements in this paper to [19].

One can also drop the $\mathbb{G}_{\mathrm{m}}$-equivariance entirely and carry out all the constructions in the preceding sections in $D^{\mathrm{b}}\mathrm{Coh}^G(\widetilde{\mathcal{N}})$ or $D^{\mathrm{b}}\mathrm{Coh}^G(\mathcal{N})$. Simple objects in $\mathrm{ExCoh}^G(\widetilde{\mathcal{N}})$ are parametrized by $\mathbf{X}$ rather than by $\mathbf{X} \times \mathbb{Z}$, and likewise for $\mathrm{PCoh}^G(\mathcal{N})$. Almost all results in the paper have analogues in this setting, obtained just by omitting grading shifts $\langle n \rangle$ and by replacing all occurrences of $\underline{\mathrm{Hom}}$ by Hom. In most cases, we will not treat these analogues explicitly.

However, there are a handful of exceptions. The proof of Theorem 3.3 is different in the graded and ungraded cases. Theorem 4.3 does not (yet?) have an ungraded version in positive characteristic. Although Theorems 4.1, 4.7, and 4.10 have graded analogues, the focus in the literature and in applications is on the ungraded case, and their statements here reflect that.

3 Structure theory I

3.1 Properly stratified categories

In this subsection, we let $\Bbbk$ be an arbitrary field. Let $\mathcal{A}$ be a $\Bbbk$-linear abelian category that is equipped with an automorphism $\langle 1 \rangle : \mathcal{A} \to \mathcal{A}$, called the *Tate twist*. Assume that this category has the following properties:

1. Every object has finite length.
2. For every simple object $L \in \mathcal{A}$, we have $\underline{\mathrm{End}}(L) \cong \Bbbk$.

Let Ω be the set of isomorphism classes of simple objects up to Tate twist. For each $\gamma \in \Omega$, choose a representative L_γ. Assume that Ω is equipped with a partial order $\leq$. For any finite order ideal $\Gamma \subset \Omega$, let $\mathcal{A}_\Gamma \subset \mathcal{A}$ be the full subcategory consisting of objects all of whose composition factors are isomorphic to some $L_\gamma\langle n\rangle$ with $\gamma \in \Gamma$.

Such a category is of particular interest when for each $\gamma \in \Gamma$, there exist four morphisms $\overline{\Delta}_\gamma \to L_\gamma$, $L_\gamma \to \overline{\nabla}_\gamma$, $\Delta_\gamma \to L_\gamma$, $L_\gamma \to \nabla_\gamma$ such that:

3. If $\xi \not\geq \gamma$, then $\underline{\mathrm{Hom}}(\overline{\Delta}_\gamma, L_\xi) = \underline{\mathrm{Ext}}^1(\overline{\Delta}_\gamma, L_\xi) = 0$ and $\underline{\mathrm{Hom}}(L_\xi, \overline{\nabla}_\gamma) = \underline{\mathrm{Ext}}^1(L_\xi, \overline{\nabla}_\gamma) = 0$.
4. The kernel of $\overline{\Delta}_\gamma \to L_\gamma$ and the cokernel of $L_\gamma \to \overline{\nabla}_\gamma$ both lie in $\mathcal{A}_{<\gamma}$.
5. If γ is a maximal element in an order ideal $\Gamma \subset \Omega$, then in $\mathcal{A}_\Gamma$, $\Delta_\gamma \to L_\gamma$ is projective cover of L_γ, and $L_\gamma \to \nabla_\gamma$ is an injective envelope. Moreover, Δ_γ has a filtration whose subquotients are of the form $\overline{\Delta}_\gamma\langle n\rangle$, and ∇_γ has a filtration whose subquotients are of the form $\overline{\nabla}_\gamma\langle n\rangle$.
6. $\underline{\mathrm{Ext}}^2(\Delta_\gamma, \overline{\nabla}_\xi) = 0$ and $\underline{\mathrm{Ext}}^2(\overline{\Delta}_\gamma, \nabla_\xi) = 0$ for all $\gamma, \xi \in \Omega$.

A category satisfying (1)–(6) is called a *graded properly stratified category*. If it happens that $\Delta_\gamma \cong \overline{\Delta}_\gamma$ and $\nabla_\gamma = \overline{\nabla}_\gamma$ for all $\gamma \in \Gamma$, then instead we call it a *graded quasihereditary category* or a *highest-weight category*.

Objects of the form $\overline{\Delta}_\gamma\langle n\rangle$ (resp. $\overline{\nabla}_\gamma\langle n\rangle$) are called *proper standard* (resp. *proper costandard*) objects. Those of the form $\Delta_\gamma\langle n\rangle$ (resp. $\nabla_\gamma\langle n\rangle$) are called *true standard* (resp. *true costandard*) objects. In the quasihereditary case, there is no distinction between proper standard objects and true standard objects; we simply call them *standard*, and likewise for *costandard*.

There are obvious ungraded analogues of these notions: we omit the Tate twist, and replace all occurences of $\underline{\mathrm{Hom}}$ and $\underline{\mathrm{Ext}}$ above by ordinary Hom and Ext.

Remark 3.1. Some sources, such as [3, 18], use the term *quasihereditary* to refer to a category that only satisfies properties (1)–(4) above.

3.2 Quasihereditarity and derived equivalences

Let us now return to the assumptions on $\Bbbk$ from Section 2.1. Our next goal is to see that the exotic and perverse-coherent t-structures fit the framework introduced above. Because of a subtlety in the perverse-coherent case, the statements in this subsection explicitly mention the $\mathbb{G}_\mathrm{m}$-equivariant and non-$\mathbb{G}_\mathrm{m}$-equivariant cases separately.

Theorem 3.2. *The category* $\mathrm{ExCoh}^G(\widetilde{\mathcal{N}})$ *is quasihereditary, and* $\mathrm{ExCoh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})$ *is graded quasihereditary. There are equivalences of categories*

$$D^\mathrm{b}\mathrm{ExCoh}^G(\widetilde{\mathcal{N}}) \xrightarrow{\sim} D^\mathrm{b}\mathrm{Coh}^G(\widetilde{\mathcal{N}}), \quad D^\mathrm{b}\mathrm{ExCoh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}}) \xrightarrow{\sim} D^\mathrm{b}\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}}).$$

Proof (sketch). The fact that these categories are (graded) quasihereditary is an immediate consequence of Theorem 2.3 and basic properties of exceptional sets. For the derived equivalences, one can imitate the argument of [16, Corollary 3.3.2].

Specifically, both $D^{\mathrm{b}}\mathrm{ExCoh}^{G\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$ and $D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$ are generated by both standard objects and the costandard objects. To establish the derived equivalence, it suffices to show that the natural map

$$\mathrm{Ext}^k_{\mathrm{ExCoh}^{G\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})}(\widehat{\Delta}_\lambda, \widehat{\nabla}_\mu\langle n\rangle) \to \mathrm{Hom}_{D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})}(\widehat{\Delta}_\lambda, \widehat{\nabla}_\mu\langle n\rangle[k]) \tag{7}$$

is an isomorphism for all $\lambda, \mu \in \mathbf{X}$, $k \geq 0$, and $n \in \mathbb{Z}$. When $k = 0$, this map is an isomorphism by general properties of t-structures. When $k > 0$, the left-hand side vanishes by general properties of quasihereditary categories, while the right-hand side vanishes by general properties of exceptional sets. Thus, (7) is always an isomorphism, as desired. The same argument applies to $\mathrm{ExCoh}^G(\widetilde{\mathcal{N}})$. □

Theorem 3.3. *The category* $\mathrm{PCoh}^G(\mathcal{N})$ *is properly stratified, and the category* $\mathrm{PCoh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$ *is graded properly stratified. There are equivalences of categories*

$$D^{\mathrm{b}}\mathrm{PCoh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N}) \xrightarrow{\sim} D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N}), \qquad D^{\mathrm{b}}\mathrm{PCoh}^{G}(\mathcal{N}) \xrightarrow{\sim} D^{\mathrm{b}}\mathrm{Coh}^{G}(\mathcal{N}).$$

Proof (sketch). It was shown in [18, 3] that $\mathrm{PCoh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$ satisfies axioms (1)–(4) of Section 3.1 (cf. Remark 3.1), with the objects of (5) playing the roles of the proper standard and proper costandard objects.

The proof of axioms (5) and (6) is due to Minn-Thu-Aye [37, Theorem 4.3]. His argument includes a recipe for constructing the true standard and true costandard objects. This recipe is reminiscent of Proposition 2.2: specifically, by [37, Definition 4.2], there are canonical distinguished triangles

$$\Delta_\lambda \to V(\lambda) \otimes \mathcal{O}_{\mathcal{N}}\langle\delta^*_\lambda\rangle \to \mathcal{K}_\lambda \to \quad \text{and} \quad \mathcal{K}'_\lambda \to H^0(\lambda) \otimes \mathcal{O}_{\mathcal{N}}\langle-\delta^*_\lambda\rangle \to \nabla_\lambda \to$$

with $\mathcal{K}_\lambda, \mathcal{K}'_\lambda \in D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})_{<\lambda}$.

The derived equivalences are more difficult here than in the exotic case, mainly because there may be nontrivial higher Ext-groups between proper standard and proper costandard objects. In the $(G \times \mathbb{G}_{\mathrm{m}})$-equivariant case, the result is proved in [3]. The proof makes use of the $\mathbb{G}_{\mathrm{m}}$-action in an essential way; it cannot simply be copied in the ungraded case. However, by [13, Lemma A.7.1], the following diagram commutes up to natural isomorphism:

$$\begin{array}{ccc} D^{\mathrm{b}}\mathrm{PCoh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N}) & \xrightarrow{\sim} & D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N}) \\ {\scriptstyle U}\downarrow & & \downarrow{\scriptstyle U} \\ D^{\mathrm{b}}\mathrm{PCoh}^{G}(\mathcal{N}) & \longrightarrow & D^{\mathrm{b}}\mathrm{Coh}^{G}(\mathcal{N}). \end{array}$$

Here, both vertical arrows are the functors that forget the $\mathbb{G}_{\mathrm{m}}$-equivariance. It is not difficult to check that these vertical arrows are *degrading functors* as in [16, §4.3]. Thus, for any $\lambda, \mu \in \mathbf{X}$, we have a commutative diagram

$$\begin{array}{ccc} \bigoplus_{n\in\mathbb{Z}} \operatorname{Ext}^k_{\mathrm{PCoh}^{G\times\mathbb{G}_\mathrm{m}}(\mathcal{N})}(\overline{\Delta}_\lambda, \overline{\nabla}_\mu\langle n\rangle) & \xrightarrow{\sim} & \bigoplus_{k\in\mathbb{Z}} \operatorname{Hom}_{D^\mathrm{b}\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\mathcal{N})}(\overline{\Delta}_\lambda, \overline{\nabla}_\mu\langle k\rangle[n]) \\ \downarrow\wr & & \downarrow\wr \\ \operatorname{Ext}^k_{\mathrm{PCoh}^G(\mathcal{N})}(U(\overline{\Delta}_\lambda), U(\overline{\nabla}_\mu)) & \longrightarrow & \operatorname{Hom}_{D^\mathrm{b}\mathrm{Coh}^G(\mathcal{N})}(U(\overline{\Delta}_\lambda), U(\overline{\nabla}_\mu)[k]). \end{array}$$

Since the top arrow and both vertical arrows are isomorphisms, the bottom arrow must be as well. That map is analogous to (7), and, as in the proof of the preceding theorem, it implies that $D^\mathrm{b}\mathrm{PCoh}^G(\mathcal{N}) \cong D^\mathrm{b}\mathrm{Coh}^G(\mathcal{N})$. □

3.3 Costandard and tilting objects

The abstract categorical framework of Section 3.1 places standard and costandard objects on an equal footing, but in practice, the following result makes costandard objects considerably easier to work with explicitly.

Theorem 3.4. *1. For all $\lambda \in \mathbf{X}$, $\widehat{\nabla}_\lambda$ is a coherent sheaf.*
2. For all $\lambda \in \mathbf{X}^+$, $\overline{\nabla}_\lambda$ is a coherent sheaf.

In contrast, even for SL_2, many standard objects are complexes with cohomology in more than one degree.

Proof (sketch). The first assertion is proved in [8], using Theorem 4.3 below to translate it into a question about the dual affine Grassmannian. The second assertion is due to Kumar–Lauritzen–Thomsen [29], although in many cases it goes back to much older work of Andersen–Jantzen [9]. □

Recall that a *tilting object* in a quasihereditary category is one that has both a standard filtration and a costandard filtration. The isomorphism classes of indecomposable tilting objects are in bijection with the isomorphism classes of simple (or standard, or costandard) objects. Let

$$\hat{\mathfrak{T}}_\lambda \in \mathrm{ExCoh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})$$

denote the indecomposable tilting object corresponding to $\mathfrak{E}_\lambda$.

In a properly stratified category that is not quasihereditary, there are two distinct versions of this notion: an object is called *tilting* if it has a true standard filtration and a proper costandard filtration, and *cotilting* if it has a proper standard filtration and a true costandard filtration. These notions need not coincide in general. See [7, §2.2] for general background on (co)tilting objects in this setting.

Proposition 3.5 ([37]). *In $\mathrm{PCoh}^{G\times\mathbb{G}_\mathrm{m}}(\mathcal{N})$, the indecomposable tilting and cotilting objects coincide, and they are all of the form $T(\lambda) \otimes \mathcal{O}_\mathcal{N}\langle n\rangle$.*

Proposition 3.6. *In $\mathrm{ExCoh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})$, every tilting object is a coherent sheaf. For $\lambda \in \mathbf{X}^+$, we have $\hat{\mathfrak{T}}_\lambda \cong T(\lambda) \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}$.*

Proof (sketch). This can be deduced by adjunction from Proposition 3.5 using the fact that $\pi^*\mathcal{O}_{\mathcal{N}} \cong \pi^!\mathcal{O}_{\mathcal{N}} \cong \mathcal{O}_{\widetilde{\mathcal{N}}}$ and the criterion from [19, Lemma 4]. □

At the moment, there is no comparable statement describing $\hat{\mathfrak{T}}_\lambda$ for nondominant λ. A better understanding of tilting exotic sheaves is highly desirable; in particular, it would shed light on the question below. An affirmative answer would have significant consequences for the geometry of affine Grassmannians and for modular representation theory. In Section 6, we will answer this question for SL_2.

Question 3.7 (Positivity for tilting exotic sheaves). Is it true that $\underline{\mathrm{Hom}}(\hat{\mathfrak{T}}_\lambda, \hat{\mathfrak{T}}_\mu)$ is concentrated in nonnegative degrees for all $\lambda, \mu \in \mathbf{X}$?

4 Applications

Many of the applications of exotic and perverse-coherent sheaves rely on the fact that these t-structures can be constructed in several rather different ways.

Quasiexceptional sets This refers to the construction that was carried out in Section 2. (For an explanation of this term and additional context, see [18, §2.2] and [19, §2.1].)

Whittaker sheaves In this approach, we transport the natural t-structure across a derived equivalence relating our category of coherent sheaves to a suitable category of Iwahori–Whittaker perverse sheaves on the affine flag variety for the Langlands dual group $\check{G}$.

Affine Grassmannian This approach, Koszul dual to the preceding one, involves Iwahori-monodromic perverse sheaves on the affine Grassmannian for $\check{G}$.

Local cohomology In the perverse-coherent case, there is an algebro-geometric construction in terms of local cohomology that superficially resembles the definition of ordinary (constructible) perverse sheaves. Indeed, this description is the reason for the name "perverse-coherent."

Braid positivity In the exotic case, the t-structure is uniquely determined by certain exactness properties, the most important of which involves the affine braid group action of [24].

In this section, we briefly review these various approaches and discuss some of their applications. Some of these are (for now?) available only in characteristic zero.

4.1 Whittaker sheaves and quantum group cohomology

Let p be a prime number, and let $\mathbf{K} = \bar{\mathbb{F}}_p((t))$ and $\mathbf{O} = \bar{\mathbb{F}}_p[[t]]$. Let $\check{G}$ be the Langlands dual group to G over $\bar{\mathbb{F}}_p$, and let $\check{B}^-, \check{B} \subset \check{G}$ be opposite Borel subgroups corresponding to negative and positive roots, respectively. (Note that our convention for $\check{G}$ differs from that for G, where $B \subset G$ denotes a negative Borel subgroup.)

These groups determine a pair $I^- = e^{-1}(\check{B}^-)$, $I = e^{-1}(\check{B})$ of oppposite Iwahori subgroups, where $e : \check{G}_{\mathbf{O}} \to \check{G}$ is the map induced by $t \mapsto 0$. Recall that the *affine flag variety* is the space $\mathcal{F}l = \check{G}_{\mathbf{K}}/I$.

Let $\check{U}^- \subset \check{B}^-$ be the unipotent radical, and for each simple root α, let $\check{U}^-_\alpha \subset \check{U}^-$ be the root subgroup corresponding to $-\alpha$. The quotient $\check{U}^-/[\check{U}^-, \check{U}^-]$ can be identified with the product $\prod_\alpha \check{U}^-_\alpha$. For each α, fix an isomorphism $\psi_\alpha : \check{U}^-_\alpha \cong \mathbb{G}_{\mathrm{a}}$. Let $I^-_u = e^{-1}(\check{U}^-)$ be the pro-unipotent radical of I^-, and let $\psi : I^-_u \to \mathbb{G}_{\mathrm{a}}$ be the composition

$$I^-_u \xrightarrow{e} \check{U}^- \to \check{U}^-/[\check{U}^-, \check{U}^-] \cong \prod_\alpha U^-_\alpha \xrightarrow{\prod \psi_\alpha} \prod_\alpha \mathbb{G}_{\mathrm{a}} \xrightarrow{\sum} \mathbb{G}_{\mathrm{a}}.$$

Finally, let $\mathcal{X} = \psi^* \mathrm{AS}$, where AS denotes an Artin–Schreier local system on $\mathbb{G}_{\mathrm{a}}$.

Let ℓ be a prime number different from p. The *Iwahori–Whittaker derived category* of $\mathcal{F}l$, denoted $D^{\mathrm{b}}_{\mathrm{IW}}(\mathcal{F}l, \bar{\mathbb{Q}}_\ell)$, is defined to be the $(I^-_u, \mathcal{X})$-equivariant derived category of $\bar{\mathbb{Q}}_\ell$-sheaves on $\mathcal{F}l$. (In many sources, this is simply called the (I^-_u, ψ)-equivariant derived category. For background on this kind of equivariant derived category, see, for instance, [5, Appendix A].) We also have the abelian category $\mathrm{Perv}_{\mathrm{IW}}(\mathcal{F}l, \bar{\mathbb{Q}}_\ell)$ of *Iwahori–Whittaker perverse sheaves* on $\mathcal{F}l$.

Theorem 4.1 ([11, 19]). *Assume that $\Bbbk = \bar{\mathbb{Q}}_\ell$. There is an equivalence of triangulated categories*

$$D^{\mathrm{b}}\mathrm{Coh}^G(\widetilde{\mathcal{N}}) \cong D^{\mathrm{b}}_{\mathrm{IW}}(\mathcal{F}l, \bar{\mathbb{Q}}_\ell).$$

This equivalence is t-exact for the exotic t-structure on the left-hand side and the perverse t-structure on the right-hand side. In particular, there is an equivalence of abelian categories

$$\mathrm{ExCoh}^G(\widetilde{\mathcal{N}}) \cong \mathrm{Perv}_{\mathrm{IW}}(\mathcal{F}l, \bar{\mathbb{Q}}_\ell).$$

There is an equivalence of categories $D^{\mathrm{b}}\mathrm{Perv}_{\mathrm{IW}}(\mathcal{F}l, \bar{\mathbb{Q}}_\ell) \xrightarrow{\sim} D^{\mathrm{b}}_{\mathrm{IW}}(\mathcal{F}l, \bar{\mathbb{Q}}_\ell)$ (see [11, Lemma 1]), so Theorem 4.1 can be restated in a way that matches the exotic t-structure with the natural t-structure on $D^{\mathrm{b}}\mathrm{Perv}_{\mathrm{IW}}(\mathcal{F}l, \bar{\mathbb{Q}}_\ell)$.

This equivalence plays a key role in Bezrukavnikov's computation of the cohomology of tilting modules for quantum groups at a root of unity [19]. Specifically, after relating $D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$ to the derived category of the principal block of the quantum group, the desired facts about quantum group cohomology are reduced to the following statement about exotic sheaves, called the positivity lemma [19, Lemma 9]:

$$\mathrm{Ext}^i(\widehat{\Delta}_\lambda\langle n\rangle, \mathfrak{E}_\mu) = \mathrm{Ext}^i(\mathfrak{E}_\mu, \widehat{\nabla}_\lambda\langle -n\rangle) = 0 \qquad \text{if } i > n. \tag{8}$$

To prove the positivity lemma, one uses Theorem 4.1 to translate it into a question about Weil perverse sheaves on $\mathcal{F}l$. The latter can be answered using the powerful and well-known machinery of [14].

We will not discuss the proof of Theorem 4.1, but as a plausibility check, let us review the parametrization of simple objects in $\mathrm{Perv}_{\mathrm{IW}}(\mathcal{F}l, \bar{\mathbb{Q}}_\ell)$. Iwahori–Whittaker perverse sheaves are necessarily constructible along the I^--orbits on $\mathcal{F}l$, which, like the I-orbits, are naturally parametrized by the extended affine Weyl group W_{ext} for G. However, not every I^--orbit supports an $\mathcal{X}$-equivariant local system: according to [11, Lemma 2], those that do correspond to the set ${}^f W_{\mathrm{ext}} \subset W_{\mathrm{ext}}$ of minimal-length coset representatives for $W \backslash W_{\mathrm{ext}}$. Thus, simple objects in $\mathrm{Perv}_{\mathrm{IW}}(\mathcal{F}l, \bar{\mathbb{Q}}_\ell)$ are parametrized by ${}^f W_{\mathrm{ext}}$, which is naturally in bijection with $\mathbf{X}$.

For $w \in {}^f W_{\mathrm{ext}}$, let $L_w \in \mathrm{Perv}_{\mathrm{IW}}(\mathcal{F}l, \bar{\mathbb{Q}}_\ell)$ denote the corresponding simple object. Let ${}^f W_{\mathrm{ext}}^f$ be the set of minimal-length representatives for the double cosets $W \backslash W_{\mathrm{ext}} / W$, and form the Serre quotient

$$\mathrm{Perv}_{\mathrm{IW}}^f(\mathcal{F}l, \bar{\mathbb{Q}}_\ell) = \mathrm{Perv}_{\mathrm{IW}}(\mathcal{F}l, \bar{\mathbb{Q}}_\ell) \Big/ \begin{pmatrix} \text{the Serre subcategory generated} \\ \text{by the } L_w \text{ with } w \notin {}^f W_{\mathrm{ext}}^f \end{pmatrix}.$$

Theorem 4.2 ([20]). *Assume that $\Bbbk = \bar{\mathbb{Q}}_\ell$. There is an equivalence of triangulated categories*

$$D^{\mathrm{b}}\mathrm{Coh}^G(\mathcal{N}) \cong D^{\mathrm{b}}\mathrm{Perv}_{\mathrm{IW}}^f(\mathcal{F}l, \bar{\mathbb{Q}}_\ell).$$

This equivalence is t-exact for the perverse-coherent t-structure on the left-hand side and the natural t-structure on the right-hand side. In particular, there is an equivalence of abelian categories

$$\mathrm{PCoh}^G(\mathcal{N}) \cong \mathrm{Perv}_{\mathrm{IW}}^f(\mathcal{F}l, \bar{\mathbb{Q}}_\ell).$$

It seems likely that analogous statements to the theorems in this subsection hold when $\Bbbk$ has positive characteristic.

4.2 The affine Grassmannian and the Mirković–Vilonen conjecture

Recall that the *affine Grassmannian* is the space $\mathcal{G}r = \check{G}_{\mathbf{K}}/\check{G}_{\mathbf{O}}$. Here, we may either define $\mathbf{K}$ and $\mathbf{O}$ as in the previous subsection, and work with étale sheaves on $\mathcal{G}r$, or we may instead put $\mathbf{K} = \mathbb{C}((t))$ and $\mathbf{O} = \mathbb{C}[[t]]$ and equip $\mathcal{G}r$ with the classical topology. (For a discussion of how to compare the two settings, see, e.g., [40, Remark 7.1.4(2)].) In this subsection, we will work with a certain category of "mixed" I-monodromic perverse $\Bbbk$-sheaves on $\mathcal{G}r$, denoted $\mathrm{Perv}^{\mathrm{mix}}_{(I)}(\mathcal{G}r, \Bbbk)$. If $\Bbbk$ has characteristic zero, this category should be defined following the pattern of [16, Theorem 4.4.4] or [4, §6.4]: $\mathrm{Perv}^{\mathrm{mix}}_{(I)}(\mathcal{G}r, \bar{\mathbb{Q}}_\ell)$ is *not* the category of all mixed perverse sheaves in the sense of [14], but rather the full subcategory in which we allow only Tate local systems and require the associated graded of the weight filtration to be semisimple. For $\Bbbk$ of positive characteristic, this category is defined in [6] in terms of the homological algebra of parity sheaves.

In both cases, the additive category $\mathrm{Parity}_{(I)}(\mathcal{G}r, \Bbbk)$ of Iwahori-constructible parity sheaves can be identified with a full subcategory of $D^{\mathrm{b}}\mathrm{Perv}^{\mathrm{mix}}_{(I)}(\mathcal{G}r, \Bbbk)$. In the case where $\Bbbk = \bar{\mathbb{Q}}_\ell$, $\mathrm{Parity}_{(I)}(\mathcal{G}r, \Bbbk)$ is identified with the category of pure semisimple complexes of weight 0.

Theorem 4.3 ([12] for $\Bbbk = \bar{\mathbb{Q}}_\ell$; [8, 36] in general). *There is an equivalence of triangulated categories*

$$P : D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}}) \xrightarrow{\sim} D^{\mathrm{b}}\mathrm{Perv}^{\mathrm{mix}}_{(I)}(\mathcal{G}r, \Bbbk)$$

such that $P(\mathcal{F}\langle n\rangle) \cong P(\mathcal{F})(\frac{n}{2})[n]$. This equivalence is not t-exact, but it does induce an equivalence of additive categories

$$\mathrm{Tilt}(\mathrm{ExCoh}^{G\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})) \xrightarrow{\sim} \mathrm{Parity}_{(I)}(\mathcal{G}r, \Bbbk).$$

Note that the exotic t-structure can be recovered from the class of tilting objects in its heart. When $\Bbbk = \bar{\mathbb{Q}}_\ell$, there is also an "unmixed" version of this theorem [12]. In positive characteristic, a putative unmixed statement is equivalent to a "modular formality" property for $\mathcal{G}r$ that is currently still open.

The next theorem is a similar statement for the perverse-coherent t-structure. This result does not, however, extend to an equivalence involving the full derived category $D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$.

Theorem 4.4 ([7]). *There is an equivalence of additive categories*

$$\mathrm{Tilt}(\mathrm{PCoh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})) \xrightarrow{\sim} \mathrm{Parity}_{(\check{G}_{\mathrm{O}})}(\mathcal{G}r, \Bbbk).$$

An important consequence of the preceding theorem is the following result, known as the Mirković–Vilonen conjecture (see [38, Conjecture 6.3] or [39, Conjecture 13.3]). In bad characteristic, the conjecture is false [30].

Theorem 4.5 ([7]). *Under the geometric Satake equivalence, the stalks (resp. costalks) of the perverse sheaf on $\mathcal{G}r$ corresponding to a Weyl module (resp. dual Weyl module) of G vanish in odd degrees.*

Proof (sketch). The statement we wish to prove can be rewritten as a statement about the vanishing of certain Ext-vanishing groups in the derived category of constructible complexes of $\Bbbk$-sheaves on $\mathcal{G}r$. Theorem 4.4 lets us translate that question into one about Hom-groups in the abelian category $\mathrm{PCoh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$ instead. The latter question turns out to be quite easy; it is an exercise using basic properties of properly stratified categories. □

4.3 Local cohomology and the Lusztig–Vogan bijection

The following theorem describes $\mathrm{PCoh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$ in terms of cohomology-vanishing conditions on a complex $\mathcal{F}$ and on its Serre–Grothendieck dual $\mathbb{D}(\mathcal{F})$, given by $\mathbb{D}(\mathcal{F}) = R\mathcal{H}om(\mathcal{F}, \mathcal{O}_{\mathcal{N}})$. These conditions closely resemble the definition of ordinary (constructible) perverse sheaves; indeed, this theorem is the justification for the term "perverse-coherent."

Theorem 4.6 ([18]; see also [3]). *Let $\mathcal{F} \in D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$. The following conditions are equivalent:*

1. *$\mathcal{F}$ lies in $\mathrm{PCoh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$.*
2. *We have $\dim\operatorname{supp}\mathcal{H}^i(\mathcal{F}) \le \dim\mathcal{N} - 2i$ and $\dim\operatorname{supp}\mathcal{H}^i(\mathbb{D}\mathcal{F}) \le \dim\mathcal{N} - 2i$ for all $i \in \mathbb{Z}$.*
3. *Whenever $x \in \mathcal{N}$ is a generic point of a G-orbit, we have $H^i(\mathcal{F}_x) = 0$ if $i > \frac{1}{2}\operatorname{codim}\bar{x}$, and $H^i_x(\mathcal{F}) = 0$ if $i < \frac{1}{2}\operatorname{codim}\bar{x}$.*

(In the last assertion, $\mathcal{F}_x$ is just the stalk of $\mathcal{F}$ at x, while $H^i_x(-)$ is cohomology with support at x.)

Proof (sketch). In [18], condition (3) was taken as the definition of the category $\mathrm{PCoh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$, following [17, 10], while the t-structure of Theorem 2.4 is considered separately and initially given no name. According to [18, Corollary 3], the two t-structures coincide; the proof consists of showing that the A_λ satisfy condition (3).

Condition (3) can be used to define "perverse-coherent" t-structures on varieties or stacks in considerable generality, not just on the nilpotent cone of a reductive group. This theory is developed in [17, 10]. The equivalence of conditions (2) and (3) holds in this general framework; see [10, Lemma 2.18]. □

As with ordinary perverse sheaves, there is a special class of perverse-coherent sheaves satisfying stronger dimension bounds. Let $C \subset \mathcal{N}$ be a nilpotent orbit, and let $\mathcal{E}$ be a $(G \times \mathbb{G}_{\mathrm{m}})$-equivariant vector bundle on C. There is an object

$$\mathfrak{IC}(C,\mathcal{E}) \in \mathrm{PCoh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N}),$$

called a *coherent intersection cohomology complex*, that is uniquely characterized by the following two conditions:

1. $\mathfrak{IC}(C,\mathcal{E})$ is supported on $\overline{C}$, and $\mathfrak{IC}(C,\mathcal{E})|_C \cong \mathcal{E}[-\frac{1}{2}\operatorname{codim} C]$.
2. We have $\dim\operatorname{supp}\mathcal{H}^i(\mathcal{F}) < \dim\mathcal{N} - 2i$ and $\dim\operatorname{supp}\mathcal{H}^i(\mathbb{D}\mathcal{F}) < \dim\mathcal{N} - 2i$ for all $i > \frac{1}{2}\operatorname{codim} C$.

Moreover, when $\mathcal{E}$ is an irreducible vector bundle, $\mathfrak{IC}(C,\mathcal{E})$ is a simple object of $\mathrm{PCoh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$, and every simple object arises in this way.

Theorem 4.6 has an obvious G-equivariant analogue (omitting the $\mathbb{G}_{\mathrm{m}}$-equivariance), as does the notion of coherent intersection cohomology complexes. The latter yields a bijection

$$\left\{\begin{matrix}\text{simple objects}\\ \text{in } \mathrm{PCoh}^G(\mathcal{N})\end{matrix}\right\} \overset{\sim}{\longleftrightarrow} \left\{(C,\mathcal{E}) \;\middle|\; \begin{matrix}C \text{ a } G\text{-orbit, } \mathcal{E} \text{ an irreducible}\\ G\text{-equivariant vector bundle on } C\end{matrix}\right\} \tag{9}$$

that looks very different from the parametrization of simple objects in §2.3. Comparing the two yields the following result of Bezrukavnikov.

Theorem 4.7 ([18]). *There is a canonical bijection*

$$\mathbf{X}^+ \overset{\sim}{\longleftrightarrow} \{(C,\mathcal{E})\}.$$

The existence of such a bijection was independently conjectured by Lusztig [33] and Vogan. For $G = \mathrm{GL}_n$, the Lusztig–Vogan bijection was established earlier [1] (see also [2]) by an argument that provided an explicit combinatorial description of the bijection.

In general, it is rather difficult to carry out computations with coherent $\mathfrak{IC}$'s, and the problem of computing the Lusztig–Vogan bijection explicitly remains open in most cases. The extreme cases corresponding to the regular and zero nilpotent orbits are discussed below, following [2, Proposition 2.8]. Let

$$\mathcal{N}_{\mathrm{reg}} \subset \mathcal{N} \qquad \text{and} \qquad C_0 \subset \mathcal{N}$$

denote the regular and zero nilpotent orbits, respectively.

Proposition 4.8. *The bijection of Theorem 4.7 restricts to a bijection*

$$\mathbf{X}_{\min} \overset{\sim}{\longleftrightarrow} \{(\mathcal{N}_{\mathrm{reg}}, \mathcal{E})\}$$

Proof (sketch). Since the elements of $\mathbf{X}_{\min}$ are precisely the minimal elements of $\mathbf{X}^+$ with respect to $\preceq$ (or $\leq$), the proper costandard objects $\{\overline{\nabla}_\lambda \mid \lambda \in \mathbf{X}_{\min}\}$ are simple. Every A_λ has nonzero restriction to $\mathcal{N}_{\mathrm{reg}}$ (since π is an isomorphism over $\mathcal{N}_{\mathrm{reg}}$), so for $\lambda \in \mathbf{X}_{\min}$, the simple object $\mathfrak{IC}_\lambda = \overline{\nabla}_\lambda$ must coincide with some $\mathfrak{IC}(\mathcal{N}_{\mathrm{reg}}, \mathcal{E})$. Thus, the bijection of Theorem 4.7 at least restricts to an injective map $\mathbf{X}_{\min} \hookrightarrow \{(\mathcal{N}_{\mathrm{reg}}, \mathcal{E})\}$. The fact that it is also surjective can be deduced from the well-known relationship between minuscule weights and representations of the center of G. □

Proposition 4.9. *The bijection of Theorem 4.7 restricts to a bijection*

$$\mathbf{X}^+ + 2\rho \overset{\sim}{\longleftrightarrow} \{(C_0, \mathcal{E})\}.$$

Here, $2\rho = \sum_{\alpha \in \Phi^+} \alpha$, as in Section 2.4. The proof of this will be briefly discussed at the end of Section 5.2.

4.4 Affine braid group action and modular representation theory

In [21], Bezrukavnikov and Mirković proved a collection of conjectures of Lusztig [34] involving the equivariant K-theory of Springer fibers and the representation theory of semisimple Lie algebras in positive characteristic. In this work, which builds on the localization theory developed in [22, 23], a key ingredient is the *non-commutative Springer resolution*, a certain $\Bbbk[\mathcal{N}]$-algebra A^0 equipped with a G-action, along with a derived equivalence

$$D^{\mathrm{b}}(A^0\text{-mod}^G) \cong D^{\mathrm{b}}\mathrm{Coh}^G(\widetilde{\mathcal{N}}). \qquad (10)$$

Here, we will just discuss one small aspect of the argument. At a late stage in [21], one learns that Lusztig's conjectures follow from a certain positivity statement about graded A^0-modules, and that, moreover, it is enough to prove that positivity statement in characteristic 0. From then on, the proof follows the pattern we saw with (8): by composing (10) with Theorem 4.1, one can translate the desired positivity statement into a statement about Weil perverse sheaves on the affine flag variety $\mathcal{F}l$, and then use the machinery of weights from [14].

To carry out the "translation" step, one needs an appropriate description of the t-structure on $D^{\mathrm{b}}(A^0\text{-mod}^G)$ that corresponds to the natural t-structure on $D^{\mathrm{b}}\mathrm{Perv}_{\mathrm{IW}}(\mathcal{F}l, \bar{\mathbb{Q}}_\ell)$, or, equivalently, to the exotic t-structure[2] on $D^{\mathrm{b}}\mathrm{Coh}^G(\widetilde{\mathcal{N}})$. Unfortunately, the exceptional set construction of Section 2 is ill-suited to this purpose.

The theorem below gives a new characterization of $\mathrm{ExCoh}^G(\widetilde{\mathcal{N}})$ that does adapt well to the setting of A^0-modules. Its key feature is the prominent role it gives to the *affine braid group action* on $D^{\mathrm{b}}\mathrm{Coh}^G(\widetilde{\mathcal{N}})$ that is constructed in [24]. Specifically, it involves the following notion: a t-structure on $D^{\mathrm{b}}\mathrm{Coh}^G(\widetilde{\mathcal{N}})$ is said to be *braid-positive* if, in the aforementioned affine braid group action, the action of positive words in the braid group is right t-exact. (The definition of the ring A^0 also involves braid positivity, and the t-structure on $D^{\mathrm{b}}\mathrm{Coh}^G(\widetilde{\mathcal{N}})$ corresponding to the natural t-structure on $D^{\mathrm{b}}(A^0\text{-mod}^G)$ is braid-positive.)

Before stating the theorem, we need some additional notation and terminology. Given a closed G-stable subset $Z \subset \mathcal{N}$, let $D^{\mathrm{b}}_Z\mathrm{Coh}^G(\widetilde{\mathcal{N}}) \subset D^{\mathrm{b}}\mathrm{Coh}^G(\widetilde{\mathcal{N}})$ be the full subcategory consisting of objects supported set-theoretically on $\pi^{-1}(Z)$. For a nilpotent orbit $C \subset \mathcal{N}$, let $D^{\mathrm{b}}_C\mathrm{Coh}^G(\widetilde{\mathcal{N}})$ be the quotient category

$$D^{\mathrm{b}}_{\overline{C}}\mathrm{Coh}^G(\widetilde{\mathcal{N}})/D^{\mathrm{b}}_{\overline{C}\smallsetminus C}\mathrm{Coh}^G(\widetilde{\mathcal{N}}).$$

A t-structure on $D^{\mathrm{b}}\mathrm{Coh}^G(\widetilde{\mathcal{N}})$ is said to be *compatible with the support filtration* if for every nilpotent orbit C, there are induced t-structures on $D^{\mathrm{b}}_{\overline{C}}\mathrm{Coh}^G(\widetilde{\mathcal{N}})$ and $D^{\mathrm{b}}_C\mathrm{Coh}^G(\widetilde{\mathcal{N}})$ such that the inclusion and quotient functors, respectively, are t-exact.

Theorem 4.10 ([21, §6.2.2]). *Assume that the characteristic of* $\Bbbk$ *is zero or larger than the Coxeter number of* G. *The exotic* t*-structure on* $D^{\mathrm{b}}\mathrm{Coh}^G(\widetilde{\mathcal{N}})$ *is the unique* t*-structure with all three of the following properties:*

1. *It is braid-positive.*
2. *It is compatible with the support filtration.*
3. *The functor* π_* *is* t*-exact with respect to this* t*-structure and the perverse-coherent* t*-structure on* $D^{\mathrm{b}}\mathrm{Coh}^G(\mathcal{N})$.

See [28, 35] for comprehensive accounts of the role of the affine braid group action in the study of the exotic t-structure.

[2] A caveat about terminology: most of [21] is concerned with *nonequivariant* coherent sheaves or A^0-modules. In that paper, the term *exotic* t*-structure* refers to a certain t-structure in the nonequivariant setting, and not to the t-structure of Theorem 2.3. In [21], the latter t-structure is instead called *perversely exotic*.

5 Structure theory II

5.1 Minuscule objects

As noted above, exotic sheaves do not, in general, admit a "local" description like that of perverse-coherent sheaves in Section 4.3. Nevertheless, when we look at the set of regular elements

$$\widetilde{\mathcal{N}}_{\mathrm{reg}} := \pi^{-1}(\mathcal{N}_{\mathrm{reg}}),$$

compatibility with the support filtration from Theorem 4.10 lets us identify a handful of simple exotic sheaves. Recall that π is an isomorphism over $\mathcal{N}_{\mathrm{reg}}$, so $\widetilde{\mathcal{N}}_{\mathrm{reg}}$, like $\mathcal{N}_{\mathrm{reg}}$, is a single G-orbit.

Lemma 5.1. *1. If $\lambda \in -\mathbf{X}_{\min}$, then $\mathfrak{E}_\lambda \cong \mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda)\langle\delta_\lambda\rangle$.*
2. If $\lambda \notin -\mathbf{X}_{\min}$, then $\mathfrak{E}_\lambda|_{\widetilde{\mathcal{N}}_{\mathrm{reg}}} = 0$.

This lemma says that we can detect antiminuscule composition factors in an exotic sheaf by restricting to $\widetilde{\mathcal{N}}_{\mathrm{reg}}$.

Proof. The first assertion follows from the observation that elements of $-\mathbf{X}_{\min}$ are minimal with respect to $\leq$, so the standard objects $\widehat{\Delta}_\lambda = \mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda)\langle\delta_\lambda\rangle$ are simple.

Now take an arbitrary $\lambda \in \mathbf{X}$, and suppose that $\mathfrak{E}_\lambda|_{\widetilde{\mathcal{N}}_{\mathrm{reg}}} \neq 0$. Since π is an isomorphism over $\mathcal{N}_{\mathrm{reg}}$, $\pi_*\mathfrak{E}_\lambda$ has nonzero restriction to $\mathcal{N}_{\mathrm{reg}}$. A fortiori, $\pi_*\mathfrak{E}_\lambda$ is nonzero. By Proposition 2.6, λ must be antidominant, and $\pi_*\mathfrak{E}_\lambda \cong \mathfrak{IC}_{w_0\lambda}$. Then Proposition 4.8 tells us that $w_0\lambda \in \mathbf{X}_{\min}$, so $\lambda \in -\mathbf{X}_{\min}$, as desired. □

Lemma 5.2. *Let $\lambda, \mu \in \mathbf{X}$, and suppose that $\lambda - \mu$ lies in the root lattice. Then $\mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda)\langle(2\rho^\vee, \mu - \lambda)\rangle|_{\widetilde{\mathcal{N}}_{\mathrm{reg}}} \cong \mathcal{O}_{\widetilde{\mathcal{N}}}(\mu)|_{\widetilde{\mathcal{N}}_{\mathrm{reg}}}$.*

Proof. Since $\lambda - \mu$ is a linear combination of simple roots, it suffices to prove the lemma in the special case where $\mu = 0$ and λ is a simple root, say α. In this case, $(2\rho^\vee, -\alpha) = -2$.

Let $\widetilde{\mathcal{N}}_\alpha$ be as in Section 2.4. As explained in [19, Lemma 6] or [3, Lemma 5.3], there is a short exact sequence of coherent sheaves

$$0 \to \mathcal{O}_{\widetilde{\mathcal{N}}}(\alpha)\langle -2\rangle \to \mathcal{O}_{\widetilde{\mathcal{N}}} \to i_{\alpha *}\mathcal{O}_{\widetilde{\mathcal{N}}_\alpha} \to 0. \tag{11}$$

Recall that $\widetilde{\mathcal{N}}_\alpha$ does not meet $\widetilde{\mathcal{N}}_{\mathrm{reg}}$—indeed, its image under π is the closure of the subregular nilpotent orbit. So when we restrict to $\widetilde{\mathcal{N}}_{\mathrm{reg}}$, that short exact sequence gives us the desired isomorphism $\mathcal{O}_{\widetilde{\mathcal{N}}}(\alpha)\langle -2\rangle|_{\widetilde{\mathcal{N}}_{\mathrm{reg}}} \xrightarrow{\sim} \mathcal{O}_{\widetilde{\mathcal{N}}}|_{\widetilde{\mathcal{N}}_{\mathrm{reg}}}$. □

Proposition 5.3. *For all $\lambda \in \mathbf{X}^+$, $\overline{\nabla}_\lambda$ is a torsion-free coherent sheaf on $\mathcal{N}$.*

Proof. Let $Y = \mathcal{N} \smallsetminus \mathcal{N}_{\mathrm{reg}}$, and let $s : Y \hookrightarrow \mathcal{N}$ be the inclusion map. Since $\mathcal{N}_{\mathrm{reg}}$ is the unique open G-orbit, any coherent sheaf with torsion must have a subsheaf supported on Y. If $\overline{\nabla}_\lambda$ had such a subsheaf, then $\mathcal{H}^0(s^!\overline{\nabla}_\lambda)$ would be nonzero. But this contradicts the local description of $\mathrm{PCoh}^{G\times\mathbb{G}_{\mathrm{m}}}(\mathcal{N})$ from Section 4.3. □

Proposition 5.4. *Let $\mu \in \mathbf{X}_{\min}$. For any $\lambda \in \mathbf{X}$, we have*

$$[A_\lambda : \mathfrak{IC}_\mu\langle n\rangle] = \begin{cases} 1 & \text{if } \mu = \overset{+}{\mathsf{m}}(\lambda) \text{ and } n = (2\rho^\vee, \lambda), \\ 0 & \text{otherwise.} \end{cases}$$

Proof. Proposition 4.8 implies that we can determine the multiplicities of minuscule objects in any perverse-coherent sheaf by considering its restriction to $\mathcal{N}_{\mathrm{reg}}$. By Lemmas 5.1 and 5.2, we have

$$\begin{aligned} A_\lambda|_{\mathcal{N}_{\mathrm{reg}}} \cong \pi_*\mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda)|_{\mathcal{N}_{\mathrm{reg}}} &\cong \pi_*\mathcal{O}_{\widetilde{\mathcal{N}}}(\bar{\mathsf{m}}(\lambda))\langle(2\rho^\vee, \lambda - \bar{\mathsf{m}}(\lambda))\rangle|_{\mathcal{N}_{\mathrm{reg}}} \\ &\cong \pi_*\mathfrak{E}_{\bar{\mathsf{m}}(\lambda)}\langle(2\rho^\vee, \lambda - \bar{\mathsf{m}}(\lambda)) - \delta_{\bar{\mathsf{m}}(\lambda)}\rangle|_{\mathcal{N}_{\mathrm{reg}}} \\ &\cong \mathfrak{IC}_{\overset{+}{\mathsf{m}}(\lambda)}\langle(2\rho^\vee, \lambda - \bar{\mathsf{m}}(\lambda)) - \delta^*_{\overset{+}{\mathsf{m}}(\lambda)}\rangle|_{\mathcal{N}_{\mathrm{reg}}}. \end{aligned}$$

Consider the special case where $\lambda \in \mathbf{X}_{\min}$. In other words, $\lambda = \overset{+}{\mathsf{m}}(\lambda)$, and $A_{\overset{+}{\mathsf{m}}(\lambda)} \cong \overline{\nabla}_{\overset{+}{\mathsf{m}}(\lambda)}\langle\delta^*_{\overset{+}{\mathsf{m}}(\lambda)}\rangle \cong \mathfrak{IC}_{\overset{+}{\mathsf{m}}(\lambda)}\langle\delta^*_{\overset{+}{\mathsf{m}}(\lambda)}\rangle$. Comparing with the formula above, we see that

$$(2\rho^\vee, \overset{+}{\mathsf{m}}(\lambda) - \bar{\mathsf{m}}(\lambda)) - \delta^*_{\overset{+}{\mathsf{m}}(\lambda)} = \delta^*_{\overset{+}{\mathsf{m}}(\lambda)}$$

for any $\lambda \in \mathbf{X}$. Since $(2\rho^\vee, \overset{+}{\mathsf{m}}(\lambda)) = -(2\rho^\vee, \bar{\mathsf{m}}(\lambda))$, we deduce that

$$(2\rho^\vee, \overset{+}{\mathsf{m}}(\lambda)) = (2\rho^\vee, -\bar{\mathsf{m}}(\lambda)) = \delta_{\bar{\mathsf{m}}(\lambda)} = \delta^*_{\overset{+}{\mathsf{m}}(\lambda)}. \tag{12}$$

The result follows. □

Corollary 5.5. *1. Let $\lambda \in \mathbf{X}^+$ and $\mu \in \mathbf{X}_{\min}$. We have*

$$[\overline{\nabla}_\lambda : \mathfrak{IC}_\mu\langle n\rangle] = \begin{cases} 1 & \text{if } \mu = \overset{+}{\mathsf{m}}(\lambda) \text{ and } n = (2\rho^\vee, \lambda) - \delta^*_\lambda, \\ 0 & \text{otherwise.} \end{cases}$$

2. Let $\lambda \in \mathbf{X}$ and $\mu \in -\mathbf{X}_{\min}$. We have

$$[\widehat{\nabla}_\lambda : \mathfrak{E}_\mu\langle n\rangle] = \begin{cases} 1 & \text{if } \mu = \bar{\mathsf{m}}(\lambda) \text{ and } n = (2\rho^\vee, \mathsf{dom}(\lambda)) - \delta_\lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. The first part is just a restatement of Proposition 5.4. Next, Proposition 2.6 implies that $[\widehat{\nabla}_\lambda : \mathfrak{E}_\mu\langle n\rangle] = [\overline{\nabla}_{\mathsf{dom}(\lambda)}\langle\delta^*_\lambda\rangle : \mathfrak{IC}_{w_0\mu}\langle n\rangle]$. Using the observation that $\delta^*_\lambda - \delta^*_{\mathsf{dom}(\lambda)} = -\delta_\lambda$, the second part follows from the first. □

5.2 Character formulas

In this subsection, we work with the group ring $\mathbb{Z}[\mathbf{X}]$, along with its extension $\mathbb{Z}[\mathbf{X}][[q]][q^{-1}] = \mathbb{Z}[\mathbf{X}] \otimes_{\mathbb{Z}} \mathbb{Z}[[q]][q^{-1}]$. For $\lambda \in \mathbf{X}$, we denote by e^{λ} the corresponding element of $\mathbb{Z}[\mathbf{X}]$ or $\mathbb{Z}[\mathbf{X}][[q]][q^{-1}]$. If $V = \bigoplus_{n\in\mathbb{Z}} V_n$ is a graded T-representation (or a representation of some larger group, such as B or G) with $\dim V_n < \infty$ for all n and $V_n = 0$ for $n \ll 0$, we put

$$\operatorname{ch} V = \sum_{n\in\mathbb{Z}} \sum_{\nu\in\mathbf{X}} (\dim V_n^{\nu}) q^n e^{\nu}$$

where V_n^{ν} is the ν-weight space of V_n. More generally, if V is a chain complex of graded representations, we put

$$\operatorname{ch} V = \sum_{i\in\mathbb{Z}} (-1)^i \operatorname{ch} H^i(V).$$

Next, for any $\lambda \in \mathbf{X}^+$, we put

$$\chi(\lambda) = \operatorname{ch} H^0(\lambda) = \frac{\sum_{w\in W} (-1)^{\ell(w)} e^{w(\lambda+\rho)-\rho}}{\prod_{\alpha\in\Phi^+} (1 - e^{-\alpha})} \qquad \text{for } \lambda \in \mathbf{X}^+.$$

(The right-hand side is, of course, the Weyl character formula.)

Let $\lambda \in \mathbf{X}^+$ and $\mu \in \mathbf{X}$, and let $M_\lambda^\mu(q)$ be Lusztig's q-analogue of the weight multiplicity. Recall (see [32, (9.4)] or [27, (3.3)]) that this is given by

$$M_\lambda^\mu(q) = \sum_{w\in W} (-1)^{\ell(w)} P_{w(\lambda+\rho)-(\mu+\rho)}(q),$$

where $P_\nu(q)$ is a q-analogue of Kostant's partition function, determined by

$$\prod_{\alpha\in\Phi^+} \frac{1}{1 - q e^{\alpha}} = \sum_{\nu\in\mathbf{X}} P_\nu(q) e^{\nu}.$$

Kostant's multiplicity formula says that $M_\lambda^\mu(1)$ is the dimension of the μ-weight space of the dual Weyl module $H^0(\lambda)$.

It is clear from the definition that $P_\nu(q) = 0$ unless $\nu \succeq 0$, and of course $P_0(q) = 1$. From this, one can deduce that

$$M_\lambda^\mu(q) = 0 \qquad \text{if } \mu \not\preceq \lambda. \tag{13}$$

It is known that when $\mu \in \mathbf{X}^+$, all coefficients in $M_\lambda^\mu(q)$ are nonnegative, but this is not true for general μ. Indeed, for nondominant μ, it may happen that $M_\lambda^\mu(q)$ is nonzero but $M_\lambda^\mu(1) = 0$.

Lemma 5.6 ([27, Lemma 6.1]). *Let* $\lambda \in \mathbf{X}$. *We have*

$$\operatorname{ch} A_\lambda = \sum_{\mu \in \mathbf{X}^+,\ \mu \succeq \lambda} M_\mu^\lambda(q)\chi(\mu).$$

The proof of this lemma in [27] seems to assume that $\Bbbk = \mathbb{C}$, but this actually plays no role in the proof.

Theorem 5.7. *Let* $\lambda, \mu \in \mathbf{X}^+$. *As a* G*-module,* A_λ *has a good filtration, and*

$$\sum_{n \ge 0} [A_\lambda : H^0(\mu)\langle -2n\rangle] q^n = M_\mu^\lambda(q).$$

Proof. For dominant λ, recall that A_λ is actually a coherent sheaf on $\mathcal{N}$. The fact that A_λ has a good filtration is due to [29]. The character of any G-module with a good filtration is, of course, a linear combination of various $\chi(\mu)$ (with $\mu \in \mathbf{X}^+$), and the coefficient of $\chi(\mu)$ is the multiplicity of $H^0(\mu)$. □

If M is a G-module with a good filtration, then we also have

$$\dim \operatorname{Hom}_G(V(\mu), M) = [M : H^0(\mu)].$$

This observation can be used to reformulate the preceeding theorem: for $\lambda, \mu \in \mathbf{X}^+$,

$$M_\mu^\lambda(q) = \sum_{n \ge 0} \dim \operatorname{Hom}_{G \times \mathbb{G}_\mathrm{m}}(V(\mu)\langle -2n\rangle, A_\lambda) q^n.$$

This can be generalized to arbitrary $\lambda \in \mathbf{X}$, using Lemma 5.6 and the fact that ch gives an embedding of the Grothendieck group of $\mathrm{PCoh}^{G \times \mathbb{G}_\mathrm{m}}(\mathcal{N})$ in $\mathbb{Z}[\mathbf{X}][[q]][q^{-1}]$.

Theorem 5.8. *Let* $\lambda \in \mathbf{X}$. *For any* $\mu \in \mathbf{X}^+$, *we have*

$$\sum_{i \ge 0} (-1)^i \sum_{n \ge 0} \dim \operatorname{Ext}^i_{G \times \mathbb{G}_\mathrm{m}}(V(\mu)\langle -2n\rangle, A_\lambda) q^n = M_\mu^\lambda(q).$$

S. Riche has communicated to me another proof of this fact, based on Broer's treatment [26] of the $M_\mu^\lambda(q)$ rather than Brylinski's.

We conclude this subsection with a sketch of the proof of Proposition 4.9. We begin with a lemma about characters of Andersen–Jantzen sheaves.

Lemma 5.9. *Let* $\lambda \in \mathbf{X}^+$. *We have*

$$\chi(\lambda) = \left. \left(\sum_{w \in W} (-1)^{\ell(w)} \operatorname{ch} A_{\lambda + \rho - w\rho} \right) \right|_{q=1}.$$

Proof. Fix a dominant weight $\mu \in \mathbf{X}^+$, and consider the following calculation:

$$\sum_{w\in W}(-1)^{\ell(w)}M_\mu^{\lambda+\rho-w\rho}(q) = \sum_{w,v\in W}(-1)^{\ell(w)}(-1)^{\ell(v)}P_{v(\mu+\rho)-(\lambda+2\rho-w\rho)}(q)$$
$$= \sum_{v\in W}(-1)^{\ell(v)}\left(\sum_{w\in W}(-1)^{\ell(w)}P_{w\rho-(\lambda+\rho-v(\mu+\rho)+\rho)}(q)\right)$$
$$= \sum_{v\in W}(-1)^{\ell(v)}M_0^{\lambda+\rho-v(\mu+\rho)}(q).$$

Now evaluate this at $q = 1$. We have $M_0^{\lambda+\rho-v(\mu+\rho)}(1) = 0$ unless $\lambda + \rho - v(\mu + \rho) = 0$. But since λ and μ are both dominant, $\lambda + \rho$ and $\mu + \rho$ are both dominant regular, and the condition $\lambda + \rho - v(\mu + \rho) = 0$ implies that $v = 1$ and $\mu = \lambda$. Thus,

$$\left.\left(\sum_{w\in W}(-1)^{\ell(w)}M_\mu^{\lambda+\rho-w\rho}(q)\right)\right|_{q=1} = \begin{cases}1 & \text{if } \mu = \lambda,\\ 0 & \text{otherwise.}\end{cases}$$

The left-hand side is the coefficient of $\chi(\mu)$ in $\left(\sum(-1)^{\ell(w)}\operatorname{ch} A_{\lambda+\rho-w\rho}\right)\big|_{q=1}$. □

Proof (sketch for Proposition 4.9). We first describe a way to interpret the expression "$(\operatorname{ch}\mathfrak{IC}_\mu)|_{q=1}$" for arbitrary $\mu \in \mathbf{X}^+$. Although there are typically infinitely many q^n with nonzero coefficient in $\operatorname{ch}\mathfrak{IC}_\mu$, it can be shown that there is a (possibly infinite) sum

$$\operatorname{ch}\mathfrak{IC}_\mu = \sum_{\nu\in\mathbf{X}^+} c_\nu(q)\chi(\nu),$$

where each $c_\nu(q)$ is a Laurent polynomial in $\mathbb{Z}[q, q^{-1}]$. The collection of integers $\{c_\nu(1)\}_{\nu\in\mathbf{X}^+}$ can be regarded as a function $\mathbf{X}^+ \to \mathbb{Z}$. In an abuse of notation, we let $(\operatorname{ch}\mathfrak{IC}_\mu)|_{q=1}$ denote that function.

A key point is that in the space of functions $\mathbf{X}^+ \to \mathbb{Z}$, the various $\{(\operatorname{ch}\mathfrak{IC}_\mu)|_{q=1}\}$ remain linearly independent. (This fact was explained to me in 1999 by David Vogan. It is closely related to the ideas in [41, Lecture 8].)

Since $\mathfrak{IC}_{\lambda+2\rho}\langle\ell(w_0)\rangle$ occurs as a composition factor in $A_{\lambda+2\rho}$ but not in any $A_{\lambda+\rho-w\rho}$ with $w \neq w_0$, Lemma 5.9 implies that for some integers a_μ, we have

$$\chi(\lambda) = (-1)^{\ell(w_0)}(\operatorname{ch}\mathfrak{IC}_{\lambda+2\rho})|_{q=1} + \sum_{\mu<\lambda+2\rho} a_\mu(\operatorname{ch}\mathfrak{IC}_\mu)|_{q=1}. \tag{14}$$

On the other hand, any simple G-representation $L(\nu)$ gives rise to a coherent intersection cohomology complex $\mathfrak{IC}(C_0, L(\nu))$. For some $b_\mu \in \mathbb{Z}$, we have

$$\chi(\lambda) = (-1)^{\ell(w_0)}(\operatorname{ch}\mathfrak{IC}(C_0, L(\lambda)))|_{q=1} + \sum_{\mu<\lambda} b_\mu(\operatorname{ch}\mathfrak{IC}(C_0, L(\mu)))|_{q=1}. \tag{15}$$

An induction argument comparing (14) and (15) yields the result. □

5.3 Socles and morphisms

In this subsection, we study the socles of standard objects, the cosocles of costandard objects, and Hom-spaces between them. The results for $\mathrm{PCoh}^{G\times\mathbb{G}_m}(\mathcal{N})$ strongly resemble classical facts about category $\mathcal{O}$ for a complex semisimple Lie algebra, or about perverse sheaves on a flag variety (see, for instance, [15, §2.1]). In the case of $\mathrm{ExCoh}^{G\times\mathbb{G}_m}(\widetilde{\mathcal{N}})$, the corresponding picture is partly conjectural.

Proposition 5.10. *Let $\lambda \in \mathbf{X}^+$.*

1. *The socle of $\overline{\Delta}_\lambda$ is isomorphic to $\mathfrak{IC}_{\mathsf{m}(\lambda)}^{+}\langle -(2\rho^\vee,\lambda)+\delta_\lambda^*\rangle$, and the cokernel of $\mathfrak{IC}_{\mathsf{m}(\lambda)}^{+}\langle -(2\rho^\vee,\lambda)+\delta_\lambda^*\rangle \hookrightarrow \overline{\Delta}_\lambda$ contains no composition factor of the form $\mathfrak{IC}_\mu\langle m\rangle$ with $\mu \in \mathbf{X}_{\mathrm{min}}$.*
2. *The cosocle of $\overline{\nabla}_\lambda$ is isomorphic to $\mathfrak{IC}_{\mathsf{m}(\lambda)}^{+}\langle (2\rho^\vee,\lambda)-\delta_\lambda^*\rangle$, and the kernel of $\overline{\nabla}_\lambda \twoheadrightarrow \mathfrak{IC}_{\mathsf{m}(\lambda)}^{+}\langle (2\rho^\vee,\lambda)-\delta_\lambda^*\rangle$ contains no composition factor of the form $\mathfrak{IC}_\mu\langle m\rangle$ with $\mu \in \mathbf{X}_{\mathrm{min}}$.*

Proof. Because $\overline{\nabla}_\lambda$ is a coherent sheaf, the local description of $\mathrm{PCoh}^{G\times\mathbb{G}_m}(\mathcal{N})$ from Section 4.3 implies that it has no quotient supported on $\mathcal{N}\smallsetminus\mathcal{N}_{\mathrm{reg}}$. (See [17, Lemma 6] or [10, Lemma 4.1] for details.) Therefore, its cosocle must contain only composition factors of the form $\mathfrak{IC}(\mathcal{N}_{\mathrm{reg}},\mathcal{E})$. The claims about $\overline{\nabla}_\lambda$ then follow from Propositions 4.8 and 5.4. Finally, we apply Serre–Grothendieck duality to deduce the claims about $\overline{\Delta}_\lambda$. □

Lemma 5.11. *Let $\lambda,\mu \in \mathbf{X}$. We have*

$$\dim \mathrm{Hom}(\mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda),\mathcal{O}_{\widetilde{\mathcal{N}}}(\mu)\langle n\rangle) = \begin{cases} 1 & \text{if } \lambda \succeq \mu \text{ and } n=(2\rho^\vee,\lambda-\mu),\\ 0 & \text{otherwise.}\end{cases}$$

Proof. We have already seen in (2) that this Hom-group vanishes unless $\lambda \succeq \mu$. Assume henceforth that $\lambda \succeq \mu$. We may also assume without loss of generality that $\mu = 0$. Because $\mathcal{O}_{\widetilde{\mathcal{N}}}$ is a torsion-free coherent sheaf, the restriction map

$$\mathrm{Hom}(\mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda),\mathcal{O}_{\widetilde{\mathcal{N}}}\langle n\rangle) \to \mathrm{Hom}(\mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda)|_{\widetilde{\mathcal{N}}_{\mathrm{reg}}},\mathcal{O}_{\widetilde{\mathcal{N}}}\langle n\rangle|_{\widetilde{\mathcal{N}}_{\mathrm{reg}}}) \tag{16}$$

is injective. The latter is a Hom-group between two equivariant line bundles on a $(G\times\mathbb{G}_m)$-orbit. This group has dimension 1 if those line bundles are isomorphic, and 0 otherwise. In particular, $\mathrm{Hom}(\mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda)|_{\widetilde{\mathcal{N}}_{\mathrm{reg}}},\mathcal{O}_{\widetilde{\mathcal{N}}}\langle n\rangle|_{\widetilde{\mathcal{N}}_{\mathrm{reg}}})$ can be nonzero for at most one value of n, and hence likewise for $\mathrm{Hom}(\mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda),\mathcal{O}_{\widetilde{\mathcal{N}}}\langle n\rangle)$.

Note that if $\mathrm{Hom}(\mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda_1),\mathcal{O}_{\widetilde{\mathcal{N}}}\langle n_1\rangle)$ and $\mathrm{Hom}(\mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda_2),\mathcal{O}_{\widetilde{\mathcal{N}}}\langle n_2\rangle)$ are known to be nonzero (and hence 1-dimensional), then taking their tensor product shows that

$$\mathrm{Hom}(\mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda_1+\lambda_2),\mathcal{O}_{\widetilde{\mathcal{N}}}\langle n_1+n_2\rangle)$$

is nonzero. Therefore, we can reduce to the case where λ is a simple positive root, say α. Note that $(2\rho^\vee, \alpha) = 2$. Thus, to finish the proof, it suffices to exhibit a nonzero map $\mathcal{O}_{\widetilde{\mathcal{N}}}(\alpha) \to \mathcal{O}_{\widetilde{\mathcal{N}}}\langle 2\rangle$. We have seen such a map in (11). ☐

Theorem 5.12. *Let $\lambda, \mu \in \mathbf{X}^+$. We have*

$$\dim \operatorname{Hom}(\overline{\nabla}_\lambda, \overline{\nabla}_\mu\langle n\rangle) = \begin{cases} 1 & \text{if } \lambda \geq \mu \text{ and } n = (2\rho^\vee, \lambda - \mu) - \delta^*_\lambda + \delta^*_\mu, \\ 0 & \text{otherwise.} \end{cases}$$

Proof. It is clear that this Hom-group vanishes if $\lambda \not\geq \mu$. If $\lambda \geq \mu$ but $n \neq (2\rho^\vee, \lambda - \mu) - \delta^*_\lambda + \delta^*_\mu$, then by Proposition 5.10, $\overline{\nabla}_\mu\langle n\rangle$ has no composition factor isomorphic to the cosocle of $\overline{\nabla}_\lambda$, and again the Hom-group vanishes.

Assume henceforth that $\lambda \geq \mu$ and $n = (2\rho^\vee, \lambda - \mu) - \delta^*_\lambda + \delta^*_\mu$. Let $\mathcal{K}$ be the kernel of the map $\overline{\nabla}_\mu\langle n\rangle \to \mathfrak{E}^+_{\mathsf{m}(\mu)}\langle (2\rho^\vee, \lambda) - \delta^*_\lambda\rangle$, and consider the exact sequence

$$\cdots \to \operatorname{Hom}(\overline{\nabla}_\lambda, \mathcal{K}) \to \operatorname{Hom}(\overline{\nabla}_\lambda, \overline{\nabla}_\mu\langle n\rangle) \xrightarrow{c} \operatorname{Hom}(\overline{\nabla}_\lambda, \mathfrak{E}^+_{\mathsf{m}(\mu)}\langle (2\rho^\vee, \lambda) - \delta^*_\lambda\rangle) \to \cdots .$$

The first term vanishes because $\mathcal{K}$ contains no composition factor isomorphic to the cosocle of $\overline{\nabla}_\lambda$. Therefore, the map labeled c is injective. The last term clearly has dimension 1, so $\dim \operatorname{Hom}(\overline{\nabla}_\lambda, \overline{\nabla}_\mu\langle n\rangle) \leq 1$. To finish the proof, it suffices to show that $\operatorname{Hom}(\overline{\nabla}_\lambda, \overline{\nabla}_\mu\langle n\rangle) \neq 0$.

By Lemma 5.11, there is a nonzero map $\mathcal{O}_{\widetilde{\mathcal{N}}}(\lambda)\langle -\delta^*_\lambda\rangle \to \mathcal{O}_{\widetilde{\mathcal{N}}}(\mu)\langle n - \delta^*_\mu\rangle$. Recall from (16) that that map has nonzero restriction to $\widetilde{\mathcal{N}}_{\mathrm{reg}}$. Applying π_*, we obtain a map $\overline{\nabla}_\lambda \to \overline{\nabla}_\mu\langle n\rangle$ that is nonzero, because its restriction to $\mathcal{N}_{\mathrm{reg}}$ is nonzero. ☐

It is likely that statements of a similar flavour hold in the exotic case. Corollary 5.5 lets us predict what the socles of standard objects and cosocles of costandard objects should look like. In Section 6, we will confirm the following statement for $G = \mathrm{SL}_2$.

Conjecture 5.13. Let $\lambda \in \mathbf{X}$.

1. The socle of $\widehat{\Delta}_\lambda$ is isomorphic to $\mathfrak{E}_{\bar{\mathsf{m}}(\lambda)}\langle -(2\rho^\vee, \mathsf{dom}(\lambda)) + \delta_\lambda\rangle$, and the cokernel of $\mathfrak{E}_{\bar{\mathsf{m}}(\lambda)}\langle -(2\rho^\vee, \mathsf{dom}(\lambda)) + \delta_\lambda\rangle \hookrightarrow \widehat{\Delta}_\lambda$ contains no composition factor of the form $\mathfrak{E}_\mu\langle m\rangle$ with $\mu \in -\mathbf{X}_{\min}$.
2. The cosocle of $\widehat{\nabla}_\lambda$ is isomorphic to $\mathfrak{E}_{\bar{\mathsf{m}}(\lambda)}\langle (2\rho^\vee, \mathsf{dom}(\lambda)) - \delta_\lambda\rangle$, and the kernel of $\widehat{\nabla}_\lambda \twoheadrightarrow \mathfrak{E}_{\bar{\mathsf{m}}(\lambda)}\langle (2\rho^\vee, \mathsf{dom}(\lambda)) - \delta_\lambda\rangle$ contains no composition factor of the form $\mathfrak{E}_\mu\langle m\rangle$ with $\mu \in -\mathbf{X}_{\min}$.

It may be possible to prove this conjecture using the affine braid group technology developed in [24, 35]. Below is an outline of another possible approach:

1. Consider the pair of functors

$$D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}}) \underset{\Pi^\alpha}{\overset{\Pi_\alpha}{\rightleftarrows}} D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}}_\alpha)$$

given by $\Pi_\alpha(\mathcal{F}) = \pi_{\alpha *} i_\alpha^*(\mathcal{F}(\widehat{\rho}-\alpha))$ and $\Pi^\alpha(\mathcal{F}) = (i_{\alpha *}\pi_\alpha^*\mathcal{F})(-\widehat{\rho})\langle 1\rangle$, where $\widehat{\rho}$ is as in Section 2.4. Note that $\Psi_\alpha \cong \Pi^\alpha \circ \Pi_\alpha$. Check that Π_α is left adjoint to $\Pi^\alpha\langle -1\rangle[1]$ and right adjoint to $\Pi^\alpha\langle 1\rangle[1]$.
2. Define an "exotic t-structure" on $D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}}_\alpha)$. Its heart should be a graded quasihereditary category whose standard (resp. costandard) objects are $\Pi_\alpha(\widehat{\Delta}_\lambda)$ (resp. $\Pi_\alpha(\widehat{\nabla}_\lambda)$) with $\lambda \prec s_\alpha\lambda$. The functor Π^α should be t-exact.
3. Now imitate the strategy of [15, §2.1] or [25, Lemma 4.4.7], with the functors Π_α and Π^α playing the role of push-forward or pullback along the projection from the full flag variety to a partial flag variety associated to a simple root.

One would likely have to show along the way that the distinguished triangles of Proposition 2.5(4) are actually short exact sequences in $\mathrm{ExCoh}^{G\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$:

$$\begin{array}{c} 0 \to \widehat{\Delta}_{s_\alpha\lambda} \to \widehat{\Delta}_\lambda\langle 1\rangle \to \Psi_\alpha(\widehat{\Delta}_\lambda)[1] \to 0, \\ 0 \to \Psi_\alpha(\widehat{\nabla}_\lambda)[-1] \to \widehat{\nabla}_\lambda\langle -1\rangle \to \widehat{\nabla}_{s_\alpha\lambda} \to 0 \end{array} \qquad \text{if } s_\alpha\lambda \prec \lambda. \tag{17}$$

There should also be an equivalence like that in Theorem 4.1 relating $\mathrm{ExCoh}^G(\widetilde{\mathcal{N}}_\alpha)$ to Iwahori–Whittaker sheaves on a partial affine flag variety $\check{G}_{\mathbf{K}}/J_\alpha$, where $J_\alpha \subset \check{G}_{\mathbf{O}}$ is the parahoric subgroup corresponding to α.

If these expectations hold, we would obtain the following analogue of Theorem 5.12.

Theorem 5.14. *Assume that Conjecture 5.13 holds, and that the sequences in* (17) *are exact. Let $\lambda, \mu \in \mathbf{X}$. Then* $\dim \mathrm{Hom}(\widehat{\nabla}_\lambda, \widehat{\nabla}_\mu\langle n\rangle) \le 1$*, and*

$$\dim \mathrm{Hom}(\widehat{\nabla}_\lambda, \widehat{\nabla}_\mu\langle n\rangle) = 0 \qquad \begin{array}{l} \text{if } \lambda \not\geq \mu, \text{ or} \\ \text{if } n \neq (2\rho^\vee, \mathsf{dom}(\lambda) - \mathsf{dom}(\mu)) - \delta_\lambda + \delta_\mu. \end{array}$$

If $\lambda \in \mathbf{X}^+$ and $\lambda \geq \mu$, then $\dim \mathrm{Hom}(\widehat{\nabla}_\lambda, \widehat{\nabla}_\mu\langle (2\rho^\vee, \lambda - \mathsf{dom}(\mu)) + \delta_\mu\rangle) = 1$.

In contrast with Theorem 5.12, we do not expect the Hom-group to be nonzero for arbitrary weights $\lambda \geq \mu$. Rather, it should only be nonzero when μ is smaller than λ in the finer partial order coming from the geometry of $\mathcal{F}l$ or $\mathcal{G}r$. See, for instance, [19, Footnote 5].

Proof. To show that this Hom-group vanishes if $\lambda \not\geq \mu$ or $n \neq (2\rho^\vee, \mathsf{dom}(\lambda) - \mathsf{dom}(\mu)) - \delta_\lambda + \delta_\mu$, and that it always has dimension at most 1, one can repeat the arguments from the proof of Theorem 5.12.

Suppose now that $\lambda \in \mathbf{X}^+$, $\lambda \geq \mu$, and $n = \langle 2\rho^\vee, \lambda - \mathsf{dom}(\mu)\rangle + \delta_\mu$. We must show that $\mathrm{Hom}(\widehat{\nabla}_\lambda, \widehat{\nabla}_\mu\langle n\rangle) \neq 0$. If μ happens to be dominant as well, then the claim follows from Lemma 5.11. Otherwise, note that $\lambda \geq \mathsf{dom}(\mu) \geq \mu$. By the previous case, we have a nonzero map $u : \widehat{\nabla}_\lambda \to \widehat{\nabla}_{\mathsf{dom}(\mu)}\langle\langle 2\rho^\vee, \lambda - \mathsf{dom}(\mu)\rangle\rangle$. That map must be surjective, as can be seen by considering cosocles. Next, the exact sequences in (17) imply that there is a surjective map $v : \widehat{\nabla}_{\mathsf{dom}(\mu)} \to \widehat{\nabla}_\mu\langle\delta_\mu\rangle$. The composition $v\langle\langle 2\rho^\vee, \lambda - \mathsf{dom}(\mu)\rangle\rangle \circ u$ is the desired nonzero map $\widehat{\nabla}_\lambda \to \widehat{\nabla}_\mu\langle n\rangle$.
□

6 Explicit computations for $\mathbf{SL_2}$

For the remainder of the paper, we focus on $G = \mathrm{SL}_2$. In keeping with the assumptions of Section 2.1, we assume that the characteristic of $\Bbbk$ is not 2. We identify $\mathbf{X} = \mathbb{Z}$ and $\mathbf{X}^+ = \mathbb{Z}_{\geq 0}$. Note that neither of the partial orders of Section 2.1 agrees with the usual order on $\mathbb{Z}$. In this section, $\leq$ will mean the usual order on $\mathbb{Z}$. We write $\preceq_{\mathbf{X}}$ and $\leq_{\mathbf{X}}$ for those from Section 2.1. Thus, for $n, m \in \mathbb{Z}$, we have

$$n \preceq_{\mathbf{X}} m \text{ if } m - n \in 2\mathbb{Z}_{\geq 0},$$
$$n \leq_{\mathbf{X}} m \text{ if } |n| < |m|, \text{ or else if } |n| = |m| \text{ and } n \leq m.$$

6.1 Standard and costandard exotic sheaves

Throughout, we will work in terms of the left-hand side of the equivalence (1). Typically, “writing down an object of $\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})$” will mean writing down the underlying graded B-module for an object of $\mathrm{Coh}^{B\times\mathbb{G}_\mathrm{m}}(\mathfrak{u})$. For instance, the structure sheaf $\mathcal{O}_{\widetilde{\mathcal{N}}}$ looks like

$$\begin{array}{rcccccccccccc} \text{grading degree:} & \cdots & -2 & -1 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \cdots \\ B\text{-representation:} & \cdots & - & - & \Bbbk_0 & - & \Bbbk_2 & - & \Bbbk_4 & - & \Bbbk_6 & - & \Bbbk_8 & \cdots \end{array}$$

Of course, an indecomposable object of $\mathrm{Coh}^{B\times\mathbb{G}_\mathrm{m}}(\mathfrak{u})$ must be concentrated either in even degrees or in odd degrees. In the computations below, we will often omit grading labels for degrees in which the given module vanishes.

We will also make use of notation from Section 2.4 such as $\widetilde{\mathcal{N}}_\alpha$, P_α, etc., where $\alpha = 2$ is the unique positive root of G. Note that $\widetilde{\mathcal{N}}_\alpha$ can be identified with the zero section $G/B \subset \widetilde{\mathcal{N}}$. As in (1), we have an equivalence

$$\mathrm{Coh}^{B\times\mathbb{G}_\mathrm{m}}(\mathrm{pt}) \cong \mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}}_\alpha).$$

The composition of $\pi_{\alpha*} : D^\mathrm{b}\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}}_\alpha) \to D^\mathrm{b}\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\mathrm{pt})$ with this equivalence is the induction functor $R\,\mathrm{ind}_B^G : D^\mathrm{b}\mathrm{Coh}^{B\times\mathbb{G}_\mathrm{m}}(\mathrm{pt}) \to D^\mathrm{b}\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\mathrm{pt})$.

If V is a B-representation, then $i_{\alpha*}V$ denotes the object

$$i_{\alpha*}V \cong \begin{array}{cccccccc} 0 & 2 & 4 & 6 & 8 & 10 & \cdots \\ V & - & - & - & - & - & \cdots \end{array}$$

in $\mathrm{Coh}^{B\times\mathbb{G}_{\mathrm{m}}}(\mathfrak{u})$. In this section, we will generally suppress the notation for res^G_B and tensor products. For instance, in the following statement, $H^0(-n-1)\Bbbk_{-1}$ should be understood as the B-representation $\mathrm{res}^G_B\, H^0(-n-1) \otimes \Bbbk_{-1}$.

Lemma 6.1. *If $n < 0$, then*

$$\begin{aligned} \Psi_\alpha(\mathcal{O}_{\widetilde{\mathcal{N}}}(n)) &\cong i_{\alpha*}(V(-n-1)\Bbbk_{-1})\langle 1\rangle[-1] \\ &\cong \left(\begin{array}{ccccccc} -1 & 1 & 3 & 5 & 7 & 9 & \cdots \\ V(-n-1)\Bbbk_{-1} & - & - & - & - & - & \cdots \end{array} \right)[-1]. \end{aligned}$$

If $n > 0$, then

$$\Psi_\alpha(\mathcal{O}_{\widetilde{\mathcal{N}}}(n)) \cong i_{\alpha*}(H^0(n-1)\Bbbk_{-1})\langle 1\rangle \cong \begin{array}{ccccccc} -1 & 1 & 3 & 5 & 7 & 9 & \cdots \\ H^0(n-1)\Bbbk_{-1} & - & - & - & - & - & \cdots \end{array}.$$

Finally, $\Psi_\alpha(\mathcal{O}_{\widetilde{\mathcal{N}}}) = 0$.

Proof. Recall that $\Psi_\alpha(\mathcal{O}_{\widetilde{\mathcal{N}}}(n)) \cong i_{\alpha*}\pi_\alpha^*\pi_{\alpha*}i_\alpha^*(\mathcal{O}_{\widetilde{\mathcal{N}}}(n-1)) \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}(-1)\langle 1\rangle$. In particular, we have

$$\pi_{\alpha*}i_\alpha^*(\mathcal{O}_{\widetilde{\mathcal{N}}}(n-1)) \cong R\,\mathrm{ind}_B^G\,\Bbbk_{n-1} \cong \begin{cases} H^0(n-1) & \text{if } n > 0, \\ V(-n-1)[-1] & \text{if } n < 0, \\ 0 & \text{if } n = 0. \end{cases}$$

The result follows. □

Proposition 6.2 (Costandard exotic sheaves). *If $n < 0$, then*

$$\widehat{\nabla}_n \cong \begin{array}{ccccccc} -1 & 1 & 3 & 5 & 7 & 9 & \cdots \\ H^0(-n-1)\Bbbk_{-1} & \Bbbk_{-n} & \Bbbk_{-n+2} & \Bbbk_{-n+4} & \Bbbk_{-n+6} & \Bbbk_{-n+8} & \cdots \end{array}$$

If $n \geq 0$, then

$$\widehat{\nabla}_n \cong \begin{array}{ccccccc} 0 & 2 & 4 & 6 & 8 & 10 & \cdots \\ \Bbbk_n & \Bbbk_{n+2} & \Bbbk_{n+4} & \Bbbk_{n+6} & \Bbbk_{n+8} & \Bbbk_{n+10} & \cdots \end{array}.$$

Proof. For dominant weights $n \geq 0$, this is just a restatement of the fact from (3) that $\widehat{\nabla}_n \cong \mathcal{O}_{\widetilde{\mathcal{N}}}(n)$. Suppose now that $n < 0$, and consider the distinguished triangle $\widehat{\nabla}_{-n}\langle -1\rangle \to \widehat{\nabla}_n \to \Psi_\alpha(\widehat{\nabla}_{-n}) \to$ from Proposition 2.5(4). We have already determined the first term, and the last term is given in Lemma 6.1. Combining those gives the result. □

Proposition 6.3 (Standard exotic sheaves). *If $n \le 0$, then $\widehat{\Delta}_n$ is a coherent sheaf, given by*

$$\widehat{\Delta}_0 \cong \begin{matrix} 0 & 2 & 4 & 6 & 8 & 10 & \cdots \\ \Bbbk_0 & \Bbbk_2 & \Bbbk_4 & \Bbbk_6 & \Bbbk_8 & \Bbbk_{10} & \cdots \end{matrix}$$

$$\widehat{\Delta}_n \cong \begin{matrix} -1 & 1 & 3 & 5 & 7 & 9 & \cdots \\ \Bbbk_n & \Bbbk_{n+2} & \Bbbk_{n+4} & \Bbbk_{n+6} & \Bbbk_{n+8} & \Bbbk_{n+10} & \cdots \end{matrix} \qquad \text{if } n < 0.$$

If $n > 0$, there is a distinguished triangle $\mathcal{H}^0(\widehat{\Delta}_n) \to \widehat{\Delta}_n \to \mathcal{H}^1(\widehat{\Delta}_n)[-1] \to$ with

$$\begin{matrix} & -2 & 0 & 2 & 4 & 6 & 8 & \cdots \\ \mathcal{H}^1(\widehat{\Delta}_n) \cong & V(n-1)\Bbbk_{-1} & - & - & - & - & - & \cdots \\ \mathcal{H}^0(\widehat{\Delta}_n) \cong & - & \Bbbk_{-n} & \Bbbk_{-n+2} & \Bbbk_{-n+4} & \Bbbk_{-n+6} & \Bbbk_{-n+8} & \cdots \end{matrix}$$

Proof. For $n \le 0$, this is again just a restatement of (3), while for $n > 0$, it follows from the distinguished triangle $\widehat{\Delta}_{-n}\langle -1\rangle \to \widehat{\Delta}_n \to \Psi(\widehat{\Delta}_{-n}) \to$ of Proposition 2.5(4). □

6.2 Auxiliary calculations

In this subsection, we collect a number of minor results that will be needed later for the study of simple and tilting objects.

Lemma 6.4. *For any $V, V' \in \mathrm{Rep}(G)$, we have $R\mathrm{Hom}_{\mathrm{Rep}(B)}(V, V'\Bbbk_{-1}) = 0$.*

Proof. By adjunction, $R\mathrm{Hom}(V, V'\Bbbk_{-1}) \cong R\mathrm{Hom}_{\mathrm{Rep}(G)}(V, R\,\mathrm{ind}_B^G(V'\Bbbk_{-1}))$. The latter vanishes because $R\,\mathrm{ind}_B^G(V'\Bbbk_{-1}) \cong V' \otimes R\,\mathrm{ind}_B^G \Bbbk_{-1} = 0$. □

Lemma 6.5. *For any $V \in \mathrm{Rep}(G)$, $i_{\alpha *}(V\Bbbk_{-2})$ lies in $\mathrm{ExCoh}^{G\times\mathbb{G}_{\mathrm{m}}}(\widetilde{\mathcal{N}})$.*

Proof. It suffices to show that

$$\underline{\mathrm{Hom}}(\widehat{\Delta}_m[-k], i_{\alpha *}V\Bbbk_{-2}) = \underline{\mathrm{Hom}}(i_{\alpha *}V\Bbbk_{-2}, \widehat{\nabla}_m[k]) = 0 \qquad \text{for all } k < 0.$$

The vanishing of the latter is obvious, since $i_{\alpha *}V\Bbbk_{-2}$ and $\widehat{\nabla}_m$ are both coherent sheaves. Likewise, the vanishing of the former is obvious when $m \le 0$, or when $k < -1$. When $m > 0$ and $k = -1$, using Lemma 6.4, we have

$$\begin{aligned} \underline{\mathrm{Hom}}(\widehat{\Delta}_m[1], i_{\alpha *}V\Bbbk_{-2}) &\cong \underline{\mathrm{Hom}}(i_{\alpha *}V(m-1)\Bbbk_{-1}\langle 2\rangle, i_{\alpha *}V\Bbbk_{-2}) \\ &\cong \mathrm{Hom}_{\mathrm{Rep}(B)}(V(m-1)\Bbbk_{-1}, V\Bbbk_{-2})\langle -2\rangle \\ &\cong \mathrm{Hom}_{\mathrm{Rep}(B)}(V(m-1), V\Bbbk_{-1})\langle -2\rangle = 0. \end{aligned}$$ □

Lemma 6.6. *For $n > 0$, there are short exact sequences of B-representations*

$$0 \to H^0(n-1)\Bbbk_{-1} \to H^0(n) \to \Bbbk_n \to 0,$$
$$0 \to \Bbbk_{-n} \to V(n) \to V(n-1)\Bbbk_1 \to 0.$$

Proof. This can be checked by direct computation using, say, the realization of $H^0(n)$ as the space of homogeneous polynomials of degree n on $\mathbb{A}^2$. □

Lemma 6.7. *For $n \leq -2$, there is a short exact sequence in $\mathrm{ExCoh}^{G\times\mathbb{G}_m}(\widetilde{\mathcal{N}})$:*

$$0 \to i_{\alpha*}H^0(-n-2)\Bbbk_{-2}\langle 1\rangle \to \widehat{\nabla}_n \to \widehat{\nabla}_{-n-2}\langle 1\rangle \to 0.$$

Proof. Note that for any G-representation V, there are natural isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathrm{Coh}^{G\times\mathbb{G}_m}(\widetilde{\mathcal{N}})}(V\otimes \mathcal{O}_{\widetilde{\mathcal{N}}}\langle k\rangle, \mathcal{O}_{\widetilde{\mathcal{N}}}(m)) &\cong \mathrm{Hom}_{\mathrm{Rep}(B\times\mathbb{G}_m)}(V\langle k\rangle, \Bbbk_m\otimes \Bbbk[u]) \\ &\cong \begin{cases}\mathrm{Hom}_{\mathrm{Rep}(B)}(V,\Bbbk_{m-k}) & \text{if } k\leq 0 \text{ and } k \text{ is even,}\\ 0 & \text{otherwise.}\end{cases}\end{aligned} \tag{18}$$

For instance, we can see in this way that

$$\begin{gathered}\mathrm{Hom}(H^0(-n-1)\otimes\mathcal{O}_{\widetilde{\mathcal{N}}}(1)\langle -2\rangle, \mathcal{O}_{\widetilde{\mathcal{N}}}(-n-2)),\\ \mathrm{Hom}(H^0(-n-1)\otimes\mathcal{O}_{\widetilde{\mathcal{N}}}(-1), \mathcal{O}_{\widetilde{\mathcal{N}}}(-n-2))\end{gathered} \tag{19}$$

are both 1-dimensional. Consider the following exact sequence, induced by (11):

$$\begin{aligned}0 \to H^0(-n-1)\otimes\mathcal{O}_{\widetilde{\mathcal{N}}}(1)\langle -2\rangle \xrightarrow{i} H^0(-n-1)\otimes\mathcal{O}_{\widetilde{\mathcal{N}}}(-1)&\\ \to i_{\alpha*}(H^0(-n-1)\Bbbk_{-1}) \to 0.&\end{aligned} \tag{20}$$

The map i induces an isomorphism between the two Hom-groups in (19).

By similar reasoning, we find that

$$\begin{aligned}&\mathrm{Ext}^1(H^0(-n-1)\otimes\mathcal{O}_{\widetilde{\mathcal{N}}}(-1), \mathcal{O}_{\widetilde{\mathcal{N}}}(-n-2))\\ &\quad\cong \mathrm{Ext}^1_{\mathrm{Rep}(B)}(H^0(-n-1)\Bbbk_{-1}, \Bbbk_{-n-2}) \cong \mathrm{Ext}^1_{\mathrm{Rep}(B)}(H^0(-n-1), \Bbbk_{-n-1})\\ &\quad\cong \mathrm{Ext}^1_{\mathrm{Rep}(G)}(H^0(-n-1), H^0(-n-1)) = 0.\end{aligned} \tag{21}$$

Now apply $\mathrm{Ext}^\bullet(-, \mathcal{O}_{\widetilde{\mathcal{N}}}(-n-2))$ to (20) to obtain a long exact sequence. From (19) and (21), we deduce that

$$\mathrm{Ext}^1(i_{\alpha*}(H^0(-n-1)\Bbbk_{-1}), \mathcal{O}_{\widetilde{\mathcal{N}}}(-n-2)) = 0. \tag{22}$$

Now apply $\mathrm{Hom}(-, \mathcal{O}_{\widetilde{\mathcal{N}}}(-n-2)\langle 1\rangle)$ to the distinguished triangle

$$\mathcal{O}_{\widetilde{\mathcal{N}}}(-n)\langle -1\rangle \to \widehat{\nabla}_n \to i_{\alpha*}H^0(-n-1)\Bbbk_{-1}\langle 1\rangle \to \tag{23}$$

from Proposition 2.5(4). It is clear that $\mathrm{Hom}(i_{\alpha*}H^0(-n-1)\Bbbk_{-1}\langle 1\rangle, \mathcal{O}_{\widetilde{\mathcal{N}}}(-n-2)\langle 1\rangle)$ vanishes. Combining this with (22), we find that

$$\mathrm{Hom}(\widehat{\nabla}_n, \mathcal{O}_{\widetilde{\mathcal{N}}}(-n-2)\langle 1\rangle) \xrightarrow{\sim} \mathrm{Hom}(\mathcal{O}_{\widetilde{\mathcal{N}}}(-n)\langle -1\rangle, \mathcal{O}_{\widetilde{\mathcal{N}}}(-n-2)\langle 1\rangle)$$

is an isomorphism. The latter is 1-dimensional (by (18)), so the former is as well.

We have constructed a nonzero map $h : \widehat{\nabla}_n \to \widehat{\nabla}_{-n-2}\langle 1\rangle$. We claim that as a map of coherent sheaves, h is surjective. By construction, its image at least contains the image of $\widehat{\nabla}_{-n}\langle -1\rangle \cong \mathcal{O}_{\widetilde{\mathcal{N}}}(-n)\langle -1\rangle$, i.e., the submodule containing all homogeneous elements in degrees ≥ 1. The only question is whether h is surjective in grading degree -1. If it were not, then $\widehat{\nabla}_{-n}\langle -1\rangle$ would be a quotient of $\widehat{\nabla}_n$ as a coherent sheaf. This would imply the splitting of the distinguished triangle (23), contradicting the indecomposability of $\widehat{\nabla}_n$. Thus, h is surjective.

Restricting h to the space of homogeneous elements of degree -1, we get a surjective map of B-representations $H^0(-n-1)\Bbbk_{-1} \to \Bbbk_{-n-2}$. Lemma 6.6 identifies the kernel of that map for us. We therefore have a short exact sequence of coherent sheaves

$$0 \to i_{\alpha*}H^0(-n-2)\Bbbk_{-2}\langle 1\rangle \to \widehat{\nabla}_n \to \widehat{\nabla}_{-n-2}\langle 1\rangle \to 0.$$

Lemma 6.5 tells us that this is also a short exact sequence in $\mathrm{ExCoh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})$. □

Lemma 6.8. *For $n \leq -2$, there is a short exact sequence in* $\mathrm{ExCoh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})$*:*

$$0 \to \widehat{\Delta}_{-n-2}\langle -1\rangle \to \widehat{\Delta}_n \to i_{\alpha*}V(-n-2)\Bbbk_{-2} \to 0.$$

We omit the proof of this lemma, which is quite similar to Lemma 6.7. Note that in grading degree -1, we have the distinguished triangle of B-representations

$$V(-n-3)\Bbbk_{-1}[-1] \to \Bbbk_n \to V(-n-2)\Bbbk_{-2} \to$$

obtained from Lemma 6.6 by tensoring with $\Bbbk_{-2}$.

Lemma 6.9. *For any $V \in \mathrm{Rep}(G)$, $i_{\alpha*}(V\Bbbk_{-1})[-1]$ lies in* $\mathrm{ExCoh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})$.

Proof. As in Lemma 6.5, it suffices to show that for $k < 0$, we have

$$\underline{\mathrm{Hom}}(\widehat{\Delta}_m[-k], i_{\alpha*}V\Bbbk_{-1}[-1]) = \underline{\mathrm{Hom}}(i_{\alpha*}V\Bbbk_{-1}[-1], \widehat{\nabla}_m[k]) = 0.$$

The vanishing of the former is easily seen in terms of the natural t-structure on $D^\mathrm{b}\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})$. The vanishing of the latter is also clear when $k \leq -2$. If $k = -1$ and $m \geq -1$, then this Hom-group vanishes because $\widehat{\nabla}_m$ is a torsion-free coherent sheaf, while $i_{\alpha*}V\Bbbk_{-1}$ is torsion. Finally, if $k = -1$ and $m \leq -2$, we use Lemma 6.7. Consider the exact sequence

$$\begin{aligned}0 \to \underline{\mathrm{Hom}}(i_{\alpha*}V\Bbbk_{-1}, i_{\alpha*}H^0(-m-2)\Bbbk_{-2}) \to \underline{\mathrm{Hom}}(i_{\alpha*}V\Bbbk_{-1}, \widehat{\nabla}_m)\\ \to \underline{\mathrm{Hom}}(i_{\alpha*}V\Bbbk_{-1}, \widehat{\nabla}_{-m-2}\langle 1\rangle) \to \cdots . \quad (24)\end{aligned}$$

We have already seen that the last term vanishes. The first term is isomorphic to

$$\underline{\mathrm{Hom}}_{\mathrm{Rep}(B\times\mathbb{G}_\mathrm{m})}(V\Bbbk_{-1}, H^0(-m-2)\Bbbk_{-2}) \cong \underline{\mathrm{Hom}}_{\mathrm{Rep}(B\times\mathbb{G}_\mathrm{m})}(V, H^0(-m-2)\Bbbk_{-1}),$$

and this vanishes by Lemma 6.4. So the middle term in (24) vanishes as well, and $i_{\alpha *}V\Bbbk_{-1}\langle 2\rangle[-1]$ lies in $\mathrm{ExCoh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})$, as desired. □

Lemma 6.10. *We have*

$$\pi_*(\widehat{\Delta}_n \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}(1)) \cong \begin{cases} \pi_*(\widehat{\Delta}_{n+1}) & \text{if } n \le -2, \\ \pi_*(\widehat{\Delta}_{-n-1})\langle 1\rangle & \text{if } n \ge -1. \end{cases}$$

$$\pi_*(\widehat{\nabla}_n \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}(1)) \cong \begin{cases} \pi_*(\widehat{\nabla}_{-n-1})\langle 1\rangle & \text{if } n < 0, \\ \pi_*(\widehat{\nabla}_{n+1}) & \text{if } n \ge 0. \end{cases}$$

Proof. If $n \ge 0$, then it is clear that $\widehat{\nabla}_n \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}(1) \cong \widehat{\nabla}_{n+1}$. Similarly, for $n = -1$, we have $\widehat{\nabla}_{-1} \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}(1) \cong \widehat{\nabla}_0\langle 1\rangle$. If $n \le -2$, we use Lemma 6.7 together with the fact that $\pi_*(i_{\alpha *}H^0(-n-2)\Bbbk_{-1}) = 0$ to deduce that

$$\pi_*(\widehat{\nabla}_n \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}(1)) \cong \pi_*(\widehat{\nabla}_{-n-2}\langle 1\rangle \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}(1)) \cong \pi_*(\widehat{\nabla}_{-n-1})\langle 1\rangle.$$

The proof for standard objects is similar. □

6.3 Socles and morphisms

In this subsection, we verify Conjecture 5.13 and the conclusions of Theorem 5.14 for $G = \mathrm{SL}_2$.

Proposition 6.11. *In* $\mathrm{ExCoh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})$, *we have the following short exact sequences for each* $n > 0$*:*

$$0 \to \widehat{\Delta}_{-n} \to \widehat{\Delta}_n\langle 1\rangle \to \Psi_\alpha(\widehat{\Delta}_n)[1] \to 0,$$
$$0 \to \Psi_\alpha(\widehat{\nabla}_n)[-1] \to \widehat{\nabla}_n\langle -1\rangle \to \widehat{\nabla}_{-n} \to 0.$$

Proof. In view of Proposition 2.5, all we need to do is check that $\Psi_\alpha(\widehat{\Delta}_n)[1]$ and $\Psi_\alpha(\widehat{\nabla}_n)[-1]$ lie in $\mathrm{ExCoh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})$. By Lemma 6.1, we have

$$\Psi_\alpha(\widehat{\Delta}_n)[1] \cong \Psi_\alpha(\widehat{\Delta}_{-n})\langle 1\rangle \cong \Psi_\alpha(\mathcal{O}_{\widetilde{\mathcal{N}}}(-n))\langle 2\rangle \cong i_{\alpha *}V(n-1)\Bbbk_{-1}\langle 3\rangle[-1],$$
$$\Psi_\alpha(\widehat{\nabla}_n)[-1] \cong \Psi_\alpha(\mathcal{O}_{\widetilde{\mathcal{N}}}(n))[-1] \cong i_{\alpha *}H^0(n-1)\Bbbk_{-1}\langle 1\rangle[-1].$$

By Lemma 6.9, these both lie in $\mathrm{ExCoh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})$. □

Proposition 6.12. *Let $m, n \in \mathbb{Z}$. Then*

$$\dim \operatorname{Hom}(\widehat{\nabla}_m, \widehat{\nabla}_n\langle k\rangle) = \begin{cases} 1 & \text{if } m \geq_{\mathbf{X}} n \text{ and } k = |m| - |n| - \delta_m + \delta_n, \\ 0 & \text{otherwise.} \end{cases}$$

Moreover, any nonzero map $\widehat{\nabla}_m \to \widehat{\nabla}_n\langle k\rangle$ is surjective.

Proof. The dimension of $\operatorname{Hom}(\widehat{\nabla}_m, \widehat{\nabla}_n\langle k\rangle)$ can be determined by direct computation using Proposition 6.2. Note that each costandard object is generated as a $\Bbbk[\mathfrak{u}]$-module by a single homogeneous component (lying in grading degree 0 or -1). Moreover, the costandard objects associated to dominant weights are free over $\Bbbk[\mathfrak{u}]$. With these observations, the problem of computing $\operatorname{Hom}(\widehat{\nabla}_m, \widehat{\nabla}_n\langle k\rangle)$ can be reduced to that of computing Hom-groups between certain B-representations. The latter is quite straightforward.

We now consider the surjectivity claim. Suppose $m \geq_{\mathbf{X}} n$. If $m \geq 0$, then

$$m \geq_{\mathbf{X}} -m \geq_{\mathbf{X}} m-2 \geq_{\mathbf{X}} -m+2 \geq_{\mathbf{X}} \cdots \geq_{\mathbf{X}} n.$$

Lemma 6.7 and Proposition 6.11 together give us a collection of surjective maps

$$\widehat{\nabla}_m \twoheadrightarrow \widehat{\nabla}_{-m}\langle 1\rangle \twoheadrightarrow \widehat{\nabla}_{m-2}\langle 2\rangle \twoheadrightarrow \widehat{\nabla}_{-m+2}\langle 3\rangle \twoheadrightarrow \cdots \twoheadrightarrow \widehat{\nabla}_n\langle m - |n| + \delta_n\rangle.$$

Their composition is a nonzero element of $\operatorname{Hom}(\widehat{\nabla}_m, \widehat{\nabla}_n\langle m - |n| + \delta_n\rangle)$, and it is surjective. Similar reasoning applies if $m < 0$. □

Proposition 6.13. *Let $n \in \mathbb{Z}$.*

1. *The socle of $\widehat{\Delta}_n$ is isomorphic to $\mathfrak{E}_{\bar{\mathsf{m}}(n)}\langle -|n| + \delta_n\rangle$, and the cokernel of the inclusion map $\mathfrak{E}_{\bar{\mathsf{m}}(n)}\langle -|n| + \delta_n\rangle \hookrightarrow \widehat{\Delta}_n$ contains no composition factor of the form $\mathfrak{E}_m\langle k\rangle$ with $m \in \{0, -1\}$.*
2. *The cosocle of $\widehat{\nabla}_n$ is isomorphic to $\mathfrak{E}_{\bar{\mathsf{m}}(n)}\langle |n| - \delta_n\rangle$, and the kernel of the surjective map $\widehat{\nabla}_n \twoheadrightarrow \mathfrak{E}_{\bar{\mathsf{m}}(n)}\langle |n| - \delta_n\rangle$ contains no composition factor of the form $\mathfrak{E}_m\langle k\rangle$ with $m \in \{0, -1\}$.*

Proof. We will treat only the costandard case. Proposition 6.12 tells us that there is a surjective map $\widehat{\nabla}_n \twoheadrightarrow \widehat{\nabla}_{\bar{\mathsf{m}}(n)}\langle |n| - \delta_n\rangle \cong \mathfrak{E}_{\bar{\mathsf{m}}(n)}\langle |n| - \delta_n\rangle$. Corollary 5.5 already tells us that $\widehat{\nabla}_n$ can have no other antiminuscule composition factor. To finish the proof, we must show that $\widehat{\nabla}_n$ has no simple quotient $\mathfrak{E}_m\langle k\rangle$ with $m \notin \{0, -1\}$. If it did, then the composition $\widehat{\nabla}_n \to \mathfrak{E}_m\langle k\rangle \to \widehat{\nabla}_m\langle k\rangle$ would be a nonzero, nonsurjective map, contradicting Proposition 6.12. □

6.4 Simple and tilting exotic sheaves

We are now ready to determine the simple exotic sheaves $\mathfrak{E}_n$ and the indecomposable tilting objects $\hat{\mathfrak{T}}_n$ for all $n \in \mathbb{Z}$.

Proposition 6.14 (Simple exotic sheaves). *We have*

$$\mathfrak{E}_n \cong \begin{cases} i_{\alpha*}L(-n-2)\Bbbk_{-2}\langle 1\rangle & \text{if } n \le -2, \\ \mathcal{O}_{\widetilde{\mathcal{N}}}(-1)\langle 1\rangle & \text{if } n = -1, \\ \mathcal{O}_{\widetilde{\mathcal{N}}} & \text{if } n = 0, \\ i_{\alpha*}L(n-1)\Bbbk_{-1}\langle 2\rangle[-1] & \text{if } n \ge 1. \end{cases}$$

Proof. The weights $n = 0$ and $n = -1$ are antiminuscule, so in those cases, $\mathfrak{E}_n$ is given by Lemma 5.1.

By Lemmas 6.5 and 6.9, respectively, we know that $i_{\alpha*}L(-n-2)\Bbbk_{-2}\langle 1\rangle$ and $i_{\alpha*}L(n-1)\Bbbk_{-1}\langle 2\rangle[-1]$ belong to $\mathrm{ExCoh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})$. One can show by induction with respect to the partial order $\le_{\mathbf{X}}$ that they are simple, using our explicit description of the standard and costandard objects. We omit further details. ◻

Proposition 6.15 (Tilting exotic sheaves). *We have*

$$\hat{\mathfrak{T}}_n \cong \begin{cases} T(-n-1) \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}(-1)\langle 1\rangle & \text{if } n < 0, \\ T(n) \otimes \mathcal{O}_{\widetilde{\mathcal{N}}} & \text{if } n \ge 0. \end{cases}$$

Proof. For $n \ge 0$, this is just a restatement of Proposition 3.6. Assume henceforth that $n < 0$. To show that the $T(-n-1) \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}(-1)$ are tilting objects, we will use the criterion of [19, Lemma 4], which says that it is enough to check that for all $k > 0$, we have

$$\underline{\mathrm{Hom}}(\widehat{\Delta}_m[-k], T(-n-1)\otimes\mathcal{O}_{\widetilde{\mathcal{N}}}(-1)) = \underline{\mathrm{Hom}}(T(-n-1)\otimes\mathcal{O}_{\widetilde{\mathcal{N}}}(-1), \widehat{\nabla}_m[k]) = 0,$$

or, equivalently,

$$\underline{\mathrm{Hom}}(\widehat{\Delta}_m \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}(1)[-k], T(-n-1) \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}) = \underline{\mathrm{Hom}}(T(-n-1) \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}, \widehat{\nabla}_m \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}(1)[k]) = 0.$$

By adjunction and the fact that $\pi^*\mathcal{O}_{\mathcal{N}} \cong \pi^!\mathcal{O}_{\mathcal{N}} \cong \mathcal{O}_{\widetilde{\mathcal{N}}}$, this is in turn equivalent to the vanishing of the following Hom-groups in $D^{\mathrm{b}}\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\mathcal{N})$:

$$\underline{\mathrm{Hom}}(\pi_*(\widehat{\Delta}_m \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}(1))[-k], T(-n-1) \otimes \mathcal{O}_{\mathcal{N}}) = \underline{\mathrm{Hom}}(T(-n-1) \otimes \mathcal{O}_{\mathcal{N}}, \pi_*(\widehat{\nabla}_m \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}(1))[k]) = 0.$$

These equalities hold because $T(n-1) \otimes \mathcal{O}_{\mathcal{N}}$ is a tilting object in $\mathrm{PCoh}^{G\times\mathbb{G}_\mathrm{m}}(\mathcal{N})$ (Proposition 3.5), while $\pi_*(\widehat{\Delta}_m \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}(1))$ is proper standard and $\pi_*(\widehat{\nabla}_m \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}(1))$ is proper costandard (by Lemma 6.10 and Proposition 2.6).

There is an obvious morphism $\widehat{\Delta}_n \to T(-n-1) \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}(-1)\langle 1\rangle$, and this shows that $\hat{\mathfrak{T}}_n \cong T(-n-1) \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}(-1)\langle 1\rangle$, as desired. ◻

Proposition 6.16. *If* $\Bbbk = \mathbb{C}$*, then every standard object and every costandard object in* $\mathrm{ExCoh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})$ *is uniserial.*

Proof. This holds by induction with respect to $\leq_{\mathbf{X}}$, using the short exact sequences in Lemmas 6.7 and 6.8 and Proposition 6.11. □

For example, the composition series of $\widehat{\nabla}_n$ looks like this:

$$n \geq 0: \quad \begin{array}{r|c|} \cline{2-2} \text{cosocle:} & \mathfrak{E}_{\bar{\mathsf{m}}(n)}\langle n\rangle \\ \cline{2-2} & \vdots \\ \cline{2-2} & \mathfrak{E}_{-n+2}\langle 3\rangle \\ \cline{2-2} & \mathfrak{E}_{n-2}\langle 2\rangle \\ \cline{2-2} & \mathfrak{E}_{-n}\langle 1\rangle \\ \cline{2-2} \text{socle:} & \mathfrak{E}_{n} \\ \cline{2-2} \end{array} \qquad n < 0: \quad \begin{array}{r|c|} \cline{2-2} \text{cosocle:} & \mathfrak{E}_{\bar{\mathsf{m}}(n)}\langle -n-1\rangle \\ \cline{2-2} & \vdots \\ \cline{2-2} & \mathfrak{E}_{-n-4}\langle 3\rangle \\ \cline{2-2} & \mathfrak{E}_{n+2}\langle 2\rangle \\ \cline{2-2} & \mathfrak{E}_{-n-2}\langle 1\rangle \\ \cline{2-2} \text{socle:} & \mathfrak{E}_{n} \\ \cline{2-2} \end{array}.$$

We conclude by answering Question 3.7 for $G = \mathrm{SL}_2$.

Proposition 6.17 (Positivity for tilting exotic sheaves). *For any $n, m \in \mathbb{Z}$, the graded vector space $\underline{\mathrm{Hom}}(\hat{\mathfrak{T}}_n, \hat{\mathfrak{T}}_m)$ is concentrated in nonnegative degrees.*

Proof. We must show that $\mathrm{Hom}(\hat{\mathfrak{T}}_n, \hat{\mathfrak{T}}_m\langle k\rangle) = 0$ for all $k < 0$. From Proposition 6.15, this is obvious if $n < 0$ or if $m \geq 0$. It is also obvious if $n \geq 0$, $m < 0$, and $k \leq -2$. It remains to consider the case where $n \geq 0$, $m < 0$, and $k = -1$. Using Lemma 6.4, we find that

$$\begin{aligned} &\mathrm{Hom}(T(n) \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}, T(-m-1) \otimes \mathcal{O}_{\widetilde{\mathcal{N}}}(-1)) \\ &\qquad\qquad \cong \mathrm{Hom}_{\mathrm{Rep}(B)}(T(n), T(-m-1)\Bbbk_{-1}) = 0, \end{aligned}$$

as desired. □

6.5 *Perverse-coherent sheaves*

After the hard work of the exotic case, the calculations in the perverse coherent case are relatively easy.

Proposition 6.18. *For $n \in \mathbb{Z}_{\geq 0}$, $\overline{\nabla}_n$ is given by*

$$\overline{\nabla}_0 \cong \begin{array}{ccccccc} 0 & 2 & 4 & 6 & 8 & 10 & \cdots \\ H^0(0) & H^0(2) & H^0(4) & H^0(6) & H^0(8) & H^0(10) & \cdots \end{array}$$

$$\overline{\nabla}_n \cong \begin{array}{ccccc} 1 & 3 & 5 & 7 & \cdots \\ H^0(n) & H^0(n+2) & H^0(n+4) & H^0(n+6) & \cdots \end{array} \qquad \textit{if } n > 0.$$

For $n \in \{0, 1\}$, we have $\overline{\Delta}_n \cong \overline{\nabla}_n$, whereas for $n \geq 2$, we have

$$\begin{array}{rccccccccc} & & & & n-3 & n-1 & n+1 & n+3 & \\ & -1 & 1 & \cdots & -\overset{+}{\mathsf{m}}(n) & -\overset{+}{\mathsf{m}}(n) & -\overset{+}{\mathsf{m}}(n) & -\overset{+}{\mathsf{m}}(n) & \cdots \\ \mathcal{H}^1(\overline{\Delta}_n) \cong & V(n-2) & V(n-4) & \cdots & V(\overset{+}{\mathsf{m}}(n)) & - & - & \cdots & \\ \mathcal{H}^0(\overline{\Delta}_n) \cong & - & - & \cdots & - & H^0(\bar{\mathsf{m}}(n)) & H^0(\bar{\mathsf{m}}(n)+2) & H^0(\bar{\mathsf{m}}(n)+4) & \cdots \end{array}$$

Proof. Apply $R\operatorname{ind}_B^G$ to the formulas from Propositions 6.2 and 6.3. □

With a bit more effort, it is possible to give a finer description of the $\Bbbk[\mathcal{N}]$-action on these modules. Recall that for SL_2, the nilpotent cone $\mathcal{N}$ is isomorphic as an SL_2-variety to the quotient $\mathbb{A}^2/(\mathbb{Z}/2)$, where the nontrivial element of $\mathbb{Z}/2$ acts by negation. This gives rise to an isomorphism

$$\Bbbk[\mathcal{N}] \cong \Bbbk[x^2, xy, y^2],$$

where the right-hand side is the subring of $(\mathbb{Z}/2)$-invariant elements in the polynomial ring $\Bbbk[x, y]$. For $n \in \mathbb{Z}_{\geq 0}$, let

$$M_n = \Bbbk[x^2, xy, y^2] \cdot (x^n, x^{n-1}y, \ldots, y^n) \subset \Bbbk[x, y].$$

Thus, M_n consists of polynomials whose terms have degrees $\geq n$ and $\equiv n \pmod 2$.

Lemma 6.19. *There is an isomorphism of* $\Bbbk[\mathcal{N}]$*-modules* $\widehat{\nabla}_n \cong M_n\langle n - \delta_n^*\rangle$.

Proof. For $n = 0$, we have $M_0 \cong \mathcal{O}_{\mathcal{N}} \cong \overline{\nabla}_0$, and there is nothing to prove. For $n = 1$, recall that $\overline{\nabla}_1$ is a simple perverse-coherent sheaf, and up to grading shift, it is the unique simple object that is a torsion-free coherent sheaf not isomorphic to $\mathcal{O}_{\mathcal{N}}$. It is easy to check from Proposition 6.18 that M_1 is isomorphic as a $(G \times \mathbb{G}_\mathrm{m})$-representation to $\overline{\nabla}_1$ (and that it is *not* isomorphic to $\overline{\nabla}_0$), so to prove that $M_1 \cong \overline{\nabla}_1$, it suffices to show that M_1 is a simple perverse-coherent sheaf. This can be done by computing its local cohomology at 0, and then using the criterion described after Theorem 4.6.

For $n \geq 2$, Proposition 5.10 gives us a map

$$\overline{\nabla}_n \to \overline{\nabla}_{\overset{+}{\mathsf{m}}(n)}\langle n - 1\rangle$$

that is surjective as a morphism in $\mathrm{PCoh}^{G\times\mathbb{G}_\mathrm{m}}(\mathcal{N})$. We claim that it is also injective as a morphism in $\mathrm{Coh}^{G\times\mathbb{G}_\mathrm{m}}(\mathcal{N})$. Indeed, the map is an isomorphism over $\mathcal{N}_{\mathrm{reg}}$, so its kernel would have to be supported on $\mathcal{N} \smallsetminus \mathcal{N}_{\mathrm{reg}}$. But $\overline{\nabla}_n$ is a torsion-free coherent sheaf (see Proposition 5.3), so that kernel must be trivial. In other words, as a coherent sheaf, $\overline{\nabla}_n$ can be identified with a certain submodule of $\overline{\nabla}_{\overset{+}{\mathsf{m}}(n)}\langle n - 1\rangle$. Proposition 6.18 shows us that the desired submodule is precisely M_n. □

Let $i_0 : \{0\} \hookrightarrow \mathcal{N}$ be the inclusion map of the origin into the nilpotent cone.

Proposition 6.20 (Simple perverse-coherent sheaves). *We have*

$$\mathfrak{IC}_n \cong \begin{cases} \mathcal{O}_{\mathcal{N}} & \text{if } n = 0, \\ \overline{\Delta}_1 \cong \overline{\nabla}_1 \cong M_1 & \text{if } n = 1, \\ i_{0*}L(n-2)\langle 1\rangle[-1] & \text{if } n \geq 2. \end{cases}$$

Proof. The local description of $\mathrm{PCoh}^{G\times\mathbb{G}_\mathrm{m}}(\mathcal{N})$ from Section 4.3 makes it clear that the list of objects above is an exhaustive list of simple perverse-coherent sheaves up to grading shift. To check the parametrization in the cases where $n \geq 2$, we simply note that there is a nonzero map $\overline{\Delta}_n \to i_{0*}L(n-2)\langle 1\rangle[-1]$. □

Recall that the tilting objects in $\mathrm{PCoh}^{G\times\mathbb{G}_\mathrm{m}}(\mathcal{N})$ have been completely described in Proposition 3.5. We will not repeat that description here.

Proposition 6.21. *In* $\mathrm{PCoh}^{G\times\mathbb{G}_\mathrm{m}}(\mathcal{N})$*, we have the following short exact sequences for each* $n \geq 2$*:*

$$0 \to i_{0*}H^0(n-2)\langle 1\rangle[-1] \to \overline{\nabla}_n \to \overline{\nabla}_{n-2}\langle 1+\delta^*_{n-2}\rangle \to 0,$$
$$0 \to \overline{\Delta}_{n-2}\langle -1-\delta^*_{n-2}\rangle \to \overline{\Delta}_n \to i_{0*}V(n-2)\langle 1\rangle[-1] \to 0.$$

Proof. We know from Theorem 5.12 that $\dim \mathrm{Hom}(\overline{\nabla}_n, \overline{\nabla}_{n-2}\langle 1+\delta^*_{n-2}\rangle) = 1$. Lemma 6.19 lets us identify the cone of any such map (up to grading shift) with the space of homogeneous polynomials in $\Bbbk[x,y]$ of degree $n-2$: in other words, with $H^0(n-2)$. The local description of $\mathrm{PCoh}^{G\times\mathbb{G}_\mathrm{m}}(\mathcal{N})$ implies that $i_{0*}H^0(n-2)\langle 1\rangle[-1]$ is indeed a perverse-coherent sheaf, and this gives us the first short exact sequence above. The second is then obtained by applying the Serre–Grothendieck duality functor $\mathbb{D}$. □

Proposition 6.22. *If* $\Bbbk = \mathbb{C}$*, then every standard object and every costandard object in* $\mathrm{PCoh}^{G\times\mathbb{G}_\mathrm{m}}(\widetilde{\mathcal{N}})$ *is uniserial.*

Proof. This is immediate from Proposition 6.21. □

For example, the composition series of $\overline{\nabla}_n$ for $n > 0$ looks like this:

n even and ≥ 2:

cosocle:	$\mathfrak{IC}_0\langle n-1\rangle$
	$\mathfrak{IC}_2\langle n-2\rangle$
	$\mathfrak{IC}_4\langle n-4\rangle$
	$\vdots$
	$\mathfrak{IC}_{n-4}\langle 4\rangle$
	$\mathfrak{IC}_{n-2}\langle 2\rangle$
socle:	$\mathfrak{IC}_n$

n odd:

cosocle:	$\mathfrak{IC}_1\langle n-1\rangle$
	$\mathfrak{IC}_3\langle n-3\rangle$
	$\mathfrak{IC}_5\langle n-5\rangle$
	$\vdots$
	$\mathfrak{IC}_{n-4}\langle 4\rangle$
	$\mathfrak{IC}_{n-2}\langle 2\rangle$
socle:	$\mathfrak{IC}_n$

Finally, since $\mathrm{PCoh}^{G\times\mathbb{G}_\mathrm{m}}(\mathcal{N})$ is properly stratified but not quasihereditary, it also has true standard and true costandard objects. The following proposition describes them.

Proposition 6.23. *We have* $\Delta_0 \cong \nabla_0 \cong \mathcal{O}_{\mathcal{N}}$*. If* $n > 0$*, there are short exact sequences*

$$0 \to \overline{\nabla}_n \to \nabla_n \to \overline{\nabla}_n\langle 2\rangle \to 0, \qquad 0 \to \overline{\Delta}_n\langle -2\rangle \to \Delta_n \to \overline{\Delta}_n \to 0.$$

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Parameters for twisted representations

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The first author dedicates this to the second on the occasion of his 60^{th} birthday

Abstract The main result of [4] is the description of an algorithm to compute the signature of the Hermitian form on an irreducible representation of a real reductive Lie group G, and therefore determine if it is unitary. This paper concerns an important ingredient of the algorithm. If the inner class of G is defined by an outer automorphism δ, so that G does not have discrete series representations, it is necessary to compute a new class of Kazhdan–Lusztig–Vogan polynomials for G. These were defined and studied by Lusztig and Vogan in [10]. In order to carry out the computation, we introduce new class of *twisted* parameters, and study the Hecke algebra action in the resulting basis.

Key words: unitary representation, Kazhdan–Lusztig polynomial, Hermitian form

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1 Introduction

One of the central problems in representation theory is understanding irreducible unitary representations. The reason is that in many applications of linear algebra (like those of representation theory to harmonic analysis) the notion of *length* of vectors is fundamentally important. Unitary representations are exactly those preserving a good notion of length.

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The paper [4] provides an algorithm for calculating the irreducible unitary representations of a real reductive Lie group G. The starting point for this algorithm is the Langlands classification, which provides a parameter space for the irreducible admissible representations of G. In order to determine the unitary representations, it is necessary to pass to a larger extended group ${}^{\delta}G$ containing G of index 2, and construct a parameter space for the representations of ${}^{\delta}G$. The purpose of this paper is to address the following problem: when a parameter for G extends to ${}^{\delta}G$ in two ways, there is no canonical way to choose one of the extensions. Consequently the theory for G does not carry over to ${}^{\delta}G$ in a simple way, and it is necessary to define parameters for ${}^{\delta}G$ and study their properties in some detail.

In order to explain this we need to describe briefly (or at least more briefly than [4]) the nature of the unitarity algorithm. In order to minimize technicalities, we will provide in the introduction complete details only for *finite-dimensional* representations. For a real reductive Lie group, the theory of Harish-Chandra modules provides a complete way to deal with the complications attached to infinite-dimensional representations.

To study unitary representations it is natural to study the larger class of representations with invariant Hermitian forms. Here is the underlying formalism.

Definition 1.1. Suppose V and W are complex vector spaces. A *sesquilinear pairing* is a map

$$\langle \cdot, \cdot \rangle : V \times W \to \mathbb{C}$$

that is linear in V and conjugate-linear in W:

$$\langle av_1 + bv_2, w \rangle = a\langle v_1, w\rangle + b\langle v_2, w\rangle, \quad \langle v, cw_1 + dw_2\rangle = \overline{c}\langle v, w_1\rangle + \overline{d}\langle v, w_2\rangle.$$

In case $V = W$, the pairing is called *Hermitian* if in addition

$$\langle v_1, v_2\rangle = \overline{\langle v_2, v_1\rangle}.$$

If $\langle , \rangle$ is a *nondegenerate* Hermitian pairing on a *finite-dimensional* vector space V, then there is a one-to-one correspondence between linear maps $A \in \mathrm{Hom}(V, V)$ and sesquilinear pairings $\langle , \rangle_A$ on V, defined by

$$\langle v, w\rangle_A = \langle v, Aw\rangle.$$

In this correspondence, $\langle , \rangle_A$ is Hermitian if and only if A is self-adjoint with respect to $\langle , \rangle$.

Definition 1.2. Suppose (π, V) is a representation of a group G_1 on a finite-dimensional complex vector space V. An *invariant Hermitian form* on V is a Hermitian pairing

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$$

with the property that

$$\langle \pi(g)v, \pi(g)w\rangle = \langle v, w\rangle \qquad (v, w \in V,\ g \in G_1).$$

The representation π is *Hermitian* if it is endowed with a nondegenerate invariant Hermitian form, and *unitary* if in addition this form is positive definite.

If G_1 is a connected real Lie group with Lie algebra $\mathfrak{g}_1$, then π is determined by its differential (still called π)

$$\pi\colon \mathfrak{g}_1 \to \mathrm{End}(V),$$

a Lie algebra representation. The condition for the Hermitian form to be invariant is equivalent to

$$\langle \pi(X)v, w\rangle + \langle v, \pi(X)w\rangle = 0 \qquad (v, w \in V,\ X \in \mathfrak{g}_1);$$

that is, that the real Lie algebra $\mathfrak{g}_1$ acts by skew-Hermitian operators.

A Hermitian form on a finite-dimensional vector space V has a *signature* which for us will be a triple $(p, q, z) \in \mathbb{N}^3$: here p is the dimension of a maximal positive-definite subspace of V, q is the dimension of a maximal negative-definite subspace, and z is the dimension of the radical. Sylvester's law of inertia says that p, q, and z are well-defined, and that

$$p + q + z = \dim(V). \tag{1}$$

Proposition 1.3 (Schur's Lemma). *Suppose* $(\pi, V) \in (\widehat{G_1})_{fin}$ *(notation* (4)*). Then any two nonzero invariant Hermitian forms on* V *are nondegenerate, and differ by a real nonzero scalar. In particular, the signature* $(p(\pi), q(\pi))$ *is well-defined up to interchanging* p *and* q.

Here is an outline of the algorithm in [4] for determining the unitary irreducible representations of a real reductive group.

Algorithm. Suppose G_1 is the group of real points of a complex connected reductive algebraic group.

1. *List* all the irreducible representations of G_1 admitting a nonzero invariant Hermitian form.
2. For each such irreducible π, *choose* a nonzero invariant form $\langle , \rangle_\pi$.
3. For each form $\langle , \rangle_\pi$, *calculate* the signature $(p(\pi), q(\pi))$.
4. *Check* whether one of $p(\pi)$ and $q(\pi)$ is zero; in this case, π is an irreducible unitary representation.

We have explained this algorithm in the case of finite-dimensional representations. For infinite-dimensional representations step 1 is the Langlands classification, and what it means to calculate the signature of an invariant form on an infinite-dimensional representation is discussed in [4].

Of these steps, (1) was carried out by Knapp and Zuckerman about 1976; there is an account in [7, Chapter 16]. Their argument was a reduction of the problem to the special case of *representations with real infinitesimal character*. We will not recall the precise definition (see [4, Definition 5.5] or [14, Definition 5.4.11]). The nature of the reduction provided at the same time a reduction of (2)–(4): the entire problem of understanding unitary irreducible representations was reduced to the case of real infinitesimal character. We will therefore concentrate henceforth on this case. (If G_1 is real semisimple, then every finite-dimensional representation of G_1 has real infinitesimal character; so the reduction is invisible on the level of finite-dimensional representations.)

Before we look at an example, one more general idea is useful. A fundamental idea in the representation theory of a real Lie group (or Lie algebra) is to *complexify* the group (or Lie algebra), and take advantage of the (simpler and stronger) structural results available for complex Lie algebras and groups. This is particularly easy for Lie algebras: any real Lie algebra $\mathfrak{g}_1$ has a natural complexification

$$\mathfrak{g} =_{\text{def}} \mathfrak{g}_1 \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{g}_1 \oplus i\,\mathfrak{g}_1. \tag{2a}$$

(The distinction between $\mathfrak{g}_1$ and $\mathfrak{g}$ in this notation seems a little obscure and hard to remember. In the body of the paper, G_1 will usually be something like $G(\mathbb{R})$, and $\mathfrak{g}_1$ will be $\mathfrak{g}(\mathbb{R})$.) The extra structure on the complex Lie algebra $\mathfrak{g}$ that remembers $\mathfrak{g}_1$ is a *real form*: a conjugate-linear real Lie algebra automorphism of order two

$$\sigma_1 : \mathfrak{g} \to \mathfrak{g}, \qquad \sigma_1(X + iY) = X - iY \quad (X, Y \in \mathfrak{g}_1). \tag{2b}$$

Any real Lie algebra representation $\pi_{\mathbb{R}}$ of $\mathfrak{g}_1$ on a complex vector space V gives rise to a *complex* Lie algebra representation

$$\pi_{\mathbb{C}}(X + iY) = \pi_{\mathbb{R}}(X) + i\pi_{\mathbb{R}}(Y) \qquad (X, Y \in \mathfrak{g}_1); \tag{2c}$$

and of course $\pi_{\mathbb{R}}$ can be recovered from $\pi_{\mathbb{C}}$ by restriction. This is so elementary and fundamental that it usually goes unsaid, and the subscripts $\mathbb{R}$ and $\mathbb{C}$ on π are not used.

The reason we make this explicit now is that an invariant Hermitian form for $\pi_{\mathbb{R}}$ is almost *never* invariant for $\pi_{\mathbb{C}}$; if $\langle \pi_{\mathbb{R}}(X)v, w\rangle + \langle v, \pi_{\mathbb{R}}(X)w\rangle = 0$ $(x \in \mathfrak{g}_1)$, then

$$\langle \pi_{\mathbb{C}}(iX)v, w\rangle + \langle v, \pi_{\mathbb{C}}(iX)w\rangle = i(\langle \pi_{\mathbb{R}}(X)v, w\rangle - \langle v, \pi_{\mathbb{R}}(X)w\rangle)$$

and there is no reason for this to be 0. What is true is that *the Hermitian form* $\langle\cdot,\cdot\rangle$ *on* V *is* $\pi_{\mathbb{R}}$*-invariant (see 1.2) if and only if*

$$\langle \pi_{\mathbb{C}}(Z)v, w\rangle + \langle v, \pi_{\mathbb{C}}(\sigma_1(Z))w\rangle = 0 \qquad (v, w \in V,\ Z \in \mathfrak{g}). \tag{2d}$$

That is, we require that $\pi_{\mathbb{C}}$ *should carry the complex conjugation on* $\mathfrak{g}$ *to minus Hermitian transpose on operators*. In this case we call $\langle\cdot,\cdot\rangle$ a σ_1*-invariant form* for the representation $\pi_{\mathbb{C}}$ of $\mathfrak{g}$.

The point to remember is that the definition of invariant Hermitian form on a complex representation of a complex Lie algebra $\mathfrak{g}$ *requires* a choice of real form on $\mathfrak{g}$. Changing the real form changes everything: whether an invariant form exists, and what its signature is.

Example 1.4. Suppose $G_1 = \mathrm{SL}(3,\mathbb{R})$. The finite-dimensional representations of G_1 are precisely those of the complex Lie algebra $\mathfrak{sl}(3,\mathbb{C})$. The corresponding real form of $\mathfrak{sl}(3,\mathbb{C})$ is

$$\sigma_1(Z) = \overline{Z} \qquad (Z \in \mathfrak{sl}(3,\mathbb{C})),$$

complex conjugation of matrices.

Irreducible finite-dimensional representations of $\mathfrak{sl}(3,\mathbb{C})$ are indexed by highest weights

$$\lambda = (\lambda_1, \lambda_2, \lambda_3), \quad \lambda_1 + \lambda_2 + \lambda_3 = 0, \quad \lambda_p - \lambda_q \in \mathbb{Z}, \quad \lambda_1 \geq \lambda_2 \geq \lambda_3.$$

For example, $E_{(2/3,-1/3,-1/3)}$ is the tautological representation on $\mathbb{C}^3$ and $E_{(1,0,-1)}$ is the 8-dimensional adjoint representation. It turns out that the only representations with a nonzero invariant σ_1-invariant Hermitian form are the "Cartan powers of the adjoint representation":

$$E_{(m,0,-m)} = \text{irreducible representation of dimension } (m+1)^3.$$

(This follows from the Knapp–Zuckerman result explained in [7, Chapter 16], but for finite-dimensional representations is probably much older.)

We would like to understand σ_1-invariant Hermitian forms on $E_{(m,0,-m)}$. According to the program described after Proposition 1.3, we need first to *choose* one of the two possible forms. For this (and for much more!) we will use the restriction of representations of G_1 to the maximal compact subgroup

$$K_1 = \mathrm{SO}(3).$$

Because each irreducible representation of a compact group has a positive-definite invariant Hermitian form, the positive and negative parts of an invariant form for G_1 may be understood not just as vector spaces (with dimensions) but as representations of K_1 (sums of irreducible representations with multiplicity). It turns out that $E_{(m,0,-m)}$ contains *either* the trivial representation F_1 of K_1 (if m is even), *or* the tautological three-dimensional representation F_3 (if m is odd), but not both. This representation appears with multiplicity one, so any invariant Hermitian form is either positive or negative definite on the subspace F_1 or F_3. We fix our choice of σ_1-invariant form on $E_{(m,0,-m)}$ by requiring

form is *positive* on F_1 and *negative* on F_3.

For example, the adjoint representation $E_{(1,0,-1)} \simeq \mathfrak{g}$ has a Cartan decomposition (more precisely, the complexification of the Cartan decomposition of $\mathfrak{g}_1$)

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = F_3 \oplus F_5$$

(skew-symmetric and symmetric traceless matrices), the sum of irreducible representations of K_1 of dimensions 3 and 5. We can choose for our invariant Hermitian form the trace form

$$\langle X, Y \rangle = \mathrm{tr}(XY^*) \qquad (X, Y \in \mathfrak{sl}(3, \mathbb{C})).$$

This form is easily seen to be positive definite on the space of real symmetric matrices (since these have real eigenvalues), and negative definite on the real skew-symmetric matrices (since these have purely imaginary eigenvalues). In particular, it is negative on F_3. In this way we see that the form on $E_{(1,0,-1)}$ has signature $(5, 3)$; even better, the signature is (F_5, F_3) as a representation of SO(3).

Here are a few more signatures. We are for the moment simply claiming that these formulas are correct, not explaining where they come from. Always we write F_{2k+1} for the unique irreducible representation of SO(3) of dimension $2k + 1$, endowed with a positive-definite invariant Hermitian form.

$$\text{signature of } E_{(3,0,-3)} = ([F_{13} + F_9 + F_7] + F_5, [F_{11} + F_9 + F_7] + F_3);$$

that is,

$$\mathrm{sig}(E_{(3,0,-3)}) = ([F_{13} + F_9 + F_7], [F_{11} + F_9 + F_7]) + \mathrm{sig}(E_{(1,0,-1)}).$$

Similarly we can compute

$$\begin{aligned}\mathrm{sig}(E_{(5,0,-5)}) = ([&F_{21} + F_{17} + F_{15} + F_{13} + F_{11}],\\ &[F_{19} + F_{17} + F_{15} + F_{13} + F_{11}]) + \mathrm{sig}(E_{(3,0,-3)}).\end{aligned}$$

At this point perhaps the pattern is evident: we get the signature for $E_{(2m+1,0,-2m-1)}$ from that for $E_{(2m-1,0,-2m+1)}$ by adding to the positive and negative parts sums of $2m+1$ irreducible representations of K_1. The two added strings are identical except for the first terms, which differ in dimension by two. (The pattern applies even to getting the signature of $E_{(1,0,-1)}$ from that of the (zero) representation $E_{(-1,0,1)}$.)

In the same way, it turns out that

$$\begin{aligned}\mathrm{sig}(E_{(0,0,0)}) &= (F_1, 0)\\ \mathrm{sig}(E_{(2,0,-2)}) &= ([F_9 + F_5], [F_7 + F_5]) + \mathrm{sig}(E_{(0,0,0)})\\ \mathrm{sig}(E_{(4,0,-4)}) &= ([F_{17} + F_{13} + F_{11} + F_9],\\ &\qquad [F_{15} + F_{13} + F_{11} + F_9]) + \mathrm{sig}(E_{(2,0,-2)}).\end{aligned}$$

The pattern is essentially the same as in the odd case: we get the signature for $E_{(2m+2,0,-2m-2)}$ from that for $E_{(2m,0,-2m)}$ by adding to the positive and negative parts sums of $2m + 2$ irreducible representations of K_1. The two added strings are identical except for the first terms, which differ in dimension by two.

As a consequence of this inductive description of the signature as a representation of K_1, or more directly, one can show that

$$\mathrm{sig}(E_{(m,0,-m)}) = ([(m + 1)^3 + (m + 1)]/2, [(m + 1)^3 - (m + 1)]/2).$$

In particular, $E_{(m,0,-m)}$ is unitary if and only if $m = 0$: the trivial representation is the only finite-dimensional unitary representation of $\mathrm{SL}(3, \mathbb{R})$. (The last statement of course has many extremely short proofs; the point of explaining this long argument is that the ideas apply to infinite-dimensional representations of general real reductive groups.)

This is the shape of the calculation made possible by [4]: we find enormous detail about the precise signatures of invariant Hermitian forms, and then (for the purposes of questions about unitarity) throw almost all of this information away. Of course we would be very happy to learn what interesting questions this discarded information is actually addressing.

We now describe how the calculation of signatures is related to a more classical representation-theoretic problem of Clifford theory: how to extend an irreducible representation of a normal subgroup.

We do not wish to use all of the somewhat complicated and delicate hypotheses under which we finally work (involving real algebraic groups and L-groups just as a point of departure). On the other hand, we would like to use notation that is close to being consistent with that of the body of the paper. Here is a compromise.

A key object to consider will be a group extension

$$1 \to G \to {}^{\mathrm{ex}}G \to \{1, \delta\} \to 1, \tag{3a}$$

which we call an *extended group for* G. In the body of the paper, G will very often be a complex connected reductive algebraic group. Perhaps the most familar example of such an extension, and one that we will certainly use (often behind the scenes), is Langlands L-group (12h); there the role of G is played by a (complex connected reductive algebraic) dual group, and $\{1, \delta\}$ is the Galois group of $\mathbb{C}/\mathbb{R}$.

Here is a concrete way to construct such a group extension. Begin with an automorphism $\theta \in \mathrm{Aut}(G)$, with the property that θ^2 is inner:

$$\theta^2 = \mathrm{Int}(g_0) \qquad (g \in G). \tag{3b}$$

(Only the coset $g_0 Z(G)$ is determined by θ.) Then we can define ${}^{\mathrm{ex}}G$ by generators and relations, as the group generated by G and a single additional element h_0, satisfying

$$h_0^2 = g_0, \qquad h_0 g h_0^{-1} = \theta(g) \quad (g \in G). \tag{3c}$$

(This presentation *does* depend on the choice of representative g_0 for the coset $g_0 Z(G)$.) Two automorphisms θ and θ' of G are said to be *inner* to each other if $\theta' \circ \theta^{-1}$ is an inner automorphism. Now it is clear that

$$\{\mathrm{Int}(h)|_G \mid h \in {}^{\mathrm{ex}}G - G\} \quad \text{is an inner class in } \mathrm{Aut}(G). \tag{3d}$$

This is how we will use extended groups: as a place to keep track of and compare representatives of various automorphisms of G.

An *extended subgroup* of ${}^{\mathrm{ex}}G$ is a subgroup ${}^{\mathrm{ex}}G_1$ mapping surjectively to $\{1, \delta\}$. In this case we define $G_1 = G \cap {}^{\mathrm{ex}}G_1$, so that

$$1 \to G_1 \to {}^{\mathrm{ex}}G_1 \to \{1, \delta\} \to 1. \tag{3e}$$

Often we will consider several such subgroups ${}^{\mathrm{ex}}G_1$, ${}^{\mathrm{ex}}G_2$, and so on.

In the body of the paper, the (complex) group G will be a very useful tool, but we will be interested actually in representations only of a real form $G(\mathbb{R})$; then this will be a typical G_1. A little more precisely, if ${}^{\mathrm{ex}}G$ is a complex Lie group, then a *real form* means an antiholomorphic automorphism of order two of real extended Lie groups

$$\sigma_1 \colon {}^{\mathrm{ex}}G \to {}^{\mathrm{ex}}G, \quad \sigma_1(G) = G, \qquad {}^{\mathrm{ex}}G_1 =_{\mathrm{def}} [{}^{\mathrm{ex}}G]^{\sigma_1}. \tag{3f}$$

("Antiholomorphic" means that if f is a local holomorphic function on ${}^{\mathrm{ex}}G$, then $\overline{f \circ \sigma_1}$ is holomorphic as well. This implies in particular that the differential of σ_1 (still denoted σ_1) is a real form of $\mathfrak{g}$ in the sense of (2b)).

Our main results are about the problems of Clifford theory (Proposition 1.5): the relationship between representations of G and of ${}^{\mathrm{ex}}G$. Here is a classical statement of Clifford theory for finite-dimensional representations; in the world of Harish-Chandra modules, the extension to infinite-dimensional representations is easy. Write

$$(\widehat{G})_{\mathrm{fin}} = \text{equiv. classes of fin.-diml. irreducible representations.} \tag{4}$$

Proposition 1.5 (Clifford). *Suppose $G \subset {}^{ex}G$ is an extended group* (3a).

1. *The quotient ${}^{ex}G/G = \{1, \delta\}$ acts on $\widehat{G}_{fin}$, by*

$$\delta \cdot (\pi, E) = (\pi^h, E), \qquad \pi^h(g) = \pi(hgh^{-1});$$

here h is any fixed element of ${}^{ex}G - G$ (the non-identity coset of G in ${}^{ex}G$). The equivalence class of π^h is independent of the choice of h.

2. *Define*

$$\epsilon \colon {}^{ex}G \to \{\pm 1\}, \quad \epsilon(h) = \begin{cases} 1 & (h \in G) \\ -1 & (h \notin G). \end{cases}$$

Then the group of characters $\{1, \epsilon\}$ of ${}^{ex}G/G$ acts on $(\widehat{{}^{ex}G})_{fin}$ by

$$\epsilon \cdot \Pi = \Pi \otimes \epsilon.$$

3. *Because these are actions of two-element groups, we have*

$$\pi \in \widehat{G}_{fin} \textbf{ either } \textit{is fixed by } \delta, \textbf{ or } \textit{has a two-element orbit } \{\pi, \delta \cdot \pi\}.$$

Similarly,

$$\Pi \in (\widehat{{}^{ex}G})_{fin} \textbf{ either } \textit{has a two-element orbit } \{\Pi, \epsilon \cdot \Pi\}, \textbf{ or } \textit{is fixed by } \epsilon.$$

4. *The two-element orbits of δ on $\widehat{G}_{fin}$ are in one-to-one correspondence (by induction from G to ${}^{ex}G$) with the ϵ-fixed elements of $(\widehat{{}^{ex}G})_{fin}$. These are the representations Π of ${}^{ex}G$ whose characters vanish on ${}^{ex}G - G$; their characters on G are the sum of the two corresponding characters of G.*

5. *The two-element orbits of ϵ on $(\widehat{{}^{ex}G})_{fin}$ are in one-to-one correspondence (by restriction to G) with the δ-fixed elements of $\widehat{G}_{fin}$. The two extensions Π and Π' of such a π have characters on ${}^{ex}G - G$ differing by sign; their characters on G agree with that of π.*
6. *Suppose (π, E) is a δ-fixed element of $\widehat{G}_{fin}$. Then an extension of π to ${}^{ex}G$ may be constructed as follows. Fix any element $h_0 \in {}^{ex}G - G$. Let A_π be a nonzero intertwining operator from π to π^h:*

$$A_\pi \pi(g) = \pi(h_0 g h_0^{-1}) A_\pi.$$

This requirement determines A_π up to a multiplicative scalar. Write $g_0 = h_0^2 \in G$. After modifying A_π by an appropriate scalar, we may arrange

$$A_\pi^2 = \pi(g_0);$$

with this additional condition, A_π is determined up to multiplication by ± 1. Each choice of A_π determines an extension Π of π, by the requirement

$$\Pi(h_0) = A_\pi.$$

Suppose we understand the character theory of the smaller group G. Because of this proposition, in order to understand the character theory of ${}^{\mathrm{ex}}G$, we must understand, for each δ-fixed irreducible representation of G, the character of *some extension* of it on ${}^{\mathrm{ex}}G - G$. There are always exactly two such extensions, whose characters on ${}^{\mathrm{ex}}G - G$ differ by sign; our task will be to find a way (for the particular groups of interest) to specify one of these two extensions.

Suppose now (as we will for the body of this paper) that G is a complex connected reductive algebraic group, and that

$$\sigma : G \to G \tag{5a}$$

is a real form: an antiholomorphic automorphism of order two of real Lie groups. The corresponding real reductive algebraic group is

$$G(\mathbb{R}, \sigma) = G(\mathbb{R}) =_{\mathrm{def}} G^\sigma, \tag{5b}$$

a real Lie group with Lie algebra

$$\mathfrak{g}(\mathbb{R}) =_{\mathrm{def}} \mathfrak{g}^\sigma. \tag{5c}$$

Now G has one particularly interesting (conjugacy class of) real form(s), the *compact real form* σ_c. It is characterized up to conjugation by G by the requirement that

$$G(\mathbb{R}, \sigma_c) \text{ is compact.} \tag{5d}$$

Elie Cartan showed that σ_c may be chosen to *commute* with the real form σ, and that this requirement determines σ_c up to conjugation by $G(\mathbb{R}, \sigma)$. Because of the commutativity, the composition

$$\theta = \sigma \circ \sigma_c = \sigma_c \circ \sigma \tag{5e}$$

is an algebraic involution of G of order two called the *Cartan involution*; it is determined by σ up to conjugation by $G(\mathbb{R}, \sigma)$. The group of fixed points

$$K = G^\theta \tag{5f}$$

is a (possibly disconnected) complex reductive algebraic subgroup of G. The two real forms σ and σ_c of G both preserve K, and act the same way there; the corresponding real form

$$K(\mathbb{R}) = G(\mathbb{R}, \sigma) \cap K = G(\mathbb{R}, \sigma) \cap G(\mathbb{R}, \sigma_c) = G(\mathbb{R}, \sigma_c) \cap K \tag{5g}$$

is a maximal compact subgroup of $G(\mathbb{R}, \sigma)$ and a maximal compact subgroup (the compact real form) of K.

We wish to understand σ-invariant Hermitian forms on representations of $G(\mathbb{R})$. The next proposition recalls the classical solution (by Cartan and Weyl) of a related problem, and then relates the two problems.

Proposition 1.6. *Suppose we are in the setting* (5).

1. *Finite-dimensional algebraic representations of K may be identified with finite-dimensional continuous representations of the compact real form $K(\mathbb{R})$.*
2. *Finite-dimensional algebraic representations of G may be identified with finite-dimensional continuous representations of the compact real form $G(\mathbb{R}, \sigma_c)$.*
3. *Every finite-dimensional irreducible algebraic representation (π, E) of G admits a positive-definite σ_c-invariant Hermitian form $\langle \cdot, \cdot \rangle_c$, unique up to positive scalar multiple:*
$$\langle \pi(g)v, w \rangle_c = \langle v, \pi(\sigma_c(g))^{-1} w \rangle_c.$$
4. *The finite-dimensional irreducible algebraic representation (π, E) of G admits a σ-invariant Hermitian form $\langle \cdot, \cdot \rangle$ if and only if there is a nonzero linear operator, self-adjoint with respect to the form $\langle , \rangle_c$,*
$$A_\pi \colon E \to E, \qquad A_\pi^* = A_\pi$$
with the property that
$$A_\pi \pi(g) = \pi(\theta(g)) A_\pi \qquad (g \in G).$$
In particular, A_π commutes with the action of K. These requirements determine A_π up to a real multiplicative scalar. In this case the σ-invariant form is
$$\langle v, w \rangle = \langle v, A_\pi w \rangle_c.$$
5. *In the setting of (4), A_π^2 must commute with the action of π, and so must be a nonzero scalar. Because A_π is self-adjoint with respect to the positive Hermitian form $\langle , \rangle_c$, the scalar is necessarily positive real:*
$$A_\pi^2 = r_\pi I_E, \qquad r_\pi \in \mathbb{R}^{+,\times}.$$

6. The signature of this σ-invariant form is

$$\mathrm{sig}(E) = (+1 \textit{ eigenspace of } (r_\pi^{-1/2})A_\pi)), -1 \textit{ eigenspace of } (r_\pi^{-1/2})A_\pi);$$

here the positive and negative parts are representations of K.

The conditions on A_π in Proposition 1.6 look like conditions in Proposition 1.5 for defining a representation of an extended group. We deduce easily

Corollary 1.7. *In the setting (5), suppose also that G is part of an extended group as in (3); and that*

$$\theta = \mathrm{Int}(\xi), \quad \xi \in {}^{ex}G - G, \quad \xi^2 = z \in Z(G).$$

Then a finite-dimensional irreducible algebraic representation (π, E) of G admits a σ-invariant Hermitian form if and only if π has an extension Π to ${}^{ex}G$. In that case define a nonzero complex scalar z_π so that

$$\pi(z) = z_\pi I_E,$$

and choose a square root ω_π of z_π. Then the σ-invariant Hermitian form on E may be taken to be

$$\langle v, w\rangle = \langle v, \omega_\pi^{-1}\Pi(\xi)w\rangle_c.$$

The signature of this σ-invariant form is

$$\mathrm{sig}(E) = (+1 \textit{ eigenspace of } \omega_\pi^{-1}\Pi(\xi), -1 \textit{ eigenspace of } \omega_\pi^{-1}\Pi(\xi)).$$

In particular, the difference between the dimensions of the positive and negative parts is equal to ω_π^{-1} times the character value $\mathrm{tr}(\Pi(\xi))$.

There is no difficulty in finding the extended group needed in this corollary: one can use for example

$$^{\mathrm{ex}}G = G \rtimes \{1, \xi\}, \tag{6}$$

with ξ acting on G by the Cartan involution θ. In this case $z = 1$, so the statement of the corollary simplifies a bit. We allow for more general extended groups because those will turn out to be useful for the bookkeeping we want to do.

Corollary 1.7 says that understanding the existence and signatures of σ-invariant Hermitian forms on algebraic representations of G is equivalent to understanding the algebraic representations of the disconnected (complex reductive algebraic) group ${}^{\mathrm{ex}}G$. Here is what happens in the case of SL(3).

Proposition 1.8. *Suppose $G = \mathrm{SL}(3)$, with the real form $G(\mathbb{R}, \sigma) = \mathrm{SL}(3, \mathbb{R})$ given by complex conjugation of matrices. Then a compact real form of* SL(3) *is* SU(3)*, with complex conjugation given by inverse Hermitian transpose:*

$$\sigma_c(g) = {}^t\overline{g}^{-1} \qquad (g \in \mathrm{SL}(3, \mathbb{C})).$$

Then σ_c and σ commute, so the Cartan involution is

$$\theta(g) = {}^t g^{-1}, \quad K = \mathrm{SO}(3, \mathbb{C}), \quad K(\mathbb{R}) = \mathrm{SO}(3).$$

Twisting by θ carries the representation of highest weight $(\lambda_1, \lambda_2, \lambda_3)$ (see Example 1.4) to the one of highest weight $(-\lambda_3, -\lambda_2, -\lambda_1)$. In particular, the only representations fixed are the various $(\pi_m, E_{(m,0,-m)})$.

For such a representation, we can therefore find an operator

$$A_m \colon E_{(m,0,-m)} \to E_{(m,0,-m)}, \quad A_m \pi_m(g) = \pi_m({}^t g^{-1}) \quad (g \in \mathrm{SL}(3)).$$

This requirement specifies A_m up to a scalar; we can specify it precisely by requiring that A_m act by $+1$ on the unique largest SO(3) *representation F_{4m+1} inside $E_{(m,0,-m)}$.*

With this choice, we can extend π_m to a representation Π_m of the disconnected group ${}^{ex}\mathrm{SL}(3)$ of (6), by defining

$$\Pi_m(\xi) = A_m.$$

The semisimple conjugacy classes of SL(3) *on the non-identity component of ${}^{ex}\mathrm{SL}(3)$ are represented by*

$$\left\{ h(z) = \begin{pmatrix} 0 & 0 & z \\ 0 & 1 & 0 \\ -z^{-1} & 0 & 0 \end{pmatrix} \xi \right\};$$

here $h(z)$ is conjugate to $h(z^{-1})$. By a version of the Weyl character formula (for example [16, Theorem 1.43]), the trace of this element in the extended representation Π_m is

$$\mathrm{tr}(\Pi_m(h(z))) = (-1)^m (z^{2m+2} - z^{-2m-2})/(z^2 - z^{-2})$$

for z not a fourth root of 1*. The element ξ is conjugate to $h(\pm i)$; so by L'Hôpital's rule,*

$$\mathrm{tr}(\Pi_m(\xi)) = \mathrm{tr}(\Pi_m(h(i))) = m + 1.$$

The difference between the positive and negative parts of the signature of the σ-invariant Hermitian form defined by Π_m is therefore $m+1$, so this is the same form described in Example 1.4.

Perhaps the most challenging part of proving this proposition is to verify that ξ is conjugate to $h(\pm i)$, but this can be done.

Kazhdan–Lusztig theory for computing irreducible characters typically takes place in a free $\mathbb{Z}[q]$-module with basis indexed by the irreducible characters of interest. This $\mathbb{Z}[q]$ modules carries a representation of the Hecke algebra, and the Kazhdan–Lusztig polynomials are determined by the Hecke module structure.

In something like the setting of Proposition 1.5 (where we already understand characters on G, and so wish to understand just characters on ${}^{\mathrm{ex}}G - G$) this suggests that we will be interested in a free $\mathbb{Z}[q]$-module having a basis $\{m_\Pi\}$ indexed by *one* irreducible representation Π from each pair $\Pi \neq \Pi' = \Pi \otimes \epsilon$. In this module, we will think of

$$m_{\Pi'} = -m_\Pi$$

(corresponding to the fact that the characters of Π and Π' sum to zero on ${}^{\mathrm{ex}}G - G$). All of this is explained more precisely in Section 7.

The computational problem in implementing the Kazhdan–Lusztig algorithm is that we know precisely how to parametrize the δ-fixed irreducibles π of the smaller group G; but a δ-fixed irreducible corresponds only to a pair $\{\Pi, \Pi'\}$, and so only to a basis vector of the Hecke module *defined up to sign*. We need an equally precise parametrization of irreducibles of ${}^{\mathrm{ex}}G$; that is, of how to specify one of the two possible extensions of π to ${}^{\mathrm{ex}}G$. In Proposition 1.8 this happened with the requirement that A_m act by $+1$ on F_{4m+1}. This amounts to a condition involving the action of a particular element of the larger group ${}^{\mathrm{ex}}G$ on a highest weight vector.

Corollary 1.7 shows that (in the setting (5)) understanding σ-invariant Hermitian forms on finite-dimensional representations is closely related to understanding the extensions to ${}^{\mathrm{ex}}G(\mathbb{R}, \sigma)$ of irreducible representations of $G(\mathbb{R}, \sigma)$.

An important special case is when ξ of (6) acts by an inner involution of G. In this case write $\xi(g) = xgx^{-1}$ for some $x \in G$. Then the map $\xi \to (x, \epsilon)$ induces an isomorphism

$$ {}^{\mathrm{ex}}G = G \rtimes \{1, \xi\} \simeq G \times \mathbb{Z}/2\mathbb{Z} = G \times \{1, \epsilon\}. \tag{7}$$

In this, the *equal rank case*, there is no essential new information in the representation theory of ${}^{\mathrm{ex}}G$, and it is enough to work with G itself.

With the appropriate generalizations, this can be made to work for infinite-dimensional representations as well. This is discussed in detail in [4]. Just as in the case of finite-dimensional representations, it is not necesary to use the extended group in the case of an equal rank group. See [4, Section 11].

In the unequal rank case, this requires (at least implicitly) understanding the analogues of highest weights—Lie algebra cohomology for maximal nilpotent subalgebras $\mathfrak{n}$—by which infinite-dimensional representations (π, E) of real reductive groups are classified. A little more precisely, one looks at the normalizer $G_{\mathfrak{n}}$ of $\mathfrak{n}$ in G. This group acts by a character χ_π on a Lie algebra cohomology space $H^*(\mathfrak{n}, E)$, and the character χ_π determines the representation π. To specify an extension (Π, E) of π to ${}^{\mathrm{ex}}G$, one needs an extension χ_Π of χ_π to the normalizer ${}^{\mathrm{ex}}G_{\mathfrak{n}}$ of $\mathfrak{n}$ in ${}^{\mathrm{ex}}G$. To get that, we can fix any element

$$h_{\mathfrak{n}} \in {}^{\mathrm{ex}}G_{\mathfrak{n}} - G_{\mathfrak{n}}. \tag{8a}$$

Necessarily

$$h_{\mathfrak{n}}^2 = g_{\mathfrak{n}} \in G_{\mathfrak{n}}. \tag{8b}$$

An extension Π of of π to ${}^{ex}G$ is specified by specifying the single character value $\chi_\Pi(h_{\mathfrak{n}})$, which may be either square root of $\chi_\pi(g_{\mathfrak{n}})$:

$$\text{extension } \Pi \text{ of } \pi \quad \longleftrightarrow \quad \text{square root } \chi_\Pi(h_{\mathfrak{n}}) \text{ of } \chi_\pi(g_{\mathfrak{n}}). \tag{8c}$$

What makes matters difficult is that the cohomology classes needed for different representations involve different maximal nilpotent subalgebras, and (as it turns out) necessarily different elements $h_{\mathfrak{n}}$. Even worse, for a single $\mathfrak{n}$, there may be no preferred choice of $h_{\mathfrak{n}}$. We need to have a way to keep track of choices of these elements $h_{\mathfrak{n}}$, and of the square roots $\chi_{\Pi}(h_{\mathfrak{n}})$.

A natural way to reduce choices would be to try to arrange for $h_{\mathfrak{n}}$ to have order two; in that case $g_{\mathfrak{n}} = 1$, so $\chi_{\pi}(g_{\mathfrak{n}}) = 1$, and the choice $\chi_{\Pi}(h_{\mathfrak{n}})$ must be ± 1. This is more or less what happened in Proposition 1.8, and we were then able to make the "natural" choice $\chi_{\Pi}(h_{\mathfrak{n}}) = 1$. But in general we cannot always arrange for $h_{\mathfrak{n}}$ to have order 2. It turns out that there is behavior like the example of $G = \mathbb{Z}/4\mathbb{Z} = \{\pm 1, \pm i\}$ sitting inside the quaternion group ${}^{\mathrm{ex}}G$ of order 8: every element $\{\pm j, \pm k\}$ of the non-identity coset has order exactly 4. Once we are forced to consider a case when $\chi_{\pi}(g_{\mathfrak{n}}) = -1$, it is easy to believe that there can be no preferred choice of square root.

This gives a hint at the difficulties we face. To explain in more detail their resolution, we begin with the extension of the Cartan–Weyl highest weight theory to parametrize representations. This is provided by the Langlands classification, which is phrased in terms of the complex reductive dual group. Langlands' results in their original form parametrize not individual representations but "L-packets," which are collections of finite sets of irreducible representations for each of several different real forms of G. To use the construction of (6) would require introducing a different extended group for each of these different real forms. This is inconvenient at best, and is inconsistent with the cleanest formulation of the Langlands classification.

A glimpse of this inconvenience is the description of conjugacy classes in the extended group given in Proposition 1.8. What is good about the elements $h(z)$ defined there is that they normalize the standard Borel subgroup (consisting of upper triangular matrices) in SL(3); the element ξ does not. It is this good property that allows one to write a nice Weyl character formula for the elements $h(z)$. We recall next the notion of *pinning* for a reductive algebraic group, and the derived notion of *distinguished automorphism*; these are required for the formulation of the Langlands classification made in Section 3.

Definition 1.9. Suppose G is a complex connected reductive algebraic group. A *pinning* of G consists of

1. a Borel subgroup $B \subset G$;
2. a maximal torus $H \subset B$; and
3. for each simple root α, a choice of basis vector $X_\alpha \in \mathfrak{g}_\alpha$.

The pair $H \subset B$ is determined by $\{X_\alpha\}$, so we can just write $(G, \{X_\alpha\})$ for the pinning.

An algebraic automorphism δ_0 of G is called *distinguished* (with respect to this pinning) if the differential of δ_0 permutes the chosen simple root vectors X_α. (As a consequence, δ_0 must preserve H and B.)

If θ_0 is a distinguished automorphism of order one or two, we define the *distinguished extended group* to be the algebraic group ${}^{\Gamma}G$ generated by G and one more element ξ_0, subject to the relations

$$\xi_0^2 = 1, \qquad \xi_0 g = \theta_0(g)\xi_0 \quad (\mathfrak{g} \in G).$$

Recall that two automorphisms δ and δ' of G are said to be *inner* to each other if $\delta' \circ \delta^{-1}$ is an inner automorphism.

Proposition 1.10 ([11, Corollary 2.14]). *Suppose $(G, \{X_\alpha\})$ is a complex connected reductive algebraic group with a pinning. Then any automorphism δ of G is inner to a* unique *distinguished automorphism δ_0. Necessarily the order of δ_0 divides the order of δ (where we make the conventions that any nonzero natural number divides infinity, and infinity divides itself). If in addition δ is semisimple (for example, if δ has finite order), then δ is conjugate by G to an automorphism* $\mathrm{Ad}(h)\delta_0$*, for some (usually not unique) $h \in H$.*

In case θ_0 has order one or two, the proposition says that *every automorphism θ of G inner to θ_0 may be realized by the conjugation action of an element ξ of the nonidentity coset $G\xi_0$ of the corresponding distinguished extended group*. The difference from (6) is that, even if $\theta^2 = 1$, the element ξ^2 may be a *nontrivial* element of $Z(G)$. This turns out to be a small price to pay for having a single extended group to work with (as θ varies over an inner class).

The Cartan involution θ (and therefore the extended group ${}^{\mathrm{ex}}G$) is playing a double role in Corollary 1.7: first, specifying the real form $G(\mathbb{R})$; and second, specifying an automorphism of $G(\mathbb{R})$ by which we wish to twist representations. It will be convenient to separate these two roles: to study the twisting of representations of $G(\mathbb{R})$ by a second automorphism δ.

Section 2 establishes the required notation for "doubly extended groups," and recalls also Langlands' L-group. Section 3 recalls from [3] a formulation of the Langlands classification well-suited to calculation. Section 4 computes the twisting action of δ on representations. The idea here (exactly as in the original work of Knapp and Zuckerman recorded in [7]) is that this is a fairly elementary inspection of the twisting action on *parameters* for representations.

Section 5 describes a way to add information to a δ-fixed parameter—essentially choices of elements $h_{\mathfrak{n}}$ and $\chi_\Pi(h_{\mathfrak{n}})$ discussed in (8)—to specify a representation of the corresponding extended group ${}^\delta G(\mathbb{R})$. Particularly because the extended group element $h_{\mathfrak{n}}$ is not unique, the question of when two of these extended representations are equivalent is a bit subtle; Section 6 answers this question.

In this way we are able to write explicitly a basis (not just a basis defined up to sign) for the Hecke module considered in [10], and the Kazhdan–Lusztig polynomials are determined by this Hecke module. Precise formulas for the action of Hecke algebra generators on the basis are written in Section 7. Each such formula involves one to four basis vectors in the module. In [10] it was shown that these one to four basis vectors could be chosen so that the action of the generator was given by a specified matrix (of size one to four). A typical example (the *only* example in the original paper [6]) is

$$\begin{pmatrix} 0 & q \\ 1 & q-1 \end{pmatrix}. \tag{9}$$

The technical problem that led to this paper, mentioned at the beginning of the introduction, is that *these nice choices of basis vectors cannot be made consistently as the Hecke algebra generator varies*. The result is that if we fix a single choice of basis for the Hecke module, then the actions of some of the Hecke algebra generators will be given by matrices made of blocks not only like (9), but also by conjugates of such a matrix by a diagonal matrix with entries ± 1. A typical example is

$$\begin{pmatrix} 0 & -q \\ -1 & q-1 \end{pmatrix}. \tag{10}$$

The point of the formulas in Section 7 is to say precisely where the minus signs must go. In order to do this, one needs to say how to manipulate our extended parameters to get the nice basis vectors discussed in [10]. There are two cases where this manipulation is somewhat more complicated, and they are described in detail in Sections 8 and 9. Ultimately this gives an explicit algorithm for computing the polynomials of [10], which is being implemented in the `atlas of Lie groups and Representations` software [17]. The application to our computation of Hermitian forms is [4, Theorem 19.4].

A guiding principle in formulating these results is the fundamental duality theorem originating in [6, Theorem 3.1], and extended to Harish-Chandra modules in [15]. Section 11 describes how to prove this for the Hecke modules in the twisted setting. The heart of the proof in every case is that a "transpose" of one Hecke algebra action is equal to another Hecke algebra action; explicitly, that the transpose of the matrix giving an action of a generator is equal to the matrix giving the action of the same generator on a different module. That such a statement is true up to signs was clear from [10]; with the specification of the signs in this paper we are able to prove it completely.

2 Setting

Our first goal is to understand which representations are fixed by a given outer automorphism, and how to to write down the corresponding representations of the extended group. We begin by setting up some notation in this section, discuss the `atlas` parametrization of representations in Section 3, and the action of twisting on these parameters in Section 4.

We start with a connected complex reductive algebraic group G, equipped with a pinning (Definition 1.9). Acting on this we have two commuting distinguished involutive automorphisms:

$$\xi_0\colon (G,B,H) \to (G,B,H), \qquad \delta_0\colon (G,B,H) \to (G,B,H), \tag{11a}$$

satisfying

$$\xi_0(X_\alpha) = X_{\xi_0(\alpha)}, \quad \delta_0(X_\alpha) = X_{\delta_0(\alpha)} \qquad (\alpha,\ \xi(\alpha),\ \delta_0(\alpha) \in \Pi). \tag{11b}$$

See Definition 1.9 and [1, p. 34 or p. 51].

The automorphism ξ_0 defines the inner class of real forms under consideration; it is the unique Cartan involution in the inner class which is distinguished, and is the Cartan involution of the "most compact" real form in the inner class. The automorphism δ_0 defines the twisting of representations that we will consider. Since any automorphism is inner to a distinguished one there is no loss in assuming δ_0 is distinguished.

We will abuse notation and use these automorphisms to define a semidirect product of G with the Klein 4-group $(\mathbb{Z}/2\mathbb{Z})^2$:

$$^{\Delta}G = G \rtimes \{1, \xi_0, \delta_0, \xi_0\delta_0\}. \tag{11c}$$

The superscript Δ is supposed to suggest "double." The abuse of notation is that from now on ξ_0 may denote an element of $^{\Delta}G$ (which by definition is never the identity) *or* an automorphism of G (which is the identity exactly when ξ_0 defines the equal rank inner class).

It is helpful to use also the corresponding *large* ([1, p. 51]) involutive automorphism. As in [1] we write

$$e\colon \mathfrak{h} \to H, \qquad e(X) = \exp(2\pi i X); \tag{11d}$$

this is a surjective homomorphism from the Lie algebra onto H, with kernel equal to $X_*(H)$. Also we write

$$\rho = \frac{1}{2}\sum_{\beta\in R^+(G,H)} \beta, \quad \rho^\vee = \frac{1}{2}\sum_{\beta\in R^+(G,H)} \beta^\vee. \tag{11e}$$

Then $\alpha(e(\rho^\vee/2)) = -1$ for every simple root α; so if we define

$$\xi_1 = e(\rho^\vee/2)\xi_0 \in H\xi_0, \tag{11f}$$

then this element of $^{\Delta}G$ acts on G as an involutive automorphism satisfying

$$\xi_1|_H = \xi_0|H, \quad \xi_1(X_\alpha) = -X_{\xi(\alpha)}, \qquad (\alpha \in \Pi). \tag{11g}$$

This element satisfies

$$\xi_1^2 = e(\rho^\vee) =_{\text{def}} z(\rho^\vee) \in Z(G), \tag{11h}$$

a central element of order (one or) two.

Our torus $H \subset G$ has a well-defined (that is, uniquely defined up to unique isomorphism) dual torus

$$^\vee H = X^*(H) \otimes_{\mathbb{Z}} \mathbb{C}^\times. \tag{12a}$$

The characters and cocharacters of $^\vee H$ are naturally identified with the cocharacters and characters of H:

$$X^*(^\vee H) \simeq X_*(H), \qquad X_*(^\vee H) \simeq X^*(H). \tag{12b}$$

The isomorphisms here are canonical, and respect the pairings into $\mathbb{Z}$.

The automorphisms ξ_0 and δ_0 of H (cf. (11)) define automorphisms ${}^t\xi_0$ and ${}^t\delta_0$ of $X^*(H)$, and therefore

$$^\vee\xi_0 =_{\text{def}} -w_0{}^t\xi_0, \qquad {}^\vee\delta_0 =_{\text{def}} {}^t\delta_0 \tag{12c}$$

of $X^*(H)$ and of ${}^\vee H$. Here we write

$$w_0 \in W(G,H) \simeq W({}^\vee G, {}^\vee H) \tag{12d}$$

for the unique longest element, which carries $R^+(G,H)$ to $-R^+(G,H)$. *Notice the presence of a minus sign in the definition of* ${}^\vee\xi_0$ *(partly "corrected" by the factor of* w_0*) and its absence in the definition of* ${}^\vee\delta_0$. This is the way things are. One way to understand it is that ξ is related to the Cartan involution for G, which is less fundamental and natural than the Galois action for a real form. The Cartan involution acts on the root datum (with respect to a real θ-stable Cartan) by the *negative* of the Galois action on the root datum; and it is this minus sign which accounts for the minus sign in (12c).

Now we construct a dual group ${}^\vee G \supset {}^\vee H$, whose root datum is dual to that of G:

$$^\vee G \supset {}^\vee B = {}^\vee H\,{}^\vee N, \quad R^+({}^\vee G, {}^\vee H) = \{\beta^\vee \mid \beta \in R^+(G,H)\}. \tag{12e}$$

We choose also a pinning: nonzero root vectors

$$\{X_{\alpha^\vee} \mid \alpha^\vee \in \Pi^\vee\} \subset {}^\vee\mathfrak{n}. \tag{12f}$$

Such a choice of dual group and pinning is unique up to unique isomorphism. Because the automorphisms ${}^\vee\xi_0$ and ${}^\vee\delta_0$ respect the based root datum, they extend uniquely to (distinguished) automorphisms

$$\begin{aligned} &{}^\vee\xi_0 \colon ({}^\vee G, {}^\vee G, {}^\vee H) \to ({}^\vee G, {}^\vee B, {}^\vee H), \quad {}^\vee\xi_0(X_{\alpha^\vee}) = X_{-w_0\xi_0(\alpha)^\vee} \\ &{}^\vee\delta_0 \colon ({}^\vee G, {}^\vee B, {}^\vee H) \to ({}^\vee G, {}^\vee B, {}^\vee H), \quad {}^\vee\delta_0(X_{\alpha^\vee}) = X_{\delta_0(\alpha)^\vee}. \end{aligned} \tag{12g}$$

Automatically ${}^\vee\delta_0$ and ${}^\vee\xi_0$ commute. By definition the *L-group of G* is the semidirect product

$$^L G = {}^\vee G \rtimes \{1, {}^\vee\xi_0\}. \tag{12h}$$

(A little more precisely, it is this group endowed with the ${}^\vee G$-conjugacy class of $({}^\vee B, \{X_{\alpha^\vee}\}, {}^\vee\xi_0)$.)

Just as for G, it is convenient to have in hand also the *large* representative

$$^\vee\xi_1 = e(\rho/2)\xi_0, \qquad {}^\vee\xi_1(X_{\alpha^\vee}) = -X_{-w_0\xi_0(\alpha)^\vee}. \tag{12i}$$

Again this element satisfies

$$^\vee\xi_1^2 = e(\rho) =_{\text{def}} z(\rho) \in Z({}^\vee G), \tag{12j}$$

a central element of order (one or) two.

We say a little more about the identification of Weyl groups in (12d). Define

$$\begin{aligned} &s_\alpha \in \mathrm{Aut}(X_*(H)), \quad s_\alpha(t) = t - \langle \alpha, t\rangle \alpha^\vee \\ &W(G,H) = \langle s_\alpha \mid \alpha \in \Pi\rangle \subset \mathrm{Aut}(X_*(H)). \end{aligned} \tag{12k}$$

Then the identification

$$\mathrm{Aut}(X_*(H)) \supset W(G,H) \simeq W({}^\vee G, {}^\vee H) \subset \mathrm{Aut}(X^*(H))$$

is given by

$$s_\alpha \mapsto s_{\alpha^\vee}, \qquad w \mapsto {}^t w^{-1}. \tag{12l}$$

3 Atlas parameters

The basic reference for this section is [3].

As explained after Proposition 1.10, we are going to represent involutive automorphisms of G (briefly, *involutions*) by the conjugation action of elements of $G\xi_0$. For this purpose we introduce the set of *strong involutions*:

$$\mathcal{I} = \{\xi \in G\xi_0 \mid \xi^2 \in Z(G)\}. \tag{13a}$$

If $\xi \in \mathcal{I}$, then

$$\theta_\xi = \mathrm{int}(\xi), \qquad K_\xi = G^{\theta_\xi} = \mathrm{Cent}_G(\xi). \tag{13b}$$

is an involutive automorphism of G, in the inner class of ξ_0; and every such involutive automorphism arises this way. We need to allow $\xi^2 \in Z(G)$ (and not merely $\xi^2 = 1$) because not every involution in the inner class of ξ_0 arises from an element ξ of order 2. (But we *can* easily arrange for ξ to have order a power of 2.) The central element

$$z = \xi^2 \in Z(G) \tag{13c}$$

is called the *central cocharacter* of the strong involution ξ.

A *strong real form* of G is a G-conjugacy class $\mathcal{C} \subset \mathcal{I}$. The central cocharacter is constant on $\mathcal{C}$, so we may write it as

$$z(\mathcal{C}) = \xi^2 \in Z(G) \qquad (\xi \in \mathcal{C}). \tag{13d}$$

The various involutions $\{\theta_\xi \mid \xi \in \mathcal{C}\}$ form a single G-conjugacy class of involutive automorphisms of G, so the subgroups $\{K_\xi \mid \xi \in \mathcal{C}\}$ are a single G-conjugacy class as well. If G is adjoint, then these three G-conjugacy classes (strong involutions, involutions, and fixed point subgroups) are identified by the natural maps

$$\xi \to \theta_\xi \to K_\xi.$$

If G is not adjoint, however, the first of these maps need not be one-to-one: choosing a strong involution is more restrictive than choosing an involution.

Here is the reason that strong involutions and strong real forms are useful.

Proposition 3.1. *Suppose ξ and ξ' are strong involutions in the same strong real form—that is, conjugate by G ((13)). Then there is a* canonical *bijection from equivalence classes of irreducible $(\mathfrak{g}, K_\xi)$-modules to equivalence classes of irreducible $(\mathfrak{g}, K_{\xi'})$-modules.*

Proof. Suppose $g \in G$ conjugates ξ to ξ'. Then twisting by g carries $(\mathfrak{g}, K_\xi)$-modules to $(\mathfrak{g}, K_{\xi'})$-modules. So far this would have worked using just involutive automorphisms θ and θ'. What is special about strong involutions is that *the stabilizer of ξ in G is precisely K_ξ* (whereas the stabilizer of θ_ξ can be bigger). This means that the coset gK_ξ is uniquely determined. Because twisting by K_ξ acts trivially on equivalence classes of $(\mathfrak{g}, K_\xi)$-modules, it follows that the bijection we have defined is unique. □

Using these unique bijections, one can make a well-defined set of equivalence classes of irreducible modules attached to each strong real form $\mathcal{C}$. These equivalence classes are what we will study.

In classical representation theory, one fixes once and for all a Cartan involution θ of G, defining a single symmetric subgroup $K = G^\theta$. The theory of $(\mathfrak{g}, K)$-modules proceeds by defining and studying (for example) various maximal tori preserved by θ. A central idea in the `atlas` algorithms is instead to fix the maximal torus $H \subset G$, and to study various Cartan involutions preserving it. There are hints of this idea in the classical theory. For example, it is common in introductory texts to describe the principal series representations of $\mathrm{SL}(2, \mathbb{R})$, because these are closely related to the standard (diagonal) split maximal torus. When discussing the discrete series, it is common to consider instead the (isomorphic) real group $\mathrm{SU}(1, 1)$, because the discrete series are closely related to the standard (diagonal) compact maximal torus of $\mathrm{SU}(1, 1)$.

In order to pursue this idea, we need to single out the strong involutions preserving our fixed H. These are

$$\begin{aligned} \widetilde{\mathcal{X}} &= \mathcal{I} \cap \mathrm{Norm}_{G\xi_0}(H) = \{\xi \in \mathrm{Norm}_{G\xi_0}(H) \mid \xi^2 \in Z(G)\} \\ \mathcal{X} &= \widetilde{\mathcal{X}}/H \quad \text{(quotient by conjugation action of } H). \end{aligned} \tag{14a}$$

If $z \in Z(G)$, we write

$$\begin{aligned} \widetilde{\mathcal{X}}_z &= \{\xi \in \mathrm{Norm}_{G\xi_0}(H) \mid \xi^2 = z\} \\ \mathcal{X}_z &= \widetilde{\mathcal{X}}_z/H \quad \text{(quotient by conjugation action of } H) \end{aligned} \tag{14b}$$

for the subset of elements of central cocharacter z.

Write $p : \widetilde{\mathcal{X}} \to \mathcal{X}$ for the projection map.

For $x \in \mathcal{X}$ let θ_x be the restriction of θ_ξ to H for any $\xi \in p^{-1}(x)$. The central technical difficulty we face is that *the involution θ_x of H only depends on x, but the extension θ_ξ to G depends on the choice of representative ξ.*

It is easy to check that

$$\theta_x = w_x\xi_0 \in \mathrm{Aut}(H) \quad (w_x \in W(G,H)) \tag{14c}$$

for some *twisted involution* w_x with respect to ξ_0:

$$w_x\xi_0(w_x) = 1. \tag{14d}$$

Conversely, if $w \in W$ is any twisted involution with respect to ξ_0, then

$$\theta_w =_{\mathrm{def}} w\xi_0 \in \mathrm{Aut}(H) \tag{14e}$$

is an involutive automorphism of H (or, equivalently, of $X_*(H)$). We define

$$\mathcal{X}^w = \{x \in \mathcal{X} \mid w_x = w\}, \qquad \widetilde{\mathcal{X}}^w = p^{-1}\mathcal{X}^w, \tag{14f}$$

so that $\mathcal{X}$ is the disjoint union over twisted involutions w of the various $\mathcal{X}^w$.

The definition (14c) of w_x can be restated as

$$\xi = s_1\sigma_{w_x}\xi_0 \qquad (\text{some } s_1 \in H). \tag{14g}$$

Here σ_{w_x} is the Tits group representative of w_x (see (53f)). We call s_1 the *unnormalized torus part* of ξ. We compute

$$\begin{aligned}\xi^2 &= s_1\sigma_{w_x}\xi_0 s_1\sigma_{w_x}\xi_0\\ &= s_1\theta_{w_x}(s_1)\sigma_{w_x}\sigma_{\xi_0(w_x)}\\ &= s_1\theta_{w_x}(s_1)\sigma_{w_x}\sigma_{w_x^{-1}}\\ &= s_1\theta_{w_x}(s_1)e((\rho^\vee - \theta_x\rho^\vee)/2) \quad (\text{by Proposition 12.1})\\ &= (s_1e(-\rho^\vee/2))\theta_{w_x}(s_1e(-\rho^\vee/2))e(\rho^\vee).\end{aligned} \tag{14h}$$

We call $s = s_1e(-\rho^\vee/2)$ the *normalized torus part* of ξ:

$$\begin{aligned}\xi &= se(\rho^\vee/2)\sigma_{w_x}\xi_0 = s\xi_w \qquad (\text{some } s \in H).\\ \xi^2 &= s\theta_{w_x}(s)z(\rho^\vee).\end{aligned} \tag{14i}$$

Here we have used the definition of ξ_w in the following proposition.

Proposition 3.2. *For every ξ_0-twisted involution $w \in W(G,H)$ there is a basepoint (the one with trivial* normalized *torus part)*

$$\xi_w =_{def} e(\rho^\vee/2)\sigma_w\xi_0 \in \widetilde{\mathcal{X}}$$

of central cocharacter $z(\rho^\vee)$ (see (13c)):

$$\xi_w^2 = e(\rho^\vee) = z(\rho^\vee).$$

This basepoint is conjugate by G to the large representative ξ_1 of (11f).

Proof. The formula for ξ_w^2 is immediate from (14h). We omit the argument that ξ_w is conjugate to ξ_1. □

Fix a set S of representatives of the set of strong real forms:

$$\widetilde{\mathcal{X}} \supset S \overset{1-1}{\longleftrightarrow} \mathcal{I}/G. \tag{15}$$

Proposition 3.3 ([3, Corollary 9.9]). *There is a canonical bijection*

$$\mathcal{X} \longleftrightarrow \prod_{\xi' \in S} K_{\xi'} \backslash G/B.$$

The bijection restricts to classes on both sides of any fixed central cocharacter (see (13c)), in which case both sides are finite sets.

Because of this proposition, we refer to $\mathcal{X}$ as the KGB-space, and say $x \in \mathcal{X}$ is a KGB-class.

The KGB classes are parametrized first by a twisted involution $w \in W$ (see (14c)), and then (for each w) by the allowed (twisted H-conjugacy classes of) normalized torus parts. Our next task is to describe those torus parts. It is convenient to fix also a central element $z \in Z(G)$, and to restrict attention to strong involutions of central cocharacter z. According to (14i), we are therefore seeking to solve the equation

$$s\theta_w(s) = zz(-\rho^\vee) \qquad (\xi = s\xi_w). \tag{16a}$$

Conjugation by $h \in H$ replaces the torus part s by

$$s(h\theta_w(h)^{-1}),$$

so the solutions we want—elements of the KGB space $\mathcal{X}$—are cosets of the connected torus

$$(1-\theta_w)H = \text{identity component of } H^{-\theta_w} = H_0^{-\theta_w}. \tag{16b}$$

In order to keep track of such elements, we would like to have nice representatives for the cosets $H/(1-\theta_w)H$. Because the Lie algebra is the direct sum of the $+1$ and -1 eigenspaces of θ_w, we get

$$H = [H_0^{\theta_w}][H_0^{-\theta_w}], \qquad [H_0^{\theta_w}] \cap [H_0^{-\theta_w}] \subset [H_0^{\theta_w}](2) \tag{16c}$$

(the elements of order 2).

This says that every coset of $H_0^{-\theta_w}$ has a representative in $H_0^{\theta_w}$; and that this representative is unique up to multiplication by the finite 2-group $[H_0^{\theta_w}] \cap [H_0^{-\theta_w}]$. We call a coset representative in $H_0^{\theta_w}$ *preferred*. Our immediate goal is therefore to write down all solutions $s \in H_0^{\theta_w}$ of (16a).

As with many calculations in Lie theory, solving this equation is easier on the Lie algebra. We will use the exponential map isomorphisms

$$e\colon \mathfrak{h}/X_*(H) \to H, \qquad e\colon \mathfrak{h}^{\theta_w}/X_*(H)^{\theta_w} \to H_0^{\theta_w} \tag{16d}$$

of (11d). In order to do that, we first choose a logarithm g of the central cocharacter z:

$$z = e(g) \qquad (g \in \mathfrak{h} = X_*(H) \otimes_{\mathbb{Z}} \mathbb{C}). \tag{16e}$$

We say that a strong real form of central cocharacter z has *infinitesimal cocharacter* g. It is convenient (and easy) to arrange also

$$\langle \alpha, g \rangle \in \mathbb{Z}_{>0} \quad (\alpha \in R^+(G, H)). \tag{16f}$$

(Because z is assumed central, roots take integer values on g.)

Next, we choose a logarithm v for the normalized torus part s:

$$s = e(v) \qquad (v \in \mathfrak{h}^{\theta_w}). \tag{16g}$$

Now (16a) can be written

$$2v = v + \theta_w(v) = g - \rho^\vee - \ell \qquad (\text{some } \ell \in X_*(H)), \tag{16h}$$

or

$$v = (g - \rho^\vee - \ell)/2. \tag{16i}$$

Conversely, if $\ell \in X_*(H)$ has the property that

$$g - \rho^\vee + \ell \in \mathfrak{h}^{\theta_w}, \tag{16j}$$

then $e((g - \rho^\vee - \ell)/2)$ is a preferred representative for a normalized torus part (of some $\xi \in \widetilde{\mathcal{X}}$ of central cocharacter z).

We have proven the following proposition.

Proposition 3.4. *Fix an infinitesimal cocharacter g and a ξ_0-twisted involution w. Let $\theta_w = w \circ \xi_0 \in \mathrm{Aut}(H)$. The set $\mathcal{X}_g^w$ of* KGB *classes of infinitesimal cocharacter g (equivalently, of central cocharacter $z = e(g)$) with $w_x = w$ (cf. (14c)) is in one-to-one correspondence with*

$$\left\{\overline{\ell} \in X_*(H)/(1+\theta_w)X_*(H) \mid (1-\theta_w)\ell = (1-\theta_w)(g - \rho^\vee)\right\}.$$

This set is either empty (if $(1-\theta_w)(g-\rho^\vee)$ does not belong to $(1-\theta_w)X_*$), or has a simply transitive action of

$$X_*^{\theta_w}/(1+\theta_w)X_*.$$

This latter group is a vector space over $\mathbb{Z}/2\mathbb{Z}$, of dimension at most the rank of X_*.

The corresponding x has a preferred representative (cf. (16c)*) ξ with unnormalized torus part*

$$s_1 = e((g - \ell)/2),$$

(see (14g)*) or normalized torus part*

$$s = e((g - \rho^\vee - \ell)/2)$$

(see (14i)*). Here $\ell \in X_*(H)$ is a representative of $\overline{\ell}$. If we modify the element ℓ in its coset by adding $(1+\theta_w)f$ (for some $f \in X_*(H)$), then s (or s_1) is multiplied by $e((1+\theta_w)f/2)$. That is, this preferred choice of torus part is unique up to the image of $1+\theta_w$ acting on the elements $H(2)$ of order 2 in H. Another formulation is that these preferred representatives ξ of x are a single conjugacy class under $H(2)$.*

The KGB classes in this proposition usually represent several *different* strong real forms (all of a fixed central cocharacter); that is, they are usually *not* conjugate by G. The parametrization of KGB classes is so beautiful and simple precisely because of this inclusion of several real forms. For example, if $G = \mathrm{GL}(n)$, $\xi_0 = 1$, and $w = 1$ (so that we are talking about compact maximal tori in equal rank real forms), then the KGB classes amount to discrete series for strong real forms. If we choose $g = \rho^\vee$, then the proposition says that the KGB classes are indexed by $X_*(H)/2X_*(H)$, an n-dimensional vector space over $\mathbb{Z}/2\mathbb{Z}$. There are $n+1$ different strong real forms appearing in this list: the various $\mathrm{U}(p, n-p)$ with $0 \leq p \leq n$. Such a strong real form has $\binom{n}{p}$ discrete series; only when we take the union over p do we get something as simple as 2^n.

We turn now to writing down Langlands parameters for representations of real forms of G, in the form described in [1]. These are constructed in a manner roughly parallel to the strong involutions above, but in the L-group of (12h) rather than in the extended group for G.

Definition 3.5. A *Langlands parameter* for representations of real forms of G is a pair $({}^\vee\xi, \gamma)$ such that

(a) ${}^\vee\xi \in {}^\vee G\,{}^\vee\xi_0$;
(b) $\gamma \in {}^\vee\mathfrak{g}$ is semisimple; and
(c) ${}^\vee\xi^2 = e(\gamma)$.

Two Langlands parameters are called *equivalent* if they are conjugate by ${}^\vee G$. The semisimple group element ${}^\vee z = {}^\vee\xi^2 \in {}^\vee G$ is called the *central character* of the Langlands parameter, and the Lie algebra element γ is called the *infinitesimal character*.

Because Langlands parameters matter only up to conjugation by ${}^\vee G$, it is convenient to consider representatives aligned with our fixed ${}^\vee H \subset {}^\vee B$. The Langlands parameter is said to be of *type* ${}^\vee H$ if

$${}^\vee\xi \in \mathrm{Norm}_{{}^\vee G\,{}^\vee\xi_0}({}^\vee H), \quad \text{and} \quad \gamma \in {}^\vee\mathfrak{h}.$$

Finally, a Langlands parameter of type ${}^\vee H$ is said to be *integrally dominant* if it is dominant for the integral root system:

$$\langle \gamma, {}^\vee\alpha \rangle \in \mathbb{Z} \implies \langle \gamma, {}^\vee\alpha \rangle \geq 0 \quad (\alpha \in R^+(G, H)). \tag{17}$$

Harish-Chandra's theorem guarantees that representation-theoretic infinitesimal characters—homomorphisms from the center of $U(\mathfrak{g})$ to $\mathbb{C}$—are in one-to-one correspondence with ${}^\vee G$ orbits of semisimple elements in ${}^\vee\mathfrak{g}$. The infinitesimal character defined here of the Langlands parameter will turn out to correspond exactly to the representation-theoretic infinitesimal characters of the corresponding representations of real forms of G. Unfortunately the central character defined here bears no such simple relationship to the representation-theoretic central characters.

Here is the original statement of the Langlands classification (with the notion of Langlands parameter modified in accordance with [1]).

Theorem 3.6 ([9, Proposition 4.1]). *In the setting of Definition 3.5, fix a strong real form ξ of G. Attached to each equivalence class of Langlands parameters $({}^\vee\xi, \gamma)$ for G there is a finite set $\Pi_{{}^\vee\xi,\gamma}(\xi)$ of equivalence classes of irreducible $(\mathfrak{g}, K_\xi)$-modules of infinitesimal character γ. These finite sets partition the full set of equivalence classes of such representations.*

Langlands called the finite sets $\Pi_{{}^\vee\xi,\gamma}(\xi)$ *L-packets*, because of their role in automorphic representation theory.

Because of this theorem, we want to understand in more detail what Langlands parameters can look like; and for a fixed Langlands parameter, we want to understand the structure of the L-packet $\Pi_{{}^\vee\xi,\gamma}$.

Proposition 3.7. *Any Langlands parameter is equivalent to an integrally dominant one of type ${}^\vee H$. If the infinitesimal character $\gamma \in {}^\vee\mathfrak{h}$ is regular, then two Langlands parameters of type ${}^\vee H$ and infinitesimal character γ are equivalent (that is, conjugate by ${}^\vee G$) if and only if they are conjugate by ${}^\vee H$. In other words, a collection of all equivalent Langlands parameters of type ${}^\vee H$ and infinitesimal character γ is a single ${}^\vee H$-conjugacy class.*

This is an elementary consequence of the definition, and we omit the proof.

Here is some structure theory for Langlands parameters analogous to that given for strong involutions in (14).

We begin with a dual torus element—not assumed central as in (13c)—

$${}^\vee z = e(\gamma) \in {}^\vee H. \tag{18a}$$

We always wish to assume that γ is integrally dominant (17). Almost all of our results will be about the case of *regular* infinitesimal character, so we will assume

$$\langle \gamma, {}^\vee\alpha \rangle \in \mathbb{Z} \implies \langle \gamma, {}^\vee\alpha \rangle > 0 \quad (\alpha \in R^+(G, H)) \tag{18b}$$

or in other words

$$\langle \gamma, \alpha^\vee \rangle \notin \{0, -1, -2, -3, \ldots\} \qquad (\alpha \in R^+(G, H)). \tag{18c}$$

Define

$$ {}^\vee G({}^\vee z) = \text{centralizer of } {}^\vee z \text{ in } {}^\vee G \supset {}^\vee H. \tag{18d}$$

(This closed reductive subgroup of ${}^\vee G$ may be disconnected, a point which will require some attention; we write ${}^\vee G({}^\vee z)_0$ for its identity component.) An `atlas` *dual strong involution of central character* ${}^\vee z$ is an element

$$ {}^\vee\xi \in {}^\vee G\,{}^\vee\xi_0, \qquad {}^\vee\xi^2 = {}^\vee z, $$

and an `atlas` *dual strong real form of central character* ${}^\vee z$ is a ${}^\vee G({}^\vee z)$-conjugacy class ${}^\vee\mathcal{C}$ of such elements. The automorphism

$$ {}^\vee\theta_{{}^\vee\xi} = \text{int}({}^\vee\xi) \tag{18e}$$

of ${}^\vee G$ preserves ${}^\vee G({}^\vee z)$, and acts on this group (*not* in general on all of ${}^\vee G$) as an involutive automorphism. *It is therefore real forms of* ${}^\vee G({}^\vee z)$ *that are under discussion.*

Keeping in mind the case ${}^\vee z \in {}^\vee H$, we define the *dual* KGB *space*—now a space of equivalence classes of Langlands parameters—by

$$ \begin{aligned} {}^\vee\widetilde{\mathcal{X}} &= \{{}^\vee\xi \in \text{Norm}_{{}^\vee G\,{}^\vee\xi_0}({}^\vee H) \mid {}^\vee\xi^2 \in {}^\vee H\} \\ {}^\vee\mathcal{X} &= {}^\vee\widetilde{\mathcal{X}}/{}^\vee H; \end{aligned} \tag{18f}$$

just as in (14a), we are dividing by the conjugation action of ${}^\vee H$. We also write

$$ \begin{aligned} {}^\vee\widetilde{\mathcal{X}}_{{}^\vee z} &= {}^\vee\widetilde{\mathcal{X}}_\gamma = \{{}^\vee\xi \in \text{Norm}_{{}^\vee G\,{}^\vee\xi_0}({}^\vee H) \mid {}^\vee\xi^2 = {}^\vee z = e(\gamma)\}, \\ {}^\vee\mathcal{X}_{{}^\vee z} &= {}^\vee\mathcal{X}_\gamma = {}^\vee\widetilde{\mathcal{X}}_\gamma/{}^\vee H. \end{aligned} \tag{18g}$$

According to Definition 3.5, a Langlands parameter of type ${}^\vee H$ and infinitesimal character γ is a pair $({}^\vee\xi, \gamma)$, with ${}^\vee\xi \in {}^\vee\widetilde{\mathcal{X}}_\gamma$. According to Proposition 3.7, an equivalence class of Langlands parameters of infinitesimal character γ is a pair (y, γ), with $y \in {}^\vee\mathcal{X}_\gamma$.

Associated to $y \in {}^\vee\mathcal{X}$ is an involution of ${}^\vee H$ (conjugation by the element ${}^\vee\xi \in \text{Norm}_{{}^\vee G\,{}^\vee\xi_0}({}^\vee H)$—that is, the restriction to ${}^\vee H$ of ${}^\vee\theta_{{}^\vee\xi}$—for any representative ${}^\vee\xi$ of y). We denote this involution ${}^\vee\theta_y$:

$$ {}^\vee\theta_y = {}^\vee w_y\,{}^\vee\xi_0 \in \text{Aut}({}^\vee H) \simeq \text{Aut}(X^*(H)) \quad ({}^\vee w_y \in W({}^\vee G, {}^\vee H)). $$

Write

$$ {}^\vee\xi = {}^\vee s_1 \sigma_{{}^\vee w_y}\,{}^\vee\xi_0 \tag{18h}$$

(compare (14g)). The fact that $({}^\vee\theta_y)^2 = 1$ is equivalent to

$$ {}^\vee w_y\,{}^\vee\xi_0({}^\vee w_y) = 1, \tag{18i}$$

i.e., ${}^\vee w_y$ is a twisted involution (in W) with respect to the automorphism ${}^\vee\xi_0$. Just as in (14f), we define

$$ {}^\vee\mathcal{X}^{{}^\vee w} = \{y \in {}^\vee\mathcal{X} \mid {}^\vee w_y = {}^\vee w\}. \tag{18j}$$

Exactly as in Proposition 3.4, we can now describe the set of Langlands parameters attached to a given twisted involution.

Proposition 3.8. *Fix an infinitesimal character γ and a ${}^\vee\xi_0$-twisted involution ${}^\vee w$. Let ${}^\vee\theta_{{}^\vee w} = {}^\vee w \circ \xi_0 \in \mathrm{Aut}(H)$. The set $\mathcal{X}_\gamma^{{}^\vee w}$ of dual* KGB *classes of infinitesimal character γ (equivalently, of central character ${}^\vee z = e(\gamma)$) with ${}^\vee w_y = {}^\vee w$ (cf.* (18h)*) is in one-to-one correspondence with*

$$\left\{\overline{\lambda} \in X^*(H)/(1 + {}^\vee\theta_{{}^\vee w})X^*(H) \mid (1 - {}^\vee\theta_{{}^\vee w})\lambda = (1 - {}^\vee\theta_{{}^\vee w})(\gamma - \rho)\right\}.$$

This set is either empty (if $(1 - {}^\vee\theta_{{}^\vee w})(\gamma - \rho)$ does not belong to $(1 - {}^\vee\theta_{{}^\vee w})X^*$), or has a simply transitive action of

$$(X^*)^{{}^\vee\theta_{{}^\vee w}}/(1 + {}^\vee\theta_{{}^\vee w})X^*.$$

This latter group is a vector space over $\mathbb{Z}/2\mathbb{Z}$, of dimension at most the rank of X^*.

The corresponding y has a preferred representative (defined by analogy with (16c)*) ${}^\vee\xi$ with unnormalized torus part*

$${}^\vee s_1 = e((\gamma - \lambda)/2),$$

(see (18h)*) or normalized torus part*

$${}^\vee s = e((\gamma - \rho - \lambda)/2)$$

(defined by analogy with (14i)*). Here $\lambda \in X^*(H)$ is a representative of $\overline{\lambda}$. If we modify the element λ in its coset by adding $(1 + {}^\vee\theta_{{}^\vee w})\phi$ (for some $\phi \in X^*(H)$), then ${}^\vee s$ (or ${}^\vee s_1$) is multiplied by $e((1 + {}^\vee\theta_{{}^\vee w})\phi/2)$. That is, this preferred choice of torus part is unique up to the image of $1 + {}^\vee\theta_{{}^\vee w}$ acting on the elements ${}^\vee H(2)$ of order 2 in ${}^\vee H$. Another formulation is that these preferred representatives ${}^\vee\xi$ of y are a single conjugacy class under ${}^\vee H(2)$.*

The proof is identical to that of Proposition 3.4, and we omit it.

Because the set is parametrized by certain (cosets of) characters of H, it is easy and useful to reformulate the result as follows.

Corollary 3.9. *In the setting of Proposition 3.8, the set of dual* KGB *classes ${}^\vee\mathcal{X}_\gamma^{{}^\vee w}$ is naturally in bijection with the set of (automatically one-dimensional) irreducible $(\mathfrak{h}, H^{\theta_w})$-modules of differential equal to $\gamma - \rho$.*

Proof. By definition, an $(\mathfrak{h}, H^{\theta_w})$-module is a vector space carrying an algebraic action of the group H^{θ_w}, and a representation of the abelian Lie algebra $\mathfrak{h}$, so that the differential of the former is the restriction to $\mathfrak{h}^{\theta_w}$ of the latter. In the corollary, we want $\mathfrak{h}$ to act by $\gamma - \rho$, so there is nothing to say about that. The characters of H^{θ_w} are the restrictions to H^{θ_w} of characters of H; so they are indexed by

$$\overline{\lambda} \in X^*(H)/(1 - {}^t\theta_w)X^*(H), \tag{19}$$

the denominator being the characters trivial on H^{θ_w}. Now it is clear that the modules we want are indexed exactly by such cosets $\overline{\lambda}$, subject to the requirement

$$(1 + {}^t\theta_w)\lambda = (1 + {}^t\theta_w)(\gamma - \rho)$$

(that the differential of λ is the restriction of $\gamma - \rho$). This is exactly the condition on λ written in Proposition 3.8. □

Definition 3.10. A `KGB` class x and a dual `KGB` class y are said to be *aligned* if

$$-{}^t\theta_x = {}^\vee\theta_y \in \mathrm{Aut}(X^*(H))$$

((14c), (18h)). Equivalently, we require the twisted involutions to satisfy

$$w_x w_0 = {}^\vee w_y.$$

In this case we call the pair (x, y) an `atlas` *parameter* for G. We write

$$\mathcal{Z} = \{(x, y) \in \mathcal{X} \times {}^\vee\mathcal{X} \mid -{}^t\theta_x = {}^\vee\theta_y\},$$

for the set of all `atlas` parameters. If $z = e(g) \in Z(G)$ and ${}^\vee z = e(\gamma) \in {}^\vee H$ (with g and γ regular and integrally dominant), we write

$$\mathcal{Z}_{z,{}^\vee z} = \mathcal{Z}_{g,\gamma} = \{(x, y) \in \mathcal{Z} \mid x^2 = z,\ y^2 = {}^\vee z\}$$

for the subset of parameters of infinitesimal cocharacter g and infinitesimal character γ. If $w \in W$ is a ξ_0-twisted involution, we define ${}^\vee w = w w_0$ (a ${}^\vee\xi_0$-twisted involution, and write

$$\mathcal{Z}^w = \mathcal{X}^w \times {}^\vee\mathcal{X}^{{}^\vee w},$$

so that $\mathcal{Z}$ is the disjoint union over ξ_0-twisted involutions w of the subsets $\mathcal{Z}^w$.

We are now in a position to sharpen the Langlands classification Theorem 3.6, by parametrizing each L-packet.

Theorem 3.11 ([1, Theorem 1.18]). *In the setting of Definition 3.5, fix a regular and integrally dominant infinitesimal character $\gamma \in {}^\vee\mathfrak{h}$, and a regular integral dominant $g \in \mathfrak{h}$ (so that $e(g) \in Z(G)$). Then there is a natural bijection between irreducible admissible representations of infinitesimal character γ of strong real forms of G having infinitesimal cocharacter g; and the set of pairs $(x, y) \in \mathcal{Z}_{g,\gamma}$ of* `atlas` *parameters of infinitesimal cocharacter g and infinitesimal character γ. In this bijection, the strong real form may be taken to be any representative ξ of the first factor x. The Langlands parameter (Definition 3.5) may be taken to be $({}^\vee\xi, \gamma)$, with ${}^\vee\xi$ any representative of the second factor y. We write*

$$J(x, y, \gamma)$$

for the irreducible module of infinitesimal character γ attached to (x, y).

The notation requires some explanation, because we have not even said of what group $J(x, y, \gamma)$ is a representation. If ξ is any representative of x, then $\theta_\xi = \mathrm{int}(\xi)$ is a well-defined involutive automorphism of G, with fixed point group K_ξ as in (13b). Then $J(x, y, \gamma)$ is an irreducible $(\mathfrak{g}, K_\xi)$-module. A different choice ξ' of representative of x gives rise to a (necessarily different, because $K_{\xi'}$ is different) $(\mathfrak{g}, K_{\xi'})$-module $J'(x, y, \gamma)$. Part of what the theorem means is that these two different modules are identified by the canonical bijection of Proposition 3.1.

Corollary 3.12. *Suppose $({}^\vee\xi, \gamma)$ is a Langlands parameter of type ${}^\vee H$ and regular infinitesimal character γ. Fix a dominant regular infinitesimal cocharacter g as in* (16f)*. Then the union (over strong real forms of infinitesimal cocharacter g) of the L-packets $\Pi_{{}^\vee\xi,\gamma}(\xi)$ (Theorem 3.6) may be identified with the set $\mathcal{X}_g^w$ of Proposition 3.4. (Here w is the twisted involution dual to the one for ${}^\vee\xi$.) In particular, this L-packet is either empty (if $(1-\theta_w)(g-\rho^\vee)$ does not belong to $(1-\theta_w)X_*$), or has a simply transitive action of*

$$X_*^{\theta_w}/(1+\theta_w)X_*.$$

Notice that if we consider real forms of infinitesimal cocharacter $\rho^\vee$ (which includes the quasisplit real form), then this union of L-packets is never empty. This is consistent with Langlands' result that every L-packet is nonempty for the quasisplit real form.

This classical corollary is the most familiar way of thinking about ambiguity in the Langlands classification: starting with a Langlands parameter, and enumerating the various (strong) real forms where it can give a representation. We are in fact going to be interested mostly in the *dual* problem: starting with a strong real form of type H and and an infinitesimal character γ, and enumerating the various Langlands parameters giving a representation. For example, if we start with a split maximal torus, then the Langlands parameters in question just index the characters of the split maximal torus of differential γ. These admit a simply transitive action of the group $(\mathbb{Z}/2\mathbb{Z})^n$ of characters of the component group of the split torus.

Corollary 3.13. *Suppose ξ is a strong real form of type H and dominant regular infinitesimal cocharacter g. Fix a dominant regular infinitesimal character $\gamma \in \mathfrak{h}^*$. Then the collection of Langlands parameters $({}^\vee\xi, \gamma)$ of type ${}^\vee H$ aligned with ξ (Definition 3.10) may be identified with the set ${}^\vee\mathcal{X}_\gamma^{{}^\vee w}$ of Proposition 3.4. (Here ${}^\vee w = ww_0$ is the twisted involution dual to $w = w_\xi$.) In particular, this set of parameters is either empty (if $(1-{}^\vee\theta_{{}^\vee w})(\gamma-\rho)$ does not belong to $(1-{}^\vee\theta_{{}^\vee w})X^*$), or has a simply transitive action of*

$$(X^*)^{{}^\vee\theta_{{}^\vee w}}/(1+{}^\vee\theta_{{}^\vee w})X^*.$$

We are going to need a slightly more precise understanding of how the parametrization of representations in Theorem 3.11 actually works. So let us fix an `atlas` parameter (x, y) (Definition 3.10) of (integrally dominant regular) infinitesimal character $\gamma \in \mathfrak{h}^*$. Choose a strong real form representative ξ for x, so that what we are seeking to construct is an irreducible $(\mathfrak{g}, K_\xi)$-module $J(x, y, \gamma)$. The construction begins with the θ_ξ-stable Cartan subgroup H. The Cartan involution θ_ξ acts on H by θ_w, so

$$H \cap K_\xi = H^{\theta_w}. \tag{20a}$$

By definition of `atlas` parameter, $y \in {}^\vee\mathcal{X}_\gamma^{{}^\vee w}$; so by Corollary 3.9, y defines

$$\mathbb{C}(y, \gamma) = \text{irreducible } (\mathfrak{h}, H \cap K_\xi)\text{-module of differential } \gamma - \rho. \tag{20b}$$

We want to construct a $(\mathfrak{g}, K_\xi)$-module using the character $\mathbb{C}(y, \gamma)$. This is a large and complicated problem, solved by work of Zuckerman reported in [14], but here is a sketch. (Shorthand for this construction is *cohomological induction*, and we will use that phrase to refer to it.)

Now we extend $\mathbb{C}(y, \gamma)$ to a $(\mathfrak{b}, H \cap K_\xi)$-module by making $\mathfrak{n}$ act trivially, and then form the (dual to Verma) $(\mathfrak{g}, H \cap K_\xi)$-module

$$M(y, \gamma) = \mathrm{Hom}_{\mathfrak{b}}(U(\mathfrak{g}), \mathbb{C}(y, \gamma) \otimes \mathbb{C}(2\rho)). \tag{20c}$$

Here $\mathbb{C}(2\rho)$ is the representation of B on the top exterior power of the Lie algebra: the sum of the positive roots. The weight of $\mathfrak{h}$ by which we are "producing" is $\gamma + \rho$, so this is the lowest weight of $M(y, \gamma)$. By the theory of Verma modules, $M(y, \gamma)$ has infinitesimal character γ. Now we apply the Zuckerman right derived functor ([8, (2.113)])

$$\left(\Gamma^{\mathfrak{g}, K_\xi}_{\mathfrak{g}, H \cap K_\xi}\right)^S : (\mathfrak{g}, H \cap K_\xi)\text{-modules} \to (\mathfrak{g}, K_\xi)\text{-modules}, \tag{20d}$$

with $S = \dim \mathfrak{n} \cap \mathfrak{k}_\xi$, obtaining what is called the *standard* $(\mathfrak{g}, K_\xi)$*-module*

$$I^{\text{quo}}(x, y, \gamma) = \left(\Gamma^{\mathfrak{g}, K_\xi}_{\mathfrak{g}, H \cap K_\xi}\right)^S (M(y, \gamma)). \tag{20e}$$

This module has finite length, and has a unique irreducible quotient $J(x, y, \gamma)$. Proofs may be found in [8, Theorem 11.129].

4 Twisting parameters

We want to consider the action of δ_0 on representations. In terms of parameters, we need to study the action of δ_0 on $\mathcal{X}$ (and ${}^\vee\delta_0$ on ${}^\vee\mathcal{X}$). We do this in the setting of Propositions 3.8 and 3.4.

Of course a δ_0-fixed representation of infinitesimal character γ can exist only if the infinitesimal character γ is itself fixed by δ_0; that is, if and only if

$$ {}^\vee\delta_0(\gamma) = z \cdot \gamma \qquad (\text{some } z \in W). \tag{21}$$

Because of the integrally dominant condition (18b) that we impose on γ (and which is automatically inherited by ${}^\vee\delta_0(\gamma)$), it follows that

$$z \cdot R^+(\gamma) = R^+({}^\vee\delta_0(\gamma)) \subset R^+; \tag{22}$$

here $R^+(\gamma)$ is the set of positive integral roots defined by γ. If γ is integral (so that $R^+(\gamma) = R^+$), such a condition forces $z = 1$, i.e., ${}^\vee\delta_0(\gamma) = \gamma$. If γ is *not* integral, however, we can draw no such conclusion. Here is a convenient substitute.

Lemma 4.1. *Suppose $\gamma \in \mathfrak{h}^*$, and that ${}^\vee\delta_0$ preserves the W orbit of γ. Then this orbit has an integrally dominant representative γ' with the property that*

$${}^\vee\delta_0(\gamma') = \gamma'.$$

We omit the (elementary) proof. Because of this lemma, it is sufficient to study representations of infinitesimal character represented by a ${}^\vee\delta_0$-fixed integrally dominant weight

$${}^\vee\delta_0(\gamma) = \gamma, \quad \langle \gamma, \beta^\vee \rangle \notin \{0, -1, -2, \dots\} \quad (\beta \in R^+(G, H)). \tag{23}$$

The situation for real forms is a bit more subtle, because the infinitesimal cocharacter is not an invariant of a real form, but merely a useful extra parameter that we attach to the real form. Here is what we would like to know.

Conjecture 4.2. Suppose ξ is a strong involution for G (see (13a)), of (dominant regular integral) infinitesimal cocharacter g. Assume that the involution $\delta_0 \circ \theta_\xi \circ \delta_0$ is equivalent (that is, conjugate by G) to θ_ξ. Then there is a δ_0-fixed regular integral g', and a strong involution ξ' of infinitesimal cocharacter g', such that $\theta_\xi = \theta_{\xi'}$.

Unfortunately this statement is false.

Example 4.3. Suppose $G = \mathrm{SL}(4)$, endowed with the trivial distinguished automorphism ξ_0 (so that we are considering equal rank real forms) and the nontrivial distinguished automorphism δ_0. On the diagonal torus,

$$\delta_0(a_1, a_2, a_3, a_4) = (a_4^{-1}, a_3^{-1}, a_2^{-1}, a_1^{-1}).$$

Let $\omega = \exp(2\pi i/8)$ be a primitive eighth root of 1, and define

$$\xi = \mathrm{diag}(\omega, \omega, \omega, \omega^5) = \omega[\mathrm{diag}(1, 1, 1, -1)].$$

Then $\xi^2 = \omega^2 I = iI$, a central element of order 4; and

$$K_\xi = S(\mathrm{GL}(3) \times \mathrm{GL}(1)),$$

the complexified maximal compact subgroup for the real form $\mathrm{SU}(3, 1)$ of G. The infinitesimal cocharacter of this strong real form is any weight of the form

$$g = (g_1, g_2, g_3, g_4) \in \mathbb{Q}^4,$$
$$\sum g_j = 0, \quad g_1 > g_2 > g_3 > g_4,$$
$$\exp(2\pi i g_j) = \xi_j;$$

that is,

$$g_j \cong \begin{cases} 1/8 \pmod{\mathbb{Z}} & (g = 1, 2, 3) \\ 5/8 \pmod{\mathbb{Z}} & (g = 4), \end{cases}$$

These conditions are easily satisfied (for example by $(17/8, 9/8, 1/8, -27/8)$); but it is easy to see that they *cannot* be satisfied by a δ_0-fixed g. The reason is that

$$\delta_0(g) = (-g_4, -g_3, -g_2, -g_1).$$

If this is equal to g, then $g_3 = -g_2$, which contradicts the requirements $g_3 \cong g_2 \cong 1/8 \pmod{\mathbb{Z}}$. On the other hand, $\delta_0 \circ \theta_\xi \circ \delta_0$ *is* conjugate to θ_ξ (by a cyclic permutation matrix).

One might hope that in this example none of the representations of SU(3, 1) is fixed by δ_0, and indeed none of the four discrete series representations is fixed; but there is a spherical principal series representation (of infinitesimal character ρ) which *is* fixed.

In any case, we are going to consider only cases when Conjecture 4.2 is true; that is, we are going to consider only real forms of G of infinitesimal cocharacter g satisfying

$$\delta_0(g) = g, \quad \langle \beta, g \rangle \notin \{0, -1, -2, \dots\} \quad (\beta \in R^+(G, H)). \tag{24}$$

In general there will be an extra twist by the central element z, ($-I$ in the example), satisfying $\delta_0(\xi^2) = z\xi^2$.

Suppose $x \in \mathcal{X}$, and $\xi \in p^{-1}(x)$. Then $\delta_0(x) = x$ if and only if $\delta_0(\xi) = h^{-1}\xi h$ for some $h \in H$ (we cannot necessarily choose ξ so that $h = 1$). This is equivalent to

$$\xi(h\delta_0)\xi^{-1} = h\delta_0,$$

i.e.,

$$({}^{\delta_0}H)^{\theta_\xi} = \langle H^{\theta_x}, h\delta_0 \rangle.$$

Suppose ξ corresponds to $\overline{\ell} \in X_*(H)/(1 + \theta_x)X_*(H)$ by Proposition 3.4, and $\ell \in X_*(H)$ is a representative. Then x is δ_0-fixed if and only if

$$\delta_0 \ell \in \ell + (1 + \theta_x)X_*(H).$$

Here is a precise statement.

Proposition 4.4. *Suppose g is an infinitesimal cocharacter as in* (16f). *Suppose $x \in \mathcal{X}$ has infinitesimal cocharacter g, and let $w = w_x$ be the underlying twisted involution* (14c). *Assume*

$$\delta_0(g) = g, \quad \delta_0(w) = w. \tag{24a}$$

Suppose that x corresponds via Proposition 3.4 to $\overline{\ell} \in X_*(H)/(1+\theta_x)X_*(H)$. Choose $\ell \in X_*(H)$ representing $\overline{\ell}$, so x has a representative with unnormalized torus part $s_1 = e((g-\ell)/2)$, or normalized torus part $s = e((g-\ell-\rho^\vee)/2)$.

1. *The class x is fixed by δ_0 if and only if*

$$(\delta_0 - 1)\ell = (1+\theta_x)t \qquad (\text{some } t \in X_*(H)). \tag{24b}$$

2. *The element t is uniquely defined by ℓ up to adding $X_*(H)^{-\theta_x}$; if also ℓ is modified in its coset, then t changes by $(1-\delta_0)X_*(H)$.*
3. *The corresponding special representative*

$$\xi = e((g-\ell)/2)\sigma_w \xi_0 \tag{24c}$$

satisfies

$$\delta_0 \xi \delta_0^{-1} = e((1-\theta_x)t/2)\xi = e(t/2)\xi e(-t/2); \tag{24d}$$

that is, ξ is conjugate to its δ_0 twist using the element $e(t/2)$ of $H(2)$.

4. *Condition (24d) is equivalent to*

$$({}^{\delta_0}H)^{\theta_\xi} = \langle H^{\theta_x}, e(-t/2)\delta_0\rangle. \tag{24e}$$

Here is the version for ${}^\vee G$.

Proposition 4.5. *Suppose γ is an infinitesimal character as in (18b). Suppose $y \in {}^\vee\mathcal{X}$ has infinitesimal character γ, and let $w = w_y$ be the underlying twisted involution (18h). Assume*

$$^\vee\delta_0(\gamma) = \gamma, \quad ^\vee\delta_0(w) = w. \tag{25a}$$

Suppose that y corresponds via Proposition 3.8 to $\overline{\lambda} \in X^*(H)/(1+\theta_x)X^*(H)$. Choose $\lambda \in X^*(H)$ representing $\overline{\lambda}$, so y has a representative with unnormalized torus part $e((\gamma-\lambda)/2)$, and normalized torus part $e((\gamma-\lambda-\rho)/2)$.

1. *The class y is fixed by ${}^\vee\delta_0$ if and only if*

$$({}^\vee\delta_0 - 1)(\lambda) = (1+{}^\vee\theta_y)\tau \qquad (\text{some } \tau \in X^*(H)). \tag{25b}$$

2. *The element τ is uniquely defined by λ up to adding $X^*(H)^{-{}^\vee\theta_y}$; if also ℓ is modified in its coset, then t changes by $(1-\delta_0)X^*(H)$.*
3. *The corresponding special representative*

$$^\vee\xi = e((\gamma-\lambda)/2)^\vee\sigma_w {}^\vee\xi_0 \tag{25c}$$

satisfies

$$^\vee\delta_0({}^\vee\xi)^\vee\delta_0^{-1} = e((1-{}^\vee\theta_y)\tau/2) = e(\tau/2)^\vee\xi e(-\tau/2)\xi; \tag{25d}$$

that is, ${}^\vee\xi$ is conjugate to its ${}^\vee\delta_0$ twist using the element $e(\tau/2)$ of ${}^\vee H(2)$.

4. Condition (24d) is equivalent to

$$({}^{\vee\delta_0\vee}H)^{\vee\theta_{\vee\xi}} = \langle H^{\vee\theta_y}, e(-\tau/2)^{\vee}\delta_0\rangle. \tag{25e}$$

5 Extended parameters

We now define parameters for $(\mathfrak{g}, {}^{\delta_0}K)$-modules. Suppose $(x, \overline{\lambda}, \gamma)$ is a δ_0-fixed parameter. If $\xi \in p^{-1}(x) \in \widetilde{\mathcal{X}}$, then $J(x, \overline{\lambda}, \gamma)$ is a δ_0-fixed $(\mathfrak{g}, K_\xi)$-module. As discussed in the introduction this can be extended in two ways to give a $(\mathfrak{g}, {}^{\delta_0}K_\xi)$-module.

Lemma 5.1. *Suppose $(x, \overline{\lambda}, \gamma)$ is a δ_0-fixed parameter. Choose $h\delta_0 \in ({}^{\delta_0}H)^{\theta_\xi}$ as in Proposition 4.4. The two extensions of $J(x, \overline{\lambda}, \nu)$ to a $(\mathfrak{g}, {}^{\delta_0}K_\xi)$-module are parametrized by the two extensions of the character $\overline{\lambda}$ of H^{θ_x} to*

$$({}^{\delta_0}H)^{\theta_\xi} = \langle H^{\theta_x}, h\delta_0\rangle,$$

whose values at $h\delta_0$ are the two square roots of $\overline{\lambda}(h\delta_0(h))$.

We now begin to assemble the data—the *extended parameters* of Definition 5.4—that we will use to construct one of the square roots required in Lemma 5.1. We will consider representations with a fixed regular infinitesimal character, for real forms with a fixed infinitesimal cocharacter. So fix an integrally dominant infinitesimal character γ:

$$\gamma \in X^*(H)_{\mathbb{C}} \subset \mathfrak{h}^*, \quad \langle \gamma, {}^{\vee}\alpha\rangle \notin \mathbb{Z}_{<0} \quad (\alpha \in R^+(G, H)) \tag{26a}$$

and an integral dominant infinitesimal cocharacter g:

$$g \in X_*(H)_{\mathbb{Q}} \subset \mathfrak{h}, \quad \langle g, \alpha\rangle \in \mathbb{Z}_{>0} \quad (\alpha \in R^+(G, H)). \tag{26b}$$

We require (see Lemma 4.1 and Conjecture 4.2)

$$\delta_0(g) = g, \quad {}^t\delta_0(\gamma) = \gamma. \tag{27}$$

Definition 5.2. Suppose (x, y, γ) is a parameter for a δ_0-fixed representation. Define $\overline{\lambda} \in X^*(H)/(1-\theta_x)X^*(H) \simeq X^*(H^{\theta_x})$ (from y) by Proposition 3.8. Choose a preferred representative ξ for x, and define $\overline{\ell} \in X_*(H)/(1+\theta_x)X_*(H)$ corresponding to ξ, by Proposition 3.4. Choose a representative $\ell \in X_*(H)$ for $\overline{\ell}$, and choose $t \in X_*(H)$ satisfying (24b). Set $h = e(t/2)$ so $h\delta_0 \in ({}^{\delta_0}H)^{\theta_\xi}$ and $(h\delta_0)^2 = h\delta_0(h) \in H^{\theta_x}$. Define

$$\epsilon(x, y) = \overline{\lambda}(h\delta_0(h)). \tag{27a}$$

Lemma 5.3.

1. *$\epsilon(x,y)=(-1)^{\langle\overline{\lambda},(1+\delta_0)t\rangle}$,*
2. *$\epsilon(x,y)$ is independent of the choices of ξ, $\overline{\ell}$, ℓ, and t (for fixed g and γ).*

Proof. The first statment is immediate. By (24b), t is determined by ℓ up to adding elements of $X_*(H)^{-\theta_x}$, and by $\overline{\ell}$ up to $(1-\delta_0)X_*(H)$. Therefore

$$t \text{ is determined by } x \text{ up to adding } X_*(H)^{-\theta_x}+(1-\delta_0)X_*(H).$$

We have

$$\begin{aligned}(-1)^{\langle\lambda,(1+\delta_0)t\rangle} &= (-1)^{\langle\lambda,(1\pm\delta_0)t\rangle}\\ &= (-1)^{\langle(1\pm{}^t\delta_0)\lambda,t\rangle}\\ &= (-1)^{\langle(1\pm{}^\vee\theta_y)\tau,t\rangle}\\ &= (-1)^{\langle\tau,(1\pm\theta_x)t\rangle}.\end{aligned}\tag{28}$$

The second equality shows this sign is unchanged by adding to t an element of $(1-\delta_0)X_*(H)$, and the last one shows it is unaffected by adding elements of $X_*(H)^{-\theta_x}$. ◻

We need to choose a square root of $\epsilon(x,y)$. Just as for the parameters (x,y) for representations of real forms of G, it is helpful to symmetrize the picture with respect to G and ${}^\vee G$.

Definition 5.4. Fix γ, g as in (26), and a ξ_0-twisted involution $w\in W$. Let $\theta=\theta_w=w\xi_0\in\mathrm{Aut}(H)$ and ${}^\vee\theta={}^\vee\theta_{ww_0}=-{}^t\theta$.

An *extended parameter* (for the twisted involution w and the specified infinitesimal character and cocharacter) is a set

$$E=(\lambda,\tau,\ell,t)$$

where

1. $\lambda\in X^*(H)$ satisfies $(1-{}^\vee\theta)\lambda=(1-{}^\vee\theta)(\gamma-\rho)$;
2. $\ell\in X_*(H)$ satisfies $(1-\theta)\ell=(1-\theta)(g-{}^\vee\rho)$;
3. $\tau\in X^*(H)$ satisfies $({}^\vee\delta_0-1)\lambda=(1+{}^\vee\theta)\tau$;
4. $t\in X_*(H)$ satisfies $(\delta_0-1)\ell=(1+\theta)t$.

Associated to an extended parameter $E=(\lambda,\tau,\ell,t)$ are the following elements:

(a) $\xi(E)\in\widetilde{\mathcal{X}}$ corresponds to λ by Proposition 3.8;
(b) ${}^\vee\xi(E)\in{}^\vee\widetilde{\mathcal{X}}$ corresponds to ℓ by Proposition 3.4;
(c) $x(E)=_{\mathrm{def}}p(\xi(E))\in\mathcal{X}$, $x(E)^2=e(g)$;
(d) $y(E)=_{\mathrm{def}}p({}^\vee\xi(E))\in{}^\vee\mathcal{X}$, $y(E)^2=e(\gamma)$;
(e) $h(E)\delta_0=e(t/2)\delta_0\in({}^{\delta_0}H)^{\theta_\xi}$ (cf. (24e));
(f) ${}^\vee h(E){}^\vee\delta_0=e(\tau/2){}^\vee\delta_0\in({}^{{}^\vee\delta_0}{}^\vee H)^{{}^\vee\theta_{{}^\vee\xi}}$ (cf. (25e)).

We say E is an *extended parameter for* $(x(E),y(E))$.

Definition 5.5. Suppose (λ, τ, ℓ, t) is an extended parameter for (x, y). Define

$$z(\lambda, \tau, \ell, t) = i^{\langle \tau, (1+\theta_x)t\rangle}(-1)^{\langle \lambda, t\rangle}. \tag{29}$$

By (28) we have

$$z(\lambda, \tau, \ell, t)^2 = \epsilon(x, y). \tag{30}$$

Associated to (λ, τ, ℓ, t) is an extension of $J(x, y, \gamma)$ defined as follows.

Definition 5.6. Suppose (λ, τ, ℓ, t) is an extended parameter for (x, y). Set $\xi = \xi(\lambda, \tau, \ell, t)$ and $h = h(\lambda, \tau, \ell, t) = e(t/2)$. Define an extension of $\overline{\lambda}$ to $({}^{\delta_0}H)^{\theta_\xi}$ (see Lemma 5.1) by having it take the value $z(\lambda, \tau, \ell, t)$ at $h\delta_0$. This defines an extension of $J(x, y, \gamma)$ to a $(\mathfrak{g}, {}^{\delta_0}K_\xi)$-module, denoted $J_z(\lambda, \tau, \ell, t)$. (The subscript z refers to the particular formula chosen in Definition 5.5.)

We deal with the question of equivalence of parameters in the next section.

For later use we record precisely how these elements depend on the various choices. Suppose we are given $(x, y) \in \mathcal{Z}$. Choose representatives ξ for x and ${}^\vee\xi$ for y by Propositions 3.4 and 3.8, respectively. That is

$$\begin{aligned} \xi &= e((g-\ell)/2)\sigma_w \xi_0 \\ {}^\vee\xi &= e((\gamma-\lambda)/2)\,{}^\vee\sigma_{ww_0}\,{}^\vee\xi_0. \end{aligned} \tag{31}$$

Then

$$\begin{aligned} &\ell \text{ is determined by } \xi \text{ up to } 2X_*^{\theta_x} \\ &\ell \text{ is determined by } x \text{ up to } (1+\theta_x)X_* \\ &\lambda \text{ is determined by } {}^\vee\xi \text{ up to } 2(X^*)^{{}^\vee\theta_y} \\ &\lambda \text{ is determined by } y \text{ up to } (1+{}^\vee\theta_y)X^*. \end{aligned} \tag{32}$$

It is helpful to write in addition

$$\begin{aligned} f &= (\delta_0 - 1)\ell = (1+\theta_x)t \\ \phi &= ({}^\vee\delta_0 - 1)\lambda = (1+{}^\vee\theta_y)\tau. \end{aligned} \tag{33}$$

Because (for example) t is evidently determined by f up to $X_*^{-\theta_x}$, the corresponding uniqueness statements are

$$\begin{aligned} &f \text{ is determined by } \xi \text{ up to } 2(1-\delta_0)X_*^{\theta_x} \\ &f \text{ is determined by } x \text{ up to } (1-\delta_0)(1+\theta_x)X_* \\ &t \text{ is determined by } \xi \text{ up to } (1-\delta_0)(1+\theta_x)X_*^{\theta_x} + X_*^{-\theta_x} \\ &t \text{ is determined by } x \text{ up to } (1-\delta_0)X_* + X_*^{-\theta_x} \\ &\phi \text{ is determined by } {}^\vee\xi \text{ up to } 2(1-{}^\vee\delta_0)(X^*)^{{}^\vee\theta_y} \\ &\phi \text{ is determined by } y \text{ up to } (1-{}^\vee\delta_0)(1+{}^\vee\theta_y)X^* \\ &\tau \text{ is determined by } {}^\vee\xi \text{ up to } (1-{}^\vee\delta_0)(X^*)^{{}^\vee\theta_y} + (X^*)^{-{}^\vee\theta_y} \\ &\tau \text{ is determined by } y \text{ up to } (1-{}^\vee\delta_0)(1+{}^\vee\theta_y)X^* + (X^*)^{-{}^\vee\theta_y}. \end{aligned} \tag{34}$$

Eventually we will want a parallel choice of square root of ϵ related to the dual group ${}^\vee G$. This is

$$\begin{aligned}\zeta(\lambda,\tau,\ell,t) &=_{\text{def}} i^{\langle\tau,f\rangle}(-1)^{\langle\tau,\ell\rangle}\\ &= z(\lambda,\tau,\ell,t)(-1)^{\langle\lambda,t\rangle}(-1)^{\langle\tau,\ell\rangle}.\end{aligned} \tag{35}$$

6 Equivalences of extended parameters

In this section we record how to tell when two of the extended modules defined in Definition 5.6 are equivalent.

Fix γ and g as usual, and suppose (x, y) is a δ_0-fixed parameter. Choose two extended parameters

$$E = (\lambda,\tau,\ell,t), \qquad E' = (\lambda',\tau',\ell',t') \tag{36a}$$

for (x, y) (Definition 5.4). Set $\xi = \xi(E)$, $\xi' = \xi(E')$, and define

$$K_\xi = \mathrm{Cent}_G(\xi), \qquad {}^{\delta_0}K_\xi = \mathrm{Cent}_{\langle G,\delta_0\rangle}(\xi), \tag{36b}$$

and similarly with primes. Because ξ and ξ' are assumed to be conjugate by G, Proposition 3.1 provides a *canonical* identification

$$\text{irreducible } (\mathfrak{g}, K_\xi)\text{-modules} \simeq \text{irreducible } (\mathfrak{g}, K_{\xi'})\text{-modules} \tag{36c}$$

(by twisting the action by $\mathrm{Ad}(g)$). Exactly the same argument applies to irreducible $(\mathfrak{g}, {}^{\delta_0}K_\xi)$-modules.

Definition 6.1. We say E *is equivalent to* E' if $J_z(E)$ and $J_z(E')$ correspond by this canonical identification.

Define $\mathrm{sgn}(E, E') = 1$ if $J_z(E) \sim J_z(E')$, or -1 otherwise.

In other words, if [ ] denotes the image of a representation of an extended group in the module $\mathcal{M}$ (see the Introduction or Section 7), then

$$[J_z(E)] = \mathrm{sgn}(E, E')[J_z(E')]. \tag{36d}$$

The Langlands classification attaches to (ξ, y) an irreducible $(\mathfrak{g}, K_\xi)$-module $J(\xi, y)$. The construction of $J(\xi, y, \gamma)$ begins with a one-dimensional $(\mathfrak{h}, H^{\theta_\xi})$-module $\mathbb{C}_{y,\gamma}$. Cohomological induction produces a "standard" $(\mathfrak{g}, K_\xi)$-module $I(\xi, y, \gamma)$, with unique irreducible quotient $J(\xi, y, \gamma)$. The nature of this construction makes it obvious that the identification of (36c) carries $I(\xi, y, \gamma)$ to $I(\xi', y, \gamma)$, and consequently $J(\xi, y, \gamma)$ to $J(\xi', y, \gamma)$.

Here is more detail on how the extended group representation of Definition 5.6 is constructed. First, the element $e(t/2)\delta_0$ is a generator for the extended Cartan:

$$({}^{\delta_0}H)^{\theta_\xi} = \langle e(t/2)\delta_0, H^{\theta_\xi}\rangle. \tag{37a}$$

The one-dimensional module $\mathbb{C}_{y,\gamma}$ extends to a one-dimensional $(\mathfrak{h}, ({}^{\delta_0}H)^{\theta_\xi})$-module by declaring

$$e(t/2)\delta_0 \text{ acts by the scalar } z(\lambda,\tau,\ell,t). \tag{37b}$$

Cohomological induction from this one-dimensional representation provides an extension of $I(\xi, y, \gamma)$ to a $(\mathfrak{g}, {}^{\delta_0}K_\xi)$-module $I_z(\lambda,\tau,\ell,t)$, and then $J_z(\lambda,\tau,\ell,t)$ is its unique irreducible quotient. Of course exactly the same words describe $J_z(\lambda',\tau',\ell',t')$.

So how do we decide whether these two modules are equivalent? According to (32), we can find $u \in X_*(H)$ so that

$$\begin{aligned} \ell' &= \ell + (\theta_x + 1)u \\ f' &= f + (\theta_x + 1)(\delta_0 - 1)u. \end{aligned} \tag{37c}$$

It follows that

$$e(u/2)\cdot\xi\cdot e(-u/2) = \xi'. \tag{37d}$$

If we define

$$t_2 = t + (\delta_0 - 1)u, \tag{37e}$$

then (ℓ', t_2) is another choice of representative for x as in (26), and in fact conjugate to (ℓ, t) by $e(u/2)$:

$$e(u/2)\cdot e(t/2)\delta_0 \cdot e(-u/2) = e(t_2/2)\delta_0. \tag{37f}$$

Consequently,

$$i =_{\mathrm{def}} t' - t_2 \in X_*^{-\theta_x}, \qquad t' = t + (\delta_0 - 1)u + i. \tag{37g}$$

In exactly the same way, we find

$$\begin{aligned} \lambda' &= \lambda + ({}^\vee\theta_y + 1)\omega \qquad (\text{some } \omega \in X^*(H)) \\ \tau' &= \tau + ({}^\vee\delta_0 - 1)\omega + \iota \qquad (\text{some } \iota \in X^*(H)^{-\theta_y}) \\ \phi' &= \phi + ({}^\vee\delta_0 - 1)({}^\vee\theta_y + 1)\omega. \end{aligned} \tag{37h}$$

Proposition 6.2. *Suppose $E = (\lambda,\tau,\ell,t)$ and $E' = (\lambda',\ell',\tau',t')$ are extended parameters for (x, y). Then*

$$\mathrm{sgn}(E, E') = (-1)^{\langle (1+{}^\vee\delta_0)\tau, u\rangle}(-1)^{\langle \iota, t'\rangle}.$$

Here u and ι are defined in (37c), (37g), and (37h).

Proof. We change the parameter (λ,τ,ℓ,t) to (λ',ℓ',τ',t) in three steps:

$$\begin{aligned} E = (\lambda,\tau,\ell,t) \to F = (\lambda,\tau,\ell',t_2) \to \\ G = (\lambda,\tau,\ell',t') \to E' = (\lambda',\tau',\ell',t'). \end{aligned} \tag{38a}$$

In the first step we have conjugated by $e(u/2)$. It follows easily that the extended representations correspond if and only if the scalars chosen for the actions of $e(t/2)\delta_0$ and $e(t_2/2)\delta_0$ agree. That is,

$$\operatorname{sgn}(E,F) = z(E)/z(F). \tag{38b}$$

At the second step of (38a), we are keeping the group ${}^{\delta_0}K_{\xi'}$ the same, but changing the representative of the extended Cartan from $e(t_2/2)\delta_0$ to $e(t'/2)\delta_0$. This gives an equivalent extended parameter exactly if we multiply the scalar by

$$(-1)^{\langle\lambda,t'-t_2\rangle} = (-1)^{\langle\lambda,i\rangle}.$$

Therefore

$$\operatorname{sgn}(F,G) = \frac{z(F)}{z(G)}(-1)^{\langle\lambda,i\rangle}. \tag{38c}$$

Finally, in the last step of (38a) the group and the extended Cartan representative remain the same; all that may change is the scalar z. Therefore

$$\operatorname{sgn}(G,E') = z(G)/z(E'). \tag{38d}$$

Combining (38b)–(38d), we find

$$\operatorname{sgn}(E,E') = \frac{z(E)}{z(F)}\frac{z(F)}{z(G)}(-1)^{\langle\lambda,i\rangle}\frac{z(G)}{z(E')} = \frac{z(E)}{z(E')}(-1)^{\langle\lambda,i\rangle}. \tag{38e}$$

It remains to compute $z(E)/z(E')$. We do this in two steps. First of all we have from (29)

$$z(E)/z(G) = i^{\langle\tau,(1+\theta_x)(t-t')\rangle}(-1)^{\langle\lambda,t-t'\rangle}. \tag{38f}$$

With u and i given by (37g) this gives

$$z(E)/z(G) = i^{\langle\tau,(1+\theta_x)[(1-\delta_0)w+i]\rangle}(-1)^{\langle\lambda,(\delta_0-1)u+i\rangle} \tag{38g}$$

and a short computation using the identities gives

$$\begin{aligned} z(E)/z(G) &= (-1)^{\langle({}^\vee\delta_0+{}^\vee\theta_y)\tau,u\rangle}(-1)^{\langle(1+{}^\vee\theta_y)\tau,u\rangle}(-1)^{\langle\lambda,i\rangle} \\ &= (-1)^{\langle(1+{}^\vee\delta_0)\tau,u\rangle}(-1)^{\langle\lambda,i\rangle}. \end{aligned} \tag{38h}$$

Next we compute

$$z(G)/z(E') = i^{\langle\tau-\tau',(1+\theta_x)t'\rangle}(-1)^{\langle\lambda-\lambda',t'\rangle}. \tag{38i}$$

Using (37h) this gives

$$\begin{aligned} z(G)/z(E') &= i^{\langle({}^\vee\delta_0-1)\omega+\iota,(1+\theta_x)t'\rangle}(-1)^{\langle(1+{}^\vee\theta_y)\omega,t'\rangle} \\ &= (-1)^{\langle\omega,(1+\theta_x)t'\rangle}(-1)^{\langle\iota,t'\rangle}(-1)^{\langle\omega,(1+\theta_x)t'\rangle} \\ &= (-1)^{\langle\iota,t'\rangle}. \end{aligned} \tag{38j}$$

Multiplying (h) and (i) gives

$$z(E)/z(E') = (-1)^{\langle(1+{}^\vee\delta_0)\tau,u\rangle}(-1)^{\langle\lambda,i\rangle}(-1)^{\langle\iota,t'\rangle}. \tag{38k}$$

Multiplying both sides by $(-1)^{\langle\lambda,i\rangle\rangle}$ and using (38e) gives the result. □

6.1 Duality for extended parameters

We offer some remarks about duality in the sense of [15]. Define a group (called ${}^\vee G(e(\gamma))_0$ in (18d)):

$$^\vee G(\gamma) = [\mathrm{Cent}_{^\vee G}(e(\gamma))]_0 \supset {}^\vee H, \tag{39a}$$

a connected reductive group with root system

$$^\vee R(\gamma) = \{\alpha^\vee \in R^\vee \mid \langle\gamma,\alpha^\vee\rangle \in \mathbb{Z}\}, \tag{39b}$$

the integral roots for the infinitesimal character γ. The adjoint action of the representative

$$^\vee\xi = e((\gamma-\lambda)/2)^\vee\sigma_y{}^\vee\xi_0 \tag{39c}$$

defines an involutive automorphism of ${}^\vee G(\gamma)$, so

$$^\vee K_{^\vee\xi} = \mathrm{Cent}_{^\vee G(\gamma)}(^\vee\xi) \tag{39d}$$

is a symmetric subgroup of ${}^\vee G(\gamma)$. By symmetry, the parameter (y,x) defines an irreducible $({}^\vee\mathfrak{g}(\gamma), {}^\vee K_{^\vee\xi})$-module ${}^\vee J(x,y)$, with infinitesimal character g. (To be precise, we need to introduce a covering group related to the difference in ρ-shifts between ${}^\vee G$ and ${}^\vee G(\gamma)$, but we will overlook this technicality.) As in (24e), we find that

$$^\vee h^\vee\delta_0 = e(\tau/2)^\vee\delta_0 \tag{39e}$$

is a representative for an extended Cartan. As in Definition 5.2 we need to take a square root of

$$\overline{\ell}((^\vee h^\vee\delta_0)^2) = (-1)^{\langle\ell,(1+^\vee\delta_0)\tau\rangle} \tag{39f}$$

which by (28) is precisely the sign $\epsilon(x,y)$ of Definition 5.2.

Therefore we may define an extended representation by making $e(\tau/2)^\vee\delta_0$ act by any desired square root of ϵ. It turns out that duality dictates choosing a different square root than we did earlier; we choose ζ as in (35):

$$\zeta(\lambda,\tau,\ell,t) = i^{\langle\tau,f\rangle}(-1)^{\langle\tau,\ell\rangle} = z(\lambda,\tau,\ell,t)(-1)^{\langle\lambda,t\rangle}(-1)^{\langle\ell,\tau\rangle}. \tag{39g}$$

Then define an extended representation ${}^\vee J_\zeta(\lambda,\tau,\ell,t)$ by

$$e(\tau/2)^\vee\delta_0 \mapsto \zeta(\lambda,\tau,\ell,t). \tag{39h}$$

The point of this choice of sign is that it makes the next result hold. Recall that if E, E' are parameters for (x, y), then $\mathrm{sgn}(E, E')$ is defined by the identity

$$[J_z(E)] = \mathrm{sgn}(E, E')[J_z(E')],$$

and a formula for it is given in Proposition 6.2.

Proposition 6.3. *Suppose E, E' are extended parameters for (x, y). Then*

$$[{}^\vee J_\zeta(E)] = \mathrm{sgn}(E, E')[{}^\vee J_\zeta(E')],$$

where $\mathrm{sgn}(E, E')$ *is defined in Definition 6.1. Equivalently,*

$$J_z(E) \simeq J_z(E') \text{ if and only if } {}^\vee J_\zeta(E) \simeq {}^\vee J_\zeta(E'). \tag{40}$$

The proof is identical to that of Proposition 6.2. What matters for us, and what is by no means automatic, is that the sign is the same as the sign in Definition 6.1. We deduce

Corollary 6.4. *In the setting (26), there is a natural bijection from δ_0-fixed extended representations (of strong real forms of infinitesimal cocharacter g) of G, of infinitesimal character γ; to ${}^\vee\delta_0$-fixed representations of (strong real forms of infinitesimal cocharacter γ) of ${}^\vee G(\gamma)$, of infinitesimal character g. The bijection sends $J_z(\lambda, \tau, \ell, t)$ to ${}^\vee J_\zeta(\lambda, \tau, \ell, t)$.*

The fact that this map is well defined on equivalence classes is precisely (40). In Section 11 we use this to extend the duality of [15] to the twisted setting.

The formulations of these results are designed to allow a theoretical analysis of all possible parameters for extended representations. For computational purposes, one may simply want to ask when two given parameters are equivalent. To answer that question using the results above requires calculating elements u and ω by solving their defining equations (37c) and (37h). This is not enormously difficult, but it is not necessary. We therefore conclude this section with a simpler formula for $\mathrm{sgn}(E, E')$.

Proposition 6.5. *Suppose E and E' are extended parameters for (x, y). Then*

$$\begin{aligned}\mathrm{sgn}(E, E') &\\ &= i^{\langle({}^\vee\delta_0-1)\lambda, t'-t\rangle+\langle\tau'-\tau,(\delta_0-1)\ell'\rangle}(-1)^{\langle\tau,\ell'-\ell\rangle}(-1)^{\langle\lambda'-\lambda,t'\rangle}(-1)^{\langle\tau,t'-t\rangle}\\ &= i^{\langle\tau',(\delta_0-1)\ell'\rangle-\langle\tau,(\delta_0-1)\ell\rangle}(-1)^{\langle\tau,\ell'-\ell\rangle}(-1)^{\langle\lambda'-\lambda,t'\rangle}.\end{aligned}$$

Here the two expressions on the right are automatically equal, and the powers of i appearing are automatically even.

The proof is similar to the proofs of Propositions 6.2 and 6.3. We omit the details.

7 Hecke algebra action

Our goal is to compute the Hecke algebra action defined in [10]. We begin by summarizing the definition of this Hecke algebra module. We then explain what extra information is needed, beyond the formulas of [10, Sections 7.5–7.7], to carry out the computation.

In Sections 7 through 9 we consider the case of integral infinitesimal character. In Section 10 we discuss the modification necessary to treat the general case.

We start with our group G and a pair of commuting involutions δ_0, ξ_0 as in Section 2. Fix a regular, integral infinitesimal character $\gamma \in \mathfrak{h}^*$. As always we assume γ is integrally dominant as in Definition 3.5; since γ is integral this means γ is dominant: ${}^\vee\alpha(\gamma) \in \mathbb{Z}_{>0}$ for all $\alpha > 0$. Let $\mathcal{H}$ be the twisted Hecke algebra of [10, Section 4], and set $\mathcal{A} = \mathbb{Z}[q^{\frac{1}{2}}, q^{-\frac{1}{2}}]$. Fix a strong involution ξ inner to ξ_0, and set $K = K_\xi$. Associated to γ is an $\mathcal{H}$-module M, defined in [10, Section 2.3]. In our setting this is a quotient of the Grothendieck group over $\mathcal{A}$ of $(\mathfrak{g}, {}^{\delta_0}K)$-modules with infinitesimal character γ. Write $[X]$ for the image in M of a $(\mathfrak{g}, {}^{\delta_0}K)$-module X. Let χ be the non-trivial extension of the trivial representation of a one-dimensional $(\mathfrak{g}, {}^{\delta_0}K)$-module. In M we have the relation

$$[X] + [X \otimes \chi] \equiv 0.$$

Therefore M has a basis consisting of one extension to $(\mathfrak{g}, {}^{\delta_0}K)$ of each irreducible δ_0-fixed $(\mathfrak{g}, K)$-module with infinitesimal character γ. Furthermore if J is irreducible, and is not the extension of an irreducible $(\mathfrak{g}, K)$-module, then $[J] \equiv 0$.

Associated to a δ_0-orbit κ of simple roots is a generator T_κ of $\mathcal{H}$. Suppose I is a standard, δ_0-fixed $(\mathfrak{g}, K)$-module with infinitesimal character γ, and $\widetilde{I}$ is an extension of I to a $(\mathfrak{g}, {}^{\delta_0}K)$-module. Then formulas for $T_\kappa([\widetilde{I}])$ given in [10, Sections 7.5–7.7] are of the following form. There is a set $\{I_i \mid 1 \leq i \leq n\}$ (with $n \leq 3$) of standard, δ_0-fixed $(\mathfrak{g}, K)$-modules, such that the appropriate formula

$$T_\kappa([\widetilde{I}]) = \sum_i a_i[\widetilde{I}_i] \tag{41}$$

of [10] holds *for some choices of extension of each I_i to a $(\mathfrak{g}, {}^{\delta_0}K)$-module $\widetilde{I}_i$*. If we choose each extension $\widetilde{I}_i$ arbitrarily, then (41) holds with a factor of ± 1 in front of each term on the right.

It is natural to ask if it is possible to choose the $\widetilde{I}_i$ uniformly, so that the formulas (41) hold for all I and κ. The fact that in the `2i12` and `2r21` cases there is a term with a negative sign is a hint that this might not be the case, and it turns out not to be possible in general.

Instead, we carry over the Hecke module structure to our extended parameters, and compute the Hecke operators in this setting, keeping the extra information of which extensions (i.e., signs) appear in the formulas. This is straightforward except when κ is of type `2i12f, 2r21f, 2Ci` or `2Cr`.

Definition 7.1. Let $\mathcal{M}$ be the $\mathcal{A}$-module spanned by the extended parameters of infinitesimal character γ, modulo the relation

$$[E] \equiv \mathrm{sgn}(E, E')[E'].$$

By (36d) the map $[J_z(E)] \to [E]$ is a well-defined $\mathcal{A}$-module isomorphism. Using this we carry over the $\mathcal{H}$-module structure on M to define $\mathcal{M}$ as an $\mathcal{H}$-module.

To interpret the formulas of [10, Sections 7.5–7.7] in terms of $\mathcal{M}$ we need the notion of Cayley transform (defined only for certain particular κ) and cross action (defined for *every* κ) of extended parameters (defined in that reference). The rows of Tables 2–4 corresponding to Cayley transforms are labeled *Cay*, and those for cross action *crx*.

In addition, when κ is of type `2i12, 2r21, 2Ci` or `2Cr`, the formulas in [10] make use of one more transform, given (on the level of parameters for G) by the cross action of just *one* of the two simple roots comprising κ. On most parameters, this cross action will not give a δ_0-fixed parameter; like the Cayley transforms, the definition makes sense only when κ is of one of these four special types. The corresponding rows of Table 3 are labeled *cr1x*.

These formulas are given in Tables 2–4. Except in the cases noted above, this gives the formulas for the Hecke algebra action (see Proposition 7.2).

Here are some notes for interpreting the tables. Always we start with a δ_0-fixed representation of (${}^\vee\delta_0$-fixed) infinitesimal character γ, for a strong real form of δ_0-fixed infinitesimal cocharacter g, with `atlas` parameter (x, y). Let (λ, τ, ℓ, t) be an extended parameter for (x, y) (Definition 5.4).

We also fix a ${}^\vee\delta_0$-orbit κ on the set of simple roots, consisting of either

$$\begin{aligned} &\text{one root } \{\alpha = {}^\vee\delta_0(\alpha)\} && && \text{(type 1); or}\\ &\text{two roots } \{\alpha, \beta = {}^\vee\delta_0(\alpha)\}, && \langle\alpha, \beta^\vee\rangle = 0 && \text{(type 2); or}\\ &\text{two roots } \{\alpha, \beta = {}^\vee\delta_0(\alpha)\}, && \langle\alpha, \beta^\vee\rangle = -1 && \text{(type 3).}\end{aligned} \tag{42a}$$

We will sometimes write

$$\kappa =_{\text{def}} \alpha + \beta \in X^*, \qquad \kappa^\vee = \alpha^\vee + \beta^\vee \in X_* \tag{42b}$$

in types `2` and `3`. (The weight κ is a root in type `3`, but not in type `2`.) Let

$$w_\kappa = \begin{cases} s_\alpha & \text{type 1} \\ s_\alpha s_\beta & \text{type 2} \\ s_\alpha s_\beta s_\alpha = s_\kappa & \text{type 3.} \end{cases} \tag{42c}$$

Then W^{δ_0} is a Coxeter group with these elements as Coxeter generators.

We will write (x_1, y_1) for the `atlas` parameters defining (one of) the other δ_0-fixed representations appearing in the action of the Hecke algebra generator T_κ on (x, y), given by a (possibly iterated) cross action or Cayley transform. The point of the tables is to calculate new extended parameters, denoted $E_1 = (\lambda_1, \tau_1, \ell_1, t_1)$, for (x_1, y_1) in terms of $E = (\lambda, \tau, \ell, t)$.

Write $w_\kappa \times E$ for the cross action on E (described in the crx rows of Tables 2–4). Write

$$w_\kappa \times_1 E \qquad (\kappa \text{ of type } \mathtt{2i12} \text{ or } \mathtt{2r21}) \tag{42d}$$

for the element extending $s_\alpha \times (\lambda, \ell)$ defined in the cr1x rows of Table 3. Finally, write

$$c_\kappa(E) = E_\kappa \quad \text{or} \quad c_\kappa(E) = \{E_\kappa, E'_\kappa\} \tag{42e}$$

for the (possibly multi-valued) Cayley transform defined by the Cay rows of Tables 2–4.

We will write

$$\gamma_\alpha =_{\text{def}} \langle \gamma, \alpha^\vee \rangle, \quad g_\alpha =_{\text{def}} \langle \alpha, g \rangle, \tag{42f}$$

and similarly for λ and ℓ; these quantities are all integers. The δ_0-fixed requirement means that

$$\gamma_\alpha = \gamma_\beta, \quad g_\alpha = g_\beta \qquad (\text{types } \mathtt{2} \text{ and } \mathtt{3}). \tag{42g}$$

The δ_0-fixed requirement on λ and ℓ is more subtle, and with the details depending on the case. For example, we have

$$\lambda_\alpha + \lambda_\beta = 2(\gamma_\alpha - 1), \quad \ell_\alpha = \ell_\beta \qquad (\text{type } \mathtt{2Ci}). \tag{42h}$$

A few (but not many) such conditions are recorded in the notes column.

The notes column of the tables includes additional notation peculiar to some cases. For example, the case 1i1 corresponds to a discrete series in a block for A_1 with two discrete series and just one principal series. This turns out to mean that the root α must be trivial on the fixed points $H^{\theta_{x_1}}$ for the more split Cartan; and this in turn is equivalent to the existence of $\sigma \in X^*(H)$ so that

$$\alpha = (1 + {}^\vee\theta_{y_1})\sigma. \tag{42i}$$

That is the meaning of the note in the 1i1 row; the weight σ (which one needs to find by solving (42i) to implement the algorithm) appears in the formula for τ_1.

The terminology here is more compact than that of [10]. See Table 1.

Proposition 7.2. *Suppose (x, y) is a δ_0-fixed parameter, and E is an extended parameter for (x, y). Let κ be a δ_0-orbit of simple roots.*

Suppose κ is not of type 2i12, 2r21, 2Ci *or* 2Cr. *Then the formulas for the action of the Hecke operator T_κ from [10] apply, using the Cayley transforms and cross actions from Tables 2–4, to give a formula for $T_\kappa([E])$.*

This is a direct translation of the calculations of [10, Sections 7.5–7.7] to our setting. We treat the excluded cases in the next two sections.

Example 7.3. Suppose κ is a *two-imaginary noncompact type I-I ascent* for (x, y) (in the terminogy of [10]), i.e., of type 2i11 (in our terminology). Suppose E_1 is an extended parameter for (x, y), and set $E_2 = w_\kappa \times E_1$, and set $E' = c_\kappa(E_1)$, as defined by Table 3. Then formula [10, (7.6)(e′)] gives:

$$T_\kappa([E_1]) = [E_2] + [E']$$
$$T_\kappa([E_2]) = [E_1] + [E']$$
$$T_\kappa([E']) = (q-1)([E_1] + [E_2]) + (q-2)[E'].$$

8 The `2i12` case

If κ is of type `2i12f` or `2r21f`, then the formulas for the Hecke operator T_κ do not carry over directly from [10, (7.6)(i″) and (j″)]. We start with a special case.

Lemma 8.1. *Suppose κ is of type* `2i12f` *for $E_0 = (\lambda, \tau, \ell, t)$. Assume*

$$\begin{aligned} \tau_\alpha &= \tau_\beta = 0 \\ t_\alpha &= t_\beta = 0 \\ g_\alpha - \ell_\alpha &= g_\beta - \ell_\beta = 1 \\ \gamma_\alpha - \lambda_\alpha &= \gamma_\beta - \lambda_\beta = 1. \end{aligned} \tag{43}$$

Let $E_0' = w_\kappa \times_1 E_0$ as given by Table 3. (Recall that this is a certain extension of the parameter $s_\alpha \times (\lambda, \ell)$ for G. The Cayley transform $c_\kappa(E_0)$ is double-valued; write $c_\kappa(E_0) = \{F_0, F_0'\}$, where the parameter for F_0 is (λ, τ, ℓ, t), and $F_0' = w_\kappa \times_1 F_0$.

The action of T_κ on the space spanned by E_0, E_0', F_0, F_0' is

$$\begin{aligned} T_\kappa(E_0) &= E_0 + F_0 + F_0' \\ T_\kappa(E_0') &= E_0' + F_0 - F_0' \\ T_\kappa(F_0) &= (q^2-1)(E_0 + E_0') + (q^2-2)F_0 \\ T_\kappa(F_0') &= (q^2-1)(E_0 - E_0') + (q^2-2)F_0'. \end{aligned} \tag{44}$$

Explicitly the extended parameters are:

$$\begin{aligned} &E_0 : (\lambda, \tau, \ell, t) \qquad && E_0' : (\lambda, \tau, \ell + \alpha^\vee, t - s) \\ &F_0 : (\lambda, \tau, \ell, t) \qquad && F_0' : (\lambda + \alpha, \tau - \sigma, \ell, t) \end{aligned} \tag{45}$$

with σ and s given in Table 3.

When the parameters are in this form, this is simply a direct translation of the proof of [10, (7.6)(i″)].

Lemma 8.2. *Suppose E is an extended parameter for (x, y), and κ is of type* `2i12fr` *for E. Set $E' = w_\kappa \times_1 E$. Write $c_\kappa(E) = c_\kappa(E') = \{F, F'\}$. Possibly after switching E and E', and possibly also switching F and F', we can find E_0, E_0', F_0, F_0' as in the previous lemma, such that E and E_0 are extensions of the same parameter, and similarly $(E', E_0'), (F, F_0)$ and (F', F_0').*

Proof. Write $E = (\lambda, \tau, \ell, t)$, so $E' = (\lambda, \tau, \ell + \alpha, t - s)$. After replacing τ with a different solution of its defining equation:

$$\tau \to \tau + \tau_\beta \sigma + \frac{1}{2}(\tau_\alpha + \tau_\beta)\alpha$$

where $\alpha - \beta = (1 + {}^\vee\theta_{y_1})\sigma$, we can assume $\tau_\alpha = \tau_\beta = 0$.

Since $2\alpha^\vee, 2\beta^\vee$ and $\alpha^\vee - \beta^\vee$ are all in $(1+\theta_x)X_*$, $a\alpha^\vee + b\beta^\vee$ is in $(1+\theta_x)X_*$ provided $a+b \in 2\mathbb{Z}$. By adding such a term to ℓ we can arrange that $g_\alpha - \ell_\alpha - 1 = 0$ and $g_\beta - \ell_\beta - 1 = 0$ or 2. Make the corresponding change $t \to t + \frac{1}{2}(b-a)(\alpha^\vee - \beta^\vee)$. If $g_\beta - \ell_\beta - 1 = 2$, replace E with $E' = w_\kappa \times_1 E$, and now we have

$$g_\alpha - \ell_\alpha - 1 = g_\beta - \ell_\beta - 1 = 0.$$

Since $g_\alpha = g_\beta$ this implies $\ell_\alpha = \ell_\beta$. Then Conditions (a) and (d) of Definition 5.4 imply $\lambda_\alpha - \gamma_\alpha - 1 = \lambda_\beta - \gamma_\beta - 1 = 0$ and $t_\alpha = t_\beta = 0$. Table 3 then says that $F = (\lambda, \tau, \ell, t)$ is one of the two Cayley transforms of E, and that our parameters now have the form (45). ◻

Proposition 8.3. *In the setting of the previous lemma we have*

$$\begin{aligned}
T_\kappa(E) &= E + \mathrm{sgn}(E, E_0)(\mathrm{sgn}(F, F_0)F + \mathrm{sgn}(F', F_0')F') \\
T_\kappa(E') &= E' + \mathrm{sgn}(E', E_0')(\mathrm{sgn}(F, F_0)F - \mathrm{sgn}(F', F_0')F') \\
T_\kappa(F) &= (q^2-1)\mathrm{sgn}(F, F_0)(\mathrm{sgn}(E, E_0)E + \mathrm{sgn}(E', E_0')E') + (q^2-2)F \\
T_\kappa(F') &= (q^2-1)\mathrm{sgn}(F', F_0')(\mathrm{sgn}(E, E_0)E - \mathrm{sgn}(E', E_0')E') + (q^2-2)F'.
\end{aligned}$$

This formula is independent of the choice of E_0, E_0', F_0, F_0'.

This is immediate.

9 The `2Ci` case

Now we describe the Hecke algebra action in the `2Ci` case. So fix a type 2 root $\kappa = \{\alpha, \beta\}$, and an extended parameter

$$E = (\lambda,\ \tau,\ \ell,\ t) \tag{46a}$$

as in 5.4. Assume that κ is of type `2Ci` for E: that is, that α and β are complex roots interchanged by

$$\theta_x = \mathrm{Ad}(e((g-\ell)/2)\sigma_w \xi_0). \tag{46b}$$

This means in turn that

$$w\xi_0\alpha = \beta, \qquad w\xi_0\beta = \alpha. \tag{46c}$$

Proposition 12.2 says that

$$\sigma_w \xi_0 X_\alpha = X_\beta, \qquad \sigma_w \xi_0 X_\beta = X_\alpha, \tag{46d}$$

and therefore that

$$\theta_x(X_\alpha) = (-1)^{g_\beta - \ell_\beta} X_\beta, \quad \theta_x(X_\beta) = (-1)^{g_\alpha - \ell_\alpha} X_\beta. \tag{46e}$$

The requirement (27) implies that

$$\gamma_\alpha = \gamma_\beta, \qquad g_\alpha = g_\beta. \tag{46f}$$

Similarly, the requirements for an extended parameter to be δ_0-fixed imply among other things that

$$\lambda_\alpha + \lambda_\beta = 2(\gamma_\alpha - 1), \quad \lambda_\alpha - \lambda_\beta = \tau_\beta - \tau_\alpha, \quad \ell_\alpha = \ell_\beta, \quad t_\alpha = -t_\beta. \tag{46g}$$

In particular, we can define a sign

$$\epsilon = \epsilon(E) = (-1)^{g_\alpha - \ell_\alpha} = (-1)^{g_\beta - \ell_\beta}. \tag{46h}$$

Writing

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{s} \tag{46i}$$

for the eigenspace decomposition under θ_x, we get from (46e)

$$X_\alpha + \epsilon X_\beta =_{\text{def}} X_{\mathfrak{k}} \in \mathfrak{k}, \qquad X_\alpha - \epsilon X_\beta =_{\text{def}} X_{\mathfrak{s}} \in \mathfrak{s}, \tag{46j}$$

and also

$$\theta_x(\sigma_\alpha) = \sigma_\beta^\epsilon. \tag{46k}$$

The Weyl group element $s_\alpha s_\beta$ is represented by

$$\sigma_E = \sigma_\alpha \sigma_\beta^\epsilon \in G^{\theta_x} = K. \tag{46l}$$

Finally, the extended group ${}^{\delta_0}K$ is generated by K and the element

$$h = e(t/2)\delta_0 \tag{46m}$$

(Definition 5.4(e)).

Table 3 constructs from E a second extended parameter:

$$\begin{aligned} E_1 &= (\lambda_1, \tau_1, \ell_1, t_1) \\ &= (s_\alpha\lambda + (\gamma_\alpha - 1)\alpha, s_\alpha\tau - (\lambda_\alpha - \gamma_\alpha + 1)\alpha, \\ &\qquad s_\alpha\ell + (g_\alpha - 1)\alpha^\vee, s_\alpha t + (\ell_\alpha - g_\alpha + 1)\alpha^\vee). \end{aligned} \tag{46n}$$

The root κ is of type `2Cr` for the parameter E_1. The element ℓ_1 is chosen so that the corresponding Cartan involution (on all of G, not just H) is

$$\theta_{x_1} = \sigma_\alpha^{-1} \theta_x \sigma_\alpha. \tag{46o}$$

Proposition 9.1. *Suppose we are in the setting* (46).

1. *Applying the formula in Table 3 to the* `2Cr` *parameter* E_1 *gives exactly the same parameter* E *with which we started.*
2. *The action of the Hecke algebra generator* T_κ *([10, 7.6(c″)]) is*

$$T_\kappa(E) = qE + (-1)^{[(\tau_\alpha+\tau_\beta)/2](g_\alpha-\ell_\alpha-1)}(q+1)E_1.$$

Here the sign may be regarded as specifying a renormalization of E_1 *(whose existence is asserted in [10]).*

3. *The corresponding formula for the case* `2Cr` *is*

$$T_\kappa(E_1) = (q^2-q-1)E_1 + (-1)^{(\gamma_{1,\alpha}-\lambda_{1,\alpha}+\tau_{1,\alpha}-1)[(t_{1,\beta}-t_{1,\alpha})/2]}(q^2-q)E.$$

The sign is exactly the same as the one for $T_\kappa(E)$*, written in terms of the parameter* E_1*.*

Proof. The first assertion can be verified by applying the formulas for passing from `2Ci` to `2Cr` and from `2Cr` to `2Ci` in succession, then simplifying; we omit the details.

For the second assertion, we need to understand representation-theoretically the relationship between the extended parameters E and E_1, and how this relates to the Hecke algebra action. For this question it is easiest to think of $(\mathfrak{g}, K)$-modules with a fixed K; that is, to conjugate θ_{x_1} back to θ_x, and to correspondingly change E_1 into a parameter

$$\begin{aligned} E_2 &= (\lambda_2, \tau_2, \ell_2, t_2) \\ &= (\lambda - (\gamma_\alpha - 1)\alpha, \tau + (\lambda_\alpha - \gamma_\alpha + 1)\alpha, \\ &\qquad \ell - (g_\alpha - 1)\alpha^\vee, t - (\ell_\alpha - g_\alpha + 1)\alpha^\vee) \end{aligned} \tag{47a}$$

related to the Borel subgroup

$$B' = \sigma_\alpha B \sigma_\alpha^{-1}. \tag{47b}$$

(The `atlas` decision to prefer E_1 to E_2 is just a bookkeeping convenience. Everything about representation theory, and also most things about perverse sheaves, are calculated with a fixed Cartan involution, and so refer to the relationship between E and E_2. The `atlas` formulas for Hecke algebra actions index bases by E_1 rather than E_2, so we will occasionally mention E_1 below; but mostly we will be concerned about E and E_2.)

The distinguished automorphism corresponding to δ_0 for B' is

$$\delta_0' = \sigma_\alpha \delta_0 \sigma_\alpha^{-1} = \sigma_\alpha \sigma_\beta^{-1} \delta_0. \tag{47c}$$

The generator for the extended Cartan defined by E_2 is

$$h_2 = e(t_2/2)\delta_0' = e(t/2) m_\alpha^{\ell_\alpha - g_\alpha + 1} \sigma_\alpha \sigma_\beta^{-1} \delta_0 = e(t/2)\sigma_E^{-\epsilon}\delta_0 \tag{47d}$$

(with $\sigma_E \in K$ as in (46l). Now we can start to talk about representation theory: that is, about $(\mathfrak{g}, K)$-modules M and their extensions to $(\mathfrak{g}, {}^{\delta_0}K)$-modules M'. Write

$$P = LU \supset B,\ B' \tag{47e}$$

for the parabolic subgroup with L generated by H and the simple roots α and β. Then

$$M_j = H_j(\mathfrak{u}, M), \qquad M'_j = H_j(\mathfrak{u}, M') \tag{47f}$$

are $(\mathfrak{l}, L\cap K)$- and $(\mathfrak{l}, {}^{\delta_0}(L\cap K))$-modules respectively; and the relationship between representations and parameters (which uses $\mathfrak{n}$-homology) factors through this construction by means of the Hochschild–Serre spectral sequence. In this way (omitting details) one can reduce the questions we are studying to the case

$$G = L, \quad R = \{\pm\alpha, \pm\beta\}. \tag{47g}$$

In the setting (47g), here is what the representation theory looks like. The group L is locally SL(2) × SL(2), and K is approximately a "diagonal" copy of SL(2). (More precisely, the "diagonal" copy is

$$\mathrm{SL}(2)_K = \left\{ \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \begin{pmatrix} a & \epsilon b \\ \epsilon c & d \end{pmatrix} \right) \middle| \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}(2,\mathbb{C}) \right\}, \tag{47h}$$

with ϵ as in (46h). Furthermore K, and even its intersection with the derived group of L, may be disconnected.

Attached to E is an irreducible principal series $(\mathfrak{g}, {}^{\delta_0}K)$-module $I(E)$. The restriction of $I(E)$ to ${}^{\delta_0}K$ is

$$I(E) = \mathrm{Ind}_{{}^{\delta_0}(H\cap K)}^{{}^{\delta_0}K}(\Lambda_E + 2\rho_n). \tag{47i}$$

Here Λ_E is the character of $H \cap K$ defined by the first term λ in E, extended to ${}^{\delta_0}(H \cap K)$ by making h (from (46m)) act by (29). The twist $2\rho_n$ is the character by which ${}^{\delta_0}(H \cap K)$ acts on (the top exterior power of) $\mathfrak{n} \cap \mathfrak{s}$; that is, on the vector $X_{\mathfrak{s}}$ from (46j).

The reason for the last twist is that for fundamental series modules M, the character of $H \cap K$ in the parameter is a weight on $H_{\dim(\mathfrak{n}^{\mathrm{op}}\cap\mathfrak{s})}(\mathfrak{n}^{\mathrm{op}}, M)$, specifically appearing in the image of a natural map

$$H_0(\mathfrak{n}^{\mathrm{op}} \cap \mathfrak{k}, M) \otimes \bigwedge^{\dim(\mathfrak{n}^{\mathrm{op}}\cap\mathfrak{s})}(\mathfrak{n}^{\mathrm{op}} \cap \mathfrak{s}) \to H_{\dim(\mathfrak{n}^{\mathrm{op}}\cap\mathfrak{s})}(\mathfrak{n}^{\mathrm{op}}, M).$$

The conclusion is that *the weight of $H \cap K$ on the parameter is equal to the $\mathfrak{n} \cap \mathfrak{k}$-highest weight of the lowest K-type, minus $2\rho_n$*. The coroot for K is

$$\alpha^\vee + \beta^\vee, \tag{47j}$$

which acts on $X_{\mathfrak{s}}$ by 2. The dimension of the lowest K-type is therefore

$$\lambda_\alpha + \lambda_\beta + 2 + 1 = 2\gamma_\alpha + 1; \tag{47k}$$

the 1 comes from the ρ-shift in the Weyl dimension formula, and we have used (46g) to convert λ to γ. In particular, we find that

$$I(E)|_{\mathrm{SL}(2)_K} = \text{sum of irreducibles of dimensions } 2\gamma_\alpha + 1, 2\gamma_\alpha + 3, \ldots. \tag{47l}$$

In the same fashion, attached to E_2 is a reducible principal series $(\mathfrak{g}, {}^{\delta_0}K)$-module $I(E_2)$. The restriction of $I(E_2)$ to ${}^{\delta_0}K$ is

$$I(E_2) = \mathrm{Ind}_{\delta_0(H\cap K)}^{\delta_0 K}(\Lambda_{E_2}). \tag{47m}$$

The reason for the absence of a twist on Λ_{E_2} is that for principal series modules M_2 for quasisplit groups, the parameter appears as a weight on $H_0(\mathfrak{n}^{\mathrm{op}}, M_2)$; and for (almost) spherical representations, this weight space is precisely the image of the (almost) spherical vector. In particular,

$$I(E_2)|_{\mathrm{SL}(2)_K} = \text{sum of irreducibles of dims } 1, 3, \ldots. \tag{47n}$$

The principal series representation $I(E_2)$ has a unique irreducible quotient representation $J(E_2)$:

$$\begin{aligned} J(E_2)|_{\mathrm{SL}(2)_K} &= \text{sum of irreducibles of dims } 1, 3, \ldots, 2\gamma_\alpha - 1,\\ \dim J(E_2) &= \gamma_\alpha^2. \end{aligned} \tag{47o}$$

We get a short exact sequence

$$\begin{aligned} 0 \to & I(E') \to I(E_2) \to J(E_2) \to 0,\\ & I(E')|_{\mathrm{SL}(2)_K} = \text{sum of irreducibles of dims } 2\gamma_\alpha + 1, 2\gamma_\alpha + 3, \ldots. \end{aligned} \tag{47p}$$

This extended parameter E' with $I(E')$ appearing as a composition factor of $I(E_2)$ is the one on which the Hecke algebra action gives E_1 (remember that this is essentially just another label for E_2) with positive coefficient. (This is a consequence of the Beilinson–Bernstein localization theory relating perverse sheaves to representations, and the perverse sheaf definition of the Hecke algebra action in [10].) So we need to understand the relationship between the extended parameters E and E'.

Because the spherical composition factor $J(E_2)$ is a unique quotient of $I(E_2)$, the spherical vector in $I(E_2)$ is cyclic. The action of $X_{\mathfrak{s}}$ carries highest weight vectors for K to highest weight vectors for K; so we deduce

$$\begin{aligned} & X_{\mathfrak{s}}^{\gamma_\alpha}(\text{spherical vector in } I(E_2))\\ &\qquad = \text{highest weight vector for lowest } K\text{-type of } I(E'). \end{aligned} \tag{48a}$$

Because $\sigma_E \in \mathrm{SL}(2)_K$ acts trivially on the (one-dimensional) lowest K-type of $J(E_2)$, the formula (47d) shows that

$$\Lambda_{E_2}(h_2) = \text{action of } e(t/2)\delta_0 \text{ on } J(E_2) \text{ lowest } K\text{-type}. \tag{48b}$$

It is easy to calculate

$$\mathrm{Ad}(e(t/2)\delta_0)(X_{\mathfrak{s}}) = -\epsilon(-1)^{t_\alpha} = (-1)^{g_\alpha-\ell_\alpha-1+t_\alpha}.$$

Combining (48b) with (48a), we find that the

$$\begin{aligned} &\text{action of } h = e(t/2)\delta_0 \text{ on } I(E') \text{ lowest } K\text{-type} \\ &= \Lambda_{E_2}(h_2)(-\epsilon)^{\gamma_\alpha}(-1)^{\gamma_\alpha t_\alpha} \\ &= \Lambda_{E_2}(h_2)(-1)^{\gamma_\alpha((g_\alpha-\ell_\alpha-1)+t_\alpha)}. \end{aligned} \tag{48c}$$

Using the description of the parameter for E' given before (47j), we get

$$\Lambda_{E'}(h) = \Lambda_{E_2}(h_2)(-1)^{(\gamma_\alpha-1)((g_\alpha-\ell_\alpha-1)+t_\alpha)}. \tag{48d}$$

Now we compare this "desired" relationship between $\Lambda_{E'}(h)$ and $\Lambda_{E_2}(h_2)$ with the actual relationship between $\Lambda_E(h)$ and $\Lambda_{E_2}(h_2)$. We find (using (29) and the formulas in Table 3 for E_1)

$$\begin{aligned} \Lambda_E(h)\Lambda_{E_2}^{-1}(h_2) = \Lambda_E(h)\Lambda_{E_1}^{-1}(h_1) \\ &= i^{\langle\tau,(\delta_0-1)\ell\rangle}(-1)^{\langle\lambda,t\rangle} \\ &\quad i^{-\langle\tau-[(\tau_\alpha+\tau_\beta)/2]]\alpha,(\delta_0-1)(\ell+(g_\alpha-\ell_\alpha-1)\alpha^\vee\rangle} \\ &\quad (-1)^{\langle(\lambda+(\gamma_\alpha-\lambda_\alpha-1)\alpha,t+(\ell_\alpha-g_\alpha-t_\alpha+1)\alpha^\vee\rangle} \\ &= i^{\langle[(\tau_\alpha+\tau_\beta)/2]\alpha,(\delta_0-1)(\ell+(g_\alpha-\ell_\alpha-1)\alpha^\vee\rangle} \\ &\quad i^{-\langle\tau,(\delta_0-1)(g_\alpha-\ell_\alpha-1)\alpha\rangle} \\ &\quad (-1)^{\langle(\gamma_\alpha-\lambda_\alpha-1)\alpha,t+(\ell_\alpha-g_\alpha-t_\alpha+1)\alpha^\vee\rangle} \\ &\quad (-1)^{\langle\lambda,(\ell_\alpha-g_\alpha-t_\alpha+1)\alpha^\vee\rangle}. \end{aligned} \tag{48e}$$

There are four factors on the right. In the first,

$$\langle\alpha,(\delta_0-1)\ell\rangle = \ell_\alpha-\ell_\beta = 0$$

by (46g). In the third, $\langle\alpha,\alpha^\vee\rangle = 2$ contributes an even power of (-1), so can be dropped. We are left with

$$\begin{aligned} \Lambda_E(h)\Lambda_{E_2}^{-1}(h_2) = &\ i^{\langle[(\tau_\alpha+\tau_\beta)/2]\alpha,(\delta_0-1)((g_\alpha-\ell_\alpha-1)\alpha^\vee)\rangle} i^{-\langle\tau,(\delta_0-1)((g_\alpha-\ell_\alpha-1)\alpha)\rangle} \\ &\ (-1)^{\langle(\gamma_\alpha-\lambda_\alpha-1)\alpha,t\rangle}(-1)^{\langle\lambda,(\ell_\alpha-g_\alpha-t_\alpha+1)\alpha^\vee\rangle} \\ = &\ (-1)^{[(\tau_\alpha+\tau_\beta)/2](g_\alpha-\ell_\alpha-1)]}(-1)^{[(\tau_\alpha-\tau_\beta)/2](g_\alpha-\ell_\alpha-1)} \\ &\ (-1)^{(\gamma_\alpha-\lambda_\alpha-1)t_\alpha}(-1)^{\lambda_\alpha(\ell_\alpha-g_\alpha-t_\alpha+1)} \\ = &\ (-1)^{\tau_\alpha(g_\alpha-\ell_\alpha-1)}(-1)^{(\gamma_\alpha-\lambda_\alpha-1)t_\alpha}(-1)^{\lambda_\alpha(\ell_\alpha-g_\alpha-t_\alpha+1)}. \end{aligned} \tag{48f}$$

Splitting the last factor between the first two gives

$$\Lambda_E(h)\Lambda_{E_2}^{-1}(h_2) = \begin{array}{l}(-1)^{(\lambda_\alpha+\tau_\alpha)(g_\alpha-\ell_\alpha-1)} \\ (-1)^{(\gamma_\alpha-1)t_\alpha}.\end{array} \tag{48g}$$

Now use the first two formulas from (46g) to write $\lambda_\alpha = (\gamma_\alpha - 1) + (\tau_\beta - \tau_\alpha)/2$. We get

$$\begin{aligned}\Lambda_E(h)\Lambda_{E_2}^{-1}(h_2) &= (-1)^{[(\gamma_\alpha-1)+(\tau_\alpha+\tau_\beta)/2](g_\alpha-\ell_\alpha-1)}(-1)^{(\gamma_\alpha-1)t_\alpha} \\ &= (-1)^{(\gamma_\alpha-1)(g_\alpha-\ell_\alpha+t_\alpha-1)}(-1)^{[(\tau_\alpha+\tau_\beta)/2](g_\alpha-\ell_\alpha-1)}.\end{aligned} \tag{48h}$$

The first factor here is exactly the one from (48d), so we deduce

$$\Lambda_E(h) = \Lambda_{E'}(h)(-1)^{[(\tau_\alpha+\tau_\beta)/2][g_\alpha-\ell_\alpha-1]}. \tag{48i}$$

The sign on the right has to appear in front of the [10] Hecke algebra formula for the coefficient of E_1 in $T_\kappa E$. This proves the second assertion of the proposition. For the third, we just rewrite exactly the same formula in terms of the parameter E_1; by the first assertion of the proposition, the formulas in Table 3 tell us how to do that. We omit the algebraic details. □

We summarize the results of Sections 7–9.

Theorem 9.2. *If the infinitesimal character γ is integral, then the action of $\mathcal{H}$ on $\mathcal{M}$ (Definition 7.1) is given by Propositions 7.2, 8.3 and 9.1.*

10 Nonintegral infinitesimal character

Suppose the infinitesimal character γ is not necessarily integral. As always we assume it is integrally dominant (17). Set

$${}^\vee R(\gamma) = \{{}^\vee\alpha \in {}^\vee R \mid \langle \gamma, {}^\vee\alpha\rangle \in \mathbb{Z}\} \tag{49a}$$

as in (39b), and set

$$\begin{aligned} R(\gamma) &= \{\alpha \in R \mid {}^\vee\alpha \in {}^\vee R(\gamma)\} \\ R(\gamma)^+ &= R^+ \cap R(\gamma). \end{aligned} \tag{49b}$$

We say $\alpha \in R$ is *integral* if $\alpha \in R(\gamma)$. We say an integral root is *simple* (respectively *integral-simple*) if it is simple for R^+ (respectively $R(\gamma)^+$).

The Weyl group $W(\gamma)$ of $R(\gamma)$ satisfies

$$W(\gamma) = \{w \in W \mid w\gamma - \gamma \in \mathbb{Z}R\}. \tag{49c}$$

We now assume ${}^\vee\delta_0(\gamma) = \gamma$ (see Lemma 4.1), so ${}^\vee\delta_0$ acts on $R(\gamma)$. Then ${}^\vee\delta_0$ preserves both the simple and integral-simple roots, so the notions of integral and integral-simple apply to a ${}^\vee\delta_0$-orbit $\kappa = \{\alpha, {}^\vee\delta_0(\alpha)\}$ of roots. Let $\mathcal{H}(\gamma)$ be the Hecke algebra of [10, (4.7)] applied to $(R(\gamma), \delta_0)$.

Let $\mathcal{M}_\gamma$ be the module of Definition 7.1. The construction of [10] gives a representation of $\mathcal{H}(\gamma)$ on $\mathcal{M}_\gamma$. (More precisely, the construction of [10] concerns geometry related by base change (to compare a base field of finite characteristic with $\mathbb{C}$) and Beilinson–Bernstein localization (to relate K-equivariant perverse sheaves to $(\mathfrak{g}, K)$-modules) to the module of Definition 7.1. In order to make a parallel identification in the case of nonintegral infinitesimal character, one needs a discussion like that in [1, Chapter 17]. We omit the details.)

Suppose κ is a ${}^\vee\delta_0$-orbit of roots that are integral (for γ) and simple (for G). Then the formulas of Tables 2–4 apply to give a formula for the action T_γ on $\mathcal{M}$. The technical issue we have to deal with here is what to do if κ is integral-simple (for γ) but not simple (for G).

Definition 10.1. Let $\mathcal{ID}$ be the set of integrally dominant elements of $\mathfrak{h}^*$:

$$\mathcal{ID} = \{\gamma \in \mathfrak{h}^* \mid \alpha \in R(\gamma)^+ \implies \langle \gamma, {}^\vee\alpha \rangle \geq 0\}.$$

If $\gamma \in \mathfrak{h}^*$, then γ is $W(\gamma)$-conjugate to a unique element of $\mathcal{ID}$. If $\gamma \in \mathcal{ID}$ and $w \in W$, let $w * \gamma$ be the unique element of $\mathcal{ID}$ which is $W(w\gamma)$ conjugate to $w\gamma$.

It is easy to see that $w * \gamma$ is the unique element satisfying

(a) $w * \gamma \in \mathcal{ID}$
(b) $w * \gamma$ is W-conjugate to γ
(c) $w * \gamma \in w\gamma + \mathbb{Z}R$.

Condition (c) is equivalent (in the presence of (a) and (b)) to

(c′) $w * \lambda = xw\lambda$ for some $x \in W(w\lambda)$.

Lemma 10.2. *The map $(w, \gamma) \to w * \gamma$ is an action of W on $\mathcal{ID}$. It satisfies:*

1. *$Stab_W(\gamma) = W(\gamma)$;*
2. *The W-orbit of γ under $*$ is in bijection with $W/W(\gamma)$;*
3. *$w * \gamma = xw\gamma$ for some $x \in W(w\gamma)$;*
4. *Suppose α is simple for R^+. Then*

$$s_\alpha * \gamma = \begin{cases} \gamma & \alpha \in R(\gamma) \\ s_\alpha(\gamma) & \alpha \notin R(\gamma). \end{cases}$$

Proof. If $x, y \in W$, then $(xy) * \lambda$ is the unique element satisfying conditions (a–c) above with respect to xy. On the other hand $x * (y * \lambda)$ obviously satisfies (a) and (b). Condition (c) holds as well:

$$\begin{aligned} x * (y * \gamma) &\in x(y * \gamma) + \mathbb{Z}R \\ &\in x(y\gamma + \mathbb{Z}R) + \mathbb{Z}R \quad = (xy)\gamma + \mathbb{Z}R. \end{aligned}$$

Assertions (1–3) are straightforward, and (4) is clear if α is integral for γ, so assume this is not the case. Obviously $s_\alpha(\gamma)$ satisfies the conditions (b) and (c) for $s_\alpha * \gamma$, and (a) follows from the fact that s_α permutes $R^+ - \{\alpha\}$. We leave the details to the reader. □

If γ is integral, the formulas for the cross action in Tables 2–4 define an action of W^{δ_0} on $\mathcal{M}$. With a small change the same holds in general. To indicate the role of γ, write $(\lambda, \tau, \ell, t, \gamma)$ for an extended parameter.

Definition 10.3. Suppose κ is a ${}^\vee\delta_0$-orbit of simple roots. Suppose $\gamma \in \mathcal{ID}$ and $(\lambda, \tau, \ell, t, \gamma)$ is an extended parameter. Use the formulas for the cross action of κ from Tables 2–4, applied to (λ, τ, ℓ, t), to define $(\lambda_1, \tau_1, \ell_1, t_1)$. Then define

$$w_\kappa \times (\lambda, \tau, \ell, t, \gamma) = (\lambda_1 + (w_\kappa * \gamma - \gamma), \tau_1, \ell_1, t_1, w_\kappa * \gamma).$$

Define the cross action of any element of W^{δ_0} by writing it as a product of w_κs.

If κ is integral then $w_\kappa * \gamma = \gamma$, and Definition 10.3 agrees with the definition of the cross action in Tables 2–4. The main point is that even if κ is not integral, the formula in Definition 10.3 gives a valid extended parameter. In particular the relation

$$(1 - {}^\vee\theta_1)(\lambda_1) = (1 - {}^\vee\theta_1)(\gamma - \rho)$$

holds exactly as in the integral case. What we need to know is that

$$(1 - {}^\vee\theta_1)(\lambda_1 + (w_\kappa * \gamma - \gamma)) = (1 - {}^\vee\theta_1)(w_\kappa * \gamma - \rho)$$

which follows immediately. Furthermore in the integral case $\lambda_1 \in X^*$. In the non-integral case it follows readily from the definitions that $\lambda_1 + (w_\kappa * \gamma - \gamma) \in X^*$ (even though this doesn't hold separately for λ_1 and $w_\kappa * \gamma - \gamma$).

Proposition 10.4. *Suppose $\gamma \in \mathcal{ID}$ is not necessarily integral. Then the action of $\mathcal{H}(\gamma)$ on $\mathcal{M}_\gamma$ is given by the formulas in Table 5, with the following changes.*

Suppose κ is integral-simple, $w \in W^{\delta_0}$, and these satisfy: $w\kappa$ is integral, simple (for R^+), and the Cayley transform $c_{w\kappa}(w \times E)$ is defined by Tables 2–4. Then define $c_\kappa(E)$ to be

$$c_\kappa(E) = w^{-1} \times c_{w\kappa}(w \times E)$$

where the cross action is that of Definition 10.3.

On the other hand, suppose $w\kappa$ is of type `2i11`, `2i12`, `2r11`, `2r12` *for $w \times E$, so $w\kappa \times (w \times E)$ is defined by Table 3. Define*

$$w_\kappa \times E = w^{-1} \times [w\kappa \times (w \times E)].$$

It is helpful to reformulate the action of W.

Lemma 10.5. *Suppose $E = (\lambda, \tau, \ell, t, \gamma)$ is an extended parameter, and $w \in W^{\delta_0}$. Then $w \times E = (\lambda', \tau', \ell', t', w * \gamma)$ where*

$$\lambda' = w * \gamma - w(\gamma - \lambda) + (w\rho - \rho) - (w\rho_r(x) - \rho_r(wxw^{-1}))$$
$$\tau' = w\tau - ({}^\vee\delta_0 - 1)(w\rho_r(x) - \rho_r(wxw^{-1}))/2$$
$$\ell' = g - w(g - \ell) + (w{}^\vee\rho - {}^\vee\rho) - (w\rho_r(y) - \rho_r(wyw^{-1})$$
$$t' = wt - (\delta_0 - 1)(w\rho_r(y) - \rho_r(wyw^{-1}))/2.$$

The proof is that these formulas agree with those Definition 10.3 when $w = w_\kappa$. We omit the details.

To apply the proposition we need the following lemma.

Lemma 10.6. *Suppose κ is a ${}^\vee\delta_0$-orbit of integral-simple roots. Then there exists $w \in W^{\delta_0}$ such that $w\kappa$ is simple, unless $\kappa = \{\alpha\}$ is of length* 1 *and (the simple factor of) G is locally isomorphic to* $\mathrm{SL}(2n+1, \mathbb{R})$.

This follows from the facts that the "quotient" root system R/δ_0 [12] consisting of the restrictions of roots to H^{δ_0}, is a (possibly non-reduced) root system, with Weyl group W^{δ_0}; and in a reduced root system every root is conjugate to a simple root. The excluded case in the lemma is type A_{2n}, in which case R/δ_0 is the non-reduced system of type BC_n, and a δ_0-fixed root restricts to twice a root.

Extending Proposition 10.4 to this excluded case requires just a calculation in $\mathrm{SL}(3, \mathbb{R})$, which we omit.

11 Duality

Definition 11.1. Let τ be the anti-automorphism of $\mathcal{H}$ given by

$$q\tau(T_\kappa) = -q^\ell T_\kappa^{-1} = -T_\kappa + (q^\ell - 1) \quad (\ell = \mathrm{length}(\kappa)). \tag{50}$$

Suppose π is a representation of $\mathcal{H}$ on an $\mathcal{A}$-module V. The dual representation π^*, on $\mathrm{Hom}_{\mathcal{A}}(V, \mathcal{A})$ is given by

$$\pi^*(T_\kappa)(\lambda)(v) = \lambda(\pi(\tau(T_\kappa)v).$$

In the setting of Section 2 let $\mathcal{H}$ be the Hecke algebra for (G, δ_0) (see [10] and Section 7). Let ${}^\vee\mathcal{H}$ be the algebra given by the same construction applied to $({}^\vee G, {}^\vee\delta_0)$. If κ is a δ_0-orbit of simple roots for G, then ${}^\vee\kappa$ is a ${}^\vee\delta_0$-orbit of simple roots for ${}^\vee G$, and the map $T_\kappa \to T_{{}^\vee\kappa}$ induces a Hecke algebra isomorphism.

Fix (regular, rational) infinitesimal character γ and (regular, integral) infinitesimal character cocharacter g as in (26).

We now assume that γ is integral. Let $\mathcal{M}$ be the $\mathcal{H}$-module of Definition 7.1, applied to G, δ_0, and γ. Recall $\mathcal{M}$ is spanned by equivalence classes $[E]$, for E an extended parameter with infinitesimal character γ, and $[I_z(E)] \to [E]$ is an isomorphism of Hecke modules.

Let ${}^\vee\mathcal{M}$ be the ${}^\vee\mathcal{H}$-module obtained by applying the same construction to ${}^\vee G, {}^\vee\delta_0$ and g. If E is an extended parameter, write ${}^\vee E$ for the same parameter, viewed as an extended parameter for ${}^\vee G$. The map $[I_\xi({}^\vee E)] \to [{}^\vee E]$ is an isomorphism of ${}^\vee\mathcal{H}$-modules. Write $[E]' \in \mathrm{Hom}_{\mathcal{A}}(V, \mathcal{A})$ for the dual basis vector.

Proposition 11.2. *The map* $[E]' \to (-1)^{length(E)}[{}^\vee E]$ *is an isomorphism of* $\mathcal{H} \simeq {}^\vee\mathcal{H}$*-modules.*

Proof. The statement is equivalent to the following assertion. For all κ, and extended parameters E, F:

$$\text{the coefficient of } [E] \text{ in } -T_\kappa([F]) + (u^{\ell(\kappa)} - 1)\mathrm{sgn}(E, F) \tag{51a}$$

is equal to

$$(-1)^{\ell(E)-\ell(F)} * \text{the coefficient of } [{}^\vee F] \text{ in } T_{{}^\vee\kappa}([{}^\vee E]). \tag{51b}$$

In (a) $\mathrm{sgn}(E, F)$ is defined to be 0 if E, F are not extensions of the same parameter.

Up to signs, all of these formulas can be read off easily from the formulas for the Hecke algebra action on parameters. See Table 5. The fact that the signs are correct is due to the symmetry of Table 5. This is best illustrated by an example.

Example 11.3. Suppose κ is type `1i1` for an extended parameter F. Then κ is also of type `1i1` for $w_\kappa \times F$. According to Table 5,

$$\text{the coefficient of } [w_\kappa \times F] \text{ in } -T_\kappa([F]) \text{ is } -1. \tag{52a}$$

We need to show this equals

$$-1(\text{the coefficient of } [{}^\vee F] \text{ in } T_{{}^\vee\kappa}([{}^\vee(w_\kappa \times F)]). \tag{52b}$$

From the same line in Table 5, applied to ${}^\vee G$, we know that

$$-(\text{the coefficient of } [{}^\vee F] \text{ in } T_{{}^\vee\kappa}([w_\kappa \times {}^\vee F]) = -1. \tag{52c}$$

So we need to know that

$$(w_\kappa \times F)^\vee \equiv w_\kappa \times {}^\vee F. \tag{52d}$$

This identity reflects a symmetry of the tables. Here $w_\kappa \times F$ is a cross action of type `1i1`, $w_\kappa \times {}^\vee F$ is of type `1r1`. Switching the roles of $\lambda \leftrightarrow \ell$, and $\tau \leftrightarrow t$ interchanges these two formulas.

The necessary symmetry holds for all Cayley transforms and cross actions; in Table 5 the dual operations are listed on the same line. This completes the proof of the proposition. □

12 Appendix

We collect a few technical results about the Tits group [13], which will be needed for our study of parameters for representations in Section 3. We continue with the notation of (11). For each simple root α, the pinning defines a canonical homomorphism

$$\phi_\alpha \colon (\mathrm{SL}(2), \mathrm{diag}) \to (G, H) \qquad d\phi_\alpha \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = X_\alpha. \tag{53a}$$

Similarly,

$$\phi_{\alpha^\vee} \colon (\mathrm{SL}(2), \mathrm{diag}) \to ({}^\vee G, {}^\vee H) \qquad d\phi_{\alpha^\vee} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = X_{\alpha^\vee}. \tag{53b}$$

It is sometimes convenient to define also

$$H_\alpha = d\phi_\alpha \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad X_{-\alpha} = d\phi_\alpha \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \tag{53c}$$

the first element (because $\alpha(H_\alpha) = 2$) "is" the coroot $\alpha^\vee$. The second is a preferred root vector for $-\alpha$, characterized by the last of the three relations

$$[H_\alpha, X_\alpha] = 2X_\alpha, \quad [H_\alpha, X_{-\alpha}] = -2X_{-\alpha}, \quad [X_\alpha, X_{-\alpha}] = H_\alpha. \tag{53d}$$

In this way we get a distinguished representative

$$\begin{aligned} \sigma_\alpha &=_{\mathrm{def}} \phi_\alpha \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \exp(\frac{\pi}{2}(X_\alpha - X_{-\alpha})) \\ \sigma_\alpha^2 &= m_\alpha =_{\mathrm{def}} \alpha^\vee(-1) \end{aligned} \tag{53e}$$

for the simple reflection s_α. These representatives satisfy the braid relations (see [13]) and therefore define distinguished representatives

$$\sigma_w =_{\mathrm{def}} \sigma_{\alpha_1}\sigma_{\alpha_2}\cdots\sigma_{\alpha_r} \qquad (w = s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_r} \text{ reduced}) \tag{53f}$$

for each Weyl group element w. (That σ_w is independent of the choice of reduced decomposition is a consequence of the fact that the σ_α satisfy the braid relations.) If γ is any distinguished (that is, pinning-preserving) automorphism of (G, B, H), then

$$\gamma(\sigma_w) = \sigma_{\gamma(w)}. \tag{53g}$$

The braid relations imply, for any $w \in W$ and simple root α

$$\sigma_w\sigma_\alpha = \begin{cases} \sigma_{ws_\alpha} & \mathrm{length}(ws_\alpha) = \mathrm{length}(w) + 1 \\ \sigma_{ws_\alpha} m_\alpha & \mathrm{length}(ws_\alpha) = \mathrm{length}(w) - 1 \end{cases} \tag{53h}$$

and a similar result for $\sigma_\alpha\sigma_w$ (with m_α on the left).

In exactly the same way, we get a distinguished representative in ${}^\vee G$:

$$^\vee\sigma_w =_{\text{def}} \sigma_{\alpha_1^\vee}\sigma_{\alpha_2^\vee}\cdots\sigma_{\alpha_r^\vee} \qquad (w = s_{\alpha_1}s_{\alpha_2}\cdots s_{\alpha_r} \text{ reduced}). \tag{53i}$$

The main fact we need about these representatives is

Proposition 12.1. *In the setting of* (53),

$$\begin{aligned}\sigma_w\sigma_{w^{-1}} &= (w\rho^\vee - \rho^\vee)(-1)\\ &= e((\rho^\vee - w\rho^\vee)/2)\\ &= \prod_{\substack{\beta\in R^+(G,H)\\ w^{-1}\beta\notin R^+(G,H)}} m_\beta.\end{aligned}$$

The proof is an easy induction on $\ell(w)$. See [2, Lemma 5.4].

Proposition 12.2. *In the setting* (11)*, suppose $w \in W$, $\alpha, \beta \in \Pi$ are simple roots, and $w\alpha = \beta$. Write X_α and X_β for the simple root vectors given by the pinning, and $\sigma_w \in N(H)$ for the Tits representative of w defined in* (53f)*. Then*

$$\sigma_w\sigma_\alpha\sigma_w^{-1} = \sigma_\beta$$

and

$$\mathrm{Ad}(\sigma_w)(X_\alpha) = X_\beta, \quad \mathrm{Ad}(\sigma_w)(X_{-\alpha}) = X_{-\beta}. \tag{54}$$

Proof. Since $\beta = w\alpha$, $s_\beta w = ws_\alpha$. If $\text{length}(ws_\alpha) = \text{length}(s_\beta w) = \text{length}(w) + 1$, then the first case of (53h) implies

$$\sigma_w\sigma_\alpha = \sigma_{ws_\alpha} = \sigma_{s_\beta w} = \sigma_\beta\sigma_w.$$

If the lengths are decreasing, we see

$$\begin{aligned}\sigma_w\sigma_\alpha &= \sigma_{ws_\alpha}m_\alpha\\ &= \sigma_{s_\beta w}m_\alpha\\ &= m_\beta\sigma_\beta\sigma_w m_\alpha\\ &= m_\beta m_{s_\beta w\alpha}\sigma_\beta\sigma_w\\ &= \sigma_\beta\sigma_w \quad (\text{since } s_\beta w\alpha = -\beta).\end{aligned}$$

For the second statement we observe that $\mathrm{Ad}(\sigma_w)(X_\alpha)$ is some multiple of X_β. The Tits group preserves the $\mathbb{Z}$-form of $\mathfrak{g}$ generated by the various $X_{\pm\alpha}$, so this scalar is ± 1; we need to show it is 1. We compute

$$\begin{aligned}\sigma_w\sigma_\alpha\sigma_w^{-1} &= \sigma_w(\exp\frac{\pi}{2}(X_\alpha - X_{-\alpha}))\sigma_w^{-1}\\ &= \exp(\frac{\pi}{2}\,\mathrm{Ad}(\sigma_w)(X_\alpha - X_{-\alpha})).\end{aligned}$$

On the other hand, by what we just proved this equals

$$\sigma_\beta = \exp(\frac{\pi}{2}(X_\beta - X_{-\beta})).$$

Setting these equal gives the two equalities in the second statement. □

Corollary 12.3. *In the setting (11), suppose $w \in W$, $\alpha, \beta \in \Pi$ are simple roots, and $w\alpha = -\beta$. Write X_α and X_β for the simple root vectors given by the pinning, and $\sigma_w \in N(H)$ for the Tits representative of w defined in (53f). Then*

$$\sigma_w \sigma_\alpha \sigma_w^{-1} = \sigma_\beta$$

and

$$\mathrm{Ad}(\sigma_w)(X_\alpha) = -X_{-\beta}, \qquad \mathrm{Ad}(\sigma_w)(X_{-\alpha}) = -X_\beta.$$

Proof. Let $w' = ws_\alpha$. The first assertion follows from the previous lemma applied to $\sigma_{w'}$, using the fact that $\sigma_{w'} = \sigma_w\sigma_\alpha$ (since $w' = ws_\alpha$ is a reduced expression). As in the proof of the previous proposition we conclude that

$$\exp(\frac{\pi}{2}(\mathrm{Ad}(\sigma_w)(X_\alpha - X_{-\alpha})) = \exp(\frac{\pi}{2}(X_\beta - X_{-\beta})),$$

and in this case this implies $\mathrm{Ad}(\sigma_w)(X_\alpha) = -X_{-\beta}$ and $\mathrm{Ad}(\sigma_w)(X_{-\alpha}) = -X_\beta$. □

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Table 1 Types of roots, and their associated Cayley transforms

type	terminology of [10]	definition	Cayley transform
`1C+`	complex ascent	α complex, $\theta\alpha > 0$	
`1C-`	complex descent	α complex, $\theta\alpha < 0$	
`1i1`	imaginary noncpt type I ascent	α imaginary, noncpt, type 1	$\gamma^\kappa = \gamma^\alpha$
`1i2f`	imaginary noncpt type II ascent	α imaginary, noncpt, type 2 δ fixes both terms of γ^α	$\gamma^\kappa = \gamma^\alpha = \{\gamma_1^\kappa, \gamma_2^\kappa\}$
`1i2s`	imaginary noncpt type II ascent	α imaginary, noncpt, type 2 δ switches the two terms of γ^α	
`1ic`	cpt imaginary descent	α cpt imaginary	
`1r1f`	real type I descent	α real, parity, type 1 δ fixes both terms of γ_α	$\gamma_\kappa = \gamma_\alpha = \{\gamma_\kappa^1, \gamma_\kappa^2\}$
`1r1s`	real type I descent	α real, parity, type 1 δ switches the two terms of γ_α	
`1r2`	real type II descent	α real, parity, type 2	$\gamma_\kappa = \gamma_\alpha$
`1rn`	real nonparity ascent	α real, non-parity	
`2C+`	two-complex ascent	α, β complex $\theta\,\alpha > 0, \theta\alpha \neq \beta$	
`2C-`	two-complex descent	α, β complex $\theta\,\alpha < 0, \theta\alpha \neq \beta$	
`2Ci`	two-semiimaginary ascent	α, β complex, $\theta\,\alpha = \beta$	$\gamma^\kappa = s_\alpha \times \gamma = s_\beta \times \gamma$
`2Cr`	two-semireal descent	α, β complex, $\theta\,\alpha = -\beta$	$\gamma_\kappa = s_\alpha \times \gamma = s_\beta \times \gamma$
`2i11`	two-imaginary noncpt type I-I ascent	α, β noncpt imaginary, type 1 $(\gamma^\alpha)^\beta$ single valued	$\gamma^\kappa = (\gamma^\alpha)^\beta$
`2i12f`	two-imaginary noncpt type I-II ascent	α, β noncpt imaginary, type 1 $(\gamma^\alpha)^\beta$ double valued	$\gamma^\kappa = \{\gamma_1^\kappa, \gamma_2^\kappa\} = (\gamma^\alpha)^\beta$
`2i12s`	two-imaginary noncpt type I-II ascent	α, β noncpt imaginary, type 1 $(\gamma^\alpha)^\beta$ double valued, switched by δ	

(continued)

Table 1 (continued)

type	terminology of [10]	definition	Cayley transform
`2i22`	two-imaginary noncpt type II-II ascent	α, β noncpt imaginary, type 1 $(\gamma^\alpha)^\beta$ has 4 values	$\gamma^\kappa = \{\gamma_1^\kappa, \gamma_2^\kappa\} = \{\gamma^{\alpha,\beta}\}^\sigma$
`2r22`	two-real type II-II descent	α, β real, parity, type 2 $(\gamma_\alpha)_\beta$ single valued	$\gamma_\kappa = (\gamma_\alpha)_\beta$
`2r21f`	two-real type II-I descent	α, β real, parity, type 2 $(\gamma_\alpha)_\beta$ double valued	$\gamma_\kappa = \{\gamma_\kappa^1, \gamma_\kappa^2\} = (\gamma_\alpha)_\beta$
`2r21s`	two-real type II-I descent	α, β real, parity, type 2 $(\gamma_\alpha)_\beta$ double valued, switched by δ	
`2r11`	two-real type I-I descent	α, β real, parity, type 2 $(\gamma_\alpha)_\beta$ has 4 values	$\gamma_\kappa = \{\gamma_\kappa^1, \gamma_\kappa^2\} = \{(\gamma_\alpha)_\beta\}^\sigma$
`2rn`	two-real nonparity ascent	α, β real, nonparity	
`2ic`	two-imaginary cpt descent	α, β cpt imaginary	
`3C+`	three-complex ascent	α, β complex $\theta\alpha > 0$, $\theta\alpha \neq \beta$	
`3C-`	three-complex descent	α, β complex $\theta\alpha < 0$, $\theta\alpha \neq \beta$	
`3Ci`	three-semiimaginary ascent	α, β complex, $\theta\alpha = \beta$	$\gamma^\kappa = (s_\alpha \times \gamma)^\beta \cap (s_\beta \times \gamma)^\alpha$
`3Cr`	three-semireal descent	α, β complex, $\theta\alpha = -\beta$	$\gamma_\kappa = (s_\alpha \times \gamma)_\beta \cap (s_\beta \times \gamma)_\alpha$
`3i`	three imaginary noncpt ascent	α, β noncpt imaginary, type 1	$\gamma^\kappa = s_\alpha \times \gamma^\beta = s_\beta \times \gamma^\alpha$
`3r`	three-real descent	α, β real, parity, type 2	$\gamma_\kappa = s_\alpha \times \gamma_\beta = s_\beta \times \gamma_\alpha$
`3rn`	three-real non-parity ascent	α, β real, nonparity	
`3ic`	three-imaginary cpt descent	α, β noncpt imaginary	

Table 2 Cayley and cross actions on extended parameters: type `1`

	$\lambda\|_{\mathfrak{t}} = (\gamma - \rho)\|_{\mathfrak{t}}$	$({}^\vee\delta_0 - 1)\lambda = (1 + {}^\vee\theta_y)\tau$	$\ell\|_{\mathfrak{a}} = (g - \rho^\vee)\|_{\mathfrak{a}}$	$(\delta_0 - 1)\ell = (1 + \theta_x)t$	
type	λ_1	τ_1	ℓ_1	t_1	notes
`1C` crx	$s_\alpha\lambda + (\gamma_\alpha - 1)\alpha$	$s_\alpha\tau$	$s_\alpha\ell + (g_\alpha - 1)\alpha^\vee$	$s_\alpha t$	
`1i1` crx	λ	τ	$\ell + \alpha^\vee$	t	
`1i1` Cay	λ	$\tau - \tau_\alpha\sigma$	$\ell + \frac{g_\alpha - \ell_\alpha - 1}{2}\alpha^\vee$	t	$\alpha = (1 + {}^\vee\theta_{y_1})\sigma$
`1i2f` Cay	$\lambda, \lambda + \alpha$	$\tau - \frac{\tau_\alpha}{2}\alpha$	$\ell + \frac{g_\alpha - \ell_\alpha - 1}{2}\alpha^\vee$	t	τ_α even
`1i2s` Cay	$\lambda, \lambda + \alpha$	[none]	$\ell + \frac{g_\alpha - \ell_\alpha - 1}{2}\alpha^\vee$	t	τ_α odd
`1r1f` Cay	$\lambda + \frac{\gamma_\alpha - \lambda_\alpha - 1}{2}\alpha$	τ	$\ell, \ell + \alpha^\vee$	$t - \frac{t_\alpha}{2}\alpha^\vee$	t_α even
`1r1s` Cay	$\lambda + \frac{\gamma_\alpha - \lambda_\alpha - 1}{2}\alpha$	τ	$\ell, \ell + \alpha^\vee$	[none]	t_α odd
`1r2` crx	$\lambda + \alpha$	τ	ℓ	t	
`1r2` Cay	$\lambda + \frac{\gamma_\alpha - \lambda_\alpha - 1}{2}\alpha$	τ	ℓ	$t - t_\alpha s$	$\alpha^\vee = s + \theta_{x_1}s$

Table 3 **Cayley and cross actions on extended parameters: type 2**

type	$\lambda\vert_{\mathfrak{t}} = (\gamma-\rho)\vert_{\mathfrak{t}}$ λ_1	$({}^\vee\delta_0-1)\lambda = (1+{}^\vee\theta_y)\tau$ τ_1	$\ell\vert_{\mathfrak{a}} = (g-\rho^\vee)\vert_{\mathfrak{a}}$ ℓ_1	$(\delta_0-1)\ell = (1+\theta_x)t$ t_1	notes
2C crx	$w_\kappa\lambda + (\gamma_\alpha-1)\kappa$	$w_\kappa\tau$	$w_\kappa\ell + (g_\alpha-1)\kappa^\vee$	$w_\kappa t$	
2Ci Cay	$s_\alpha\lambda + (\gamma_\alpha-1)\alpha$	$s_\alpha\tau - (\lambda_\alpha-\gamma_\alpha+1)\alpha$	$s_\alpha\ell + (g_\alpha-1)\alpha^\vee$	$s_\alpha t + (\ell_\alpha-g_\alpha+1)\alpha^\vee$	$\ell_\alpha = \ell_\beta$
2Cr Cay	$s_\alpha\lambda + (\gamma_\alpha-1)\alpha$	$s_\alpha\tau + (\lambda_\alpha-\gamma_\alpha+1)\alpha$	$s_\alpha\ell + (g_\alpha-1)\alpha^\vee$	$s_\alpha t - (\ell_\alpha-g_\alpha+1)\alpha^\vee$	$\lambda_\alpha = \lambda_\beta$
2i11 crx	λ	τ	$\ell+\kappa^\vee$	t	
2i11 Cay	λ	$\tau - \tau_\alpha\sigma(\alpha) - \tau_\beta\sigma(\beta)$	$\ell + \frac{g_\alpha-\ell_\alpha-1}{2}\alpha^\vee + \frac{g_\beta-\ell_\beta-1}{2}\beta^\vee$	t	$\alpha = (1+{}^\vee\theta_{y_1})\sigma(\alpha)$
2i12 cr1x	λ	τ	$\ell+\alpha^\vee$	$t-s$	$\alpha^\vee-\beta^\vee = s+\theta_x s$
2i12f Cay	$\lambda, \lambda+\alpha$	$\tau+\tau_\beta\sigma - \frac{\tau_\alpha+\tau_\beta}{2}\alpha$, $\tau_1-\sigma$	$\ell + \frac{g_\alpha-\ell_\alpha-1}{2}\alpha^\vee + \frac{g_\beta-\ell_\beta-1}{2}\beta^\vee$	t	$\tau_\alpha+\tau_\beta$ even $\alpha-\beta = \sigma+{}^\vee\theta_{y_1}\sigma$
2i12s Cay	$\lambda, \lambda+\alpha$	[none]	$\ell + \frac{g_\alpha-\ell_\alpha-1}{2}\alpha^\vee + \frac{g_\beta-\ell_\beta-1}{2}\beta^\vee$	t	$\tau_\alpha+\tau_\beta$ odd
2i22 Cay	$\lambda,\ \lambda+\kappa$ **OR** $\lambda+\alpha,\ \lambda+\beta$	$\tau - \frac{\tau_\alpha}{2}\alpha - \frac{\tau_\beta}{2}\beta$ **OR** $\tau - \frac{\tau_\alpha\pm1}{2}\alpha - \frac{\tau_\beta\mp1}{2}\beta$	$\ell + \frac{g_\alpha-\ell_\alpha-1}{2}\alpha^\vee + \frac{g_\beta-\ell_\beta-1}{2}\beta^\vee$	t	τ_α, τ_β even **OR** τ_α, τ_β odd
2r22 crx	$\lambda+\kappa$	τ	ℓ	t	
2r22 Cay	$\lambda + \frac{\gamma_\alpha-\lambda_\alpha-1}{2}\alpha + \frac{\gamma_\beta-\lambda_\beta-1}{2}\beta$	τ	ℓ	$t - t_\alpha s(\alpha^\vee) - t_\beta s(\beta^\vee)$	$\alpha^\vee = (1+\theta_{x_1})s(\alpha^\vee)$
2r21 cr1x	$\lambda+\alpha$	$\tau-\sigma$	ℓ	t	$\alpha-\beta = \sigma+{}^\vee\theta_y\sigma$
2r21f Cay	$\lambda + \frac{\gamma_\alpha-\lambda_\alpha-1}{2}\alpha + \frac{\gamma_\beta-\lambda_\beta-1}{2}\beta$	τ	$\ell, \ell+\alpha^\vee$	$t+t_\beta s - \frac{t_\alpha+t_\beta}{2}\alpha^\vee$, t_1-s	$t_\alpha+t_\beta$ even $\alpha^\vee-\beta^\vee = s+\theta_{x_1}s$
2r21s Cay	$\lambda + \frac{\gamma_\alpha-\lambda_\alpha-1}{2}\alpha + \frac{\gamma_\beta-\lambda_\beta-1}{2}\beta$	τ	$\ell, \ell+\alpha^\vee$	[none]	$t_\alpha+t_\beta$ odd
2r11 Cay	$\lambda + \frac{\gamma_\alpha-\lambda_\alpha-1}{2}\alpha + \frac{\gamma_\beta-\lambda_\beta-1}{2}\beta$	τ	$\ell,\ \ell+\kappa^\vee$ **OR** $\ell+\alpha^\vee,\ \ell+\beta^\vee$	$t - \frac{t_\alpha}{2}\alpha^\vee - \frac{t_\beta}{2}\beta^\vee$ **OR** $t - \frac{t_\alpha\pm1}{2}\alpha^\vee - \frac{t_\beta\mp1}{2}\beta^\vee$	t_α, t_β even **OR** t_α, t_β odd

Table 4 Cayley and cross actions on extended parameters: type 3

type	$\lambda\vert_{\mathfrak{t}} = (\gamma - \rho)\vert_{\mathfrak{t}}$ λ_1	$({}^\vee\delta_0 - 1)\lambda = (1 + {}^\vee\theta_y)\tau$ τ_1	$\ell\vert_{\mathfrak{a}} = (g - \rho^\vee)\vert_{\mathfrak{a}}$ ℓ_1	$(\delta_0 - 1)\ell = (1 + \theta_x)t$ t_1	notes
`3C` crx	$w_\kappa\lambda + (\gamma_\kappa - 2)\kappa$	$w_\kappa\tau$	$w_\kappa\ell + (g_\kappa - 2)\kappa^\vee$	$w_\kappa t$	
`3Ci` Cay	λ **OR** $\lambda + \kappa$	$\tau - \frac{\tau_\kappa}{2}\kappa$	$\ell + (g_\alpha - 1 - \ell_\alpha)\kappa^\vee$	t	$\gamma_\alpha - 1 - \lambda_\alpha$ even **OR** $\gamma_\alpha - 1 - \lambda_\alpha$ odd
`3Cr` Cay	$\lambda + (\gamma_\alpha - 1 - \lambda_\alpha)\kappa$	τ	ℓ **OR** $\ell + \kappa^\vee$	$t - \frac{t_\kappa}{2}\kappa^\vee$	$g_\alpha - 1 - \ell_\alpha$ even **OR** $g_\alpha - 1 - \ell_\alpha$ odd
`3i` Cay	λ	$\tau - \tau_\kappa\alpha$	$\ell + \left(g_\alpha - 1 - \frac{\ell_\kappa}{2}\right)\kappa^\vee$	t	
`3r` Cay	$\lambda + \left(\gamma_\alpha - 1 - \frac{\lambda_\kappa}{2}\right)\kappa$	τ	ℓ	$t - t_\kappa\alpha^\vee$	

Table 5 Action of Hecke operators

κ-type(E)	$T_\kappa(E)$	κ-type(E)	$T_\kappa(E)$
1C+	$w_\kappa \times E$	1C-	$(q-1)E + q(w_\kappa \times E)$
1i1	$w_\kappa \times E + E_\kappa$	1r2	$(q-1)E - w_\kappa \times E + (q-1)E_\kappa$
1i2f	$E + E_\kappa^1 + E_\kappa^2$	1r1f	$(q-2)E + (q-1)(E_\kappa^1 + E_\kappa^2)$
1i2s	$-E$	1r1s	qE
1ic	qE	1rn	$-E$
2C+	$w_\kappa \times E$	2C-	$(q^2-1)E + q^2(w_\kappa \times E)$
2Ci	$qE \pm (q+1)E_\kappa$ (see Section 9)	2Cr	$(q^2-q-1)E \pm (q^2-q)E_\kappa$ (see Section 9)
2i22	$E + E_\kappa^1 + E_\kappa^2$	2r11	$(q^2-2)E + (q^2-1)(E_\kappa^1 + E_\kappa^2)$
2i11	$w_\kappa \times E + E_\kappa$	2r22	$(q^2-1)E - w_\kappa \times E + (q^2-1)E_\kappa$
2i12f	$w_\kappa \times E \pm E_\kappa^1 \pm E_\kappa^2$ (see Section 8)	2r21f	$(q^2-2)E + (q^2-1)(\pm E_\kappa^1 \pm E_\kappa^2)$ (see Section 8)
2i12s	−E	2r21s	q^2E
2ic	q^2E	2rn	$-E$
3C+	$w_\kappa \times E$	3C-	$(q^3-1)E + q^3(w_\kappa \times E)$
3Ci	$qE + (q+1)E_\kappa$	3Cr	$(q^3-q-1)E + (q^3-q)E_\kappa$
3i	$qE + (q+1)E_\kappa$	3r	$(q^3-q-1)E + (q^3-q)E_\kappa$
3ic	q^3E	3rn	$-E$

Ladder representations of $\mathbf{GL}(n, \mathbb{Q}_p)$

Dan Barbasch and Dan Ciubotaru

To David Vogan with admiration

Abstract In this paper, we recover certain known results about the ladder representations of $\mathrm{GL}(n, \mathbb{Q}_p)$ defined and studied by Lapid, Mínguez, and Tadić. We work in the equivalent setting of graded Hecke algebra modules. Using the Arakawa–Suzuki functor from category O to graded Hecke algebra modules, we show that the determinantal formula proved by Lapid–Mínguez and Tadić is a direct consequence of the BGG resolution of finite-dimensional simple $\mathfrak{gl}(n)$-modules. We make a connection between the semisimplicity of Hecke algebra modules, unitarity with respect to a certain hermitian form, and ladder representations.

Key words: ladder representations, unitary representations, graded affine Hecke algebra, nilpotent orbits

MSC (2010): Primary 22E50; Secondary 20C08

1 Introduction

In this paper we study a class of representations of the graded affine Hecke algebra which are unitary for a star operation which we call $\circledast$. The $\circledast$-unitary dual for type A is determined completely. In this case, the unitary dual corresponds via the

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Borel–Casselmann equivalence of categories [Bo] composed with the reduction to the graded affine Hecke algebra of [Lu1] to the *ladder representations* defined and studied in [LM] and [Ta] for $\mathrm{GL}(n, \mathbb{Q}_p)$.

The classification of the unitary dual of real and p-adic reductive groups is one of the central problems of representation theory. Typically, by results of Harish-Chandra, this problem is reduced to an algebraic one, the study of *admissible representations* of an algebra endowed with a *star operation*. In the case of real groups, this algebra is the Hecke algebra $R(\mathfrak{g}, K)$ introduced in [KV]. In the case of p-adic groups, it is an Iwahori–Hecke type algebra with parameters. In both cases, the star operation is derived from the anti-automorphism $g \mapsto g^{-1}$. In the real case, David Vogan and his collaborators [ALTV] made a deep study of signatures of hermitian forms of admissible modules by exploiting the relationship between two different star operations, one related to the real form of the reductive group, the other related to the compact form of the group. Motivated by this, we study the analogues of these star operations for graded affine Hecke algebras. The star operation coming from the p-adic group is made explicit in [BM2]. In [BC1], we introduce and study another star operation which we denote by $\circledast$, the analogue of the star operation for a compact form. The problem of the unitarity of representations for $\circledast$ seemed an artificial one. Motivated by the study of spherical eigenfunctions for invariant differential operators on symmetric spaces, Opdam [Op] studied two graded Hecke algebra modules $C_c^\infty(\mathfrak{a})$ ($\mathfrak{a}$ being a Euclidean space) and $C^\infty(T)$ (T being a compact torus). The modules $C^\infty(T)$ are related to the compact symmetric case, and are unitary for $\circledast$ (denoted $+$ in [Op]). The modules $C_c^\infty(\mathfrak{a})$ are related to the noncompact symmetric spaces, are unitary for $*$, and are studied further in [Od].

The unitary dual for $\circledast$ is a central topic of this paper. The first set of results is a connection between $\circledast$-unitary representations, and representations which are $\mathbb{A}$-semisimple. This is the content of Theorem 2.11. This provides a connection to the work of [Ch],[KR], and [Ra].

In ongoing research we are planning to determine the entire $\circledast$-unitary dual for graded affine Hecke algebras of arbitrary type. The most complete results to date are for type A. In the process we found the links to the ladder representations in the title, and the results of [LM], [CR], and [Ta].

A seminal idea, pioneered by D. Vogan, was to try to make a connection between the unitary dual of real and p-adic groups via intertwining operators, *petite* K-types and W-types. This was developed systematically by the authors of this paper, jointly and separately, in particular to determine the full spherical unitary dual of split p-adic (and split real) classical groups. We follow this approach in this paper. We relate the unitary (star for the compact form of the Lie algebra) dual of Verma modules to the $\circledast$-unitary dual of the graded affine Hecke algebra using the functors introduced by Arakawa and Suzuki, [AS, Su]. The advantage of this method is that it provides interesting connections between the Bernstein–Gelfand–Gelfand resolution and results about character formulas of ladder representations.

Some time ago, motivated by conjectures of Arthur concerning unipotent representations, D. Barbasch, S. Evens, and A. Moy conjectured the existence of an action of the Iwahori–Hecke algebra on the cohomology of the incidence variety of a pair of nilpotent elements whose $\mathfrak{sl}(2)$-triples commute. More details are at the end of Section 5.2. In Section 5 we provide new evidence for this conjecture, and establish connections to the work of [Gi] and [EP].

2 The star operation ⊛

2.1 Graded affine Hecke algebra

Let $\Phi = (V, R, V^\vee, R^\vee, \Pi)$ be a reduced based $\mathbb{R}$-root system. In particular, V and $V^\vee$ are finite-dimensional $\mathbb{R}$-vector spaces in perfect duality $(\, , \,) : V \times V^\vee \to \mathbb{R}$. Let $W \subset GL(V)$ be the Weyl group generated by the simple reflections $\{s_\alpha : \alpha \in \Pi\}$. The positive roots are R^+ and the positive coroots are $R^{\vee,+}$. The complexifications of V and $V^\vee$ are denoted by $V_\mathbb{C}$ and $V_\mathbb{C}^\vee$, respectively, and we denote by $\bar{\ }$ the complex conjugations of $V_\mathbb{C}$ and $V_\mathbb{C}^\vee$ induced by V and $V^\vee$, respectively. Extend linearly the pairing $(\, , \,)$ to $V_\mathbb{C} \times V_\mathbb{C}^\vee$.

Let $k : \Pi \to \mathbb{R}_{>0}$ be a function such that $k_\alpha = k_{\alpha'}$ whenever $\alpha, \alpha' \in \Pi$ are W-conjugate. Let $\mathbb{C}[W]$ denote the group algebra of W and $S(V_\mathbb{C})$ the symmetric algebra over $V_\mathbb{C}$. The group W acts on $S(V_\mathbb{C})$ by extending the action on V. For every $\alpha \in \Pi$, denote the difference operator by

$$\Delta : S(V_\mathbb{C}) \to S(V_\mathbb{C}), \quad \Delta_\alpha(a) = \frac{a - s_\alpha(a)}{\alpha}, \text{ for all } a \in S(V_\mathbb{C}). \tag{1}$$

If $a \in V_\mathbb{C}$, then $\Delta_\alpha(a) = (a, \alpha^\vee)$.

Definition 2.1 ([Lu1]). The graded affine Hecke algebra $\mathbb{H} = \mathbb{H}(\Phi, k)$ is the unique associative unital algebra generated by $\{a : a \in S(V_\mathbb{C})\}$ and $\{t_w : w \in W\}$ such that

(i) the assignment $t_w a \mapsto w \otimes a$ gives an isomorphism of $(\mathbb{C}[W], S(V_\mathbb{C}))$-bimodules between $\mathbb{H}$ and $\mathbb{C}[W] \otimes S(V_\mathbb{C})$;
(ii) $a t_{s_\alpha} = t_{s_\alpha} s_\alpha(a) + k_\alpha \Delta_\alpha(a)$, for all $\alpha \in \Pi$, $a \in S(V_\mathbb{C})$.

The center of $\mathbb{H}$ is $S(V_\mathbb{C})^W$ ([Lu1]). By Schur's Lemma, the center of $\mathbb{H}$ acts by scalars on each irreducible $\mathbb{H}$-module. The central characters are parameterized by W-orbits in $V_\mathbb{C}^\vee$. If X is an irreducible $\mathbb{H}$-module, denote by $\mathsf{cc}(X) \in V_\mathbb{C}^\vee$ (actually in $W \backslash V_\mathbb{C}^\vee$) its central character.

2.2 Star operations

Let w_0 denote the long Weyl group element, and let δ be the involutive automorphism of $\mathbb{H}$ determined by

$$\delta(t_w) = t_{w_0 w w_0},\ w \in W, \quad \delta(\omega) = -w_0(\omega),\ \omega \in V_{\mathbb{C}}. \tag{2}$$

When w_0 is central in W, $\delta = \mathrm{Id}$.

Definition 2.2. Let $\kappa : \mathbb{H} \to \mathbb{H}$ be a conjugate linear involutive algebra anti-automorphism. An $\mathbb{H}$-module (π, X) is said to be κ-hermitian if X has a hermitian form $(\ ,\ )$ which is κ-invariant, i.e.,

$$(\pi(h)x, y) = (x, \pi(\kappa(h))y), \quad x, y \in X,\ h \in \mathbb{H}.$$

A hermitian module X is κ-unitary if the κ-hermitian form is positive definite.

Definition 2.3. Define a conjugate linear algebra anti-involution $*$ of $\mathbb{H}$ by

$$t_w^* = t_{w^{-1}},\ w \in W, \quad \omega^* = \overline{\mathrm{Ad}\, t_{w_0}(\delta(a))} = -t_{w_0} \cdot \overline{w_0(\omega)} \cdot t_{w_0},\ \omega \in V_{\mathbb{C}}. \tag{3}$$

Similarly, define $\circledast$ by

$$t_w^{\circledast} = t_{w^{-1}},\ w \in W, \quad \omega^{\circledast} = \overline{\omega},\ \omega \in V_{\mathbb{C}}. \tag{4}$$

The operations $*$ and $\circledast$ are related by

$$* = \mathrm{Ad}\, t_{w_0} \circ \circledast \circ \delta, \qquad \text{for all } h \in \mathbb{H}. \tag{5}$$

Remark 2.4. In [BC1], it is proved that $*$ and $\circledast$ are the only star operations of $\mathbb{H}$ that satisfy certain natural conditions. When $\mathbb{H}$ is obtained by grading the Iwahori–Hecke algebra of a reductive p-adic group, $*$ corresponds to the natural star operation of the p-adic group. The operation $\circledast$ is the analogue of the compact star operation defined for real reductive groups in [ALTV].

2.3 Semisimplicity

In the rest of this section, suppose the parameters k_α are positive, but arbitrary. Let (π, X) be a finite-dimensional $\mathbb{H}$-module. For every $\lambda \in V_{\mathbb{C}}^\vee$, define

$$\begin{aligned} X_\lambda &= \{x \in X : \pi(\omega)x = (\omega, \lambda)x,\ \text{for all } \omega \in V_{\mathbb{C}}\}, \\ X_\lambda^{\mathsf{gen}} &= \{x \in X : (\pi(\omega) - (\omega, \lambda))^n x = 0 \text{ for some } n \in \mathbb{N},\ \text{for all } \omega \in V_{\mathbb{C}}\}. \end{aligned} \tag{6}$$

A functional $\lambda \in V_{\mathbb{C}}^\vee$ is called a weight of X if $X_\lambda \neq 0$. Let $\mathsf{Wt}(X)$ denote the set of weights of X. It is straightforward that $\mathsf{Wt}(X) \subset W \cdot \mathsf{cc}(X)$.

Definition 2.5. The module (π, X) is called $\mathbb{A}$-semisimple if $X_\lambda = X_\lambda^{\mathsf{gen}}$ for all λ.

Lemma 2.6 (see [KR]). *If X is $\mathbb{A}$-semisimple, then $(\alpha, \lambda) \neq 0$ for all $\alpha \in \Pi$ and $\lambda \in \mathsf{Wt}(X)$.*

Proof. We actually prove more. For every $\alpha \in \Pi$, define (as in [Lu1]) the element

$$R_\alpha = t_{s_\alpha}\alpha - k_\alpha, \tag{7}$$

with the properties

$$a \cdot R_\alpha = R_\alpha \cdot s_\alpha(a),\ a \in S(V_\mathbb{C}), \quad R_\alpha^2 = k_\alpha^2 - \alpha^2. \tag{8}$$

In particular, if $x_\lambda \in X_\lambda$, then $R_\alpha x_\lambda \in X_{s_\alpha(\lambda)}$ (possibly zero).

Let $\alpha \in \Pi$ be given, and let $\mathbb{H}_\alpha$ be the subalgebra generated by t_α, α. This is a graded Hecke algebra of type A_1 with parameter k_α. Given the $\mathbb{H}$-module (π, X) and a vector $0 \neq x_\lambda \in X_\lambda$, the span of $\{x_\lambda, \pi(t_{s_\alpha})x_\lambda\}$ is an $\mathbb{H}_\alpha$-module. There are two cases:

1. Suppose $\pi(R_\alpha)x_\lambda = 0$. Also $\pi(R_\alpha^2)x_\alpha = 0$, which implies that $(\alpha, \lambda) = \epsilon_\alpha k_\alpha$ for some $\epsilon_\alpha \in \{\pm 1\}$. Moreover $\pi(t_{s_\alpha})x_\lambda = \epsilon_\alpha x_\lambda$. The $\mathbb{H}_\alpha$-module generated by x_λ is either the trivial (when $\epsilon_\alpha = 1$) or the Steinberg module (when $\epsilon_\alpha = -1$.)
2. Suppose $\pi(R_\alpha)x_\lambda \neq 0$. Then there are two subcases:
   a. $(\alpha, \lambda) = 0$, in which case $\pi(R_\alpha)x_\lambda = -k_\alpha x_\lambda$. Set $y_\lambda = \pi(t_{s_\alpha})x_\lambda$. Then $\pi(\alpha)y_\lambda = 2k_\alpha x_\lambda$, so $\mathrm{span}\{x_\lambda, y_\lambda\}$ is a two-dimensional $\mathbb{H}_\alpha$-module isomorphic to the irreducible tempered principal series at 0. In particular, $\mathbb{A}$ does not act semisimply.
   b. $(\alpha, \lambda) \neq 0$, in which case $\pi(R_\alpha)x_\lambda$ is a weight vector with weight $s_\alpha(\lambda)$. The $\mathrm{span}\{x_\lambda, y_\lambda\}$ is a two dimensional $\mathbb{H}_\alpha$-module isomorphic to an irreducible non-tempered principal series, and $\mathbb{A}$ acts semisimply on it.

□

Proposition 2.7. *Assume (π, X) is a $\circledast$-unitary finite-dimensional $\mathbb{H}$-module. Then X is $\mathbb{A}$-semisimple.*

Proof. Let $(\ ,\ )_X$ be the positive definite $\circledast$-form on X. Let λ be a weight of X and $x_\lambda \neq 0$ a weight vector. Define

$$\{x_\lambda\}^\perp = \{y \in X : (x_\lambda, y)_X = 0\}.$$

Let $y \in \{x_\lambda\}^\perp$ be given. We get, for $\omega \in V_\mathbb{C}$,

$$0 = (\omega, \lambda)(x_\lambda, y)_X = (\pi(\omega)x_\lambda, y)_X = (x_\lambda, \pi(\omega^\circledast)y)_X = (x_\lambda, \pi(\overline{\omega})y)_X,$$

so $(x_\lambda, \pi(\overline{\omega})y)_X = 0$. It follows that $\{x_\lambda\}^\perp$ is $\mathbb{A}$-invariant. Since the form $(\ ,\ )_X$ is positive definite, we have $X = \mathbb{C}x_\lambda \oplus \{x_\lambda\}^\perp$ as $\mathbb{A}$-modules. By induction, it follows that X is a direct sum of one-dimensional $\mathbb{A}$-modules, thus $\mathbb{A}$-semisimple. □

Remark 2.8. The above proposition can also be interpreted as the following linear algebra statement: if J is a hermitian matrix, and N a nonzero nilpotent matrix such that

$$JN = N^*J, \quad \text{where } N^* \text{ the transpose conjugate of } N,$$

then J is not positive definite.

Remark 2.9. The proof and statement of Proposition 2.7 can be easily generalized by replacing $\mathbb{A}$ with any parabolic subalgebra of $\mathbb{H}$.

In light of Proposition 2.7, we need to record certain facts about $\mathbb{A}$-semisimple modules. The following easy properties can be found in [KR].

Theorem 2.10 ([KR]). *Assume (π, X) is a finite-dimensional $\mathbb{A}$-semisimple $\mathbb{H}$-module.*

1. *If (π, X) is irreducible, then* $\dim X_\lambda \leq 1$.
2. *Suppose (π, X) is a simple $\mathbb{A}$-semisimple module. Then there exists a basis $\{x_\lambda : \lambda \in \mathsf{Wt}(X)\}$ of X such that the W-action is*

$$\pi(t_{s_\alpha})x_\lambda = \begin{cases} \frac{k_\alpha}{(\alpha,\lambda)} x_\lambda, & \text{if } s_\alpha(\lambda) \notin \mathsf{Wt}(X), \\ \frac{k_\alpha}{(\alpha,\lambda)} x_\lambda + \left(1 + \frac{k_\alpha}{(\alpha,\lambda)}\right) x_{s_\alpha(\lambda)}, & \text{if } s_\alpha(\lambda) \in \mathsf{Wt}(X), \end{cases} \tag{9}$$

for every $\alpha \in \Pi$. (Recall that $(\alpha, \lambda) \neq 0$ by Lemma 2.6.)

We can now prove a characterization of $\circledast$-unitary modules.

Theorem 2.11. *A simple $\mathbb{H}$-module (π, X) with real central character is $\circledast$-unitary if and only if it is $\mathbb{A}$-semisimple, and*

$$|(\alpha, \lambda)| \geq k_\alpha \tag{10}$$

for every $\alpha \in \Pi$ and $\lambda \in \mathsf{Wt}(X)$.

In particular, if $\mathbb{H}$ has equal parameters $k_\alpha = 1$ and X has integral central character, then X is $\circledast$-unitary if and only if it is $\mathbb{A}$-semisimple.

Proof. Suppose first that (π, X) is simple $\circledast$-unitary with form $(\ ,\ )_X$. By Proposition 2.7, it is $\mathbb{A}$-semisimple. Let $\lambda \in \mathsf{Wt}(X)$ and let x_λ a corresponding weight vector be given. Set $y_\lambda := \pi(R_\alpha)x_\lambda$. Using $R_\alpha^\circledast = -R_\alpha$, we have

$$\begin{aligned} 0 \leq (y_\lambda, y_\lambda)_X = (\pi(R_\alpha)x_\alpha, \pi(R_\alpha)x_\alpha)_X &= -(\pi(R_\alpha^2)x_\lambda, x_\lambda)_X \\ &= \big((\alpha,\lambda)^2 - k_\alpha^2\big)(x_\lambda, x_\lambda)_X, \end{aligned} \tag{11}$$

which implies (10). Notice that in fact, if $y_\lambda \neq 0$, which is the case when $s_\alpha(\lambda) \in \mathsf{Wt}(X)$, the inequality is strict, and so $|(\alpha, \lambda)| > k_\alpha$.

For the converse, suppose that X is an irreducible $\mathbb{A}$-semisimple and that (10) holds. We construct a $\circledast$-hermitian form $(\ ,\ )$ on X which is positive definite. Let $\{x_\lambda\}$ be a basis of X as in Theorem 2.10(3). Clearly, we need to set

$$(x_\lambda, x_\mu) = 0, \text{ if } \lambda \neq \mu,$$

and with this condition, the form is automatically $\mathbb{A}$-invariant. Set $(x_\lambda, x_\lambda) = a_\lambda$. We need to check using the W-invariance that the scalars a_λ can be chosen to be all positive in a consistent way.

Let α be a simple root. We use the formula in Theorem 2.10 repeatedly. If $\mu \neq \lambda, s_\alpha(\lambda)$, then

$$(\pi(t_{s_\alpha})x_\lambda, x_\mu) = 0 = (x_\lambda, \pi(t_{s_\alpha})x_\mu).$$

Next

$$(\pi(t_{s_\alpha})x_\lambda, x_\lambda) = \frac{k_\alpha}{(\alpha, \lambda)} a_\lambda = (x_\lambda, \pi(t_{s_\alpha})x_\lambda).$$

It remains to check that $(\pi(t_{s_\alpha})x_\lambda, x_{s_\alpha(\lambda)}) = (x_\lambda, \pi(t_{s_\alpha})x_{s_\alpha(\lambda)})$. If $s_\alpha(\lambda)$ is not a weight, there is nothing to check, so suppose both λ and $s_\alpha(\lambda)$ are weights. Then

$$\begin{aligned}(\pi(t_{s_\alpha})x_\lambda, x_{s_\alpha(\lambda)}) &= \left(1 + \frac{k_\alpha}{(\alpha, \lambda)}\right) a_{s_\alpha(\lambda)} \\ (x_\lambda, \pi(t_{s_\alpha})x_{s_\alpha(\lambda)}) &= \left(1 + \frac{k_\alpha}{(\alpha, s_\alpha(\lambda))}\right) a_\lambda = \left(1 - \frac{k_\alpha}{(\alpha, \lambda)}\right) a_\lambda.\end{aligned} \tag{12}$$

So we need that

$$a_{s_\alpha(\lambda)} = \frac{(\alpha, \lambda) - k_\alpha}{(\alpha, \lambda) + k_\alpha} a_\lambda,$$

and this is positive if $a_\lambda > 0$ by (10). □

3 Ladder representations: definitions

We consider the graded Hecke algebra of type A. More precisely the data $(V, R, V^\vee, R^\vee)$ correspond to the Hecke algebra of GL(n). We will denote this algebra by $\mathbb{H}_n$. More generally, if $P = MN \subset G$ is a standard parabolic subgroup, we will write $\mathbb{H}_M$ for the Hecke algebra corresponding to $(V, R(M), R(M)^\vee, V^\vee)$ viewed as a subalgebra of the Hecke algebra $\mathbb{H}$ for G.

In the case of GL(n), we will classify the $\circledast$-unitary dual. We begin by recalling Zelevinsky's classification [Ze1] of the simple modules. We will phrase the classification "with quotients" rather than "submodules", cf. [Ze1, §10].

3.1 Multisegments

We restrict to $\mathbb{H}$-modules with real central character. By [BC2, Corollary 4.3.2 or Corollary 5.1.3], every simple $\mathbb{H}$-module with real central character admits a non-degenerate $\circledast$-invariant hermitian form.

A segment is a set $\Delta = \{a, a+1, a+2, \dots, b\}$, where $a, b \in \mathbb{R}$ and $a \equiv b$ (mod $\mathbb{Z}$). We will write $\Delta = [a, b]$ and $|\Delta| = b - a + 1$ for the length. A multisegment is an ordered collection $(\Delta_1, \Delta_2, \dots, \Delta_r)$ of segments. Following [Ze1, §4.1], two segments Δ_1 and Δ_2 are called

(a) linked, if $\Delta_1 \not\subset \Delta_2$, $\Delta_2 \not\subset \Delta_1$, and $\Delta_1 \cup \Delta_2$ is a segment;
(b) juxtaposed, if Δ_1, Δ_2 are linked and $\Delta_1 \cap \Delta_2 = \emptyset$.

One says that

(c) Δ_1 precedes Δ_2 if Δ_1, Δ_2 are linked and $a_1 < a_2$.

For every segment Δ with $m = b - a + 1$, let $\langle\Delta\rangle$ denote the one-dimensional $\mathbb{H}_m$-module which extends the sign W-representation and on which $\mathbb{A}$ acts by the character $\mathbb{C}_{[a,b]}$. If $(\Delta_1, \Delta_2, \dots, \Delta_r)$ is a multisegment, denote by

$$\langle\Delta_1\rangle \times \langle\Delta_2\rangle \times \cdots \times \langle\Delta_r\rangle \tag{13}$$

the induced module $\mathbb{H}_n \otimes_{\mathbb{H}_{m_1} \times \mathbb{H}_{m_2} \times \cdots \times \mathbb{H}_{m_r}} (\langle\Delta_1\rangle \boxtimes \langle\Delta_2\rangle \boxtimes \cdots \boxtimes \langle\Delta_r\rangle)$, where $m_i = b_i - a_i + 1$ and $n = \sum m_i$. Here $\mathbb{H}_{m_1} \times \cdots \times \mathbb{H}_{m_r} \subset \mathbb{H}_m$ is the subalgebra $\mathbb{H}_M \subset \mathbb{H}_n$ corresponding to the standard parabolic subgroup $P = MN \subset G$ with Levi component $\mathrm{GL}(m_1) \times \cdots \times \mathrm{GL}(m_r) \subset \mathrm{GL}(n)$.

We need two of the main results from [Ze1].

Theorem 3.1 ([Ze1, Theorem 4.2]). *The following conditions are equivalent:*

1. *The module $\langle\Delta_1\rangle \times \cdots \times \langle\Delta_r\rangle$ is irreducible.*
2. *For each $i, j = 1, \dots, r$, the segments Δ_i and Δ_j are not linked.*

Theorem 3.2 ([Ze1, Theorem 6.1]).

(a) *Let $(\Delta_1, \dots, \Delta_r)$ be a multisegment. Suppose that for each $i < j$, Δ_i does not precede Δ_j. Then the representation $\langle\Delta_1\rangle \times \cdots \times \langle\Delta_r\rangle$ has a unique irreducible quotient denoted by $\langle\Delta_1, \dots, \Delta_r\rangle$.*
(b) *The modules $\langle\Delta_1, \dots, \Delta_r\rangle$ and $\langle\Delta'_1, \dots, \Delta'_s\rangle$ are isomorphic if and only if the corresponding multisegments are equal up to a rearrangement.*
(c) *Every simple $\mathbb{H}_n$-module with real central character is isomorphic to one of the form $\langle\Delta_1\rangle \times \cdots \times \langle\Delta_r\rangle$.*

Remark 3.3. For the most part, the above results are instances of the Langlands classification. A multisegment corresponds to data (M, σ, ν) where

$$M = \mathrm{GL}(b_1 - a_1 + 1) \times \cdots \times \mathrm{GL}(b_r - a_r + 1)$$

is a Levi component, the tempered representation σ is the Steinberg representation, and the (a_i, b_i) determine the ν. The fact that Δ_i precedes Δ_j is the usual dominance condition for ν. The remaining results are sharpenings of the Langlands classification in the case of $\mathrm{GL}(n)$.

Definition 3.4 (Ladder representations [LM]). Let $\Delta_i = [a_i, b_i]$ $1 \leq i \leq r$ be Zelevinsky segments. If $a_1 > a_2 > \cdots > a_r$ and $b_1 > b_2 > \cdots > b_r$, call the irreducible representation $\langle \Delta_1, \Delta_2, \ldots, \Delta_r \rangle$ a *ladder representation*.

Example 3.5 (Speh representations [BM1]). Let Δ_i, $1 \leq i \leq r$ be segments as in Definition 3.4, such that $b_i - a_i + 1 = d$ for a fixed d and $a_i - a_{i+1} = 1$ for all i. Then $\langle \Delta_1, \ldots, \Delta_r \rangle$ is irreducible as an S_n-representation, isomorphic to the S_n-representation parameterized by the rectangular Young diagram with r rows and d columns. These modules are both $\circledast$-unitary and $*$-unitary ([BM1, CM]) and correspond to the (I-fixed vectors) of Speh representations.

3.2 Cherednik's construction

As in Definition 3.4, let $\langle \Delta_1, \ldots, \Delta_r \rangle$, $a_1 > a_2 > \cdots > a_r$, $b_1 > b_2 > \cdots > b_r$ be a ladder representation. The interesting case is when Δ_i is linked to Δ_{i+1} for all i. In fact, since tensoring with a character of the center of $\mathbb{H}$ does not change $\mathbb{A}$-semisimplicity, we may even assume that $a_i, b_i \in \mathbb{Z}$ for all i. From now on, this type of ladder representations will be called *integral*.

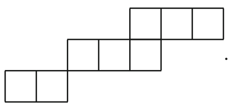
Following [Ch], we give a combinatorial construction of integral ladder representations. Let $\langle \Delta_1, \ldots, \Delta_r \rangle$ be an integral ladder representation with the notation as above. Set

$$\lambda = (a_1, \ldots, b_1, a_2, \ldots, b_2, \ldots, a_r, \ldots, b_r) \in \mathbb{Z}^n \tag{14}$$

viewed as an element of $V_{\mathbb{C}}^{\vee} \cong \mathbb{C}^n$. The underlying multisegment $(\Delta_1, \ldots, \Delta_r)$ gives a skew-Young diagram, where each box in the Young diagram corresponds to an integer in one of the multisegments. More precisely, the underlying skew diagram is formed as follows. The first segment Δ_1 gives the top row with $|\Delta_1|$ boxes, each box for one of the integers in Δ_1 in order. The segment Δ_2 gives the second row with $|\Delta_2|$ boxes, immediately below the first, etc. The rows are aligned so that

1. the two boxes are in the same column if and only if they correspond to the same integer in the multisegment, and
2. two boxes in two adjacent columns correspond to two consecutive integers in the multisegment.

For example, if the multisegment is $([2, 4], [0, 2], [-2, -1])$, the resulting skew-Young diagram is



(15)

Notice that the skew-Young diagram does not recover the integral ladder representation uniquely, only up to tensoring with a character of $\mathbb{H}$ that is trivial on W. However, if we specify an integer a such that the first segment starts with a, then the multisegment is determined.

We fix a skew-Young diagram as above and we will form skew-Young tableaux with that shape. Let $[1 \dots n]$ be the set of integers $1, 2, \dots, n$. Let Y_1 be the skew-Young tableau with entries in $[1 \dots n]$ such that in the first row, the entries are, in order, $1, 2, \dots, b_1 - a_1 + 1$, in the second row, $b_1 - a_1 + 2, b_1 - a_1 + 3, \dots, b_1 + b_2 - a_1 - a_2 + 2$, etc. In our example,

$$Y_1 = \begin{array}{ccccccc} & & & & \boxed{1} & \boxed{2} & \boxed{3} \\ & & \boxed{4} & \boxed{5} & \boxed{6} & & \\ \boxed{7} & \boxed{8} & & & & & \end{array}. \tag{16}$$

Consider all skew-Young tableaux with entries in $[1 \dots n]$ subject to the requirements:

1. the entries are increasing left-right on each row;
2. the entries are increasing up-down on each 45°-diagonal.

Denote every such tableau by Y_w, where $w \in S_n$ is the permutation transforming $(1, 2, \dots, n)$ to the entries of the tableau read in order from the top row to the bottom row and on each row from left to right. Let

$$\begin{aligned} W(\Delta_1, \dots, \Delta_r) = {} & \text{the set of } w \in S_n \text{ parameterizing the tableaux } Y_w \\ & \text{for } (\Delta_1, \dots, \Delta_r). \end{aligned} \tag{17}$$

Theorem 3.6 (Cherednik [Ch, Theorem 4], see also Ram [Ra]). *The set $\{Y_w\}$ defined above is a basis of a simple $\mathbb{H}$-module $\mathsf{C}(\Delta_1, \dots, \Delta_r)$ such that*

1. Y_w is an $\mathbb{A}$-weight vector with weight $w(\lambda)$.
2. the action of W on $\{Y_w\}$ is as follows:

$$\pi(t_{s_\alpha}) Y_w = \begin{cases} \frac{1}{(\alpha, w(\lambda))} Y_w, & \text{if } s_\alpha w \notin W(\Delta_1, \dots, \Delta_r) \\ \frac{1}{(\alpha, w(\lambda))} Y_w + \left(1 + \frac{1}{(\alpha, w(\lambda))}\right) Y_{s_\alpha w}, & \text{if } s_\alpha w \in W(\Delta_1, \dots, \Delta_r), \end{cases} \tag{18}$$

for every $\alpha \in \Pi$.

The weight λ is such that it determines the Langlands parameter of the irreducible ladder representation $\langle \Delta_1, \dots, \Delta_r \rangle$. See [Ev] for relevant details on the Langlands classification for representations of the graded Hecke algebras. Thus

$$\langle \Delta_1, \dots, \Delta_r \rangle \cong \mathsf{C}(\Delta_1, \dots, \Delta_r), \tag{19}$$

for every integral ladder representation.

4 Ladder representations: functors from category $\mathcal{O}$ to $\mathbb{H}$-modules

In this section, we apply some constructions of Zelevinsky [Ze2] and Arakawa and Suzuki [AS, Su] to the study of ⊛-unitary representations.

4.1 Category O

Let $\mathfrak{g}$ be a complex reductive Lie algebra with universal enveloping algebra $U(\mathfrak{g})$. Fix a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$, and a Borel subalgebra $\mathfrak{b} = \mathfrak{h} \oplus \mathfrak{n}$. Let $R \subset \mathfrak{h}^*$ denote the roots of $\mathfrak{g}$ with respect to $\mathfrak{h}$, and let R^+ be the positive roots with respect to $\mathfrak{b}$. Let $W = N_G(\mathfrak{h})/Z_G(\mathfrak{h})$ be the Weyl group with length function ℓ.

Let $\mathfrak{n}^-$ be the nilradical of the opposite Borel subalgebra. Let Π be the simple roots defined by R^+, and for every root α, let $\alpha^\vee \in \mathfrak{h}$ be the coroot. Let α_i, $i = 1, \dots, |\Pi|$ denote the simple roots, and $\omega_i^\vee$ the corresponding fundamental coweights. Set $\rho = \frac{1}{2}\sum_{\alpha \in R^+} \alpha$. We denote by $\langle\ ,\ \rangle$ the pairing between $\mathfrak{h}^*$ and $\mathfrak{h}$.

Define

$$\Lambda = \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z},\ \text{for all } \alpha \in R\};$$
$$\Lambda^+ = \{\lambda \in \mathfrak{h}^* : \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}_{\geq 0},\ \text{for all } \alpha \in R^+\}.$$

Let O denote the category of finitely generated $U(\mathfrak{g})$-modules, which are $\mathfrak{n}$-locally finite and $\mathfrak{h}$-semisimple. If X is a module in O, let $\Omega(X)$ denote the set of $\mathfrak{h}$-weights of X.

For every $\mu \in \mathfrak{h}^*$, let $M(\mu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}v_\mu$ denote the Verma module with highest weight μ and infinitesimal character $\mu + \rho$. Then $M(\mu) \in O$ has a unique simple quotient, the highest weight module $L(\mu)$. As it is well known, $L(\mu)$ is a simple finite-dimensional module if and only if $\mu \in \Lambda^+$.

For every $w \in W$, $\lambda \in \mathfrak{h}^*$, define

$$w \circ \lambda = w(\lambda + \rho) - \rho.$$

For $X \in O$, let

$$H_0(\mathfrak{n}^-, X) = X/\mathfrak{n}^- X \tag{20}$$

denote the 0-th $\mathfrak{n}^-$-homology space, viewed as an $\mathfrak{h}$-module. For every $\lambda \in \Lambda^+$ and every finite-dimensional $\mathfrak{g}$-module $\mathcal{V}$, define the functor

$$F_{\lambda,\mathcal{V}} : O \to \mathrm{Vect}, \quad F_{\lambda,\mathcal{V}}(X) = H_0(\mathfrak{n}^-, X \otimes \mathcal{V})_\lambda, \tag{21}$$

where the subscript stands for the λ-weight space, and Vect denotes the category of $\mathbb{C}$-vector spaces.

Remark 4.1. In general, $F_{\lambda,\mathcal{V}}$ need not be an exact functor. However, if one assumes that $\lambda + \rho \in \Lambda^+$, then $F_{\lambda,\mathcal{V}}$ is exact. See for example [AS, Proposition 1.4.2].

Let $L(\mu)$ be a simple finite-dimensional module. Recall that there exists a resolution of $L(\mu)$ in O defined by Bernstein–Gelfand–Gelfand:

$$0 \to C_N \to C_{N-1} \to \cdots \to C_1 \to C_0 \to L(\mu) \to 0, \tag{22}$$

where

$$C_i = \bigoplus_{w\in W,\ell(w)=i} M_{w\circ\mu}.$$

In particular, applying the Euler–Poincaré principle, the identity

$$L(\mu) = \sum_{w\in W} \mathsf{sgn}(w) M_{w\circ\mu} \tag{23}$$

holds in the Grothendieck group of O.

Proposition 4.2 ([Ze2, Proposition 1]). *Fix $\lambda \in \Lambda^+$, $\mu \in \Lambda^+$, $\chi \in \Lambda$, and a finite-dimensional representation $\mathcal{V}$.*

1. *The functor $F_{\lambda,\mathcal{V}}$ transforms the BGG resolution (22) into an exact sequence.*
2. *There are natural $\mathbb{C}$-linear isomorphisms*

$$F_{\lambda,\mathcal{V}}(M(\chi)) = \mathcal{V}_{\lambda-\chi} \quad \textit{and} \quad F_{\lambda,\mathcal{V}}(L(\mu)) = \mathcal{V}_{\lambda-\mu}[\mu], \tag{24}$$

where $\mathcal{V}_{\lambda-\chi}$ denotes the $(\lambda-\chi)$-weight space of $\mathcal{V}$, and

$$\mathcal{V}_{\lambda-\mu}[\mu] = \{v \in \mathcal{V}_{\lambda-\mu} : e_\alpha^{\langle \mu+\rho,\alpha^\vee\rangle} v = 0, \textit{ for all } \alpha \in \Pi\}.$$

Here $e_\alpha \in \mathfrak{n}$ denotes a fixed root vector for $\alpha \in \Pi$.

As a corollary, one can transfer formula (23) via F_λ. This is particularly interesting when the images of modules in O under F_λ admit actions by a different group (such as in the classical Schur–Weyl duality) or other algebras.

4.2 The Arakawa–Suzuki functor

We specialize to $\mathfrak{g} = \mathfrak{gl}(n,\mathbb{C})$. Let $E_{i,j}$ denote the matrix with 1 in the (i,j)-position and 0 elsewhere. Let $s_{ij} \in S_n$ denote the transposition. Fix a positive integer ℓ, and set

$$\mathcal{V}_\ell = (\mathbb{C}^n)^{\otimes\ell}, \tag{25}$$

with the diagonal $\mathfrak{g}$-action.

Remark 4.3. If $\ell = n$, the finite-dimensional $\mathfrak{g}$-module $\mathcal{V}_n$ has the property that its 0-weight space is naturally isomorphic to the standard representation of S_n.

For every $0 \leq i < j \leq \ell$, consider the operator

$$\Omega_{i,j} = \sum_{1 \leq k,m \leq n} (E_{k,m})_i \otimes (E_{m,k})_j \in \mathrm{End}(X \otimes \mathcal{V}_\ell), \tag{26}$$

where $(\ )_i$ means that the corresponding element acts on the i-th factor of the tensor product. It is well known that $\Omega_{i,j}$, $1 \leq i < j \leq n$ flips the i, j factors of tensor product, i.e.,

$$\Omega_{i,j}(x \otimes v_1 \otimes \cdots \otimes v_i \otimes \cdots \otimes v_j \otimes \cdots \otimes v_\ell) = x \otimes v_1 \otimes \cdots \otimes v_j \otimes \cdots \otimes v_i \otimes \cdots \otimes v_\ell.$$

Lemma 4.4 ([AS, Theorem 2.2.2], [Su, Lemma 3.1.1]). *For every $X \in O$, the assignment*

$$\begin{aligned} s_{i,i+1} &\mapsto -\Omega_{i,i+1}, & 1 \leq i \leq \ell - 1, \\ \epsilon_j^\vee &\mapsto \sum_{0 \leq i < j} \Omega_{i,j} + \frac{n-1}{2}, & 1 \leq j \leq \ell, \end{aligned}$$

extends to an action of the graded Hecke algebra $\mathbb{H}_\ell$ of $\mathfrak{gl}(\ell)$ on $X \otimes \mathcal{V}_\ell$.

Notice the presence of the minus sign in the action of $s_{i,i+1}$ which is not the convention in [AS]. We make this adjustment so that the results fit with the previous sections. This is because the standard modules for $\mathbb{H}_\ell$ are induced from Steinberg modules to conform with the Langlands classification, not the trivial modules as in [AS].

In this way, the functor $F_{\lambda,\mathcal{V}_\ell}$ from (21) maps to $\mathbb{H}_\ell$-modules. Since we will consider λ such that $\lambda + \rho \in \Lambda^+$, this will be an exact functor.

In [AS] and [Su], the images of Verma modules and highest weight modules are computed. We recall their results now.

Let $P(\mathcal{V}_\ell) \subset \mathfrak{h}^*$ denote the set of weights of $\mathcal{V}_\ell$. If we identify $\mathfrak{h}^*$ with $\mathbb{C}^n$, then these weights are of the form $(\ell_1, \dots, \ell_n)$ where $\sum \ell_i = \ell$ and $\ell_i \geq 0$.

Assume that $\lambda + \rho \in \Lambda^+$ and let $\mu \in \mathfrak{h}^*$ be such that $\lambda - \mu \in P(\mathcal{V}_\ell)$. Define the multisegment [Su, (2.2.7)]

$$\Phi_{\lambda,\mu} = (\Delta_1, \dots, \Delta_n), \quad \Delta_i = [\langle \mu + \rho, \epsilon_i^\vee \rangle, \langle \lambda + \rho, \epsilon_i^\vee \rangle - 1], \tag{27}$$

and the standard $\mathbb{H}_\ell$-module

$$\mathcal{M}(\lambda, \mu) = \langle \Delta_1, \dots, \Delta_n \rangle = \mathbb{H}_\ell \otimes_{\mathbb{H}_{\ell_1} \times \cdots \times \mathbb{H}_{\ell_n}} (\mathsf{St} \otimes \mathbb{C}_{\Delta_1}) \otimes \cdots \otimes (\mathsf{St} \otimes \mathbb{C}_{\Delta_n}). \tag{28}$$

Let $\mathcal{L}(\lambda, \mu)$ denote the unique simple quotient of $\mathcal{M}(\lambda, \mu)$. The lowest S_ℓ-type of $\mathcal{L}(\lambda, \mu)$ is parameterized by the partition of ℓ obtained by ordering $\lambda - \mu = (\ell_1, \dots, \ell_n)$ in decreasing order.

Theorem 4.5 ([Su, Theorems 3.2.1 and 3.2.2]). *Assume that* $\lambda + \rho \in \Lambda^+$ *and* $\mu \in \lambda - P(\mathcal{V}_\ell)$.

1. $F_{\lambda,\mathcal{V}_\ell}(M(\mu)) = \mathcal{M}(\lambda,\mu)$ *as* $\mathbb{H}_\ell$*-modules.*
2. *If* μ *satisfies the condition*

$$\langle \mu + \rho, \alpha^\vee \rangle \in \mathbb{Z}_{\leq 0} \text{ for all } \alpha \in R^+ \text{ satisfying } \langle \lambda + \rho, \alpha^\vee \rangle = 0, \tag{29}$$

then

$$F_{\lambda,\mathcal{V}_\ell}(L(\mu)) = \mathcal{L}(\lambda,\mu).$$

3. *If* μ *does not satisfy condition (29), then*

$$F_{\lambda,\mathcal{V}_\ell}(L(\mu)) = 0.$$

Notice that if λ in the theorem is such that $\langle \lambda + \rho, \alpha^\vee \rangle \geq 1$ for all simple roots α, then condition (29) is vacuously true.

4.3 A character formula

We apply the previous results to the ladder representations. Consider segments $\Delta_i = [a_i, b_i]$, $i = 1, n$, such that $a_1 > a_2 > \dots$ and $b_1 > b_2 > \dots$. Let $\mathsf{C}(\Delta_1, \dots, \Delta_n)$ denote the ladder representation for $\mathbb{H}_\ell$. Identify $(a_1, a_2, \dots, a_n)$ and $(b_1, b_2, \dots, b_n)$ with elements of $\mathfrak{h}^* \cong \mathbb{C}^n$. Set

$$\mu = (a_1, \dots, a_n) - \rho, \quad \lambda = (b_1 + 1, \dots, b_n + 1) - \rho. \tag{30}$$

In coordinates $\rho = (\frac{n-1}{2}, \frac{n-3}{2}, \dots, -\frac{n-1}{2})$.

Assume from now on that $(a_1, \dots, a_n) \equiv \rho \bmod \mathbb{Z}$. Then λ and μ just defined satisfy the conditions of Theorem 4.5 and, in fact, $\langle \lambda + \rho, \alpha^\vee \rangle \geq 1$ for all simple roots α.

Theorem 4.6. *With the notation as above,*

$$F_{\lambda,\mathcal{V}_\ell}(L(\mu)) = \mathsf{C}(\Delta_1, \dots, \Delta_n).$$

In other words ladder representations correspond to finite-dimensional representations highest weight modules.

Proof. This follows immediately from Theorem 4.5(2). □

Corollary 4.7 (see also [Su, §5.1]).

$$\mathsf{C}(\Delta_1, \dots, \Delta_n) = \sum_{w \in S_n} \mathsf{sgn}(w) \langle w \cdot \Delta_1, \dots, w \cdot \Delta_n \rangle,$$

where $w \cdot \Delta_i := [a_{w(i)}, b_i]$, *and the standard representation* $\langle w \cdot \Delta_1, \dots, w \cdot \Delta_n \rangle$ *is understood to be* 0 *if* $a_{w(i)} > b_i$ *for some* i.

Proof. We first apply the functor $F_{\lambda,\mathcal{V}_\ell}$ to the BGG formula (23), and then identify the images of the Verma modules as in Theorem 4.5(1). □

Remark 4.8. Corollary 4.7(2) recovers the known "determinantal" character formula for ladder representations of Tadić [Ta], and Lapid–Minguez [LM, Theorem 1], see also [CR]. This approach also provides a resolution of the ladder representations which is the image of the BGG resolution under the functor.

4.4 Invariant forms

The functor $F_{\lambda,\mathcal{V}_\ell}$ behaves well with respect to invariant hermitian (or symmetric bilinear) forms, and in fact, this is an ingredient in the proof of Theorem 4.5(2). We recall the results in the setting of hermitian rather than symmetric forms, with the obvious modifications.

Recall that $\mathfrak{g} = \mathfrak{gl}(n, \mathbb{C})$ viewed as a Lie algebra admits a complex conjugate linear anti-automorphism $* : A \mapsto \overline{A^T}$. A module $X \in O$ is called hermitian if it admits an invariant form $(\ ,\ )_X$ satisfying

$$(Ax, y)_X = (x, A^* y)_X, \qquad \text{for all } A \in \mathfrak{g} = \mathfrak{gl}(n). \tag{31}$$

The standard representation $\mathbb{C}^n$ is hermitian, the usual inner product

$$(x, y)_{\mathbb{C}^n} = \sum_i x_i \overline{y}_i$$

has property (31).

If X admits an invariant hermitian form, then $X \otimes \mathcal{V}_\ell = X \otimes (\mathbb{C}^n)^{\otimes \ell}$ can be endowed with the product form. The following lemma is straightforward.

Lemma 4.9 ([Su, Lemma 4.1.4]). *Suppose X admits a $\mathfrak{g}$-invariant form as in (31). Then the form on $X \otimes \mathcal{V}_\ell$ is $\mathbb{H}_\ell$-invariant with respect to the star operation $\circledast$ of $\mathbb{H}_\ell$ (see Definition 2.2).*

If the form on X is nondegenerate (positive definite), then the form obtained on $F_{\lambda,\mathcal{V}_\ell}(X)$ is nondegenerate (positive definite).

Combining Lemma 4.9 with Theorem 4.6, we obtain as a consequence the known semisimplicity result for ladder representations [LM].

Proposition 4.10. *Every ladder representation $\mathsf{C}(\Delta_1, \ldots, \Delta_n)$ is $\circledast$-unitary, and therefore $\mathbb{H}_M$-semisimple for every parabolic Hecke subalgebra $\mathbb{H}_M$.*

Proof. Apply Lemma 4.9 with $X = L(\mu)$, where μ and λ are as in Theorem 4.6(1). □

Remark 4.11. The $\mathbb{H}_M$-semisimplicity of ladder representations from Proposition 4.10 is the Hecke algebra equivalent of the semisimplicity of the Jacquet modules of ladder representations proved in [LM].

5 Ladder representations: pairs of commuting nilpotent elements

We relate the $\mathbb{A}$-semisimple $\mathbb{H}$-modules to the geometry of pairs of commuting nilpotent elements considered by [Gi]. Let $\mathfrak{g}$ be a complex semisimple Lie algebra and $G = \mathrm{Ad}(\mathfrak{g})$.

Definition 5.1 ([Gi], [EP]). A pair $\underline{e} = (e_1, e_2) \in \mathfrak{g} \times \mathfrak{g}$ is called a nilpotent pair if $[e_1, e_2] = 0$ and for all $(t_1, t_2) \in \mathbb{C}^\times \times \mathbb{C}^\times$, there exists $g \in G$ such that $\mathrm{Ad}(g)(e_1, e_2) = (t_1 e_1, t_2 e_2)$. In addition:

1. $\underline{e}$ is called principal if $\dim \mathfrak{z}_\mathfrak{g}(\underline{e}) = \mathrm{rank}\ \mathfrak{g}$;
2. $\underline{e}$ is called distinguished if

   a. $\mathfrak{z}_\mathfrak{g}(\underline{e})$ contains no semisimple elements, and
   b. there exists a semisimple pair $\underline{h} = (h_1, h_2) \in \mathfrak{g} \times \mathfrak{g}$ such that $\mathrm{ad}(h_1), \mathrm{ad}(h_2)$ have rational eigenvalues,
   $$[h_1, h_2] = 0,\ [h_i, e_j] = \delta_{ij} e_j,$$
   and $\mathfrak{z}_\mathfrak{g}(\underline{h})$ is a Cartan subalgebra.

3. $\underline{e}$ is called rectangular if e_1, e_2 can be embedded in commuting $\mathfrak{sl}(2)$ triples.

By [Gi, Theorem 1.2], every principal nilpotent pair $\underline{e}$ is distinguished, and in fact, the associated semisimple pair $\underline{h}$ has the property that the eigenvalues of $\mathrm{ad}\, h_i$ are integral.

5.1 Principal nilpotent pairs

We summarize some of the results from [Gi].

Theorem 5.2 ([Gi, Theorem 3.7, Theorem 3.9, Corollary 3.6]).

1. *Any two principal nilpotent pairs $\underline{e}$ and $\underline{e}'$ with the same associated semisimple pair $\underline{h}$ are conjugate to each other by the maximal torus $T = Z_G(\underline{h})$.*
2. *There are finitely many adjoint G-orbits of principal nilpotent pairs.*
3. *For every principal nilpotent pair $\underline{e}$, the centralizer $Z_G(\underline{e})$ is connected.*

The construction of principal pairs is as follows. Let $\mathfrak{p} = \mathfrak{l} \oplus \mathfrak{u}$ be a parabolic subalgebra and $e_1 \in \mathfrak{l}$ a principal nilpotent element. Assume that $e_2 \in \mathfrak{z}_{\mathfrak{u}}(e_1)$ is a Richardson element for $\mathfrak{p}$. Set $\underline{e} = (e_1, e_2)$. The following are equivalent:

1. $\underline{e}$ is a principal nilpotent pair.
2. The orbit Ad $Z_P(e_1) \cdot e_2$ is Zariski open dense in $\mathfrak{z}_{\mathfrak{u}}(e_1)$.

Every principal nilpotent pair is of this form. More precisely, for a given principal pair $\underline{e}$, let $\underline{h} = (h_1, h_2)$ be the associated semisimple pair. Let $\mathfrak{g} = \oplus_{p,q} \mathfrak{g}_{p,q}$ be the bigradation of $\mathfrak{g}$ defined by the adjoint action of $\underline{h}$. Define

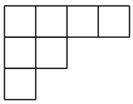
$$\mathfrak{g}_{p,*} = \oplus_q \mathfrak{g}_{p,q} \quad \text{and} \quad \mathfrak{g}_{*,q} = \oplus_p \mathfrak{g}_{p,q}, \tag{32}$$

and the parabolic subalgebras

$$\mathfrak{p}^{\text{east}} = \oplus_{p \geq 0} \mathfrak{g}_{p,*} \quad \text{and} \quad \mathfrak{p}^{\text{south}} = \oplus_{q \geq 0} \mathfrak{g}_{*,q} \tag{33}$$

with Levi subalgebras $\mathfrak{g}^1 = \mathfrak{g}_{*,0}$ and $\mathfrak{g}^2 = \mathfrak{g}_{0,*}$, respectively. Then (e_1, e_2) are given by the above construction for $\mathfrak{p} = \mathfrak{p}^{\text{south}}$ and $\mathfrak{l} = \mathfrak{g}^1$.

The notation is motivated by the example $\mathfrak{g} = \mathfrak{sl}(n)$. In this case let σ be a Young diagram visualized as in the following example:



Enumerate the boxes $1, 2, \dots, n$ in some order and label the basis of $\mathbb{C}^n$ by the box with the corresponding number. Let $e_1, e_2 \in \mathrm{End}(\mathbb{C}^n)$ be defined as follows:

e_1: sends a basis vector corresponding to a box to the vector corresponding to the next box on the row (to the east) or 0 if it's the last row box;

e_2: same as e_1 except the direction is down (south) on the columns.

Theorem 5.3 ([Gi]). *Suppose $\mathfrak{g} = \mathfrak{sl}(n)$. Every adjoint G-orbit of principal nilpotent pairs has a representative obtained from a Young diagram by the procedure described above.*

The classification of the larger class of distinguished nilpotent pairs has a similar flavor. Consider σ to be a skew-Young diagram, i.e., the set difference of two Young diagrams as before with the same corner. Moreover, assume that σ is connected, i.e., for every square there is another square which shares a corner with the first one. Define $\underline{e} = (e_1, e_2)$ as in the Young diagram case, but for the skew diagram σ.

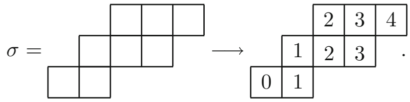
Theorem 5.4 ([Gi, Theorem 5.6]). *The adjoint G-orbits of distinguished nilpotent pairs are in one to one correspondence, via the construction above, with connected skew diagrams σ.*

The rectangular distinguished nilpotent pairs (in the sense of Definition 5.1(3)) correspond to rectangular Young diagrams, i.e., usual Young diagrams in the shape of rectangles (Example 3.5).

5.2 Weights and ladder representations

We make the connection with ladder representations. Given an $a \in \mathbb{Z}$ and σ a connected skew diagram, we associate an integral ladder representation $C(\sigma, a)$ as follows.

Form a skew-Young tableau as follows: the leftmost box of the first row of σ gets content (the number in the box) a, then the contents increase to the right and decrease to the left on rows, and stay constant on the columns. In the following example, $a = 2$:



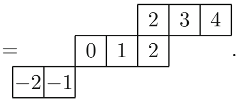
(34)

Suppose a_i' is the leftmost content in row i, while b_i' is the rightmost content. Define the segments:

$$\Delta_i = [a_i, b_i], \text{ where } a_i = -(i-1) + a_i' \text{ and } b_i = -(i-1) + b_i'. \tag{35}$$

In other words, move the i-th row $(i-1)$-units to the left, for every i. In our example,

$(\Delta_1, \Delta_2, \Delta_3) = ([2, 4], [0, 2], [-2, -1]) =$



(36)

Definition 5.5. The integral ladder representation defined above will be called

$$\mathsf{C}(\sigma, a) := \langle \Delta_1, \ldots, \Delta_r \rangle. \tag{37}$$

Consider the variety $\mathcal{B}(\underline{e}, \underline{h})$ of Borel subalgebras of $\mathfrak{g}$ containing the elements (e_1, e_2, h_1, h_2). When $\underline{e}$ is distinguished, $\mathcal{B}(\underline{e}, \underline{h})$ is 0-dimensional. More precisely, suppose $\mathfrak{b} \in \mathcal{B}(\underline{e}, \underline{h})$. Since $h_1, h_2 \in \mathfrak{b}$, also $\mathfrak{z}_{\mathfrak{g}}(\underline{h}) \subset \mathfrak{b}$. As $\underline{e}$ is distinguished, $\mathfrak{h} := \mathfrak{z}_{\mathfrak{g}}(\underline{h})$ is a Cartan subalgebra. This means that every $\mathfrak{b} \in \mathcal{B}(\underline{e}, \underline{h})$ contains the Cartan subalgebra $\mathfrak{h}$. Let W be the Weyl group of $\mathfrak{h}$ in $\mathfrak{g}$. If $\mathfrak{b}_0$ is a Borel subalgebra containg $\mathfrak{h}$ such that $e_1, e_2 \in \mathfrak{b}_0$, then

$$\begin{aligned} \mathcal{B}(\underline{e}, \underline{h}) &= \{w\mathfrak{b}_0 : w \in W(\underline{e}, \mathfrak{b}_0)\}, \text{ where} \\ W(\underline{e}, \mathfrak{b}_0) &= \{w \in W : \ w^{-1}e_1 \in \mathfrak{b}_0, \ w^{-1}e_2 \in \mathfrak{b}_0\}. \end{aligned} \tag{38}$$

Clearly, if $\mathfrak{b}_0' = u\mathfrak{b}_0$ is another Borel subalgebra in $\mathcal{B}(\underline{e}, \underline{h})$, with $u \in W$, then $W(\underline{e}, \mathfrak{b}_0') = W(\underline{e}, \mathfrak{b}_0)u^{-1}$.

Proposition 5.6. *Suppose* $\mathfrak{g} = \mathfrak{sl}(n, \mathbb{C})$. *Let* σ *be a connected skew diagram. Let* $\underline{e}$ *be a distinguished nilpotent pair with associated semisimple pair* $\underline{h}$, *such that* $\underline{e}$ *is attached to* σ *by Theorem 5.4. Let* $(\Delta_1, \ldots, \Delta_r)$ *be the multisegment constructed from* σ *by procedure (35). Then, for every Borel subalgebra* $\mathfrak{b}_0 \in \mathcal{B}(\underline{e}, \underline{h})$, *we have*

$$W(\underline{e}, \mathfrak{b}_0) = W(\Delta_1, \ldots, \Delta_r)u^{-1}, \text{ for some } u \in W, \tag{39}$$

where $W(\underline{e})$*,* $W(\Delta_1, \ldots, \Delta_r)$ *are defined in (38) and (17), respectively.*

Proof. It is sufficient to prove that if $\mathfrak{b}_0$ is the lower triangular Borel subalgebra and $\mathfrak{h}$ is the diagonal Cartan subalgebra, then $W(\underline{e}, \mathfrak{b}_0) = W(\Delta_1, \ldots, \Delta_r)$. If we assign to the boxes of σ the standard basis elements of $\mathbb{C}^n$ in row order, e.g., the boxes of the first row correspond to $x_1, x_2, \ldots, x_{m_1}$, where $m_1 = |\Delta_1|$, etc., then the nilpotent element $e_1 \in \mathfrak{b}_0$ is a sum

$$e_1 = \sum_{i=1}^{r} X_i, \quad \text{where } X_1 = E_{21} + E_{32} + \cdots + E_{m_1, m_1 - 1}, \quad \text{etc.} \tag{40}$$

Since $w \cdot E_{ij} = E_{w(i), w(j)}$, for $w \in S_n$, it is clear that the condition $w^{-1} \cdot e_1 \in \mathfrak{b}_0$ translates to the same rule as the "row rule" (1) used in defining $W(\Delta_1, \ldots, \Delta_r)$.

Similarly, e_2 is defined using the columns of σ. Then the restrictions imposed by the condition $w^{-1} \cdot e_2 \in \mathfrak{b}_0$ are the same as the "45°-diagonal rule" (2) used in the definition of $W(\Delta_1, \ldots, \Delta_r)$. Recall that $(\Delta_1, \ldots, \Delta_r)$ is obtained from σ by shifting each row to the left and therefore the column relations become diagonal relations. □

Proposition 5.6 has the following immediate corollary.

Corollary 5.7. *Keep the notation in Proposition 5.6. For every* $a \in \mathbb{Z}$*, the* $\mathbb{A}$*-weights of the ladder representation* $\mathsf{C}(\sigma, a)$ *defined in (37) are in one-to-one correspondence with the points of the variety* $\mathcal{B}(\underline{e}, \underline{h})$*.*

Remark 5.8. In [KL] and [CG], a standard module (possibly zero) for the Hecke algebra is defined for each conjugacy class of pairs (s, e, ϕ) where e is a nilpotent element, s is semisimple satisfying $\mathrm{Ad}\, s(e) = qe$, and ϕ a character of the component group of the centralizer $G(e)$. An analogue for the graded affine Hecke algebra is defined by [Lu2] on the homology $H_\bullet(\mathcal{B}_e)^\phi$. The $\mathbb{A}$-character satisfies an analogue of Theorem 8.2.1 in [CG]. In unpublished work, D. Barbasch, S. Evens and A. Moy considered the incidence variety $H_\bullet(\mathcal{B}_{\underline{e}})$ for a pair of nilpotent elements $\underline{e} = (e_1, e_2)$ belonging to commuting $\mathfrak{sl}(2)$-triples, and defined an action of $\mathbb{A}$, analogous to the aforementioned one, Theorem 8.2.1 in [CG]. They conjectured that this action can be extended to an action of $\mathbb{H}$. More details and relations to the Arthur conjectures were presented by D. Barbasch in a series of seminar talks at IAS in 1995, entitled *The local Langlands conjectures and characteristic cycles*.

Corollary 5.7 provides evidence for these conjectures for more general pairs of commuting nilpotent elements.

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Arithmetic invariant theory II: Pure inner forms and obstructions to the existence of orbits

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To David Vogan on his 60th birthday

Abstract Let k be a field, let G be a reductive group, and let V be a linear representation of G. Let $V/\!\!/G = \mathrm{Spec}(\mathrm{Sym}^*(V^*))^G$ denote the geometric quotient and let $\pi : V \to V/\!\!/G$ denote the quotient map. Arithmetic invariant theory studies the map π on the level of k-rational points. In this article, which is a continuation of the results of our earlier paper "Arithmetic invariant theory", we provide necessary and sufficient conditions for a rational element of $V/\!\!/G$ to lie in the image of π, assuming that generic stabilizers are abelian. We illustrate the various scenarios that can occur with some recent examples of arithmetic interest.

Key words: Representation theory, Galois cohomology

MSC (2010): Primary 11E72; Secondary 14L24

1 Introduction

Geometric invariant theory involves in particular the study of invariant polynomials for the action of a reductive algebraic group G on a linear representation V over a field k, and the relation between these invariants and the G-orbits on V, usually

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under the hypothesis that the base field k is separably closed. In favorable cases, one can determine the geometric quotient $V/\!/G = \mathrm{Spec}(\mathrm{Sym}^*(V^\vee))^G$ and identify certain fibers of the morphism $V \to V/\!/G$ with certain G-orbits on V. For general fields k the situation is more complicated. The additional complexity in the orbit picture, when k is not separably closed, is what we refer to as *arithmetic invariant theory*.

In a previous paper [4], we studied the arithmetic invariant theory of a reductive group G acting on a linear representation V over a general field k. Let k^s denote a separable closure of k. When the stabilizer G_v of a vector v is smooth, the k-orbits inside of the k^s-orbit of v are parametrized by classes in the kernel of the map of pointed sets in Galois cohomology $\gamma : H^1(k, G_v) \to H^1(k, G)$ (cf. [23]).

We produced elements in the kernel of γ for three representations of the split odd orthogonal group $G = \mathrm{SO}(W) = \mathrm{SO}(2n+1)$: the standard representation $V = W$, the adjoint representation $V = \wedge^2(W)$, and the symmetric square representation $V = \mathrm{Sym}^2 W$. For all three representations the ring of G-invariant polynomials on V is a polynomial ring and the categorical quotient $V/\!/G$ is isomorphic to an affine space. Furthermore, in each case there is a natural section of the morphism $\pi : V \to V/\!/G$, so the k-rational points of $V/\!/G$ lift to k-rational orbits of G on V.

Such a section may not exist for the action of the odd orthogonal groups $G' = \mathrm{SO}(W')$ that are not split over k. The corresponding representations $V' = W'$, $\wedge^2(W')$, and $\mathrm{Sym}^2 W'$ have the same ring of polynomial invariants, so $V'/\!/G' = V/\!/G$, but there may be rational points in this affine space that do not lift to rational orbits of G' on V'.

The groups $G' = \mathrm{SO}(W')$ are the *pure inner forms* of G. These are the forms of G over k corresponding to cohomology classes c in the pointed set $H^1(k, G)$, as opposed to inner forms of G which correspond to classes in $H^1(k, G^{\mathrm{ad}})$. We show that any representation V of G determines a representation V' of G' which becomes isomorphic to V over k^s (this is not true for general inner forms). Suppose that the image of v in $V/\!/G$ is equal to f, and that $G(k^s)$ acts transitively on the k^s-rational points of the fiber above f. Then we show that the k-orbits for G' on V' with invariant f are parametrized by the elements in the fiber of the map $\gamma : H^1(k, G_v) \to H^1(k, G)$ above the class c.

We also consider representations where there is an obstruction to lifting k-rational invariants in $V/\!/G$ to k-rational orbits on V, for *all* pure inner forms of G. Let f be a rational invariant in $V/\!/G$, and assume that there is a single orbit over k^s with invariant f, whose stabilizers G_v are *abelian*. We show that these stabilizers are canonically isomorphic to a fixed commutative group scheme G_f, which is determined by f and is defined over k. We then construct a class d_f in the cohomology group $H^2(k, G_f)$, whose non-vanishing obstructs the descent of the orbit to k, for all pure inner forms of G. On the other hand, if $d_f = 0$, we show that there is least one pure inner form of G that has k-rational orbits with invariant f.

When the stabilizer G_v is trivial, so the action of $G(k^s)$ on elements with invariant f over k^s is simply transitive, the obstruction d_f clearly vanishes. In this case, we show that there is a unique pure inner form G' for which there exists a unique k-rational orbit on V' with invariant f. We give a number of examples of such representations, such as the action of $\mathrm{SO}(W) = \mathrm{SO}(n+1)$ on n copies of the standard representation W, and the action of $\mathrm{SL}(W) = \mathrm{SL}(5)$ on three copies of the exterior square representation $\wedge^2(W)$.

It is also possible that the stabilizer G_v is abelian and nontrivial, and yet the obstruction d_f still vanishes. This scenario occurs frequently; for example, it occurs for all representations arising in Vinberg's theory of θ-groups (see [20] and [18]). These representations are remarkable in that the morphism $\pi : V \to V/\!\!/G$ has an (algebraic) section (called the *Kostant section*). This implies that the obstruction d_f vanishes. The representations $\wedge^2(W)$ and $\mathrm{Sym}^2 W$ of the odd split orthogonal group $\mathrm{SO}(W)$ studied in [4] indeed shared this property. (For a treatment of many such representations of arithmetic interest, involving rational points and Selmer groups of Jacobians of algebraic curves, see [7], [5], [11], [25], and [26].)

Finally, it is possible that the stabilizer G_v of a stable vector v is abelian and nontrivial, and the obstruction class d_f is also nontrivial in $H^2(k, G_v)$. Fewer such representations occur in the literature, but they too appear to be extremely rich arithmetically especially when the generic stabilizers are naturally subgroup schemes of abelian varieties. In this paper, we give a detailed study of such a representation, namely the action of $G = \mathrm{SL}(W) = \mathrm{SL}(n)$ on the vector space $V = \mathrm{Sym}_2 W^* \oplus \mathrm{Sym}_2 W^*$ of pairs of symmetric bilinear forms on W. Like the representation $\mathrm{Sym}^2 W$ of $\mathrm{SO}(W)$, the ring of polynomial invariants is a polynomial ring, and there are stable orbits in the sense of geometric invariant theory. In fact, the stabilizer G_v of any vector v in one of the stable orbits is a finite commutative group scheme isomorphic to $(\mathbb{Z}/2\mathbb{Z})^{n-1}$ over k^s, and $G(k^s)$ acts transitively on the vectors in $V(k^s)$ with the same invariant f as v. However, when the dimension $n = 2g + 2$ of W is even, it may not be possible to lift k-rational points f of the quotient $V/\!\!/G$ to k-rational orbits of G on V. We relate this obstruction to the arithmetic of 2-coverings of Jacobians of hyperellipic curves of genus g over k.

In [3], this connection with hyperelliptic curves was used to show that most hyperelliptic curves over $\mathbb{Q}$ of genus $g \geq 2$ have no rational points. In a forthcoming paper [6], we will use the full connection with 2-coverings of Jacobians of hyperelliptic curves to study the arithmetic of hyperelliptic curves; in particular, we will prove that a positive proportion of hyperelliptic curves over $\mathbb{Q}$ have points locally over $\mathbb{Q}_\nu$ for all places ν of $\mathbb{Q}$, but have no points globally over *any* odd degree extension of $\mathbb{Q}$.

This paper is organized as follows. In Section 2, we describe the notion of a pure inner form G' of a reductive group G over a field k, and the corresponding twisted form V' of a given representation V of G. We also discuss in detail the problem of lifting k-rational points of $V/\!\!/G$ to k-rational orbits of G (and its pure inner forms) in the case where the generic stabilizer G_v is abelian, and we describe the cohomological obstruction to lifting invariants lying in $H^2(k, G_f)$. The obstruction element in $H^2(k, G_f)$ can also be deduced from the theory of residual gerbes on algebraic stacks (see [10] and [15, Chapter 11]). Since we have not seen any concise reference to the specific results needed in this context, we felt it would be useful to give a self-contained account here.

In Section 3, we then consider three examples of representations where the stabilizer G_v is trivial. These representations are:

1. the split orthogonal group $\mathrm{SO}(W)$ acting on n copies of W, where $\dim(W) = n + 1$;
2. $\mathrm{SL}(W)$ acting on three copies of $\wedge^2 W$, where $\dim(W) = 5$;
3. the unitary group $\mathrm{U}(n)$ acting on the adjoint representation of $\mathrm{U}(n + 1)$.

In each of these three cases, the cohomological obstruction clearly vanishes and we see explicitly how the orbits, over all pure inner forms of the group G, are classified by the elements of the space $V/\!\!/G$ of invariants. The third representation and its orbits have played an important role in the work of Jacquet–Rallis [14] and Wei Zhang [30] in connection with the relative trace formula approach to the conjectures of Gan, Gross, and Prasad [12].

In Section 4, we study three examples of representations where the stabilizer G_v is nontrivial and abelian, and where there are cohomological obstructions to lifting invariants. These representations are:

1. $\mathrm{Spin}(W)$ acting on n copies of W, where $\dim(W) = n + 1$;
2. $\mathrm{SL}(W)$ acting on $\mathrm{Sym}_2 W^* \oplus \mathrm{Sym}_2 W^*$;
3. $(\mathrm{SL}/\mu_2)(W)$ acting on $\mathrm{Sym}_2 W^* \oplus \mathrm{Sym}_2 W^*$ (this group acts only when $\dim(W)$ is even).

In the first case, we show that the obstruction is the Brauer class of a Clifford algebra determined by the invariants. In the second and third cases, we show that when n is odd, there is no cohomological obstruction to lifting invariants, but when n is even, the obstruction can be nontrivial. We parametrize the orbits for both groups in terms of arithmetic data over k, and describe the resulting criterion for the existence of orbits over k. We describe the connection between the cohomological obstruction and the arithmetic of two-covers of Jacobians and hyperelliptic curves over k, which will play an important role in [6]. Finally we give a description of the integral orbits for the second case and, as we will see, new techniques are required to study them.

As in [4], the heart of this paper lies in the examples that illustrate the various scenarios that can occur, and how one can treat each scenario in order to classify the orbits, over a field that is not necessarily separably closed, in terms of suitable arithmetic data.

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2 Lifting results

In this section, we assume that G is a reductive group with a linear representation V over the field k. We will study the general problem of lifting k-rational points of $V/\!\!/G$ to k-rational orbits of pure inner forms G' of G on the corresponding twists V'

of V. For stable orbits over the separable closure k^s with smooth abelian stabilizers G_v, we will show how these stabilizers descend to a group scheme G_f over k and describe a cohomological obstruction to the lifting problem lying in $H^2(k, G_f)$.

2.1 Pure inner forms

We begin by recalling the notion of a pure inner form G^c of G and the action of G^c on a twisted representation V^c ([23, Ch 1 §5]).

Suppose $(\sigma \to c_\sigma)$ is a 1-cocycle on $\mathrm{Gal}(k^s/k)$ with values in the group $G(k^s)$. That is, $c_{\sigma\tau} = c_\sigma \cdot {}^\sigma c_\tau$ for any $\sigma, \tau \in \mathrm{Gal}(k^s/k)$. We define the pure inner form G^c of G over k by giving its k^s-points and describing a Galois action. Let $G^c(k^s) = G(k^s)$ with action

$$\sigma(h) = c_\sigma {}^\sigma h c_\sigma^{-1} \tag{1}$$

for any $\sigma \in \mathrm{Gal}(k^s/k)$ and any $h \in G(k^s)$. Since c is a cocycle, we have $\sigma\tau(h) = \sigma(\tau(h))$.

Let g be an element of $G(k^s)$. If $b_\sigma = g^{-1} c_\sigma {}^\sigma g$ is a cocycle in the same cohomology class as c, then the map on k^s-points $G^b \to G^c$ defined by $h \to ghg^{-1}$ commutes with the respective Galois actions, so defines an isomorphism over k. Hence the isomorphism class of the pure inner form G^c over k depends only on the image of c in the pointed set $H^1(k, G)$.

2.2 Twisting the representation

If we compose the cocycle c with values in $G(k^s)$ with the homomorphism $\rho : G \to \mathrm{GL}(V)$, we obtain a cocycle $\rho(c)$ with values in $\mathrm{GL}(V)(k^s)$. By the generalization of Hilbert's Theorem 90, we have $H^1(k, \mathrm{GL}(V)) = 1$ ([24, Ch X]). Hence there is an element g in $\mathrm{GL}(V)(k^s)$, well-defined up to left multiplication by $\mathrm{GL}(V)(k)$, such that

$$\rho(c_\sigma) = g^{-1\sigma} g \tag{2}$$

for all σ in $\mathrm{Gal}(k^s/k)$.

We use the element g to define a twisted representation of the group G^c on the vector space V over k. The homomorphism

$$\rho_g : G^c(k^s) \to \mathrm{GL}(V)(k^s),$$

defined by $\rho_g(h) = g\rho(h)g^{-1}$, commutes with the respective Galois actions, and so defines a representation over k. We emphasize that the Galois action on $G^c(k^s)$ is as defined in (1), whereas the Galois action on $\mathrm{GL}(V)(k^s)$ is the usual action.

The isomorphism class of the representation $\rho_g : G^c \to \mathrm{GL}(V)$ over k is independent of the choice of g in (2) which trivializes the cocycle. If $g' = ag$ is another choice, with a in $\mathrm{GL}(V)(k)$, then conjugation by a gives an isomorphism from ρ_g to ρ'_g. Since the isomorphism class of this representation depends only on the cocycle c, we will write V^c for the representation ρ_g of G^c.

The fact that the cocycle c_σ takes values in G, and not in the adjoint group, is crucial to defining the twist V^c of the representation V. For 1-cocycles c with values in $G^{\mathrm{ad}} \hookrightarrow \mathrm{Aut}(G)$, one can define the inner form G^c, but one does not always obtain a twisted representation V^c. For example, consider the case of $G = \mathrm{SL}_2$ with V the standard two-dimensional representation. The nontrivial inner forms of G are obtained from nontrivial cohomology classes in $H^1(k, \mathrm{PGL}_2)$. These are the groups G^c of invertible elements of norm 1 in quaternion division algebras D over k. The group G^c does not have a faithful two-dimensional representation over k—this representation is obstructed by the quaternion algebra D. Since $H^1(k, \mathrm{SL}_2)$ is trivial, there are no nontrivial *pure* inner forms of G.

2.3 Rational orbits in the twisted representation

We now fix a rational point f in the canonical quotient $V /\!/ G$, and let V_f be the fiber in V. For the rest of this subsection, we assume that the set $V_f(k)$ of rational points in the fiber is nonempty, and that $G(k^s)$ acts transitively on the points in $V_f(k^s)$. In particular, this orbit is closed (as it is defined by the values of the invariant polynomials). Let v be a point in $V_f(k)$ and let G_v denote its stabilizer in G.

The group $G(k)$ acts on the rational points of the fiber over f. In Proposition 1 of [4] we showed that the orbits of $G(k)$ on the set $V_f(k)$ correspond bijectively to elements in the kernel of the map

$$\gamma : H^1(k, G_v) \to H^1(k, G)$$

of pointed sets in Galois cohomology. In this section, we will generalize this to a parametrization of certain orbits of $G^c(k)$, where $c \in H^1(k, G)$. Note that by our hypothesis and the definition of G^c, the group $G^c(k^s) = G(k^s)$ acts transitively on the set $gV_f(k^s)$ in $V(k^s)$, where g is as in (2). We define the set

$$V_f^c(k) := V(k) \cap gV_f(k^s),$$

which admits an action of the rational points of the pure inner form G^c.

Here is a simple example, which illustrates many elements of the theory of orbits for pure inner twists with a fixed rational invariant f. Assume that the characteristic of k is not equal to 2, and let G be the étale group scheme μ_2 of order 2 over k. Let V be the nontrivial one-dimensional representation of G on the field k. (This is the standard representation of the orthogonal group $O(1)$ over k.) The polynomial invariants of this representation are generated by $q(x) = x^2$, so the canonical quotient $V /\!/ G$ is the affine line. Let f be a rational invariant in k with $f \neq 0$. Then the fiber V_f is the subscheme of V defined by $\{x : x^2 = f\}$, so $V_f(k)$ is nonempty if and only if f is a square in $k^\times$. This is certainly true over the separable closure k^s of k, and the group $G(k^s)$ acts simply transitively on $V_f(k^s)$.

An element c in $k^\times$ defines a cocycle $c_\sigma = {}^\sigma\sqrt{c}/\sqrt{c}$ with values in $G(k^s)$, whose class in the cohomology group $H^1(k,G) = k^\times/k^{\times 2}$ depends only on the image of c modulo squares. The element $g = \sqrt{c}$ in $\mathrm{GL}(V)(k^s)$ trivializes this class in the group $H^1(k, \mathrm{GL}(V))$. Although the inner twist G^c and the representation V^c remain exactly the same, we find that

$$V_f^c(k) = V(k) \cap gV_f(k^s) = \{x \in k^\times : x^2 = fc\}.$$

Hence the set $V_f^c(k)$ is nonempty if and only if the element fc is a square in $k^\times$. Note that there is a unique inner twist G^c where the fiber V_f^c has k-rational points, and in that case the group $G^c(k)$ acts simply transitively on $V_f^c(k)$.

Returning to the general case, we have the following generalization of Proposition 1 in [4] (which is the case $c = 1$ below).

Proposition 2.1. *Let G be a reductive group with representation V. Suppose there exists $v \in V(k)$ with invariant $f \in (V/\!\!/G)(k)$ and stabilizer G_v such that $G(k^s)$ acts transitively on $V_f(k^s)$. Then there is a bijection between the set of $G^c(k)$-orbits on $V_f^c(k)$ and the fiber $\gamma^{-1}(c)$ of the map*

$$\gamma : H^1(k, G_v) \to H^1(k, G)$$

above the class $c \in H^1(k,G)$. In particular, the image of $H^1(k, G_v)$ in $H^1(k,G)$ determines the set of pure inner forms of G for which the k-rational invariant f lifts to a k-rational orbit of G^c on V^c.

Before giving the proof, we illustrate this with an example from [4]. Let W be a split orthogonal space of dimension $2n+1$ and signature $(n+1, n)$ over $k = \mathbb{R}$, let $G = \mathrm{SO}(W) = \mathrm{SO}(n+1, n)$. The pure inner forms of G are the groups $G^c = \mathrm{SO}(p,q)$ with $p + q = 2n+1$ and $q \equiv n \pmod 2$, and the representation W^c of G^c is the standard representation on the corresponding orthogonal space $W(p,q)$ of signature (p,q). The group $G = \mathrm{SO}(W)$ acts faithfully on the space $V = \mathrm{Sym}_2(W)$ of self-adjoint operators T on W. For this representation, the inner twists G^c of G are exactly the same, and the twisted representation V^c of G^c is isomorphic to $\mathrm{Sym}^2 W^c$. The polynomial invariants f in $(V/\!\!/G)(\mathbb{R})$ are given by the coefficients of the characteristic polynomial of T. Assume that this characteristic polynomial is separable, with $2m+1$ real roots. Then the stabilizer of a point $v_0 \in V_f(\mathbb{R})$ is the finite commutative group scheme $(\mu_2^{2m+1} \times (\mathrm{Res}_{\mathbb{C}/\mathbb{R}}\, \mu_2)^{n-m})_{N=1}$. Hence $H^1(\mathbb{R}, G_{v_0})$ is an elementary abelian 2-group of order 2^{2m}. This group maps under γ to the pointed set $H^1(\mathbb{R}, \mathrm{SO}(W))$, which is finite of cardinality $n+1$. The fiber over the class of $\mathrm{SO}(p,q)$ is nonempty if and only if both p and q are greater than or equal to $n-m$. In this case, write $q = n-m+a$, with $a \equiv m \pmod 2$. Then the fiber has cardinality $\binom{2m+1}{a}$. For example, the kernel has cardinality $\binom{2m+1}{m}$. When $pq = 0$, the space $W^c = W(p,q)$ is definite and there are orbits in $V_f^c(\mathbb{R})$

only in the case when $m = n$, so the characteristic polynomial splits completely over $\mathbb{R}$. In that case there is a single orbit. This is the content of the classical spectral theorem.

Proof of Proposition 2.1: Suppose c is a 1-cocycle with values in $G(k^s)$ and fix $g \in \mathrm{GL}(V)(k^s)$ such that $c_\sigma = g^{-1}{}^\sigma g$ for all $\sigma \in \mathrm{Gal}(k^s/k)$. When $V_f^c(k)$ is nonempty we must show that c is in the image of $H^1(k, G_v)$. Indeed, suppose $gw \in V_f^c(k)$ for some $w \in V_f(k^s)$. By our assumption on the transitivity of the action on k^s points, there exists $h \in G(k^s)$ such that $w = hv$. The rationality condition on gw translates into saying that, for any $\sigma \in \mathrm{Gal}(k^s/k)$, we have $c_\sigma {}^\sigma hv = hv$. That is, $h^{-1}c_\sigma {}^\sigma h \in G_v$ for any $\sigma \in \mathrm{Gal}(k^s/k)$. In other words, c is in the image of γ.

Now suppose $c \in H^1(k, G)$ is in the image of γ. Without loss of generality, assume that $c_\sigma \in G_v(k^s)$ for any $\sigma \in \mathrm{Gal}(k^s/k)$. Pick any $g \in \mathrm{GL}(V)(k^s)$ as in (2) above and set $w = gv \in V_f^c(k^s)$. Then for any $\sigma \in \mathrm{Gal}(k^s/k)$, we have

$$^\sigma w = gc_\sigma v = gv = w.$$

This shows that $w \in V_f^c(k)$. Hence there is a bijection between $G^c(k)\backslash V_f^c(k)$ and $\ker \gamma_c$ where γ_c is the natural map of sets $H^1(k, G_w^c) \to H^1(k, G^c)$. To prove Proposition 2.1, it suffices to establish a bijection between $\gamma^{-1}(c)$ and $\ker \gamma_c$. Consider the following two maps:

$$\begin{array}{cc} \gamma^{-1}(c) \to \ker \gamma_c & \ker \gamma_c \to \gamma^{-1}(c) \\ (\sigma \to d_\sigma) \mapsto (\sigma \to d_\sigma c_\sigma^{-1}) & (\sigma \to a_\sigma) \mapsto (\sigma \to a_\sigma c_\sigma). \end{array}$$

We need to check that these maps are well-defined. First, suppose $(\sigma \to d_\sigma) \in \gamma^{-1}(c)$. Then we need to show that $(\sigma \to d_\sigma c_\sigma^{-1})$ is a 1-cocycle in the kernel of γ_c. Note that, for any $\sigma, \tau \in \mathrm{Gal}(k^s/k)$, we have

$$(d_\sigma c_\sigma^{-1}) \cdot \sigma(d_\tau c_\tau^{-1}) \cdot (d_{\sigma\tau} c_{\sigma\tau}^{-1})^{-1} = d_\sigma c_\sigma^{-1}(c_\sigma {}^\sigma d_\tau {}^\sigma c_\tau^{-1} c_\sigma^{-1})(d_{\sigma\tau} c_{\sigma\tau}^{-1})^{-1} = 1.$$

Moreover, there exists $h \in G(k^s)$ such that $d_\sigma = h^{-1}c_\sigma {}^\sigma h$ for any $\sigma \in \mathrm{Gal}(k^s/k)$, and thus

$$h^{-1}\sigma(h) = h^{-1}c_\sigma {}^\sigma h c_\sigma^{-1} = d_\sigma c_\sigma^{-1}.$$

This shows that $(\sigma \to d_\sigma c_\sigma^{-1})$ is in the kernel of γ_c. Likewise, one can show that the second map is also well-defined. The composition of these two maps in either order yields the identity map, and this completes the proof. □

2.4 A cohomological obstruction to lifting invariants

Suppose $f \in (V/\!\!/G)(k)$ is a rational invariant. We continue to assume that the group $G(k^s)$ acts transitively on the set $V_f(k^s)$. In this section, we consider the problem of determining when the set $V_f^c(k)$ is nonempty for some $c \in H^1(k, G)$.

That is, when does a rational invariant lift to a rational orbit for some pure inner form of G? We resolve this problem under the additional assumption that the stabilizer G_v of any point in the orbit $V_f(k^s)$ is abelian.

For $\sigma \in \mathrm{Gal}(k^s/k)$, the vector ${}^{\sigma}v$ also lies in $V_f(k^s)$, so there is an element g_σ with $g_\sigma {}^{\sigma}v = v$. The element g_σ is well-defined up to left multiplication by an element in the subgroup G_v. Since we are assuming that the stabilizers are abelian, the homomorphism $\theta_\sigma : G_{\sigma v} \to G_v$ defined by mapping α to $g_\sigma \alpha g_\sigma^{-1}$ is independent of the choice of g_σ. This gives a collection of isomorphisms

$$\theta_\sigma : {}^{\sigma}(G_v) \to G_v$$

that satisfy the 1-cocycle condition $\theta_{\sigma\tau} = \theta_\sigma \circ {}^{\sigma}\theta_\tau$, and hence provide descent data for the group scheme G_v. We let G_f be the corresponding commutative group scheme over k which depends only on the rational invariant f. Let $\iota_v : G_f(k^s) \xrightarrow{\sim} G_v$ denote the canonical isomorphisms. More precisely, if $h \in G(k^s)$ and $v \in V_f(k^s)$, then

$$\iota_{hv}(b) = h\iota_v(b)h^{-1} \quad \forall\, b \in G_f(k^s). \tag{3}$$

The descent data translates into saying that for any $\sigma \in \mathrm{Gal}(k^s/k)$ and $v \in V_f(k^s)$, we have

$$^{\sigma}(\iota_v(b)) = \iota_{\sigma v}({}^{\sigma}b) \qquad \forall\, b \in G_f(k^s). \tag{4}$$

Before constructing a class in $H^2(k, G_f)$ whose vanishing is intimately related to the existence of rational orbits, we give an alternate method (shown to us by Brian Conrad) to obtain the finite group scheme G_f over k using fppf descent. Suppose G is a group scheme of finite type over k such that the orbit map $G \times V_f \to V_f$ is fppf. Suppose also that the stabilizer $G_v \in G(k^a)$ for any $v \in V_f(k^a)$ is abelian where k^a denotes an algebraic closure of k. Let H denote the stabilizer subscheme of $G \times V_f$. In other words, H is the pullback of the action map $G \times V_f \to V_f \times V_f$ over the diagonal of V_f. Note that H is a V_f-scheme and its descent to k will be G_f. The descent datum amounts to a canonical isomorphism $p_1^*H \simeq p_2^*H$ where p_1, p_2 denote the two projection maps $V_f \times V_f \to V_f$. The commutativity of G_v for any $v \in V_f(k^a)$ implies the commutativity of $(G_R)_x$ for any k-algebra R and any element $x \in V_f(R)$. Therefore, there are canonical isomorphisms $(G_R)_x \to (G_R)_y$ for any $x, y \in V_f(R)$. This gives canonical isomorphisms $p_1^*H \simeq p_2^*H$ locally over $V_f \times V_f$. Being canonical, these local isomorphisms patch together to a global isomorphism and hence yield the desired descent datum.

We now construct a class d_f in $H^2(k, G_f)$ that will be trivial whenever a rational orbit exists. Choose v and g_σ as above, with $g_\sigma {}^{\sigma}v = v$. Define

$$d_{\sigma,\tau} = \iota_v^{-1}(g_\sigma {}^{\sigma}g_\tau g_{\sigma\tau}^{-1}).$$

Standard arguments show that $d_{\sigma,\tau}$ is a 2-cocycle whose image d_f in $H^2(k, G_f)$ does not depend on the choice of g_σ. We also check that the 2-cochain $d_{\sigma,\tau}$ does not depend on the choice of $v \in V_f(k^s)$. Suppose $v' = hv \in V_f(k^s)$ for some $h \in G(k^s)$. For any $\sigma \in \mathrm{Gal}(k^s/k)$, we have

$$hg_\sigma{}^\sigma h^{-1\sigma}v' = hg_\sigma{}^\sigma v = hv = v'.$$

Moreover, for any $\sigma, \tau \in \mathrm{Gal}(k^s/k)$, we compute

$$hg_\sigma{}^\sigma h^{-1}\,{}^\sigma(hg_\tau{}^\tau h^{-1})\,(hg_{\sigma\tau}{}^{\sigma\tau}h^{-1})^{-1} = hg_\sigma{}^\sigma g_\tau g_{\sigma\tau}^{-1}h^{-1};$$

hence, by (3), we have

$$\iota_{v'}^{-1}(hg_\sigma{}^\sigma h^{-1}\,{}^\sigma(hg_\tau{}^\tau h^{-1})\,(hg_{\sigma\tau}{}^{\sigma\tau}h^{-1})^{-1}) = \iota_v^{-1}(g_\sigma{}^\sigma g_\tau g_{\sigma\tau}^{-1}).$$

If $V_f(k)$ is nonempty, then one can take v in $V_f(k)$. Then one can take $g_\sigma = 1$ and hence $d_f = 0$. We have therefore obtained the following necessary condition for lifting invariants to orbits.

Proposition 2.2. *Suppose that f is a rational invariant, and that $G(k^s)$ acts transitively on $V_f(k^s)$ with abelian stabilizers. If $V_f(k)$ is nonempty, then $d_f = 0$ in $H^2(k, G_f)$.*

This necessary condition is not always sufficient. As shown by the following cocycle computation, the class d_f in $H^2(k, G_f)$ does not depend on the pure inner form of G. Indeed, suppose $c \in H^1(k, G)$ and $g \in \mathrm{GL}(V)(k^s)$ such that $c_\sigma = g^{-1\sigma}g$ for all $\sigma \in \mathrm{Gal}(k^s/k)$. Note that $gv \in V_f^c(k^s)$ and

$$(gg_\sigma c_\sigma^{-1}g^{-1}) \cdot {}^\sigma(gv) = gv.$$

A direct computation then gives

$$(g_\sigma c_\sigma^{-1}) \cdot \sigma(g_\tau c_\tau^{-1}) \cdot (c_{\sigma\tau}g_{\sigma\tau}^{-1}) = g_\sigma{}^\sigma g_\tau g_{\sigma\tau}^{-1}.$$

The fact that d_f is independent of the pure inner form suggests that $d_f = 0$ might be sufficient for the existence of a rational orbit for *some* pure inner twist. Indeed, this is the case.

Theorem 2.3. *Suppose that f is a rational invariant, and that $G(k^s)$ acts transitively on $V_f(k^s)$ with abelian stabilizers. Then $d_f = 0$ in $H^2(k, G_f)$ if and only if there exists a pure inner form G^c of G such that $V_f^c(k)$ is nonempty. That is, the condition $d_f = 0$ is necessary and sufficient for the existence of rational orbits for some pure inner twist of G. In particular, when $H^1(k, G) = 1$, the condition $d_f = 0$ in $H^2(k, G_f)$ is necessary and sufficient for the existence of rational orbits of $G(k)$ on $V_f(k)$.*

Proof. Necessity has been shown in Proposition 2.2 and the above computation. It remains to prove sufficiency. Fix $v \in V_f(k^s)$ and g_σ such that $g_\sigma{}^\sigma v = v$ for any $\sigma \in \mathrm{Gal}(k^s/k)$. The idea of the proof is that if $d_f = 0$, then one can pick g_σ so that $(\sigma \to g_\sigma)$ is a 1-cocycle and that rational orbits exist for the pure inner twist associated to this 1-cocycle.

Suppose $d_f = 0$ in $H^2(k, G_f)$. Then there exists a 1-cochain $(\sigma \to b_\sigma)$ with values in $G_f(k^s)$ such that

$$g_\sigma {}^\sigma g_\tau g_{\sigma\tau}^{-1} = \iota_v(b_\sigma {}^\sigma b_\tau b_{\sigma\tau}^{-1}) \quad \forall \sigma, \tau \in \mathrm{Gal}(k^s/k).$$

Lemma 2.4. *There exists a* 1*-cochain* e_σ *with values in* $G_v(k^s)$ *such that* $(\sigma \to e_\sigma g_\sigma)$ *is a* 1*-cocycle.*

To see how Lemma 2.4 implies Theorem 2.3, we consider the twist of G and V using the 1-cocycle $c = (\sigma \to e_\sigma g_\sigma) \in H^1(k, G)$. Choose any $g \in \mathrm{GL}(V)(k^s)$ such that $g^{-1} {}^\sigma g = e_\sigma g_\sigma$ for any $\sigma \in \mathrm{Gal}(k^s/k)$. Then $gv \in V_f^c(k)$. Indeed,

$${}^\sigma(gv) = g e_\sigma g_\sigma {}^\sigma v = g e_\sigma v = gv \quad \forall\ \sigma \in \mathrm{Gal}(k^s/k).$$

We now prove Lemma 2.4. Consider $e_\sigma = \iota_v(b_\sigma^{-1})$ for any $\sigma \in \mathrm{Gal}(k^s/k)$. Since $g_\sigma {}^\sigma v = v$, we have by (3) and (4) that

$$g_\sigma {}^\sigma(\iota_v(b)) g_\sigma^{-1} = \iota_v({}^\sigma b) \quad \forall \sigma \in \mathrm{Gal}(k^s/k), b \in G_f(k^s).$$

Hence for any $\sigma, \tau \in \mathrm{Gal}(k^s/k)$, we have

$$\begin{aligned}(e_\sigma g_\sigma)^\sigma(e_\tau g_\tau)(e_{\sigma\tau} g_{\sigma\tau})^{-1} &= \iota_v(b_\sigma^{-1}) g_\sigma {}^\sigma(\iota_v(b_\tau^{-1}))^\sigma g_\tau g_{\sigma\tau}^{-1} \iota_v(b_{\sigma\tau}) \\ &= \iota_v(b_\sigma^{-1}) \iota_v({}^\sigma b_\tau^{-1}) g_\sigma {}^\sigma g_\tau g_{\sigma\tau}^{-1} \iota_v(b_{\sigma\tau}) \\ &= \iota_v(b_\sigma^{-1}) \iota_v({}^\sigma b_\tau^{-1}) \iota_v(b_\sigma {}^\sigma b_\tau b_{\sigma\tau}^{-1}) \iota_v(b_{\sigma\tau}) \\ &= 1\end{aligned}$$

where the last equality follows because $G_f(k^s)$ is abelian. □

Corollary 2.5. *Suppose that* f *is a rational orbit and that* $G(k^s)$ *acts simply transitively on* $V_f(k^s)$*. Then there is a unique pure inner form* G^c *of* G *such that* $V_f^c(k)$ *is nonempty. Moreover, the group* $G^c(k)$ *acts simply transitively on* $V_f^c(k)$*.*

Proof. Since $G_f = 1$, we have $H^2(k, G_f) = 0$, and so the cohomological obstruction d_f vanishes. We conclude that rational orbits exist for some pure inner twist G^c. Let $v_0 \in V_f^c(k)$ denote any k-rational lift. Since $H^1(k, G_f) = 0$, the image of $\gamma : H^1(k, G_{v_0}^c) \to H^1(k, G^c)$ is a single point, and hence no other pure inner twist has a rational orbit with invariant f. Since the kernel of γ has cardinality 1, there is a single orbit of $G^c(k)$ on $V_f^c(k)$. □

3 Examples with trivial stabilizer

In this section, we give several examples of representations $G \to \mathrm{GL}(V)$ over k where there are stable orbits which are determined by their invariants f in $V/\!\!/G$ and which have trivial stabilizer over k^s. Thus $G(k^s)$ acts simply transitively on the set $V_f(k^s)$. When f is rational, Corollary 2.5 implies that there is a unique pure inner form G' of G over k for which $V'_f(k)$ is nonempty, and that $G'(k)$ acts simply transitively on $V'_f(k)$.

We will describe this pure inner form, using the following results on classical groups [13]. Since $H^1(k, \mathrm{GL}(W))$ and $H^1(k, \mathrm{SL}(W))$ are both pointed sets with a single element, there are no nontrivial pure inner forms of $\mathrm{GL}(W)$ and $\mathrm{SL}(W)$. On the other hand, when the characteristic of k is not equal to 2 and W is a nondegenerate quadratic space over k, the pointed set $H^1(k, \mathrm{SO}(W))$ classifies the quadratic spaces W' with $\dim(W') = \dim(W)$ and $\mathrm{disc}(W') = \mathrm{disc}(W)$. The corresponding pure inner form is the group $G' = \mathrm{SO}(W')$. Similarly, if W is a nondegenerate Hermitian space over the separable quadratic extension E of k, then the pointed set $H^1(k, \mathrm{U}(W))$ classifies Hermitian spaces W' over E with $\dim(W') = \dim(W)$, and the corresponding pure inner form of G is the group $G' = \mathrm{U}(W')$.

3.1 SO($n+1$) *acting on the direct sum of n copies of the standard representation*

In this subsection, we assume that k is a field of characteristic not equal to 2.

We first consider the action of the split group $G = \mathrm{SO}(W) = \mathrm{SO}(4)$ on three copies of the standard representation $V = W \oplus W \oplus W$. Let $q(w) = \langle w, w\rangle/2$ be the quadratic form on W and let $v = (w_1, w_2, w_3)$ be a vector in V. The coefficients of the ternary quadratic form $f(x, y, z) = q(xw_1 + yw_2 + zw_3)$ give six invariant polynomials of degree 2 on V, which freely generate the ring of polynomial invariants, and an orbit is stable if the discriminant $\Delta(f)$ of this quadratic form is nonzero in k^s. In this case, the group $G(k^s)$ acts simply transitively on $V_f(k^s)$. Indeed, the quadratic space U_0 of dimension 3 with form f embeds isometrically into W over k^s, and the subgroup of $\mathrm{SO}(W)$ that fixes U_0 acts faithfully on its orthogonal complement, which has dimension 1. The condition that the determinant of an element in $\mathrm{SO}(W)$ is equal to 1 forces it to act trivially on the orthogonal complement.

The set $V_f(k)$ is nonempty if and only if the quadratic form f represents zero over k. Indeed, if $v = (w_1, w_2, w_3)$ is a vector in this orbit over k, then the vectors w_1, w_2, w_3 are linearly independent and span a 3-dimensional subspace of W. This subspace must have a nontrivial intersection with a maximal isotropic subspace of W, which has dimension 2. Conversely, if the quadratic form f represents zero, let U_0 be the 3-dimensional quadratic space with this bilinear form, and U the orthogonal direct sum of U_0 with a line spanned by a vector u with $\langle u, u\rangle = \det(U_0)$. Then U is a quadratic space of dimension 4 and discriminant 1 containing an isotropic line (from U_0). It is therefore split, and isomorphic over k to the quadratic space W. Choosing an isometry $\theta : U \to W$, we obtain three vectors (w_1, w_2, w_3) as the images of the basis elements of U_0, and this gives the desired element in $V_f(k)$. Note that θ is only well-defined up to composition by an automorphism of W, so we really obtain an orbit for the orthogonal group of W. Since the stabilizer of this orbit is a simple reflection, we obtain a single orbit for the subgroup $\mathrm{SO}(W)$.

If the form f does not represent zero, let W' be the quadratic space of dimension 4 that is the orthogonal direct sum of the subspace U_0 of dimension 3 with quadratic form f and a nondegenerate space of dimension 1, chosen so that the discriminant of W' is equal to 1. Then $G' = \mathrm{SO}(W')$ is the unique pure inner form of G (guaranteed to exist by Corollary 2.5) where $V'_f(k)$ is nonempty. The construction of an orbit for G' is the same as above.

The same argument works for the action of the group $G = \mathrm{SO}(W) = \mathrm{SO}(n+1)$ on n copies of the standard representation: $V = W \oplus W \oplus \cdots \oplus W$. The coefficients of the quadratic form $f(x_1, x_2, \ldots, x_n) = q(x_1 w_1 + x_2 w_2 + \cdots + x_n w_n)$ give polynomial invariants of degree 2, which freely generate the ring of invariants. The orbit of $v = (w_1, w_2, \ldots, w_n)$ is stable, with trivial stabilizer, if and only if the discriminant $\Delta(f)$ is nonzero in k^s. If W' is the quadratic space of dimension $n+1$ with $\mathrm{disc}(W') = \mathrm{disc}(W)$, that is the orthogonal direct sum of the space U_0 of dimension n with quadratic form f and a nondegenerate space of dimension 1, then $G' = \mathrm{SO}(W')$ is the unique pure inner form with $V'_f(k)$ nonempty.

3.2 SL(5) *acting on* 3 *copies of the representation* $\wedge^2$(5)

Let k be a field of characteristic not equal to 2, U a k-vector space of dimension 3, and W a k-vector space of dimension 5. In this subsection, we consider the action of $G = \mathrm{SL}(W)$ on $V = U \otimes \wedge^2 W$.

Choosing bases for U and W, we may identify $\mathrm{U}(k)$ and $W(k)$ with k^3 and k^5, respectively, and thus $V(k)$ with $\wedge^2 k^5 \oplus \wedge^2 k^5 \oplus \wedge^2 k^5$. We may then represent elements of $V(k)$ as a triple (A, B, C) of 5×5 skew-symmetric matrices with entries in k. For indeterminates x, y, and z, we see that the determinant of $Ax + By + Cz$ vanishes, being a skew-symmetric matrix of odd dimension.

To construct the G-invariants on V, we consider instead the 4×4 principal sub-Pfaffians of $Ax + By + Cz$; this yields five ternary quadratic forms $Q_1, \ldots, Q_5$ in x, y, and z, which are generically linearly independent over k. In basis-free terms, we obtain a G-equivariant map

$$U \otimes \wedge^2 W \to \mathrm{Sym}^2 U \otimes W^*. \tag{5}$$

Now an $\mathrm{SL}(W)$-orbit on $\mathrm{Sym}^2 U \otimes W^*$ may be viewed as a five-dimensional subspace of $\mathrm{Sym}^2 U$; hence we obtain a natural G-equivariant map

$$\mathrm{Sym}^2 U \otimes W^* \to \mathrm{Sym}^2 U^*. \tag{6}$$

The composite map $\pi : U \otimes \wedge^2 W \to \mathrm{Sym}^2 U^*$ is thus also G-equivariant, but since G acts trivially on the image of π, we see that the image of π gives (a 6-dimensional space of) G-invariants, and indeed we may identify $V /\!\!/ G$ with $\mathrm{Sym}^2 U^*$. A vector $v \in V$ is stable precisely when $\det(\pi(v)) \neq 0$.

Now since $\mathrm{SL}(W)$ acts with trivial stabilizer on W^*, it follows that $\mathrm{SL}(W)$ acts with trivial stabilizer on $\mathrm{Sym}^2 U \otimes W^*$ as well. Since the map (5) is G-equivariant, it follows that the generic stabilizer in $G(k)$ of an element in $V(k)$ is also trivial!

Since $\mathrm{SL}(W)$ has no other pure inner forms, by Corollary 2.5 we conclude that every $f \in \mathrm{Sym}^2 U^*$ of nonzero determinant arises as the set of G-invariants for a unique $G(k)$-orbit on $V(k)$.

3.3 $\mathbf{U}(n-1)$ *acting on the adjoint representation* $\mathfrak{u}(n)$ *of* $\mathbf{U}(n)$

In this subsection, we assume that the field k does not have characteristic 2 and that E is an étale k-algebra of rank 2. Hence E is either a separable quadratic extension field, or the split algebra $k \times k$. Let τ be the nontrivial involution of E that fixes k.

Let Y be a free E-module of rank $n \geq 2$, and let

$$\langle\ ,\ \rangle : Y \times Y \to E$$

be a nondegenerate Hermitian symmetric form on Y. In particular $\langle y, z\rangle = {}^{\tau}\langle z, y\rangle$. Let e be a vector in Y with $\langle e, e\rangle \neq 0$, and let W be the orthogonal complement of e in Y. Hence $Y = W \oplus Ee$. The unitary group $G = \mathrm{U}(W) = \mathrm{U}(n-1)$ embeds as the subgroup of $\mathrm{U}(Y)$ that fixes the vector e. In particular, it acts on the Lie algebra $\mathfrak{u}(Y) = \mathfrak{u}(n)$ via the restriction of the adjoint representation.

Define the adjoint T^* of an E-linear map $T : Y \to Y$ by the usual formula $\langle Ty, z\rangle = \langle y, T^*z\rangle$. The elements of the group $\mathrm{U}(Y)$ are the maps g that satisfy $g^* = g^{-1}$. Differentiating this identity, we see that the elements of the Lie algebra are those endomorphisms of Y that satisfy $T + T^* = 0$. The group acts on the space of skew self-adjoint operators by conjugation: $T \to gTg^{-1} = gTg^*$. If T is skew self-adjoint and δ is an invertible element in E satisfying $\delta^\tau = -\delta$, then the scaled operator δT is self-adjoint. Hence the adjoint representation of $\mathrm{U}(Y)$ on its Lie algebra is isomorphic to its action by conjugation on the vector space V, of dimension n^2 over k, consisting of the self-adjoint endomorphisms $T : Y \to Y$. In this subsection, we consider the restriction of this representation to the subgroup $G = \mathrm{U}(W)$.

The ring of polynomial invariants for $G = \mathrm{U}(W)$ on V is a polynomial ring, freely generated by the n coefficients $c_i(T)$ of the characteristic polynomial of T (which are invariants for the larger group $\mathrm{U}(Y)$) as well as the $n-1$ inner products $\langle e, T^j e\rangle$ for $j = 1, 2, \ldots, n-1$ ([30, Lemma 3.1]). Note that all of these coefficients and inner products take values in k, as T is self-adjoint. In particular, the space $V /\!\!/ G$ is isomorphic to the affine space of dimension $2n-1$. Note that the inner products $\langle T^i e, T^j e\rangle$ are all polynomial invariants for the action of G. Let D be the invariant polynomial that is the determinant of the $n \times n$ symmetric matrix with entries $\langle T^i e, T^j e\rangle$ for $0 \leq i, j \leq n-1$. Clearly D is nonzero if and only if the vectors $\{e, Te, T^2e, \ldots, T^{n-1}e\}$ form a basis for the space Y over E. Rallis and Shiffman [21, Theorem 6.1] show that the condition $D(f) \neq 0$ is equivalent to the condition that $G(k^s)$ acts simply transitively on the points of $V_f(k^s)$. We can therefore conclude that when $D(f)$ is nonzero, there is a unique pure inner form G' of $G = \mathrm{U}(W)$ that acts simply transitively on the corresponding points in $V'_f(k)$, and that these spaces are empty for all other pure inner forms. To determine the pure inner form $G' = \mathrm{U}(W')$ for which $V'_f(k)$ is nonempty, it suffices to determine the Hermitian space W' over E of rank $n-1$. The rational invariant f determines the inner products $\langle T^i e, T^j e\rangle$, and hence a Hermitian structure on $Y' = Ee + E(Te) + \cdots + E(T^{n-1}e)$. Since the nonzero value $\langle e, e\rangle$ is fixed, this gives the Hermitian structure on its orthogonal complement W' in Y', and hence the pure inner form G' such that $V'_f(k)$ is nonempty.

When the algebra E is split, the Hermitian space $Y = X + X^\vee$ decomposes as the direct sum of an n-dimensional vector space X over k and its dual. The group $\mathrm{U}(Y)$ is isomorphic to $\mathrm{GL}(X) = \mathrm{GL}(n)$. The vector e gives a nontrivial vector x in X as well as a nontrivial functional f in $X^\vee$ with $f(x) \neq 0$. Let X_0 be the kernel of f, so $X = X_0 + kx$. The subgroup $\mathrm{U}(W)$ is isomorphic to $\mathrm{GL}(X_0) = \mathrm{GL}(n-1)$. In this case, the representation of $\mathrm{U}(W)$ on the space of self-adjoint endomorphisms of Y is isomorphic to the representation of $G = \mathrm{GL}(n-1)$ by conjugation on the space $V = \mathrm{End}(X)$ of all k-linear endomorphisms of X. Since $\mathrm{GL}(n-1)$ has no pure inner forms, Corollary 2.5 implies that $\mathrm{GL}(n-1)$ acts simply transitively on the points of $V_f(k)$ whenever $D(f) \neq 0$.

Once we have chosen an invertible element δ in E of trace zero, the rational invariants for the action of $\mathrm{U}(W) = \mathrm{U}(n-1)$ on the Lie algebra of $\mathrm{U}(n)$ match the rational invariants for the action of $\mathrm{GL}(X) = \mathrm{GL}(n-1)$ on the Lie algebra of $\mathrm{GL}(n)$. Since the stable orbits for the pure inner forms $\mathrm{U}(W')$ and $\mathrm{GL}(X)$ are determined by these rational invariants, we obtain a matching of orbits. This gives a natural explanation for the matching of orbits that plays an important role in the work of Jacquet and Rallis [14] on the relative trace formula, where they establish a comparison of the corresponding orbital integrals, and in the more recent work of Wei Zhang [30] on the global conjecture of Gan, Gross, and Prasad [12].

4 Examples with nontrivial stabilizer and nontrivial obstruction

In this section, we will provide some examples of representations with a nontrivial abelian stabilizer G_f, and calculate the obstruction class d_f in $H^2(k, G_f)$. The first example is a simple modification of a case we have already considered, namely, the non-faithful representation V of $\mathrm{Spin}(W) = \mathrm{Spin}(n+1)$ on n copies of the standard representation W of the special orthogonal group $\mathrm{SO}(W)$. In this case, the stabilizer G_f of the stable orbits is the center μ_2. We will also describe the stable orbits for the groups $G = \mathrm{SL}(W)$ and $H = \mathrm{SL}(W)/\mu_2$ acting on the representation $V = \mathrm{Sym}_2 W^* \oplus \mathrm{Sym}_2 W^*$. (The group H exists and acts when the dimension of W is even.) In these cases, the stabilizer G_f is a finite elementary abelian 2-group, related to the 2-torsion in the Jacobian of a hyperelliptic curve.

4.1 Spin($n + 1$) *acting on n copies of the standard representation of* SO($n + 1$)

In this subsection, we reconsider the representation $V = W^n$ of SO(W) studied in §3.1. There we saw that the orbits of vectors $v = (w_1, w_2, \ldots, w_n)$, where the quadratic form $f = q(x_1w_1 + x_2w_2 + \cdots + x_nw_n)$ has nonzero discriminant, have trivial stabilizer. If we consider V as a representation of the two-fold covering group $G = \text{Spin}(W)$, then these orbits have stabilizer $G_f = \mu_2$.

In the former case, we found that the unique pure inner form SO(W') for which $V'_f(k)$ is nonempty corresponded to the quadratic space W' of dimension $n + 1$ and $\text{disc}(W') = \text{disc}(W)$ that is the orthogonal direct sum of the subspace U_0 with quadratic form f and a nondegenerate space of dimension 1. The group Spin(W') will have orbits with invariant f, but this group may *not* be a pure inner form of the group $G = \text{Spin}(W)$. If it is not a pure inner form, the invariant d_f must be non-trivial in $H^2(k, G_f)$.

Assume, for example, that the orthogonal space W is split and has odd dimension $2m + 1$, so that the spin representation U of $G = \text{Spin}(W)$ of dimension 2^m is defined over k. Then a necessary and sufficient condition for the group $G' = \text{Spin}(W')$ to be a pure inner form of G is that the even Clifford algebra $C^+(W')$ of W' is a matrix algebra over k. In this case, the spin representation U' of G' can also be defined over k. Hence the obstruction d_f is given by the Brauer class of the even Clifford algebra of the space W' determined by f. Note that the even Clifford algebra $C^+(W')$ has an anti-involution, so its Brauer class has order 2 and lies in the group $H^2(k, G_f) = H^2(k, \mu_2)$.

4.2 SL_n *acting on* $\text{Sym}_2(n) \oplus \text{Sym}_2(n)$

Let k be a field of characteristic not equal to 2 and let W be a vector space of dimension n over k. Let e be a basis vector of the one-dimensional vector space $\wedge^n W$. The group $G = \text{SL}_n$ acts linearly on W and trivially on $\wedge^n W$.

The action of G on the space $\text{Sym}_2 W^*$ of symmetric bilinear forms $\langle v, w\rangle$ on W is given by the formula

$$g \cdot \langle v, v'\rangle = \langle gv, gv'\rangle.$$

This action preserves the discriminant of the bilinear form $A = \langle\ ,\ \rangle$, which is defined by the formula

$$\text{disc}(A) = (-1)^{n(n-1)/2}\langle e, e\rangle_n.$$

Here $\langle\ ,\ \rangle_n$ is the induced symmetric bilinear form on $\wedge^n(W)$. If $\{w_1, w_2, \ldots, w_n\}$ is any basis of W with $w_1 \wedge w_2 \wedge \ldots \wedge w_n = e$, then $\langle e, e\rangle_n = \det(\langle w_i, w_j\rangle)$. The discriminant is a polynomial of degree $n = \dim(W)$ on $\text{Sym}_2 W^*$ which freely generates the ring of G-invariant polynomials.

Now consider the action of G on the representation $V = \mathrm{Sym}_2 W^* \oplus \mathrm{Sym}_2 W^*$. If $A = \langle\ ,\ \rangle_A$ and $B = \langle\ ,\ \rangle_B$ are two symmetric bilinear forms on W, we define the binary form of degree n over k by the formula

$$f(x, y) = \mathrm{disc}(xA - yB) = f_0 x^n + f_1 x^{n-1} y + \cdots + f_n y^n.$$

The coefficients of this form are each polynomial invariants of degree n on V, and the $n + 1$ coefficients f_j freely generate the ring of polynomial invariants for G on V. (This will follow from our determination of the orbits of G over k^s in Theorem 4.1.) We call $f(x, y)$ the *invariant binary form* associated to (the orbit of) the vector $v = (A, B)$.

The discriminant $\Delta(f)$ of the binary form f is defined by writing $f(x, y) = \prod(\alpha_i x - \beta_i y)$ over the algebraic closure of k and setting

$$\Delta(f) = \prod_{i<j}(\alpha_i \beta_j - \alpha_j \beta_i)^2.$$

Then $\Delta(f)$ is a homogeneous polynomial of degree $2n - 2$ in the coefficients f_j, so is a polynomial invariant of degree $2n(n - 1)$ on V. For example, the binary quadratic form $ax^2 + bxy + cy^2$ has discriminant $\Delta = b^2 - 4ac$ and the binary cubic form $ax^3 + bx^2y + cxy^2 + dy^3$ has discriminant $\Delta = b^2c^2 + 18abcd - 4ac^3 - 4b^3d - 27a^2d^2$.

The first result shows how the invariant form and its discriminant determine the stable orbits for G on V over k^s.

Theorem 4.1. *Let k^s be a separable closure of k, and let $f(x, y)$ be a binary form of degree n over k^s with $f_0 \neq 0$ and $\Delta(f) \neq 0$. Then there are vectors (A, B) in $V(k^s)$ with invariant form $f(x, y)$, and these vectors all lie in a single orbit for $G(k^s)$. This orbit is closed, and the stabilizer of any vector in the orbit is an elementary abelian 2-group of order 2^{n-1}.*

To begin the proof, we make a simple observation. Let A and B denote two symmetric bilinear forms on W over k^s with $\mathrm{disc}(xA - yB) = f(x, y)$. Then both A and B give k^s-linear maps $W \to W^*$. Our assumption that f_0 is nonzero implies that the linear map $A : W \to W^*$ is an isomorphism, so we obtain an endomorphism $T = A^{-1}B : W \to W$. The fact that both A and B are symmetric with respect to transpose implies that T is self-adjoint with respect to the bilinear form $\langle\ ,\ \rangle_A$ on W.

Write $f(x, 1) = f_0 g(x)$ with $g(x)$ monic of degree n. The characteristic polynomial $\det(xI - T)$ is equal to the monic polynomial $g(x)$, and our assumption that the discriminant of $f(x, y)$ is nonzero in k implies that the polynomial $g(x)$ is separable. Hence the endomorphism T of V is regular and semisimple. The group $G(k^s)$ acts transitively on the bilinear forms with discriminant f_0, and the stabilizer of A is the orthogonal group $\mathrm{SO}(W, A)$. Since the group $\mathrm{SO}(W, A)(k^s)$ acts transitively on the self-adjoint operators T with a fixed separable characteristic polynomial $g(x)$, there is a single $G(k^s)$-orbit on the vectors (A, B) with invariant form $f(x, y)$.

The stabilizer is the centralizer of T in $\mathrm{SO}(W, A)$, which is an elementary abelian 2-group of order 2^{n-1}. For proofs of these assertions, see [4, Prop. 4].

Having classified the stable orbits of G on V over the separable closure, we now turn to the problem of classifying the orbits with a fixed invariant polynomial $f(x, y)$ over k.

Theorem 4.2. *Let $f(x, y) = f_0x^n + f_1x^{n-1}y + \cdots + f_ny^n$ be a binary form of degree n over k whose discriminant Δ and leading coefficient f_0 are both nonzero in k. Write $f(x, 1) = f_0g(x)$ and let L be the étale algebra $k[x]/(g)$ of degree n over k. Then there is a canonical bijection (constructed below) between the set of orbits (A, B) of $G(k)$ on $V(k)$ having invariant binary form $f(x, y)$ and the equivalence classes of pairs (α, t) with $\alpha \in L^\times$ and $t \in k^\times$, satisfying $f_0N(\alpha) = t^2$. The pair (α, t) is equivalent to the pair (α^*, t^*) if there is an element $c \in L^\times$ with $c^2\alpha^* = \alpha$ and $N(c)t^* = t$.*

The group scheme G_f obtained by descending the stabilizers $G_{A,B}$ for $(A, B) \in V_f(k^s)$ to k is the finite abelian group scheme $(\mathrm{Res}_{L/k}\,\mu_2)_{N=1}$ of order 2^{n-1} over k.

As a corollary, we see that the set of orbits with invariant form $f(x, y)$ is nonempty if and only if the element $f_0 \in k^\times$ lies in the subgroup $N(L^\times)k^{\times 2}$. In this case, we obtain a surjective map (by forgetting t) from the set of orbits to the set $(L^\times/L^{\times 2})_{N\equiv f_0}$, where the subscript indicates that the norm is congruent to f_0 in the group $k^\times/k^{\times 2}$. This map is a bijection when there is an element $c \in L^\times$ that satisfies $c^2 = 1$ and $N(c) = -1$. Such an element c will exist if and only if the polynomial $g(x)$ has a monic factor of odd degree over k. If no such element c exists, then the two orbits (α, t) and $(\alpha, -t)$ are distinct and map to the same class α in $(L^\times/L^{\times 2})_{N\equiv f_0}$. In that case, the map is two-to-one.

When $n = 2g + 1$ is odd, the set of orbits is always nonempty and has a natural base point $(\alpha, t) = (f_0, f_0^{(n+1)/2})$. Using this base point, and the existence of an element c with $c^2 = 1$ and $N(c) = -1$, we can identify the set of orbits with invariant form $f(x, y)$ with the group $(L^\times/L^{\times 2})_{N\equiv 1}$. This group classifies the principal homogeneous spaces for the group scheme $(\mathrm{Res}_{L/k}\,\mu_2)_{N=1}$. In fact, each orbit with invariant form $f(x, y)$ gives rise to a (geometrically) abelian cover of $\mathbb{P}^1$ of degree 2^{2g} with an action of this group scheme, which is ramified to order 2 at the $2g + 1$ points cut out by the equation $f(x, y) = 0$ and unramified elsewhere. The principal homogeneous space is the fiber over the point ∞ of $\mathbb{P}^1$, which is unramified in the cover by our hypothesis that $f_0 \neq 0$.

When $n = 2g+2$ is even, f_0 may not lie in the subgroup $N(L^\times)k^{\times 2}$ of $k^\times$. In this case, there may be no orbits over k with invariant polynomial $f(x, y)$. For example, when $n = 2$ there are no orbits over $\mathbb{R}$ with invariant form $f(x, y) = -x^2 - y^2$. However, there is a close relation between the existence of an orbit with invariant $f(x, y)$ and the arithmetic of the smooth hyperelliptic curve C of genus g over k, defined by the equation $z^2 = f(x, y)$ in the weighted projective plane $\mathbb{P}(1, 1, g+1)$. For example, every k-rational point $P = (u, 1, v)$ on C with $v \neq 0$ (so P is not a Weierstrass point) gives rise to an orbit [3, §2]. Indeed, write $f(x, 1) = f_0 \cdot g(x)$

and let θ be the image of x in the algebra $L = k[x]/(g(x))$. The orbit associated to P has $\alpha = u - \theta \in L^\times$ and $t = v \in k^\times$. Then $N(\alpha) = g(u)$, so $t^2 = f_0 \cdot N(\alpha)$. This is the association used in [3] to show that most hyperelliptic curves over $\mathbb{Q}$ have no rational points.

Proof of Theorem 4.2: Assume that we have a vector (A, B) in $V(k)$ with $\mathrm{disc}(xA - yB) = f(x, y)$. Using the k-linear maps $W \to W^*$ given by the bilinear forms A and B and the assumption that f_0 is nonzero, we obtain an endomorphism $T = A^{-1}B : W \to W$ which is self-adjoint for the pairing $\langle , \rangle_A$ and has characteristic polynomial $g(x)$. Since $\Delta(f)$ is nonzero, the polynomial $g(x)$ is separable and W has the structure of a free $L = k[T] = k[x]/(g)$ module of rank one. Let β denote the image of x in L, and let $\{1, \beta, \beta^2, \cdots, \beta^{n-1}\}$ be the corresponding power basis of L over k.

The k-bilinear forms A and B both arise as the traces of L-bilinear forms on the rank-one L module W. Choose a basis vector m of W over L and consider the k-linear map $L \to k$ defined by $\lambda \to \langle m, \lambda m\rangle_A$. Since $g(x)$ is separable, the element $g'(\beta)$ is a unit in L and the trace map from L to k is nonzero. Hence there is a unique element κ in $L^\times$ such that

$$\langle m, \lambda m\rangle_A = \mathrm{Trace}(\kappa\lambda/g'(\beta))$$

for all λ in L. Since all elements of L are self-adjoint with respect to the form $\langle , \rangle_A$, we find that the formula

$$\langle \mu m, \lambda m\rangle_A = \mathrm{Trace}(\kappa\mu\lambda/g'(\beta))$$

holds for all μ and λ in L. Since the discriminant f_0 of the bilinear form $\langle , \rangle_A$ is nonzero in k, we conclude that κ is a unit in the algebra L, so is an element of the group $L^\times$. We define $\alpha = \kappa^{-1} \in L^\times$, so that

$$\langle \mu m, \lambda m\rangle_A = \mathrm{Trace}(\mu\lambda/\alpha g'(\beta)).$$

A famous formula due to Euler [24, Ch III, §6] then shows that for all μ and λ in L, the value $\langle \mu m, \lambda m\rangle_A$ is the coefficient of β^{n-1} in the basis expansion of the product $\mu\lambda/\alpha$. It follows that the value $\langle \mu m, \lambda m\rangle_B$ is the coefficient of β^{n-1} in the basis expansion of the product $\beta\mu\lambda/\alpha$.

We define the element $t \in k^\times$ by the formula

$$t(m \wedge \beta m \wedge \beta^2 m \wedge \ldots \wedge \beta^{n-1} m) = e$$

in the one-dimensional vector space $\wedge^n(W)$. Then $\langle e, e\rangle_n = t^2 \det(\langle \beta^i m, \beta^j m\rangle_A)$. Since $\langle e, e\rangle_n = (-1)^{n(n-1)/2} f_0$ and $\det(\langle \beta^i m, \beta^j m\rangle_A) = (-1)^{n(n-1)/2} N(\alpha)^{-1}$, we have that $t^2 = f_0 N(\alpha)$.

We have therefore associated to the binary form $f(x, y)$ an étale algebra L, and to the vector (A, B) with discriminant $f(x, y)$ an element $\alpha \in L^\times$ and an element $t \in k^\times$ satisfying $t^2 = f_0 N(\alpha)$. The definition of α and t required the choice of a basis vector m for W over L. If we choose instead $m^* = cm$ with c in $L^\times$, then $\alpha = c^2\alpha^*$ and $t = N(c)t^*$. Hence the vector (A, B) only determines the equivalence class of the pair (α, t) as defined above.

It is easy to see that every equivalence class (α, t) determines an orbit. Since the dimension n of L over k is equal to the dimension n of W, we can choose a linear isomorphism $\theta : L \to W$ that maps the element $1 \wedge \beta \wedge \beta^2 \ldots \wedge \beta^{n-1}$ in $\wedge^n(L)$ to the element $t^{-1}e$ in $\wedge^n(V)$. Every other isomorphism with this property has the form $h\theta$, where h is an element in the subgroup $G = \mathrm{SL}(W)$. Using θ we define two bilinear forms on W:

$$\langle\theta(\mu), \theta(\lambda)\rangle_A = \mathrm{Trace}(\mu\lambda/(\alpha g'(\beta)))$$

$$\langle\theta(\mu), \theta(\lambda)\rangle_B = \mathrm{Trace}(\beta\mu\lambda/(\alpha g'(\beta))).$$

The $G(k)$-orbit of the vector (A, B) in $V(k)$ is well-defined and has invariant polynomial $f(x, y)$.

To complete the proof, we need to determine the stabilizer of a point $(A, B) \in V(k^s)$ in an orbit with binary form $f(x, y)$. Let $L^s = k^s[x]/(g(x))$ denote the k^s-algebra of degree n. Since the bilinear form $\langle\ ,\ \rangle_A$ is nondegenerate, the stabilizer of A in G is the special orthogonal group $\mathrm{SO}(W, A)$ of this form. The stabilizer of B in the special orthogonal group $\mathrm{SO}(W, A)$ is the subgroup of those g that commute with the self-adjoint transformation T. Since T is regular and semisimple, the centralizer of T in $GL(W)$ is the subgroup $k^s[T]^\times = L^{s\times}$, and the operators in $L^{s\times}$ are all self-adjoint. Hence the intersection of $L^{s\times}$ with the special orthogonal group $\mathrm{SO}(W, A)(k^s)$ consists of those elements g that are simultaneously self-adjoint and orthogonal, so consists of those elements g in $L^{s\times}$ with $g^2 = 1$ and $N(g) = 1$. The same argument works over any k^s-algebra E. The elements in $G(E)$ stabilizing (A, B) are the elements h in $(E \otimes L^s)^\times$ with $h^2 = 1$ and $N(h) = 1$. Hence the stabilizer $G_{A,B}$ is isomorphic to the finite étale group scheme $(\mathrm{Res}_{L^s/k^s}\,\mu_2)_{N=1}$ over k^s.

To show that these group schemes descend to $(\mathrm{Res}_{L/k}\,\mu_2)_{N=1}$, it remains to construct isomorphisms $\iota_v : (\mathrm{Res}_{L/k}\,\mu_2)_{N=1}(k^s) \to G_v$ compatible with the descent data for every $v \in V_f(k^s)$, i.e., satisfying (3) and (4). Let $\alpha_1, \ldots, \alpha_n \in k^s$ denote the roots of $g(x)$. For any $i = 1, \ldots, n$, define

$$h_i(x) = \frac{g(x)}{x - \alpha_i}, \qquad g_i(x) = 1 - 2\frac{h_i(x)}{h_i(\alpha_i)}.$$

For any linear operator T on W with characteristic polynomial $g(x)$, the operator $g_i(T)$ acts as -1 on the α_i-eigenspace of T and acts trivially on all other eigenspaces. Then for any $v = (A, B) \in V_f(k^s)$, the map ι_v sends an n-tuple $(m_1, \ldots, m_n)$ of 0's or 1's, such that $\sum m_i$ is even, to

$$\iota_v(m_1, \ldots, m_n) = \prod_{i=1}^{n} g_i(T)^{m_i},$$

where $T = A^{-1}B$ as before. □

In [29], Wood classified the elements of the representation $\mathrm{Sym}_2 R^n \oplus \mathrm{Sym}_2 R^n$ for any base ring (or even any base scheme) R, in terms of suitable algebraic data involving ideal classes of "rings of rank n" over R; see §4.6 for more details on the case $R = \mathbb{Z}$. The special case where R is a field, and a description of the resulting orbits under the action of $\mathrm{SL}_n(R)$, is given by Theorem 4.2.

4.3 Some finite group schemes and their cohomology

To give a cohomological interpretation of Theorem 4.2 and to make preparations for studying the orbits of the action of SL_n/μ_2 on $\mathrm{Sym}^2(n) \oplus \mathrm{Sym}^2(n)$ in the next two subsections, we collect some results on the cohomology of $\mathrm{Res}_{L/k}\mu_2$ and other closely related finite group schemes. A good reference for much of this material is Section 6 the recent preprint [9].

Fix an integer $n \geq 1$, and consider the action of the symmetric group S_n on the vector space $N = (\mathbb{Z}/2\mathbb{Z})^n$ by permutation of the natural basis elements e_i. The nondegenerate symmetric bilinear form

$$\langle n, m\rangle = \sum n_i m_i$$

is S_n-invariant. We have the stable subspace N_0 of elements with $\sum n_i = 0$, and on this subspace the bilinear form is alternating. It is also nondegenerate when n is odd.

When n is even, the kernel of the form on N_0 is the one-dimensional subspace M spanned by the vector $n = (1, 1, \ldots, 1)$, and we obtain a nondegenerate alternating pairing

$$N_0 \times N/M \to \mathbb{Z}/2\mathbb{Z}.$$

This induces an alternating duality which is S_n-invariant on the subquotient N_0/M.

We want to translate these results on finite elementary abelian 2-groups with an action of S_n to finite étale group schemes over a field k whose characteristic is not equal to 2. Let L be an étale k-algebra of rank n, and let R be the finite group scheme $\mathrm{Res}_{L/k}\,\mu_2$. Let k^s be a fixed separable closure of k. The Galois group of k^s over k permutes the n distinct homomorphisms $L \to k^s$, and this determines a homomorphism $\mathrm{Gal}(k^s/k) \to S_n$ up to conjugacy. We have an isomorphism $R(k^s) \cong N$ of $\mathrm{Gal}(k^s/k)$ modules. If $L = k[x]/g(x) = k[\beta]$ with $g(x)$ monic and separable of degree n, then the distinct homomorphisms $L \to k^s$ are obtained by mapping β to the distinct roots β_i of $g(x)$ in k^s. Hence the points of R over an extension K of k correspond bijectively to the monic factors $h(x)$ of $g(x)$ over K.

Let $R_0 = (\mathrm{Res}_{L/k}\,\mu_2)_{N=1}$ be the subgroup scheme of elements of norm 1 to μ_2. The above isomorphism maps $R_0(k^s)$ to the Galois module N_0, and the points of R_0 over an extension K correspond to the monic factors $h(x)$ of $g(x)$ of even degree over K.

The diagonally embedded $\mu_2 \to R$ corresponds to the trivial Galois submodule M of N, and the points of R/μ_2 over K correspond to the monic factorizations $g(x) = h(x)j(x)$ that are rational over K. This means that either $h(x)$ and $j(x)$ have coefficients in K, or that they have conjugate coefficients in some quadratic extension of K.

When n is even, the subgroup μ_2 of R is actually a subgroup of R_0. The points of R_0/μ_2 over K correspond to the monic factorizations $g(x) = h(x)j(x)$ of even degree that are rational over K.

Since the pairings defined above are all S_n-invariant, we obtain Cartier dualities

$$R \times R \to \mu_2,$$

$$R_0 \times R/\mu_2 \to \mu_2.$$

Since the Cartier dual of R_0 is the finite group scheme R/μ_2, we obtain a cup product pairing

$$H^2(k, R_0) \times H^0(k, R/\mu_2) \longrightarrow H^2(k, \mathbb{G}_m)[2] = H^2(k, \mu_2).$$

When n is odd, we obtain an alternating duality on $R_0 \cong R/\mu_2$. When n is even, we obtain an alternating duality

$$R_0/\mu_2 \times R_0/\mu_2 \to \mu_2.$$

We now consider the Galois cohomology of these étale group schemes. For $R = \mathrm{Res}_{L/k}\,\mu_2$ we have

$$H^0(k, R) = L^\times[2], \quad H^1(k, R) = L^\times/L^{\times 2}, \quad H^2(k, R) = \mathrm{Br}(L)[2].$$

For $R_0 = (\mathrm{Res}_{L/k}\,\mu_2)_{N=1}$, we have $H^0(k, R_0) = L^\times[2]_{N=1}$ and the long exact sequence in cohomology gives an exact sequence

$$\begin{aligned} 1 \to \langle \pm 1\rangle/N(L^\times[2]) \to & H^1(k, R_0) \to L^\times/L^{\times 2} \\ & \to k^\times/k^{\times 2} \to H^2(k, R_0) \to \mathrm{Br}(L)[2]. \end{aligned} \tag{7}$$

The group $H^1(k, R_0)$ maps surjectively to the subgroup $(L^\times/L^{\times 2})_{N\equiv 1}$ of elements in $L^\times/L^{\times 2}$ whose norm to $k^\times/k^{\times 2}$ is a square. The kernel of this map has order one if -1 is the norm of an element of $L^\times[2]$, or equivalently if $g(x)$ has a factor of odd degree. If $g(x)$ has no factor of odd degree, then the kernel has order two.

This computation allows us to give a cohomological interpretation to Theorem 4.2. For each rational invariant $f(x, 1) = f_0 g(x)$ with nonzero $\Delta(f)$ and f_0, the stabilizer G_f is isomorphic to the finite group scheme $(\mathrm{Res}_{L/k}\,\mu_2)_{N=1} = R_0$. The quotient group $k^\times/k^{\times 2}(NL^\times)$ is the kernel of the map from $H^2(k, R_0)$ to $H^2(k, R)$. In Theorem 4.4, we will show that the class of $f_0 \in k^\times/k^{\times 2}N(L^\times)$ maps to the class $d_f \in H^2(k, R_0)$ defined in §2.4. Since $H^1(k, \mathrm{SL}_n) = 0$, by Theorem 2.3 the nontriviality of d_f in $H^2(k, R_0)$ is the only obstruction to the existence of an $\mathrm{SL}_n(k)$-orbit with invariant form $f(x, y)$. This gives another proof that rational orbits with invariant $f(x, y)$ exist if and only if $f_0 \in N(L^\times)k^{\times 2}$. When the class d_f vanishes, the orbits of $\mathrm{SL}_n(k)$ with this rational invariant f form a principal homogeneous space for the group $H^1(k, R_0)$.

4.4 SL_n/μ_2 *acting on* $\mathrm{Sym}_2(n) \oplus \mathrm{Sym}_2(n)$

When the dimension n of W is odd, we have obtained a bijection from the set of orbits for $\mathrm{SL}(W)$ with invariant f to the elements of the group $(L^\times/L^{\times 2})_{N\equiv 1}$

In this section, we consider the more interesting situation when the dimension n of W is even. In this case, the central subgroup μ_2 in $\mathrm{SL}(W)$ acts trivially on $V = \mathrm{Sym}_2 W^* + \mathrm{Sym}_2 W^*$, and we can consider the orbits of the group $H = \mathrm{SL}(W)/\mu_2$ on V over k. Since $H^1(k, \mathrm{SL}(W)) = 1$, the group of k-rational points of H lies in the exact sequence

$$1 \to \mathrm{SL}(W)(k)/\langle \pm 1\rangle \to H(k) \to k^\times/k^{\times 2} \to 1.$$

A representative in $H(k)$ of the coset of d in $k^\times/k^{\times 2}$ can be obtained as follows. Lift $d \in k^\times/k^{\times 2}$ to an element $d \in k^\times$ and let $K = k(\sqrt{d})$ be the corresponding quadratic extension. Let τ be the nontrivial involution of K over k and let $g(d)$ be any element of $\mathrm{SL}(W)(K)$ whose conjugate ${}^\tau(g(d))$ is equal to $-g(d)$. For example, one can take a diagonal matrix with $n/2$ entries equal to $\sqrt{d}$ and $n/2$ entries equal to $1/\sqrt{d}$. Then the image of $g(d)$ in the quotient group $H(K)$ gives a rational element in $H(k)$. The elements $g(d)$ for d in $k^\times/k^{\times 2}$ give coset representatives for the subgroup $\mathrm{SL}(W)(k)/\langle \pm 1\rangle$ of $H(k)$.

If v is any vector in $V_f(k)$ and d represents a coset of $k^\times/k^{\times 2}$, then

$${}^\tau(g(d)(v)) = {}^\tau(g(d))(v) = -g(d)(v) = g(d)(v),$$

so the vector $g(d)(v)$ is also an element of $V_f(k)$. Since the coset of $g(d)$ is well-defined, and $g(d)^2$ is an element of $\mathrm{SL}(W)(k)$, we see that the action of $g(d)$ gives an involution (possibly trivial) on the orbits of $\mathrm{SL}(W)(k) = G(k)$ on $V_f(k)$.

We have seen that the orbits of $G(k)$ with invariants $f(x, y)$ are determined by two invariants: $\alpha \in L^\times$ and $t \in k^\times$ that satisfy $f_0 N(\alpha) = t^2$. The pair (α, t) is equivalent to the pair $(c^2\alpha, N(c)t)$. Under this bijection, the element represented by $g(d)$ in $H(k)$ maps the equivalence class of (α, t) to the equivalence class $(d\alpha, d^{n/2}t)$. This gives the following result.

Theorem 4.3. *Assume that n is even and let* $f(x, y) = f_0x^n + f_1x^{n-1}y + \cdots + f_ny^n$ *be a binary form of degree n over k whose discriminant* Δ *and leading coefficient* f_0 *are both nonzero in k. Write* $f(x, 1) = f_0 g(x)$ *and let L be the étale algebra* $k[x]/(g)$ *of degree n over k. Then there is a bijection between the set of orbits* (A, B) *of* $H(k)$ *on* $V(k)$ *having invariant binary form* $f(x, y)$ *and the set of equivalence classes of pairs* (α, t) *with* $\alpha \in L^\times$ *and* $t \in k^\times$ *satisfying* $f_0N(\alpha) = t^2$. *The pair* (α, t) *is equivalent to the pair* (α^*, t^*) *if there is an element* $c \in L^\times$ *and an element* $d \in k^\times$ *with* $c^2 d\alpha^* = \alpha$ *and* $N(c)d^{n/2}t^* = t$.

The group scheme H_f obtained by descending the stabilizers $H_{A,B}$ for $(A, B) \in V_f(k^s)$ to k is the finite abelian group scheme $(\mathrm{Res}_{L/k}\,\mu_2)_{N=1} = R_0/\mu_2$ of order 2^{n-2} over k.

Theorem 4.3 implies that orbits for $H(k)$ exist with invariant binary form $f(x, y)$ if and only if the leading coefficient f_0 lies in the subgroup $k^{\times 2}N(L^\times)$ of $k^\times$. When orbits do exist, we can associate to each $H(k)$-orbit the class of α in the set

$$(L^\times/L^{\times 2}k^\times)_{N\equiv f_0}.$$

This is a surjective map, which is a bijection when there are elements $c \in L^\times$ and $d \in k^\times$ satisfying $c^2d = 1$ and $N(c)d^{n/2} = -1$. Such a pair (c, d) exists if and only if the monic polynomial $g(x)$ has an odd factorization over k. If $g(x)$ has a rational factor of odd degree, then there is a pair with $c^2 = 1$ and $d = 1$. On the other hand, if $g(x)$ has no rational factor of odd degree, but has a rational factorization, then $n/2$ is odd and the factorization occurs over the unique quadratic extension $K = k(\sqrt{d})$ which is a subalgebra of L. If $g(x)$ has no odd factorization, the two orbits (α, t) and $(\alpha, -t)$ are distinct and the surjective map from the set of $H(k)$-orbits to the set $(L^\times/L^{\times 2}k^\times)_{N\equiv f_0}$ is two-to-one.

We can also reinterpret this result in terms of the Galois cohomology of the stabilizer $H_f = R_0/\mu_2$. We assume that there exists rational $(A, B) \in V(k)$ with invariant binary form $f(x, y)$. In the next subsection, we study the obstruction to this existence. The set of rational orbits with invariant f is in bijection with the kernel of the composite map $\gamma : H^1(k, H_{A,B}) \to H^1(k, H) \hookrightarrow H^2(k, \mu_2)$ of pointed sets. We now give another description of γ and in particular show that it is a group homomorphism; hence the set of $H(k)$-orbits forms a principal homogenous space for $\ker\gamma$. Note that even though both the source and target of γ are groups, there is a priori no reason for γ to be a group homomorphism. The short exact sequence

$$1 \to \mu_2 \to R_0 \to R_0/\mu_2 \to 1 \tag{8}$$

of finite abelian group schemes over k gives rise to the long exact sequence in cohomology

$$\begin{aligned}1 \to \langle\pm 1\rangle \to R_0(k) \to R_0/\mu_2(k) \to & k^\times/k^{\times 2} \to \\ & \to H^1(k, R_0) \to H^1(k, R_0/\mu_2) \xrightarrow{\delta} H^2(k, \mu_2) = \mathrm{Br}(k)[2].\end{aligned}$$

By definition of the connecting homomorphism, we see that $\delta = \gamma$. Let the kernel be denoted $H^1(k, R_0/\mu_2)_{ker} := \ker\delta$. Then we have the following short exact sequence

$$1 \to k^\times/k^{\times 2}\langle H\rangle \to H^1(k, R_0) \to H^1(k, R_0/\mu_2)_{ker} \to 1, \tag{9}$$

where $\langle H\rangle$ denotes the image of $R_0/\mu_2(k)$ in $k^\times/k^{\times 2}$. The group $\langle H\rangle$ can be nontrivial only when n is divisible by 4; in this case $\langle H\rangle$ is a finite elementary abelian 2-group corresponding to the quadratic field extensions K of k that are contained in the algebra L. In that case, a factorization of $g(x)$ into two even degree polynomials conjugate over K gives a rational point of $R_0/\mu_2(k)$ which is not in the image of $R_0(k)$. Recall that $R_0 = G_f$ is the stabilizer for the action of the group $G = \mathrm{SL}(W)$ (Theorem 4.2). Therefore, (9) describes how $G(k)$-orbits combine into $H(k)$-orbits and reflects the extra relations in Theorem 4.3.

We now give a more concrete description of $H^1(k, R_0/\mu_2)_{ker}$ in terms of the algebras L and k. The above short exact sequence maps surjectively to the short exact sequence

$$1 \to k^\times/k^{\times 2}\langle I\rangle \to (L^\times/L^{\times 2})_{N\equiv 1} \to (L^\times/L^{\times 2}k^\times)_{N\equiv 1} \to 1,$$

where $\langle I\rangle$ is the finite elementary abelian subgroup corresponding to all of the quadratic extensions K of k that are contained in L. We have $\langle I\rangle = \langle H\rangle$ except in the case when n is not divisible by 4 and there is a (unique) quadratic extension field K contained in L, in which case the kernel of the map from $H^1(k, R_0)$ to $(L^\times/L^{\times 2})_{N\equiv 1}$ has order 2, whereas the map from $H^1(k, R_0/\mu_2)_{\mathrm{ker}}$ to $(L^\times/L^{\times 2}k^\times)_{N\equiv 1}$ is a bijection. In all other cases, these maps have isomorphic kernels (of order 1 or 2).

We note that the existence and surjectivity of the map from $H^1(k, R_0/\mu_2)_{\mathrm{ker}}$ to $(L^\times/L^{\times 2}k^\times)_{N\equiv 1}$ in the above paragraph follows formally from exactness. More canonically, the group $(L^\times/L^{\times 2}k^\times)_{N\equiv 1}$ can be viewed as the subgroup of $H^1(k, R/\mu_2)$ consisting of elements that map to 0 in $H^2(k, \mu_2)$ under the connecting homomorphism in Galois cohomology and to 0 in $H^1(k, \mu_2)$ under the map induced by $N : R/\mu_2 \to \mu_2$. The natural map $H^1(k, R_0/\mu_2) \to H^1(k, R/\mu_2)$ sends $H^1(k, R_0/\mu_2)_{ker}$ to this subgroup. The kernel of this map is generated by a class $W_H \in H^1(k, R_0/\mu_2)$. The points of the principal homogeneous space W_H over an extension field E are the odd factorizations of $g(x)$ that are rational over E.

Since the finite group scheme $H_{A,B} = R_0/\mu_2$ is self-dual, we obtain a cup product pairing

$$H^1(k, R_0/\mu_2) \times H^1(k, R_0/\mu_2) \to H^2(k, \mu_2).$$

The connecting homomorphism $\delta : H^1(k, R_0/\mu_2) \to H^2(k, \mu_2)$ (and hence also γ) is given by the cup product against the class W_H in $H^1(k, R_0/\mu_2)$ ([19, Proposition 10.3]).

Theorems 4.2 and 4.3 have a number of applications to the arithmetic of hyperelliptic curves, which we study in a forthcoming paper [6]. A binary form $f(x, y)$ of degree $n = 2g + 2$ with nonzero discriminant determines a smooth hyperelliptic curve $C : z^2 = f(x, y)$ of genus g. Here we view C as embedded in the weighted projective space $\mathbb{P}(1, 1, g+1)$. Denote the Jacobian of C by J. Then $J[2]$ is canonically isomorphic to R_0/μ_2. Under this isomorphism, the self-duality of R_0/μ_2 is given by the Weil pairing on $J[2]$. In [6], we use this connection to show that a positive proportion of hyperelliptic curves over $\mathbb{Q}$ of a fixed genus g have points locally at every place of $\mathbb{Q}$, but have no points over any odd degree extension of $\mathbb{Q}$.

4.5 Obstructions for the existence of orbits

In the previous section, we assumed that rational orbits with invariant f exist and studied the set of $H(k)$-orbits on $V_f(k)$. By Theorem 2.3, we know that there are two obstructions to existence of a $H(k)$-orbit with invariant f: the nonvanishing of a class $d_f \in H^2(k, H_f)$, and the possibility that we are not working with the correct pure inner form. In this subsection, we compute the class $d_f \in H^2(k, H_f)$ and when $d_f = 0$, and describe the set of pure inner forms H^c of H for which $V_f^c(k)$ is nonempty.

Theorem 4.4. *Let $f(x, y) = f_0x^n + \cdots + f_ny^n$ be a binary form of even degree n such that $\Delta(f)$ and f_0 are both nonzero. Write $f(x, 1) = f_0g(x)$ for some monic polynomial $g(x)$ and let $L = k(x)/(g) = k[\beta]$ be the associated étale algebra of rank n over k. The groups $G = \mathrm{SL}_n$ and $H = \mathrm{SL}_n/\mu_2$ act on $V = \mathrm{Sym}^2(n) \oplus \mathrm{Sym}^2(n)$. The stabilizer G_f (resp. H_f) associated to an element of $V_f(k^s)$ is the finite group scheme $(\mathrm{Res}_{L/k}\mu_2)_{N=1}$ (resp. $(\mathrm{Res}_{L/k}\mu_2)_{N=1}/\mu_2$). Let δ_0 denote the connecting homomorphism $H^1(k, \mu_2) \to H^2(k, G_f)$ appearing in (7). Let $d_f^G \in H^2(k, G_f)$ (resp. $d_f^H \in H^2(k, H_f)$) denote the obstruction class for the existence of $G(k)$– (resp. $H(k)$–) orbits with invariant f as defined in §2.4. Then d_f^G is the image of f_0 under δ_0, and the natural map $H^2(k, G_f) \to H^2(k, H_f)$ sends d_f^G to d_f^H.*

Proof. The statement regarding the stabilizer schemes G_f and H_f has been proved in Theorems 4.2 and 4.3, respectively. We now compute d_f^G following its definition given in §2.4. Let A_0 denote the matrix with 1's on the anti-diagonal and 0's elsewhere. Let $h(x) \in k^s[x]$ be a polynomial such that $N_{L^s/k^s}(h(\beta)) = f_0$. Let T be a k-rational linear operator on W that is self-adjoint with respect to A_0 and has characteristic polynomial $g(x)$ ([25, §2.2]). Then the element $v = (A_0h(T), A_0Th(T)) \in V_f(k^s)$ has invariant f. We need to pick $g_\sigma \in G(k^s)$ such that $g_\sigma {}^\sigma v = v$ for every $\sigma \in \mathrm{Gal}(k^s/k)$. We take g_σ to be of the form $g_\sigma = j_\sigma(T)$ for some polynomial $j_\sigma(x) \in k^s[x]$ such that $j_\sigma(\beta)^2 = ({}^\sigma h)(\beta)/h(\beta)$. By writing $({}^\sigma h)(\beta)$, we wish to emphasize that σ is not acting on β, and hence for any polynomial $h'(x) \in k^s[x]$, we have

$$^\sigma(N_{L^s/k^s}(h'(\beta))) = N_{L^s/k^s}(({}^\sigma h')(\beta)).$$

Let $\sqrt{h}(x) \in k^s[x]$ denote a polynomial such that $(\sqrt{h}(\beta))^2 = h(\beta)$. Set $j_\sigma(x) \in k^s[x]$ to be the polynomial such that $j_\sigma(\beta) = ({}^\sigma\sqrt{h})(\beta)/\sqrt{h}(\beta)$. By definition, d_f^G is then the 2-cocycle

$$(\sigma, \tau) \mapsto j_\sigma(\beta)^\sigma(j_\tau(\beta))j_{\sigma\tau}(\beta)^{-1}.$$

On the other hand, let $\sqrt{f_0}$ denote the square root of f_0 such that

$$\sqrt{f_0} = N_{L^s/k^s}(\sqrt{h}(\beta)).$$

Then the 1-cocycle $\sigma \mapsto {}^{\sigma}\sqrt{f_0}/\sqrt{f_0}$ corresponds to the class of $f_0 \in k^{\times}/k^{\times 2}$. To compute $\delta_0(f_0)$, for each $\sigma \in \mathrm{Gal}(k^s/k)$ we need to find an element in L^s whose norm to k^s is ${}^{\sigma}\sqrt{f_0}/\sqrt{f_0}$. A natural choice is $j_\sigma(\beta)$. The equality $d_f^G = \delta_0(f_0)$ is then clear. The second statement is also clear from the above computation for d_f^G. □

Since G has no nontrivial pure inner forms, the vanishing of d_f^G is sufficient for the existence of rational orbits. For H, there is a second (Brauer-type) obstruction coming from the pure inner forms of H.

Theorem 4.5. *Let $f(x, y) = f_0x^n + \cdots + f_ny^n$ be a binary form of even degree n such that $\Delta(f)$ is nonzero. Let $d_f^G \in H^2(k, G_f)$ (resp. $d_f^H \in H^2(k, H_f)$) denote the obstruction class for the existence of $G(k)$- (resp. $H(k)$-) orbits with invariant f. Consider the following diagram:*

$$\begin{array}{ccccc} & & H^1(k, H) & & \\ & & \downarrow{\scriptstyle \delta_2} & & \\ H^1(k, H_f) & \xrightarrow{\delta} & H^2(k, \mu_2) & \xrightarrow{\alpha} & H^2(k, G_f), \end{array}$$

where δ, δ_2 are the connecting homomorphisms in Galois cohomology and α is induced by the diagonal inclusion $\mu_2 \to G_f$. Suppose $d_f^H = 0$. Then d_f^G is the image of some $d \in H^2(k, \mu_2)$, where d lies in the image of δ_2. The pure inner forms of H for which rational orbits exist with invariant f correspond to classes $c \in H^1(k, H)$ such that $\alpha\delta_2(c) = d_f^G$ in $H^2(k, G_f)$.

Proof. Fix any $v \in V_f(k^s)$. Choose $g_\sigma \in H(k^s)$ for each $\sigma \in \mathrm{Gal}(k^s/k)$ such that $g_\sigma {}^{\sigma}v = v$. Since $d_f^H = 0$, by Lemma 2.4 we may pick g_σ such that $c = (\sigma \to g_\sigma)$ is a 1-cocycle in $H^1(k, H)$. Lift each g_σ arbitrarily to $\widetilde{g}_\sigma \in G(k^s)$. Since the center of $G(k^s)$ acts trivially on V, we have $\widetilde{g}_\sigma {}^{\sigma}v = v$ for every $\sigma \in \mathrm{Gal}(k^s/k)$. The 2-cocycle d_f^G in $H^2(k, \mu_2)$ is then given by

$$(d_f^G)_{\sigma,\tau} = \widetilde{g}_\sigma {}^{\sigma}\widetilde{g}_\tau \widetilde{g}_{\sigma\tau}^{-1}, \tag{10}$$

which is exactly the image of c under δ_2.

For the second statement, choose $g \in \mathrm{GL}(V)(k^s)$ such that $g_\sigma = g^{-1}{}^{\sigma}g$ for every $\sigma \in \mathrm{Gal}(k^s/k)$. From the definition of g_σ, we see that $gv \in V_f^c(k)$. For every $v' \in V_f(k^s)$, let $\iota_{v'} : H_f(k^s) \to H_{v'}$ denote the canonical isomorphism. Then we have a Galois invariant isomorphism $H_f(k^s) \to H_{gv}^c(k^s)$ sending $b \in H_f(k^s)$ to $\iota_v(b)$. Let ι denote the following composition:

$$\iota : H^1(k, H_f) \xrightarrow{\sim} H^1(k, H_{gv}^c) \to H^1(k, H^c) \xrightarrow{\sim} H^1(k, H),$$

where the last map is the bijection given by $(\sigma \to d_\sigma) \mapsto (\sigma \to d_\sigma g_\sigma)$.

Lemma 4.6. *For any $b \in H^1(k, H_f)$, we have*

$$\delta_2(\iota(b)) = \delta(b) + \delta_2(c).$$

This lemma follows from a direct computation similar to the proof of Lemma 2.4.

Proposition 2.1 states that the set of pure inner forms of H for which rational orbits exist with invariant f is in bijection with the image of $H^1(k, H_f)$ under ι. Since δ_2 is injective, Lemma 4.6 implies that this set is in bijection with $\delta(H^1(k, H_f)) + \delta_2(c)$, which equals $\alpha^{-1}(\alpha\delta_2(c))$ by exactness. Theorem 4.5 now follows since, by (10), we have $d_f^G = \alpha\delta_2(c)$. □

4.6 Integral orbits

In this section, we discuss the orbits of the group $G(\mathbb{Z}) = \mathrm{SL}_n(\mathbb{Z})$ on the free $\mathbb{Z}$-module $V(\mathbb{Z}) = \mathrm{Sym}_2\, \mathbb{Z}^n \oplus \mathrm{Sym}_2\, \mathbb{Z}^n$ of symmetric matrices (A, B) having entries in $\mathbb{Z}$. Even though Galois cohomology was very useful in the previous sections to study rational orbits, we will see in this section that one will generally need different techniques to study integral orbits.

Associated to an integral orbit we have the invariant binary n-ic form $f(x, y) = \mathrm{disc}(xA - yB) = f_0x^n + \cdots + f_ny^n$ with integral coefficients. We assume as above that the integers $\Delta(f)$ and f_0 are both nonzero, and write $f(x, 1) = f_0g(x)$. The polynomial $g(x)$ is separable over $\mathbb{Q}$, but its coefficients will not necessarily be integers (when $f_0 \neq \pm 1$). In this case, the image θ of x in the étale algebra $L = \mathbb{Q}[x]/g(x)$ will not necessarily be an algebraic integer.

The rational orbits with this binary form f correspond to equivalence classes of pairs (γ, t). Here γ is an invertible element in the étale algebra L and t is an invertible element of $\mathbb{Q}$ satisfying $t^2 = f_0N(\gamma)$. The equivalence relation is $(\gamma, t) \sim (c^2\gamma, N(c)t)$ for all $c \in L^\times$. In this section, we specify the additional data that determines an integral orbit in this rational orbit.

Recall that an *order* R in L is a subring that is free of rank n over $\mathbb{Z}$ and generates L over $\mathbb{Q}$. The ring $\mathbb{Z}[\theta]$ generated by θ will not be an order in L when the coefficients of $g(x)$ are not integers. However, there is a natural order R_f contained in L which is determined by the integral binary form $f(x, y)$. This order R_f as a $\mathbb{Z}$-module was first introduced by Birch and Merriman [8] and proved to be an order by Nakagawa [17]. A basis-free description was discovered by Wood [28], namely, R_f is the ring of the global sections of the structure sheaf of the subscheme S_f of $\mathbb{P}^1$ defined by the homogeneous equation $f(x, y) = 0$ of degree n. The ring R_f possesses a natural $\mathbb{Z}$-basis, namely,

$$R_f = \mathrm{Span}_{\mathbb{Z}}\{1, \zeta_1, \zeta_2, \ldots, \zeta_{n-1}\},$$

where

$$\zeta_k = f_0\theta^k + f_1\theta^{k-1} + \cdots + f_{k-1}\theta. \tag{11}$$

Note that the ζ_k are all algebraic integers, even though θ might not be. One easily checks ([8]) the remarkable equality $\mathrm{disc}(f) = \mathrm{disc}(R_f)$.

A *fractional ideal* I for an order R is a free abelian subgroup of rank n in L, which is stable under multiplication by R. The norm $N(I)$ is defined to be the positive rational number that is the quotient of the index of I in M by the index of R in M, where M is any lattice in L that contains both I and R. If the fractional ideal I is contained in R, and so defines an ideal of R in the usual sense, then $N(I)$ is its index in R. An *oriented fractional ideal* for an order R is a pair (I, ε), where I is any fractional ideal of R and $\varepsilon = \pm 1$ gives the *orientation* of I. The norm of an oriented ideal (I, ε) is defined to be the nonzero rational number $\varepsilon N(I)$. For an element $\kappa \in L^\times$, the product $\kappa(I, \varepsilon)$ is defined to be the oriented fractional ideal $(\kappa I, \mathrm{sgn}(N(\kappa))\varepsilon)$. Then $N(\kappa(I, \varepsilon)) = N(\kappa)N(I, \varepsilon)$ in $\mathbb{Q}^\times$. In practice, we denote an oriented ideal (I, ε) simply by I, with the orientation $\varepsilon = \varepsilon(I)$ on I being understood.

We say that a fractional ideal I is *based* if it comes with a fixed ordered basis over $\mathbb{Z}$. If the order R and the fractional ideal I are both based, then we can define the orientation of I as the sign of the determinant of the $\mathbb{Z}$-linear transformation taking the chosen basis of I to the basis of R. The norm of this oriented fractional ideal is then equal to the actual determinant. Changing the basis by an element of $\mathrm{SL}_n(\mathbb{Z})$ does not change the orientation ε of I or the norm $N(I)$ in $\mathbb{Q}^\times$.

The binary form $f(x, y)$ not only defines an order R_f in L, but also a collection of based fractional ideals $I_f(k)$ for $k = 0, 1, 2 \ldots, n-1$ (see [28]). The ideal $I_f(0) = R_f$ and for $k > 0$ the ideal $I_f(k)$ has a $\mathbb{Z}$-basis $\{1, \theta, \theta^2, \ldots, \theta^k, \zeta_{k+1}, \ldots, \zeta_{n-1}\}$. This gives $I_f(k)$ an orientation relative to R_f, and the norm of the oriented ideal $I_f(k)$ is equal to $1/f_0^k$. We have inclusions $R_f \subset I_f(1) \subset I_f(2) \subset \cdots \subset I_f(n-1)$.

Let $I_f = I_f(1)$. Then we find by explicit computation that $I_f(k) = I_f^k$. As shown by Wood [28], abstractly the fractional ideal I_f is the module of global sections of the pullback of the line bundle $\mathcal{O}(1)$ on $\mathbb{P}^1$ to the subscheme S_f defined by the equation $f(x, y) = 0$, and the ideals $I_f(k)$ are the global sections of the pullbacks of the line bundles $\mathcal{O}(k)$. We say that the form $f(x, y)$ is *primitive* if the greatest common divisor of its coefficients is equal to 1. When $f(x, y)$ is primitive, the scheme $S_f = \mathrm{Spec}(R_f)$ has no vertical components and is affine. In this case, the pullbacks of these line bundles have no higher cohomology, and the ideals $I_f(k) = I_f^k$ are all projective R_f-modules.

The oriented fractional ideal $I_f(n-1)$ has a power basis $\{1, \theta, \theta^2, \ldots, \theta^{n-1}\}$. When the form $f(x, y)$ is primitive, this fractional ideal is a projective, hence a proper, R_f-module. In this case, the ring R_f has a simple definition as the endomorphism ring of the lattice $\mathrm{Span}_{\mathbb{Z}}\{1, \theta, \theta^2, \ldots, \theta^{n-1}\}$ in the algebra L.

There is also a nice interpretation of the oriented fractional ideal $I_f(n-2) = I_f^{n-2}$ in terms of the trace pairing on L. Define a nondegenerate bilinear pairing $\langle\ ,\ \rangle_f : R_f \times I_f^{n-2} \to \mathbb{Z}$ by taking $\langle \lambda, \mu \rangle_f$ as the coeffiecient of ζ_{n-1} in the product $\lambda\mu$. Define $f'(\theta)$ in $L^\times$ by the formula $f'(\theta) = f_0 g'(\theta)$. Then $f'(\theta)$ lies in the fractional ideal I_f^{n-2}. A computation due to Euler shows that the above bilinear pairing is given by the formula

$$\langle \lambda, \mu \rangle_f = \mathrm{Trace}(\lambda\mu/f'(\theta)),$$

where the trace is taken from L to $\mathbb{Q}$. We have an inclusion $R_f \subset (1/f'(\theta))I_f^{n-2}$ and the index is the absolute value of $\Delta(f)$. In fact, the oriented fractional ideal $(1/f'(\theta))I_f^{n-2}$ has norm $1/\Delta(f)$. This is precisely the "inverse different"—the dual module to R_f in L under the trace pairing. When $f(x, y)$ is primitive, the dual module is projective and the ring R_f is Gorenstein.

The oriented fractional ideal $I_f(n-3) = I_f^{n-3}$ appears in the study of integral orbits. Before introducing the action of $\mathrm{SL}_n(\mathbb{Z})$, we first describe the elements in $V(\mathbb{Z})$ using a general theorem of Wood (see [29, Theorems 4.1 & 5.7], or [1, Theorem 16] and [2, Theorem 4] for the special cases $n = 2$ and $n = 3$):

Theorem 4.7 (Wood). *The elements of* $\mathrm{Sym}_2(\mathbb{Z}^n) \oplus \mathrm{Sym}_2(\mathbb{Z}^n)$ *having a given invariant binary n-ic form f with nonzero discriminant Δ and nonzero first coefficient f_0 are in bijection with the equivalence classes of pairs (I, α), where $I \subset L$ is a based fractional ideal of R_f, $\alpha \in L^\times$, $I^2 \subseteq \alpha I_f^{n-3}$ as fractional ideals, and $N(I)^2 = N(\alpha)N(I_f^{n-3}) = N(\alpha)/f_0^{n-3} \in \mathbb{Q}^\times$. Two pairs (I, α) and (I^*, α^*) are equivalent if there exists $\kappa \in L^\times$ such that $I^* = \kappa I$ and $\alpha^* = \kappa^2\alpha$.*

The way to recover a pair (A, B) of symmetric $n \times n$ matrices from a pair (I, α) above is by taking the coefficients of ζ_{n-1} and ζ_{n-2} in the image of the map

$$\frac{1}{\alpha}\times : I \times I \to I_f^{n-3} \tag{12}$$

in terms of the $\mathbb{Z}$-basis of I.

Next, note that the group $G(\mathbb{Z}) = \mathrm{SL}_n(\mathbb{Z})$ acts naturally on the set $V(\mathbb{Z}) = \mathrm{Sym}_2(\mathbb{Z}^n) \oplus \mathrm{Sym}_2(\mathbb{Z}^n)$. It also acts on the bases of the based fractional ideals I in the corresponding pairs (I, α), and preserves the norm and orientation. Thus, when considering $\mathrm{SL}_n(\mathbb{Z})$-orbits, we may drop the bases of I and view I simply as an oriented fractional ideal ideal. We thus obtain:

Corollary 4.8. *The orbits of* $\mathrm{SL}_n(\mathbb{Z})$ *on* $\mathrm{Sym}_2(\mathbb{Z}^n) \oplus \mathrm{Sym}_2(\mathbb{Z}^n)$ *having a given invariant binary n-ic form f with nonzero discriminant Δ and nonzero first coefficient f_0 are in bijection with equivalence classes of pairs (I, α), where $I \subset L$ is an oriented fractional ideal of R_f, $\alpha \in L^\times$, $I^2 \subseteq \alpha I_f^{n-3}$, and $N(I)^2 = N(\alpha)N(I_f^{n-3}) = N(\alpha)/f_0^{n-3}$. Two pairs (I, α) and (I^*, α^*) are* equivalent *if there exists $\kappa \in L^\times$ such that $I^* = \kappa I$ and $\alpha^* = \kappa^2\alpha$. The stabilizer in* $\mathrm{SL}_n(\mathbb{Z})$ *of a nondegenerate element in* $\mathrm{Sym}_2(\mathbb{Z}^n) \oplus \mathrm{Sym}_2(\mathbb{Z}^n)$ *having invariant binary form f is the finite elementary abelian 2-group $S^\times[2]_{N=1}$, where S is the endomorphism ring of I in L.*

We can specialize this result to the case when the order R_f is maximal in L (which occurs, for example, when the discriminant $\Delta(f)$ is squarefree). Then the set of oriented fractional ideals of R_f form an abelian group under multiplication, and the principal oriented ideals form a subgroup. The *oriented class group* C^* is then defined as the quotient of the group of invertible oriented ideals by the subgroup of principal oriented ideals. The elements of this group are called the invertible oriented ideal classes of R_f, and two oriented ideals (I, ε) and (I', ε') of R_f are in the same oriented ideal class if $(I', \varepsilon') = \kappa \cdot (I, \varepsilon)$ for some $\kappa \in L^\times$. Note that the oriented class group is isomorphic to the usual class group of R_f if there is an element of $R_f^\times$ with norm -1; otherwise, it is an extension of the usual class group by $\mathbb{Z}/2\mathbb{Z}$, where the generator of this $\mathbb{Z}/2\mathbb{Z}$ is given by the oriented ideal $(R_f, -1)$ of R_f. (In the case of a binary form with positive discriminant, when R_f is an order in a real number field, the oriented class group coincides with what is usually called the *narrow class group*).

When R_f is maximal, integral orbits (A, B) with invariant f will exist if and only if the class of the oriented ideal $I_f(n-3) = I_f^{n-3}$ is a square in the oriented ideal class group (this will certainly hold when n is odd). If the class of I_f^{n-3} is a square, we can find a pair (I, α) satisfying $I^2 = \alpha I_f^{n-3}$ and $N(I)^2 = N(\alpha)/f_0^{n-3}$. In this case, the set of orbits is finite and forms a principal homogeneous space for an elementary abelian 2-group that is an extension of the group of elements of order 2 in the oriented class group by the group $(R_f^\times/R_f^{\times 2})_{N=1}$. The number of distinct integral orbits with binary form $f(x, y)$ is given by the formula

$$2^{r_1+r_2-1}\#C^*[2]$$

where r_1 and r_2 are the number of real and complex places of L respectively and $C^*[2]$ is the subgroup of elements of order 2 in the oriented class group C^*.

We end with a comparison of the integral and rational orbits with a fixed invariant form f for the action of $G = \mathrm{SL}_n$ on $V = \mathrm{Sym}_2(n) \oplus \mathrm{Sym}_2(n)$. Let $f(x, y) = f_0x^n + f_1x^{n-1}y + \cdots + f_ny^n$ be an integral binary form of degree n with $\Delta(f) \neq 0$ and $f_0 \neq 0$. Write $f(x, 1) = f_0g(x)$ with $g(x) \in \mathbb{Q}[x]$ and let $L = \mathbb{Q}[x]/(g(x))$. Recall from §4.1 that the orbits $v = (A, B)$ of $\mathrm{SL}_n(\mathbb{Q})$ on $V(\mathbb{Q})$ with invariant f correspond bijectively to the equivalence classes of pairs (γ, t), with $\gamma \in L^\times$ and $t \in \mathbb{Q}^\times$ satisfying $t^2 = f_0N(\gamma)$. More precisely, the $\mathrm{SL}_n(\mathbb{Q})$-orbit of the bilinear form A is given by the pairing

$$\langle \lambda, \mu\rangle_\gamma = \mathrm{Trace}(\lambda\mu/\gamma g'(\theta)).$$

using the oriented basis $t(1 \wedge \theta \wedge \theta^2 \wedge \ldots \wedge \theta^{n-1})$ of $\wedge^n L$. It follows that $\langle \lambda, \mu\rangle_A$ is equal to the coefficient of θ^{n-1} in the expansion of the product $\lambda\mu/\gamma$ using this oriented basis.

On the other hand, an integral orbit (A, B) is given by the equivalence class of the pair (I, α) with $I^2 \subset \alpha I_f^{n-3}$ and $N(I)^2 = N(\alpha)/f_0^{n-3}$. For λ and μ in the oriented fractional ideal I, the bilinear form $\langle \lambda, \mu\rangle_A$ is equal to the coefficient of ζ_{n-1} in the expansion of the product $\lambda\mu/\alpha$ with respect to the natural basis of $I_f(n-3)$. Since $\zeta_{n-1} = f_0\theta^{n-1} + f_1\theta^{n-2} + \cdots + f_{n-2}\theta$ in L, we see that the corresponding rational orbit has parameters

$$\gamma = f_0 \alpha,$$
$$t = f_0^{n-1} N(I).$$

Similarly, the bilinear form $\langle \lambda, \mu \rangle_B$ is equal to the coefficient of ζ_{n-2} in the expansion of the product $\lambda\mu/\alpha$ with respect to the natural basis of $I_f(n-3)$. Note that we obtain Gram matrices for these two bilinear forms by using the basis of the ideal I that maps to the basis element

$$\begin{aligned} N(I)(1 \wedge \zeta_1 \wedge \zeta_2 \wedge \ldots \wedge \zeta_{n-1}) &= N(I) f_0^{n-1}(1 \wedge \theta \wedge \theta^2 \wedge \ldots \theta^{n-1}) \\ &= t(1 \wedge \theta \wedge \theta^2 \wedge \ldots \theta^{n-1}) \end{aligned}$$

of the top exterior power of I over $\mathbb{Z}$.

If we fix a rational orbit with integral form $f(x, y)$, then the parameters (γ, t) determine both α and $N(I)$ by the above formulae. The rational orbit has an integral representative if and only if one can find an oriented fractional ideal I for R_f satisfying $I^2 \subseteq \alpha I_f^{n-3}$ and $N(I) = N(\alpha)N(I_f^{n-3}) = N(\alpha)/f_0^{n-3}$. The distinct integral orbits in this rational orbit correspond to the different possible choices for the oriented fractional ideal I satisfying these two conditions. We note that there is at most one choice when the order R_f is maximal in L. In that case, the fractional ideal I is determined by the identity $I^2 = \alpha I_f^{n-3}$, and its orientation by the identity $N(I) = N(\alpha)/f_0^{n-3}$.

When n is odd, there is a canonical integral orbit with invariant binary n-ic form $f(x, y)$. This has parameters $(I, \alpha) = (I_f^{(n-3)/2}, 1)$. The corresponding rational orbit has parameters $(\gamma, t) = (f_0, f_0^{(n+1)/2})$.

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Hecke algebras with unequal parameters and Vogan's left cell invariants

Cédric Bonnafé and Meinolf Geck

To David Vogan on the occasion of his 60th birthday

Abstract In 1979, Vogan introduced a generalised τ-invariant for characterising primitive ideals in enveloping algebras. Via a known dictionary this translates to an invariant of left cells in the sense of Kazhdan and Lusztig. Although it is not a complete invariant, it is extremely useful in describing left cells. Here, we propose a general framework for defining such invariants which also applies to Hecke algebras with unequal parameters.

Key words: Coxeter groups, Hecke algebras, Kazhdan–Lusztig cells

MSC (2010): Primary 20C08; Secondary 20F55

1 Introduction

Let W be a finite Weyl group. Using the corresponding generic Iwahori–Hecke algebra and the "new" basis of this algebra introduced by Kazhdan and Lusztig [16], we obtain partitions of W into left, right, and two-sided cells. Analogous notions originally arose in the theory of primitive ideals in enveloping algebras; see Joseph [15]. This is one of the sources for the interest in knowing the cell partitions of W; there are also deep connections [19] with representations of reductive groups,

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singularities of Schubert cells, and the geometry of unipotent classes. Vogan [24], [25] introduced invariants of left cells which are computable in terms of certain combinatorially defined operators $T_{\alpha\beta}$, $S_{\alpha\beta}$ where α, β are adjacent simple roots of W. In the case where W is the symmetric group $\mathfrak{S}_n$, these invariants completely characterise the left cells; see [16, §5], [24, §6]. Although Vogan's invariants are not complete invariants in general, they have turned out to be extremely useful in describing left cells.

Now, the Kazhdan–Lusztig cell partitions are not only defined and interesting for finite Weyl groups, but also for affine Weyl groups and Coxeter groups in general; see, e.g., Lusztig [18], [20]. Furthermore, the original theory was extended by Lusztig [17] to allow for the possibility of attaching weights to the simple reflections. The original setting then corresponds to the case where all weights are equal to 1; we will refer to this case as the "equal parameter case". Our aim here is to propose analogues of Vogan's invariants which work in general, i.e., for arbitrary Coxeter groups and arbitrary (positive) weights.

In Sections 2 and 3 we briefly recall the basic setup concerning Iwahori–Hecke algebras, cells in the sense of Kazhdan and Lusztig, and the concept of induction of cells. In Section 4 we introduce the notion of *left cellular maps*; a fundamental example is given by the Kazhdan–Lusztig $*$-operations. In Section 5, we discuss the equal parameter case and Vogan's original definition of a generalised τ-invariant. As this definition relied on the theory of primitive ideals, it only applies to finite Weyl groups. In Theorem 5.2, we show that this works for arbitrary Coxeter groups satisfying a certain boundedness condition. (A similar result has also been proved by Shi [22, 4.2], but he uses a definition slightly different from Vogan's; our argument seems to be more direct.) In Sections 6 and 7, we develop an abstract setting for defining such invariants; this essentially relies on the concept of induction of cells and is inspired by Lusztig's method of *strings* [18, §10]. As a bi-product of our approach, we obtain that the $*$-operations also work in the unequal parameter case. We conclude by discussing examples and stating open problems.

Remark. In [4, Cor. 6.2], the first author implicitly assumed that the results on the Kazhdan–Lusztig $*$-operations [16, §4] also hold in the unequal parameter context — which was a serious mistake at the time. Corollary 6.4 below justifies *a posteriori* this assumption.

Notation. We fix a Coxeter system (W, S) and we denote by $\ell : W \to \mathbb{Z}_{\geqslant 0}$ the associated length function. We also fix a totally ordered abelian group $\mathscr{A}$. We use an exponential notation for the group algebra $A = \mathbb{Z}[\mathscr{A}]$:

$$A = \bigoplus_{a \in \mathscr{A}} \mathbb{Z}v^a \qquad \text{where} \qquad v^a v^{a'} = v^{a+a'} \text{ for all } a, a' \in \mathscr{A}.$$

We write $\mathscr{A}_{\leqslant 0} := \{\alpha \in \mathscr{A} \mid \alpha \leqslant 0\}$ and $A_{\leqslant 0} := \oplus_{a \in \mathscr{A}_{\leqslant 0}} \mathbb{Z}v^a$; the symbols $\mathscr{A}_{\geqslant 0}$, $A_{\geqslant 0}$ etc. have analogous meanings. We denote by $\bar{\ } : A \to A$ the involutive automorphism such that $\overline{v^a} = v^{-a}$ for all $a \in \mathscr{A}$.

2 Weight functions and cells

Let $p : W \to \mathscr{A}, w \mapsto p_w$, be a *weight function* in the sense of Lusztig [20], that is, we have $p_s = p_t$ whenever $s, t \in S$ are conjugate in W, and $p_w = p_{s_1} + \cdots + p_{s_k}$ if $w = s_1 \cdots s_k$ (with $s_i \in S$) is a reduced expression for $w \in W$. The original setup in [16] corresponds to the case where $\mathscr{A} = \mathbb{Z}$ and $p_s = 1$ for all $s \in S$; this will be called the "equal parameter case". We shall assume throughout that $p_s > 0$ for all $s \in S$. (There are standard techniques for reducing the general case to this case [3, §2].)

Let $\mathscr{H} = \mathscr{H}_A(W, S, p)$ be the corresponding generic Iwahori–Hecke algebra. This algebra is free over A with basis $(T_w)_{w\in W}$, and the multiplication is given by the rule

$$T_s T_w = \begin{cases} T_{sw} & \text{if } sw > w, \\ T_{sw} + (v^{p_s} - v^{-p_s})T_w & \text{if } sw < w, \end{cases}$$

where $s \in S$ and $w \in W$; here, $\leqslant$ denotes the Bruhat–Chevalley order on W.

Let $(C'_w)_{w\in W}$ be the "new" basis of $\mathscr{H}$ introduced in [16, (1.1.c)], [17, §2]. (These basis elements are denoted c_w in [20].) For any $x, y \in W$, we write

$$C'_x\, C'_y = \sum_{z\in W} h_{x,y,z}\, C'_z \qquad \text{where } h_{x,y,z} \in A \text{ for all } x, y, z \in W.$$

We have the following more explicit formula for $s \in S$, $y \in W$ (see [17, §6], [20, Chap. 6]):

$$C'_s\, C'_y = \begin{cases} (v^{p_s} + v^{-p_s})\, C'_y & \text{if } sy < y, \\ C'_{sy} + \displaystyle\sum_{z\in W\,:\, sz<z<y} M^s_{z,y} C'_z & \text{if } sy > y, \end{cases}$$

where $C'_s = T_s + v^{-p_s} T_1$ and $M^s_{z,y} = \overline{M}^s_{z,y} \in A$ is determined as in [17, §3].

As in [20, §8], we write $x \leftarrow_L y$ if there exists some $s \in S$ such that $h_{s,y,x} \neq 0$, that is, C'_x occurs with a nonzero coefficient in the expression of $C'_s\, C'_y$ in the C'-basis. The Kazhdan–Lusztig left pre-order $\leqslant_L$ is the transitive closure of $\leftarrow_L$. The equivalence relation associated with $\leqslant_L$ will be denoted by $\sim_L$ and the corresponding equivalence classes are called the *left cells* of W. Note that $\mathscr{H} C_w \subseteq \sum_y A C_y$ where the sum runs over all $y \in W$ with $y \leqslant_L w$.

Similarly, we can define a pre-order $\leqslant_R$ by considering multiplication by C'_s on the right in the defining relation. The equivalence relation associated with $\leqslant_R$ will be denoted by $\sim_R$ and the corresponding equivalence classes are called the *right cells* of W. We have

$$x \leqslant_R y \quad \Longleftrightarrow \quad x^{-1} \leqslant_L y^{-1}; \tag{1}$$

see [20, 5.6, 8.1]. Finally, we define a pre-order $\leqslant_{LR}$ by the condition that $x \leqslant_{LR} y$ if there exists a sequence $x = x_0, x_1, \ldots, x_k = y$ such that, for each $i \in \{1, \ldots, k\}$, we have $x_{i-1} \leqslant_L x_i$ or $x_{i-1} \leqslant_R x_i$. The equivalence relation associated with $\leqslant_{LR}$ will be denoted by $\sim_{LR}$ and the corresponding equivalence classes are called the *two-sided cells* of W.

Definition 2.1. A (non-empty) subset Γ of W is called *left-closed* if, for any $x, y \in \Gamma$, we have $\{z \in W \mid x \leqslant_L z \leqslant_L y\} \subseteq \Gamma$.

Note that any such subset is a union of left cells. A left cell itself is clearly left-closed with respect to $\leqslant_L$. It immediately follows from these definitions that, given any left-closed subset $\Gamma \subseteq W$, the A-submodules

$$\begin{aligned}\mathscr{I}_\Gamma &= \langle C'_w \mid w \leqslant_L z \text{ for some } z \in \Gamma\rangle_A,\\ \hat{\mathscr{I}}_\Gamma &= \langle C'_w \mid w \notin \Gamma, w \leqslant_L z \text{ for some } z \in \Gamma\rangle_A,\end{aligned}$$

are left ideals in $\mathscr{H}$. Hence we obtain an $\mathscr{H}$-module $[\Gamma]_A := \mathscr{I}_\Gamma / \hat{\mathscr{I}}_\Gamma$, which is free over A with basis given by $(e_x)_{x\in\Gamma}$, where e_x denotes the residue class of C'_x in $[\Gamma]_A$. The action of C'_w $(w \in W)$ is given by the formula

$$C'_w \cdot e_x = \sum_{y\in\Gamma} h_{w,x,y}\, e_y.$$

3 Cells and parabolic subgroups

A key tool in this work will be the process of *induction of cells*. Let $I \subseteq S$ and consider the parabolic subgroup $W_I \subseteq W$ generated by I. Then

$$X_I := \{w \in W \mid ws > w \text{ for all } s \in I\}$$

is the set of distinguished left coset representatives of W_I in W. The map $X_I \times W_I \to W$, $(x, u) \mapsto xu$, is a bijection and we have $\ell(xu) = \ell(x) + \ell(u)$ for all $x \in X_I$ and $u \in W_I$; see [13, §2.1]. Thus, given $w \in W$, we can write uniquely $w = xu$ where $x \in X_I$ and $u \in W_I$. In this case, we denote $\mathrm{pr}_I(w) := u$. Let $\leqslant_{L,I}$ and $\sim_{L,I}$ be respectively the pre-order and equivalence relations for W_I defined similarly as $\leqslant_L$ and $\sim_L$ are defined in W.

Theorem 3.1. *Let $I \subseteq S$. If $x, y \in W$ are such that $x \leqslant_L y$ (resp. $x \sim_L y$), then $\mathrm{pr}_I(x) \leqslant_{L,I} \mathrm{pr}_I(y)$ (resp. $\mathrm{pr}_I(x) \sim_{L,I} \mathrm{pr}_I(y)$). In particular, if Γ is a left cell of W_I, then $X_I\Gamma$ is a union of left cells of W.*

This was first proved by Barbasch–Vogan [1, Cor. 3.7] for finite Weyl groups in the equal parameter case (using the theory of primitive ideals); see [9] for the general case.

Example 3.2. Let Γ be a left-closed subset of W_I. Then the subset $X_I\Gamma$ of W is left-closed (see Theorem 3.1). Let $\mathscr{H}_I \subseteq \mathscr{H}$ be the parabolic subalgebra spanned by all T_w where $w \in W_I$. Then we obtain the $\mathscr{H}_I$-module $[\Gamma]_A$, with standard basis $(e_w)_{w\in\Gamma}$, and the $\mathscr{H}$-module $[X_I\Gamma]_A$, with standard basis $(e_{xw})_{x\in X_I, w\in\Gamma}$. By [10, 3.6], we have an isomorphism of $\mathscr{H}$-modules

$$[X_I\Gamma]_A \xrightarrow{\sim} \mathrm{Ind}_I^S([\Gamma]_A), \qquad e_{yv} \mapsto \sum_{x\in X_I, w\in\Gamma} p^*_{xu,yv}\left(T_x \otimes e_u\right),$$

where $p^*_{xu,yv} \in A$ are the *relative* Kazhdan–Lusztig polynomials of [9, Prop. 3.3] and, for any $\mathscr{H}_I$-module V, we denote by $\mathrm{Ind}_I^S(V) := \mathscr{H} \otimes_{\mathscr{H}_I} V$ the *induced module*, with basis $(T_x \otimes e_w)_{x\in X_I, w\in\Gamma}$. (In [10, §3], it is not stated explicitly that $\Gamma = X_I\Gamma$ is left-closed, but this condition is used implicitly in the discussion there.)

A first invariant of left cells is given as follows. For any $w \in W$, we denote by $\mathscr{R}(w) := \{s \in S \mid ws < w\}$ the *right descent set* of w (or τ*-invariant* of w in the language of primitive ideals; see [1]). The next result has been proved in [16, 2.4] (for the equal parameter case) and in [20, 8.6] (for the unequal parameter case).

Proposition 3.3 (Kazhdan–Lusztig, Lusztig). *Let $x, y \in W$.*

(a) If $x \leqslant_L y$, then $\mathscr{R}(y) \subseteq \mathscr{R}(x)$.
(b) If $x \sim_L y$, then $\mathscr{R}(x) = \mathscr{R}(y)$.
(c) For any $I \subseteq S$, the set $\{w \in W \mid \mathscr{R}(w) = I\}$ is a union of left cells of W.

We show how this can be deduced from Theorem 3.1. First, note that (b) and (c) easily follow from (a), so we only need to prove (a). Let $x, y \in W$ be such that $x \leqslant_L y$. Let $s \in \mathscr{R}(y)$ and set $I = \{s\}$. Then $\mathrm{pr}_I(y) = s$ and so $\mathrm{pr}_I(x) \leqslant_{L,I} \mathrm{pr}_I(y) = s \in W_I = \{1, s\}$. Since $p_s > 0$, the definitions immediately show that $s \leqslant_{I,L} 1$ but $s \not\leqslant_{L,I} 1$. Hence, we must have $\mathrm{pr}_I(x) = s$ and so $s \in \mathscr{R}(x)$. Thus we have $\mathscr{R}(y) \subseteq \mathscr{R}(x)$.

4 Left cellular maps

Definition 4.1. A map $\delta : W \to W$ is called *left cellular* if the following conditions are satisfied for every left cell $\Gamma \subseteq W$ (with respect to the given weights $\{p_s \mid s \in S\}$):

(A1) $\delta(\Gamma)$ also is a left cell.
(A2) The map δ induces an $\mathscr{H}$-module isomorphism $[\Gamma]_A \cong [\delta(\Gamma)]_A$.

A prototype of such a map is given by the Kazhdan–Lusztig $*$-operations. We briefly recall how this works. For any $s, t \in S$ such that $st \neq ts$, we set

$$\mathscr{D}_R(s,t) := \{w \in W \mid \mathscr{R}(w) \cap \{s,t\} \text{ has exactly one element}\}$$

and, for any $w \in \mathscr{D}_R(s,t)$, we set $\mathfrak{T}_{s,t}(w) := \{ws, wt\} \cap \mathscr{D}_R(s,t)$. (See [16, §4], [24, §3].) Note that $\mathfrak{T}_{s,t}(w)$ consists of one or two elements; in order to have uniform notation, we consider $\mathfrak{T}_{s,t}(w)$ as a multiset with two identical elements if $\{ws, wt\} \cap \mathscr{D}_R(s,t)$ consists of only one element.

If st has order 3, then the intersection $\{ws, wt\} \cap \mathscr{D}_R(s,t)$ consists of only one element which will be denoted by w^*. Thus, we have $\mathfrak{T}_{s,t}(w) = \{w^*, w^*\}$ in this case. With this notation, we can now state:

Proposition 4.2 (Kazhdan–Lusztig $*$-operations [16, §4]). *Assume that we are in the equal parameter case and that st has order 3. Then we obtain a left cellular map $\delta\colon W \to W$ by setting*

$$\delta(w) = \begin{cases} w^* & \text{if } w \in \mathscr{D}_R(s,t), \\ w & \text{otherwise}. \end{cases}$$

In particular, if $\Gamma \subseteq \mathscr{D}_R(s,t)$ is a left cell, then so is $\Gamma^* := \{w^* \mid w \in \Gamma\}$.

(In Corollary 6.4 below, we extend this to the unequal parameter case.)

If st has order $\geqslant 4$, then the set $\mathfrak{T}_{s,t}(w)$ may contain two distinct elements. In order to obtain a single-valued operator, Vogan [25, §4] (for the case $m = 4$; see also McGovern [21, §4]) and Lusztig [18, §10] (for any $m \geqslant 4$) propose an alternative construction, as follows.

Remark 4.3. Let $s, t \in S$ be such that st has any finite order $m \geqslant 3$. Let $W_{s,t} = \langle s, t\rangle$, a dihedral group of order $2m$. For any $w \in W$, the coset $wW_{s,t}$ can be partitioned into four subsets: one consists of the unique element x of minimal length, one consists of the unique element of maximal length, one consists of the $(m-1)$ elements $xs, xst, xsts, \ldots$ and one consists of the $(m-1)$ elements $xt, xts, xtst, \ldots$. Following Lusztig [18, 10.2], the last two subsets (ordered as above) are called *strings*. (Note that Lusztig considers the coset $W_{s,t}w$ but, by taking inverses, the two versions are clearly equivalent.) Thus, if $w \in \mathscr{D}_R(s,t)$, then w belongs to a unique string which we denote by λ_w; we certainly have $\mathfrak{T}_{s,t}(w) \subseteq \lambda_w \subseteq \mathscr{D}_R(s,t)$ for all $w \in \mathscr{D}_R(s,t)$.

We define an involution $\mathscr{D}_R(s,t) \to \mathscr{D}_R(s,t)$, $w \mapsto \tilde{w}$, as follows. Let $w \in \mathscr{D}_R(s,t)$ and $i \in \{1, \ldots, m-1\}$ be the index such that w is the ith element of the string λ_w. Then $\tilde{w}$ is defined to be the $(m-i)$th element of λ_w. Note that, if $m = 3$, then $\tilde{w} = w^*$, with w^* as in Proposition 4.2.

Let us write $T_x T_y = \sum_{z \in W} f_{x,y,z} T_z$ where $f_{x,y,z} \in A$ for all $x, y, z \in W$. Following [20, 13.2], we say that $\mathscr{H}$ is *bounded* if there exists some positive $N \in \mathscr{A}$ such that $v^{-N} f_{x,y,z} \in A_{\leqslant 0}$ for all $x, y, z \in W$. We can now state:

Proposition 4.4 (Lusztig [18, 10.7]). *Assume that we are in the equal parameter case and that $\mathscr{H}$ is bounded. If $\Gamma \subseteq \mathscr{D}_R(s,t)$ is a left cell, then so is*

$$\tilde{\Gamma} := \{\tilde{w} \mid w \in \Gamma\}.$$

(It is assumed in [18, 10.7] that W is crystallographic, but this assumption is now superfluous thanks to Elias–Williamson [6]. The boundedness assumption is obviously satisfied for all finite Coxeter groups. It also holds, for example, for affine Weyl groups; see the remarks following [20, 13.4].)

In Corollary 6.4 below, we shall show that $w \mapsto \tilde{w}$ also gives rise to a left cellular map and that this works without any assumption, as long as $p_s = p_t$.

5 Vogan's invariants

Hypothesis. *Throughout this section, and only in this section, we assume that we are in the equal parameter case.*

We recall the following definition.

Definition 5.1 (Vogan [24, 3.10, 3.12]). For $n \geqslant 0$, we define an equivalence relation $\approx_n$ on W inductively as follows. Let $x, y \in W$.

- For $n = 0$, we write $x \approx_0 y$ if $\mathscr{R}(x) = \mathscr{R}(y)$.
- For $n \geqslant 1$, we write $x \approx_n y$ if $x \approx_{n-1} y$ and if, for any $s, t \in S$ such that $x, y \in \mathscr{D}_R(s,t)$ (where st has order 3 or 4), the following holds: if $\mathfrak{T}_{s,t}(x) = \{x_1, x_2\}$ and $\mathfrak{T}_{s,t}(y) = \{y_1, y_2\}$, then either $x_1 \approx_{n-1} y_1$, $x_2 \approx_{n-1} y_2$ or $x_1 \approx_{n-1} y_2$, $x_2 \approx_{n-1} y_1$.

If $x \approx_n y$ for all $n \geqslant 0$, then x, y are said to have the same *generalised τ-invariant*.

The following result was originally formulated and proved for finite Weyl groups by Vogan [24, §3], in the language of primitive ideals in enveloping algebras. It then follows for cells as defined in Section 2 using a known dictionary (see, e.g., Barbasch–Vogan [1, §2]). The proof in general relies on Proposition 4.2 and results on *strings* as defined in Remark 4.3.

Theorem 5.2 (Kazhdan–Lusztig [16, §4], Lusztig [18, §10], Vogan [24, §3]). *Assume that $\mathscr{H}$ is bounded and recall that we are in the equal parameter case. Let Γ be a left cell of W. Then all elements in Γ have the same generalised τ-invariant.*

Proof. We prove by induction on n that if $y, w \in W$ are such that $y \sim_L w$, then $y \approx_n w$. For $n = 0$, this holds by Propositon 3.3. Now let $n > 0$. By induction, we already know that $y \approx_{n-1} w$. Then it remains to consider $s, t \in S$ such that $st \neq ts$ and $y, w \in \mathscr{D}_R(s,t)$. If st has order 3, then $\mathfrak{T}_{s,t}(y) = \{y^*, y^*\}$ and $\mathfrak{T}_{s,t}(w) = \{w^*, w^*\}$; furthermore, by Proposition 4.2, we have $y^* \sim_L w^*$ and so $y^* \approx_{n-1} w^*$, by induction. Now assume that st has order 4. In this case, the argument is more complicated (as it is also in the setting of [24, §3].) Let $I = \{s, t\}$

and Γ be the left cell containing y, w. Since all elements in Γ have the same right descent set (by Proposition 3.3), we can choose the notation such that $xs < x$ and $xt > x$ for all $x \in \Gamma$. Then, for $x \in \Gamma$, we have $x = x's$, $x = x'ts$ or $x = x'sts$ where $x' \in X_I$. This yields that

$$(\dagger) \qquad \mathfrak{T}_{s,t}(x) = \begin{cases} \{x'st, x'st\} & \text{if } x = x's, \\ \{x't, x'tst\} & \text{if } x = x'ts, \\ \{x'st, x'st\} & \text{if } x = x'sts. \end{cases}$$

We now consider the string λ_x and distinguish two cases.

Case 1. Assume that there exists some $x \in \Gamma$ such that $x = x's$ or $x = x'sts$. Then $\lambda_x = (x's, x'st, x'sts)$ and so the set $\Gamma^* := \big(\bigcup_{w \in \Gamma} \lambda_w\big) \setminus \Gamma$ contains elements with different right descent sets. On the other hand, by [18, Prop. 10.7], Γ^* is the union of at most two left cells. (Again, the assumption in [18, Prop. 10.7] that W is crystallographic is now superfluous thanks to [6].) We conclude that $\Gamma^* = \Gamma_1 \cup \Gamma_2$ where Γ_1, Γ_2 are left cells such that:

- all elements in Γ_1 have s in their right descent set, but not t;
- all elements in Γ_2 have t in their right descent set, but not s.

Now consider $y, w \in \Gamma$; we write $\mathfrak{T}_{s,t}(y) = \{y_1, y_2\} \subseteq \Gamma^*$ and $\mathfrak{T}_{s,t}(w) = \{w_1, w_2\} \subseteq \Gamma^*$. By ($\dagger$), all the elements y_1, y_1, w_1, w_2 belong to Γ_2. In particular, $y_1 \sim_L w_1$, $y_2 \sim_L w_2$ and so, by induction, $y_1 \approx_{n-1} w_1$, $y_2 \approx_{n-1} w_2$.

Case 2. We are not in Case 1, that is, all elements $x \in \Gamma$ have the form $x = x'ts$ where $x' \in X_I$. Then $\lambda_x = (x't, x'ts, x'tst)$ for each $x \in \Gamma$. Let us label the elements in such a string as x_1, x_2, x_3. Then $x = x_2$ and $\mathfrak{T}_{s,t}(x) = \{x't, x'tst\} = \{x_1, x_3\}$.

Now consider $y, w \in \Gamma$. There is a chain of elements which connect y to w via the elementary relations $\leftarrow_L$, and vice versa. Assume first that y, w are directly connected as $y \leftarrow_L w$. Using the labelling $y = y_2$, $w = w_2$ and the notation of [18, 10.4], this means that $a_{22} \neq 0$. Hence, the identities "$a_{11} = a_{33}$", "$a_{13} = a_{31}$", "$a_{22} = a_{11} + a_{13}$" in [18, 10.4.2] imply that

$$(y_1 \leftarrow_L w_1 \text{ and } y_3 \leftarrow_L w_3) \quad \text{or} \quad (y_1 \leftarrow_L w_3 \text{ and } y_3 \leftarrow_L w_1).$$

Now, in general, there is a sequence of elements $y = y^{(0)}, y^{(1)}, \ldots, y^{(k)} = w$ in Γ such that $y^{(i-1)} \leftarrow_L y^{(i)}$ for $1 \leqslant i \leqslant k$. At each step, the elements in the strings corresponding to these elements are related as above. Combining these steps, one easily sees that

$$(y_1 \leqslant_L w_1 \text{ and } y_3 \leqslant_L w_3) \quad \text{or} \quad (y_1 \leqslant_L w_3 \text{ and } y_3 \leqslant_L w_1).$$

(See also [22, Prop. 4.6].) Now, all elements in a string belong to the same right cell (see [18, 10.5]); in particular, all the elements y_i, w_j belong to the same two-sided cell. Hence, [18, Cor. 6.3] implies that either $y_1 \sim_L w_1$, $y_3 \sim_L w_3$ or $y_1 \sim_L w_3$, $y_3 \sim_L w_1$. (Again, the assumption in [18, Cor. 6.3] that W is crystallographic is now superfluous thanks to [6].) Consequently, by induction, we have either $y_1 \approx_{n-1} w_1$, $y_3 \approx_{n-1} w_3$ or $y_1 \approx_{n-1} w_3$, $y_3 \approx_{n-1} w_1$. □

One of the most striking results about this invariant has been obtained by Garfinkle [8, Theorem 3.5.9]: two elements of a Weyl group of type B_n belong to the same left cell (equal parameter case) if and only if the elements have the same generalised τ-invariant. This fails in general; a counterexample is given by W of type D_n for $n \geqslant 6$ (as mentioned in the introduction of [7]).

Remark 5.3. Vogan [25, §4] also proposed the following modification of the above invariant. Let $s, t \in S$ be such that st has finite order $m \geqslant 3$. Let us set $\tilde{\mathfrak{T}}_{s,t}(w) := \{\tilde{w}\}$ for any $w \in \mathscr{D}_R(s,t)$, with $\tilde{w}$ as in Remark 4.3. Then we obtain a new invariant by exactly the same procedure as in Definition 5.1, but using $\tilde{\mathfrak{T}}_{s,t}$ instead of $\mathfrak{T}_{s,t}$ and allowing any $s, t \in S$ such that st has finite order $\geqslant 3$. (Note that Vogan only considered the case where $m = 4$, but then Lusztig's method of strings shows how to deal with the general case.) In any case, this is the model for our more general construction of invariants below.

6 Induction of left cellular maps

We return to the general setting of Section 2, where $\{p_s \mid s \in S\}$ are any positive weights for W.

Definition 6.1. A pair (I, δ) consisting of a subset $I \subseteq S$ and a left cellular map $\delta \colon W_I \to W_I$ is called *KL-admissible*. We recall that this means that the following conditions are satisfied for every left cell $\Gamma \subseteq W_I$ (with respect to the weights $\{p_s \mid s \in I\}$):

(A1) $\delta(\Gamma)$ also is a left cell.
(A2) The map δ induces an $\mathscr{H}_I$-module isomorphism $[\Gamma]_A \cong [\delta(\Gamma)]_A$.

We say that (I, δ) is *strongly KL-admissible* if, in addition to (A1) and (A2), the following condition is satisfied:

(A3) We have $u \sim_{R,I} \delta(u)$ for all $u \in W_I$.

If $I \subseteq S$ and if $\delta : W_I \to W_I$ is a map, we obtain a map $\delta^L : W \to W$ by

$$\delta^L(xw) = x\delta(w) \qquad \text{for all } x \in X_I \text{ and } w \in W_I.$$

The map δ^L will be called the *left extension* of δ to W.

Theorem 6.2. *Let (I, δ) be a KL-admissible pair. Then the following hold.*

(a) The left extension $\delta^L : W \to W$ is a left cellular map for W.
(b) If (I, δ) is strongly admissible, then we have $w \sim_R \delta^L(w)$ for all $w \in W$.

Proof. (a) By Theorem 3.1, there is a left cell Γ' of W_I such that $\Gamma \subseteq X_I\Gamma'$. By condition (A1) in Definition 6.1, the set $\Gamma'_1 := \delta(\Gamma')$ is also a left cell of W_I and, by condition (A2), the map δ induces an $\mathscr{H}_I$-module isomorphism $[\Gamma']_A \cong [\Gamma'_1]_A$. By Example 3.2, the subsets $X_I\Gamma'$ and $X_I\Gamma'_1$ of W are left-closed and, hence, we have corresponding $\mathscr{H}$-modules $[X_I\Gamma']_A$ and $[X_I\Gamma'_1]_A$. These two $\mathscr{H}$-modules are isomorphic to the induced modules $\mathrm{Ind}_I^S([\Gamma'])$ and $\mathrm{Ind}_I^S([\Gamma'_1])$, respectively, where explicit isomorphisms are given by the formula in Example 3.2. Now, by [10, Lemma 3.8], we have

$$p^*_{xu,yv} = p^*_{xu_1,yv_1} \qquad \text{for all } x,y \in X_I \text{ and } u,v \in \Gamma',$$

where we set $u_1 = \delta(u)$ and $v_1 = \delta(v)$ for $u,v \in \Gamma'$. By [10, Prop. 3.9], this implies that δ^L maps the partition of $X_I\Gamma'$ into left cells of W onto the analogous partition of $X_I\Gamma'_1$. In particular, since $\Gamma \subseteq X_I\Gamma'$, the set $\delta^L(\Gamma) \subseteq X_I\Gamma'_1$ is a left cell of W; furthermore, [10, Prop. 3.9] also shows that δ^L induces an $\mathscr{H}$-module isomorphism $[\Gamma]_A \cong [\delta^L(\Gamma)]_A$.

(b) Since condition (A3) in Definition 6.1 is assumed to hold, this is just a restatement of [20, Prop. 9.11(b)]. □

We will now give examples in which $|I| = 2$. Let us first fix some notation. If s, $t \in S$ are such that $s \neq t$ and st has finite order, let $w_{s,t}$ denote the longest element of $W_{s,t} = \langle s,t\rangle$ and let

$$\Gamma_s^{s,t} = \{w \in W_{s,t} \mid \ell(ws) < \ell(w) \text{ and } \ell(wt) > \ell(w)\},$$
$$\Gamma_t^{s,t} = \{w \in W_{s,t} \mid \ell(ws) > \ell(w) \text{ and } \ell(wt) < \ell(w)\}.$$

Example 6.3 (Dihedral groups with equal parameters). Let $s,t \in S$ be such that $s \neq t$, $p_s = p_t$ and st has finite order. It follows from [20, §8.7] that $\{1\}$, $\{w_{s,t}\}$, $\Gamma_s^{s,t}$ and $\Gamma_t^{s,t}$ are the left cells of $W_{s,t}$. Let $\sigma_{s,t}$ be the unique group automorphism of $W_{s,t}$ which exchanges s and t. Now, let $\delta_{s,t}$ denote the map $W_{s,t} \mapsto W_{s,t}$ defined by

$$\delta_{s,t}(w) = \begin{cases} w & \text{if } w \in \{1, w_{s,t}\},\\ \sigma_{s,t}(w)w_{s,t} & \text{otherwise.}\end{cases}$$

Then, by [20, Lemma 7.2 and Prop. 7.3], the pair $(\{s,t\}, \delta_{s,t})$ is strongly KL-admissible. Therefore, by Theorem 6.2,

$$\delta_{s,t}^L \colon W \to W \text{ is a left cellular map.}$$

In particular, this means that

$$x \sim_L y \textit{ if and only if } \delta_{s,t}^L(x) \sim_L \delta_{s,t}^L(y) \tag{2}$$

for all $x, y \in W$. Note also the following facts:

- If st has odd order, then $\delta_{s,t}$ exchanges the left cells $\Gamma_s^{s,t}$ and $\Gamma_t^{s,t}$.
- If st has even order, then $\delta_{s,t}$ stabilizes the left cells $\Gamma_s^{s,t}$ and $\Gamma_t^{s,t}$.

For example, if st has order 3, then $\Gamma_s^{s,t} = \{s, ts\}$ and $\Gamma_t^{s,t} = \{t, st\}$; furthermore, $\delta_{s,t}(s) = st$ and $\delta_{s,t}(ts) = t$. The matrix representation afforded by $[\Gamma_s^{s,t}]_A$ with respect to the basis (e_s, e_{ts}) is given by

$$C'_s \mapsto \begin{bmatrix} v^{p_s} + v^{-p_s} & 1 \\ 0 & 0 \end{bmatrix}, \qquad C'_t \mapsto \begin{bmatrix} 0 & 0 \\ 1 & v^{p_t} + v^{-p_t} \end{bmatrix} \qquad (p_s = p_t).$$

The fact that $\delta_{s,t}$ is left cellular just means that we obtain exactly the same matrices when we consider the matrix representation afforded by $[\Gamma_{s,t}^t]_A$ with respect to the basis (e_{st}, e_t).

Let us explicitly relate the above discussion to the $*$-operations in Proposition 4.2 and the extension in Proposition 4.4. In particular, this yields new proofs of these two propositions and shows that they also hold in the unequal parameter case, without any further assumptions, as long as $p_s = p_t$. (Partial results in this direction are obtained in [23, Cor. 3.5(4)].)

Corollary 6.4. *Let $s, t \in S$ be such that st has finite order $\geqslant 3$ and assume that $p_s = p_t$. Then, with the notation in Remark 4.3, we obtain a left cellular map $\delta\colon W \to W$ by setting*

$$\delta(w) = \begin{cases} \tilde{w} & \textit{if } w \in \mathscr{D}_R(s,t), \\ w & \textit{otherwise}. \end{cases}$$

(If st has order 3, then this coincides with the map defined in Proposition 4.2.)

Proof. Just note that if $w \in \mathscr{D}_R(s,t)$, then $\delta_{s,t}^L(w) = \tilde{w}$. Thus, the assertion simply is a restatement of the results in Example 6.3. Furthermore, if st has order 3, then $\tilde{w} = w^*$ for all $w \in \mathscr{D}_R(s,t)$, as noted in Remark 4.3. □

Example 6.5 (Dihedral groups with unequal parameters). Let $s, t \in S$ be such that st has finite *even* order $\geqslant 4$ and that $p_s < p_t$. Then it follows from [20, §8.7] that $\{1\}$, $\{w_{s,t}\}$, $\{s\}$, $\{w_{s,t}s\}$, $\Gamma_s^{s,t} \setminus \{s\}$ and $\Gamma_t^{s,t} \setminus \{w_{s,t}s\}$ are the left cells of $W_{s,t}$. Now, let $\delta_{s,t}$ denote the map $W_{s,t} \mapsto W_{s,t}$ defined by

$$\delta_{s,t}^{<}(w) = \begin{cases} w & \text{if } w \in \{1, w_{s,t}, s, w_{s,t}s\}, \\ ws & \text{otherwise}. \end{cases}$$

Then, again by [20, Lemma 7.5 and Prop. 7.6] (or [13, Exc. 11.4]), the pair $(\{s,t\}, \delta_{s,t}^{<})$ is strongly KL-admissible. Therefore, again by Theorem 6.2,

$$\delta_{s,t}^{<,L}\colon W \to W \text{ is a left cellular map.}$$

In particular, this means that

$$x \sim_L y \textit{ if and only if } \delta_{s,t}^{<,L}(x) \sim_L \delta_{s,t}^{<,L}(y) \tag{3}$$

for all $x, y \in W$. Note also that $\delta_{s,t}^{<}$ exchanges the left cells $\Gamma_s^{s,t} \setminus \{s\}$ and $\Gamma_t^{s,t} \setminus \{w_{s,t}s\}$ while it stabilizes all other left cells in $W_{s,t}$.

For example, if st has order 4, then $\Gamma_1 := \Gamma_s^{s,t} \setminus \{s\} = \{ts, sts\}$ and $\Gamma_2 := \Gamma_t^{s,t} \setminus \{w_{s,t}s\} = \{t, st\}$; furthermore, $\delta_{s,t}^{<}(ts) = t$ and $\delta_{s,t}^{<}(sts) = st$. As before, the fact that $\delta_{s,t}^{<}$ is left cellular just means that the matrix representation afforded by $[\Gamma_1]_A$ with respect to the basis (e_{ts}, e_{sts}) is exactly the same as the matrix representation afforded by $[\Gamma_2]_A$ with respect to the basis (e_t, e_{st}).

The next example shows that left extensions from dihedral subgroups are, in general, not enough to describe all left cellular maps.

Example 6.6. Let W be a Coxeter group of type B_r ($r \geqslant 2$), with diagram and weight function as follows:

B_r 

$b > (r-1)a > 0.$

This is the *asymptotic case* studied by Iancu and the first-named author [2], [5]. In this case, the left, right and two-sided cells are described in terms of a Robinson–Schensted correspondence for bi-tableaux. Using results from [2], [5], it is shown in [10, Theorem 6.3] that the following hold:

(a) If Γ_1 and Γ_2 are two left cells contained in the same two-sided cell, then there exists a bijection $\delta : \Gamma_1 \xrightarrow{\sim} \Gamma_2$ which induces an isomorphism of $\mathscr{H}$-modules $[\Gamma_1]_A \xrightarrow{\sim} [\Gamma_2]_A$.
(b) The bijection δ in (a) is uniquely determined by the condition that $w, \delta(w)$ lie in the same right cell.

However, one can check that for $r \in \{3, 4, 5\}$, the map δ does not always arise from a left extension of a suitable left cellular map of a dihedral subgroup of W. It is probable that this observation holds for any $r \geqslant 3$.

Example 6.7. Let W be an affine Weyl group and let W_0 be the finite Weyl group associated with W. Then there is a well-defined "lowest" two-sided cell, which consists of precisely $|W_0|$ left cells; see Guilhot [14] and the references there. It is likely that these $|W_0|$ left cells are all related by suitable left cellular maps.

7 An extension of the generalised τ-invariant

Notation. *We fix in this section a set Δ of KL-admissible pairs, as well as a surjective map $\rho : W \to E$ (where E is a fixed set) such that the fibers of ρ are unions of left cells. We then denote by $\mathscr{V}_\Delta$ the group of permutations of W generated by the family $(\delta^L)_{(I,\delta)\in\Delta}$.*

Note that giving a surjective map ρ as above is equivalent to giving an equivalence relation on W which is coarser than $\sim_L$.

Then, each $w \in W$ defines a map $\tau_w^{\Delta,\rho} : \mathscr{V}_\Delta \longrightarrow E$ as follows:

$$\tau_w^{\Delta,\rho}(\sigma) = \rho(\sigma(w)) \qquad \text{for all } \sigma \in \mathscr{V}_\Delta.$$

Definition 7.1. Let $x, y \in W$. We say that x and y have the same $\tau^{\Delta,\rho}$-*invariant* if $\tau_x^{\Delta,\rho} = \tau_y^{\Delta,\rho}$ (as maps from $\mathscr{V}_\Delta$ to E). The equivalence classes for this relation are called the *left Vogan* $(\boldsymbol{\Delta}, \boldsymbol{\rho})$-*classes*.

An immediate consequence of Theorem 6.2 is the following:

Theorem 7.2. *Let $x, y \in W$ be such that $x \sim_L y$. Then x and y have the same $\tau^{\Delta,\rho}$-invariant.*

Remark 7.3. There is an equivalent formulation of Definition 7.1 which is more in the spirit of Vogan's Definition 5.1. We define by induction on n a family of equivalence relations $\approx_n^{\Delta,\rho}$ on W as follows. Let $x, y \in W$.

- For $n = 0$, we write $x \approx_0^{\Delta,\rho} y$ if $\rho(x) = \rho(y)$.
- For $n \geqslant 1$, we write $x \approx_n^{\Delta,\rho} y$ if $x \approx_{n-1}^{\Delta,\rho} y$ and $\delta^L(x) \approx_{n-1}^{\Delta,\rho} \delta^L(y)$ for all $(I, \delta) \in \Delta$.

Note that the relation $\approx_n^{\Delta,\rho}$ is finer than $\approx_{n-1}^{\Delta,\rho}$. It follows from the definition that x, y have the same $\tau^{\Delta,\rho}$-invariant if and only if $x \approx_n^{\Delta,\rho} y$ for all $n \geqslant 0$.

This inductive definition is less easy to write than Definition 7.1, but it is more efficient for computational purpose. Indeed, if one finds an n_0 such that the relations $\approx_{n_0}^{\Delta,\rho}$ and $\approx_{n_0+1}^{\Delta,\rho}$ coincide, then x and y have the same $\tau^{\Delta,\rho}$-invariant precisely when $x \approx_{n_0}^{\Delta,\rho} y$. Note that such an n_0 always exists if W is finite. Also, even in small Coxeter groups, the group $\mathscr{V}_\Delta$ can become enormous (see Example 7.6 below) while n_0 is reasonably small and the relation $\approx_{n_0}^{\Delta,\rho}$ can be computed quickly.

Example 7.4 (Enhanced right descent set). One could take for ρ the map $\mathscr{R} : W \to \mathscr{P}(S)$ (power set of S); see Proposition 3.3. Assuming that we are in the equal parameter case, we then obtain exactly the invariant in Remark 5.3. In the unequal parameter case, we can somewhat refine this, as follows. Let

$$S^p = S \cup \{sts \mid s, t \in S \text{ such that } p_s < p_t\}$$

and, for $w \in W$, let

$$\mathscr{R}^p(w) = \{s \in S^p \mid \ell(ws) < \ell(w)\} \subseteq S^p.$$

Then it follows from the description of left cells of $W_{s,t}$ in Example 6.5 and from Theorem 3.1 (by using the same argument as for the proof of Proposition 3.3 given in §3) that

$$\textit{if } x \leqslant_L y, \textit{ then } \mathscr{R}^p(y) \subseteq \mathscr{R}^p(x).$$

In particular,

$$\text{if } x \sim_L y, \text{ then } \mathscr{R}^p(x) = \mathscr{R}^p(y).$$

So one could take for ρ the map $\mathscr{R}^p : W \to \mathscr{P}(S^p)$.

Let Δ_2 be the set of all pairs (I, δ) such that $I = \{s, t\}$ with $s \neq t$ and $p_s \leqslant p_t$; furthermore, if $p_s = p_t$, then $\delta = \delta_{s,t}$ (as defined in Example 6.3) while, if $p_s < p_t$, then $\delta = \delta_{s,t}^{<}$ (as defined in Example 6.5). Then the pairs in Δ_2 are all strongly KL-admissible. With the notation in Example 7.4, we propose the following conjecture:

Conjecture 7.5. Let $x, y \in W$. Then $x \sim_L y$ if and only if $x \sim_{LR} y$ and x, y have the same $\tau^{\Delta_2, \mathscr{R}^p}$-invariant.

If W is finite and we are in the equal parameter case, then Conjecture 7.5 is known to hold except possibly in type B_n, D_n; see the remarks at the end of [12, §6]. We have checked that the conjecture also holds for F_4, B_n ($n \leqslant 7$) and all possible weights, using PyCox [11].

By considering collections Δ with subsets $I \subseteq S$ of size larger than 2, one can obtain further refinements of the above invariants. In particular, it is likely that the results of [2], [5] can be interpreted in terms of generalised $\tau^{\Delta,\rho}$-invariants for suitable Δ, ρ. This will be discussed elsewhere.

Example 7.6. Let (W, S) be of type H_4. Then it can be checked by using computer computations in GAP that

$$|\mathscr{V}_{\Delta_2}| = 2^{40} \cdot 3^{20} \cdot 5^8 \cdot 7^4 \cdot 11^2.$$

On the other hand, the computation of left Vogan $(\Delta_2, \mathscr{R}^p)$-classes using the alternative definition given in Remark 7.3 takes only a few minutes on a standard computer.

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The smooth loci of spiral Schubert varieties of type $\widetilde{A}_2$

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To David Vogan for his 60th birthday, with thanks

Abstract Spiral Schubert varieties are conjecturally the only Schubert varieties in type $\widetilde{A}_2$ for which rational smoothness at a torus-fixed point is not detected by the number of torus-invariant curves passing through that point. In this paper we determine the locus of smooth points of a spiral Schubert variety of type $\widetilde{A}_2$. This continues the study begun in [7], where the locus of rationally smooth points was determined. The main result describes the smooth locus in terms of the action of the Weyl group on $\mathbb{R}^2$; using this result, we identify the maximal singular points of these varieties. We make key use of the results of [7] relating the Bruhat order to the Weyl group action on $\mathbb{R}^2$.

Key words: Schubert variety, spiral, Weyl group, Bruhat order, smoothness

MSC (2010): Primary 14M15; Secondary 14N15

1 Introduction

This paper determines the locus of smooth points of spiral Schubert varieties of type $\widetilde{A}_2$. Although this is a special case of a general problem, there are several reasons to study it. First, the spiral Schubert varieties form a distinguished class in type

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$\widetilde{A}_2$: computer evidence indicates that they are the only Schubert varieties in $\widetilde{A}_2$ for which rational smoothness at a torus-fixed point is not detected by the number of torus-invariant curves passing through that point. Second, our results form a further application of the results of [7] relating the Bruhat order to the action of the Weyl group W on the plane $\mathbb{R}^2$. In [7], this relationship was used to describe the set of torus-fixed points in a spiral Schubert variety, as well as the locus of rationally smooth points. Here we extend these methods to calculate subtler invariants called equivariant multiplicities, which can detect smoothness. The W-action on the plane is central both to the formulation of the main result and its proof, and we believe that these methods will be useful in other types besides type $\widetilde{A}_2$.

To describe our results in more detail, we need some background. The Weyl group W of type $\widetilde{A}_2$ is generated by three elements s_1, s_2 and s_3, called simple reflections, subject to the relations $s_i^2 = (s_i s_j)^3 = 1$; the reflections in W are the W-conjugates of the simple reflections. The length $\ell(w)$ of $w \in W$ is the smallest number n such that $w = s_{i_1} \cdots s_{i_n}$. Each $w \in W$ corresponds to a Schubert variety X_w for the affine Kač–Moody group $\mathcal{G}$ of type $\widetilde{A}_2$; X_w is an algebraic variety of dimension $\ell(w)$. If $w = s_i s_j s_k s_i s_j \cdots$ (with ℓ terms), for some permutation i, j, k of $1, 2, 3$, then w and X_w are called spiral (see [7] for a discussion of this term). By symmetry, to obtain results for general spiral elements, it is enough to study the particular spiral elements $w(\ell) = s_1 s_2 s_3 s_1 s_2 \cdots$.

The group W is equipped with a partial order called the Bruhat order which can be defined algebraically (see [8]), and satisfies the property that if $w \in W$, then each element $x \in W$ satisfying $x \leq w$ corresponds to a point $x\mathcal{B} \in X_w$; moreover, the point $x\mathcal{B}$ is fixed by the standard maximal torus T of $\mathcal{G}$. In this case, let R_x^w denote the set of reflections $r \in W$ such that $rx \leq w$, and let $q_x^w = |R_x^w| - \ell(w)$. The numbers q_x^w are always nonnegative [6]. The geometric significance of q_x^w is that it is the difference of the number of T-invariant curves in X_w through $x\mathcal{B}$ and the minimum number $\ell(w)$ of such curves through any T-fixed point (cf. [9, Prop. 12.1.7]). Carrell and Peterson proved that $x\mathcal{B}$ is rationally smooth in X_w if and only if $q_y^w = 0$ for all $y \in W$ with $x \leq y \leq w$ (see [9, Theorem 12.2.14]). Computer calculations done by the authors for many elements $w \in W$ indicate that for a non-spiral Schubert variety X_w of type $\widetilde{A}_2$, the integer q_x^w by itself is enough to detect rational smoothness at $x\mathcal{B}$. This distinguishes the spiral elements, and motivated their study in [7], where the rationally smooth locus of $X_{w(\ell)}$ was described. Not all the rationally smooth points of $X_{w(\ell)}$ are smooth, and in this paper we determine the smooth locus by computing the equivariant multiplicity $e_x^{w(\ell)}$ for appropriate x. However, $e_x^{w(\ell)}$ is substantially more difficult to compute than $q_x^{w(\ell)}$. Thus, while in [7] we described $q_x^{w(\ell)}$ for any $x \leq w(\ell)$, here we compute $e_x^w(\ell)$ only for a relatively small number of x which we need to describe the smooth locus.

The Bruhat order is central to the computation of both $q_x^{w(\ell)}$ and $e_x^{w(\ell)}$, and the main idea of [7] was to use the action of the Weyl group on the plane $\mathbb{R}^2$ to better understand this order. The reflections in W act as reflections across certain lines in $\mathbb{R}^2$; the complement of these lines is a union of connected components called alcoves. There is a particular alcove A_0 called the fundamental alcove, with center denoted by q, and the map $w \mapsto wq$ gives a bijection between W and the set of center points of alcoves. The W-orbit of q is denoted by Wq; this is a discrete set in $\mathbb{R}^2$. A powerful tool for studying the Bruhat order is the endpoint theorem ([7, Theorem 5.5]), which states that if x, y and z are elements of W such that the points xq, yq and zq lie on a line parallel to a root, and if yq is between xq and zq, then $y \le x$ or $y \le z$ (or both).

The main results of [7] about spiral varieties answer the questions of how to determine when $x\mathcal{B}$ is in $X_{w(\ell)}$—that is, when $x \le w(\ell)$—and, if $x\mathcal{B} \in X_{w(\ell)}$, how to determine if $x\mathcal{B}$ is a rationally smooth point. They are formulated in terms of the W-action on $\mathbb{R}^2$. For any $\ell \ge 1$, we define a region $\overline{\Delta}(\ell) \subset \mathbb{R}^2$ consisting of a triangle with its interior, and set $\Delta(\ell) = Wq \cap \overline{\Delta}(\ell)$. We define $R(\ell)$ to equal $\Delta(\ell)$ if ℓ is even, and $\Delta(\ell)$ with two particular points removed if ℓ is odd. We answer the above questions in [7] by proving that for $x \in W$, we have $x \le w(\ell)$ if and only if $xq \in R(\ell)$, and moreover, that $x\mathcal{B}$ is rationally smooth in $X_{w(\ell)}$ if and only if $xq \in R(\ell) \setminus \Delta(\ell - 3)$. See the figures in Section 3.

Since every smooth point is rationally smooth, if $x\mathcal{B}$ is a smooth point of $X_{w(\ell)}$, then xq is in the set $R(\ell) \setminus \Delta(\ell - 3)$. To understand the smooth locus we must examine this set more closely. We observe that this set is a union of certain sets, each of which is the intersection of a line segment parallel to a root with the set Wq. We call these intersections “rationally smooth edges” (although they are finite sets, since Wq is a discrete set), and their extreme points we call endpoints. These edges are defined in Definition 3.5; there are four edges if ℓ is even, and five if ℓ is odd. The main result of this paper is the following theorem, which describes the smooth locus of $X_{w(\ell)}$ in terms of these edges. See the figures in Section 4.

Theorem 4.1 *Let $\ell \ge 6$ and let $x \le w(\ell)$. Then $x\mathcal{B}$ is smooth in $X_{w(\ell)}$ if and only if there is a rationally smooth edge of $R(\ell) \setminus \triangle(\ell - 3)$ containing xq as either an endpoint, or a point adjacent to an endpoint.*

As a consequence of our main theorem, we obtain a description of the maximal elements $x \in W$ such that $x\mathcal{B}$ is a singular point of $X_{w(\ell)}$ (see Theorem 7.1). This gives an alternative description of the singular locus of $X_{w(\ell)}$.

Here is an outline of the proof of Theorem 4.1. Since smooth points are rationally smooth, we may assume xq lies on a rationally smooth edge. Next, we claim that if xq is an endpoint or adjacent to an endpoint, then $x\mathcal{B}$ is smooth, and that if xq is the second point from an endpoint, then $x\mathcal{B}$ is not smooth. This suffices to prove the theorem. Indeed, suppose xq is on an edge but is not an endpoint or adjacent to one. Let yq and $y'q$ be the second points from the two endpoints. The claim implies that $y\mathcal{B}$ and $y'\mathcal{B}$ are not smooth. Since the singular locus is closed, the Schubert varieties X_y and $X_{y'}$ are contained in the singular locus of $X_{w(\ell)}$. Since xq is between yq and $y'q$, the endpoint theorem implies that $x \le y$ or $x \le y'$ (or both), so $x\mathcal{B}$ is in $X_y \cup X_{y'}$ and hence is singular in $X_{w(\ell)}$.

In proving the claim, we first use some properties of the invariants e^w_x (Proposition 2.5) to reduce the number of x for which we must prove $x\mathcal{B}$ is smooth or non-smooth (see Section 4). The discussion preceding Definition 4.2 implies that we only need to prove this for 8 elements of W. (To be precise, we need to consider the $x \in W$ such that xq is one of the three points nearest to either end on one of the edges $E_1(\ell)$, or nearest to one particular end of the edge $E_3(\ell)$; of these points, one is contained in both $E_1(\ell)$ and $E_3(\ell)$, yielding a total of 8.) We then need to determine the subexpressions of the unique reduced expression $\mathscr{S}$ for $w(\ell)$ which multiply to these x. It turns out that all of the x we need to consider have length ℓ, $\ell - 1$, or $\ell - 2$, except for one element $m(\ell)$ of length $\ell - 3$ (Proposition 5.1). In Section 5, we determine all the elements $x \leq w$ satisfying $\ell(x) \geq \ell - 2$, along with all subexpressions of $\mathscr{S}$ multiplying to such an x (Propositions 5.2 and 5.3), or to $m(\ell)$ (Proposition 5.5). This data shows that for each x for which we need to show $x\mathcal{B}$ is smooth, there is only one subexpression of $\mathscr{S}$ multiplying to x; a general result (Theorem 2.3) then yields smoothness. Next, there are only three elements x for which we need to show that $x\mathcal{B}$ is non-smooth. From the enumeration of subexpressions, we see that all three have the property that there are m subexpressions of $\mathscr{S}$ multiplying to x, where $m > 1$ is explicitly determined. Because these $x\mathcal{B}$ are rationally smooth in $X_{w(\ell)}$, results of Kumar (Theorem 2.2) imply that $e_x^{w(\ell)} = (-1)^{\ell(w)-\ell(x)} \frac{k}{\prod_{\beta \in \Psi_x^w} \beta}$, and moreover, that $x\mathcal{B}$ is smooth in $X_{w(\ell)}$ if and only if $k = 1$. Here Ψ_x^w is a set of roots (see Definition 2.1) which can be calculated using the results of [7]. We show (see Proposition 4.5) that for these three elements, the numerator k is equal to the number of subexpressions m; since $m > 1$, these $x\mathcal{B}$ are not smooth in $X_{w(\ell)}$, completing the proof of the claim. The proof of Proposition 4.5, by induction and explicit calculation, is given in Section 7.

2 Preliminaries

We begin by briefly recalling some results we will need about Kač–Moody algebras and groups, as well as the corresponding Schubert varieties. Our main reference for these results is [9].

2.1 Root systems and Weyl groups associated to Kač–Moody Lie algebras

Suppose $\mathfrak{g}$ is the Kač–Moody Lie algebra constructed from a generalized Cartan matrix $(a_{ij})_{1 \leq i,j \leq n}$. Thus, we have dual vector spaces $\mathfrak{h}$ and $\mathfrak{h}^*$, with subsets $\pi = \{\alpha_1, \ldots, \alpha_n\} \subset \mathfrak{h}^*$ and $\pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_n^\vee\} \subset \mathfrak{h}$ of simple roots and coroots respectively, satisfying $\alpha_j(\alpha_i^\vee) = a_{ij}$. For each $i \in 1, \ldots, n$, define a linear transformation s_i of the real vector space spanned by the α_i by the formula $s_i(\chi) = \chi - (\chi, \alpha_i^\vee)\alpha_i$. The Weyl group W can be viewed as the group of linear transformations generated by the s_i, which are called simple reflections. The set Δ_{re} of real roots is the union of the W-orbits of the α_i, and Δ_{re}^+ consists of those $\beta \in \Delta_{\mathrm{re}}$

which can be written as a nonnegative linear combination of the α_i. If $\beta = w\alpha_i$, then there is a reflection $s_\beta = ws_iw^{-1}$, and then for $u \in W$, we have $us_\beta u^{-1} = s_{u(\beta)}$. If A is a set of roots, then $\prod A$ denotes the product of the elements of A.

A reduced expression for $w \in W$ is a sequence $\mathscr{S} = (s_{i_1}, \ldots, s_{i_n})$, where n is as small as possible such that $s_{i_1} \cdots s_{i_n} = w$; in this case the length $\ell(w)$ of w is n. A subexpression of $\mathscr{S}$ is a sequence $\mathscr{T} = (\sigma_1, \ldots, \sigma_n)$, where each σ_k equals 1 or s_{i_k}. Define $\prod \mathscr{T} = \prod \sigma_i$; we say $\mathscr{T}$ multiplies to x if $\prod \mathscr{T} = x$. Let $\mathscr{S}(x)$ denote the set of subexpressions of $\mathscr{S}$ multiplying to x. The Bruhat order on W is the partial order characterized by the property that $x \leq w$ if and only for any reduced expression $\mathscr{S}$ for w, the set $\mathscr{S}(x)$ is nonempty. If $w \in W$ has a unique reduced expression $\mathscr{S}$(for example, if $w = w(\ell)$), we say $\mathscr{T}$ is a subexpression of w if $\mathscr{T}$ is a subexpression of $\mathscr{S}$.

We will be interested in the case where $\mathfrak{g}$ is the affine Lie algebra associated to a finite-dimensional semisimple Lie algebra with root system Φ contained in a Euclidean space V (that is, V is a real vector space with a positive definite inner product). We can describe the set of real roots for $\mathfrak{g}$ as follows (cf. [9, Section 13.1]). Given $\alpha \in \Phi$, the corresponding coroot is $\alpha^\vee = 2\alpha/(\alpha, \alpha)$, and $\Phi^\vee$ denotes the set of coroots. Let Φ^+ be a positive system for Φ, with simple roots $\alpha_1, \ldots, \alpha_{n-1}$, and let $\widetilde{\alpha}$ denote the highest root. Form real vector spaces $V \oplus \mathbb{R} \cdot \delta$ and $V \oplus \mathbb{R} \cdot c$, equipped with a degenerate pairing $(v_1 + a\delta, v_2 + bc) = (v_1, v_2)$. (These spaces are contained in $\mathfrak{h}^*$ and $\mathfrak{h}$, respectively, but are not equal to them.) Define $\alpha_n = \delta - \widetilde{\alpha}$ and $\alpha_n^\vee = c - \widetilde{\alpha}^\vee$. The simple roots for $\mathfrak{g}$ are the set $\pi = \{\alpha_1, \ldots, \alpha_{n-1}, \alpha_n\}$, and the simple coroots are the set $\pi^\vee = \{\alpha_1^\vee, \ldots, \alpha_{n-1}^\vee, \alpha_n^\vee\}$. The positive real roots for $\mathfrak{g}$ are $\Delta_{\mathrm{re}}^+ = \Phi^+ \cup \{\beta + j\delta \mid \beta \in \Phi, j \in \mathbb{Z}_{>0}\}$. The space $V \oplus \mathbb{R}\delta$ is the span of the simple roots, and W can be viewed as a group of linear transformations of this space. The finite Weyl group W_f is the subgroup of W generated by $s_1, \ldots, s_{n-1}$. Note that if $\alpha \in \Phi$, then s_α preserves the subspace V, and we will use the notation s_α both for the linear transformation of $V \oplus \mathbb{R}\delta$, and its restriction to V.

Let $L(\Phi^\vee)$ denote the set of $\mathbb{Z}$-linear combinations of elements of $\Phi^\vee$. We view $L(\Phi^\vee)$ as a group of translations of V; $t(\gamma) : V \to V$ denotes the translation corresponding to γ in $L(\Phi^\vee)$. The group W_f acts on $L(\Phi^\vee)$ and the affine Weyl group W_{aff} is the semidirect product $L(\Phi^\vee) \rtimes W_f$, which we view as a group of isometries of V. Given $\alpha \in \Phi$, let $H_{\alpha,n} = \{v \in V \mid (\alpha, v) = n\}$, and let $s_{\alpha,n}$ denote the reflection across the affine hyperplane $H_{\alpha,n}$. Thus, $s_{\alpha,0}$ equals s_α as defined above, and

$$s_{\alpha,n} = t(n\alpha^\vee)s_\alpha. \tag{2.1}$$

Hence each $s_{\alpha,n}$ is in W_{aff}; also, since $wt(\lambda)w^{-1} = t(w\lambda)$ for $w \in W_f$, $\lambda \in L(\Phi^\vee)$, we have

$$t(\alpha^\vee)s_{\alpha,n}t(-\alpha^\vee) = s_{\alpha,n+2}. \tag{2.2}$$

There is an isomorphism $W \to W_{\mathrm{aff}}$ defined by $s_i \mapsto s_{\alpha_i}$ for $1 \leq i \leq n-1$, and $s_n \mapsto s_{\widetilde{\alpha},1}$. Using this, we will identify W and W_{aff}. If Φ has only one root length, then under this identification, the reflections in W are exactly the affine reflections $s_{\alpha,n}$ for $\alpha \in \Phi$, $n \in \mathbb{Z}$ (see [7]). In Proposition 2.6 we describe this correspondence precisely for the affine root system of type $\widetilde{A}_2$.

The set $V \setminus (\cup H_{\alpha,n})$, where the union is over $\alpha \in \Phi^+$ and $n \in \mathbb{Z}$, is a union of connected components called alcoves. The fundamental alcove $A_\circ$ is the subset of $v \in V$ satisfying $\alpha_i(v) > 0$ for $1 \le i \le n-1$ and $\tilde{\alpha}(v) < 1$. Let q denote the center point of $A_\circ$. There are bijections $W \to \{\text{alcoves}\} \to Wq$ given by $w \mapsto xA_\circ \mapsto xq$.

2.2 Schubert varieties for Kač–Moody groups

In this section we recall some results about Schubert varieties for Kač–Moody groups. In particular we recall Kumar's criteria for smoothness and rational smoothness of such varieties at torus-fixed points. There is one new result, Theorem 2.3, concerning smoothness of X_w at rationally smooth points $x\mathcal{B}$ where there is only one subexpression multiplying to x.

Let $\mathcal{G}$ be any Kač–Moody group with Lie algebra $\mathfrak{g}$, and $\mathcal{B} \supset T$ a standard Borel subgroup and maximal torus of $\mathcal{G}$. The flag variety $\mathcal{G}/\mathcal{B}$ has the structure of an ind-variety. For any $w \in W$, there is a Schubert variety $X_w = \sqcup \mathcal{B}x\mathcal{B} \subset \mathcal{G}/\mathcal{B}$, where the union is over all $x \in W$ satisfying $x \le w$; X_w is a finite-dimensional algebraic variety of dimension $\ell(w)$.

Definition 2.1. Given $x \le w$ in W, define $\Psi_x^w = \{\beta \in \Delta_{\mathrm{re}}^+ \mid s_\beta x \le w\}$, and define $R_x^w = \{s_\alpha \mid \alpha \in \Psi_x^w\}$. Define $q_x^w = |\Psi_x^w| - \ell(w)$.

Note that the definition of Ψ_x^w is the same as in [3, (1.1)], but different from [7], where Ψ_x^w is what we have here denoted by R_x^w. Since Ψ_x^w and R_x^w have the same cardinality, either can be used to define q_x^w. Each q_x^w is nonnegative; the Carrell–Peterson criterion states that $x\mathcal{B}$ is rationally smooth in X_w if and only if $q_y^w = 0$ for all $y \in W$ satisfying $x \le y \le w$. However, the integers q_x^w cannot distinguish between smoothness and rational smoothess; for this we need certain elements e_x^w in the quotient field of $S(\mathfrak{h}^*)$, which, following [4], we call equivariant multiplicities. (In Kumar's notation [9], $e_x^w = c_{w,x}$.) Given a reduced expression $\mathscr{S} = (s_{i_1}, \ldots, s_{i_n})$ for w,

$$e_x^w = (-1)^{\ell(w)} \sum_{(\sigma_1,\ldots,\sigma_n)\in\mathscr{S}(x)} \prod_{j=1}^{n} \frac{1}{\sigma_1\cdots\sigma_j(\alpha_j)} \tag{2.3}$$

(see [9, Theorem 11.1.2]).

The following theorem is due to Kumar (see [9], Theorems 12.2.16 and 12.1.11).

Theorem 2.2. *Let $x \le w$ be elements of W.*

(1) The point $x\mathcal{B}$ is rationally smooth in X_w $\Leftrightarrow$ for all $y \in W$ with $x \le y \le w$, $e_y^w = \dfrac{k_{w,y}}{\prod \Psi_x^w}$, where $(-1)^{\ell(w)-\ell(y)} k_{w,y} \in \mathbb{Z}_+$.

(2) The point $x\mathcal{B}$ is smooth in X_w $\Leftrightarrow$ $e_x^w = \dfrac{(-1)^{\ell(w)-\ell(x)}}{\prod \Psi_x^w}$.

As a consequence, we deduce the following result about smoothness of X_w at rationally smooth points $x\mathcal{B}$ where there is only one subexpression multiplying to x.

Theorem 2.3. *Let $x \le w$ be in W. Suppose $x\mathcal{B}$ is rationally smooth in X_w. Suppose that $\mathcal{S} = (s_{i_1}, \dots, s_{i_n})$ is a reduced expression for W, and suppose that there is only one subexpression $(\sigma_1, \dots, \sigma_n)$ of $\mathcal{S}$ multiplying to x. Then $x\mathcal{B}$ is smooth in X_w.*

Proof. Suppose x and w are as in the statement of the theorem. Theorem 2.2 implies that $e_x^w = \frac{(-1)^{\ell(w)-\ell(x)}k}{\prod \Psi_x^w}$, where k is a positive integer. We must show that $k = 1$. Since $(\sigma_1, \dots, \sigma_n)$ is the only subexpression of $\mathcal{S}$ multiplying to x, (2.3) implies that $e_x^w = (-1)^{\ell(w)} \frac{1}{\prod_{\beta \in P} \beta}$, where $P = \{\sigma_1(\alpha_1), \sigma_1\sigma_2(\alpha_2), \cdots\}$. Since our two expressions for e_x^w are equal, we have in $\mathbb{C}[\alpha_1, \dots, \alpha_n]$ the equality $(-1)^{\ell(x)} k \prod_{\beta \in P} \beta = \prod_{\alpha \in \Psi_x^w} \alpha$. Each root is a degree 1 (and hence irreducible) element of the ring $\mathbb{C}[\alpha_1, \dots, \alpha_n]$. Since $\mathbb{C}[\alpha_1, \dots, \alpha_n]$ is a unique factorization domain, each α in Ψ_x^w is a multiple of some $\beta \in P$. The only multiples of α which are roots are $\pm\alpha$ ([9, Corollary 1.3.6]), so $\prod_{\beta \in P} \beta = \pm \prod_{\alpha \in \Psi_x^w} \alpha$. Therefore $k = \pm 1$. But $k > 0$, so $k = 1$. Therefore $x\mathcal{B}$ is smooth in X_w. □

Remark 2.4. We do not know if the hypothesis that $x\mathcal{B}$ is rationally smooth in X_w is necessary for the conclusion of the theorem to hold.

The next proposition will reduce the number of points where we need to compute rational smoothness or smoothness. The first two parts are essentially in [3, Section 4]; see also [7, Prop. 2.1]. Although [7, Prop. 2.1] is stated only for the case $sw < w$, as noted there, the analogous results hold in the case $ws < w$.

Recall that if s is a simple reflection and $sw < w$ (resp. $ws < w$), then $x \le w$ iff $sx \le w$ (resp. $xs \le w$) ([8, Prop. 5.9]). Also, if β is a root, $|\beta|$ denotes the positive root of the pair $\{\beta, -\beta\}$.

Proposition 2.5. *Let $x \le w$ be in W, let $s = s_\alpha$ be a simple reflection, and suppose that $sw < w$ (resp. $ws < w$). Then:*

(a) $\Psi_{sx}^w = s(\Psi_x^w) \setminus \{-\alpha\} \cup \{\alpha\}$ (resp. $\Psi_{xs}^w = \Psi_x^w$). Hence $q_x^w = q_{sx}^w$ (resp. $q_x^w = q_{xs}^w$).

(b) In X_w, $x\mathcal{B}$ is rationally smooth $\Leftrightarrow$ $sx\mathcal{B}$ is rationally smooth (resp. $xs\mathcal{B}$ is rationally smooth).

(c) $e_x^w = s(e_{sx}^w)$ (resp. $e_x^w = -e_{xs}^w$).

(d) In X_w, $x\mathcal{B}$ is smooth $\Leftrightarrow$ $sx\mathcal{B}$ is smooth (resp. $xs\mathcal{B}$ is smooth).

Proof. For (a) and (b), see [7, Prop. 2.1] (that $\Psi_{sx}^w = s(\Psi_x^w) \setminus \{-\alpha\} \cup \{\alpha\}$ follows from the bijection $R_x^w \to R_{sx}^w$ given by $r \mapsto srs^{-1}$).

We prove (c). Let $\mathcal{S} = (s_{i_1}, \dots, s_{i_n})$ denote a reduced expression for w. If $sw < w$, we may choose $\mathcal{S}$ so that $s_{i_1} = s$, and then there is a bijection $\mathcal{S}(x) \to \mathcal{S}(sx)$ taking $(\tau_1, \dots, \tau_n)$ to $(s\tau_1, \dots, \tau_n)$. The equality $e_x^w = s(e_{sx}^w)$ follows from the definition of the equivariant multiplicities, since

$$\tau_1(\alpha_{i_1}) \cdot \tau_1\tau_2(\alpha_{i_2}) \cdots \tau_1 \cdots \tau_j(\alpha_{i_j}) = s(s\tau_1(\alpha_{i_1}) \cdot s\tau_1\tau_2(\alpha_{i_2}) \cdots s\tau_1 \cdots \tau_j(\alpha_{i_j})).$$

Similarly, if $ws < s$, we may choose $\mathscr{S}$ so that $s_{i_n} = s$, and then there is a bijection $\mathscr{S}(x) \to \mathscr{S}(xs)$ taking $(\tau_1, \ldots, \tau_n)$ to $(\tau_1, \ldots, \tau_n s)$. Again, the equality $e^w_x = -e^w_{xs}$ follows from the definition of the equivariant multiplicities, since $s(\alpha_{i_n}) = -\alpha_{i_n}$.

We now prove (d). It is enough to prove the implication ($\Rightarrow$); the other implication follows by interchanging the roles of x and sx (resp. xs). First, consider the case $sw < w$. We have $x\mathcal{B}$ is smooth in $X_w \Leftrightarrow e^w_x = (-1)^{\ell(w)-\ell(x)} \frac{1}{\prod_{\beta \in \Psi^w_x} \beta}$ $\Leftrightarrow e^w_{sx} = (-1)^{\ell(w)-\ell(x)} \frac{1}{s(\prod_{\beta \in \Psi^w_x} \beta)} = (-1)^{\ell(w)-\ell(x)} \frac{1}{\prod_{\beta \in \Psi^w_{sx}} \beta}$ by (a) $\Leftrightarrow sx\mathcal{B}$ is smooth in X_w. Similarly, if $ws < w$, then $x\mathcal{B}$ is smooth in $X_w \Leftrightarrow -e^w_{xs} = e^w_x = (-1)^{\ell(w)-\ell(x)} \frac{1}{\prod_{\beta \in \Psi^w_x} \beta} \Leftrightarrow e^w_{xs} = (-1)^{\ell(w)-\ell(xs)} \frac{1}{\prod_{\beta \in \Psi^w_{sx}} \beta}$ by (a) $\Leftrightarrow xs$ is smooth. $\square$

2.3 Type $\widetilde{A}_2$

For the remainder of this paper, we will assume our Kač–Moody Lie algebra is the affine Lie algebra type $\widetilde{A}_2$. Thus, Δ is the affine root system constructed from the finite root system Φ of type A_2. We can take $V = \{(a_1, a_2, a_3) \in \mathbb{R}^3 \mid \sum a_i = 0\}$, with the inner product which is the restriction of the usual inner product on $\mathbb{R}^3$, and $\Phi = \{\epsilon_i - \epsilon_j \mid i \neq j, 1 \leq i, j \leq 3\}$. We will often simply write $\mathbb{R}^2$ for V. We choose the positive system for Φ such that the simple roots are $\alpha_1 = \epsilon_1 - \epsilon_2$, $\alpha_2 = \epsilon_2 - \epsilon_3$; then the highest root is $\widetilde{\alpha} = \alpha_1 + \alpha_2$. The simple roots of Δ are $\alpha_1, \alpha_2, \alpha_3$. The Weyl group W is generated by s_1, s_2, s_3, subject to the relations $s_i^2 = (s_i s_j)^3 = 1$, and we have $s_i(\alpha_i) = -\alpha_i$ and $s_i(\alpha_j) = \alpha_i + \alpha_j$ for $i \neq j$.

Under the identification of W with W_{aff}, the translations by simple coroots are expressed in terms of simple reflections by

$$t(\alpha_1^\vee) = s_3 s_2 s_3 s_1,\ t(\alpha_2^\vee) = s_3 s_1 s_3 s_2,\ t(\widetilde{\alpha}^\vee) = s_3 s_1 s_2 s_1 \tag{2.4}$$

(see [7, Section 3.2]). Moreover, the reflections s_β in W correspond to affine reflections $s_{\alpha,k}$ for $\alpha \in \Phi$ and $k \in \mathbb{Z}$, so given $\alpha \in \Phi$ and integers $a \leq b = a + n - 1$, there are positive real roots $\beta_1, \ldots, \beta_n \in \Delta^+_{\mathrm{re}}$ such that $s_{\beta_i} = s_{\alpha, a+i-1}$. We define $[a, b]_\alpha = \{\beta_1, \ldots, \beta_n\}$. (This differs from [7], where $[a, b]_\alpha$ denoted the set $\{s_{\alpha,a}, s_{\alpha,a+1}, \ldots, s_{\alpha,b}\}$.) If $a > b$, then $[a, b]_\alpha$ is the empty set. We write $\{a\}_\alpha$ for $[a, a]_\alpha$.

Proposition 2.6. *The following formulas for $\{n\}_\alpha$ (resp. $\{-n\}_\alpha$) hold when $n \geq 1$ (resp. $n \geq 0$), $\alpha \in \Phi^+$.*

$$\{n\}_{\alpha_1} = (n-1)\alpha_1 + n\alpha_2 + n\alpha_3,\ \{-n\}_{\alpha_1} = (n+1)\alpha_1 + n\alpha_2 + n\alpha_3$$
$$\{n\}_{\alpha_2} = n\alpha_1 + (n-1)\alpha_2 + n\alpha_3,\ \{-n\}_{\alpha_2} = n\alpha_1 + (n+1)\alpha_2 + n\alpha_3$$
$$\{n\}_{\widetilde{\alpha}} = (n-1)\alpha_1 + (n-1)\alpha_2 + n\alpha_3,\ \{-n\}_{\widetilde{\alpha}} = (n+1)\alpha_1 + (n+1)\alpha_2 + n\alpha_3.$$

Proof. The proposition is proved by verifying the assertion for two consecutive values of n, and then using induction together with the equation $t(\alpha^\vee)s_{\alpha,n}t(-\alpha^\vee) = s_{\alpha,n+2}$ (see (2.2)). We illustrate by proving the formulas for the root $\widetilde{\alpha}$, leaving the remaining formulas to the reader. We first consider $s_{\widetilde{\alpha},n}$ for $n \geq 1$. By definition, $s_{\widetilde{\alpha},1} = s_3 = s_{\alpha_3}$. Next,

$$\begin{aligned} s_{\widetilde{\alpha},2} = t(\widetilde{\alpha}^\vee)s_{\widetilde{\alpha},1} =& s_3s_1s_2s_1s_3 \\ =& s_3s_1(s_{\alpha_2})(s_3s_1)^{-1} = s_{s_3s_1(\alpha_2)} = s_{\alpha_1+\alpha_2+2\alpha_3}. \end{aligned}$$

This verifies the desired formula for $n = 1, 2$. Suppose the formula holds for n. Then taking $u = t(\widetilde{\alpha}^\vee) = s_3s_1s_2s_1$, we have $s_{\widetilde{\alpha},n+2} = us_\gamma u^{-1} = s_{u(\gamma)}$, where $\gamma = (n-1)\alpha_1 + (n-1)\alpha_2 + n\alpha_3$. Then $u(\gamma) = (n+1)\alpha_1 + (n+1)\alpha_2 + (n+2)\alpha_3$, as follows by direct calculation, so the result holds for $n + 2$. This proves the formula for $s_{\widetilde{\alpha},n}$ for $n \geq 1$. Next, we consider $s_{\widetilde{\alpha},-n}$ for $n \geq 0$. For $n = 0$, since $\widetilde{\alpha} = \alpha_1 + \alpha_2$, we have $s_{-\widetilde{\alpha},0} = s_{\widetilde{\alpha}} = s_{\alpha_1+\alpha_2}$. Next, for $n = 1$,

$$\begin{aligned} s_{-\widetilde{\alpha},1} = s_{\widetilde{\alpha},-1} &= t(-2\widetilde{\alpha}^\vee)s_{\widetilde{\alpha},1} = (s_1s_2s_1s_3)(s_1s_2s_1s_3)s_3 \\ &= (s_1s_2s_1)s_3(s_1s_2s_1)^{-1} = s_{s_1s_2s_1(\alpha_3)} = s_{2\alpha_1+2\alpha_2+\alpha_3}. \end{aligned}$$

This verifies the desired formula for $n = 0, 1$. Suppose the formula holds for n. Then taking $v = t(-\widetilde{\alpha}^\vee) = s_1s_2s_1s_3$, we have $s_{\widetilde{\alpha},-(n+2)} = vs_\gamma v^{-1} = s_{v(\gamma)}$, where $\gamma = (n + 1)\alpha_1 + (n + 1)\alpha_2 + n\alpha_3$. Direct calculation shows that $v(\gamma) = (n+3)\alpha_1 + (n+3)\alpha_2 + (n+2)\alpha_3$, so the formula for $s_{\widetilde{\alpha},-n}$ holds for all $n \geq 0$. □

2.4 A parametrization of the alcoves

In this section we recall from [7, Section 4] a parametrization of the set of alcoves in type $\widetilde{A}_2$, along with some useful results related to this parametrization. Recall that the alcoves are by definition the set of connected components of $\mathbb{R}^2 \setminus (\cup H_{\alpha,n})$, where the union is over $\alpha \in \Phi^+$ and $n \in \mathbb{Z}$. The parametrization of alcoves is as follows. Let $\beta_1 = \alpha_1, \beta_2 = \alpha_2, \beta_3 = -\widetilde{\alpha}$. Let

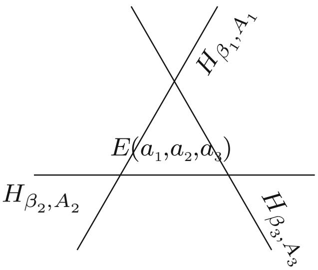
$$E(a_1, a_2, a_3) = \{p \mid (p, \beta_i) \geq A_i \text{ for } i = 1, 2, 3\},$$

$$O(a_1, a_2, a_3) = \{p \mid (p, \beta_i) \leq A_i \text{ for } i = 1, 2, 3\}$$

where $A_i = a_i + \varepsilon_i$, and $\varepsilon_1 = \varepsilon_2 = 0, \varepsilon_3 = -1$. These are pictured in Figure 1. When we write $X(a_1, a_2, a_3)$, we mean X is equal to either E or O. The alcoves are exactly the interiors of the $X(a_1, a_2, a_3)$, where $\sum a_i = 0$ if $X = E$, and $\sum a_i = 2$ if $X = O$. We will abuse terminology and refer to $X(a_1, a_2, a_3)$ as an alcove (rather than an alcove closure). The fundamental alcove is $A_\circ = E(0, 0, 0)$, and $q = \frac{1}{3}(\alpha_1^\vee + \alpha_2^\vee)$ is its center point. As noted in the previous section, we have bijections

$W \to \{\text{alcoves}\} \to Wq$ given by $w \mapsto xA_\circ \mapsto xq$. If $xA_\circ = X(a_1, a_2, a_3)$, we will say that $X(a_1, a_2, a_3)$ is the alcove formula for x. By [7, Prop. 4.8], the center point of $X(a_1, a_2, a_3)$ is

$$\frac{1}{3}(a_1\alpha_1^\vee + a_2\alpha_2^\vee - (a_3 - 1)\widetilde{\alpha}^\vee). \tag{2.5}$$



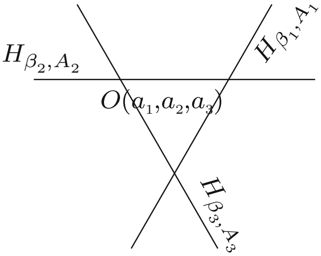


Figure 1 $E(a_1, a_2, a_3)$ and $O(a_1, a_2, a_3)$

The action of W on $\mathbb{R}^2$ restricts to an action on the set of alcoves, which can be described as follows. If $\gamma \in L(\Phi^\vee)$, then

$$t(\gamma)X(a_1, a_2, a_3) = X(a_1 + (\beta_1, \gamma), a_2 + (\beta_2, \gamma), a_3 + (\beta_3, \gamma)) \tag{2.6}$$

Also, if $\{i, j, k\} = \{1, 2, 3\}$, then

$$s_i X(a_1, a_2, a_3) = X'(b_1, b_2, b_3) \tag{2.7}$$

where X' is of opposite type to X, and $b_i = -a_i, b_j = -a_k + 1, b_k = -a_j + 1$. If $xA_\circ = X(a_1, a_2, a_3)$, we say the length of the alcove is $\ell(X(a_1, a_2, a_3)) = \ell(x)$. We have

$$\ell(X(a_1, a_2, a_3)) = \begin{cases} |a_1| + |a_2| + |a_3| & \text{if } X = E \\ |a_1 - 1| + |a_2 - 1| + |a_3 - 1| & \text{if } X = O. \end{cases} \tag{2.8}$$

See [7, Propositions 4.7 and 4.11] for these results.

Because the set Wq is in bijection with W, there is a right action of W on Wq defined by $(xq)w = xwq$ for $x, w \in W$. However, this action is not the restriction of an action on the plane by isometries. Given any subset S of Wq, and $w \in W$, define $wS = \{wxq \mid x \in W, xq \in S\}$ and $Sw = \{xwq \mid x \in W, xq \in S\}$.

Remark 2.7. The right action by any simple reflection $r = s_i$ takes an alcove $A = xA_0$ (for $x \in W$) to the alcove xrA_0. This alcove is obtained by reflecting A across one of the lines which bound A. To see this, observe that r acts on $\mathbb{R}^2$ by reflection across a line H, which is one of the lines bounding the fundamental alcove A_0.

If $x \in W$, then xH is a line bounding the alcove $A = xA_0$, and xrx^{-1} is the reflection across xH. The alcove xrA_0 is obtained by reflecting A across xH, since $xrA_0 = xrx^{-1}xA_0$. From this we see that the right action by r is not the restriction of an isometry, since different alcoves may be reflected across different lines.

3 The rationally smooth loci of spiral Schubert varieties

There are subsets $R(\ell)$ and $\Delta(\ell)$ of $\mathbb{R}^2$, such that the set of rationally smooth points $x\mathcal{B}$ in $X_{w(\ell)}$ is in bijection with $R(\ell) \setminus \Delta(\ell-3)$. In this section we recall the definitions of these sets from [7], along with some results from that paper that we will need here. We conclude the section with two results (Proposition 3.4 and Lemma 3.6) about the set $R(\ell) \setminus \Delta(\ell-3)$.

Recall from the introduction that $w(\ell)$ denotes the spiral element of length ℓ defined by $w(\ell) = s_1s_2s_3s_1\cdots$ (with ℓ factors). Our convention will be that in a statement such as "let $w(\ell) = s_1s_2\cdots s_is_js_k$", the right-hand side is assumed to be a reduced expression, and the indices i, j and k indicate the last three reflections which appear in this reduced expression (where the values of i, j and k depend on ℓ mod 3). Observe that $w(\ell)$ has a unique reduced expression, since any other reduced expression would be obtainable from the one just given by applying the relations $s_as_bs_a = s_bs_as_b$, but the reduced expression for $w(\ell)$ does not contain the string $s_as_bs_a$.

It is useful in certain proofs to know that the element $w(6n) = t(n(-2\alpha_1^\vee - \alpha_2^\vee))$ is a translation (see [7, (6.1)]). From [7, Prop. 6.2] we have the formulas

$$w(\ell)A_\circ = E(-\frac{\ell}{2}, 0, \frac{\ell}{2}) \text{ and } w(\ell)q = \frac{1-\ell}{3}\alpha_1^\vee + \frac{2-\ell}{6}\alpha_2^\vee \text{ if } \ell \text{ is even} \quad (3.1)$$

$$w(\ell)A_\circ = O(\frac{1-\ell}{2}, 1, \frac{\ell+1}{2}) \text{ and } w(\ell)q = \frac{1-\ell}{3}\alpha_1^\vee + \frac{3-\ell}{6}\alpha_2^\vee \text{ if } \ell \text{ is odd.} \quad (3.2)$$

The formal definitions of $\triangle(\ell)$ and $R(\ell)$ from [7] are recalled in Definition 3.1 below; here is an informal description. We first define a subset $\overline{\triangle(\ell)}$ of the plane consisting of a triangle and its interior. The set Wq consists of the center points of the alcoves, and $\triangle(\ell)$ consists of the points in Wq which are contained in $\overline{\triangle(\ell)}$. If ℓ is even and $x \in W$, then $x \leq w$ if and only if $xq \in \triangle(\ell)$. However, if ℓ is odd, then there are two elements $A_i(\ell)$ (for $i = 1, 2$) of W such that $A_i(\ell)q \in \triangle(\ell)$ but $A_i(\ell) \not\leq w(\ell)$, and therefore we define a set $R(\ell)$ consisting of $\triangle(\ell)$ with the two elements $A_i(\ell)q$ removed. (If ℓ is even, we let $R(\ell)$ equal $\triangle(\ell)$.) Thus, the sets $\triangle(\ell)$ and $R(\ell)$ give geometric interpretations of the set of $x \in W$ with $x \leq W(\ell)$. They are also used in describing the set of non-rationally smooth $x\mathcal{B}$ in $X_{w(\ell)}$; see Theorem 3.3.

If $z = a_1\alpha_1^\vee + a_2\alpha_2^\vee$, set $\lambda_1(z) = a_1, \lambda_2(z) = a_2, \lambda_{21}(z) = a_2 - a_1$.

Definition 3.1. Let ℓ be an integer.

(1) Define $\overline{\triangle}(\ell)$ to be the set of all $z \in \mathbb{R}^2$ satisfying the inequalities

$$(I_1(\ell)): \quad \lambda_{21}(z) \le \frac{\ell}{6} + \varepsilon, \qquad (I_2(\ell)): \quad \lambda_1(z) \le \frac{\ell}{6} + \varepsilon,$$
$$(I_3(\ell)): \quad \lambda_2(z) \ge -\frac{\ell}{6} + \frac{1}{3} + \varepsilon,$$

where $\varepsilon = 0$ if ℓ is even, $\varepsilon = \frac{1}{6}$ if ℓ is odd. Let $\triangle(\ell) = Wq \cap \overline{\triangle}(\ell)$. Note that we have $\triangle(\ell+1) \supset \triangle(\ell)$ for all $\ell \ge 1$ ([7, Prop. 7.3]).

(2) If ℓ is even, let $R(\ell) = \triangle(\ell)$. If $\ell \ge 1$ is odd, define $A_1(\ell) := t(\frac{\ell-1}{2}\widetilde{\alpha}^\vee)w(\ell) = s_{\widetilde{\alpha}}w(\ell+1)$, and $A_2(\ell) := s_1 A_1(\ell)$. If $\ell \ge 3$ is odd, let

$$R(\ell) = \triangle(\ell) \setminus \{A_1(\ell)q, A_2(\ell)q\}.$$

Let $R(1) = \triangle(1)$. Note that $R(\ell+1) \supset R(\ell)$ for all $\ell \ge 1$ ([7, Prop. 7.3]).

(3) For each $i = 1, 2, 3$, define $L_i(\ell)$ to be the line consisting of all $z \in \mathbb{R}^2$ such that z satisfies $I_k(\ell)$ with equality holding.

(4) Let $E_i(\ell) = L_i(\ell) \cap R(\ell)$ for $i = 1, 2, 3$. We will refer to the $E_i(\ell)$ as edges. If $L_i(\ell)$ is parallel to β, and $z \in L_i(\ell)$, the endpoints of $E_i(\ell)$, denoted $EP_i(\ell)$, are defined to be the points on $E_i(\ell)$ of the form $z + t_1\beta$ and $z + t_2\beta$ where t_1 and t_2 are chosen as small (respectively, large) as possible. They are given by $EP_1(\ell) = \{w(\ell)q, s_{\widetilde{\alpha}}w(\ell)q\}$, $EP_2(\ell) = s_1 EP_1(\ell)$, $EP_3(\ell) = \{w(\ell)q, s_1 w(\ell)q\}$ (see [7, Prop. 7.6]). Two points on an edge are said to be adjacent if there is no point on that edge between them.

Observe that if $\ell \le 0$, then $\overline{\triangle}(\ell)$ is empty. Hence the related sets $\triangle(\ell)$, $R(\ell)$, $E_i(\ell)$, $EP_i(\ell)$ are empty as well.

Figures 2 and 3 show $\triangle(\ell)$ and $R(\ell)$ for $\ell = 7$ and $\ell = 6$. The dots are the center points of alcoves, and $\triangle(\ell)$ consists of the center points lying in the triangular region. In case ℓ is odd, $R(\ell)$ is defined by removing two points $A_i(\ell)q$ from $\triangle(\ell)$; these appear on the top left of Figure 2. The sets $E_i(\ell)$ are the center points in $R(\ell)$ which lie on the lines $L_i(\ell)$ bounding $R(\ell)$.

Remark 3.2. We will often write xq is on an edge when we mean xq is an element of the edge (for us, an edge is a finite set, since it is the intersection of Wq with a line segment). Also, for convenience we will often omit the symbol q and write that $x \in W$ is on an edge, is an endpoint, etc., when we mean that $xq \in \mathbb{R}^2$ is on an edge, is an endpoint, etc. There is no ambiguity in doing this, since the edges are subsets of Wq and the map $W \to Wq$ given by $w \mapsto wq$ is a bijection.

Note that if $\ell = 3$ (for ℓ even) or $i = 1, 2$ (for ℓ odd), then the proof of Proposition 3.4 shows that

$$L_i(\ell-1) \cap R(\ell) = L_i(\ell-1) \cap R(\ell-1) = E_i(\ell-1).$$

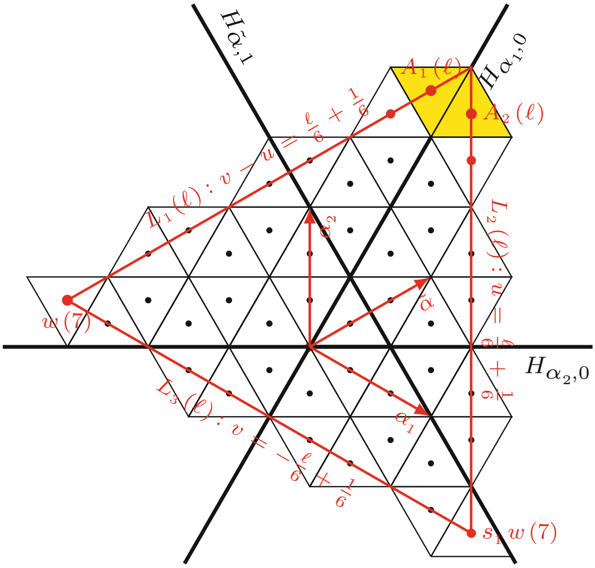


Figure 2 $\triangle(\ell)$ and $R(\ell)$ for ℓ odd ($\ell = 7$)

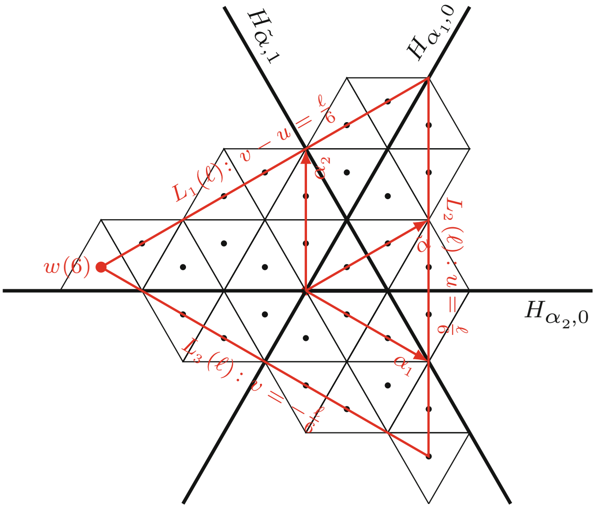


Figure 3 $\triangle(\ell) = R(\ell)$ for ℓ even ($\ell = 6$)

The next theorem restates some of the main results of [7] (see Theorems 8.1 and 10.4).

Theorem 3.3. *Let $x \in W$ and $\ell \geq 1$.*

(a) We have $x \leq w(\ell) \Leftrightarrow xq \in R(\ell)$.
(b) The point $x\mathcal{B} \in X_{w(\ell)}$ is not rationally smooth $\Leftrightarrow xq \in \triangle(\ell - 3)$.

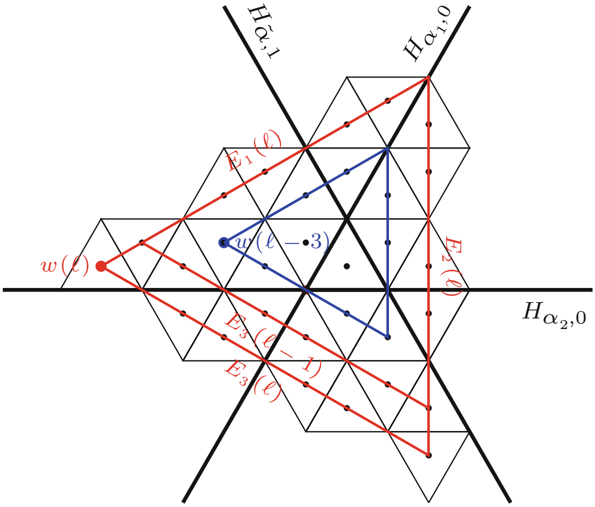


Figure 4 $R(\ell) \setminus \triangle(\ell-3)$ as a union $E_1(\ell) \cup E_2(\ell) \cup E_3(\ell) \cup E_3(\ell-1)$ of rationally smooth edges, for ℓ even

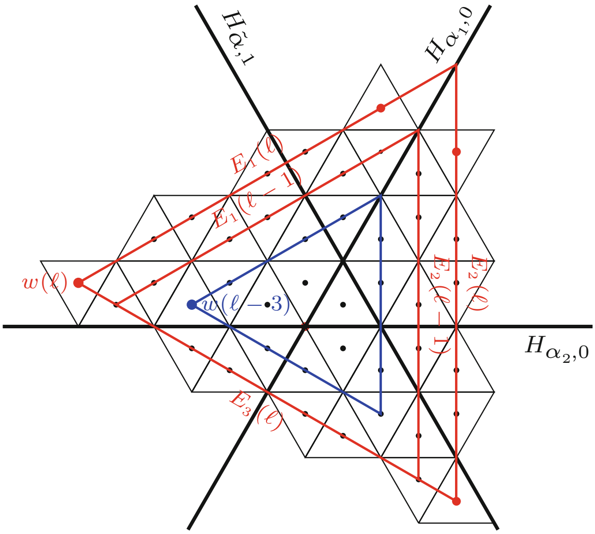


Figure 5 $R(\ell) \setminus \triangle(\ell-3)$ as a union $E_1(\ell) \cup E_2(\ell) \cup E_3(\ell) \cup E_1(\ell-1) \cup E_2(\ell-1)$ of rationally smooth edges, for ℓ odd

We now give a more precise description of the set $R(\ell) \setminus \triangle(\ell-3)$.

Proposition 3.4. *(1) If ℓ is even, then*

$$R(\ell) \setminus \triangle(\ell-3) = E_1(\ell) \cup E_2(\ell) \cup E_3(\ell) \cup E_3(\ell-1). \tag{3.3}$$

If ℓ is odd, then

$$R(\ell) \setminus \triangle(\ell-3) = E_1(\ell) \cup E_2(\ell) \cup E_3(\ell) \cup E_1(\ell-1) \cup E_2(\ell-1). \tag{3.4}$$

(2) For $i = 1, 2, 3$, we have $L_i(\ell) \cap (R(\ell) \setminus \triangle(\ell-3)) = E_i(\ell)$.
(3) Suppose $\ell \geq 2$. For $i = 3$ if ℓ is even, and $i = 1, 2$ if ℓ is odd, we have

$$L_i(\ell-1) \cap (R(\ell) \setminus \triangle(\ell-3)) = E_i(\ell-1).$$

Proof. (1) If ℓ is even, (3.3) holds by [7, Proposition 7.11(a)] and [7, Remark 7.12]. If ℓ is odd, then by [7, Proposition 7.6], $E_3(\ell) \setminus E_3(\ell-1) = EP_3(\ell) = \{w(\ell)q, s_1 w(\ell)q\} \subset E_1(\ell) \cup E_2(\ell)$, so the right-hand side of (3.4) is unchanged if we replace $E_3(\ell)$ with $E_3(\ell-1)$. The resulting statement then follows from [7, Proposition 7.11(b)].

(2) The intersection $L_i(\ell) \cap \triangle(\ell-3)$ is empty by [7, Lemma 7.8] (note that $\triangle(\ell-3) \subset \triangle(\ell-2)$). Hence $L_i(\ell) \cap (R(\ell) \setminus \triangle(\ell-3)) = L_i(\ell) \cap R(\ell) = E_i(\ell)$.

(3) If ℓ is even, then $R(\ell) = R(\ell-1) \sqcup (E_3(\ell) \cup \{A_1(\ell-1)q, A_2(\ell-1)q\})$ by [7, Proposition 7.9(2)]. Since $L_3(\ell-1)$ does not intersect $E_3(\ell)$ (as the lines $L_3(\ell-1)$ and $L_3(\ell)$ are parallel but not equal), or contain $A_i(\ell-1)q$, we have $L_3(\ell-1) \cap R(\ell) = L_3(\ell-1) \cap R(\ell-1) = E_3(\ell-1)$, as desired. If ℓ is odd, by [7, Proposition 7.9(1)], $R(\ell) = R(\ell-1) \sqcup \{E_1(\ell), E_2(\ell)\}$. For $i = 1, 2$, the line $L_i(\ell-1)$ does not intersect $E_i(\ell)$ (as $L_i(\ell-1)$ and $L_i(\ell)$ are parallel but not equal). Also, we have $L_1(\ell) \cap L_2(\ell-1) = A_1(\ell)q$ and $L_2(\ell) \cap L_1(\ell-1) = A_2(\ell)q$; since $\ell \geq 3$, $A_i(\ell)q \notin R(\ell)$, and therefore $L_i(\ell-1) \cap E_j(\ell) = \varnothing$ for $\{i, j\} = \{1, 2\}$. Therefore, we have $L_i(\ell-1) \cap R(\ell) = L_i(\ell-1) \cap R(\ell-1) = E_i(\ell-1)$, as desired. □

Definition 3.5. The edges of $R(\ell) \setminus \triangle(\ell-3)$ are $E_i(\ell)$ for $i = 1, 2, 3$, together with $E_3(\ell-1)$ (if ℓ is even), or $E_1(\ell-1)$ and $E_2(\ell-1)$ (if ℓ is odd). We will refer to these as the "rationally smooth" edges. See Figures 4 and 5.

The proof of part (1) of the next lemma is simpler than the proof of (2), because given a reflection $s \in W$, the map $Wq \to Wq$ given by $xq \mapsto sxq$ is the restriction of an isometry $V \to V$, but the map $xq \mapsto xsq$ is not (cf. Remark 2.7).

Lemma 3.6. *Let $\ell \geq 1$, and let $w(\ell) = s_1 s_2 s_3 \cdots s_k$ be a reduced expression.*

(1) There are bijections

$$\begin{aligned} E_1(\ell) &\to E_2(\ell) & z &\mapsto s_1 z \\ E_3(\ell) &\to E_3(\ell) & z &\mapsto s_1 z \\ E_1(\ell) &\to E_1(\ell) & z &\mapsto s_{\widetilde{\alpha}} z. \end{aligned}$$

(2) Let $i = 3$ if ℓ is even, and $i = 1$ if ℓ is odd. There is a bijection $E_i(\ell - 1) \to E_i(\ell)$, $z \mapsto zs_k$.

Each of the bijections in (1) and (2) preserves the set of endpoints and takes the n-th point from an endpoint to the n-th point from the other endpoint.

Proof. (1) By [7, Prop. 7.5], $s_1 E_1(\ell) = E_2(\ell)$ and $s_1 EP_1(\ell) = EP_2(\ell)$. Let $p_0, p_1, p_2, \ldots$ be the points on $E_1(\ell)$, listed in order starting from the endpoint $p \in EP_1(\ell)$.Then $s_1 p \in EP_2(\ell)$ and the points $s_1 p_0, s_1 p_1, s_1 p_2, \ldots$ are all in $E_2(\ell)$. Moreover, each point in this list is adjacent to the next, since if there were $z \in E_2(\ell)$ between $s_1 p_n$ and $s_1 p_{n+1}$, then $s_1 z \in E_1(\ell)$ would be between p_n and p_{n+1}, contradicting our assumption. This proves the part of the statement involving $E_1(\ell) \to E_2(\ell)$; the part involving $E_3(\ell) \to E_3(\ell)$ is proved similarly. Finally, although [7, Prop. 7.5] does not discuss the symmetry of $E_1(\ell)$ under $s_{\widetilde{\alpha}}$, this symmetry can be deduced from the fact that $s_{\widetilde{\alpha}}$ switches the endpoints of $E_1(\ell)$ (see Definition 3.1), and then the part of the statement involving $E_1(\ell) \to E_1(\ell)$ follows by arguments similar to the arguments for other parts.

(2) Suppose ℓ is even. We have $EP_3(\ell) = \{w(\ell)q, s_1 w(\ell)q\}$ and since $w(\ell - 1) = w(\ell)s_k$, we have $EP_3(\ell - 1) = EP_3(\ell)s_k$. Also, $E_3(\ell)$ (resp. $E_3(\ell - 1)$) consists of the points in $L_3(\ell)$ (resp. $L_3(\ell - 1)$) between the endpoints. Let q_0 (resp. q_0') denote the endpoint $w(\ell)q \in EP_3(\ell)$ (resp. $w(\ell - 1)q \in EP_3(\ell - 1)$). By (3.1) and (3.2), these correspond to the alcoves $E(-\frac{\ell}{2}, 0, \frac{\ell}{2})$ and $O(-\frac{\ell}{2} + 1, 1, \frac{\ell}{2})$, respectively, so by [7, Corollary 4.7], both q_0 and q_0' lie between $H_{\alpha_1, -\frac{\ell}{2}}$ and $H_{\alpha_1, -\frac{\ell}{2}+1}$.

Let $q_0, q_1, \ldots$ (resp. $q_0', q_1', \ldots$) be the points on $E_3(\ell)$ (resp. $E_3(\ell - 1)$) listed in order starting at q_0 (resp. q_0'). By [7, Theorem 5.3] and its proof, q_i (resp. q_i') is the unique point of the form uq (for $u \in W$) lying between $H_{\alpha_1, -\frac{\ell}{2}+i}$ and $H_{\alpha_1, -\frac{\ell}{2}+i+1}$, and moreover, $q_i = w_i q$ (resp. $q_i' = w_i s_k q$), where $w_i = s_{\alpha_1, -\frac{\ell}{2}+i} \cdots s_{\alpha_1, -\frac{\ell}{2}+2} s_{\alpha_1, -\frac{\ell}{2}+1} w(\ell)$. Hence the map $z \mapsto zs_k$ takes the n-th point q_n from q_0 to the n-th point q_n' from q_0'. A similar argument shows that the map takes the n-th point from the endpoint $s_1 w(\ell)q$ to the n-th point from the endpoint $s_1 w(\ell - 1)q$. This proves the result if ℓ is even. The proof is similar if ℓ is odd; we omit the details. □

4 Description of the smooth locus

The main result of the paper is the following theorem, which describes the smooth locus of a spiral Schubert variety in type $\widetilde{A}_2$. See Figures 6 and 7.

Theorem 4.1. *Let $\ell \geq 6$ and let $x \leq w(\ell)$. Then $x\mathcal{B}$ is smooth in $X_{w(\ell)}$ if and only if there is a rationally smooth edge of $R(\ell) \setminus \triangle(\ell - 3)$ containing xq as either an endpoint, or a point adjacent to an endpoint.*

As explained in the introduction, to prove the theorem, it suffices to show that for $x \in W$ on a rationally smooth edge (recall from Remark 3.2 that this means xq is an element of the edge), if xq is an endpoint or adjacent to an endpoint, then $x\mathcal{B}$ is smooth in $X_{w(\ell)}$, and that if xq is the second point from an endpoint, then $x\mathcal{B}$ is not smooth in $X_{w(\ell)}$. Proposition 2.5 implies that if $y \le w$, then in $X_{w(\ell)}$, the points $y\mathcal{B}, s_1 y\mathcal{B}$, and $ys_k\mathcal{B}$ (with $w(\ell) = s_1 s_2 \cdots s_k$) are either all smooth or all singular. By Lemma 3.6, $s_1 E_2(\ell) = E_1(\ell)$, $s_1 E_2(\ell-1) = E_1(\ell-1)$, and $s_1 E_3(\ell) = E_3(\ell)$. Also, if ℓ is even, $E_3(\ell-1)s_k = E_3(\ell)$; if ℓ is odd, $E_1(\ell-1)s_k = E_1(\ell)$. Combining these observations with the definition of the rationally smooth edges (Definition 3.5) shows that we may assume x is on $E_1(\ell)$ or $E_3(\ell)$, and moreover, that if x is on $E_3(\ell)$, then x is either the endpoint $w(\ell)q$ or one of the next two points from that endpoint.

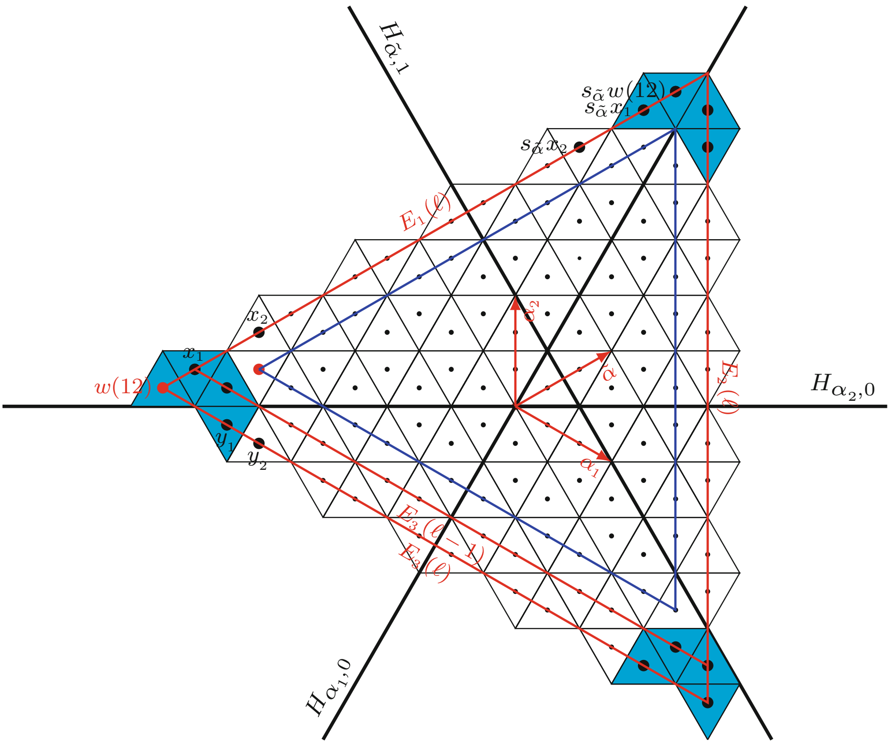


Figure 6 Smooth locus for $X_{w(\ell)}$ for ℓ even ($\ell = 12$). Here $x_i = p_i(E_1(\ell)), y_i = p_i(E_3(\ell))$. The center points of the shaded alcoves correspond to smooth points of $X_{w(\ell)}$.

It will be convenient to introduce some notation for these points.

Definition 4.2. Let $p(E_1(\ell)) = p(E_3(\ell))$ be the endpoint $w(\ell)$ of $E_1(\ell)$ and $E_3(\ell)$. (This endpoint is an element of each of the sets $EP_1(\ell)$ and $EP_3(\ell)$; see Definition 3.1.) Let the next two points from this endpoint on $E_i(\ell)$ (for

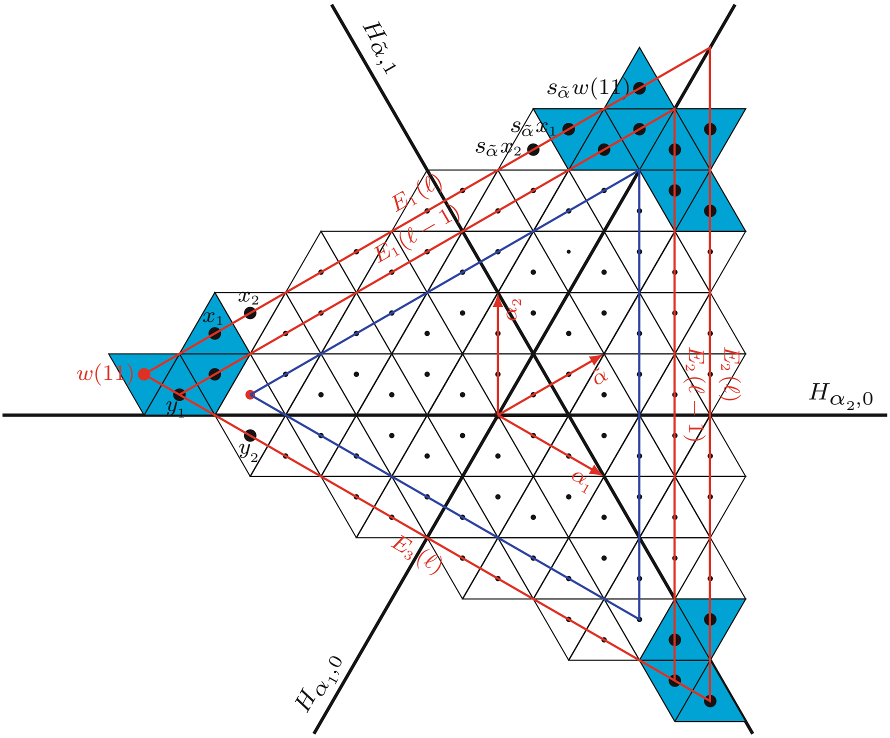


Figure 7 Smooth locus for $X_{w(\ell)}$ for ℓ odd ($\ell = 11$). Here $x_i = p_i(E_1(\ell)), y_i = p_i(E_3(\ell))$. The center points of the shaded alcoves correspond to smooth points of $X_{w(\ell)}$.

$i = 1, 3$) be denoted (in order) by $p_1(E_i(\ell))$, $p_2(E_i(\ell))$. The other endpoint of $E_1(\ell)$ is $s_{\tilde{\alpha}}p(E_1(\ell))$ and the next two points are $s_{\tilde{\alpha}}p_1(E_1(\ell))$, $s_{\tilde{\alpha}}p_2(E_1(\ell))$ (see Lemma 3.6).

Formulas for these points are given in Proposition 5.1.

Theorem 4.1 will follow from the next two propositions. In the next proposition, by a subexpression of $w(\ell)$ we mean a subexpression of the unique reduced expression for $w(\ell)$.

Remark 4.3. In the next proposition, as well as in some of the later propositions, some small values of ℓ are excluded. One reason is that for small values of ℓ, the set $R(\ell)$ may be too small for certain elements to be defined. For example, if $\ell = 3$, then $E_1(\ell)$ has only two points, so $p_2(E_1(\ell))$ is not defined. Also, some of the length formulas fail for small values of ℓ: for example, if $\ell = 4$, then $s_{\tilde{\alpha}}p_2(E_1(\ell))$ has length 3, but for $\ell \geq 6$, this element has length $\ell - 3$. Similarly, some of the subexpression formulas hold only if the reduced expression for $w(\ell)$ has enough terms.

Proposition 4.4.

(1) Let $\ell \geq 3$. Let x be the endpoint $p(E_1(\ell)) = p(E_3(\ell))$, the endpoint $s_{\widetilde{\alpha}} p(E_1(\ell))$, or a point on $E_1(\ell)$ or $E_3(\ell)$ adjacent to one of these endpoints. Then there is one subexpression of $w(\ell)$ multiplying to x.

(2) Let x be equal to $p_2(E_1(\ell))$, $s_{\widetilde{\alpha}} p_2(E_1(\ell))$, or $p_2(E_3(\ell))$, and assume that $\ell \geq$ 4, 6, or 3, respectively, depending on x. There are m subexpressions of $w(\ell)$ multiplying to x. If ℓ is even, then $m = \frac{\ell}{2} - 1$. If ℓ is odd, then $m = \frac{\ell-1}{2}$ if xq lies on $E_3(\ell)$ and $m = \frac{\ell-3}{2}$ if x lies on $E_1(\ell)$.

The second proposition concerns equivariant multiplicities.

Proposition 4.5. *Let ℓ be as in Proposition 4.4. Let x be the endpoint $p(E_1(\ell)) = p(E_3(\ell))$, the endpoint $s_{\widetilde{\alpha}} p(E_1(\ell))$, or one of the next two points on $E_1(\ell)$ or $E_3(\ell)$ starting from one of these endpoints. Then*

$$e_x^{w(\ell)} = (-1)^{\ell(w(\ell))-\ell(x)} \frac{m}{\prod_{\beta \in \Psi_x^{w(\ell)}} \beta},$$

where m is the number of subexpressions of $s_1 s_2 \cdots s_k$ multiplying to x.

These two propositions, combined with the smoothness criterion in terms of equivariant multiplicities of Theorem 2.2, imply that if x is one of the endpoints $p(E_1(\ell))$ or $s_{\widetilde{\alpha}} p(E_1(\ell))$, or adjacent to one of these endpoints on $E_1(\ell)$, (resp. x is the endpoint $p(E_3(\ell))$, or adjacent to this endpoint on $E_3(\ell)$), then $x\mathcal{B}$ is smooth in $X_{w(\ell)}$, and if x is the second point on $E_1(\ell)$ from $p(E_1(\ell))$ or $s_{\widetilde{\alpha}} p(E_1(\ell))$ (resp. the second point on $E_3(\ell)$ from $p(E_3(\ell))$), then $x\mathcal{B}$ is singular in $X_{w(\ell)}$. As explained above, this proves Theorem 4.1.

Proposition 4.4 is proved in Section 5. Observe that once we have proved this proposition, the part of Proposition 4.5 concerning the endpoints or points adjacent to endpoints follows easily. Indeed, since $x\mathcal{B}$ is rationally smooth in $X_{w(\ell)}$ and there is only one subexpression of $w(\ell)$ multiplying to such a point, Theorem 2.3 implies that $x\mathcal{B}$ is smooth in $X_{w(\ell)}$. The statement about equivariant multiplicities then follows from Theorem 2.2. Therefore, in proving Proposition 4.5, we may assume that x is one of $p_2(E_1(\ell))$, $s_{\widetilde{\alpha}} p_2(E_1(\ell))$, or $p_2(E_3(\ell))$. The proof of the proposition in the cases $p_2(E_1(\ell))$ and $p_2(E_3(\ell))$ is given in Section 7; the case $s_{\widetilde{\alpha}} p_2(E_1(\ell))$ is discussed briefly.

Remark 4.6. The part of Proposition 4.5 concerning the point $s_{\widetilde{\alpha}} p_2(E_1(\ell))$ is not necessary to prove our main theorem identifying the set of smooth points. Indeed, Proposition 5.1 implies that for $\ell \geq 6$, $s_{\widetilde{\alpha}} p_2(E_1(\ell)) < p_2(E_1(\ell))$ (since these two elements differ by a reflection, and the length of the second is greater). Since the singular locus is closed, once we know that $p_2(E_1(\ell))$ corresponds to a singular point of $X_{w(\ell)}$, this implies that $s_{\widetilde{\alpha}} p_2(E_1(\ell))$ does as well, without the need to calculate the equivariant multiplicity at that point (cf. the discussion of the proof of Theorem 4.1 in the introduction). For this reason, and because it is similar to the case of $p_2(E_1(\ell))$, we omit most of the proof of Proposition 4.5 in the case $s_{\widetilde{\alpha}} p_2(E_1(\ell))$.

5 Subexpressions

In this section we prove Proposition 4.4, which describes the number of subexpressions of $w(\ell)$ multiplying to each element x listed in Definition 4.2. In fact, we obtain more information than this. All of the elements listed in Definition 4.2 have length $\ell - 2$ or greater, except for $s_{\tilde{\alpha}} p_2(E_1(\ell))$, which has length $\ell - 3$ (see Proposition 5.1). Motivated by this, in Propositions 5.2 and 5.3 we identify all the subexpressions of $w(\ell)$ which multiply to elements of length $\ell - 2$ or greater, along with the elements to which they multiply. Proposition 4.4 follows immediately from this, except for the element $s_{\tilde{\alpha}} p_2(E_1(\ell))$, which is handled separately in Proposition 5.5.

In the next proposition we will calculate alcove formulas for some elements of W. These are used in determining elements to which subexpressions multiply, and also in computing the sets $\Psi_x^{w(\ell)}$ (the alcove formula for x yields a formula for xq, which is the main ingredient in computing $\Psi_x^{w(\ell)}$).

Before stating the proposition we briefly discuss the proofs of the alcove formulas. In general, if we know the alcove formula for $x \in W$, the alcove formulas for any $s_{\beta,k}x$ can be computed by expressing $s_{\beta,k}$ as a composition of a translation and a simple reflection (using (2.1)) and then using the description of the W-action on alcoves given in (2.6) and (2.7). Now suppose we know the alcove formula for $xA_\circ$, and let L be a line in $\mathbb{R}^2$ parallel to $\beta \in \Phi$ and passing through xq. If $x \in W$ is such that xq lies on L, we can determine from the alcove formula for x the value of i such that xq lies between the lines $H_{\beta,i}$ and $H_{\beta,i+1}$. The next two points on the line L starting from xq in the direction β are $s_{\beta,i+1}xq$ and $s_{\beta,i+2}s_{\beta,i+1}xq = t(\beta^\vee)xq$ (see the proof of Theorem 5.3 in [7]). The alcove formulas for these points can be computed from the alcove formula for x, as discussed above. Replacing β by $-\beta$ gives the results for the points in the direction of $-\beta$. We apply this method to prove the alcove formulas in the proposition, starting with the alcove formulas for $w(\ell)$ from (3.2) and (3.1).

Proposition 5.1. *Let $\ell \geq 4$. The following table gives the alcove formula for $xA_\circ$ and the length $\ell(x)$ for an element x from Definition 4.2 (the length formula for the element $s_{\tilde{\alpha}} p_2(E_3(\ell))$ requires $\ell \geq 6$). In addition, $p_1(E_i(\ell)) = w(\ell - 1)$, where $i = 3$ if ℓ is odd, and $i = 1$ if ℓ is even. Also, if ℓ is even, then $p_1(E_3(\ell)) = s_{\alpha_1, -\frac{\ell-2}{2}} w(\ell)$, and if ℓ is odd, then $p_1(E_1(\ell)) = s_{\tilde{\alpha}, -\frac{\ell-3}{2}} w(\ell)$.*

Proof. To prove the formulas in the table involving $E_1(\ell)$, write $p = p(E_1(\ell)) = w(\ell)$, $p_i = p_i(E_1(\ell)$. The alcove formulas for p in the table are those given in (3.1) and (3.2). Write $\beta_3 = -\alpha$ to be consistent with the notation of [7]. By [7, Cor. 4.7] applied to the alcove formulas for p, we have $i < (p, \beta_3) < i + 1$, where $i = \frac{\ell-2}{2}$ if ℓ is even, or $i = \frac{\ell-3}{2}$ if ℓ is odd. Since $s_{\tilde{\alpha}} p - p$ is a negative multiple of $\beta_3 = -\tilde{\alpha}$, the points p, p_1, p_2 are listed in the direction of $-\beta_3$. Now the method discussed above gives the formulas for p_1 and p_2. The proofs of the other formulas in the table as well as the expressions for $p_1(E_3(\ell))$ (ℓ even) and $p_1(E_1(\ell))$ (ℓ odd) are similar, and we omit them. The equations $p_1(E_i(\ell)) = w(\ell - 1)$, where $i = 3$

if ℓ is odd, and $i = 1$ if ℓ is even, follow from comparing the alcove formulas of the table with the corresponding formulas for $w(\ell-1)$. The lengths are computed from the alcove formulas using (2.8). □

Point	Alcove		Length
	ℓ even	ℓ odd	
On $E_1(\ell)$			
$p(E_1(\ell))$	$E(-\frac{\ell}{2}, 0, \frac{\ell}{2})$	$O(\frac{1-\ell}{2}, 1, \frac{1+\ell}{2})$	ℓ
$p_1(E_1(\ell))$	$O(\frac{2-\ell}{2}, 1, \frac{\ell}{2})$	$E(\frac{1-\ell}{2}, 1, \frac{\ell-3}{2})$	$\ell-1$
$p_2(E_1(\ell))$	$E(\frac{2-\ell}{2}, 1, \frac{\ell-4}{2})$	$O(\frac{3-\ell}{2}, 2, \frac{\ell-3}{2})$	$\ell-2$
$s_{\tilde{\alpha}}p(E_1(\ell))$	$O(0, \frac{\ell}{2}, \frac{4-\ell}{2})$	$E(-1, \frac{\ell-1}{2}, \frac{3-\ell}{2})$	$\ell-1$
$s_{\tilde{\alpha}}p_1(E_1(\ell))$	$E(-1, \frac{\ell-2}{2}, \frac{4-\ell}{2})$	$O(-1, \frac{\ell-1}{2}, \frac{7-\ell}{2})$	$\ell-2$
$s_{\tilde{\alpha}}p_2(E_1(\ell))$	$O(-1, \frac{\ell-2}{2}, \frac{8-\ell}{2})$	$E(-2, \frac{\ell-3}{2}, \frac{7-\ell}{2})$	$\ell-3 \ (\ell \geq 6)$
On $E_3(\ell)$			
$p(E_3(\ell))$	$E(-\frac{\ell}{2}, 0, \frac{\ell}{2})$	$O(\frac{1-\ell}{2}, 1, \frac{1+\ell}{2})$	ℓ
$p_1(E_3(\ell))$	$O(\frac{4-\ell}{2}, 0, \frac{\ell}{2})$	$E(\frac{1-\ell}{2}, 0, \frac{\ell-1}{2})$	$\ell-1$
$p_2(E_3(\ell))$	$E(\frac{4-\ell}{2}, -1, \frac{\ell-2}{2})$	$O(\frac{5-\ell}{2}, 0, \frac{\ell-1}{2})$	$\ell-2$

The next proposition describes the elements $x \leq w(\ell)$ of length $\ell - 1$, and the subexpressions of $w(\ell)$ multiplying to each such element.

Proposition 5.2. *Let $\ell \geq 4$. There are 4 elements $x \in W$ of length $\ell - 1$ such that $x \leq w(\ell)$. There is exactly one subexpression of $w(\ell)$ multiplying to each such x. The 4 elements and the corresponding subexpressions are:*

(a) The subexpression $\mathscr{T}_1 = (1, s_2, s_3, s_1, s_2, s_3, \ldots, s_i, s_j, s_k)$ multiplies to $s_1 w(\ell) \in s_1 E_1(\ell) = E_2(\ell)$.
(b) The subexpression $\mathscr{T}_2 = (s_1, 1, s_3, s_1, s_2, s_3, \ldots, s_i, s_j, s_k)$ multiplies to $s_{\tilde{\alpha}} w(\ell) = s_{\tilde{\alpha}} p(E_1(\ell))$.

(c) *The subexpression* $\mathscr{T}_3 = (s_1, s_2, s_3, s_1, s_2, s_3, \dots, s_i, 1, s_k)$ *multiplies to* $p_1(E_3(\ell))$ *if* ℓ *is even and* $p_1(E_1(\ell))$ *if* ℓ *is odd, and equals* $s_{\alpha_1, -\frac{\ell-2}{2}} w(\ell)$ *if* ℓ *is even,* $s_{\widetilde{\alpha}, -\frac{\ell-3}{2}} w(\ell)$ *if* ℓ *is odd.*

(d) *The subexpression* $\mathscr{T}_4 = (s_1, s_2, s_3, s_1, s_2, s_3, \dots, s_i, s_j, 1)$ *multiplies to the element* $w(\ell - 1)$*, which equals* $p_1(E_1(\ell))$ *if* ℓ *is even, and* $p_1(E_3(\ell))$ *if* ℓ *is odd.*

Proof. Write $w(\ell) = s_1 s_2 s_3 s_1 s_2 \cdots s_i s_j s_k$. If we delete a reflection s_c that is not one of the first two or last two reflections, then the resulting element has length $\leq \ell - 2$, since deleting s_c from $s_a s_b s_c s_a s_b$ yields $s_a s_b s_a s_b = s_b s_a$. Hence the only possible subexpressions of $w(\ell)$ which multiply to length $\ell - 1$ elements are $\mathscr{T}_1, \mathscr{T}_2, \mathscr{T}_3, \mathscr{T}_4$.

We verify the assertions in (a)–(d) about the elements to which these subexpressions multiply. (a) is immediate from the definitions. For (d), it is immediate that $\mathscr{T}_4$ multiplies to $w(\ell - 1)$. By (3.1) and (3.2), $w(\ell - 1)$ equals $O(\frac{2-\ell}{2}, 1, \frac{\ell}{2})$ if ℓ is even, and $E(\frac{1-\ell}{2}, 0, \frac{\ell-1}{2})$ if ℓ is odd. From Proposition 5.1, we see $w(\ell - 1)$ equals $p_1(E_1(\ell))$ if ℓ is even and $p_1(E_3(\ell))$ if ℓ is odd. This proves (d). Part (b) holds since $s_{\widetilde{\alpha}} w(\ell) = (s_1 s_2 s_1)(s_1 s_2 \cdots s_k) = s_1 s_3 w(\ell - 3)$, which is what $\mathscr{T}_3$ multiplies to.

We now prove (c). The subexpression $\mathscr{T}_3$ multiplies to $w(\ell - 2)s_k$. We must calculate the alcove $w(\ell - 2)s_k A_\circ$, and show that it agrees with the formula for $p_1(E_3(\ell))$ (if ℓ is even) or $p_1(E_1(\ell))$ (if ℓ is odd) given in Proposition 5.1. The alcove $w(\ell - 2)s_k A_\circ$ can be calculated using the fact that $w(6n) = t(n(-2\alpha_1^\vee - \alpha_2^\vee))$ is a translation (see Section 3), together with the formulas of (2.6) and (2.7) describing the W-action on alcoves. Indeed, since $\ell \geq 4$, ℓ is of the form $6n + i$, for some $n \geq 1$ and $i \in \{-2, -1, 0, 1, 2, 3\}$. Suppose first that $i = -2$ so $\ell = 6n - 2$. Then $w(\ell) = s_1 s_2 s_3 \cdots s_1 s_2 s_3 s_1$, so $k = 1$, and $w(\ell - 2)s_k = w(6n) s_3 s_2 s_1 s_3 s_1$. Then using (2.6) and (2.7), we have

$$\begin{aligned} w(\ell - 2)s_1 A_\circ &= t(n(-2\alpha_1^\vee - \alpha_2^\vee)) s_3 s_2 s_1 s_3 s_1 A_\circ \\ &= t(n(-2\alpha_1^\vee - \alpha_2^\vee)) O(3, 0, -1) \\ &= O(-3n + 3, 0, 3n - 1) = O(-\frac{\ell}{2} + 2, 0, \frac{\ell}{2}). \end{aligned}$$

which equals $p_1(E_3(\ell))$ (which equals $s_{\alpha_1, -\frac{\ell-2}{2}} w(\ell)$ as ℓ is even, by Proposition 5.1). This proves (c) in case $\ell = 6n + i$, $i = -2$; the calculation for the other values of i is similar, and we omit it.

Finally, observe that all the elements in (a)–(d) coincide with elements listed in Proposition 5.1 and from that proposition we see that they have length $\ell - 1$, as claimed. □

The next proposition identifies the length $\ell - 2$ elements $x \leq w(\ell)$ and the subexpressions of $w(\ell)$ multiplying to them.

Proposition 5.3. *Let* $\ell \geq 5$*. The number of elements* $x \in W$ *of length* $\ell - 2$ *such that* $x \leq w(\ell)$ *is* 7 *if* ℓ *is even, and* 8 *if* ℓ *is odd. If* ℓ *is even (resp. odd) the following* 5 *(resp.* 6*) elements of length* $\ell - 2$ *each have exactly one subexpression* $\mathscr{S}_i$ *multiplying* *to that element, for $i = 1, \ldots, 5$ (resp. $i = 1, \ldots, 6$). The subexpressions $\mathscr{S}_i$ and the corresponding elements are listed in the table below. (If ℓ is even, $\mathscr{S}_6$ multiplies to $s_{\widetilde{\alpha}} p_1(E_1(\ell))$, which has length $\ell - 4$, so we have omitted it from the table.)*

	subexpression	multiplies to	
		if ℓ even	if ℓ odd
$\mathscr{S}_1$	$(1, s_2, s_3, \ldots, s_i, s_j, 1)$	$s_1 p_1(E_1(\ell))$	$s_1 p_1(E_3(\ell))$
$\mathscr{S}_2$	$(1, s_2, s_3, \ldots, s_i, 1, s_k)$	$s_1 p_1(E_3(\ell))$	$s_1 p_1(E_1(\ell))$
$\mathscr{S}_3$	$(1, 1, s_3, \ldots, s_i, s_j, s_k)$	$s_1 s_{\widetilde{\alpha}} p(E_1(\ell))$	
$\mathscr{S}_4$	$(s_1, 1, s_3, \ldots, s_i, s_j, 1)$	$s_{\widetilde{\alpha}} p_1(E_1(\ell))$	$s_{\widetilde{\alpha}} p_1(E_3(\ell))$
$\mathscr{S}_5$	$(s_1, s_2, s_3, \ldots, s_i, 1, 1)$	$p_1(E_3(\ell - 1))$	$p_1(E_1(\ell - 1))$
$\mathscr{S}_6$	$(s_1, 1, s_3, \ldots, s_i, 1, s_k)$		$s_{\widetilde{\alpha}} p_1(E_1(\ell))$

The other two elements of length $\ell - 2$ are $p_2(E_3(\ell)) = t(\alpha_1^\vee)w(\ell)$ and $p_2(E_1(\ell)) = t(\widetilde{\alpha}^\vee)w(\ell)$. The set of subexpressions multiplying to $p_2(E_3(\ell))$ is

$$\{(1, s_2, 1, s_1, \ldots, s_k), (s_1, s_2, 1, s_1, 1, s_3, \ldots, s_k), \ldots\} \tag{5.1}$$

The set of subexpressions multiplying to $p_2(E_1(\ell))$ is

$$\{(s_1, 1, s_3, 1, s_2, \ldots, s_k), (s_1, s_2, s_3, 1, s_2, 1, \ldots, s_k), \ldots\}. \tag{5.2}$$

Proof. First, we show that the subexpressions listed are the only possibilities for length $\ell - 2$ elements, that is, that no other subexpression can multiply to a length $\ell - 2$ element of W. Later, we will show these subexpressions actually do multiply to length $\ell - 2$ elements (except for $\mathscr{S}_6$ when ℓ is even).

Suppose $\mathscr{S} = (\tau_1, \tau_2, \ldots, \tau_\ell)$ is a subexpression of $w(\ell)$ (so each τ_i is a reflection or 1) multiplying to an element x of length $\ell - 2$. If exactly one τ_i were equal to 1 then $\ell(x)$ would be congruent to $\ell - 1$ (mod 2), and if 3 or more of the τ_i were equal to 1, then $\ell(x) \leq \ell - 3$. Hence exactly two of the τ_i are equal to 1. If all such τ_i satisfy $i \in \{1, 2, n-1, n\}$, then $\mathscr{S}$ is one of the subexpressions $\mathscr{S}_1$-$\mathscr{S}_6$ listed in the table. Otherwise, $\tau_i = 1$ for some $i \notin \{1, 2, \ell - 1, \ell\}$. We claim that either $\tau_{i-2} = 1$ or $\tau_{i+2} = 1$. Indeed, there is exactly one $j \neq i$ with $\tau_j = 1$. If $j = i - 1$, then $\mathscr{S}$ contains the sequence $(\ldots, \tau_{i-2}, 1, 1, \tau_{i+1}, \ldots)$. Since $\tau_{i-2} = \tau_{i+1}, \ell(x) \leq \ell - 4$, contradicting our hypothesis that $\ell(x) = \ell - 2$. Hence $j \neq i - 1$; a similar argument shows $j \neq i + 1$. If $|\ j - i\ | > 2$, then $\mathscr{S}$ contains the sequence $(\ldots, \tau_{i-2}, \tau_{i-1}, 1, \tau_{i+1}, \tau_{i+2}, \ldots)$. Since $\tau_{i-2} = \tau_{i+1}$ and $\tau_{i-1} = \tau_{i+2}$, we have $\tau_{i-2}\tau_{i-1}\tau_{i+1}\tau_{i+2} = \tau_{i-1}\tau_{i-2}$, so $\ell(x) \leq \ell - 4$, again a contradiction. Thus $j \in \{i - 2, i + 2\}$, proving the claim. We conclude that $\mathscr{S}$ is one of the subexpressions listed in (5.1) and (5.2).

We conclude that the only possible subexpressions which multiply to length $\ell - 2$ elements are those listed in the statement of the proposition.

We now verify the entries in the table. Let $\mathscr{T}_i$ be as in Proposition 5.2. By inspection (noting that $s_{\widetilde{\alpha}} = s_1 s_2 s_1$), we see (with notation as in Section 2) that

$\prod \mathscr{S}_1 = s_1 w(\ell - 1)$, $\prod \mathscr{S}_2 = s_1 \prod \mathscr{T}_3$, $\prod \mathscr{S}_3 = s_2 s_1 w(\ell) = s_1 s_{\tilde{\alpha}} w(\ell)$, $\prod \mathscr{S}_4 = s_{\tilde{\alpha}} \prod \mathscr{T}_4$, $\prod \mathscr{S}_5 = w(\ell - 2)$, and $\prod \mathscr{S}_6 = s_{\tilde{\alpha}} \prod \mathscr{T}_4$. The remaining entries in the last two columns of the table follow from the above calculations and Proposition 5.2. Note that if ℓ is even, then by Proposition 5.2, this equals $s_{\tilde{\alpha}} p_1(E_3(\ell)) = s_{\tilde{\alpha}} O(\frac{4-\ell}{2}, 0, \frac{\ell}{2}) = E(0, \frac{\ell-4}{2}, \frac{-\ell+4}{2})$, which by (2.8) has length $\ell - 4$, so the entry for $\mathscr{S}_6$ is omitted.

We now verify that each of $\mathscr{S}_1, \dots, \mathscr{S}_5$ (and $\mathscr{S}_6$ if ℓ is odd) multiplies to an element of length $\ell - 2$. This holds for $\mathscr{S}_1$, $\mathscr{S}_3$ and $\mathscr{S}_5$ since these multiply to spiral elements. It holds for $\mathscr{S}_4$, and $\mathscr{S}_6$ if ℓ is odd, by Proposition 5.1. If $\mathscr{S}_2$ multiplies to x, then $\ell(x) = \ell(x^{-1})$ and x^{-1} is of the form $\mathscr{S}_4$ (under some permutation of the simple reflections s_1, s_2, s_3). Since permuting the simple reflections does not change the length, we conclude by the result for $\mathscr{S}_4$ that $\ell(x) = \ell - 2$.

We now turn to the subexpressions listed in (5.1) and (5.2). Observe that the subexpressions $(1, s_j, 1, s_i, s_j)$ and $(s_i, s_j, 1, s_i, 1)$ of $(s_i, s_j, s_k, s_i, s_j, s_k)$ both multiply to the same element. Hence the subexpressions in (5.1) all multiply to the same element, which equals $s_2 w(\ell - 3)$ (as is seen by looking at the first subexpression in the list). Since $t(\alpha_1^\vee) = s_2 s_3 s_2 s_1$, we see that $s_2 w(\ell - 3) = t(\alpha_1^\vee) w(\ell) = p_2(E_3(\ell))$. Similarly, the subexpressions in (5.2) all multiply to $s_1 s_3 s_2 s_3 w(\ell - 6)$, which equals $t(\tilde{\alpha}^\vee) w(\ell) = p_2(E_1(\ell))$ (as $t(\tilde{\alpha}^\vee) = s_3 s_1 s_2 s_1$). Both $p_2(E_3(\ell))$ and $p_2(E_1(\ell))$ have length $\ell - 2$ by Proposition 5.1. One can check that all the elements $x \in W$ listed in the statement of the proposition are distinct by computing the center points xq and observing that these points are distinct. This completes the proof. □

Remark 5.4. Proposition 5.3 remains true if $\ell = 4$, except that some of the subexpressions coincide, so there are not as many distinct elements. In particular, $\mathscr{S}_2$ coincides with (5.1) and $\mathscr{S}_4$ coincides with (5.2). Also, if $\ell = 3$, the statement about $p_2(E_3(\ell))$ remains true: indeed, $p_2(E_3(\ell)) = s_2$ and there is one subexpression $(1, s_2, 1)$ multiplying to this.

As explained in Section 4, one of the elements x such that we need to determine $e_x^{w(\ell)}$ is $x = s_{\tilde{\alpha}} p_2(E_1(\ell))$. Since this element has length $\ell - 3$, the subexpressions of $w(\ell)$ multiplying to x are not listed above. We list them in the next proposition.

Proposition 5.5. *Let $\ell \geq 5$. The element $s_{\tilde{\alpha}} p_2(E_1(\ell))$ has length $\ell - 3$. The subexpressions of $w(\ell)$ multiplying to $s_{\tilde{\alpha}} p_2(E_1(\ell))$ are as follows:*

$$(s_1, s_2, s_3, 1, s_2, s_3, s_1, s_2, s_3, s_1, \dots, s_i, s_j, s_k)$$
$$(s_1, 1, s_3, 1, s_2, 1, s_1, s_2, s_3, s_1, \dots, s_i, s_j, s_k)$$
$$(s_1, 1, s_3, s_1, s_2, 1, s_1, 1, s_3, s_1, \dots, s_i, s_j, s_k)$$
$$(s_1, 1, s_3, s_1, s_2, s_3, s_1, 1, s_3, 1, \dots, s_i, s_j, s_k)$$
$$\cdots.$$

Proof. For brevity, write $m(\ell) = s_{\tilde{\alpha}} p_2(E_1(\ell))$. We must show that all the subexpressions listed in the statement multiply to $m(\ell)$. First, we observe that they all multiply to the same element. Indeed, this holds for the first and second subexpressions, since they agree after the first 6 terms, and $s_1 s_2 s_3 s_2 s_3 = s_1 s_3 s_2$. For all the subexpressions except the first, if we remove the first two entries (that is, s_1 and 1) we are left with a subexpression that is of the form (5.2) up to a permutation of the simple reflections. Since all the subexpressions in (5.2) multiply to the same element, we see that all the subexpressions in the statement of this proposition multiply to the same element.

Since $p_2(E_1(\ell)) = t(\widetilde{\alpha}^\vee)w(\ell)$ by the discussion preceding Proposition 5.1, and $s_{\widetilde{\alpha}} = s_1s_2s_1$, $t(\widetilde{\alpha}^\vee) = s_3s_1s_2s_1$ (see (2.4)), we can compute $m(\ell)$ directly from its definition, and after simplifying, we obtain $s_1s_3s_2w(\ell-6)$. Since the second subexpression listed in the statement of the proposition multiplies to this element, we see that all the subexpressions in the statement multiply to $m(\ell)$.

To complete the proof we must show that these are the only subexpressions of $w(\ell)$ multiplying to $m(\ell)$. Observe that $\ell(s_3s_2m(\ell)) = \ell - 1$. Indeed, from the alcove formulas for $m(\ell)$ given in Proposition 5.1, we see that the alcove formula for $s_3s_2m(\ell)$ is $E(\frac{\ell-1}{2}, \frac{7-\ell}{2}, -3)$ if ℓ is odd, and $O(2, \frac{\ell}{2}-1, 1-\frac{\ell}{2})$ if ℓ is even. In either case, (2.8) shows that $\ell(x) = \ell - 1$.

Now suppose that $\mathscr{S}$ is a subexpression of $w(\ell)$ multiplying to $m(\ell)$. We want to show that $\mathscr{S}$ is one of the subexpressions in the statement of the proposition. We claim that the first two entries of $\mathscr{S}$ cannot both be 1. Indeed, suppose both are 1. Then the third is s_3: otherwise $\mathscr{S}$ would multiply to an element which is less than or equal to $w(\ell-3)$ in the Bruhat order (cf. Section 2), but the only such element of length $\ell-3$ is $w(\ell-3) \neq m(\ell)$. Hence $m(\ell) = s_3x$, where $x \le w(\ell-3)$ and $x = s_3m(\ell)$. If ℓ is even, then $s_3m(\ell)A_\circ = E(0, \frac{\ell}{2}-1, 1-\frac{\ell}{2})$ which has length $\ell-2$, contradicting $x \le w(\ell-3)$. If ℓ is odd, then $s_3m(\ell)A_\circ = O(\frac{5-\ell}{2}, 3, \frac{\ell-7}{2})$, which has length $\ell-4$. Since $x \le w(\ell-3)$, Proposition 5.2 implies x is one of the elements $s_1w(\ell-3), s_{\widetilde{\alpha}}w(\ell-3), w(\ell-4)$ or $s_{\alpha_1,-\frac{\ell-5}{2}}w(\ell-3)$. One can compute the alcove formulas for these four elements and verify that none equals $s_3m(\ell)A_\circ$; this is a contradiction. The claim follows.

Next, the first entry of $\mathscr{S}$ cannot be 1. Indeed, if this entry were 1, then by the previous claim, the second term would be s_2, so the first three entries would be $1, s_2, s_3$ or $1, s_2, 1$. Then $m(\ell)$ would equal s_2s_3x or s_2x, where $x \le w(\ell-3)$. Then $x = s_3s_2m(\ell)$ or $x = s_2m(\ell)$. In either case, $\ell(x) \ge \ell - 2$ (since $\ell(s_3s_2m(\ell)) = \ell - 1$), which is impossible, since the length of $w(\ell-3)$ is $\ell-3$.

We have shown that the first entry of $\mathscr{S}$ is s_1. The second entry is either 1 or s_2. Suppose that the second entry is 1. Then $s_1x = m(\ell)$, so $x = s_1m(\ell)$. Using the second subexpression in the statement of the proposition for $m(\ell)$, we see $x = s_31s_21s_1s_2\cdots s_is_js_k$, so $x < s_3s_1s_2\cdots s_is_js_k$. Hence $f(x) = p_2(E_1(\ell)) < f(s_3s_1s_2\cdots s_is_js_k) = w(\ell-2)$, where f is the automorphism of W such that $f(s_3) = s_1, f(s_1) = s_2, f(s_2) = s_3$. By Proposition 5.2, the only subexpressions of $w(\ell-2)$ multiplying to $p_2(E_1(\ell))$ are $(s_1, 1, s_3, 1\ldots,)$, $(s_1, s_2, s_3, 1, s_2, 1, \ldots)$, ..., so, writing $g = f^{-1}$, we see that the only subexpressions of $(s_3, s_1, s_2, \ldots)$ multiplying to x are $(g(s_1), 1, g(s_3), 1, \ldots,)$, $(g(s_1), g(s_2), g(s_3), 1, g(s_2), 1, \ldots)$, ..., that is, $(s_3, 1, s_2, 1, \ldots)$, $(s_3, s_1, s_2, 1, s_1, 1, \ldots)$, ... We conclude that $\mathscr{S}$ is one of the subexpressions in the statement of the proposition.

Finally, suppose that the second entry of $\mathscr{S}$ is s_2. Then $s_1s_2x = m(\ell)$, where $x = s_2s_1m(\ell) \le y := s_3s_1s_2s_3\cdots$. We have $s_2s_1m(\ell)A_\circ = E(\frac{\ell-3}{2}, \frac{5-\ell}{2}, -1)$ (resp. $O(\frac{\ell}{2}-1, 3-\frac{\ell}{2}, 0)$) if ℓ is even (resp. ℓ is odd), so (2.8) implies that

$\ell(x) = \ell - 3$. With f as above, we have $f(y) = w(\ell - 2)$, and, by reasoning similar to the previous paragraph, we deduce from Proposition 5.2 that all 4 length $\ell - 3$ subexpressions of $(s_3, s_1, s_2, s_3, \dots)$ multiply to different elements. Hence the only subexpression of y multiplying to x is $(s_3, 1, s_2, s_3, s_1, \dots)$. This implies that $\mathscr{S} = (s_1, s_2, s_3, 1, s_2, s_3, s_1, \dots)$, completing the proof. □

6 Some lemmas

In order to perform the calculations of equivariant multiplicities in the next section, we need to know the sets $\Psi_x^{w(\ell)}$ for various x, and also to identify certain roots of the form $x(\alpha)$, where α is simple. These results are given (respectively) in Lemma 6.4 and Lemma 6.6. The reader may wish to read the next section, where these results are used, before reading this section.

We will compute the sets $\Psi_x^{w(\ell)}$ (and thus prove Lemma 6.4) by computing related sets $\Lambda_x^{w(\ell)}$ and adjusting. The reason we use $\Lambda_x^{w(\ell)}$ is that if we know the point $xq \in \mathbb{R}^2$ (that is, the center point of the alcove corresponding to x), then we can use [7, Proposition 9.6] to compute $\Lambda_x^{w(\ell)}$. The formulas for xq are given in Lemma 6.1, and the adjustment to obtain $\Psi_x^{w(\ell)}$ in Lemma 6.2. Note that there is some duplication in the table in Lemma 6.1, since by Proposition 5.2, $w(\ell - 1)$ equals $p_1(E_1(\ell))$ if ℓ is even, and $p_1(E_3(\ell))$ if ℓ is odd.

Lemma 6.1. *The following table lists the center points xq for certain $x \in W$. For ℓ odd, among the elements listed in the table, $p_1(E_1(\ell))$ and $s_{\tilde{\alpha}} p_2(E_1(\ell))$ are of the form $s_{\tilde{\alpha},m} A_i(\ell)$ for some $i = 1, 2$ and some $m \in \mathbb{Z}$; the others are not.*

Element x	Center Point xq	
	ℓ even	ℓ odd
$w(\ell - 1)$	$\frac{2-\ell}{3}\alpha_1^\vee + \frac{4-\ell}{6}\alpha_2^\vee$	$\frac{2-\ell}{3}\alpha_1^\vee + \frac{3-\ell}{6}\alpha_2^\vee$
$p_1(E_1(\ell))$	$\frac{2-\ell}{3}\alpha_1^\vee + \frac{4-\ell}{6}\alpha_2^\vee$	$\frac{3-\ell}{3}\alpha_1^\vee + \frac{7-\ell}{6}\alpha_2^\vee$
$p_2(E_1(\ell))$	$\frac{4-\ell}{3}\alpha_1^\vee + \frac{8-\ell}{6}\alpha_2^\vee$	$\frac{4-\ell}{3}\alpha_1^\vee + \frac{9-\ell}{6}\alpha_2^\vee$
$p_1(E_3(\ell))$	$\frac{3-\ell}{3}\alpha_1^\vee + \frac{2-\ell}{6}\alpha_2^\vee$	$\frac{2-\ell}{3}\alpha_1^\vee + \frac{3-\ell}{6}\alpha_2^\vee$
$p_2(E_3(\ell))$	$\frac{4-\ell}{3}\alpha_1^\vee + \frac{2-\ell}{6}\alpha_2^\vee$	$\frac{4-\ell}{3}\alpha_1^\vee + \frac{3-\ell}{6}\alpha_2^\vee$
$s_{\tilde{\alpha}} p_1(E_1(\ell))$	$\frac{\ell-4}{6}\alpha_1^\vee + \frac{\ell-2}{3}\alpha_2^\vee$	$\frac{\ell-7}{6}\alpha_1^\vee + \frac{\ell-3}{3}\alpha_2^\vee$
$s_{\tilde{\alpha}} p_2(E_1(\ell))$	$\frac{\ell-8}{6}\alpha_1^\vee + \frac{\ell-4}{3}\alpha_2^\vee$	$\frac{\ell-9}{6}\alpha_1^\vee + \frac{\ell-4}{3}\alpha_2^\vee$

Proof. By [7, Proposition 4.8], if $\beta_1 = \alpha_1$, $\beta_2 = \alpha_2$ and $\beta_3 = -(\alpha_1 + \alpha_2)$, then the center point of $X(a_1, a_2, a_3)$ is

$$\frac{1}{3}(a_1\beta_1^\vee + a_2\beta_2^\vee + (a_3 - 1)\beta_3^\vee) = \frac{1}{3}((a_1 - a_3 + 1)\alpha_1^\vee + (a_2 - a_3 + 1)\alpha_2^\vee).$$

Applying this to the alcove formulas in Proposition 5.1 gives the center points in the table.

We now verify the assertion about elements of the form $s_{\widetilde{\alpha},m}A_i(\ell)$ for ℓ odd. Since $p_2(E_3(\ell))q - A_i(\ell)q$ is not a multiple of $\widetilde{\alpha}$, $p_2(E_3(\ell))$ is not of the form $s_{\widetilde{\alpha},m}A_i(\ell)$. Since $A_2(\ell) = t(\frac{\ell-1}{2}\widetilde{\alpha}^\vee)w(\ell-1)$ and $A_1(\ell) = s_1A_2(\ell)$, we see that $w(\ell-1)$ is not of the form $s_{\widetilde{\alpha},m}A_2(\ell)$ or $s_{\widetilde{\alpha},m}A_1(\ell)$.

The other elements in the table are all on $E_1(\ell)$. By [7, Proposition 10.2] and its proof, an element x on $E_1(\ell)$ is of the form $s_{\widetilde{\alpha},m}A_i(\ell) \Leftrightarrow x$ is of the form $s_{\widetilde{\alpha},k}w(\ell)$; because reflections and translations have opposite parity, this does not hold if x is of the form $t(c\widetilde{\alpha}^\vee)w(\ell)$. Therefore, which of the elements on $E_1(\ell)$ are of the form $s_{\widetilde{\alpha},m}A_i(\ell)$ can be determined by inspection from the equations below.

$$p_2(E_1(\ell)) = t(\widetilde{\alpha}^\vee)w(\ell)$$

$$s_{\widetilde{\alpha}}p_2(E_1(\ell)) = s_{\widetilde{\alpha},-1}w(\ell)$$

$$p_1(E_1(\ell)) = s_{\widetilde{\alpha},\frac{3-k}{2}}w(\ell)$$

$$s_{\widetilde{\alpha}}p_1(E_1(\ell)) = t(\frac{k-3}{2}\widetilde{\alpha}^\vee)w(\ell).$$

These equations can be verified by computing the alcove formulas (cf. the remarks before Proposition 5.1); we omit the calculations. □

If ℓ is odd, we define $\Lambda_x^{w(\ell)} = \Psi_x^{w(\ell)} \sqcup \{\alpha \in \Delta_{\mathrm{re}}^+ \mid s_\alpha x = A_i(\ell), i = 1, 2\}$. As with the sets $\Psi_x^{w(\ell)}$, this definition has been modified from the definition in [7] by replacing reflections by the corresponding roots. If $xq = a_1\alpha_1^\vee + a_2\alpha_2^\vee$, then $\Lambda_x^{w(\ell)}$ is a union of intervals of the form $[a, b]_\alpha$ (cf. Section 2.3), which can be computed from a_1 and a_2 by [7, Prop. 9.6]. The next lemma tells us how to obtain $\Psi_x^{w(\ell)}$ from $\Lambda_x^{w(\ell)}$ for x as in the previous lemma (except for $x = p_2(E_3(\ell))$, which is not on $E_1(\ell)$ or $E_1(\ell-1)$, and so is considered in Remark 6.3).

Lemma 6.2. *Suppose ℓ is odd. Write $w = w(\ell)$, and suppose that $x \leq w$. Suppose that $xq \in E_1(\ell)$ or $E_1(\ell-1)$, and that $\Lambda_x^w = [a_1, b_1]_{\alpha_1} \cup [a_2, b_2]_{\alpha_2} \cup [\widetilde{a}, \widetilde{b}]_{\widetilde{\alpha}}$. If $s_{\widetilde{\alpha},\widetilde{b}}x = A_i(\ell)$ for $i = 1$ or $i = 2$, then $\Psi_x^w = [a_1, b_1]_{\alpha_1} \cup [a_2, b_2]_{\alpha_2} \cup [\widetilde{a}, \widetilde{b} - 1]_{\widetilde{\alpha}}$. Otherwise, $\Psi_x^w = \Lambda_x^w$.*

Proof. If there are no reflections r such that $rx = A_i(\ell)$, then $\Lambda_x^w = \Psi_x^w$. Suppose then that $rx = A_i(\ell)$ for some $i \in \{1, 2\}$ and some reflection r. If $xq \in E_1(\ell)$, we claim that $A_1(\ell) = s_{\widetilde{\alpha},m}x$ for some m. Indeed, [7, Prop. 10.2] implies that x is of the form $s_{\widetilde{\alpha},k}w(\ell)$, but $w(\ell) = t(\frac{1-\ell}{2}\alpha^\vee)A_1(\ell)$. Substituting and simplifying (using (2.1)) implies the claim. Our hypothesis implies $m \in [\widetilde{a}, \widetilde{b}]$. Since $s_{\widetilde{\alpha},m+1}xq = t(\widetilde{\alpha}^\vee)s_{\widetilde{\alpha},m}xq = t(\widetilde{\alpha}^\vee)A_1(\ell)q \notin \triangle(\ell)$, we have $m+1 \notin [\widetilde{a}, \widetilde{b}]$. Hence $m = \widetilde{b}$, so

$$\Psi_x^w \subseteq \Lambda_x^w \setminus \{\widetilde{b}\}_{\widetilde{\alpha}}. \tag{6.1}$$

Since by [7, Lemma 9.4], $|\Psi_x^w| \geq |\Lambda_x^w| - 1$, we conclude that (6.1) is an equality.

Next suppose that $xq \in E_1(\ell-1)$. Again using [7, Prop. 10.2], we deduce that $A_2(\ell) = s_{\widetilde{\alpha},m}x$ for some m. The rest of the argument proceeds as in the previous paragraph; we omit the details. This proves the lemma. □

Remark 6.3. If ℓ is odd and $x = p_2(E_3(\ell))$, then $\Psi_x^{w(\ell)} = \Lambda_x^{w(\ell)}$. The reason is that for $i = 1, 2$, $xq - A_i(\ell)q$ is not a multiple of a root. Hence there is no reflection r with $rx = A_i(\ell)$.

Lemma 6.4. *Let* $\ell \geq 4$.

(1)

$$\Psi_{w(\ell-1)}^{w(\ell)} = \begin{cases} [1-\frac{\ell}{2}, 0]_{\alpha_1} \cup \varnothing_{\alpha_2} \cup [1-\frac{\ell}{2}, 0]_{\widetilde{\alpha}} & \textit{if } \ell \textit{ is even} \\ [\frac{1-\ell}{2}, 0]_{\alpha_1} \cup \varnothing_{\alpha_2} \cup [\frac{3-\ell}{2}, 0]_{\widetilde{\alpha}} & \textit{if } \ell \textit{ is odd.} \end{cases}$$

(2)

$$\Psi_{p_2(E_1(\ell))}^{w(\ell)} = \begin{cases} [2-\frac{\ell}{2}, 0]_{\alpha_1} \cup \{1\}_{\alpha_2} \cup [2-\frac{\ell}{2}, 1]_{\widetilde{\alpha}} & \textit{if } \ell \textit{ is even} \\ [\frac{3-\ell}{2}, 0]_{\alpha_1} \cup \{1\}_{\alpha_2} \cup [\frac{5-\ell}{2}, 1]_{\widetilde{\alpha}} & \textit{if } \ell \textit{ is odd.} \end{cases}$$

(3)

$$\Psi_{p_2(E_3(\ell))}^{w(\ell)} = \begin{cases} [2-\frac{\ell}{2}, 1]_{\alpha_1} \cup \{0\}_{\alpha_2} \cup [2-\frac{\ell}{2}, 0]_{\widetilde{\alpha}} & \textit{if } \ell \textit{ is even} \\ [\frac{3-\ell}{2}, 1]_{\alpha_1} \cup \{0\}_{\alpha_2} \cup [\frac{5-\ell}{2}, 0]_{\widetilde{\alpha}} & \textit{if } \ell \textit{ is odd.} \end{cases}$$

(4)

$$\Psi_{s_{\widetilde{\alpha}}p_1(E_1(\ell))}^{w(\ell)} = \begin{cases} \{0\}_{\alpha_1} \cup [1, \frac{\ell}{2}-1]_{\alpha_2} \cup [0, \frac{\ell}{2}-1]_{\widetilde{\alpha}}. & \textit{if } \ell \textit{ is even} \\ [-1, 0]_{\alpha_1} \cup [1, \frac{\ell-3}{2}]_{\alpha_2} \cup [0, \frac{\ell-3}{2}]_{\widetilde{\alpha}} & \textit{if } \ell \textit{ is odd.} \end{cases}$$

(5)

$$\Psi_{s_{\widetilde{\alpha}}p_2(E_1(\ell))}^{w(\ell)} = \begin{cases} [-1, 0]_{\alpha_1} \cup [1, \frac{\ell}{2}-2]_{\alpha_2} \cup [-1, \frac{\ell}{2}-2]_{\widetilde{\alpha}} & \textit{if } \ell \textit{ is even} \\ [-1, 0]_{\alpha_1} \cup [1, \frac{\ell-3}{2}]_{\alpha_2} \cup [-1, \frac{\ell-5}{2}]_{\widetilde{\alpha}}. & \textit{if } \ell \textit{ is odd.} \end{cases}$$

(6) (a) $\Psi_{p_1(E_3(\ell))}^{w(\ell)} = [1-\frac{\ell}{2}, 0]_{\alpha_1} \cup \{0\}_{\alpha_2} \cup [2-\frac{\ell}{2}, 0]_{\widetilde{\alpha}}$ *if* ℓ *is even*

(b) $\Psi_{p_1(E_1(\ell))}^{w(\ell)} = [\frac{3-\ell}{2}, 0]_{\alpha_1} \cup \{1\}_{\alpha_2} \cup [\frac{3-\ell}{2}, 0]_{\widetilde{\alpha}}$ *if* ℓ *is odd.*

Proof. Let $x \in W$ be one of the elements for which we want $\Psi_x^{w(\ell)}$. The formula for xq is given in Lemma 6.1. From this, we can calculate $\lambda_1(xq)$, $\lambda_2(xq)$ and $\lambda_{21}(xq)$, and then use [7, Proposition 9.6] to obtain Λ_x^w. Lemma 6.1 tells us if x is of the form $s_{\widetilde{\alpha},m}A_i(\ell)$, and then Lemma 6.2 and Remark 6.3 tell us how to obtain $\Psi_x^{w(\ell)}$ from Λ_x^w. To illustrate, we carry this out for the case $x = p_1(E_1(\ell))$, (ℓ odd), the other cases being similar. If $x = p_1(E_1(\ell))$, then $xq = \frac{6-2\ell}{6}\alpha_1 + \frac{7-\ell}{6}\alpha_2$, so $\lambda_1(xq) = \frac{6-2\ell}{6}$, $\lambda_2(xq) = \frac{7-\ell}{6}$, $\lambda_{21}(xq) = \frac{\ell+1}{6}$. So by [7, Proposition 9.6], $\Lambda_x^w = [-\frac{\ell-3}{2}, 0]_{\alpha_1} \cup \{1\}_{\alpha_2} \cup [-\frac{\ell-3}{2}, 1]_{\widetilde{\alpha}}$. Since x is of the form $s_{\widetilde{\alpha},m}A_i(\ell)$, we have $\Psi_x^{w(\ell)} = [-\frac{\ell-3}{2}, 0]_{\alpha_1} \cup \{1\}_{\alpha_2} \cup [-\frac{\ell-3}{2}, 0]_{\widetilde{\alpha}}$. □

Remark 6.5. The formula for $\Psi^{w(\ell)}_{p_2(E_3(\ell))}$ remains true for $\ell = 3$ (the root interval $[2, 0]_{\widetilde{\alpha}}$ is the empty set).

Lemma 6.6. *Let $w(\ell) = s_1 s_2 \cdots s_i s_j s_k$.*

(1) Let $\ell \geq 2$ be even and write $y(\ell) = t(\widetilde{\alpha}^\vee)w(\ell)$. (If $\ell \geq 6$, then $y(\ell) = p_2(E_1(\ell))$.) Then

(a) $y(\ell)(\alpha_i) = \{-\frac{\ell}{2} + 1\}_{\alpha_1}$
(b) $y(\ell)(\alpha_j) = -\{1\}_{\alpha_2}$
(c) $y(\ell)(\alpha_i + \alpha_j) = \{-\frac{\ell}{2} + 2\}_{\widetilde{\alpha}}$.

(2) Let $\ell \geq 2$ be even. Write $m(\ell) = s_{\widetilde{\alpha}}y(\ell)$. Then

(a) $m(\ell)(\alpha_i) = \{\frac{\ell}{2} - 1\}_{\alpha_2}$
(b) $m(\ell)(\alpha_j) = -\{-1\}_{\alpha_1}$.
(c) $m(\ell)(\alpha_i + \alpha_j) = \{\frac{\ell}{2} - 2\}_{\widetilde{\alpha}}$.

(3) Let $\ell \geq 3$ be odd. Write $u(\ell) = t(\alpha_1^\vee)w(\ell)$. (If $\ell \geq 7$, then $u(\ell) = p_2(E_3(\ell))$.)

(a) $u(\ell)(\alpha_i) = \{-\frac{\ell-3}{2}\}_{\widetilde{\alpha}}$
(b) $u(\ell)(\alpha_j) = -\{0\}_{\alpha_2}$.
(c) $u(\ell)(\alpha_i + \alpha_j) = \{-\frac{\ell-3}{2}\}_{\alpha_1}$.

Proof. (1) We first prove (a) and (b). First, by Proposition 2.6, $\{1\}_{\alpha_2} = \alpha_1 + \alpha_3$, and if $\ell \geq 2$ is even, $\{-\frac{\ell}{2} + 1\}_{\alpha_1} = \frac{\ell}{2}(\alpha_1 + \alpha_2 + \alpha_3) - (\alpha_2 + \alpha_3)$. Also, by (2.4), $y(\ell) = t(\widetilde{\alpha}^\vee)w(\ell) = s_3 s_1 s_2 s_3 w(\ell)$. By direct calculation, assertions (a) and (b) hold for $\ell = 2$, $\ell = 4$, and $\ell = 6$. (For $\ell = 2$, $w(\ell) = s_1 s_2$ and our convention is $i = 3$, $j = 1$, $k = 2$.) We now show that if these assertions hold for ℓ, then they hold for $\ell + 6$. Observe that

$$y(\ell + 6) = t(\widetilde{\alpha}^\vee)w(\ell + 6) = t(\widetilde{\alpha}^\vee)w(6)w(\ell) = w(6)t(\widetilde{\alpha}^\vee)w(\ell) = w(6)y(\ell),$$

since $w(6)$ is a translation and therefore commutes with $t(\widetilde{\alpha}^\vee)$. Also, in passing from ℓ to $\ell + 6$, the roles of i, j and k remain unchanged. Hence $y(\ell + 6)\alpha_i = w(6)y(\ell)\alpha_i$, which by our inductive hypothesis equals $w(6)(\frac{\ell}{2}(\alpha_1 + \alpha_2 + \alpha_3) - (\alpha_2 + \alpha_3))$. This can be calculated directly since $w(6) = s_1 s_2 s_3 s_1 s_2 s_3$, and is equal to $(\frac{\ell+6}{2})(\alpha_1+\alpha_2+\alpha_3)-(\alpha_2+\alpha_3) = \{-\frac{(\ell+6)}{2}+1\}_{\alpha_1}$, as desired. (The calculation is simplified by the fact that $s_a(\alpha_1+\alpha_2+\alpha_3) = \alpha_1+\alpha_2+\alpha_3$ for $a = 1, 2, 3$.) Similarly, we have $y(\ell+6)\alpha_j = w(6)y(\ell)\alpha_j = w(6)(-(\alpha_1+\alpha_3))$, which by direct calculation equals $-(\alpha_1+\alpha_3) = -\{1\}_{\alpha_2}$. This proves (a) and (b). Part (c) follows from (a) and (b), since by Proposition 2.6 we have $\{-\frac{\ell}{2}+1\}_{\alpha_1} + -\{1\}_{\alpha_2} = \{-\frac{\ell}{2}+2\}_{\tilde{\alpha}}$.

(2) The formula for each $m(\ell)(\beta)$ follows by applying $s_{\tilde{\alpha}}$ to the formula from (1) for $y(\ell)(\beta)$ and using Proposition 2.6; we omit the details.

(3) Observe that $\{-\frac{\ell-3}{2}\}_{\tilde{\alpha}} = \frac{\ell-1}{2}(\alpha_1+\alpha_2+\alpha_3)-\alpha_3$. We prove (a) and (b) by induction on ℓ. The result is true for $\ell = 3$, $\ell = 5$ and $\ell = 7$ by direct computation. Assume that assertions (a) and (b) hold for ℓ; we prove they hold for $\ell+6$. Observe that as in part (1), $u(\ell+6) = w(6)u(\ell)$, and the roles of i, j, and k are unchanged when we pass from ℓ to $\ell+6$. Our inductive hypothesis implies that

$$u(\ell+6)(\alpha_i) = w(6)u(\ell)(\alpha_i) = w(6)(\frac{\ell-1}{2}(\alpha_1+\alpha_2+\alpha_3)-\alpha_3),$$

which equals $\frac{\ell+5}{2}(\alpha_1+\alpha_2+\alpha_3)-\alpha_3$, so (a) holds for $\ell+6$. Similarly, since $\{0\}_{\alpha_2} = \alpha_2$, we have $u(\ell+6)\alpha_j = w(6)u(\ell)(\alpha_j) = w(6)(-\alpha_2) = -\alpha_2$, so (b) holds for $\ell+6$ as well. This proves (a) and (b); part (c) then follows by using Proposition 2.6 as in (1). □

7 Calculation of equivariant multiplicities

In this section we prove Proposition 4.5 and thereby complete the proof of Theorem 4.5. We need to calculate $e_x^{w(\ell)}$, where x is one of the elements $p_2(E_1(\ell))$, $s_{\tilde{\alpha}}p_2(E_1(\ell))$, or $p_2(E_3(\ell))$. We consider these three elements separately. The pattern of the arguments is almost identical for the first two elements $p_2(E_1(\ell))$ and $s_{\tilde{\alpha}}p_2(E_1(\ell))$, but is slightly different for the third element $p_2(E_3(\ell))$. As discussed in Remark 4.6, we omit most of the proof in the case $s_{\tilde{\alpha}}p_2(E_1(\ell))$.

To simplify the notation in the proofs, we will use the same notation for a set of roots and the product of the elements in that set (which is a polynomial in the roots). For example, $\Psi_{u(3)}^{w(3)}$ is the set $\{\alpha_1, \alpha_2, \alpha_2+\alpha_3\}$, but it will be convenient to write simply $\Psi_{u(3)}^{w(3)}$ instead of $\prod_{\beta\in\Psi_{u(3)}^{w(3)}}\beta$ for the product of the elements of this set. Similarly, $[0,1]_{\alpha_1}[1,3]_{\tilde{\alpha}}$ denotes the product of the roots in the union of these two root intervals.

If $\mathscr{S}$ is a subexpression, let $(\mathscr{S}, s_a)$ (resp. $(\mathscr{S}, s_a, s_b)$) denote $\mathscr{S}$ with s_a (resp. s_a, s_b) appended to the end.

7.1 $p_2(E_1(\ell))$

For brevity, write $y(\ell) = p_2(E_1(\ell)) = t(\widetilde{\alpha}^\vee)w(\ell)$. Since Proposition 4.4 has been proved, we can reformulate the assertion of Proposition 4.5 for $e^{w(\ell)}_{y(\ell)}$ as saying that for $n \geq 2$, if $\ell = 2n$ or $2n+1$, then

$$e^{w(\ell)}_{y(\ell)} = \frac{n-1}{\Psi^{w(\ell)}_{y(\ell)}}. \tag{7.1}$$

First, we show (7.1) for $\ell = 2n$. If $n = 2$, then $w(4) = s_1 s_2 s_3 s_1$ and by Proposition 5.3, $y(4) = s_1 1 s_3 1$, so

$$e^{w(4)}_{y(4)} = (s_1(\alpha_1)s_1(\alpha_2)s_1 s_3(\alpha_3)s_1 s_3(\alpha_1))^{-1} = (\alpha_1(\alpha_1+\alpha_2)(\alpha_1+\alpha_3)\alpha_3)^{-1}.$$

On the other hand, Lemma 6.4 and Proposition 2.6 imply that

$$\Psi^{w(4)}_{y(4)} = \{0\}_{\alpha_1}\{1\}_{\alpha_2}[0,1]_{\widetilde{\alpha}} = \alpha_1(\alpha_1+\alpha_3)(\alpha_1+\alpha_2)\alpha_3.$$

Hence the result holds for $n = 2$.

Assume now that (7.1) holds for $\ell = 2n$. We will show it holds for $\ell = 2n+2$. Write $w(\ell) = s_1 s_2 \cdots s_i s_j s_k$ and $w(\ell+2) = s_1 s_2 \cdots s_k s_i s_j$. If $\mathscr{S}$ is a subexpression of $w(\ell)$ multiplying to $y(\ell)$, then $(\mathscr{S}, s_i, s_j)$ is a subexpression of $w(\ell+2)$ multiplying to $y(\ell+2)$. Proposition 5.3 implies that all the subexpressions of $w(\ell+2)$ multiplying to $y(\ell+2)$ are of this form, except for $(s_1, \ldots, 1, s_i, 1)$. Therefore, we can write $e^{w(\ell+2)}_{y(\ell+2)}$ as a sum of two terms. The first term is the sum of all contributions from the subexpressions $(\mathscr{S}, s_i, s_j)$ and the second term is the contribution from the subexpression $(s_1, \ldots, 1, s_i, 1)$.

We now calculate these terms separately. The formula (2.3) for equivariant multiplicities implies that the first term is $e^{w(\ell)}_{y(\ell)} \cdot \dfrac{1}{y(\ell)s_i(\alpha_i)} \cdot \dfrac{1}{y(\ell)s_i s_j(\alpha_j)}$. Since $s_i(\alpha_i) = -\alpha_i$ and $s_i s_j(\alpha_j) = -(\alpha_i + \alpha_j)$, this equals

$$e^{w(\ell)}_{y(\ell)} \cdot \frac{1}{y(\ell)(\alpha_i+\alpha_j)} \cdot \frac{1}{y(\ell)(\alpha_i)}.$$

Recall from Lemma 6.6 that

$$y(\ell)(\alpha_i) = \{-n+1\}_{\alpha_1}, \quad y(\ell)(\alpha_j) = -\{1\}_{\alpha_2}$$
$$\text{and} \quad y(\ell)(\alpha_i+\alpha_j) = \{-n+2\}_{\widetilde{\alpha}}.$$

Combining this with our inductive hypothesis, we see that the first term is

$$\frac{n-1}{\Psi^{w(\ell)}_{y(\ell)}} \cdot \frac{1}{\{-n+2\}_{\widetilde{\alpha}}} \cdot \frac{1}{\{-n+1\}_{\alpha_1}}. \tag{7.2}$$

We claim that the second term, which is the contribution from the subexpression $(s_1, \ldots, 1, s_i, 1)$, is equal to

$$e_{w(\ell-1)}^{w(\ell)} \cdot \frac{1}{w(\ell-1)s_i(\alpha_i)} \cdot \frac{1}{w(\ell-1)s_i(\alpha_j)}.$$

Indeed, this second term is a product of $\ell + 2$ factors, each of the form $\frac{1}{\beta}$ for some root β. The product of the first ℓ factors is $e_{w(\ell-1)}^{w(\ell)}$, since by Proposition 5.2, the only subexpression of $w(\ell)$ multiplying to $w(\ell-1)$ is $(s_1, \ldots, s_i, s_j, 1)$. The last two factors correspond to $\beta = w(\ell-1)s_i(\alpha_i)$ and $\beta = w(\ell-1)s_i(\alpha_j)$. This proves the claim.

We can rewrite the expression for the second term as follows. Since $w(\ell-1) = y(\ell)s_i s_j s_i$, we have

$$w(\ell-1)s_i(\alpha_i) = y(\ell)(\alpha_j) \quad \text{and} \quad w(\ell-1)s_i(\alpha_j) = -y(\ell)(\alpha_i + \alpha_j).$$

Also, by Theorem 2.3, $w(\ell-1)\mathcal{B}$ is a smooth point of $X_{w(\ell)}$, so by Theorem 2.2, $e_{w(\ell-1)}^{w(\ell)} = -1/\Psi_{w(\ell-1)}^{w(\ell)}$. Hence the second term is equal to

$$-\frac{1}{\Psi_{w(\ell-1)}^{w(\ell)}} \cdot \frac{1}{\{1\}_{\alpha_2}} \cdot \frac{1}{\{-n+2\}_{\widetilde{\alpha}}}. \tag{7.3}$$

Let $A = [-n+2, 0]_{\alpha_1}[-n+2, 0]_{\widetilde{\alpha}}$. By Lemma 6.4,

$$\begin{aligned}
\Psi_{y(\ell)}^{w(\ell)} &= [-n+2, 0]_{\alpha_1}\{1\}_{\alpha_2}[-n+2, 1]_{\widetilde{\alpha}} = A\{1\}_{\alpha_2}\{1\}_{\widetilde{\alpha}} \\
\Psi_{w(\ell-1)}^{w(\ell)} &= [-n+1, 0]_{\alpha_1}[-n+1, 0]_{\widetilde{\alpha}} = A\{-n+1\}_{\alpha_1}\{-n+1\}_{\widetilde{\alpha}} \\
\Psi_{y(\ell+2)}^{w(\ell+2)} &= [-n+1, 0]_{\alpha_1}\{1\}_{\alpha_2}[-n+1, 1]_{\widetilde{\alpha}} \\
&= \{-n+1\}_{\alpha_1}\{1\}_{\alpha_2}\{-n+1\}_{\widetilde{\alpha}}\{1\}_{\widetilde{\alpha}}A.
\end{aligned}$$

Since $e_{y(\ell+2)}^{w(\ell+2)}$ is the sum of (7.2) and (7.3), we have

$$e_{y(\ell+2)}^{w(\ell+2)} = \frac{1}{A\{1\}_{\alpha_2}\{-n+1\}_{\alpha_1}\{-n+2\}_{\widetilde{\alpha}}}\left(\frac{n-1}{\{1\}_{\widetilde{\alpha}}} - \frac{1}{\{-n+1\}_{\widetilde{\alpha}}}\right). \tag{7.4}$$

We want to show that

$$e_{y(\ell+2)}^{w(\ell+2)} = \frac{n}{\Psi_{y(\ell+2)}^{w(\ell+2)}} = \frac{n}{A\{-n+1\}_{\alpha_1}\{1\}_{\alpha_2}\{-n+1\}_{\widetilde{\alpha}}\{1\}_{\widetilde{\alpha}}}. \tag{7.5}$$

To prove this, we show that the right-hand sides of (7.4) and (7.5) are equal. Clearing denominators, we see that we must show

$$(n-1)\{-n+1\}_{\widetilde{\alpha}} - \{1\}_{\widetilde{\alpha}} = n\{-n+2\}_{\widetilde{\alpha}}.$$

This follows by substituting the formulas of Proposition 2.6. This proves (7.1) in case $\ell = 2n$.

We now prove (7.1) for $\ell = 2n+1$. With $w(2n)$ as above, we have $w(2n+1) = s_1 s_2 \cdots s_j s_k s_i$. By Proposition 5.3, the map $\mathscr{S} \mapsto (\mathscr{S}, s_i)$ is a bijection between subexpressions of $w(2n)$ multiplying to $y(2n)$ and subexpressions of $w(2n+1)$ multiplying to $y(2n+1)$. Hence using the formula (2.3) for equivariant multiplicities, (7.1) for $\ell = 2n$, and the fact that $y(2n+1)(\alpha_i) = -y(2n)(\alpha_i) = \{-n+1\}_{\alpha_1}$, we have

$$e^{w(2n+1)}_{y(2n+1)} = -e^{w(2n)}_{y(2n)} \cdot \frac{1}{y(2n+1)(\alpha_i)} = \frac{n-1}{\Psi^{w(2n)}_{y(2n)}} \cdot \frac{1}{\{-n+1\}_{\alpha_1}}.$$

Lemma 6.4 implies that $\Psi^{w(2n+1)}_{y(2n+1)} = \Psi^{w(2n)}_{y(2n)} \cdot \{-n+1\}_{\alpha_1}$; (7.1) follows.

7.2 $x = s_{\widetilde{\alpha}} p_2(E_1(\ell))$

Write $m(\ell) = s_{\widetilde{\alpha}} p_2(E_1(\ell)) = s_{\widetilde{\alpha}} t(\widetilde{\alpha}^\vee) w(\ell)$. Using Proposition 4.4, we can reformulate the assertion of Proposition 4.5 for $e^{w(\ell)}_{y(\ell)}$ as saying that for $n \geq 3$, if $\ell = 2n$ or $2n+1$,

$$e^{w(\ell)}_{m(\ell)} = -\frac{n-1}{\Psi^{w(\ell)}_{m(\ell)}}.$$

As discussed in Remark 4.6, we omit most of the proof, giving only a brief outline. The statement is first proved for $\ell = 2n$ by induction on n, the case $\ell = 2n+1$ being deduced from this. The inductive step involves knowing $\Psi^{w(\ell)}_{m(\ell)}$ and $\Psi^{w(\ell)}_{z(\ell)}$, where $z(\ell) = s_{\widetilde{\alpha}} p_1(E_1(\ell))$. These are given in Lemma 6.4. We also need the calculations given in part (2) of Lemma 6.6. We omit further details.

7.3 $x = p_2(E_3(\ell))$

Write $u(\ell) = p_2(E_3(\ell)) = t(\alpha_1^\vee) w(\ell)$. Using Proposition 4.4, we can reformulate the assertion of Theorem 4.5 for $e^{w(\ell)}_{u(\ell)}$ as saying that for $n \geq 2$, if $\ell = 2n-1$ or $\ell = 2n$, then

$$e^{w(\ell)}_{u(\ell)} = \frac{n-1}{\prod_{\beta \in \Psi^{w(\ell)}_{u(\ell)}} \beta}. \tag{7.6}$$

Although this computation is similar to the computation of the previous subsections, there are two main differences. First, here we show the result for odd ℓ by induction, and from this deduce the result for even ℓ; in the other subsections the roles of odd and even were reversed. Second, when we break up $e^{w(\ell+2)}_{u(\ell+2)}$ into two terms, in the analogous places in the previous subsections, we would express the second term using $e^{w(\ell)}_x$ for some x (here x would be $x = w(\ell-1)$). In this case, that would lead to more complicated calculations, so instead we express the second term using $e^{w(\ell+1)}_{v(\ell+1)}$, where by definition $v(\ell+1) = p_1(E_3(\ell+1))$.

We begin by showing the result for $\ell = 2n-1$. The case $n = 2$ can be verified as in the previous subsections (using Remark 6.5); we omit the details. Now we assume the result holds for $\ell = 2n-1$, and show it holds for $\ell = 2n+1$. Write $w(\ell) = s_1 s_2 \cdots s_i s_j s_k$ and $w(\ell+2) = s_1 s_2 \cdots s_k s_i s_j$. As in the previous subsections, we can write $e^{w(\ell+2)}_{u(\ell+2)}$ as a sum of two terms. The first term is the sum of all contributions from the subexpressions $(\mathscr{S}, s_i, s_j)$ where $\mathscr{S}$ is a subexpression of $w(\ell)$ multiplying to $u(\ell)$, and the second term is the contribution from the subexpression $(s_1, \ldots, s_j, 1, s_i, 1)$.

We now calculate these terms separately. The formula for equivariant multiplicities implies that the first term is

$$e^{w(\ell)}_{u(\ell)} \cdot \frac{1}{u(\ell)s_i(\alpha_i)} \cdot \frac{1}{u(\ell)s_i s_j(\alpha_j)} = e^{w(\ell)}_{u(\ell)} \cdot \frac{1}{u(\ell)(\alpha_i+\alpha_j)} \cdot \frac{1}{u(\ell)(\alpha_i)}.$$

Recall from Lemma 6.6 that

$$u(\ell)(\alpha_i) = \{-n+2\}_{\tilde{\alpha}}, \;\; u(\ell)(\alpha_j) = -\{0\}_{\alpha_2}$$
$$\text{and } \; u(\ell)(\alpha_i+\alpha_j) = \{-n+2\}_{\alpha_1}.$$

Combining this with our inductive hypothesis, we see that the first term is

$$\frac{n-1}{\Psi^{w(\ell)}_{u(\ell)}} \cdot \frac{1}{\{-n+2\}_{\tilde{\alpha}}} \cdot \frac{1}{\{-n+2\}_{\alpha_1}}. \tag{7.7}$$

The second term corresponds to the subexpression $(s_1, \ldots, s_j, 1, s_i, 1)$. By Proposition 5.2, $(s_1, \ldots, s_j, 1, s_i)$ is the unique subexpression of $w(\ell+1)$ multiplying to $v(\ell+1) = p_1(E_3(\ell+1))$. Hence, from the formula (2.3) for equivariant multiplicities, the second term is equal to $-e^{w(\ell+1)}_{v(\ell+1)} \frac{1}{v(\ell+1)(\alpha_j)}$. (The negative sign occurs because the lengths of $w(\ell+2)$ and $w(\ell+1)$ have opposite parity.) Since $v(\ell+1) = u(\ell)s_i s_j$, we have $v(\ell+1)(\alpha_j) = -u(\ell)(\alpha_i+\alpha_j)$. Since $v(\ell+1)\mathcal{B}$ is a smooth point of $X_{w(\ell+1)}$, Theorem 2.2 implies that $e^{w(\ell+1)}_{v(\ell+1)} = -1/\Psi^{w(\ell+1)}_{v(\ell+1)}$. Hence, the second term is

$$\frac{-1}{\Psi^{w(\ell+1)}_{v(\ell+1)}\{-n+2\}_{\alpha_1}}. \tag{7.8}$$

Let $A = [-n+2, 0]_{\alpha_1}\{0\}_{\alpha_2}[-n+3, 0]_{\tilde{\alpha}}$. Then Lemma 6.4 implies that

$$\begin{aligned}
\Psi^{w(\ell)}_{u(\ell)} &= [-n+2, 1]_{\alpha_1}\{0\}_{\alpha_2}[-n+3, 0]_{\tilde{\alpha}} = A\{1\}_{\alpha_1} \\
\Psi^{w(\ell+1)}_{v(\ell+1)} &= [-n+1, 0]_{\alpha_1}\{0\}_{\alpha_2}[-n+2, 0]_{\tilde{\alpha}} = A \cdot \{-n+1\}_{\alpha_1} \cdot \{-n+2\}_{\tilde{\alpha}} \\
\Psi^{w(\ell+2)}_{u(\ell+2)} &= [-n+1, 1]_{\alpha_1}\{0\}_{\alpha_2}[-n+2, 0]_{\tilde{\alpha}} = A\{1\}_{\alpha_1}\{-n+1\}_{\alpha_1}\{-n+2\}_{\tilde{\alpha}}.
\end{aligned}$$

Therefore adding (7.7) and (7.8), we see that

$$e_{u(\ell+2)}^{w(\ell+2)} = \frac{1}{A\{-n+2\}_{\alpha_1}\{-n+2\}_{\widetilde{\alpha}}} \cdot \left(\frac{n-1}{\{1\}_{\alpha_1}} - \frac{1}{\{-n+1\}_{\alpha_1}}\right). \tag{7.9}$$

We want to show that

$$e_{u(\ell+2)}^{w(\ell+2)} = \frac{n}{\Psi_{u(\ell+2)}^{w(\ell+2)}} = \frac{n}{A\{1\}_{\alpha_1}\{-n+1\}_{\alpha_1}\{-n+2\}_{\widetilde{\alpha}}}. \tag{7.10}$$

In other words, we must show that the right-hand sides of (7.9) and (7.10) are equal. Clearing denominators, we see that we must show

$$(n-1)\{-n+1\}_{\alpha_1} - \{1\}_{\alpha_1} = n\{-n+2\}_{\alpha_1},$$

which follows from the formulas of Proposition 2.6. This proves (7.6) for $\ell = 2n-1$.

We now prove (7.6) for $\ell = 2n$. With $w(2n-1)$ as above, we have $w(2n) = s_1 s_2 \cdots s_j s_k s_i$. Proposition 5.3 implies that the map $\mathscr{S} \mapsto (\mathscr{S}, s_i)$ is a bijection between subexpressions of $w(2n-1)$ multiplying to $u(2n-1)$ and subexpressions of $w(2n)$ multiplying to $u(2n)$. Hence by (7.6) for $\ell = 2n-1$, (2.3), and the fact that $u(2n)(\alpha_i) = -u(2n-1)\alpha_i = \{-n+2\}_{\widetilde{\alpha}}$, we have

$$e_{u(2n)}^{w(2n)} = -e_{u(2n-1)}^{w(2n-1)} \cdot \frac{1}{u(2n)(\alpha_i)} = \frac{n-1}{\Psi_{u(2n-1)}^{w(2n-1)}} \frac{1}{\{-n+2\}_{\widetilde{\alpha}}}.$$

Lemma 6.4 implies that $\Psi_{u(2n)}^{w(2n)} = \Psi_{u(2n-1)}^{w(2n-1)}\{-n+2\}_{\widetilde{\alpha}}$; (7.6) follows. This completes the proof of Proposition 4.5, and with it, Theorem 4.5. □

7.4 *The maximal singular points of $X_{w(\ell)}$*

In this section we describe the set of elements $x \in W$ which are maximal in the Bruhat order subject to the condition that $x\mathcal{B}$ is a singular point of $X_{w(\ell)}$. For simplicity, we will refer to such an $x \in W$ as a maximal singular point in $X_{w(\ell)}$. We will also follow the convention of Remark 3.2 by identifying $x \in W$ with the point $xq \in \mathbb{R}^2$.

Recall that we defined $p_2(E_i(\ell))$ for $i = 1, 3$ as the second point on $E_i(\ell)$ from the endpoint $w(\ell)q$. We have $p_2(E_1(\ell)) = t(\widetilde{\alpha}^\vee)w(\ell)$ and $p_2(E_3(\ell)) = t(\alpha_1^\vee)w(\ell)$, as can be shown using arguments similar to those used in the proof of Proposition 5.1. These elements each have length $\ell - 2$ by Proposition 5.1; expressions for these elements in terms of simple reflections can be obtained by using (2.4). Recall also the element $A_1(\ell) = s_{\widetilde{\alpha}} w(\ell-2)$ from Definition 3.1.

For later use, we observe that

$$w(\ell-3) < p_2(E_1(\ell)). \tag{7.11}$$

To see this, note that because the length of $w(\ell-3)$ is $\ell-3$, and the length of $p_2(E_1(\ell))$ is $\ell-2$, it suffices to show that

$$w(\ell-3) = s_{\alpha_2,1}p_2(E_1(\ell)). \tag{7.12}$$

This is apparent in Figure 6. For an algebraic proof, observe that the right-hand side equals

$$s_{\alpha_2,1}t(\widetilde{\alpha}^\vee)w(\ell) = t(\alpha_2^\vee)s_2t(\widetilde{\alpha}^\vee)w(\ell),$$

and by (2.4),

$$t(\alpha_2^\vee)s_2t(\widetilde{\alpha}^\vee) = (s_3s_1s_3s_2)s_2(s_3s_1s_2s_1) = s_3s_2s_1,$$

from which (7.12) follows.

The maximal singular points of $X_{w(\ell)}$ given in the next theorem are located in Figures 6 and 7 as follows. In Figure 7, they are the points labelled x_2 and y_2. In Figure 6, they are the points labelled x_2 and y_2, along with the top point on the top edge of the inner triangle.

Theorem 7.1. *Let $\ell \geq 6$. The set of maximal singular points of $X_{w(\ell)}$ is equal to*

$$\{p_2(E_1(\ell)), p_2(E_3(\ell)), A_1(\ell-3)\}$$

if ℓ is even, and

$$\{p_2(E_1(\ell)), p_2(E_3(\ell))\} \tag{7.13}$$

if ℓ is odd.

Proof. Write $w(\ell) = s_1s_2s_3\cdots s_k$. Let $\mathcal{A}$ denote the set of $y \in W$ such that y is either a maximal nonrationally smooth (nrs) point of $X_{w(\ell)}$, or y is the second point from the endpoint on a rationally smooth edge. (For simplicity, we will refer to these points simply as “second points”.) The elements $y \in \mathcal{A}$ correspond to singular points $y\mathcal{B}$ of $X_{w(\ell)}$, and moreover, any $x \in W$ such that $x\mathcal{B}$ is singular in $X_{w(\ell)}$ satisfies $x \leq y$ for some $y \in \mathcal{A}$ (cf. the discussion of the proof of Theorem 4.1 in the introduction). Hence, the set of maximal singular points of $X_{w(\ell)}$ is the set of maximal elements of $\mathcal{A}$.

First, suppose ℓ is even. The maximal nrs points of $X_{w(\ell)}$ are $w(\ell-3)$ and $A_1(\ell-3)$ (see [7, Corollary 10.5]). By Proposition 3.4 and Lemma 3.6, the rationally smooth edges are $E_1(\ell)$, $E_2(\ell) = s_1E_1(\ell)$, $E_3(\ell)$, and $E_3(\ell-1) = E_3(\ell)s_k$. Among the second points on these edges, we must identify the maximal elements.

We claim that among the subset of these points which lie on $E_1(\ell)$ or $E_2(\ell)$, the maximal element is $p_2(E_1(\ell))$. Indeed, as noted in Remark 4.6, this element is greater than $s_{\widetilde{\alpha}}p_2(E_1(\ell))$, which is the other second point on $E_1(\ell)$. Next, by

Lemma 3.6, the map $E_1(\ell) \to E_2(\ell)$, $z \to s_1 z$, takes the second points on $E_1(\ell)$ to the second points on $E_2(\ell)$. From the alcove formulas for $p_2(E_1(\ell))$ and $s_\alpha p_2(E_1(\ell))$ given in Proposition 5.1, along with the formulas for the W-action on alcoves given in Section 2.4, one can see that the alcove formulas for these elements are given (respectively) by $O(\frac{\ell-2}{2}, \frac{6-\ell}{2}, 0)$ and $E(1, \frac{\ell}{2} - 3, 2 - \frac{\ell}{2})$. By (2.8), the lengths of these elements are (respectively) $\ell - 3$ and $\ell - 4$. Comparing with Proposition 5.1, we see that $\ell(s_1 p_2(E_1(\ell))) < \ell(p_2(E_1(\ell)))$, and hence in the Bruhat order, $s_1 p_2(E_1(\ell)) < p_2(E_1(\ell))$. Similarly, $s_1 s_\alpha p_2(E_1(\ell)) < s_\alpha p_2(E_1(\ell))$. This proves the claim.

Next, we claim that among the subset of points of $\mathcal{A}$ on $E_3(\ell)$ or $E_3(\ell - 1) = E_3(\ell)s_k$, the maximal element is $p_2(E_3(\ell))$. The proof of this claim is very similar to the proof in the preceding paragraph, so we only sketch the calculations. We need alcove formulas for $p_2(E_3(\ell))$ and $p_2(E_3(\ell - 1))$, and for the other second points on $E_3(\ell)$ and $E_3(\ell - 1)$, which by Lemma 3.6, are given by applying s_1 to these two points. The alcove formulas for the first two points are given in Proposition 5.1; the formulas for the second two points can be calculated from these as in the preceding paragraph. We can then calculate the lengths of each of these elements, and reasoning as in the preceding paragraph, shows that $p_2(E_3(\ell))$ is greater in the Bruhat order than the other three second points, proving the claim.

What we have shown so far implies that the set of maximal elements of $\mathcal{A}$ is the set of maximal elements among the 4 elements $p_2(E_1(\ell))$, $p_2(E_3(\ell))$, $w(\ell - 3)$ and $A_1(\ell - 3)$. The first two elements each have length $\ell - 2$ and are therefore incomparable. We have $w(\ell - 3) < p_2(E_1(\ell))$ by (7.11). We claim that $A_1(\ell - 3)$ is not less than $p_2(E_1(\ell))$ or $p_2(E_3(\ell))$; this suffices to complete the proof of the theorem for ℓ even. Since $A_1(\ell - 3)$ has length $\ell - 3$, and the other two elements have length $\ell - 2$, the element $A_1(\ell - 3)$ could be less than $p_2(E_1(\ell))$ or $p_2(E_3(\ell))$ only if it were equal to a reflection times one of these elements. Since the reflections in W are the elements $s_{\alpha,k}$ for $\alpha \in \{\alpha_1, \alpha_2, \widetilde{\alpha}\}$ (cf. Section 2.3), this would mean that the point $A_1(\ell - 3)q \in \mathbb{R}^2$ would differ from the center points corresponding to $p_2(E_1(\ell))$ or $p_2(E_3(\ell))$ by a multiple of one of the roots α_1, α_2 or $\widetilde{\alpha}$. However, this is not true, as can be seen by calculating the relevant points using the alcove formulas and (2.5). This proves the claim, and with it the theorem in case ℓ is even.

If ℓ is odd, the argument is similar, and we only sketch it. The maximal nrs point of $X_{w(\ell)}$ is $w(\ell - 3)$. The rationally smooth edges are $E_1(\ell)$, $E_1(\ell)s_k$, $E_2(\ell) = s_1 E_1(\ell)$, $E_2(\ell)s_k$, and $E_3(\ell)$. As in the case where ℓ is even, we show that every point on a rationally smooth edge is less than either $p_2(E_1(\ell))$ or $p_2(E_3(\ell))$. Therefore the set of maximal elements of $\mathcal{A}$ is the set of maximal elements among the elements $p_2(E_1(\ell))$, $p_2(E_3(\ell))$ and $w(\ell - 3)$. Since the first two elements have length $\ell - 2$, they are incomparable. Since $w(\ell - 3) < p_2(E_1(\ell))$ by (7.11), the set of maximal elements of $\mathcal{A}$ is equal to the set in (7.13), proving the theorem. $\square$

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Centers and cocenters of 0-Hecke algebras

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Dedicated to David Vogan on his 60th birthday

Abstract In this paper, we give explicit descriptions of the centers and cocenters of 0-Hecke algebras associated to finite Coxeter groups.

Key words: finite Coxeter groups, 0-Hecke algebras, conjugacy classes

MSC (2010): Primary 20C08; Secondary 20C20

1 Introduction

Iwahori–Hecke algebras H_q are deformations of the group algebras of finite Coxeter groups W (with nonzero parameters q). They play an important role in the study of representations of finite groups of Lie type.

In 1993, Geck and Pfeiffer [4] discovered some remarkable properties of the minimal length elements in their conjugacy classes in W (see Theorem 2.2). Based on these properties, they defined the "character table" for Iwahori–Hecke algebras. They also gave a basis of the cocenter of Iwahori–Hecke algebras, using minimal length elements. Later, Geck and Rouquier [6] gave a basis of the center of Iwahori–Hecke algebras. It is interesting that both centers and cocenters of Iwahori–Hecke algebras are closely related to minimal length elements in the finite Coxeter groups and their dimensions both equal the number of conjugacy classes of the finite Coxeter groups.

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The 0-Hecke algebra H_0 was used by Carter and Lusztig in [2] in the study of p-modular representations of finite groups of Lie type. It is a deformation of the group algebras of finite Coxeter groups (with zero parameter). In this paper, we study the center and cocenter of 0-Hecke algebras H_0. We give a basis of the center of H_0 in Theorem 5.4 and a basis of the cocenter of H_0 in Theorem 6.5.

It is interesting to compare the (co)centers of H_q and H_0. Let $W_{\min}$ be the set of minimal length elements in their conjugacy classes in W. There are two equivalence relations $\sim$ and $\approx$, on $W_{\min}$ (see §2.1 for the precise definition). Hence we have the partition of $W_{\min}$ into $\sim$-equivalence classes and $\approx$-equivalence classes. The second partition is finer than the first one.

The center and cocenter of H_q have basis sets indexed by the set of conjugacy classes of W, which are in natural bijection with $W_{\min}/\sim$. The cocenter of H_0 has a basis set indexed by $W_{\min}/\approx$ and the center of H_0 has a basis set indexed by $W_{\max}/\approx$. Here $W_{\max}/\approx$ is defined using maximal length elements instead and there is a natural bijection between $W_{\max}/\approx$ with the set of $\approx$-equivalence classes of minimal length elements in their "twisted" conjugacy classes in W. In general, the number of elements in $W_{\max}/\approx$ is different from the number of elements in $W_{\min}/\approx$.

The paper is organized as follows. In Section 2, we recall some properties of the minimal length and maximal length elements. In Section 3, we recall the results on the center and cocenter of H_q. We give parameterizations of $W_{\min}/\approx$ and $W_{\max}/\approx$ in Section 4. In Section 5, we give a basis of the center of H_0 and in Section 6, we give a basis of the cocenter of H_0. In Section 7, we describe the image of a standard element t_w in the cocenter of H_0 and discuss some applications to the class polynomials of H_q.

2 Finite Coxeter groups

2.1 Definitions

Let S be a finite set. A Coxeter matrix $(m_{s,s'})_{s,s' \in S}$ is a matrix with entries in $\mathbb{N} \cup \{\infty\}$ such that $m_{ss} = 1$ and $m_{s,s'} = m_{s',s} \geqslant 2$ for all $s \neq s'$ in S. The Coxeter group W associated to the Coxeter matrix $(m_{s,s'})$ is the group generated by S with relations $(ss')^{m_{s,s'}} = 1$ for $s, s' \in S$ with $m_{s,s'} < \infty$. The Coxeter group W is equipped with the length function $\ell : W \to \mathbb{N}$ and the Bruhat order $\leqslant$.

For any $J \subseteq S$, let W_J be the subgroup of W generated by elements in J. Then W_J is also a Coxeter group.

Let δ be an automorphism of W with $\delta(S) = S$. We say that the elements $w, w' \in W$ are δ-conjugate if there exists $x \in W$ such that $w' = xw\delta(x)^{-1}$. Let $\mathrm{cl}(W)_\delta$ be the set of δ-conjugacy classes of W. We say that a δ-conjugacy class $\mathcal{O}$ is *elliptic* if $\mathcal{O} \cap W_J = \emptyset$ for any $J = \delta(J) \subsetneq S$.

For any $w \in W$, let $\mathrm{supp}(w)$ be the set of simple reflections that appear in some (or equivalently, any) reduced expression of w. Set $\mathrm{supp}_\delta(w) = \bigcup_{i \geqslant 0} \delta^i(\mathrm{supp}(w))$. Then $\mathcal{O} \in \mathrm{cl}(W)_\delta$ is elliptic if and only if $\mathrm{supp}_\delta(w) = S$ for any $w \in \mathcal{O}$.

For $w, w' \in W$ and $s \in S$, we write $w \xrightarrow{s}_\delta w'$ if $w' = sw\delta(s)$ and $\ell(w') \leqslant \ell(w)$. We write $w \to_\delta w'$ if there exists a sequence $w = w_0, w_1, \dots, w_n = w'$ of elements in W such that for any k, $w_{k-1} \xrightarrow{s}_\delta w_k$ for some $s \in S$. We write $w \approx_\delta w'$ if $w \to_\delta w'$ and $w' \to_\delta w$.

We say that the two elements $w, w' \in W$ are *elementarily strongly δ-conjugate* if $\ell(w) = \ell(w')$ and there exists $x \in W$ such that $w' = xw\delta(x)^{-1}$, and $\ell(xw) = \ell(x) + \ell(w)$ or $\ell(w\delta(x)^{-1}) = \ell(x) + \ell(w)$. We say that w, w' are *strongly δ-conjugate* if there exists a sequence $w = w_0, w_1, \dots, w_n = w'$ such that for each i, w_{i-1} is elementarily strongly δ-conjugate to w_i. We write $w \sim_\delta w'$ if w and w' are strongly δ-conjugate. It is easy to see the following.

Lemma 2.1. *If $w, w' \in W$ with $w \approx_\delta w'$, then $w \sim_\delta w'$.*

Note that $\sim_\delta$ and $\approx_\delta$ are both equivalence relations. For any $\mathcal{O} \in \mathrm{cl}(W)$, let $\mathcal{O}_{\min}$ be the set of minimal length elements in $\mathcal{O}$ and let $\mathcal{O}_{\max}$ be the set of maximal length elements in $\mathcal{O}$. Since $\sim_\delta$ and $\approx_\delta$ are compatible with the length function, both $\mathcal{O}_{\min}$ and $\mathcal{O}_{\max}$ are unions of $\sim_\delta$-equivalence classes and unions of $\approx_\delta$-equivalence classes.

Let $W_{\delta,\min} = \bigsqcup_{\mathcal{O} \in \mathrm{cl}(W)_\delta} \mathcal{O}_{\min}$ and let $W_{\delta,\max} = \bigsqcup_{\mathcal{O} \in \mathrm{cl}(W)_\delta} \mathcal{O}_{\max}$. Let $W_{\delta,\min}/\sim_\delta$ be the set of $\sim_\delta$-equivalence classes in $W_{\min}$. We define $W_{\delta,\min}/\approx_\delta$, $W_{\delta,\max}/\sim_\delta$ and $W_{\delta,\max}/\approx_\delta$ in a similar way.

If δ is the identity map, then we may omit δ in the subscript.

The following result is proved in [4, Theorem 1.1], [3, Theorem 2.6] and [7, Theorem 7.5] (see also [9] for a case-free proof).

Theorem 2.2. *Let W be a finite Coxeter group and let $\mathcal{O}$ be a δ-conjugacy class of W. Then*

(1) For any $w \in \mathcal{O}$, there exists $w' \in \mathcal{O}_{\min}$ such that $w \to_\delta w'$.
(2) $\mathcal{O}_{\min}$ is a single strongly δ-conjugate class.
(3) If $\mathcal{O}$ is elliptic, then $\mathcal{O}_{\min}$ is a single $\approx_\delta$-equivalence class.

As a consequence of Theorem 2.2, it is proved in [7, Corollary 4.5] that the set of minimal length elements in $\mathcal{O}$ coincides with the set of minimal elements in $\mathcal{O}$ with respect to the Bruhat order $\leqslant$.

Corollary 2.3. *Let W be a finite Coxeter group and $\mathcal{O}$ be a δ-conjugacy class of W. Then $\mathcal{O}_{\min} = \{w \in \mathcal{O}; w' \not< w \text{ for any } w' \in \mathcal{O}\}$.*

2.2 A variation

One may transfer the results on minimal length elements to results on maximal length elements via the trick in [3, §2.9]. Let w_0 be the longest element in W and let $\delta' = \mathrm{Ad}(w_0) \circ \delta$ be the automorphism on W. Then the map

$$W \to W, \quad w \mapsto w w_0$$

reverses the Bruhat order and sends a δ-conjugacy class $\mathcal{O}$ to a δ'-conjugacy class $\mathcal{O}'$. Moreover, $w_1 \to_\delta w_2$ if and only if $w_2 w_0 \to_{\delta'} w_1 w_0$. Thus

Theorem 2.4. *Let W be a finite Coxeter group and $\mathcal{O}$ be a δ-conjugacy class of W. Then*

(1) For any $w \in \mathcal{O}$, there exists $w' \in \mathcal{O}_{\max}$ such that $w' \to_\delta w$.
(2) $\mathcal{O}_{\max} = \{w \in \mathcal{O}; w \not\to w' \text{ for any } w' \in \mathcal{O}\}$.

3 Finite Hecke algebras

In the rest of this paper, we assume that W is a finite Coxeter group.

Let $\mathbf{q}$ be an indeterminate and $\Lambda = \mathbb{C}[\mathbf{q}]$. The generic Hecke algebra (with equal parameters) $\mathbb{H}$ of W is the Λ-algebra generated by $\{T_w; w \in W\}$ subject to the relations:

1. $T_w \cdot T_{w'} = T_{ww'}$, if $\ell(ww') = \ell(w) + \ell(w')$.
2. $(T_s + 1)(T_s - \mathbf{q}) = 0$ for $s \in S$.

Given $q \in \mathbb{C}$, let $\mathbb{C}_q$ be the Λ-module where $\mathbf{q}$ acts by q. Let $H_q = \mathbb{H} \otimes_\Lambda \mathbb{C}_q$ be a specialization of $\mathbb{H}$.

In particular, $H_1 = \mathbb{C}[W]$ is the group algebra. The algebra H_0 is called the 0-*Hecke algebra*. We will discuss it in details in the next section.

For any $w \in W$, we denote by $T_{w,q} = T_w \otimes 1 \in H_q$. We simply write t_w for $T_{w,0}$.

Let $[\mathbb{H}, \mathbb{H}]_\delta$ be the δ-commutator of $\mathbb{H}$, that is, the Λ-submodule of $\mathbb{H}$ spanned by $[h, h'] = hh' - h'\delta(h)$ for $h, h' \in \mathbb{H}$. Let $\overline{\mathbb{H}}_\delta = \mathbb{H}/[\mathbb{H}, \mathbb{H}]_\delta$ be the δ-cocenter of $\mathbb{H}$.

For any $q \in \mathbb{C}$, we define the δ-cocenter $\overline{H_{q,\delta}}$ in the same way. Notice that if $q \neq 0$, then $T_{w,q}$ is invertible in H_q for any $w \in W$. However, if $q = 0$, then t_w is invertible in H_q if and only if $w = 1$. This makes a difference in the study of the cocenter of H_q (for $q \neq 0$) and the cocenter of H_0.

Proposition 3.1. *Let $w, w' \in W$. If $w \approx_\delta w'$, then the image of T_w and $T_{w'}$ in $\overline{\mathbb{H}}_\delta$ are the same.*

Proof. It suffices to prove the case where $w \xrightarrow{s}_\delta w'$ and $\ell(w) = \ell(w')$. Without loss of generality, we may assume furthermore that $sw < w$. Then $T_w = T_s T_{sw}$ and $T_{w'} = T_{sw} T_{\delta(s)}$. Hence the image of T_w and $T_{w'}$ are the same. □

For $q \neq 0$, a similar argument shows that if $w \sim_\delta w'$, then the image of $T_{w,q}$ and $T_{w',q}$ in $\overline{H_{q,\delta}}$ are the same. By Theorem 2.2 (2), for any δ-conjugacy class $\mathcal{O}$ of W, $\mathcal{O}_{\min}$ is a single strongly δ-conjugacy class. Thus

Proposition 3.2 ([4, §1] and [3, 7.2]). *If $q \neq 0$, then for any $\mathcal{O} \in \mathrm{cl}(W)_\delta$ and $w, w' \in \mathcal{O}_{\min}$, the image of $T_{w,q}$ and $T_{w',q}$ in $\overline{H_{q,\delta}}$ are the same.*

Remark 3.3. We denote this image by $T_{\mathcal{O},q}$.

Theorem 3.4 ([4, §1] and [3, Theorem 7.4 (1)]). *If $q \neq 0$, then $\{T_{\mathcal{O},q}\}_{\mathcal{O} \in \mathrm{cl}(W)_\delta}$ form a basis of $\overline{H_{q,\delta}}$.*

Proposition 3.5 ([4, §1.2] and [3, Theorem 7.4 (2)]). *If $q \neq 0$, then there exists a unique polynomial $f_{w,\mathcal{O}} \in \mathbb{Z}[q]$ for any $w \in W$ and $\mathcal{O} \in \mathrm{cl}(W)_\delta$ such that the image of T_w in $\overline{H_{q,\delta}}$ equals $\sum_{\mathcal{O} \in \mathrm{cl}(W)_\delta} f_{w,\mathcal{O}} T_{\mathcal{O},q}$.*

Remark 3.6. The polynomials $f_{w,\mathcal{O}}$ are called the *class polynomials*. They play an important role in the study of characters of Hecke algebras.

Theorem 3.7 ([6, Theorem 5.2]). *Let $q \neq 0$. Let*

$$Z(H_q)_\delta = \{h \in H_q; h'h = h\delta(h') \text{ for any } h' \in H_q\}$$

be the δ-center of H_q. For any $\mathcal{O} \in \mathrm{cl}(W)_\delta$, set

$$z_{\mathcal{O}} = \sum_{w \in W} q^{-\ell(w)} f_{w,\mathcal{O}} T_{w^{-1}}.$$

Then $\{z_{\mathcal{O}}\}_{\mathcal{O} \in \mathrm{cl}(W)_\delta}$ form a basis of $Z(H_q)_\delta$.

As a consequence of the results above, we have

Corollary 3.8. *If $q \neq 0$, then*

$$\dim Z(H_q)_\delta = \dim \overline{H_{q,\delta}} = \sharp \mathrm{cl}(W)_\delta.$$

4 Parameterizations of $W_{\delta,\min}/\approx_\delta$ and $W_{\delta,\max}/\approx_\delta$

4.1 Parameterizations

Notice that for $q \neq 0$, both $\overline{H_{q,\delta}}$ and $Z(H_q)_\delta$ have basis sets indexed by $\mathrm{cl}(W)_\delta$, which is in natural bijection with $W_{\delta,\min}/\sim_\delta$. As we will see later in this paper, for $\overline{H_{0,\delta}}$ and $Z(H_0)_\delta$, we need to use $W_{\delta,\min}/\approx_\delta$ and $W_{\delta,\max}/\approx_\delta$ instead. We give parameterizations of these sets here.

Let $\Gamma_\delta = \{(J, C); J = \delta(J) \subseteq S, C \in \mathrm{cl}(W_J)_\delta$ is elliptic$\}$. There is a natural map

$$f : \Gamma_\delta \to \mathrm{cl}(W)_\delta, \qquad (J, C) \mapsto \mathcal{O},$$

where $\mathcal{O}$ is the unique δ-conjugacy class of W that contains C.

We say that (J, C) is equivalent to (J', C') if there exists $x \in W^\delta$ and the conjugation by x sends J to J' and sends C to C'. By [1, Proposition 5.2.1], f induces a bijection from the equivalence classes of Γ_δ to $\mathrm{cl}(W)_\delta$.

Proposition 4.1. *Let $\mathcal{O} \in \mathrm{cl}(W)_\delta$. Then*

$$\mathcal{O}_{\min} = \bigsqcup_{\substack{(J,C)\in\Gamma_\delta \\ \text{with } f(J,C)=\mathcal{O}}} C_{\min}.$$

Proof. If $(J, C) \in \Gamma_\delta$ with $f(J, C) = \mathcal{O}$, we have $C_{\min} \subseteq \mathcal{O}_{\min}$ by [7, Lemma 7.3].

Let $w \in \mathcal{O}_{\min}$. Let $J = \mathrm{supp}_\delta(w)$ and $C \in \mathrm{cl}(W_J)_\delta$ with $w \in C$. By [7, Theorem 7.5 (P1)], C is an elliptic δ-conjugacy class of W_J. Since $w \in \mathcal{O}_{\min}$ and $w \in C$, $w \in C_{\min}$. □

Corollary 4.2. *The map*

$$f : \Gamma_\delta \to W_{\delta,\min}/ \approx_\delta, \qquad (J, C) \mapsto C_{\min}$$

is a bijection.

Proof. Let $(J, C) \in \Gamma_\delta$ and $w \in C_{\min}$. If $w \xrightarrow{s}_\delta w'$, then $w' = w$ or $sw < w$ or $w\delta(s) < w$. In the latter two cases, $s \in J$. Therefore $w' \in C$. Since $w \in C_{\min}$ and $\ell(w') \leqslant \ell(w)$, $w' \in C_{\min}$.

By definition of $\approx_\delta$, $v \in C_{\min}$ for any $v \in W$ with $w \approx_\delta v$. On the other hand, by Theorem 2.2, $C_{\min}$ is a single $\approx_\delta$-equivalence class. Hence the map $(J, C) \mapsto C_{\min} \in W_{\delta,\min}/ \approx_\delta$ is well-defined.

It is obvious that this map is injective. The surjectivity follows from Proposition 4.1. □

Using the argument in §2.2, we also obtain

Corollary 4.3. *Set $\delta' = \mathrm{Ad}(w_0) \circ \delta$. The map*

$$\Gamma_{\delta'} \to W_{\delta,\max}/ \approx_\delta, \qquad (J, C) \mapsto C_{\min} w_0$$

is a bijection.

Example 4.4. Let $W = S_3$. Then $\sharp\mathrm{cl}(W) = 3$, $\sharp\Gamma = 4$ and $\sharp\Gamma_{\mathrm{Ad}(w_0)} = 3$. Therefore $\sharp(W_{\min}/ \approx) \neq \sharp\mathrm{cl}(W)$ and $\sharp(W_{\min}/ \approx) \neq \sharp(W_{\max}/ \approx)$ for $W = S_3$.

5 Centers of 0-Hecke algebras

Let $\Sigma \in W_{\delta,\max}/\approx_\delta$. Set

$$W_{\leqslant \Sigma} = \{x \in W; x \leqslant w \text{ for some } w \in \Sigma\},$$
$$t_{\leqslant \Sigma} = \sum_{x \in W_{\leqslant \Sigma}} t_x.$$

Now we recall the following known result on the Bruhat order (see, for example, [12, Lemma 2.3]).

Lemma 5.1. *Let $x, y \in W$ with $x \leqslant y$. Let $s \in S$. Then*

(1) $\min\{x, sx\} \leqslant \min\{y, sy\}$ *and* $\max\{x, sx\} \leqslant \max\{y, sy\}$.
(2) $\min\{x, xs\} \leqslant \min\{y, ys\}$ *and* $\max\{x, xs\} \leqslant \max\{y, ys\}$.

Lemma 5.2. *Let $\Sigma \in W_{\delta,\max}/\approx_\delta$ and $s \in S$. Then*

$$\{x \in W; x \notin W_{\leqslant \Sigma}, sx \in W_{\leqslant \Sigma}\} = \{x \in W; x \notin W_{\leqslant \Sigma}, x\delta(s) \in W_{\leqslant \Sigma}\}.$$

Proof. Let $x \in W$ with $x \notin W_{\leqslant \Sigma}, sx \in W_{\leqslant \Sigma}$. By definition, $sx \leqslant w$ for some $w \in \Sigma$. Since $x \nleqslant w$, we have $sx < x$ and $sw > w$ by Lemma 5.1. Thus $\ell(sw\delta(s)) \geqslant \ell(sw) - 1 = \ell(w)$. Since $w \in W_{\delta,\max}$, $\ell(sw\delta(s)) = \ell(w)$ and $sws \in \Sigma$. Moreover, $sw\delta(s) < sw$.

Since $sx \leqslant w$ and $w < sw$, $x \leqslant sw$. By Lemma 5.1, $\min\{x, x\delta(s)\} \leqslant sw\delta(s)$. Since $x \notin W_{\leqslant \Sigma}$, $x\delta(s) \in W_{\leqslant \Sigma}$. ◻

Lemma 5.3. *Let $\Sigma \in W_{\delta,\max}/\approx_\delta$. Then $t_{\leqslant \Sigma} \in Z(H_0)_\delta$.*

Proof. Let $s \in S$. Then

$$t_s t_{\leqslant \Sigma} = \sum_{x \in W_{\leqslant \Sigma}} t_s t_x = \sum_{x, sx \in W_{\leqslant \Sigma}} t_s t_x + \sum_{\substack{y \in W_{\leqslant \Sigma}, \\ sy \notin W_{\leqslant \Sigma}}} t_s t_x.$$

If $x, sx \in W_{\leqslant \Sigma}$, then $t_s t_x + t_s t_{sx} = 0$. If $y \in W_{\leqslant \Sigma}, sy \notin W_{\leqslant \Sigma}$, then $y < sy$ and $t_s t_y = t_{sy}$. Therefore

$$t_s t_{\leqslant \Sigma} = \sum_{\substack{x \in W; x \notin W_{\leqslant \Sigma}, \\ sx \in W_{\leqslant \Sigma}}} t_x.$$

Similarly,

$$t_{\leqslant \Sigma} t_{\delta(s)} = \sum_{\substack{x \in W; x \notin W_{\leqslant \Sigma}, \\ x\delta(s) \in W_{\leqslant \Sigma}}} t_x.$$

By Lemma 5.2, $t_s t_{\leqslant \Sigma} = t_{\leqslant \Sigma} t_{\delta(s)}$ for any $s \in S$. Thus $t_{\leqslant \Sigma} \in Z(H_0)_\delta$. ◻

Theorem 5.4. *The elements* $\{t_{\leqslant \Sigma}\}_{\Sigma \in W_{\delta,\max}/\approx_\delta}$ *form a basis of* $Z(H_0)_\delta$.

Proof. For any $h = \sum_{w \in W} a_w t_w \in H_0$, we write $\text{supp}(h) = \{w \in W; a_w \neq 0\}$. Let $\text{supp}(h)_{\max}$ be the set of maximal length elements in $\text{supp}(h)$. We show:

(a) If $h \in Z(H_0)_\delta$ and $w \in \text{supp}(h)_{\max}$, then $sw\delta(s) \in \text{supp}(h)_{\max}$ and $a_{sw\delta(s)}=a_w$ for any $s \in S$ with $sw > w$ or $ws > w$.

Without loss of generality, we assume that $sw > w$. Then $sw \in \text{supp}(t_s h) = \text{supp}(h t_{\delta(s)})$ and

$$\text{supp}(t_s h)_{\max} = \{sx; x \in \text{supp}(h)_{\max}, sx > x\},$$
$$\text{supp}(h t_{\delta(s)})_{\max} = \{y\delta(s); y \in \text{supp}(h)_{\max}, y\delta(s) > y\}.$$

Therefore $sw\delta(s) \in \text{supp}(h)_{\max}$ and $\ell(sw\delta(s)) = \ell(w)$. The coefficient of t_{sw} in $t_s h$ is a_w and the coefficient of $t_{sw} = t_{(sw\delta(s))\delta(s)}$ in $h t_{\delta(s)}$ is $a_{sw\delta(s)}$. Thus $a_w = a_{sw\delta(s)}$.
(a) is proved.

Now we show the following:

(b) If $h \in Z(H_0)_\delta$, then $\text{supp}(h)_{\max} \subseteq W_{\delta,\max}$.

If $w \notin W_{\delta,\max}$, then by Theorem 2.4, there exists w' with $\ell(w') = \ell(w) + 2$ and $s \in S$ with $w' \to_\delta sw'\delta(s) \approx_\delta w$. By (a), $sw'\delta(s) \in \text{supp}(h)_{\max}$ since $sw'\delta(s) \approx_\delta w$. Since $sw' < w'$, by (a) again, $w' \in \text{supp}(h)_{\max}$. This is a contradiction.
(b) is proved.

Now suppose that $\bigoplus_{\Sigma \in W_{\delta,\max}/\approx_\delta} \mathbb{C} t_{\leqslant \Sigma} \subsetneqq Z(H_0)_\delta$. Let h be an element in $Z(H_0)_\delta - \bigoplus_{\Sigma \in W_{\delta,\max}/\approx_\delta} \mathbb{C} t_{\leqslant \Sigma}$ and $\max_{w \in \text{supp}(h)} \ell(w) \leqslant \max_{w \in \text{supp}(h')} \ell(w)$ for any $h' \in Z(H_0)_\delta - \bigoplus_{\Sigma \in W_{\delta,\max}/\approx_\delta} \mathbb{C} t_{\leqslant \Sigma}$.

By (a) and (b), $\text{supp}(h)_{\max}$ is a union of Σ with $\Sigma \in W_{\delta,\max}/\approx_\delta$. By (a), if $\Sigma \subseteq \text{supp}(h)_{\max}$, then $a_w = a_{w'}$ for any $w, w' \in \Sigma$. We set $a_\Sigma = a_w$ for any $w \in \Sigma$. Set $h' = h - \sum_{\Sigma \subseteq \text{supp}(h)_{\max}} a_\Sigma t_{\leqslant \Sigma}$. Then $h' \in Z(H_0)_\delta - \bigoplus_{\Sigma \in W_{\delta,\max}/\approx_\delta} \mathbb{C} t_{\leqslant \Sigma}$. But $\max_{w \in \text{supp}(h')} \ell(w) < \max_{w \in \text{supp}(h)} \ell(w)$. This is a contradiction. □

In fact, Theorem 5.4 also holds for the 0-Hecke algebras associated to any affine Weyl group and the proof is similar (the only difference is that one uses [14, Main Theorem 1.1] instead of Theorem 2.4).

On the other hand, there are other explicit descriptions of the centers of finite and affine Hecke algebras.

- Geck and Rouquier [6, Theorem 5.2] gave a basis of the centers of finite Hecke algebras with parameter $q \neq 0$.
- Bernstein, and Lusztig [11, Proposition 3.11] gave a basis of the centers of affine Hecke algebras with parameter $q \neq 0$.
- Vignéras [15, Theorem 1.2] gave a basis of the centers of affine 0-Hecke algebras and pro-p Hecke algebras.

It is interesting to compare Theorem 5.4 (for finite and affine 0-Hecke algebras) with the above results.

6 Cocenters of 0-Hecke algebras

For any $\Sigma \in W_{\delta,\min}/\approx_\delta$, we denote by T_Σ the image of T_w in $\overline{\mathbb{H}}_\delta$ for any $w \in \Sigma$. By Proposition 3.1, the element T_Σ is well-defined. Similar to the proof of Theorem 3.4, we have

Proposition 6.1. *The set* $\{T_\Sigma\}_{\Sigma \in W_{\delta,\min}/\approx_\delta}$ *spans* $\overline{\mathbb{H}}_\delta$.

Via the natural bijection $f : \Gamma_\delta \to W_{\delta,\min}/\approx_\delta$ in Corollary 4.2, we may write $T_{(J,C)}$ for $T_{f(J,C)}$. We also write $t_{(J,C)} = t_{f(J,C)}$ for $T_{(J,C)} \otimes 1 \in \overline{H_{0,\delta}} = \overline{\mathbb{H}}_\delta \otimes_\Lambda \mathbb{C}_0$.

It is worth mentioning that $\overline{\mathbb{H}}_\delta$ is not a free module over Λ by Theorem 3.4 and Theorem 6.5 we will prove later. This is because $\dim \overline{H_{q,\delta}} = \sharp \mathrm{cl}(W)_\delta$ for any $q \neq 0$ and $\dim \overline{H_{0,\delta}} = \sharp W_{\delta,\min}/\approx_\delta$. These numbers do not match in general (see Example 4.4).

6.1 Cocenter

Now we come to the cocenter of 0-Hecke algebras.

We first recall the Demazure product.

By [8], for any $x, y \in W$, the set $\{uv; u \leqslant x, v \leqslant y\}$ contains a unique maximal element. We denote this element by $x * y$ and call it the *Demazure product* of x and y. It is easy to see that $\mathrm{supp}(x * y) = \mathrm{supp}(x) \cup \mathrm{supp}(y)$. The following result is proved in [8, Lemma 1].

Lemma 6.2. *Let* $x, y \in W$. *Then*

$$t_x t_y = (-1)^{\ell(x)+\ell(y)-\ell(x*y)} t_{x*y}.$$

Lemma 6.3. *For any* $J = \delta(J) \subseteq S$, *set* $H_0^{\mathrm{supp}_\delta = J} = \bigoplus_{\mathrm{supp}_\delta(w)=J} \mathbb{C} t_w$. *Then*

$$[H_0, H_0]_\delta = \bigoplus_{J=\delta(J)\subseteq S} \left(H_0^{\mathrm{supp}_\delta = J} \cap [H_0, H_0]_\delta\right).$$

Proof. By Lemma 6.2, for any $x, y \in W$,

$$t_x t_y = (-1)^{\ell(x)+\ell(y)-\ell(x*y)} t_{x*y},$$
$$t_y t_{\delta(x)} = (-1)^{\ell(x)+\ell(y)-\ell(y*(\delta(x))} t_{y*\delta(x)}.$$

Also, $\text{supp}_\delta(x * y) = \text{supp}_\delta(x) \cup \text{supp}_\delta(y) = \text{supp}_\delta(y * (\delta(x))$. Thus

$$t_x t_y, t_y t_{\delta(x)} \in H_0^{\text{supp}_\delta = \text{supp}_\delta(x*y)} \text{ and } t_x t_y - t_y t_{\delta(x)} \in H_0^{\text{supp}_\delta = \text{supp}_\delta(x*y)}.$$

□

Another result we need here is that the elliptic conjugacy classes never "fuse".

Theorem 6.4 ([5, Theorem 3.2.11] and [1, Theorem 5.2.2]). [1] *Let $J = \delta(J) \subseteq S$. Let C, C' be two distinct elliptic δ-conjugacy classes of W_J. Then C and C' are not δ-conjugate in W.*

Now we come to the main theorem of this section.

Theorem 6.5. *The elements $\{t_{(J,C)}\}_{(J,C)\in\Gamma_\delta}$ form a basis of $\overline{H_{0,\delta}}$.*

Proof. Suppose that $\sum_{(J,C)\in\Gamma_\delta} a_{(J,C)} t_{(J,C)} = 0$ in $\overline{H_{0,\delta}}$ for some $a_{(J,C)} \in \mathbb{C}$. Then by Lemma 6.3, for any $J = \delta(J) \subseteq S$,

$$\sum_{C\in \text{cl}(W_J)_\delta \text{ is elliptic}} a_{(J,C)} t_{(J,C)} = 0.$$

Fix $J = \delta(J) \subseteq S$. We show that

(a) The set $\{T_{(J,C)}\}_{C\in \text{cl}(W_J)_\delta \text{ is elliptic}}$ is a linearly independent set in $\overline{\mathbb{H}}_\delta$.

Suppose that

$$\sum_{C\in \text{cl}(W_J)_\delta \text{ is elliptic}} b_C T_{(J,C)} = 0 \in \overline{\mathbb{H}}_\delta$$

for some $b_C \in \Lambda$. Then

$$\sum_{C\in \text{cl}(W_J)_\delta \text{ is elliptic}} b_C|_{\mathbf{q}=q} T_{(J,C)} = 0 \in \overline{H_{q,\delta}}$$

for any $q \neq 0$. By Theorem 3.4, the set $\{T_{(J,C),q}\}_{C\in \text{cl}(W_J)_\delta \text{ is elliptic}}$ is a linearly independent set in $\overline{H_{q,\delta}}$ for any $q \neq 0$. Hence $b_C|_{\mathbf{q}=q} = 0$ for any $q \neq 0$. Thus $b_C = 0$.

(a) is proved.

In other words, $\sum_{C\in \text{cl}(W_J)_\delta \text{ is elliptic}} \Lambda T_{(J,C)}$ is a free submodule of $\overline{\mathbb{H}}$ with basis $T_{(J,C)}$. Thus $\sum_{C\in \text{cl}(W_J)_\delta \text{ is elliptic}} \mathbb{C} t_{(J,C)}$ is a free submodule of $\overline{H_{0,\delta}}$ with basis $t_{(J,C)}$. Therefore $a_{J,C} = 0$. □

[1] The proof in [5] and [1] are based on a characterization of elliptic conjugacy classes using characteristic polynomials [5, Theorem 3.2.7 (P3)] and [7, Theorem 7.5 (P3)], which is proved via a case-by-case analysis. It is interesting to find a case-free proof of these results.

6.2 Cocenter and representations

Now we relate the cocenter of H_0 to the representations of H_0.

For any $J \subseteq S$, let λ_J be the one-dimensional representation of H_0 defined by

$$\lambda_J(t_s) = \begin{cases} -1, & \text{if } s \in J; \\ 0, & \text{if } s \notin J. \end{cases}$$

By [13], the set $\{\lambda_J\}_{J \subseteq S}$ is the set of all the irreducible representations of H_0.

Let $R(H_0)$ be the Grothendieck group of finite-dimensional representations of H_0. Then $R(H_0)$ is a free group with basis $\{\lambda_J\}_{J \subseteq S}$. Consider the trace map

$$\text{Tr} : \overline{H_0} \to R(H_0)^*, \qquad h \mapsto (V \mapsto \text{tr}(h, V)).$$

It is easy to see that for any $(J, C) \in \Gamma$ and $K \subseteq S$,

$$\text{tr}(t_{J,C}, \lambda_K) = \begin{cases} (-1)^{\ell(C)}, & \text{if } J \subseteq K; \\ 0, & \text{otherwise.} \end{cases}$$

Here $\ell(C)$ is the length of any minimal length element in C.

By [10, Proposition 6.10], for any $J \subseteq S$ and any two elliptic conjugacy classes C and C' of W_J, $\ell(C) \equiv \ell(C') \mod 2$. Therefore,

Proposition 6.6. *The trace map* $\text{Tr} : \overline{H_0} \to R(H_0)^*$ *is surjective and the kernel equals*

$$\bigoplus_{\substack{J \subseteq S, \\ C, C' \in \text{cl}(W_J) \text{ are elliptic}}} \mathbb{C}\{t_{(J,C)} - t_{(J,C')}\}.$$

7 A partial order on $W_{\delta,\min}/\approx_\delta$

Let $w \in W$ and $\Sigma \in W_{\delta,\min}/\approx_\delta$, we write $\Sigma \preceq w$ if there exists $w' \in \Sigma$ with $w' \leqslant w$. For $w \in W$ and $\mathcal{O} \in \text{cl}(W)_\delta$, we define $\mathcal{O} \preceq w$ in the same way.

We define a partial order on $W_{\delta,\min}/\approx_\delta$ as follows.

For $\Sigma, \Sigma' \in W_{\delta,\min}/\approx_\delta$, we write $\Sigma' \preceq \Sigma$ if $\Sigma' \preceq w$ for some $w \in \Sigma$. By [7, Corollary 4.6], $\Sigma' \preceq \Sigma$ if and only if $\Sigma' \preceq w$ for any $w \in \Sigma$. In particular, $\preceq$ is transitive. This defines a partial order on $W_{\delta,\min}/\approx_\delta$.

We define a partial order on $\text{cl}(W)_\delta$ in a similar way.

Proposition 7.1. *Let* $\mathcal{O}, \mathcal{O}' \in \text{cl}(W)_\delta$. *The following conditions are equivalent:*

(1) For any $w \in \mathcal{O}_{\min}$, *there exists* $w' \in \mathcal{O}'_{\min}$ *such that* $w' \leqslant w$.
(2) There exists $w \in \mathcal{O}_{\min}$ *and* $w' \in \mathcal{O}'_{\min}$ *such that* $w' \leqslant w$.

Remark 7.2. We write $\mathcal{O}' \preceq \mathcal{O}$ if the conditions above are satisfied. Then the map $W_{\delta,\min}/\approx_\delta \to \mathrm{cl}(W)_\delta$ is compatible with the partial orders $\preceq$.

Proof. Let $w, w_1 \in \mathcal{O}_{\min}$ and $w' \in \mathcal{O}'_{\min}$ with $w' \leqslant w$. Let $J = \mathrm{supp}_\delta(w)$, $J_1 = \mathrm{supp}_\delta(w_1)$ and $J' = \mathrm{supp}_\delta(w')$. Let $C \in \mathrm{cl}(W_J)_\delta$ and $C_1 \in \mathrm{cl}(W_{J_1})_\delta$ with $w \in C$ and $w'_1 \in C_1$. By §4.1, there exists $x \in W^\delta$ and the conjugation of x sends J to J_1 and sends C to C_1. Since $w' \leqslant w$, $J' \subseteq J$. As the conjugation by x sends simple reflections in J to simple reflections in J_1, we have $xw'x^{-1} \leqslant xwx^{-1}$. Moreover, $xwx^{-1} \in C_1$ is a minimal length element. By Theorem 2.2, $xwx^{-1} \approx_\delta w'$. By [7, Lemma 4.4], there exists $w'_1 \in \mathcal{O}'_{\min}$ with $w'_1 \leqslant w_1$. □

Proposition 7.3. *Let $w \in W$. Then*

(1) The set $\{\Sigma \in W_{\delta,\min}/\approx_\delta; \Sigma \preceq w\}$ contains a unique maximal element Σ_w.
(2) The image of t_w in $\overline{H_{0,\delta}}$ equals $(-1)^{\ell(w)-\ell(\Sigma_w)} t_{\Sigma_w}$.

Remark 7.4. By Theorem 6.5, part (2) of the Proposition gives another characterization of Σ_w.

Proof. We argue by induction on $\ell(w)$.

If $w \in W_{\delta,\min}$, we denote by Σ_w the $\approx_\delta$-equivalence class that contains w. By definition, for any $\Sigma \in W_{\delta,\min}/\approx_\delta$ with $\Sigma \preceq w$, $\Sigma \preceq \Sigma_w$. Also by definition, the image of t_w in $\overline{H_{0,\delta}}$ is t_{Σ_w}.

Now suppose that $w \in W_{\delta,\min}$. By Theorem 2.2 (1), there exists $w' \in W$ and $s \in S$ such that $w \approx w'$ and $\ell(sw'\delta(s)) < \ell(w')$. Let $\Sigma \in W_{\delta,\min}/\approx_\delta$ with $\Sigma \preceq w$. By [7, Lemma4.4], $\Sigma \preceq w'$. In other words, there exists $x \in \Sigma$ with $x \leqslant w'$.

Now we prove that

(a) $\Sigma \preceq \Sigma_{sw'}$.

If $x < sx$, then by Lemma 5.1, $x \leqslant sw'$ and $\Sigma \preceq sw'$.

If $sx < x$, then $\ell(sx\delta(s)) \leqslant \ell(sx) + 1 = \ell(x)$. Hence $sx\delta(s) \in \Sigma$. By Lemma 5.1, $sx \leqslant sw'$. Since $sw'\delta(s) < sw'$, by Lemma 5.1 again, we have $sx\delta(s) \leqslant sw'$. Thus $\Sigma \preceq sw'$.

Since $\ell(sw') < \ell(w)$, by inductive hypothesis, $\Sigma_{sw'}$ is defined and $\Sigma \preceq \Sigma_{sw'}$.

(a) is proved.

Since $\Sigma_{sw'} \preceq sw'$, $\Sigma_{sw'} \preceq w'$. By [7, Lemma 4.4], $\Sigma_{sw'} \preceq w$. Thus $\Sigma_{sw'}$ is the unique maximal element in $\{\Sigma \in W_{\delta,\min}/\approx_\delta; \Sigma \preceq w\}$.

We also have

$$t_w \equiv t_{w'} \equiv t_s t_{sw'} = t_{sw'} t_{\delta(s)} = -t_{sw'} \mod [H_0, H_0]_\delta.$$

By inductive hypothesis, the image of $t_{sw'}$ in $\overline{H_{0,\delta}}$ is $(-1)^{\ell(sw')-\ell(\Sigma_{sw'})} t_{\Sigma_{sw'}}$. Hence the image of t_w in $\overline{H_{0,\delta}}$ is $(-1)^{\ell(w)-\ell(\Sigma_{sw'})} t_{\Sigma_{sw'}}$. □

7.1 Class polynomials

For any $w \in W$, we denote by $\mathcal{O}_w$ the image of Σ_w under the map $W_{\delta,\min}/\approx_\delta \to \mathrm{cl}(W)_\delta$. Then $\mathcal{O}_w$ is the maximal element in $\{\mathcal{O} \in \mathrm{cl}(W)_\delta; \mathcal{O} \preceq w\}$.

Now we discuss some application to class polynomials.

Let $w \in W$. By Proposition 3.5, for any $q \neq 0$,

$$T_{w,q} = \sum_{\mathcal{O} \in \mathrm{cl}(W)_\delta} f_{w,\mathcal{O}} T_{\mathcal{O},q} \in \overline{H_{q,\delta}}.$$

By the same argument as in Proposition 7.3, $f_{w,\mathcal{O}} = 0$ unless $\mathcal{O} \preceq \mathcal{O}_w$. Moreover, by Proposition 6.1, there exists $a_{w,\Sigma} \in \Lambda$ such that

$$T_w = \sum_{\Sigma \in W_{\delta,\min}/\approx_\delta} a_{w,\Sigma} T_\Sigma \in \overline{\mathbb{H}}_\delta.$$

Let $p : W_{\delta,\min}/\approx_\delta \to \mathrm{cl}(W)_\delta$ be the natural map. Then for any $q \neq 0$,

$$T_{w,q} = \sum_{\Sigma \in W_{\delta,\min}/\approx_\delta} a_{w,\Sigma}|_{\mathbf{q}=q} T_{p(\Sigma),q} \in \overline{H_{q,\delta}}.$$

Therefore for any $\mathcal{O} \in \mathrm{cl}(W)_\delta$, $\sum_{p(\Sigma)=\mathcal{O}} a_{w,\Sigma} = f_{w,\mathcal{O}}$.

By Proposition 7.3,

$$a_{w,\Sigma} \in \begin{cases} (-1)^{\ell(w)-\ell(\Sigma_w)} + \mathbf{q}\Lambda, & \text{if } \Sigma = \Sigma_w; \\ \mathbf{q}\Lambda, & \text{otherwise.} \end{cases}$$

Therefore

$$f_{w,\mathcal{O}} \in \begin{cases} (-1)^{\ell(w)-\ell(\Sigma_w)} + q\mathbb{Z}[q], & \text{if } \Sigma_w \subseteq \mathcal{O}; \\ q\mathbb{Z}[q], & \text{otherwise.} \end{cases}$$

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Dirac cohomology, elliptic representations and endoscopy

Jing-Song Huang

To David Vogan for his 60th birthday

Abstract The first part (Sections 2–7) of this paper is a survey of some of the recent developments in the theory of Dirac cohomology, especially the relationship of Dirac cohomology with $(\mathfrak{g}, K)$-cohomology and nilpotent Lie algebra cohomology; the second part (Sections 8–13) is devoted to understanding the unitary elliptic representations and endoscopic transfer by using the techniques in Dirac cohomology. A few problems and conjectures are proposed for further investigations.

Key words: Dirac cohomology, Harish-Chandra module, elliptic representation, pseudo-coefficient, endoscopy

MSC (2010): Primary 22E46; Secondary 22E47

1 Introduction

Since its appearance in the literature [HP1], Dirac cohomology has been playing an active role in many of the recent developments in representation theory. Back in late 1990s, Vogan made a conjecture on the property of the Dirac operator in the setting of a reductive Lie algebra and its associated Clifford algebra [V3]. This property implies that the standard parameter of the infinitesimal character of a Harish-Chandra module X and the infinitesimal character of its Dirac cohomology $H_D(X)$ are conjugate under the Weyl group. Vogan's conjecture was consequently verified in [HP1], and it has been extended to several other settings by many authors (see the remark at the end of Section 2).

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Dirac cohomology of various classes of representations is intimately related to several classical subjects of representation theory like global characters and geometric construction of the discrete series (see [HP2]). The Dirac cohomology of several families of Harish-Chandra modules has been determined. These modules include finite-dimensional modules and irreducible unitary $A_{\mathfrak{q}}(\lambda)$-modules [HKP]. It was proved that if X is a unitary Harish-Chandra module, then

$$H^*(\mathfrak{g}, K; X \otimes F^*) \cong \mathrm{Hom}(H_D(F), H_D(X))$$

for any irreducible finite-dimensional module F. It is evident that unitary representations with nonzero Dirac cohomology are closely related to automorphic representations. In [HP2] we used Dirac cohomology to extend the Langlands formula on dimensions of automorphic forms [L1] to a slightly more general setting.

Another aspect of Dirac cohomology is its connection with $\mathfrak{u}$-cohomology. In particular, when G is Hermitian symmetric and $\mathfrak{u}$ is unipotent radical of a parabolic subalgebra with Levi subgroup K, [HPR] showed that for a unitary representation its Dirac cohomology is isomorphic to its $\mathfrak{u}$-cohomology up to a twist of a one-dimensional character. In particular, Enright's calculation of $\mathfrak{u}$-cohomology [E] gives the Dirac cohomology of the irreducible unitary highest weight modules. The Dirac cohomology of unitary lowest weight modules of scalar type is calculated more explicitly in [HPP]. The Euler characteristic of Dirac cohomology gives the K-character of the Harish-Chandra module. As an application, we generalized the classical theorem of Littlewood on branching rules in [HPZ] and some of the other classical branching rules in [H2].

Kostant extended Vogan's conjecture to the setting of the cubic Dirac operator and proved a nonvanishing result on Dirac cohomology for highest weight modules in the most general setting [Ko3]. He also determined the Dirac cohomology of finite-dimensional modules in the equal rank case. The Dirac cohomology for all irreducible highest weight modules was determined in [HX] in terms of coefficients of Kazhdan–Lusztig polynomials. The general formula relating the Dirac cohomology and $\mathfrak{u}$-cohomology for irreducible highest weight modules is also proved in [HX].

The aim of this paper is twofold: First, we review some of the recent developments of Dirac cohomology, in particular its relationship with $(\mathfrak{g}, K)$-cohomology and $\mathfrak{u}$-cohomology. Second, we use Dirac cohomology as a tool to study a class of irreducible unitary representations, called elliptic representations. Harish-Chandra showed that the characters of irreducible or more generally admissible representations are locally integrable functions and smooth on the open dense subset of regular elements [HC1]. An elliptic representation has a global character that does not vanish on the elliptic elements in the set of regular elements. For real reductive Lie groups with compact Cartan subgroups, the irreducible tempered elliptic representations are showed to be representations with nonzero Dirac cohomology, and they are precisely the discrete series and some of the limits of discrete series. The characters of the irreducible tempered elliptic representations are associated in a natural way to the supertempered distributions defined by Harish-Chandra [HC4]. We conjecture that in general the elliptic unitary representations are precisely the unitary representations with nonzero Dirac cohomology.

We show that an irreducible admissible (not necessarily unitary) representation is elliptic if and only if its Dirac index is not zero. We prove that under the condition of regular infinitesimal character, the Dirac index is zero if and only if the Dirac cohomology is zero. We conjecture that this equivalence holds in general without the regularity condition. We also show that the Harish-Chandra modules of irreducible elliptic unitary representations with regular infinitesimal characters are $A_{\mathfrak{q}}(\lambda)$-modules for a real reductive algebraic group $G(\mathbb{R})$.

We also observe a connection between Labesse's calculation of the endoscopic transfer of pseudo-coefficients of discrete series and the calculation of the characters of the Dirac index of discrete series. It offers a new point of view for understanding the endoscopic transfer in the framework of Dirac cohomology and the Dirac index. To classify irreducible unitary representations with nonzero Dirac cohomology remains an open and interesting problem. We conjecture at the end of the paper that any irreducible unitary representation which does not have nonzero Dirac cohomology is induced from one with nonzero Dirac cohomology.

2 Vogan's conjecture on Dirac cohomology

For a real reductive group G with a Cartan involution θ, denote by $\mathfrak{g}_0$ its Lie algebra and assume that $K = G^\theta$ is a maximal compact subgroup of G. Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the Cartan decomposition for the complexified Lie algebra of G. Let B be a nondegenerate invariant symmetric bilinear form on $\mathfrak{g}$, which restricts to the Killing form on the semisimple part $[\mathfrak{g}, \mathfrak{g}]$ of $\mathfrak{g}$.

Let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$ and $C(\mathfrak{p})$ the Clifford algebra of $\mathfrak{p}$ with respect to B. Then one can consider the following version of the Dirac operator:

$$D = \sum_{i=1}^{n} Z_i \otimes Z_i \in U(\mathfrak{g}) \otimes C(\mathfrak{p});$$

here $Z_1, \ldots, Z_n$ is an orthonormal basis of $\mathfrak{p}$ with respect to the symmetric bilinear form B. It follows that D is independent of the choice of the orthonomal basis $Z_1, \ldots, Z_n$ and it is invariant under the diagonal adjoint action of K.

The Dirac operator D is a square root of the Laplace operator associated to the symmetric pair $(\mathfrak{g}, \mathfrak{k})$. To explain this, we start with a Lie algebra map

$$\alpha : \mathfrak{k} \to C(\mathfrak{p}),$$

which is defined by the adjoint map $\mathrm{ad} : \mathfrak{k} \to \mathfrak{so}(\mathfrak{p})$ composed with the embedding of $\mathfrak{so}(\mathfrak{p})$ into $C(\mathfrak{p})$ using the identification $\mathfrak{so}(\mathfrak{p}) \simeq \bigwedge^2 \mathfrak{p}$. The explicit formula for α is (see [HP2, §2.3.3])

$$\alpha(X) = -\frac{1}{4} \sum_j [X, Z_j] Z_j. \tag{2.1}$$

Using α we can embed the Lie algebra $\mathfrak{k}$ diagonally into $U(\mathfrak{g}) \otimes C(\mathfrak{p})$, by

$$X \mapsto X_\Delta = X \otimes 1 + 1 \otimes \alpha(X).$$

This embedding extends to $U(\mathfrak{k})$. We denote the image of $\mathfrak{k}$ by $\mathfrak{k}_\Delta$, and then the image of $U(\mathfrak{k})$ is the enveloping algebra $U(\mathfrak{k}_\Delta)$ of $\mathfrak{k}_\Delta$.

Let $\Omega_\mathfrak{g}$ be the Casimir operator for $\mathfrak{g}$, given by $\Omega_\mathfrak{g} = \sum Z_i^2 - \sum W_j^2$, where W_j is an orthonormal basis for $\mathfrak{k}_0$ with respect to the inner product $-B$, where B is the Killing form. Let $\Omega_\mathfrak{k} = -\sum W_j^2$ be the Casimir operator for $\mathfrak{k}$. The image of $\Omega_\mathfrak{k}$ under Δ is denoted by $\Omega_{\mathfrak{k}_\Delta}$.

Then

$$D^2 = -\Omega_\mathfrak{g} \otimes 1 + \Omega_{\mathfrak{k}_\Delta} + (\|\rho_c\|^2 - \|\rho\|^2) 1 \otimes 1, \tag{2.2}$$

where ρ and ρ_c are half sums of positive roots and compact positive roots respectively.

The Vogan conjecture says that every element $z \otimes 1$ of $Z(\mathfrak{g}) \otimes 1 \subset U(\mathfrak{g}) \otimes C(\mathfrak{p})$ can be written as

$$\zeta(z) + Da + bD$$

where $\zeta(z)$ is in $Z(\mathfrak{k}_\Delta)$, and $a, b \in U(\mathfrak{g}) \otimes C(\mathfrak{p})$.

A main result in [HP1] is introducing a differential d on the K-invariants in $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ defined by a super bracket with D, and determining the cohomology of this differential complex. As a consequence, Pandžić and I proved the following theorem. In the following we denote by $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$ containing a Cartan subalgebra $\mathfrak{t}$ of $\mathfrak{k}$ so that $\mathfrak{t}^*$ is embedded into $\mathfrak{h}^*$, and by W and W_K the Weyl groups of $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{k}, \mathfrak{t})$ respectively.

Theorem 2.1 ([HP1]). *Let $\zeta : Z(\mathfrak{g}) \to Z(\mathfrak{k}) \cong Z(\mathfrak{k}_\Delta)$ be the algebra homomorphism that is determined by the following commutative diagram:*

$$\begin{array}{ccc} Z(\mathfrak{g}) & \xrightarrow{\zeta} & Z(\mathfrak{k}) \\ {\scriptstyle\eta}\downarrow & & \downarrow{\scriptstyle\eta_\mathfrak{k}} \\ P(\mathfrak{h}^*)^W & \xrightarrow{\text{Res}} & P(\mathfrak{t}^*)^{W_K}, \end{array}$$

where P denotes the polynomial algebra, and vertical maps η and $\eta_\mathfrak{k}$ are Harish-Chandra isomorphisms. Then for each $z \in Z(\mathfrak{g})$ one has

$$z \otimes 1 - \zeta(z) = Da + aD, \textit{ for some } a \in U(\mathfrak{g}) \otimes C(\mathfrak{p}).$$

For any admissible $(\mathfrak{g}, K)$-module X, Vogan [V3, HP1]) introduced the notion of *Dirac cohomology* $H_D(X)$ of X. Consider the action of the Dirac operator D on $X \otimes S$, with S the spinor module for the Clifford algebra $C(\mathfrak{p})$. The Dirac cohomology is defined as follows:

$$H_D(X) := \operatorname{Ker} D / \operatorname{Im} D \cap \operatorname{Ker} D.$$

It follows from the identity (2.2) that $H_D(X)$ is a finite-dimensional module for the spin double cover $\widetilde{K}$ of K. In case X is unitary, $H_D(X) = \ker D = \ker D^2$ since D is self-adjoint with respect to a natural Hermitian inner product on $X \otimes S$. As a consequence of the above theorem, we have that $H_D(X)$, if nonzero, determines the infinitesimal character of X.

Theorem 2.2 ([HP1]). *Let X be an admissible $(\mathfrak{g}, K)$-module with standard infinitesimal character parameter $\Lambda \in \mathfrak{h}^*$. Suppose that $H_D(X)$ contains a representation of $\widetilde{K}$ with infinitesimal character λ. Then Λ and $\lambda \in \mathfrak{t}^* \subseteq \mathfrak{h}^*$ are conjugate under W.*

The above theorem is proved in [HP1] for a connected semisimple Lie group G. It is straightforward to extend the result to a possibly disconnected reductive Lie group in Harish-Chandra's class [DH2].

Vogan's conjecture implies a refinement of Parthasarathy's celebrated Dirac inequality, which is an extremely useful tool for the classification of irreducible unitary representations of reductive Lie groups.

Theorem 2.3 (Extended Dirac Inequality [P], [HP1]). *Let X be an irreducible unitary $(\mathfrak{g}, K)$-module with infinitesimal character Λ. Fix a representation of K occurring in X with a highest weight $\mu \in \mathfrak{t}^*$, and a positive root system $\Delta^+(\mathfrak{g})$ for $\mathfrak{t}$ in $\mathfrak{g}$. Here $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$. Write*

$$\rho_c = \rho(\Delta^+(\mathfrak{k})),\ \rho_n = \rho(\Delta^+(\mathfrak{p})).$$

Fix an element $w \in W_K$ such that $w(\mu - \rho_n)$ is dominant for $\Delta^+(\mathfrak{k})$. Then

$$\langle w(\mu - \rho_n) + \rho_c, w(\mu - \rho_n) + \rho_c\rangle \geq \langle \Lambda, \Lambda\rangle.$$

The equality holds if and only if some W conjugate of Λ is equal to $w(\mu - \rho_n) + \rho_c$.

Remark 2.4. Dirac cohomology becomes a useful tool in representation theory and related areas with Vogan's conjecture being extended to various different settings, most notably Kostant's generalization to the setting of the cubic Dirac operator, which will be discussed in detail in Section 5. We also mention the following extensions.

(i) Alekseev and Meinrenken proved a version of Vogan's conjecture in their study of *Lie theory and the Chern–Weil homomorphism* [AM].
(ii) Kumar proved a similar version of Vogan's conjecture in *Induction functor in non-commutative equivariant cohomology and Dirac cohomology* [Ku].
(iii) Pandžić and I extended Vogan's conjecture to the symplectic Dirac operator in Lie superalgebras [HP3].
(iv) Kac, Möseneder Frajria and Papi extended Vogan's conjecture to the affine cubic Dirac operator in affine Lie algebras [KMP].
(v) Barbasch, Ciubotaru and Trapa extended Vogan's conjecture to the setting of graded affine Hecke algebras [BCT].
(vi) Ciubotaru and Trapa proved a version of Vogan's conjecture for studying Weyl group representations in connection with Springer theory [CT].

3 Dirac cohomology of Harish-Chandra modules

We now describe the Dirac cohomology of finite-dimensional modules and irreducible unitary representations with strongly regular infinitesimal characters which are $A_{\mathfrak{q}}(\lambda)$-modules. These results are proved in [HKP].

Recall that $\mathfrak{t}_0$ is a Cartan subalgebra of $\mathfrak{k}_0$ and $\mathfrak{h}_0 \supseteq \mathfrak{t}_0$ is a fundamental Cartan subalgebra of $\mathfrak{g}_0$. Then $\mathfrak{h}_0 = \mathfrak{t}_0 \oplus \mathfrak{a}_0$ with $\mathfrak{a}_0$ the centralizer of $\mathfrak{t}_0$ in $\mathfrak{p}_0$. Passing to complexifications, we will view $\mathfrak{t}^*$ as a subspace of $\mathfrak{h}^*$ by extending the functionals to act as 0 on $\mathfrak{a}$.

We denote by $\Delta(\mathfrak{g},\mathfrak{h})$ (respectively $\Delta(\mathfrak{g},\mathfrak{t})$) the root system of $\mathfrak{g}$ with respect to $\mathfrak{h}$ (respectively $\mathfrak{t}$). The root system of $\mathfrak{k}$ with respect to $\mathfrak{t}$ will be denoted by $\Delta(\mathfrak{k},\mathfrak{t})$. Note that $\Delta(\mathfrak{g},\mathfrak{h})$ and $\Delta(\mathfrak{k},\mathfrak{t})$ are reduced, while $\Delta(\mathfrak{g},\mathfrak{t})$ is in general not reduced. The Weyl groups corresponding to the above root systems are denoted by

$$W = W(\mathfrak{g},\mathfrak{h}), \quad W(\mathfrak{g},\mathfrak{t}), \quad \text{and} \quad W_K = W(\mathfrak{k},\mathfrak{t}).$$

Throughout this section we fix compatible choices of positive roots $\Delta^+(\mathfrak{g},\mathfrak{h})$, $\Delta^+(\mathfrak{g},\mathfrak{t})$ and $\Delta^+(\mathfrak{k},\mathfrak{t})$. As usual, we denote by ρ the half sum of positive roots for $(\mathfrak{g},\mathfrak{h})$, by ρ_c the half sum of positive roots for $(\mathfrak{k},\mathfrak{t})$, and by ρ_n the difference $\rho-\rho_c$. Then $\rho,\rho_c,\rho_n \in \mathfrak{t}^*$.

We let $\mathfrak{t}^*_{\mathbb{R}} = i\mathfrak{t}^*_0$ and let $\mathfrak{h}^*_{\mathbb{R}} = i\mathfrak{t}^*_0 + \mathfrak{a}^*_0$. Our fixed form B on $\mathfrak{g}$ induces inner products on $\mathfrak{t}^*_{\mathbb{R}}$ and $\mathfrak{h}^*_{\mathbb{R}}$.

We denote by $C_{\mathfrak{g}}(\mathfrak{h}^*_{\mathbb{R}})$ (respectively $C_{\mathfrak{g}}(\mathfrak{t}^*_{\mathbb{R}})$, $C_{\mathfrak{k}}(\mathfrak{t}^*_{\mathbb{R}})$) the closed Weyl chamber corresponding to $\Delta^+(\mathfrak{g},\mathfrak{h})$ (respectively $\Delta^+(\mathfrak{g},\mathfrak{t})$, $\Delta^+(\mathfrak{k},\mathfrak{t})$). Then $C_{\mathfrak{g}}(\mathfrak{t}^*_{\mathbb{R}})$ is contained in $C_{\mathfrak{g}}(\mathfrak{h}^*_{\mathbb{R}})$. Namely, if $\mu \in \mathfrak{t}^*_{\mathbb{R}} \subset \mathfrak{h}^*_{\mathbb{R}}$ has nonnegative inner product with every element of $\Delta^+(\mathfrak{g},\mathfrak{t})$, then for any $\alpha \in \Delta^+(\mathfrak{g},\mathfrak{h})$

$$\langle \mu,\alpha\rangle = \langle \mu,\alpha|_{\mathfrak{t}}\rangle + \langle \mu,\alpha|_{\mathfrak{a}}\rangle \geq 0,$$

because μ is orthogonal to $\mathfrak{a}^*$.

We define

$$W(\mathfrak{g},\mathfrak{t})^1 = \{w \in W(\mathfrak{g},\mathfrak{t}) \mid w(C_{\mathfrak{g}}(\mathfrak{t}^*_{\mathbb{R}})) \subset C_{\mathfrak{k}}(\mathfrak{t}^*_{\mathbb{R}})\}.$$

It is clear that $W(\mathfrak{k},\mathfrak{t})$ is a subgroup of $W(\mathfrak{g},\mathfrak{t})$, and that the multiplication map induces a bijection from $W(\mathfrak{k},\mathfrak{t}) \times W(\mathfrak{g},\mathfrak{t})^1$ onto $W(\mathfrak{g},\mathfrak{t})$. Thus the set $W(\mathfrak{g},\mathfrak{t})^1$ is in bijection with $W(\mathfrak{k},\mathfrak{t})\backslash W(\mathfrak{g},\mathfrak{t})$. Let E_μ denote the irreducible representation of $\mathfrak{k}$ with highest weight μ. The following fact can be found in [P] (see also [BW, II, Lemma 6.9], or [W, Lemma 9.3.2]):

Lemma 3.1. *We have the following isomorphism for $\mathfrak{k}$-modules:*

$$S \cong \bigoplus_{w\in W(\mathfrak{g},\mathfrak{t})^1} 2^{[l_0/2]} E_{w\rho-\rho_c},$$

where $l_0 = \dim \mathfrak{a}$ and mE_μ means a direct sum of m copies of E_μ.

Clearly, S is isomorphic to the Dirac cohomology $H_D(\mathbb{C})$ of the trivial representation $\mathbb{C}$.

Let V_λ be the irreducible finite-dimensional $(\mathfrak{g}, K)$-module with highest weight $\lambda \in \mathfrak{h}^*$. If Dirac cohomology of V_λ is nonzero, then $\lambda + \rho \in \mathfrak{t}^*$ and thus $\lambda \in \mathfrak{t}^*$. We have to identify highest weights γ of $\tilde{K}$-submodules of $V_\lambda \otimes S$ that satisfy $\|\gamma + \rho_c\| = \|\lambda + \rho\|$.

Theorem 3.2 (Theorem 4.2 [HKP]). *Let V_λ be an irreducible finite-dimensional $(\mathfrak{g}, K)$-module with highest weight λ. If $\lambda \neq \theta\lambda$, then the Dirac cohomology of V_λ is zero. If $\lambda = \theta\lambda$, then as a $\mathfrak{k}$-module the Dirac cohomology of V_λ is*

$$H_D(V_\lambda) = \bigoplus_{w \in W(\mathfrak{g},\mathfrak{t})^1} 2^{[l_0/2]} E_{w(\lambda+\rho)-\rho_c}.$$

We now describe the Dirac cohomology of a unitary $A_\mathfrak{q}(\lambda)$-module. Recall that a θ-stable parabolic subalgebra

$$\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$$

of $\mathfrak{g}$ is by definition the sum of nonnegative eigenspaces of $\mathrm{ad}(H)$, where H is some fixed element of $i\mathfrak{t}_0$ (and consequently $\mathrm{ad}(H)$ is semisimple with real eigenvalues). The Levi subalgebra $\mathfrak{l}$ of $\mathfrak{q}$ is the zero eigenspace of $\mathrm{ad}(H)$, while the nilradical $\mathfrak{u}$ of $\mathfrak{q}$ is the sum of positive eigenspaces of $\mathrm{ad}(H)$. Note that clearly $\mathfrak{l} \supseteq \mathfrak{h}$. Since $\theta(H) = H$, $\mathfrak{l}, \mathfrak{u}$ and $\mathfrak{q}$ are all invariant under θ. Furthermore, $\mathfrak{l}$ is real, i.e., $\mathfrak{l}$ is the complexification of a subalgebra $\mathfrak{l}_0$ of $\mathfrak{g}_0$. Let L denote the connected subgroup of G corresponding to $\mathfrak{l}_0$. We will assume that our fixed choice of positive roots $\Delta^+(\mathfrak{g}, \mathfrak{h})$ is compatible with $\mathfrak{q}$ in the sense that the set of roots

$$\Delta(\mathfrak{u}) = \{\alpha \in \Delta(\mathfrak{g}, \mathfrak{h}) \mid \mathfrak{g}_\alpha \subset \mathfrak{u}\}$$

is contained in $\Delta^+(\mathfrak{g}, \mathfrak{h})$. Note that $\Delta(\mathfrak{l}, \mathfrak{h}) \subseteq \Delta(\mathfrak{g}, \mathfrak{h})$, and we set $\Delta^+(\mathfrak{l}, \mathfrak{h}) = \Delta(\mathfrak{l}, \mathfrak{h}) \cap \Delta^+(\mathfrak{g}, \mathfrak{h})$. Likewise, $\Delta(\mathfrak{l}, \mathfrak{t}) \subseteq \Delta(\mathfrak{g}, \mathfrak{t})$, and we set $\Delta^+(\mathfrak{l}, \mathfrak{t}) = \Delta(\mathfrak{l}, \mathfrak{t}) \cap \Delta^+(\mathfrak{g}, \mathfrak{t})$.

Let $\lambda \in \mathfrak{l}^*$ be admissible. In other words, λ is the complexified differential of a unitary character of L, satisfying the following positivity condition:

$$\langle \alpha, \lambda|_\mathfrak{t} \rangle \geq 0, \qquad \text{for all} \quad \alpha \in \Delta(\mathfrak{u}).$$

Then λ is orthogonal to all roots of $\mathfrak{l}$, so we can view λ as an element of $\mathfrak{h}^*$.

Given $\mathfrak{q}$ and λ as above, define

$$\mu(\mathfrak{q}, \lambda) = \lambda|_\mathfrak{t} + 2\rho(\mathfrak{u} \cap \mathfrak{p}).$$

Here $\rho(\mathfrak{u} \cap \mathfrak{p}) = \rho(\Delta(\mathfrak{u} \cap \mathfrak{p}))$ is the half sum of all elements of $\Delta(\mathfrak{u} \cap \mathfrak{p})$, i.e., of all $\mathfrak{t}$-weights of $\mathfrak{u} \cap \mathfrak{p}$, counted with multiplicity. We will use analogous notation for other $\mathfrak{t}$-stable subspaces of $\mathfrak{g}$.

The following result of Vogan and Zuckerman characterizes the $A_\mathfrak{q}(\lambda)$-modules we wish to consider.

Theorem 3.3 ([VZ], [V2]). *Let $\mathfrak{q}$ be a θ-stable parabolic subalgebra of $\mathfrak{g}$ and let $\lambda \in \mathfrak{h}^*$ be admissible as defined above. Then there is a unique unitary $(\mathfrak{g}, K)$-module $A_{\mathfrak{q}}(\lambda)$ with the following properties:*

(i) The restriction of $A_{\mathfrak{q}}(\lambda)$ to $\mathfrak{k}$ contains the representation with highest weight $\mu(\mathfrak{q}, \lambda)$ defined as above;
(ii) $A_{\mathfrak{q}}(\lambda)$ has infinitesimal character $\lambda + \rho$;
(iii) If the representation of $\mathfrak{k}$ occurs in $A_{\mathfrak{q}}(\lambda)$, then its highest weight is of the form

$$\mu(\mathfrak{q}, \lambda) + \sum_{\beta \in \Delta(\mathfrak{u} \cap \mathfrak{p})} n_\beta \beta \tag{3.1}$$

with n_β nonnegative integers. In particular, $\mu(\mathfrak{q}, \lambda)$ is the lowest K-type of $A_{\mathfrak{q}}(\lambda)$ (and its multiplicity is 1).

We denote the Weyl groups for $\Delta(\mathfrak{l}, \mathfrak{t})$ and $\Delta(\mathfrak{l}, \mathfrak{h})$ by $W(\mathfrak{l}, \mathfrak{t})$ and $W(\mathfrak{l}, \mathfrak{h})$ respectively. Clearly, these are subgroups of $W(\mathfrak{g}, \mathfrak{t})$, respectively $W(\mathfrak{g}, \mathfrak{h})$.

Theorem 3.4 (Theorem 5.1 [HKP]). *If $\lambda \neq \theta\lambda$, then the Dirac cohomology of $A_{\mathfrak{q}}(\lambda)$ is zero. If $\lambda = \theta\lambda$, then the Dirac cohomology of the unitary irreducible $(\mathfrak{g}, K)$-module $A_{\mathfrak{q}}(\lambda)$ is*

$$H_D(A_{\mathfrak{q}}(\lambda)) = \ker D = \bigoplus_{w \in W(\mathfrak{l}, \mathfrak{t})^1} 2^{[l_0/2]} E_{w(\lambda+\rho)-\rho_c}.$$

Remark 3.5. Dirac cohomology has been calculated for other families of representations (see [BP1, BP2, MP]).

4 Dirac cohomology and $(\mathfrak{g}, K)$-cohomology

Let F be an irreducible finite-dimensional G-module with highest weight λ. By results of Vogan and Zuckerman [VZ], the irreducible unitary $(\mathfrak{g}, K)$-modules X such that $H^*(\mathfrak{g}, K; X \otimes F^*) \neq 0$ are certain $A_{\mathfrak{q}}(\lambda)$-modules with the same infinitesimal character as F. Moreover, if X is such an $A_{\mathfrak{q}}(\lambda)$-module, then

$$H^i(\mathfrak{g}, K; X \otimes F^*) = \mathrm{Hom}_{L \cap K}\big(\textstyle\bigwedge^{i-\dim(\mathfrak{u}\cap\mathfrak{p})}(\mathfrak{l} \cap \mathfrak{p}), \mathbb{C}\big),$$

where L is the Levi subgroup of G corresponding to $\mathfrak{q}$.

Recall that the above $(\mathfrak{g}, K)$-cohomology can be defined as the cohomology of the complex

$$\mathrm{Hom}_K^\bullet(\textstyle\bigwedge^\bullet(\mathfrak{p}), X \otimes F^*),$$

with differential

$$df(X_1 \wedge \cdots \wedge X_k) = \sum_i (-1)^{i-1} X_i \cdot f(X_1 \wedge \cdots \widehat{X}_i \cdots \wedge X_k).$$

To show how this is related to our results, let us first show that $(\mathfrak{g}, K)$-cohomology is related to Dirac cohomology, as stated in the introduction. As we mentioned above, if $(\mathfrak{g}, K)$-cohomology is nonzero, then X must have the same infinitesimal character as F. We assume this in the following.

Consider first the case when $\dim \mathfrak{p}$ is even. Then we can write $\mathfrak{p}$ as a direct sum of isotropic subspaces U and $\bar{U} \cong U^*$. Then we have the spinor spaces $S = \bigwedge^\bullet U$ and $S^* = \bigwedge^\bullet \bar{U}$, and

$$S \otimes S^* \cong \textstyle\bigwedge^\bullet(U \oplus \bar{U}) = \bigwedge^\bullet \mathfrak{p}.$$

It follows that we can identify the $(\mathfrak{g}, K)$-cohomology of $X \otimes F^*$ with

$$H^*(\mathrm{Hom}^\bullet_{\tilde{K}}(S \otimes S^*, X \otimes F^*)) \cong H^*(\mathrm{Hom}^\bullet_{\tilde{K}}(F \otimes S, X \otimes S)).$$

If X is unitary, Wallach has proved that the differential of this complex is 0 (see [W, Proposition 9.4.3], or [BW]). So taking the cohomology can be omitted in the above formula. It follows that

$$H^*(\mathfrak{g}, K; X \otimes F^*) = \mathrm{Hom}^\bullet_{\tilde{K}}(H_D(F), H_D(X)).$$

Namely, the eigenvalues of D^2 are nonpositive on $F \otimes S$ and nonnegative on $X \otimes S$ (see [W, 9.4.6]). Also, since the infinitesimal characters of X and F are the same, the eigenvalue of D^2 on a $\widetilde{K}$-type in either of the two variables depends only on the value of the Casimir element $\Omega_{\mathfrak{k}_\Delta}$ on that $\widetilde{K}$-type. In particular, the action of D^2 on isomorphic $\widetilde{K}$-types must have the same eigenvalue. It follows from the Dirac inequality that the same $\widetilde{K}$-type can appear in both $F \otimes S$ and $X \otimes S$ only if it is in the kernel of D^2 in each variable, and $\mathrm{Ker}\, D^2$ is equal to the Dirac cohomology for these cases.

Now we consider the case when $\dim \mathfrak{p}$ is odd. In this case, $\bigwedge^\bullet \mathfrak{p}$ is isomorphic to the direct sum of two copies of $S \otimes S^*$. Therefore, $H^*(\mathfrak{g}, K; X \otimes F^*)$ is isomorphic to the direct sum of two copies of $\mathrm{Hom}^\bullet_{\tilde{K}}(H_D(F), H_D(X))$.

If we now use the formulas for $H_D(A_\mathfrak{q}(\lambda))$ and $H_D(F)$ from Section 3, we immediately get

$$\dim H^*(\mathfrak{g}, K; X \otimes F^*) = 2^{l_0} |W(\mathfrak{l}, \mathfrak{t}) / W(\mathfrak{l} \cap \mathfrak{k}, \mathfrak{t})|.$$

This agrees with the results of [VZ].

5 Dirac cohomology of highest weight modules

We describe Dirac cohomology of irreducible highest weight modules. As mentioned in Section 3, Kostant extended Vogan's conjecture to the setting of the cubic Dirac operator [Ko3]. Fix a Cartan subalgebra $\mathfrak{h}$ in a Borel subalgebra $\mathfrak{b}$. The category $\mathcal{O}$ introduced by Bernstein, Gelfand and Gelfand [BGG] is the category of all $\mathfrak{g}$-modules, which are finitely generated, locally $\mathfrak{b}$-finite and semisimple under the $\mathfrak{h}$-action. Kostant proved a nonvanishing result on Dirac cohomology for highest weight modules in the most general setting. His theorem implies that for the equal rank case all highest weight modules have nonzero Dirac cohomology. He also determined the Dirac cohomology of finite-dimensional modules in this case. The connection of Dirac cohomology of $(\mathfrak{g}, K)$-modules and that of highest weight modules was studied in [DH1] using the Jacquet functor. In [HX] we determined the Dirac cohomology of all irreducible highest weight modules in terms of Kazhdan–Lusztig polynomials.

We first recall the definition of Kostant's cubic Dirac operator and the basic properties of the corresponding Dirac cohomology. Let $\mathfrak{g}$ be a semisimple complex Lie algebra with Killing form B. Let $\mathfrak{r} \subset \mathfrak{g}$ be a reductive Lie subalgebra such that $B|_{\mathfrak{r}\times\mathfrak{r}}$ is nondegenerate. Let $\mathfrak{g} = \mathfrak{r} \oplus \mathfrak{s}$ be the orthogonal decomposition with respect to B. Then the restriction $B|_{\mathfrak{s}}$ is also nondegenerate. Denote by $C(\mathfrak{s})$ the Clifford algebra of $\mathfrak{s}$ with

$$uu' + u'u = -2B(u, u')$$

for all $u, u' \in \mathfrak{s}$. The above choice of sign is the same as in [HP2], but different from the definition in [Ko1], as well as in [HPR]. The two different choices of signs make no essential difference since the two bilinear forms are equivalent over $\mathbb{C}$. Now fix an orthonormal basis $Z_1, \ldots, Z_m$ of $\mathfrak{s}$. Kostant [Ko1] defines the cubic Dirac operator D by

$$D = \sum_{i=1}^{m} Z_i \otimes Z_i + 1 \otimes v \in U(\mathfrak{g}) \otimes C(\mathfrak{s}).$$

Here $v \in C(\mathfrak{s})$ is the image of the fundamental 3-form $w \in \bigwedge^3(\mathfrak{s}^*)$,

$$w(X, Y, Z) = \frac{1}{2} B(X, [Y, Z]),$$

under the Chevalley map $\bigwedge(\mathfrak{s}^*) \to C(\mathfrak{s})$ and the identification of $\mathfrak{s}^*$ with $\mathfrak{s}$ by the Killing form B. Explicitly,

$$v = \frac{1}{2} \sum_{1 \le i < j < k \le m} B([Z_i, Z_j], Z_k) Z_i Z_j Z_k.$$

The cubic Dirac operator has a good square in analogy with the Dirac operator associated with the symmetric pair $(\mathfrak{g}, \mathfrak{k})$ in Section 3. We have a similar Lie algebra map

$$\alpha : \mathfrak{r} \to C(\mathfrak{s})$$

which is defined by the adjoint map ad : $\mathfrak{r} \to \mathfrak{so}(\mathfrak{s})$ composed with the embedding of $\mathfrak{so}(\mathfrak{s})$ into $C(\mathfrak{s})$ using the identification $\mathfrak{so}(\mathfrak{s}) \simeq \bigwedge^2 \mathfrak{s}$. The explicit formula for α is (see [HP2, §2.3.3])

$$\alpha(X) = -\frac{1}{4}\sum_j [X, Z_j]Z_j, \qquad X \in \mathfrak{r}. \tag{5.1}$$

Using α we can embed the Lie algebra $\mathfrak{r}$ diagonally into $U(\mathfrak{g}) \otimes C(\mathfrak{s})$, by

$$X \mapsto X_\Delta = X \otimes 1 + 1 \otimes \alpha(X).$$

This embedding extends to $U(\mathfrak{r})$. We denote the image of $\mathfrak{r}$ by $\mathfrak{r}_\Delta$, and then the image of $U(\mathfrak{r})$ is the enveloping algebra $U(\mathfrak{r}_\Delta)$ of $\mathfrak{r}_\Delta$. Let $\Omega_\mathfrak{g}$ (resp. $\Omega_\mathfrak{r}$) be the Casimir elements for $\mathfrak{g}$ (resp. $\mathfrak{r}$). The image of $\Omega_\mathfrak{r}$ under Δ is denoted by $\Omega_{\mathfrak{r}_\Delta}$.

Let $\mathfrak{h}_\mathfrak{r}$ be a Cartan subalgebra of $\mathfrak{r}$ which is contained in $\mathfrak{h}$. It follows from Kostant's calculation ([Ko1, Theorem 2.16]) that

$$D^2 = -\Omega_\mathfrak{g} \otimes 1 + \Omega_{\mathfrak{r}_\Delta} - (\|\rho\|^2 - \|\rho_\mathfrak{r}\|^2)1 \otimes 1, \tag{5.2}$$

where $\rho_\mathfrak{r}$ denote the half sum of positive roots for $(\mathfrak{r}, \mathfrak{h}_\mathfrak{r})$. We also note the sign difference with Kostant's formula due to our choice of bilinear form for the definition of the Clifford algebra $C(\mathfrak{s})$.

We denote by W the Weyl group associated to the root system $\Delta(\mathfrak{g}, \mathfrak{h})$ and $W_\mathfrak{r}$ the Weyl group associated to the root system $\Delta(\mathfrak{r}, \mathfrak{h}_\mathfrak{r})$. The following theorem due to Kostant is an extension of Vogan's conjecture on the symmetric pair case which is proved in [HP1]. (See [Ko3, Theorems 4.1 and 4.2] or [HP2, Theorem 4.1.4]).

Theorem 5.1. *There is an algebra homomorphism $\zeta : Z(\mathfrak{g}) \to Z(\mathfrak{r}) \cong Z(\mathfrak{r}_\Delta)$ such that for any $z \in Z(\mathfrak{g})$ one has*

$$z \otimes 1 - \zeta(z) = Da + aD \text{ for some } a \in U(\mathfrak{g}) \otimes C(\mathfrak{s}).$$

Moreover, ζ is determined by the following commutative diagram:

$$\begin{array}{ccc} Z(\mathfrak{g}) & \xrightarrow{\zeta} & Z(\mathfrak{r}) \\ \eta \downarrow & & \downarrow \eta_\mathfrak{r} \\ P(\mathfrak{h}^*)^W & \xrightarrow{\text{Res}} & P(\mathfrak{h}_\mathfrak{r}^*)^{W_\mathfrak{r}}. \end{array}$$

Here the vertical maps η and $\eta_\mathfrak{r}$ are Harish-Chandra isomorphisms.

Definition 5.2. Let S be a spin module of $C(\mathfrak{s})$. Consider the action of D on $V \otimes S$

$$D : V \otimes S \to V \otimes S \tag{5.3}$$

with $\mathfrak{g}$ acting on V and $C(\mathfrak{s})$ on S. The *Dirac cohomology* of V is defined to be the $\mathfrak{r}$-module

$$H_D(V) := \operatorname{Ker} D / \operatorname{Ker} D \cap \operatorname{Im} D.$$

The following theorem is a consequence of the above theorem.

Theorem 5.3 ([Ko3], [HP2]). *Let V be a $\mathfrak{g}$-module with $Z(\mathfrak{g})$ infinitesimal character χ_Λ. Suppose that an $\mathfrak{r}$-module N is contained in the Dirac cohomology $H_D(V)$ and has $Z(\mathfrak{r})$ infinitesimal character χ_λ. Then $\lambda = w\Lambda$ for some $w \in W$.*

Suppose that V_λ is a finite-dimensional representation with highest weight $\lambda \in \mathfrak{h}^*$. Kostant [Ko2] calculated the Dirac cohomology of V_λ with respect to any equal rank quadratic subalgebra $\mathfrak{r}$ of $\mathfrak{g}$. Assume that $\mathfrak{h} \subset \mathfrak{r} \subset \mathfrak{g}$ is the Cartan subalgebra for both $\mathfrak{r}$ and $\mathfrak{g}$. Define $W(\mathfrak{g}, \mathfrak{h})^1$ to be the subset of the Weyl group $W(\mathfrak{g}, \mathfrak{h})$ defined by

$$W(\mathfrak{g}, \mathfrak{h})^1 = \{w \in W(\mathfrak{g}, \mathfrak{h}) \mid w(\rho) \text{ is } \Delta^+(\mathfrak{r}, \mathfrak{h})\text{-dominant}\}.$$

This is the same as the subset of elements $w \in W(\mathfrak{g}, \mathfrak{h})$ that map the positive Weyl $\mathfrak{g}$-chamber into the positive $\mathfrak{r}$-chamber. There is a bijection

$$W(\mathfrak{r}, \mathfrak{h}) \times W(\mathfrak{g}, \mathfrak{h})^1 \to W(\mathfrak{g}, \mathfrak{h})$$

given by $(w, \tau) \mapsto w\tau$. Kostant proved [Ko2] that

$$H_D(V_\lambda) = \bigoplus_{w \in W(\mathfrak{g}, \mathfrak{h})^1} E_{w(\lambda+\rho)-\rho_{\mathfrak{r}}}.$$

The above result of Kostant on Dirac cohomology of finite-dimensional modules has been extended to the unequal rank case by Mehdi and Zierau [MZ]. We now show how to calculate the Dirac cohomology of a simple highest weight module of possibly infinite dimension. We need to recall the definition and some of the basic properties of the category $\mathcal{O}^{\mathfrak{q}}$ associated with an arbitrary parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$.

Recall that if $\mathfrak{g}$ is a complex semisimple Lie algebra with Cartan subalgebra $\mathfrak{h}$, we denote by $\Phi = \Delta(\mathfrak{g}, \mathfrak{h}) \subseteq \mathfrak{h}^*$ the root system of $(\mathfrak{g}, \mathfrak{h})$. For $\alpha \in \Phi$, let $\mathfrak{g}_\alpha$ be the root subspace of $\mathfrak{g}$ corresponding to α. We fix a choice of the set of positive roots Φ^+ and let Δ be the corresponding subset of simple roots in Φ^+. Note that each subset $I \subset \Delta$ generates a root system $\Phi_I \subset \Phi$, with positive roots $\Phi_I^+ = \Phi_I \cap \Phi^+$.

The parabolic subalgebras of $\mathfrak{g}$ up to conjugation are in one-to-one correspondence with the subsets of Δ. We let

$$\mathfrak{l}_I = \mathfrak{h} \oplus \sum_{\alpha \in \Phi_I} \mathfrak{g}_\alpha$$

be the Levi subalgebra and let

$$\mathfrak{u}_I = \sum_{\alpha \in \Phi^+ \setminus \Phi_I^+} \mathfrak{g}_\alpha, \qquad \bar{\mathfrak{u}}_I = \sum_{\alpha \in \Phi^+ \setminus \Phi_I^+} \mathfrak{g}_{-\alpha}$$

be the nilpotent radical and its dual space with respect to the Killing form B. Then $\mathfrak{q}_I = \mathfrak{l}_I \oplus \mathfrak{u}_I$ is the standard parabolic subalgebra associated with I. We set

$$\rho = \rho(\mathfrak{g}) = \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha,\ \rho(\mathfrak{l}_I) = \frac{1}{2}\sum_{\alpha\in\Phi_I^+}\alpha,\ \text{and}\ \rho(\mathfrak{u}_I) = \frac{1}{2}\sum_{\alpha\in\Phi^+\setminus\Phi_I^+}\alpha.$$

Then we have $\rho(\bar{\mathfrak{u}}_I) = -\rho(\mathfrak{u}_I)$. We note that once I is fixed, there is little use for other subsets of Δ. We will omit the subscript if a subalgebra is clearly associated with I.

Definition 5.4. The category $\mathcal{O}^{\mathfrak{q}}$ is defined to be the full subcategory of $U(\mathfrak{g})$-modules M that satisfy the following conditions:

(i) M is a finitely generated $U(\mathfrak{g})$-module;
(ii) M is a direct sum of finite-dimensional simple $U(\mathfrak{l})$-modules;
(iii) M is locally finite as a $U(\mathfrak{q})$-module.

We adopt the notation of [Hum2]. Let Λ_I^+ be the set of Φ_I^+-dominant integral weights in $\mathfrak{h}^*$, namely,

$$\Lambda_I^+ := \{\lambda\in\mathfrak{h}^* \mid \langle\lambda,\alpha^\vee\rangle\in\mathbb{Z}^{\geq 0} \text{ for all } \alpha\in\Phi_I^+\}.$$

Here $\langle,\rangle$ is the bilinear form on $\mathfrak{h}^*$ (induced from the Killing form B) and $\alpha^\vee = 2\alpha/\langle\alpha,\alpha\rangle$.

Let $F(\lambda)$ be the finite-dimensional simple $\mathfrak{l}$-module with highest weight λ. Then $\lambda\in\Lambda_I^+$. We consider $F(\lambda)$ as a $\mathfrak{q}$-module by letting $\mathfrak{u}$ act trivially on it. Then the *parabolic Verma module* with highest weight λ is the induced module

$$M_I(\lambda) = U(\mathfrak{g})\otimes_{U(\mathfrak{q})} F(\lambda).$$

The module $M_I(\lambda)$ is a quotient of the ordinary Verma module $M(\lambda)$. Using Theorem 1.2 in [Hum2], we can write unambiguously $L(\lambda)$ for the unique simple quotient of $M_I(\lambda)$ and $M(\lambda)$. Furthermore, since every nonzero module in $\mathcal{O}^{\mathfrak{q}}$ has at least one nonzero vector of maximal weight, Proposition 9.3 in [Hum2] implies that every simple module in $\mathcal{O}^{\mathfrak{q}}$ is isomorphic to $L(\lambda)$ for some $\lambda\in\Lambda_I^+$ and is therefore determined uniquely up to isomorphism by its highest weight.

Recall that $M_I(\lambda)$ and all its subquotients including $L(\lambda)$ have the same infinitesimal character

$$\chi_\lambda\colon Z(\mathfrak{g})\to\mathbb{C}.$$

Here χ_λ is an algebra homomorphism such that $z\cdot v = \chi_\lambda(z)v$ for all $z\in Z(\mathfrak{g})$ and all $v\in M(\lambda)$. We note that the standard parameter for the infinitesimal character χ_λ is the Weyl group orbit of $\lambda+\rho\in\mathfrak{h}^*$ due to the ρ-shift in the Harish-Chandra isomorphism $Z(\mathfrak{g})\to S(\mathfrak{h})^W$.

It follows from Corollary 1.2 in [Hum2] that every nonzero module $M\in\mathcal{O}^{\mathfrak{q}}$ has a finite filtration with nonzero quotients each of which is a highest weight module in $\mathcal{O}^{\mathfrak{q}}$. Then the action of $Z(\mathfrak{g})$ on M is finite. We set

$$M^\chi := \{v\in M \mid (z-\chi(z))^n v = 0 \text{ for some } n>0 \text{ depending on } z\}.$$

Then $z - \chi(z)$ acts locally nilpotently on M^χ for all $z \in Z(\mathfrak{g})$ and M^χ is a $U(\mathfrak{g})$-submodule of M. Let $\mathcal{O}^{\mathfrak{q}}_\chi$ denote the full subcategory of $\mathcal{O}^{\mathfrak{q}}$ whose objects are the modules M for which $M = M^\chi$. By the above discussion we have the following direct sum decomposition:

$$\mathcal{O}^{\mathfrak{q}} = \bigoplus_\chi \mathcal{O}^{\mathfrak{q}}_\chi,$$

where χ is of the form $\chi = \chi_\lambda$ for some $\lambda \in \mathfrak{h}^*$.

Let W be the Weyl group associated to the root system Φ. We define the dot action of W on $\mathfrak{h}^*$ by $w \cdot \lambda = w(\lambda + \rho) - \rho$ for $\lambda \in \mathfrak{h}^*$. Then $\chi_\lambda = \chi_\mu$ if and only if $\lambda \in W \cdot \mu$ by the Harish-Chandra isomorphism $Z(\mathfrak{g}) \to S(\mathfrak{h})^W$. An element $\lambda \in \mathfrak{h}^*$ is called *regular* if the isotropy group of λ in W is trivial. In other words, λ is regular if $\langle \lambda + \rho, \alpha^\vee \rangle \neq 0$ for all $\alpha \in \Phi$. A nonregular element in $\mathfrak{h}^*$ will be called *singular*.

Denote by Γ the set of all $\mathbb{Z}^{\geq 0}$-linear combinations of simple roots in Δ. Let $\mathcal{X}$ be the additive group of functions $f : \mathfrak{h}^* \to \mathbb{Z}$ whose support lies in a finite union of sets of the form $\lambda - \Gamma$ for $\lambda \in \mathfrak{h}^*$. Define the convolution product on $\mathcal{X}$ by

$$(f * g)(\lambda) := \sum_{\mu+\nu=\lambda} f(\mu)g(\nu).$$

We regard $e(\lambda)$ as a function in $\mathcal{X}$ which takes value 1 at λ and value 0 at $\mu \neq \lambda$. Then $e(\lambda) * e(\mu) = e(\lambda + \mu)$. It is clear that $\mathcal{X}$ is a commutative ring under convolution, with $e(0)$ as its multiplicative identity. Let

$$M_\lambda := \{v \in M \mid h \cdot v = \lambda(h)v \text{ for all } h \in \mathfrak{h}\}.$$

We say that a weight module (semisimple $\mathfrak{h}$-module) M *has a character* if

$$\operatorname{ch} M := \sum_{\lambda \in \mathfrak{h}^*} \dim M_\lambda \, e(\lambda) \tag{5.4}$$

is contained in $\mathcal{X}$. In this case, $\operatorname{ch} M$ is called the *formal character* of M. Notice that all the modules in $\mathcal{O}^{\mathfrak{q}}$ have characters, as do all finite-dimensional semisimple $\mathfrak{h}$-modules. In particular, if M has a character and $\dim L < \infty$, then $M \otimes L$ has a character

$$\operatorname{ch}(M \otimes L) = \operatorname{ch} M * \operatorname{ch} L.$$

In addition, for semisimple $\mathfrak{h}$-modules which have characters, their direct sums, submodules and quotients also have characters.

As a consequence of the established Vogan's conjecture for the cubic Dirac operators, we have the following proposition (see also [DH1, Theorem 4.3]).

Proposition 5.5. *Suppose that V is in $\mathcal{O}^{\mathfrak{q}}_{\chi_\mu}$. Then the Dirac cohomology $H_D(V)$ is a completely reducible finite-dimensional $\mathfrak{l}$-module. Moreover, if the finite-dimensional $\mathfrak{l}$-module $F(\lambda)$ is contained in $H_D(V)$, then $\lambda + \rho_{\mathfrak{l}} = w(\mu + \rho)$ for some $w \in W$.*

It is shown in [HX] that determining $H_D(L(\lambda))$ is equivalent to determining $\operatorname{ch} L(\lambda)$ in terms of $\operatorname{ch} M_I(\mu)$, which is solved by the Kazhdan–Lusztig algorithm. Namely, if

$$\operatorname{ch} L(\lambda) = \sum (-1)^{\epsilon(\lambda,\mu)} m(\lambda,\mu) \operatorname{ch} M_I(\mu),$$

then we have

$$H_D(L(\lambda)) = \bigoplus m(\lambda,\mu) F(\mu) \otimes \mathbb{C}_{\rho(\mathfrak{u})}.$$

Using the known results on Kazhdan–Lusztig polynomials, we can calculate explicitly the Dirac cohomology of all irreducible highest weight modules. We recall the main result from [HX] here. Recall that $W = W(\mathfrak{g},\mathfrak{h})$ is the Weyl group associated to the root system Φ. We define

$$\Phi_{[\lambda]} := \{\alpha \in \Phi \mid \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z}\}.$$

Then it is the root system of integral roots associated to λ. We also set

$$W_{[\lambda]} := \{w \in W \mid w\lambda - \lambda \in \Lambda_r\},$$

where Λ_r is the $\mathbb{Z}$-span of Φ. Then W_I is contained in $W_{[\lambda]}$. We also define

$$W^I = \{w \in W_{[\lambda]} \mid w < s_\alpha w \text{ for all } \alpha \in I\},$$

where the ordering on W is given by the Bruhat ordering. Denote by $\Delta_{[\lambda]}$ the simple system corresponding to the positive system $\Phi_{[\lambda]} \cap \Phi^+$ in $\Phi_{[\lambda]}$. Let μ be the unique anti-dominant weight in $W_{[\lambda]}{\cdot}\lambda$. The set of singular simple roots in $\Delta_{[\lambda]}$ is defined by

$$J = \{\alpha \in \Delta_{[\lambda]} \mid \langle \mu + \rho, \alpha^\vee \rangle = 0\}.$$

Then $W_J = \{w \in W \mid w(\mu+\rho) = \mu+\rho\} \subset W_{[\lambda]}$ is the isotropy group of μ. Put

$${}^J W^I = \{w \in W^I \mid w < ws_\alpha \text{ and } ws_\alpha \in W^I \text{ for all } \alpha \in J\}.$$

Following Boe and Hunziker [BH] we define

$${}^J P^I_{x,w}(q) = \sum_{i \geq 0} q^{\frac{l(w)-l(x)-i}{2}} \dim \operatorname{Ext}^i_{\mathcal{O}^{\mathfrak{p}}}(M_x, L_w), \text{ for all } x, w \in {}^J W^I.$$

It is shown to be a polynomial and is called the relative *Kazhdan–Lusztig–Vogan polynomial.*

Theorem 5.6 (Theorem 6.16 [HX]). *If $L(\lambda)$ is the simple highest weight module in $\mathcal{O}^{\mathfrak{p}}_\mu$ of weight $\lambda = w_I w \cdot \mu$ with w_I the longest element in W_I, then one has an $\mathfrak{l}$-module decomposition*

$$H_D(L(\lambda)) \simeq \bigoplus_{x \in {}^J W^I} {}^J P^I_{x,w}(1) F(w_I x \cdot \mu + \rho(\mathfrak{u})).$$

Remark 5.7. Applying the action of the Chevalley automorphism (see Section 7) on Dirac cohomology, we can also determine the Dirac cohomology of simple lowest weight modules.

6 Dirac cohomology and $\mathfrak{u}$-cohomology

In this section we review the results on Dirac cohomology and $\mathfrak{p}^+$-cohomology of unitary representations for the Hermitian symmetric case. Then we discuss the simple highest weight modules in $\mathcal{O}^{\mathfrak{q}}$. We use quite different techniques for these two cases.

Suppose that G is simple and Hermitian symmetric, with maximal compact subgroup K. In this case the $\mathfrak{k}$-module $\mathfrak{g}$ decomposes as $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} = \mathfrak{k} \oplus \mathfrak{p}^+ \oplus \mathfrak{p}^-$. We can choose the basis b_i of $\mathfrak{p}$ in the following special way. Let $\Delta_n^+ = \{\beta_1, \dots, \beta_m\}$. For each β_i we choose a root vector $e_i \in \mathfrak{p}^+$. Let $f_i \in \mathfrak{p}^-$ be the root vector for the root $-\beta_i$ such that $B(e_i, f_i) = 1$. Then for the basis b_i of $\mathfrak{p}$ we choose $e_1, \dots, e_m; f_1, \dots, f_m$. The dual basis is then $f_1, \dots, f_m; e_1, \dots, e_m$. Thus the Dirac operator is

$$D = \sum_{i=1}^{m} e_i \otimes f_i + f_i \otimes e_i.$$

We also note that in this case G is of equal rank and in particular $\mathfrak{p}$ is even-dimensional. Therefore, there is a unique irreducible $C(\mathfrak{p})$-module, the spin module S, which we choose to construct as $S = \bigwedge \mathfrak{p}^+$. It is also a module for the double cover $\widetilde{K}$ of K. Let X be a $(\mathfrak{g}, K)$-module. Since $\mathfrak{p}^+ \cong (\mathfrak{p}^-)^*$, we have

$$X \otimes S \cong X \otimes \bigwedge \mathfrak{p}^+ \cong \mathrm{Hom}(\bigwedge \mathfrak{p}^-, X) \tag{6.1}$$

as vector spaces. Note that the underlying vector space $\bigwedge \mathfrak{p}^+$ of the spin module S carries the adjoint action of $\mathfrak{k}$, but the relevant $\mathfrak{k}$-action on S is the spin action defined using the map (5.1). The spin action is equal to the adjoint action shifted by the character $-\rho_n$ of $\mathfrak{k}$ (see [Ko2, Proposition 3.6]). So as a $\mathfrak{k}$-module, $X \otimes S$ differs from $X \otimes \bigwedge \mathfrak{p}^+$ and $\mathrm{Hom}(\bigwedge \mathfrak{p}^-, X)$ by a twist of the 1-dimensional $\mathfrak{k}$-module $\mathbb{C}_{-\rho_n}$.

Let $C = \sum_{i=1}^m f_i \otimes e_i$ and $C^- = \sum_{i=1}^m e_i \otimes f_i$; so $D = C + C^-$. Then, under the identifications (6.1), C acts on $X \otimes S$ as the $\mathfrak{p}^-$-cohomology differential, while C^- acts by 2 times the $\mathfrak{p}^+$-homology differential (see [HP2, Proposition 9.1.6] or [HPR]). Furthermore, C and C^- are adjoints of each other with respect to the Hermitian inner product on $X \otimes S$ mentioned above (see [HP2, Lemma 9.3.1] or [HPR]). It was proved that Dirac cohomology is isomorphic to $\mathfrak{p}^-$-cohomology up to a one-dimensional character by using a version of Hodge decomposition.

Theorem 6.1 ([HPR], Theorem 7.11). *Let X be a unitary $(\mathfrak{g}, K)$-module. Then*

$$H_D(X) \cong H^*(\mathfrak{p}^-, X) \otimes \mathbb{C}_{-\rho(\mathfrak{p}^+)} \cong H_*(\mathfrak{p}^+, X) \otimes \mathbb{C}_{-\rho(\mathfrak{p}^+)}$$

as $\mathfrak{k}$-modules. Moreover, the above isomorphisms hold for a parabolic subalgebra $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ as long as $\mathfrak{l} \subseteq \mathfrak{k}$ and $\mathfrak{u} \supseteq \mathfrak{p}^+$, that is

$$H_D(X) \cong H^*(\mathfrak{u}^-, X) \otimes \mathbb{C}_{-\rho(\mathfrak{u})} \cong H_*(\mathfrak{u}, X) \otimes \mathbb{C}_{-\rho(\mathfrak{u})}$$

as $\mathfrak{l}$-modules.

Note that we may use $\bigwedge \mathfrak{p}^-$ instead of $\bigwedge \mathfrak{p}^+$ to construct the spin module S. Then we have

$$H_D(X) \cong H^*(\mathfrak{p}^+, X) \otimes \mathbb{C}_{\rho(\mathfrak{p}^+)} \cong H_*(\mathfrak{p}^-, X) \otimes \mathbb{C}_{\rho(\mathfrak{p}^+)}. \tag{6.2}$$

Namely, the Dirac operator is independent of the choice of positive roots. Thus, we also have

$$H^*(\mathfrak{p}^+, X) \otimes \mathbb{C}_{\rho(\mathfrak{p}^+)} \cong H^*(\mathfrak{p}^-, X) \otimes \mathbb{C}_{-\rho(\mathfrak{p}^+)},$$

and

$$H_*(\mathfrak{p}^+, X) \otimes \mathbb{C}_{-\rho(\mathfrak{p}^+)} \cong H_*(\mathfrak{p}^-, X) \otimes \mathbb{C}_{\rho(\mathfrak{p}^+)}.$$

It also follows that we know the Dirac cohomology of all irreducible unitary highest weight modules explicitly from Enright's calculation of $\mathfrak{p}^+$-cohomology [E].

Now suppose $\mathfrak{q} = \mathfrak{l} + \mathfrak{u}$ is a parabolic subalgebra of $\mathfrak{g}$ as in Section 5. We recall the result from [HX] on the relation between Dirac cohomology with respect to $D(\mathfrak{g}, \mathfrak{l})$ and $\mathfrak{u}$-cohomology. We note that the spin action $\alpha(\mathfrak{l})$ on S makes it a finite-dimensional $\mathfrak{l}$-module. If $V \in \mathcal{O}^{\mathfrak{q}}$, then $V \otimes S$ is a direct sum of finite-dimensional simple $\mathfrak{l}$-modules. Hence, any submodule, quotient or subquotient of $V \otimes S$ is also a direct sum of finite-dimensional simple $\mathfrak{l}$-modules.

Then the Casimir element $\Omega_{\mathfrak{g}}$ acts semisimply on V. We have shown that $H_D(V)$ is isomorphic to $\mathfrak{u}$-homology up to a character in [HX]. We recall here the main steps in the proof of this isomorphism ([HX, Theorem 5.12]). The $\mathfrak{u}$-homology is $\mathbb{Z}_2$-graded as follows:

$$H_+(\mathfrak{u}, V) = \bigoplus_{i=0} H_{2i}(\mathfrak{u}, V) \text{ and } H_-(\mathfrak{u}, V) = \bigoplus_{i=0} H_{2i+1}(\mathfrak{u}, V).$$

Then there are injective $\mathfrak{l}$-module homomorphisms ([HX, Proposition 4.8]):

$$H_D^{\pm}(V) \to H_{\pm}(\mathfrak{u}, V) \otimes \mathbb{C}_{\rho(\bar{\mathfrak{u}})}.$$

Note that we also have ([HX, Proposition 5.2])

$$\operatorname{ch} H_D^+(V) - \operatorname{ch} H_D^-(V) = (\operatorname{ch} H_+(\mathfrak{u}, V) - \operatorname{ch} H_-(\mathfrak{u}, V)) * \operatorname{ch} \mathbb{C}_{\rho(\bar{\mathfrak{u}})}.$$

Then the properties of KLV polynomials (see [HX, Proposition 6.14]) imply the following parity condition: $H_+(\mathfrak{u}, V)$ and $H_-(\mathfrak{u}, V)$ have no common $\mathfrak{l}$-submodules, namely,

$$\operatorname{Hom}_{\mathfrak{l}}(H_+(\mathfrak{u}, V), H_-(\mathfrak{u}, V)) = 0.$$

This is the key lemma in [HX] (Lemma 5.11). It follows that the embeddings

$$H_D^{\pm}(V) \to H_{\pm}(\mathfrak{u}, V) \otimes \mathbb{C}_{\rho(\bar{\mathfrak{u}})}$$

are isomorphisms.

Theorem 6.2 (Theorem 5.12, Corollary 5.13 [HX]). *Let V be a simple highest weight module in $\mathcal{O}^{\mathfrak{p}}$. Then we have the following $\mathfrak{l}$-module isomorphisms:*

$$H_D(V) \simeq H_*(\mathfrak{u}, V) \otimes \mathbb{C}_{-\rho(\mathfrak{u})} \simeq H^*(\bar{\mathfrak{u}}, V) \otimes \mathbb{C}_{-\rho(\mathfrak{u})}.$$

As mentioned, in the special case when $\mathfrak{q}$ contains $\mathfrak{p}^+$ in the Hermitian symmetric setting, the above theorem is proved for unitary Harish-Chandra modules in [HPR]. The argument in [HPR] depends on the existence of the positive definite Hermitian form on these modules. This argument cannot be extended to any simple highest weight modules.

For a Harish-Chandra module X for G of Hermitian symmetric type, we have similar injective homomorphisms of

$$H_D^{\pm}(X) \to H_{\pm}(\mathfrak{p}^+, X) \otimes \mathbb{C}_{-\rho(\mathfrak{p}^+)}.$$

Conjecture 6.3. Suppose that X is a simple Harish-Chandra module. Then

$$\mathrm{Hom}_{\widetilde{K}}(H_+(\mathfrak{p}^+, X), H_-(\mathfrak{p}^+, X)) = 0.$$

In particular, it implies the above injective homomorphisms are actually isomorphisms. In other words, we conjecture that Theorem 6.1 holds for any simple $(\mathfrak{g}, K)$-module X.

We note that Dirac cohomology for unitary lowest weight modules is an important ingredient for generalizing classical branching rules [HPZ, H2].

Remark 6.4. It is obvious that all the proofs for the theorems in this section on highest weight modules can be done for lowest weight modules.

We review a result about the action of an automorphism on Dirac cohomology [HPZ]. The above remark is a consequence of taking the Chevalley automorphism. Let τ be an automorphism of G preserving K. Then $\tau\big|_K$ is an automorphism of K. Also, τ induces automorphisms of $\mathfrak{g}_0$ and $\mathfrak{g}$, denoted again by τ, and τ preserves the Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$. Finally, $\tau\big|_{\mathfrak{p}}$ extends to an automorphism of the Clifford algebra $C(\mathfrak{p})$, denoted again by τ. Let X be a $(\mathfrak{g}, K)$-module. If we set $X^\tau = X$, then $(\pi \circ \tau, X^\tau)$ is also a $(\mathfrak{g}, K)$-module. Similarly, for any K-module (φ, E), if we set $E^\tau = E$, then $(\varphi \circ \tau, E^\tau)$ is also a K-module. The same is true if we replace K by $\widetilde{K}$. The following property of Dirac cohomology was proved for any unitary $(\mathfrak{g}, K)$-module in [HPZ] (Prop. 5.1 of [HPZ]). The same proof extends straightforwardly to any $(\mathfrak{g}, K)$-module ([H2, Prop. 3.12]).

Proposition 6.5. *Let X be a $(\mathfrak{g}, K)$-module. Then*

$$H_D(X^\tau) \cong (H_D(X))^\tau.$$

7 Calculation of Dirac cohomology in stages

In this section we review another technique for calculating Dirac cohomology, namely calculation in stages. This technique is needed to study the Dirac cohomology of elliptic representations. Recall that $\mathfrak{g}_0$ is the Lie algebra of G with a compact subalgebra $\mathfrak{k}_0$, the Lie algebra of K. We assume $\mathfrak{k}_0$ is of equal rank with $\mathfrak{g}_0$. Then a Cartan subalgebra $\mathfrak{t}_0$ of $\mathfrak{k}_0$ is also a Cartan subalgebra of $\mathfrak{g}_0$. We drop the subscripts for their complexification. The bilinear form B on $\mathfrak{g}$ is nondegenerate and the restriction $B|_{\mathfrak{t}}$ remains nondegenerate. Then we have an orthogonal decomposition $\mathfrak{g} = \mathfrak{t} \oplus \mathfrak{s}$ with $\mathfrak{s}$ the orthogonal complement of $\mathfrak{t}$ with respect to B. It follows that $B|_{\mathfrak{s}}$ is also nondegenerate. The cubic Dirac operator $D(\mathfrak{g}, \mathfrak{t})$ due to Kostant [Ko2] is in $U(\mathfrak{g}) \otimes C(\mathfrak{s})$. Let $\{Y_i\}_{i=1}^n$ be an orthonormal basis of $\mathfrak{s}$. Then we can write (see [HP2, 4.1.1])

$$D(\mathfrak{g}, \mathfrak{t}) = \sum_{i=1}^{n} Y_i \otimes Y_i + \frac{1}{2} \sum_{i<j<k} B([Y_i, Y_j], Y_k) \otimes Y_i Y_j Y_k.$$

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the complexification of the Cartan decomposition and let $\mathfrak{s}_1$ be the orthogonal complement of $\mathfrak{t}$ in $\mathfrak{k}$. Then $\mathfrak{s} = \mathfrak{s}_1 \oplus \mathfrak{p}$. As in [HP2, §9.3] we write the Dirac operator $D(\mathfrak{g}, \mathfrak{t})$ in terms of $D(\mathfrak{g}, \mathfrak{k})$ and $D(\mathfrak{k}, \mathfrak{t})$ by using an orthonormal basis for $\mathfrak{s}$ formed by orthonormal bases Z_i for $\mathfrak{p}$ and Z'_j for $\mathfrak{s}_1$.

Identifying $U(\mathfrak{g}) \otimes C(\mathfrak{s})$ with $U(\mathfrak{g}) \otimes C(\mathfrak{p}) \bar{\otimes} C(\mathfrak{s}_1)$, where $\bar{\otimes}$ denotes the $\mathbb{Z}_2$-graded tensor product, we can write

$$\begin{aligned} D(\mathfrak{g}, \mathfrak{t}) = &\textstyle\sum_i Z_i \otimes Z_i \otimes 1 + \sum_j Z'_j \otimes 1 \otimes Z'_j \\ &+\tfrac{1}{2} \textstyle\sum_{i<j} \sum_k B([Z_i, Z_j], Z'_k) \otimes Z_i Z_j \otimes Z'_k \\ &+\tfrac{1}{2} \textstyle\sum_{i<j<k} B([Z'_i, Z'_j], Z'_k) \otimes 1 \otimes Z'_i Z'_j Z'_k. \end{aligned} \tag{7.1}$$

Regarding $U(\mathfrak{g}) \otimes C(\mathfrak{k})$ as the subalgebra $U(\mathfrak{g}) \otimes C(\mathfrak{k}) \otimes 1$ of $U(\mathfrak{g}) \otimes C(\mathfrak{k}) \bar{\otimes} C(\mathfrak{s}_1)$, the first summand in (7.1) gives $D(\mathfrak{g}, \mathfrak{k})$ and the remaining three summands in (7.1) come from the cubic Dirac operator corresponding to $\mathfrak{t} \subset \mathfrak{k}$. However, this is an element of the algebra $U(\mathfrak{k}) \otimes C(\mathfrak{s}_1)$, and this algebra has to be embedded into $U(\mathfrak{g}) \otimes C(\mathfrak{p}) \bar{\otimes} C(\mathfrak{s}_1)$ diagonally, by

$$\Delta : U(\mathfrak{k}) \otimes C(\mathfrak{s}_1) \cong U(\mathfrak{k}_\Delta) \bar{\otimes} C(\mathfrak{s}_1) \subset U(\mathfrak{g}) \otimes C(\mathfrak{p}) \bar{\otimes} C(\mathfrak{s}_1).$$

Here $U(\mathfrak{k}_\Delta)$ is embedded into $U(\mathfrak{g}) \otimes C(\mathfrak{p})$ by a diagonal embedding while the factor $C(\mathfrak{s}_1)$ remains unchanged. We denote the image $\Delta(D(\mathfrak{k}, \mathfrak{t}))$ by $D_\Delta(\mathfrak{k}, \mathfrak{t})$.

Theorem 7.1 (Theorem 3.2 [HPR], Theorem 9.4.1 [HP2]). *With notation as above, $D(\mathfrak{g},\mathfrak{t})$ decomposes as $D(\mathfrak{g},\mathfrak{k}) + D_\Delta(\mathfrak{k},\mathfrak{t})$. Moreover, the summands $D(\mathfrak{g},\mathfrak{k})$ and $D_\Delta(\mathfrak{k},\mathfrak{t})$ anti-commute.*

The above decomposition holds in a slightly more general setting as it is stated and proved in Theorem 3.2 of [HPR] and in an even more general setting in [HP2, Theorem 9.4.1]. The anti-commuting property given here can be applied to calculate Dirac cohomology in stages. For convenience, we define the cohomology of any linear operator A on a vector space V to be the vector space

$$H(A) = \operatorname{Ker} A/(\operatorname{Im} A \cap \operatorname{Ker} A).$$

We also denote by $H(A;V)$ the cohomology when we emphasize the space V. We call the operator A semisimple if V is the (algebraic) direct sum of the eigenspaces of A.

Let S be the simple module of the Clifford algebra $C(\mathfrak{s})$. If X is a $(\mathfrak{g},K)$-module, then $D(\mathfrak{g},\mathfrak{t})$ acts on $X\otimes S$. We denote by $H_D(\mathfrak{g},\mathfrak{t};X)$ the cohomology of $X\otimes S$ with respect to $D(\mathfrak{g},\mathfrak{t})$; analogous notation will be used for other Dirac operators. We note that $H_D(\mathfrak{g},\mathfrak{t};X)$ is in fact the cohomology of the operator $D(\mathfrak{g},\mathfrak{t})$ on $X\otimes S$, namely

$$H_D(\mathfrak{g},\mathfrak{t};X) = H(D(\mathfrak{g},\mathfrak{t}); X\otimes S).$$

Lemma 7.2 (Lemma 5.3 [HPR]). *Let A and B be anti-commuting linear operators on an arbitrary vector space V. Assume that A^2 and B are semisimple. Then $H(A+B)$ is the cohomology (i.e., the kernel) of B acting on $H(A)$.*

The above theorem and lemma imply the following theorem for calculating Dirac cohomology in stages.

Theorem 7.3 (Theorem 6.1 [HPR], Theorem 9.4.4 [HP2]). *Let X be an admissible $(\mathfrak{g},K)$-module with $\Omega_\mathfrak{g}$ acting semisimply. Then we have*

$$H_D(\mathfrak{g},\mathfrak{t};X) = H_D(\mathfrak{k},\mathfrak{t};H_D(\mathfrak{g},\mathfrak{k};X)).$$

Also, we can reverse the order to have

$$H_D(\mathfrak{g},\mathfrak{t};X) = H(D(\mathfrak{g},\mathfrak{k})|_{H_D(\mathfrak{k},\mathfrak{t};X)}).$$

The above theorem is proved in [HPR] and in [HP2] for a slightly more general case when $\mathfrak{t}$ is any subalgebra of $\mathfrak{k}$. In the following we use the theorem to calculate the Dirac cohomology of the discrete series as an example.

Example 7.4. Suppose that G is a connected semisimple Lie group with finite center. Let X_λ be the Harish-Chandra module of a discrete series representation with Harish-Chandra parameter λ. Then the Dirac cohomology of X_λ with respect to $D(\mathfrak{g},\mathfrak{k})$ consists of a single $\widetilde{K}$-module E_μ, whose highest weight is $\mu = \lambda - \rho_c$. Here ρ_c is half the sum of roots of $\mathfrak{t}$ in $\mathfrak{k}$ positive on λ. We note that the highest weight μ is obtained from the highest weight $\lambda - \rho_c + \rho_n$ of the lowest K-type of X_λ by adding $-\rho_n$ (the lowest weight of the spin module S for $C(\mathfrak{p})$). The fact $H_D(X_\lambda) = E_\mu$ follows from Theorem 3.4, since $X_\lambda = A_\mathfrak{b}(\lambda - \rho)$ for a θ-stable Borel subalgebra $\mathfrak{b}$. This fact can also be proved directly without using Theorem 3.4 as follows. By [HP1, Prop.5.4], the $\widetilde{K}$-type μ is clearly contained in the Dirac cohomology. Since X_λ has a unique lowest K-type, and since $-\rho_n$ is the lowest weight of the spin module S for $C(\mathfrak{p})$, with multiplicity one, it follows that any other $\widetilde{K}$-type has strictly larger highest weight, and thus cannot contribute to the Dirac cohomology. We now apply Kostant's formula from Section 5 to calculate the Dirac cohomology of E_μ with respect to $D(\mathfrak{k}, \mathfrak{t})$:

$$H_D(\mathfrak{k}, \mathfrak{t}; E_\mu) = \ker D(\mathfrak{k}, \mathfrak{t}) = \bigoplus_{w \in W_K} \mathbb{C}_{w(\mu+\rho_c)}.$$

It follows from $\mu + \rho_c = \lambda$ that

$$H_D(\mathfrak{g}, \mathfrak{t}; X_\lambda) = \bigoplus_{w \in W_K} \mathbb{C}_{w\lambda}.$$

Remark 7.5. In [CH] a modified Dirac operator is defined as follows:

$$\widetilde{D}(\mathfrak{g}, \mathfrak{t}) = D(\mathfrak{g}, \mathfrak{k}) + i D_\Delta(\mathfrak{k}, \mathfrak{t}).$$

This is used for the geometric quantization and construction of models of discrete series. Let X be a unitary $(\mathfrak{g}, K)$-module. There is a Hermitian form on the spin module S [HP2, §2.3.9]. Together with the $\mathfrak{g}_0$ invariant Hermitian form on V, it induces a Hermitian form on $X \otimes S$. It follows from the unitarity of V and the property of the Hermitian form on S [HP2, Prop. 2.3.10] that $D(\mathfrak{g}, \mathfrak{k})$ is symmetric and $D_\Delta(\mathfrak{k}, \mathfrak{t})$ is skew-symmetric with respect to this form. Then the modified Dirac operator $\widetilde{D}(\mathfrak{g}, \mathfrak{t})$ is symmetric. Note that both $D(\mathfrak{g}, \mathfrak{k})^2$ and $-D_\Delta(\mathfrak{k}, \mathfrak{t})^2$ are positive definite, so $\widetilde{D}(\mathfrak{g}, \mathfrak{t})^2 = D(\mathfrak{g}, \mathfrak{k})^2 - D_\Delta(\mathfrak{k}, \mathfrak{t})^2$ is also positive definite. Then $\widetilde{D}(\mathfrak{g}, \mathfrak{t})$ is an elliptic differential operator. This is the purpose of introducing this modified version of the Dirac operator. We also note that $iD_\Delta(\mathfrak{k}, \mathfrak{t})$ and $D_\Delta(\mathfrak{k}, \mathfrak{t})$ define the same Dirac cohomology.

8 Elliptic representations and Dirac Index

Suppose that $G(F)$ is a real or p-adic group. That is, G is a connected reductive algebraic group over a local field F of characteristic 0. Arthur [A1] studied a subset $\Pi_{\text{temp,ell}}(G(F))$ of tempered representations of $G(F)$, namely elliptic tempered representations. The set of tempered representations $\Pi_{\text{temp}}(G(F))$ includes the discrete series and in general the irreducible constituents of representations induced from the discrete series. These are exactly the representations which occur in the Plancherel formula for $G(F)$. In Harish-Chandra's theory, the character of an infinite-dimensional representation π is defined as a distribution

$$\Theta(\pi, f) = \mathrm{tr}\Big(\int_{G(F)} f(x)\pi(x)dx \Big), \quad f \in C_c^\infty(G(F)),$$

which can be identified with a function on $G(F)$. In other words,

$$\Theta(\pi, f) = \int_{G(F)} f(x)\Theta(\pi, x)dx, \quad f \in C_c^\infty(G(F)),$$

where $\Theta(\pi, x)$ is a locally integrable function on $G(F)$ that is smooth on the open dense subset $G_{\mathrm{reg}}(F)$ of regular elements. A representation π is called elliptic if $\Theta(\pi, x)$ does not vanish on the set of elliptic elements in $G_{\mathrm{reg}}(F)$.

The central objects in [A1] are the normalized characters $\Phi(\pi, \gamma)$, namely the functions defined by

$$\Phi(\pi, \gamma) = |D(\gamma)|^{\frac{1}{2}}\Theta(\pi, \gamma),\ \pi \in \Pi_{\mathrm{temp,ell}}(G(F)),\ \gamma \in G_{\mathrm{reg}}(F),$$

where

$$D(\gamma) = \det(1 - \mathrm{Ad}(\gamma))_{\mathfrak{g}/\mathfrak{g}_\gamma},$$

is the Weyl discriminant. We will show how this normalized character $\Phi(\pi, \gamma)$ is related to the Dirac cohomology of the Harish-Chandra module of π for a real group $G(\mathbb{R})$.

From now on we only deal with the real group $G(\mathbb{R})$. Note that $G(\mathbb{R})$ has elliptic elements if and only if it is of equal rank with $K(\mathbb{R})$. We also assume this equal rank condition. Induced representations from proper parabolic subgroups are not elliptic. Consider the quotient of the Grothendieck group of the category of finite length Harish-Chandra modules by the subspace generated by induced representations. Let us call this quotient group the elliptic Grothendieck group. Arthur [A1] found an orthonormal basis of this elliptic Grothendieck group in terms of elliptic tempered (possibly virtual) characters. Those characters are the supertempered distributions defined by Harish-Chandra [HC4].

We assume that $G(\mathbb{R})$ is connected. The tempered elliptic representations for the real group $G(\mathbb{R})$ are the representations with nonzero Dirac index, which are studied in [Lab1]. Labesse shows that the tempered elliptic representations are precisely the fundamental series. We deal with the general elliptic representations and show that any elliptic representation has nonzero Dirac index.

Recall that if X is an admissible $(\mathfrak{g}, K)$-module with K-type decomposition $X = \bigoplus_\lambda m_\lambda E_\lambda$, then the K-character of X is the formal series

$$\mathrm{ch}\, X = \sum_\lambda m_\lambda\, \mathrm{ch}\, E_\lambda,$$

where $\mathrm{ch}\, E_\lambda$ is the character of the irreducible K-module E_λ. Moreover, this definition also makes sense for virtual $(\mathfrak{g}, K)$-modules X; in that case, the integers m_λ can be negative. In the following we will often deal with representations of the spin double cover $\widetilde{K}$ of K, and not K, but we will still denote the corresponding character by ch.

Since $\mathfrak{p}$ is even-dimensional, the spin module S decomposes as $S^+ \oplus S^-$, with the $\mathfrak{k}$-submodules $S^\pm$ being the even respectively odd part of $S \cong \bigwedge \mathfrak{p}^+$. Let $X = X_\pi$ be the Harish-Chandra module of an irreducible admissible representation π of $G(\mathbb{R})$. We consider the following difference of $\widetilde{K}$-modules, the spinor index of X:

$$I(X) = X \otimes S^+ - X \otimes S^-.$$

This is a virtual $\widetilde{K}$-module, an integer combination of finitely many $\widetilde{K}$-modules. The Dirac operator D induces the action of the following $\widetilde{K}$-equivariant operators:

$$D^\pm : X \otimes S^\pm \to X \otimes S^\mp.$$

Since D^2 acts by a scalar on each $\widetilde{K}$-type, most of the $\widetilde{K}$-modules in $X \otimes S^+$ are the same as in $X \otimes S^-$.

Lemma 8.1. *The spinor index is equal to the Euler characteristic of Dirac cohomology, i.e.,*

$$I(X) = H_D^+(X) - H_D^-(X).$$

Proof. As we mentioned above, $X \otimes S$ is decomposed into a direct sum of eigenspaces for D^2:

$$X \otimes S = \sum_\lambda (X \otimes S)_\lambda = (X \otimes S^+)_\lambda \oplus (X \otimes S^-)_\lambda.$$

It follows that

$$X \otimes S^+ - X \otimes S^- = (X \otimes S^+)_0 - (X \otimes S^-)_0.$$

Since D is a differential on Ker D^2 and the corresponding cohomology is exactly the Dirac cohomology $H_D(X)$, the lemma follows from the Euler–Poincaré principle. □

The spinor index $I(X)$ is also called the Dirac index of X, since it is equal to the index of D^+, in the sense of the index for a Fredholm operator. It is also identical to the Euler characteristic of the Dirac cohomology $H_D(X)$. We denote by $\theta(X)$ the character of $I(X)$. In terms of characters, this reads as

$$\theta(X) = \operatorname{ch} I(X) = \operatorname{ch} X(\operatorname{ch} S^+ - \operatorname{ch} S^-) = \operatorname{ch} H_D^+(X) - \operatorname{ch} H_D^-(X).$$

If we view $\operatorname{ch} E_\lambda$ as functions on K, then the series

$$\operatorname{ch} X = \sum_\lambda m_\lambda \operatorname{ch} E_\lambda$$

converges to a distribution on K and it coincides with $\Theta(X)$ on $K \cap G_{\text{reg}}$, according to Harish-Chandra [HC1]. Then the absolute value $|\theta_\pi|$ coincides with the absolute value $|\Phi(\pi, \gamma)| = |D(\gamma)|^{\frac{1}{2}} |\Theta(\pi, \gamma)|$ on regular elliptic elements. We write it as a lemma.

Lemma 8.2. *For any regular elliptic elements γ, we have*

$$|\theta_\pi(\gamma)| = |\Phi(\pi, \gamma)|.$$

Theorem 8.3. *Let π be an irreducible admissible representation of $G(\mathbb{R})$ with Harish-Chandra module X_π. Then π is elliptic if and only if the Dirac index $I(X_\pi) \neq 0$.*

Proof. The theorem follows immediately from the lemma. □

9 Orthogonality relations and supertempered distributions

We keep the notation from the previous section. We assume that $G(\mathbb{R})$ is cuspidal, in the sense that the (regular) elliptic set G_{ell} is nonempty. Let $A_G(\mathbb{R})$ be the split part of the center of $G(\mathbb{R})$. The cuspidal condition on $G(\mathbb{R})$ amounts to the condition that $G(\mathbb{R})$ has a maximal torus $T_{\text{ell}}(\mathbb{R})$ that is compact modulo $A_G(\mathbb{R})$. Suppose Θ_π and $\Theta_{\pi'}$ are two irreducible characters with the same central character on $A_G(\mathbb{R})$. We form the elliptic inner product by the following formula:

$$(\Theta_\pi, \Theta_{\pi'})_{\text{ell}} = |W(G(\mathbb{R}), T_{\text{ell}}(\mathbb{R}))|^{-1} \int_{T_{\text{ell}}(\mathbb{R})/A_G(\mathbb{R})} |D(\gamma)| \Theta_\pi(\gamma)\overline{\Theta_{\pi'}(\gamma)} d\gamma,$$

where $W(G(\mathbb{R}), T_{\text{ell}}(\mathbb{R}))$ is the Weyl group of $(G(\mathbb{R}), T_{\text{ell}}(\mathbb{R}))$, and $d\gamma$ is the normalized Haar measure on the compact Abelian group $T_{\text{ell}}(\mathbb{R})/A_G(\mathbb{R})$. The inner product (bilinear over $\mathbb{R}$)) is extended linearly to any two characters of admissible representations. Then we have the following theorem.

Theorem 9.1. *We have*

$$(\Theta_\pi, \Theta_{\pi'})_{\text{ell}} = (\theta_\pi, \theta_{\pi'}),$$

where the pairing on the right-hand side is the standard pairing of virtual characters on K or $\widetilde{K}$ defined by

$$(\theta_\pi, \theta_{\pi'}) = \int_K \theta_\pi \cdot \overline{\theta_{\pi'}}\, dk.$$

Proof. It follows from Lemma 8.2 that

$$(\Theta_\pi, \Theta_\pi)_{\text{ell}} = (\theta_\pi, \theta_\pi)$$

for irreducible characters and therefore for all admissible characters, in particular for any sum of two irreducible characters. Then the theorem follows from a standard argument of polarization for the inner product. □

In [DH2], the set of equivalence classes of irreducible tempered representations π with nonzero Dirac cohomology is determined. It turns out the irreducible tempered representations with nonzero Dirac cohomology are exactly those with nonzero Dirac index. Therefore, this set of representations coincides with the set of irreducible tempered elliptic representations, and it consists of the discrete series and some of limits of the discrete series, Moreover, any elliptic tempered representation is isomorphic to an $A_{\mathfrak{b}}(\lambda)$-module for some θ-stable Borel subalgebra $\mathfrak{b}$ and its Dirac cohomology is a single $\widetilde{K}$-module. As a consequence, we have the following corollary.

Corollary 9.2. *Elliptic tempered characters satisfy orthogonality relations. That is for any two tempered irreducible elliptic representations π and π',*

$$(\Theta_\pi, \Theta_{\pi'})_{\text{ell}} = \pm 1 \text{ or } 0.$$

It is also clear that the characters of the discrete series form an orthonormal set on the (regular) elliptic set $G_{\text{ell}}(\mathbb{R})$ in $G(\mathbb{R})$. Harish-Chandra defined the space of supertempered distributions in [HC4] (the last paper of his Collected Papers Volume IV). If D is a distribution on G, we denote by D_e the restriction of D on G_{ell} ($D_e = 0$ by convention when G_{ell} is empty).

Theorem 9.3 (Theorem 5 [HC4]). *Let Θ be a $Z(\mathfrak{g})$-finite tempered distribution. Suppose that Θ is supertempered. Then $\Theta_e = 0$ implies that $\Theta = 0$.*

Theorem 9.4 (Theorem 9 [HC4]). *For $\mu \in \widehat{T(\mathbb{R})}$, there is a unique supertempered distribution Θ_μ, such that*

$${}'\Delta(\gamma)\Theta_\mu(\gamma) = \sum_{w \in W_K} \epsilon(w) e^{w\mu},$$

where ${}'\Delta$ is the Weyl denominator (see [HC2, Section 27]).

Theorem 9.5 (Theorem 14 [HC4]). *If π_1, π_2 are irreducible tempered elliptic representations, then either $(\Theta_{\pi_1}, \Theta_{\pi_2})_{\text{ell}} = 0$ or $\Phi_{\pi_1} = \pm\Phi_{\pi_2}$.*

As mentioned earlier in the previous section, Arthur found an orthonormal basis for the space of supertempered distributions consisting of the virtual characters of tempered representations. It is clear from the orthogonality relation (Corollary 9.2) and the above theorems of Harish-Chandra (Theorems 9.3–9.5) that Arthur's basis consists of characters of the discrete series and appropriate linear combinations of the characters of the limits of discrete series with the same Dirac index up to a sign. We summarize it as the following corollary.

Corollary 9.6. *The characters of the discrete series and the appropriate linear combinations of the characters of the limits of discrete series with the same Dirac index (up to a sign) form an orthonormal basis of the space of supertempered distributions.*

10 Elliptic representations with regular infinitesimal characters

In this section we still assume that $G(\mathbb{R}) \supset K(\mathbb{R})$ is of equal rank and $T(\mathbb{R})$ is a maximal torus for $K(\mathbb{R})$. We consider the case where X is a simple Harish-Chandra module with regular infinitesimal character.

Theorem 10.1. *Suppose that an irreducible Harish-Chandra module X has regular infinitesimal character. Then we have*

$$\mathrm{Hom}_{\widetilde{K}}(H_D^+(X), H_D^-(X)) = 0. \tag{10.1}$$

In particular, it follows that the Dirac index $I(X) = 0$ (equivalently, its character $\theta_X = 0$) if and only if the Dirac cohomology $H_D(X) = 0$.

Proof. Let $\mathfrak{b} = \mathfrak{t} + \mathfrak{u}$ be a θ-stable Borel subalgebra. Then $\mathfrak{t}$ is contained in $\mathfrak{k}$. We need to consider the Dirac cohomology $H_D(\mathfrak{g}, \mathfrak{t}; X)$ of X with respect to the cubic Dirac operator $D(\mathfrak{g}, \mathfrak{t})$. Calculating in stages (see Section 7), we have

$$H_D(\mathfrak{g}, \mathfrak{t}; X) = H_D(\mathfrak{k}, \mathfrak{t}; H_D(X)).$$

It follows that

$$H_D^+(\mathfrak{g}, \mathfrak{t}; X) \supseteq H_D^+(\mathfrak{k}, \mathfrak{t}; H_D^+(X))$$

and

$$H_D^-(\mathfrak{g}, \mathfrak{t}; X) \supseteq H_D^+(\mathfrak{k}, \mathfrak{t}; H_D^-(X)).$$

Clearly, the following condition

$$\mathrm{Hom}_{\widetilde{T}}(H_D^+(\mathfrak{g}, \mathfrak{t}; X), H_D^-(\mathfrak{g}, \mathfrak{t}; X)) = 0 \tag{10.2}$$

implies (10.1). It remains to prove (10.2). We note that it follows from a theorem of Vogan ([V1, Theorem 7.2]) that

$$\mathrm{Hom}_T(H^+(\mathfrak{u}, X), H^-(\mathfrak{u}, X)) = 0,$$

for any irreducible Harish-Chandra module X with regular infinitesimal character. Then (10.2) follows from the above parity condition on $\mathfrak{u}$-cohomology if we have the following embeddings

$$H_D^{\pm}(\mathfrak{g}, \mathfrak{t}; X) \subseteq H^{\pm}(\mathfrak{u}, X) \otimes Z_{\rho(\bar{\mathfrak{u}})}. \tag{10.3}$$

Indeed, this can be done with slightly more deliberation using the same argument as in [HX, Proposition 4.8]. There are only finitely many $\widetilde{K}$-types in $X \otimes S$ that can possibly contribute to the Dirac cohomology with respect to $D(\mathfrak{g}, \mathfrak{k})$, and therefore also finitely many to the Dirac cohomology with respect to

$$D = D(\mathfrak{g}, \mathfrak{t}) = D(\mathfrak{g}, \mathfrak{k}) + D_{\Delta}(\mathfrak{k}, \mathfrak{t})$$

by calculating in stages. Recall in the proof of [HX, Proposition 4.8] one has $D = d + 2\partial$ and $D^2 = 2\partial d + 2d\partial$ and the decomposition

$$X \otimes S = \operatorname{Ker} D^2 \oplus \operatorname{Im} D^2,$$

where ∂ and d are the differentials for $\mathfrak{u}$-homology and $\mathfrak{u}$-cohomology. We note that an irreducible $(\mathfrak{g}, \mathfrak{k})$-module may not be $\mathfrak{t}$-admissible, and $\operatorname{Ker} D^2$ can be infinite-dimensional. To make the argument in [HX, Proposition 4.8] still works in this case, we consider

$$\widetilde{D}(\mathfrak{g}, \mathfrak{t}) = D(\mathfrak{g}, \mathfrak{k}) + iD_\Delta(\mathfrak{k}, \mathfrak{t})$$

as in Remark 7.5. It follows from $[\widetilde{D}, D^2] = 0$ that $\operatorname{Ker} D^2$ is stable under the action of $\widetilde{D}$. We restrict $\widetilde{D}$ to $\operatorname{Ker} D^2$ and have the following decomposition:

$$\operatorname{Ker} D^2 = (\ker \widetilde{D}^2 \cap \operatorname{Ker} D^2) \oplus (\operatorname{Im} \widetilde{D}^2 \cap \operatorname{Ker} D^2).$$

It is clear that $U = \operatorname{Ker} \widetilde{D}^2 \cap \operatorname{Ker} D^2 = \operatorname{Ker} D(\mathfrak{g}, \mathfrak{k})^2 \cap \operatorname{Ker} D(\mathfrak{k}, \mathfrak{t})^2$ is finite-dimensional. We set

$$V = U \oplus \partial U \oplus dU \oplus \partial dU.$$

Then V is finite-dimensional. It follows from $\partial d = -d\partial$ on $\operatorname{Ker} D^2$ that V is stable under the action of ∂ and d, as well as $D = d + 2\partial$. If we replace $\operatorname{Ker} D^2$ by V in the final step of the proof of [HX, Proposition 4.8], then the same argument works here. Precisely, we restrict all operators D, ∂ and d to V. We have $D^2 = 0$ and thus $\operatorname{Im} D \subset \operatorname{Ker} D$. Note that $\operatorname{Ker} D^2 / \operatorname{Ker} D \simeq \operatorname{Im} D$. Denote by ∂' the map ∂ restricted to V. Then $\operatorname{Ker} D^2 / \operatorname{Ker} \partial' \simeq \operatorname{Im} \partial'$. Recall that ch denotes the formal $\mathfrak{t}$-characters. We obtain

$$\operatorname{ch} \operatorname{Im} D + \operatorname{ch} \operatorname{Ker} D = \operatorname{ch} \operatorname{Im} \partial' + \operatorname{ch} \operatorname{Ker} \partial'.$$

Moreover, one has $\operatorname{Im} \partial' \subseteq \operatorname{Ker} \partial'$ since $\partial^2 = 0$. Therefore,

$$\operatorname{ch} \operatorname{Ker} \partial' / \operatorname{Im} \partial' - \operatorname{ch} \operatorname{Ker} D / \operatorname{Im} D = 2(\operatorname{ch} \operatorname{Ker} \partial' - \operatorname{ch} \operatorname{Ker} D). \tag{10.4}$$

Then all the modules here are direct sums of finite-dimensional $\mathfrak{t}$-modules. It follows from Lemma 4.6 of [HX] and (10.4) there is an injective $\mathfrak{t}$-module homomorphism

$$\operatorname{Ker} D / \operatorname{Im} D \to \operatorname{Ker} \partial' / \operatorname{Im} \partial'.$$

Note that the right side can be naturally embedded into $\operatorname{Ker} \partial / \operatorname{Im} \partial$. This gives the embedding of Dirac cohomology into $\mathfrak{u}$-homology and we get the embedding into $\mathfrak{u}$-cohomology similarly. □

We note that in the above proof we conclude that the embeddings (10.3) are actually isomorphisms

$$H_D^{\pm}(\mathfrak{g}, \mathfrak{t}; X) \cong H^{\pm}(\mathfrak{u}, X) \otimes Z_{\rho(\bar{\mathfrak{u}})}.$$

Remark 10.2. We remark that the parity condition

$$\mathrm{Hom}_T(H^+(\mathfrak{u},X),H^-(\mathfrak{u},X))=0$$

may fail if the infinitesimal character of X is not regular. I learned the following example from the lecture by Wilfried Schmid at the Vogan Conference at MIT (in May of 2014) and the lecture notes by Dragan Miličić at a recent conference (in the summer of 2014) at Jacobs University in Bremen. Let G be SU$(2,1)$ and $\mathfrak{b}$ a θ-stable parabolic which contains neither $\mathfrak{p}^+$ nor $\mathfrak{p}^-$. Let X be the degenerate limit of discrete series with infinitesimal character 0. Then X fails to satisfy the parity condition.

We note that in the above example, if X is the limit of discrete series of $G=$ SU$(2,1)$ with infinitesimal character 0, then the Dirac cohomology of X is zero and the embeddings

$$H_D^{\pm}(\mathfrak{g},\mathfrak{t};X)\subseteq H^{\pm}(\mathfrak{u},X)\otimes Z_{\rho(\bar{\mathfrak{u}})}$$

are not isomorphisms. However, the parity condition for Dirac cohomology is still true. All the examples we know indicate that this is perhaps true in general.

Conjecture 10.3. Let X be an irreducible $(\mathfrak{g},K)$-module. Then

$$\mathrm{Hom}_{\widetilde{K}}(H_D^+(X),H_D^-(X))=0.$$

As we have already mentioned in the previous section, the above conjecture is true if X is a tempered Harish-Chandra module.

It is a natural question to classify irreducible unitary elliptic representations of $G(\mathbb{R})$ and to classify irreducible unitary representations with nonzero Dirac cohomology. We can solve this problem under the condition that the infinitesimal character be regular. We first recall a theorem of Salamanca-Riba.

Theorem 10.4 (Salamanca-Riba [S]). *Let G be a connected reductive Lie group. If X is an irreducible unitary $(\mathfrak{g},K)$-module with strongly regular infinitesimal character, then $X\cong A_{\mathfrak{q}}(\lambda)$ for certain θ-stable parabolic $\mathfrak{q}$ and λ.*

Theorem 10.5. *Suppose π is an irreducible unitary elliptic representation of $G(\mathbb{R})$ with a regular infinitesimal character. Then $X_\pi\cong A_{\mathfrak{q}}(\lambda)$.*

Proof. Since X_π has nonzero Dirac cohomology, its infinitesimal character is analytically integral for $K(\mathbb{R})$ as well as for a compact real form of $G(\mathbb{C})$, and hence it is integral in $\Delta(\mathfrak{g},\mathfrak{t})$. Then the regular infinitesimal character of X_π is strongly regular, and $X_\pi\cong A_{\mathfrak{q}}(\lambda)$ follows from Salamanca-Riba's theorem. □

Suppose that π is an irreducible unitary representation. It is a natural question to ask to what extent the Dirac cohomology $H_D(X_\pi)$ determines the representation π itself. For representations with singular infinitesimal characters, it is easy to give examples of two non-isomorphic limits of discrete series π_1 and π_2 such that $H_D(X_{\pi_1})=H_D(X_{\pi_2})$. The above theorem says under the condition of regular infinitesimal character, the question is reduced to $A_{\mathfrak{q}}(\lambda)$-modules. Now the question is: if two unitary $A_{\mathfrak{q}}(\lambda)$-modules have isomorphic Dirac cohomology, would these two modules be isomorphic? The answer is not always affirmative. For example, when G is SO$(2n, 1)$, there are many non-isomorphic $A_{\mathfrak{q}}(\lambda)$-modules that have isomorphic Dirac cohomology.

11 Pseudo-coefficients of the discrete series

Many important questions on non-commutative Lie groups boil down to questions in invariant harmonic analysis: the study of distributions on groups that are invariant under conjugacy. The fundamental objects of invariant harmonic analysis are orbital integrals as the geometric objects and characters of representations as the spectral objects. The correspondence of these two kinds of objects reflects the core idea of harmonic analysis.

The orbital integrals are parameterized by the set of regular semisimple conjugacy classes in G. Recall for such a γ the orbital integral is defined as

$$\mathcal{O}_\gamma(f) = \int_{G/G_\gamma} f(x^{-1}\gamma x)dx, \quad f \in C_c^\infty(G),$$

and the stable orbital integral is defined as

$$S\mathcal{O}_\gamma(f) = \sum_{\gamma' \in S(\gamma)} \mathcal{O}_{\gamma'}(f),$$

where $S(\gamma)$ is the stable conjugacy class.

Let $\mathbb{1}$ denote the trivial representation of G and $\theta_{\mathbb{1}}$ the character of the Dirac index of the trivial representation. That is

$$\theta_{\mathbb{1}} = \operatorname{ch} H_D^+(\mathbb{1}) - \operatorname{ch} H_D^-(\mathbb{1}) = \operatorname{ch} S^+ - \operatorname{ch} S^-.$$

We note that

$$\overline{\theta_{\mathbb{1}}} = (-1)^q(\operatorname{ch} S^+ - \operatorname{ch} S^-) = (-1)^q\theta_{\mathbb{1}},$$

where $q = \frac{1}{2}\dim G(\mathbb{R})/K(\mathbb{R})$.

Recall that θ_π denotes the character of the Dirac index of π. If π is the discrete series representation with Dirac cohomology E_μ, then

$$\theta_\pi = (-1)^q\chi_\mu.$$

Labesse showed that there exists a function f_π so that for any admissible representations π',

$$\operatorname{tr}\pi'(f_\pi) = \int_K \Theta_{\pi'}(k)\overline{\theta_{\mathbb{1}} \cdot \theta_\pi}dk.$$

Let π' be a discrete series representation with Dirac cohomology $E_{\mu'}$. It follows that

$$\operatorname{tr} \pi'(f_\pi) = (\chi_{\mu'}, \chi_\mu) = \dim \operatorname{Hom}_K(E_{\mu'}, E_\mu).$$

Then we have the following theorem.

Theorem 11.1 (Labesse [Lab1]). *The function f_π is a pseudo-coefficient for the discrete series π, i.e., for any irreducible tempered representation π',*

$$\operatorname{tr} \pi'(f_\pi) = \begin{cases} 1 & \text{if } \pi \cong \pi' \\ 0 & \text{otherwise.} \end{cases}$$

Remark 11.2. The orbital integrals of the pseudo-coefficient f_π are easily computed for γ regular semisimple:

$$\mathcal{O}_\gamma(f_\pi) = \begin{cases} \Theta_\pi(\gamma^{-1}) & \text{if } \gamma \text{ is elliptic} \\ 0 & \text{if } \gamma \text{ is not elliptic.} \end{cases}$$

12 Endoscopic transfer

In the Langlands program a cruder form of conjugacy called stable conjugacy plays an important role. The study of Langlands functoriality often leads to correspondence that is defined only up to stable conjugacy. The endoscopy theory investigates the difference between ordinary and stable conjugacy and how to understand ordinary conjugacy within stable conjugacy. The aim is to recover orbital integrals and characters from endoscopy groups.

Recall that G is a connected reductive algebraic group defined over $\mathbb{R}$. Denote by $G^\vee$ the complex dual group and ${}^LG$ the L-group which is the semidirect product of $G^\vee$ and the Weil group $W_\mathbb{R}$. A Langlands parameter is an L-homomorphism

$$\phi: W_\mathbb{R} \to {}^LG.$$

Two Langlands parameters are equivalent if they are conjugated by an inner automorphism of $G^\vee$. An equivalence class of Langlands parameters is associated to a packet of irreducible admissible representations of $G(\mathbb{R})$ [L2]. The L-packets of Langlands parameters with bounded image consist of tempered representations. Temperedness is respected by L-packets, but not unitarity.

The discrete series L-packets are in bijection with the irreducible finite-dimensional representations of the same infinitesimal character. One can construct all tempered irreducible representations using unitary parabolic induction and by taking subrepresentations. Two tempered irreducible representations π and π' are in the same L-packet if up to equivalence, π and π' are subrepresentations of parabolically induced representations from discrete series σ and σ' in the same L-packets.

A stable distribution is any element of the closure of the space spanned by all distributions of the form $\sum_{\pi \in \Pi} \Theta_\pi$ for Π any tempered L-packet. Such distributions can be transferred to inner forms of G via the matching of the stable orbital integrals, while unstable distributions cannot be.

For the non-tempered case we need Arthur packets, which are parameterized by mappings

$$\psi: W_{\mathbb{R}} \times \mathrm{SL}(2, \mathbb{C}) \to {}^L G$$

for which the projection onto the dual group $G^\vee$ of $\psi(W_{\mathbb{R}})$ is relatively compact. Adams and Johnson [AJ] constructed some A-packets consisting of unitary $A_{\mathfrak{q}}(\lambda)$-modules. Determining the Dirac cohomology of $A_{\mathfrak{q}}(\lambda)$-modules may have some bearing on answering Arthur's questions (see [A2, Section 9]) on the Arthur packet Π_ψ.

In the setting of the endoscopic embedding

$$\xi : {}^L H \to {}^L G,$$

one has a map from Langlands parameters for H to that for G. The Langlands functoriality principle asserts that there should be a map from the Grothendieck group of virtual representations of $H(\mathbb{R})$ to that of $G(\mathbb{R})$, compatible with L-packets.

The endoscopy theory for real groups has been established by Shelstad in a series of papers [Sh1-5]. Recasting Shelstad's work explicitly in terms of the general transfer factors defined later by Langlands and Shelstad [LS] is the first of the *Problems for Real Groups* proposed by Arthur [A2].

We follow Labesse [Lab2, §6.7] for the description of the endoscopic transfer. Let T be an elliptic torus of G and κ an endoscopic character. Let H be the endoscopic group defined by (T, κ). Let B_G be a Borel subgroup of G containing T. Set

$$\Delta_B(\gamma) = \prod_{\alpha > 0} (1 - \gamma^{-\alpha}),$$

where the product is over the positive roots defined by B. There is only one choice of a Borel subgroup B_H in H, containing T_H and compatible with the isomorphism $j: T_H \cong T$.

Assume $\eta: {}^L H \to {}^L G$ is an admissible embedding (see [Lab2, §6.6]). Then for any pseudo-coefficent f of a discrete series of G, there is a linear combination f^H of pseudo-coefficents of discrete series of H such that for $\gamma = j(\gamma_H)$ regular in $T(\mathbb{R})$ (see [Lab2, Prop. 6.7.1]), one has

$$\mathcal{SO}_{\gamma_H}(f^H) = \Delta(\gamma_H, \gamma_G) \mathcal{O}^\kappa_{\gamma_G}(f), \tag{12.1}$$

where the transfer factor

$$\Delta(\gamma_H, \gamma_G) = (-1)^{q(G)-q(H)} \chi_{G,H}(\gamma) \Delta_B(\gamma^{-1}) \Delta_{B_H}(\gamma_H^{-1})^{-1}. \tag{12.2}$$

The transfer $f \mapsto f^H$ of the pseudo-coefficents of discrete series can be extended to all of functions in $C_c^\infty(G(\mathbb{R}))$ with extension of the correspondence $\gamma \mapsto \gamma_H$ (see [Lab2, Theorem 6.7.2]) so that the above identity (12.1) holds for all f.

The geometric transfer $f \mapsto f^H$ is dual of a transfer for representations. Given any admissible irreducible representation σ of $H(\mathbb{R})$, it corresponds to an element σ_G in the Grothendieck group of virtual representations of $G(\mathbb{R})$ as follows. Let ϕ be the Langlands parameter for σ. Let Σ be the L-packet of the admissible irreducible representations of $H(\mathbb{R})$ corresponding to a Langlands parameter ϕ and Π the L-packet of representations of $G(\mathbb{R})$ corresponding to $\eta \circ \phi$ (that can be an empty set if this parameter is not relevant for G).

Theorem 12.1 (Theorem 4.1.1 [Sh5], Theorem 6.7.3 [Lab2]). *There is a function*

$$\epsilon : \Pi \to \pm 1$$

such that, if we consider σ_G in the Grothendieck group defined by

$$\sigma_G = \sum_{\pi \in \Pi} \epsilon(\pi)\pi,$$

then the transfer $\sigma \mapsto \sigma_G$ satisfies

$$\operatorname{tr} \sigma_G(f) = \operatorname{tr} \sigma(f^H).$$

In the following we suppose that $G(\mathbb{R})$ has a compact maximal torus $T(\mathbb{R})$, and that $\rho - \rho_H$, the difference of half sum of positive roots for G and H respectively, defines a character of $T(\mathbb{R})$. In [Lab2, §7.2] Labesse shows that the canonical transfer factor

$$\Delta(\gamma^{-1}) = (-1)^{q(G)-q(H)} \frac{\sum_{w \in W(\mathfrak{g})} \epsilon(w)\gamma^{w\rho}}{\sum_{w \in W(\mathfrak{h})} \epsilon(w)\gamma^{w\rho_H}}$$

is a well-defined function. Then the transfer factor can be expressed more explicitly if H is a subgroup of G. Suppose that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{s}$ is the orthogonal decomposition with respect to a nondegenerate invariant bilinear form so that the form is nondegenerate on $\mathfrak{s}$. We write $S(\mathfrak{g}/\mathfrak{h})$ for the spin-module of the Clifford algebra $C(\mathfrak{s})$. Then

$$\Delta(\gamma^{-1}) = \operatorname{ch} S^+(\mathfrak{g}/\mathfrak{h}) - \operatorname{ch} S^-(\mathfrak{g}/\mathfrak{h}).$$

In other words, $\Delta(\gamma^{-1})$ is equal to the character of the Dirac index of the trivial representation with respect to the Dirac operator $D(\mathfrak{g}, \mathfrak{h})$. If Θ_π is the character of a finite-dimensional representation π, then

$$\Delta(\gamma^{-1})\Theta_\pi$$

is the character of the Dirac index of π. This character can be calculated easily from the Kostant formula in Section 5. We denote by F_λ the irreducible finite-dimensional representation of $G(\mathbb{R})$ with highest weight λ and by E_μ irreducible finite-dimensional representation of $H(\mathbb{R})$ with highest weight μ. Then

$$\Delta(\gamma^{-1})\Theta_{F_\lambda} = \sum_{w \in W^1} \Theta_{E_w(\lambda+\rho)-\rho_{\mathfrak{h}}}.$$

Here W^1 is a subset of elements in W corresponding to $W_{\mathfrak{h}}\backslash W$ as before.

In view of Remark 11.2, the right-hand side of (12.1) is the Dirac index of a combination of the discrete series of $G(\mathbb{R})$ and the left-hand side is a linear combination of the discrete series of $H(\mathbb{R})$. It follows from the Harish-Chandra formula for the character of discrete series and supertempered distributions (see Theorem 9.4) that the Dirac index of a discrete series π_λ with Harish-Chandra parameter λ is

$$\Delta(\gamma^{-1})\Theta\pi_\lambda = \sum_{w \in W_K^1} \operatorname{sign}(w)\Theta_{\tau_{w\lambda}}.$$

Here $\tau_{w\lambda}$ denotes the discrete series for $H(\mathbb{R})$ with Harish-Chandra parameter $w\lambda$, and W_K^1 is a subset of elements in W_K corresponding to $W_{H\cap K}\backslash W_K$. This calculation is compatible with Labesse's calculation of the transfer of the pseudo-coefficients of the discrete series in [Lab2, §7.2].

The above interpretation of the transfer factors in certain cases of endoscopy as the difference of the even and odd parts of the spin modules is clearly useful for calculation. It is also reminiscent of the transfer factors for the metaplectic groups, which is given by the formal difference of the metaplectic representations, in the work by Jeff Adams [A], David Renard [R] and Wen-Wei Li [Li]. It is worthwhile investigating the Dirac cohomology and Dirac index with respect to the symplectic Dirac operators in connection with the Weyl algebras and the oscillator representations of metapletic groups.

13 Hypoelliptic representations

In this final section we assume that $G(\mathbb{R}) \supset K(\mathbb{R})$ is not necessarily of equal rank. If $G(\mathbb{R})$ is indeed not of equal rank, then there is no elliptic representation for $G(\mathbb{R})$. Still, we know $G(\mathbb{R})$ has representations with nonzero Dirac cohomology. The natural generalization of the concept of elliptic representation for unequal rank $G(\mathbb{R})$ is the following.

Definition 13.1. A representation is called *hypoelliptic* if its global character is not identically zero on the set of regular elements in a fundamental Cartan subgroup.

By definition, an elliptic representation is hypoelliptic.

It is a natural question to ask the relationship between hypoelliptic representations and representations with nonzero Dirac cohomology.

Conjecture 13.2. Suppose that π is an irreducible admissible representation. Then $H_D(X_\pi) \neq 0$ implies that π is hypoelliptic.

Recall that if $G(\mathbb{R})$ is of equal rank with $K(\mathbb{R})$, then an irreducible tempered representation is either elliptic or induced from a tempered elliptic representation by parabolic induction.

Conjecture 13.3. A unitary representation either has nonzero Dirac cohomology or is induced from a unitary representation with nonzero Dirac cohomology by parabolic induction.

The above conjecture holds for $\mathrm{GL}(n,\mathbb{R})$, $\mathrm{GL}(n,\mathbb{C})$, $\mathrm{GL}(n,\mathbb{H})$ as well as $\widetilde{\mathrm{GL}}(n,\mathbb{R})$ (the two-fold covering group of $\mathrm{GL}(n,\mathbb{R})$).

A recent preprint of Adams–van Leeuwen–Trapa–Vogan [ALTV] gives an algorithm to determine the irreducible unitary representations. The above conjecture means that one may regard unitary representations with nonzero Dirac cohomology as 'cuspidal' ones.

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A program for branching problems in the representation theory of real reductive groups

Toshiyuki Kobayashi

Dedicated to David Vogan on the occasion of his 60th birthday, with admiration of his epoch-making contributions to the field

Abstract We wish to understand how irreducible representations of a group G behave when restricted to a subgroup G' (the *branching problem*). Our primary concern is with representations of reductive Lie groups, which involve both algebraic and analytic approaches. We divide branching problems into three stages: (A) abstract features of the restriction; (B) branching laws (irreducible decompositions of the restriction); and (C) construction of symmetry breaking operators on geometric models. We could expect a simple and detailed study of branching problems in Stages B and C in the settings that are *a priori* known to be "nice" in Stage A, and conversely, new results and methods in Stage C that might open another fruitful direction of branching problems including Stage A. The aim of this article is to give new perspectives on the subjects, to explain the methods based on some recent progress, and to raise some conjectures and open questions.

Key words: branching law, symmetry breaking operator, unitary representation, Zuckerman–Vogan's $A_{\mathfrak{q}}(\lambda)$ module, reductive group, spherical variety, multiplicity-free representation

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1 Program — ABC for branching problems

From the viewpoint of analysis and synthesis, one of the fundamental problems in representation theory is to classify the smallest objects (e.g., irreducible representations), and another is to understand how a given representation can be built up from the smallest objects (e.g., irreducible decomposition). A typical example of the latter is the *branching problem*, by which we mean the problem of understanding how irreducible representations π of a group G behave when restricted to subgroups G'. We write $\pi|_{G'}$ for a representation π regarded as a representation of G'. Our primary concern is with real reductive Lie groups. We propose a program for branching problems in the following three stages:

Stage A. Abstract features of the restriction $\pi|_{G'}$.
Stage B. Branching laws.
Stage C. Construction of symmetry breaking operators.

Here, by a *symmetry breaking operator* we mean a continuous G'-homomorphism from the representation space of π to that of an irreducible representation τ of the subgroup G'.

Branching problems for infinite-dimensional representations of real reductive groups involve various aspects. Stage A involves several aspects of the branching problem, among which we highlight that of multiplicity and spectrum here:

A.1. Estimates of multiplicities of irreducible representations of G' occurring in the restriction $\pi|_{G'}$ of an irreducible representation π of G. (There are several "natural" but inequivalent definitions of multiplicities, see Sections 3.1 and 4.2.) Note that:

- multiplicities of the restriction $\pi|_{G'}$ may be infinite even when G' is a maximal subgroup in G;
- multiplicities may be at most one (e.g., Howe's theta correspondence [18], Gross–Prasad conjecture [14], visible actions [39], etc.).

A.2. Spectrum of the restriction $\pi|_{G'}$:

- (discretely decomposable case) branching problems may be purely algebraic and combinatorial ([12, 13, 15, 26, 28, 29, 32, 49, 50, 59]);
- (continuous spectrum) branching problems may have analytic features [8, 52, 57, 63]. (For example, some special cases of branching laws of unitary representations are equivalent to a Plancherel-type theorem for homogeneous spaces.)

The goal of Stage A in branching problems is to analyze the aspects A.1 and A.2 in complete generality. A theorem in Stage A would be interesting on its own, but might also serve as a foundation for further detailed study of the restriction $\pi|_{G'}$ (Stages B and C). An answer in Stage A may also suggest an approach depending on specific features of the restrictions. For instance, if we know *a priori* that the restriction $\pi|_{G'}$ is discretely decomposable in Stage A, then one might use algebraic methods (e.g., combinatorics, $\mathcal{D}$-modules, etc.) to attack Stage B. If the restriction $\pi|_{G'}$ is known *a priori* to be multiplicity-free in Stage A, one might expect to find not only explicit irreducible decompositions (Stage B) but also quantitative estimates such as $L^p - L^q$ estimates, and Parseval–Plancherel type theorems for branching laws (Stage C).

In this article, we give some perspectives of the subject based on a general theory of A.1 and A.2, and recent progress in some classification theory:

- the multiplicities to be finite [bounded, one, $\cdots$],
- the spectrum to be discrete / continuous.

We also discuss a new phenomenon (*localness theorem*, Theorem 7.18) and open questions.

Stage B concerns the irreducible decomposition of the restriction. For a finite-dimensional representation such that the restriction $\pi|_{G'}$ is completely reducible, there is no ambiguity on the meaning of the irreducible decomposition. For a unitary representation π, we can consider Stage B by using the direct integral of Hilbert spaces (Fact 3.1). However, we would like to treat a more general setting where π is not necessarily a unitary representation. In this case, we may consider Stage B as the study of

$$\mathrm{Hom}_{G'}(\pi|_{G'}, \tau) \quad \text{or} \quad \mathrm{Hom}_{G'}(\tau, \pi|_{G'}) \tag{1.1}$$

for irreducible representations π and τ of G and G', respectively.

Stage C is more involved than Stage B as it asks for concrete intertwining operators (e.g., the projection operator to an irreducible summand) rather than an abstract decomposition; it asks for the decomposition of vectors in addition to that of representations. Since Stage C depends on the realizations of the representations; it often interacts with geometric and analytic problems.

We organize this article not in the natural order, Stage A $\Rightarrow$ Stage B $\Rightarrow$ Stage C, but in an opposite order, Stage C $\Rightarrow$ Stages A and B. This is because it is only recently that a complete construction of all symmetry breaking operators has been carried out in some special settings, and because such examples and new methods might yield yet another interesting direction of branching problems in Stages A to C. The two spaces in (1.1) are discussed in Sections 4–6 from different perspectives (Stage A). The last section returns to Stage C together with comments on the general theory (Stages A and B).

2 Two concrete examples of Stage C

In this section, we illustrate Stage C in the branching program with two recent examples, namely, an explicit construction and a complete classification of *differential* symmetry breaking operators (Section 2.1) and *continuous* symmetry breaking operators (Section 2.2). They have been carried out only in quite special situations until now. In this section we examine these new examples by making some observations that may contain some interesting hints for future study. In later sections, we discuss to what extent the new results and methods apply to other situations and what the limitations of the general theory for Stage A would be.

2.1 *Rankin–Cohen bidifferential operators for the tensor products of* SL_2*-modules*

Taking the SL_2-case as a prototype, we explain what we have in mind for Stage C by comparing it with Stages A and B. We focus on *differential* symmetry breaking operators in this subsection, and point out that there are some missing operators even in the classical SL_2-case ([9, 62], see also van Dijk–Pevzner [11], Zagier [76]).

First, we begin with finite-dimensional representations. For every $m \in \mathbb{N}$, there exists the unique $(m+1)$-dimensional irreducible holomorphic representation of $SL(2, \mathbb{C})$. These representations can be realized on the space $\mathrm{Pol}_m[z]$ of polynomials in z of degree at most m, by the following action of $SL(2, \mathbb{C})$ with $\lambda = -m$:

$$(\pi_\lambda(g)f)(z) = (cz+d)^{-\lambda} f(\frac{az+b}{cz+d}) \quad \text{for } g^{-1} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{2.1}$$

The tensor product of two such representations decomposes into irreducible representations of $SL(2, \mathbb{C})$ subject to the classical Clebsch–Gordan formula:

$$\mathrm{Pol}_m[z] \otimes \mathrm{Pol}_n[z] \simeq \mathrm{Pol}_{m+n}[z] \oplus \mathrm{Pol}_{m+n-2}[z] \oplus \cdots \oplus \mathrm{Pol}_{|m-n|}[z]. \tag{2.2}$$

Secondly, we recall an analogous result for infinite-dimensional representations of $SL(2, \mathbb{R})$. For this, let H_+ be the Poincaré upper half plane $\{z \in \mathbb{C} : \operatorname{Im} z > 0\}$. Then $SL(2, \mathbb{R})$ acts on the space $\mathcal{O}(H_+)$ of holomorphic functions on H_+ via π_λ ($\lambda \in \mathbb{Z}$). Further, we obtain an irreducible unitary representation of $SL(2, \mathbb{R})$ on the following Hilbert space V_λ (the *weighted Bergman space*) via π_λ for $\lambda > 1$:

$$V_\lambda := \{f \in \mathcal{O}(H_+) : \int_{H_+} |f(x+\sqrt{-1}y)|^2 y^{\lambda-2} dxdy < \infty\},$$

where the inner product is given by

$$\int_{H_+} f(x+\sqrt{-1}y)\overline{g(x+\sqrt{-1}y)} y^{\lambda-2} dxdy \quad \text{for } f, g \in V_\lambda.$$

Repka [63] and Molchanov [57] obtained the irreducible decomposition of the tensor product of two such unitary representations, namely, there is a unitary equivalence between unitary representations of $SL(2, \mathbb{R})$:

$$V_{\lambda_1} \widehat{\otimes} V_{\lambda_2} \simeq \sum_{a=0}^{\infty}{}^{\oplus} V_{\lambda_1+\lambda_2+2a}, \tag{2.3}$$

where the symbols $\widehat{\otimes}$ and $\sum^{\oplus}$ denote the Hilbert completion of the tensor product $\otimes$ and the algebraic direct sum $\oplus$, respectively. We then have:

Observation 2.1. (1) (multiplicity) *Both of the irreducible decompositions* (2.2) *and* (2.3) *are multiplicity-free.*
(2) (spectrum) *There is no continuous spectrum in either of the decompositions* (2.2) *or* (2.3).

These abstract features (Stage A) are immediate consequences of the decomposition formulæ (2.2) and (2.3) (Stage B); however, one could tell these properties without explicit formulæ from the general theory of visible actions on complex manifolds [34, 39] and a general theory of discrete decomposability [26, 28]. For instance, the following holds:

Fact 2.2. *Let π be an irreducible unitary highest weight representation of a real reductive Lie group G, and G' a reductive subgroup of G.*

(1) (multiplicity-free decomposition) *The restriction $\pi|_{G'}$ is multiplicity-free if π has a scalar minimal K-type and (G, G') is a symmetric pair.*
(2) (spectrum) *The restriction $\pi|_{G'}$ is discretely decomposable if the associated Riemannian symmetric spaces G/K and G'/K' carry Hermitian symmetric structures such that the embedding $G'/K' \hookrightarrow G/K$ is holomorphic.*

Stage C asks for a construction of the following explicit SL_2-intertwining operators (*symmetry breaking operators*):

$$\begin{aligned} \mathrm{Pol}_m[z] \otimes \mathrm{Pol}_n[z] &\to \mathrm{Pol}_{m+n-2a}[z] \qquad &&\text{for } 0 \le a \le \min(m,n), \\ V_{\lambda_1} \widehat{\otimes} V_{\lambda_2} &\to V_{\lambda_1+\lambda_2+2a} &&\text{for } a \in \mathbb{N}, \end{aligned}$$

for finite-dimensional and infinite-dimensional representations, respectively. We know a priori from Stages A and B that such intertwining operators exist uniquely (up to scalar multiplications) by Schur's lemma in this setting. A (partial) answer to this question is given by the classical Rankin–Cohen bidifferential operator, which is defined by

$$\begin{aligned} &\mathcal{RC}_{\lambda_1,\lambda_2}^{\lambda_1+\lambda_2+2a}(f_1, f_2)(z) \\ &\qquad := \sum_{l=0}^{a} \frac{(-1)^l}{l!(a-l)!} \frac{(\lambda_1+a-1)!(\lambda_2+a-1)!}{(\lambda_1+a-l-1)!(\lambda_2+l-1)!} \frac{\partial^{a-l} f_1}{\partial z^{a-l}}(z) \frac{\partial^l f_2}{\partial z^l}(z) \end{aligned}$$

for $a \in \mathbb{N}$, $\lambda_1, \lambda_2 \in \{2, 3, 4, \ldots\}$, and $f_1, f_2 \in \mathcal{O}(H_+)$. Then $\mathcal{RC}_{\lambda_1,\lambda_2}^{\lambda_1+\lambda_2+2a}$ is an operator which intertwines $\pi_{\lambda_1} \widehat{\otimes} \pi_{\lambda_2}$ and $\pi_{\lambda_1+\lambda_2+2a}$.

More generally, we treat *non-unitary* representations π_λ on $\mathcal{O}(H_+)$ of the universal covering group $SL(2,\mathbb{R})^\sim$ of $SL(2,\mathbb{R})$ by the same formula (2.1) for $\lambda \in \mathbb{C}$, and consider a continuous linear map

$$T : \mathcal{O}(H_+ \times H_+) \to \mathcal{O}(H_+) \tag{2.4}$$

that intertwines $\pi_{\lambda_1} \otimes \pi_{\lambda_2}$ and π_{λ_3}, where $SL(2,\mathbb{R})^\sim$ acts on $\mathcal{O}(H_+ \times H_+)$ via $\pi_{\lambda_1} \otimes \pi_{\lambda_2}$ under the diagonal action. We denote by $H(\lambda_1,\lambda_2,\lambda_3)$ the vector space of symmetry breaking operators T as in (2.4).

Question 2.3. (1) (Stage B) Find the dimension of $H(\lambda_1,\lambda_2,\lambda_3)$ for $(\lambda_1,\lambda_2,\lambda_3) \in \mathbb{C}^3$.
(2) (Stage C) Explicitly construct a basis of $H(\lambda_1,\lambda_2,\lambda_3)$ when it is nonzero.

Even in the SL_2-setting, we could not find a complete answer to Question 2.3 in the literature, and thus we explain our solution below.

Replacing $\mu!$ by $\Gamma(\mu+1)$, we can define

$$\mathcal{RC}^{\lambda_3}_{\lambda_1,\lambda_2}(f_1,f_2)(z) := \sum_{l=0}^{a} \frac{(-1)^l}{l!(a-l)!}\frac{\Gamma(\lambda_1+a)\Gamma(\lambda_2+a)}{\Gamma(\lambda_1+a-l)\Gamma(\lambda_2+l)}\frac{\partial^{a-l} f_1}{\partial z^{a-l}}(z)\frac{\partial^l f_2}{\partial z^l}(z), \tag{2.5}$$

where $a := \frac{1}{2}(\lambda_3-\lambda_1-\lambda_2)$ as long as $(\lambda_1,\lambda_2,\lambda_3)$ belongs to

$$\Omega := \{(\lambda_1,\lambda_2,\lambda_3) \in \mathbb{C}^3 : \lambda_3-\lambda_1-\lambda_2 = 0,2,4,\ldots\}.$$

We define a subset Ω_{sing} of Ω by

$$\Omega_{\mathrm{sing}} := \{(\lambda_1,\lambda_2,\lambda_3) \in \Omega : \lambda_1,\lambda_2,\lambda_3 \in \mathbb{Z}, \quad \lambda_3-|\lambda_1-\lambda_2| \geq 2 \geq \lambda_1+\lambda_2+\lambda_3\}.$$

Then we have the following classification of symmetry breaking operators by using the "F-method" ([51, Part II]). Surprisingly, it turns out that any symmetry breaking operator (2.4) is given by a differential operator.

Theorem 2.4. (1) $H(\lambda_1,\lambda_2,\lambda_3) \neq \{0\}$ *if and only if* $(\lambda_1,\lambda_2,\lambda_3) \in \Omega$.

From now on, we assume $(\lambda_1,\lambda_2,\lambda_3) \in \Omega$.
(2) $\dim_{\mathbb{C}} H(\lambda_1,\lambda_2,\lambda_3) = 1$ *if and only if* $\mathcal{RC}^{\lambda_3}_{\lambda_1,\lambda_2} \neq 0$, *or equivalently,* $(\lambda_1,\lambda_2,\lambda_3) \notin \Omega_{\mathrm{sing}}$. *In this case,* $H(\lambda_1,\lambda_2,\lambda_3) = \mathbb{C}\,\mathcal{RC}^{\lambda_3}_{\lambda_1,\lambda_2}$.
(3) *The following three conditions on* $(\lambda_1,\lambda_2,\lambda_3) \in \Omega$ *are equivalent:*

(i) $\dim_{\mathbb{C}} H(\lambda_1,\lambda_2,\lambda_3) = 2$.
(ii) $\mathcal{RC}^{\lambda_3}_{\lambda_1,\lambda_2} = 0$.
(iii) $(\lambda_1,\lambda_2,\lambda_3) \in \Omega_{\mathrm{sing}}$.

In this case, the two-dimensional vector space $H(\lambda_1,\lambda_2,\lambda_3)$ *is spanned by*

$$\mathcal{RC}^{\lambda_3}_{2-\lambda_1,\lambda_2} \circ ((\frac{\partial}{\partial z_1})^{1-\lambda_1} \otimes \mathrm{id}) \quad \textit{and} \quad \mathcal{RC}^{\lambda_3}_{\lambda_1,2-\lambda_2} \circ (\mathrm{id} \otimes (\frac{\partial}{\partial z_2})^{1-\lambda_2}).$$

Theorem 2.4 answers Question 2.3 (1) and (2). Here are some observations.

Observation 2.5. (1) (localness property) *Any symmetry breaking operator from* $\pi_{\lambda_1} \otimes \pi_{\lambda_2}$ *to* π_{λ_3} *is given by a differential operator in the holomorphic realization of* π_{λ_j} *(*$j = 1, 2, 3$*).*

(2) (multiplicity-two phenomenon) *The dimension of the space of symmetry breaking operators jumps up exactly when the holomorphic continuation of the Rankin–Cohen bidifferential operator vanishes.*

The localness property in Observation 2.5 (1) was recently proved in a more general setting (see Theorem 7.18 and Conjecture 7.23).

Remark 2.6 (higher multiplicities at Ω_{sing}).

(1) From the viewpoint of analysis (or the "F-method" [40, 47, 51]), the multiplicity-two phenomenon in Observation 2.5 (2) can be derived from the fact that Ω_{sing} is of codimension two in Ω and from the fact that $\{\frac{\partial}{\partial\lambda_1}\mathcal{RC}^{\lambda_3}_{\lambda_1,\lambda_2}, \frac{\partial}{\partial\lambda_2}\mathcal{RC}^{\lambda_3}_{\lambda_1,\lambda_2}\}$ forms a basis in $H(\lambda_1, \lambda_2, \lambda_3)$ when $\mathcal{RC}^{\lambda_3}_{\lambda_1,\lambda_2} = 0$, namely, when $(\lambda_1, \lambda_2, \lambda_3) \in \Omega_{\mathrm{sing}}$.

(2) The basis given in Theorem 2.4 (3) is different from the basis in Remark 2.6 (1), and clarifies the representation-theoretic reason for the multiplicity-two phenomenon as it is expressed as the composition of two intertwining operators.

(3) Theorem 2.4 (3) implies a multiplicity-two phenomenon for Verma modules $M(\mu) = U(\mathfrak{g}) \otimes_{U(\mathfrak{b})} \mathbb{C}_\mu$ for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{C})$:

$$\dim_{\mathbb{C}} \mathrm{Hom}_{\mathfrak{g}}(M(-\lambda_3), M(-\lambda_1) \otimes M(-\lambda_2)) = 2 \quad \text{for } (\lambda_1, \lambda_2, \lambda_3) \in \Omega_{\mathrm{sing}}.$$

Again, the tensor product $M(-\lambda_1) \otimes M(-\lambda_2)$ of Verma modules decomposes into a multiplicity-free direct sum of irreducible $\mathfrak{g}$-modules for generic $\lambda_1, \lambda_2 \in \mathbb{C}$, but not for singular parameters. See [51, Part II] for details.

(4) In turn, we shall get a two-dimensional space of differential symmetry breaking operators at Ω_{sing} for principal series representations with respect to $SL(2, \mathbb{R}) \times SL(2, \mathbb{R}) \downarrow \mathrm{diag}(SL(2, \mathbb{R}))$, see Remark 7.15 in Section 7.

2.2 *Symmetry breaking in conformal geometry*

In contrast to the localness property for symmetry breaking operators in the holomorphic setting (Observation 2.5 (1)), there exist non-local symmetry breaking operators in a more general setting. We illustrate Stage C in the branching problem by an explicit construction and a complete classification of all local and non-local symmetry breaking operators that arise from conformal geometry. In later sections, we explain a key idea of the proof (Section 7) and present potential settings where we could expect that this example might serve as the prototype of analogous questions (Section 6). For full details of this subsection, see the monograph [52] joint with Speh.

For $\lambda \in \mathbb{C}$ we denote by $I(\lambda)^\infty$ the smooth (unnormalized) spherical principal series representation of $G = \mathrm{O}(n+1,1)$. In our parametrization, $\lambda \in \frac{n}{2} + \sqrt{-1}\mathbb{R}$ is the unitary axis, $\lambda \in (0,n)$ gives the complementary series representations, and $I(\lambda)^\infty$ contains irreducible finite-dimensional representations as submodules for $\lambda \in \{0,-1,-2,\ldots\}$ and as quotients for $\lambda \in \{n, n+1, n+2, \ldots\}$.

We consider the restriction of the representation $I(\lambda)^\infty$ and its subquotients to the subgroup $G' := \mathrm{O}(n,1)$. As we did for $I(\lambda)^\infty$, we denote by $J(\nu)^\infty$ for $\nu \in \mathbb{C}$, the (unnormalized) spherical principal series representations of $G' = \mathrm{O}(n,1)$. For $(\lambda,\nu) \in \mathbb{C}^2$, we set

$$H(\lambda,\nu) := \mathrm{Hom}_{G'}(I(\lambda)^\infty, J(\nu)^\infty),$$

the space of (continuous) symmetry breaking operators. Similar to Question 2.3, we ask:

Question 2.7. (1) (Stage B) Find the dimension of $H(\lambda,\nu)$ for $(\lambda,\nu) \in \mathbb{C}^2$.
(2) (Stage C) Explicitly construct a basis for $H(\lambda,\nu)$.
(3) (Stage C) Determine when $H(\lambda,\nu)$ contains a differential operator.

The following is a complete answer to Question 2.7 (1).

Theorem 2.8. (1) *For all $\lambda, \nu \in \mathbb{C}$, we have $H(\lambda,\nu) \neq \{0\}$.*
(2) $\dim_{\mathbb{C}} H(\lambda,\nu) = \begin{cases} 1 & \text{if} \quad (\lambda,\nu) \in \mathbb{C}^2 \setminus L_{\mathrm{even}}, \\ 2 & \text{if} \quad (\lambda,\nu) \in L_{\mathrm{even}}, \end{cases}$

where the "exceptional set" L_{even} is the discrete subset of $\mathbb{C}^2$ defined by

$$L_{\mathrm{even}} := \{(\lambda,\nu) \in \mathbb{Z}^2 : \lambda \le \nu \le 0, \quad \lambda \equiv \nu \mod 2\}.$$

The role of L_{even} in Theorem 2.8 is similar to that of Ω_{sing} in Section 2.1. For Stage C, we use the "N-picture" of the principal series representations, namely, realize $I(\lambda)^\infty$ and $J(\nu)^\infty$ in $C^\infty(\mathbb{R}^n)$ and $C^\infty(\mathbb{R}^{n-1})$, respectively. For $x \in \mathbb{R}^{n-1}$, we set $|x| = (x_1^2 + \cdots + x_{n-1}^2)^{\frac{1}{2}}$. For $(\lambda,\nu) \in \mathbb{C}^2$ satisfying $\mathrm{Re}(\nu - \lambda) \gg 0$ and $\mathrm{Re}(\nu + \lambda) \gg 0$, we construct explicitly a symmetry breaking operator (i.e., continuous G'-homomorphism) from $I(\lambda)^\infty$ to $J(\nu)^\infty$ as an integral operator given by

$$\begin{aligned}(\mathbb{A}_{\lambda,\nu} f)(y) &:= \int_{\mathbb{R}^n} |x_n|^{\lambda+\nu-n}(|x-y|^2 + x_n^2)^{-\nu} f(x,x_n)\,dx\,dx_n \qquad (2.6)\\ &= \mathrm{rest}_{x_n=0} \circ (|x_n|^{\lambda+\nu-n}(|x|^2+x_n^2)^{-\nu} *_{\mathbb{R}^n} f).\end{aligned}$$

One might regard $\mathbb{A}_{\lambda,\nu}$ as a generalization of the Knapp–Stein intertwining operator ($G = G'$ case), and also as the adjoint operator of a generalization of the Poisson transform.

The symmetry breaking operator $\mathbb{A}_{\lambda,\nu}$ extends meromorphically with respect to the parameter (λ,ν), and if we normalize $\mathbb{A}_{\lambda,\nu}$ as

$$\widetilde{\mathbb{A}}_{\lambda,\nu} := \frac{1}{\Gamma(\frac{\lambda+\nu-n+1}{2})\Gamma(\frac{\lambda-\nu}{2})}\mathbb{A}_{\lambda,\nu},$$

then $\widetilde{\mathbb{A}}_{\lambda,\nu} : I(\lambda)^\infty \to J(\nu)^\infty$ is a continuous symmetry breaking operator that depends holomorphically on (λ,ν) in the entire complex plane $\mathbb{C}^2$, and $\widetilde{\mathbb{A}}_{\lambda,\nu} \neq 0$ if and only if $(\lambda,\nu) \notin L_{\text{even}}$ ([52, Theorem 1.5]).

The singular set L_{even} is most interesting. To construct a symmetry breaking operator at L_{even}, we renormalize $\widetilde{\mathbb{A}}_{\lambda,\nu}$ for $\nu \in -\mathbb{N}$, by

$$\widetilde{\widetilde{\mathbb{A}}}_{\lambda,\nu} := \Gamma(\frac{\lambda-\nu}{2})\widetilde{\mathbb{A}}_{\lambda,\nu} = \frac{1}{\Gamma(\frac{\lambda+\nu-n+1}{2})}\mathbb{A}_{\lambda,\nu}.$$

In order to construct *differential* symmetry breaking operators, we recall that the Gegenbauer polynomial $C_l^\alpha(t)$ for $l \in \mathbb{N}$ and $\alpha \in \mathbb{C}$ is given by

$$C_l^\alpha(t) := \sum_{k=0}^{[\frac{l}{2}]}(-1)^k \frac{\Gamma(l-k+\alpha)}{\Gamma(\alpha)\Gamma(l-2k+1)k!}(2t)^{l-2k}.$$

We note that $C_l^\alpha(t) \equiv 0$ if $l \geq 1$ and $\alpha = 0,-1,-2,\dots,-[\frac{l-1}{2}]$. We renormalize $C_l^\alpha(t)$ by setting $\widetilde{C}_l^\alpha(t) := \frac{\Gamma(\alpha)}{\Gamma(\alpha+[\frac{l+1}{2}])}C_l^\alpha(t)$, so that $\widetilde{C}_l^\alpha(t)$ is a nonzero polynomial in t of degree l for all $\alpha \in \mathbb{C}$ and $l \in \mathbb{N}$. We inflate it to a polynomial of two variables u and v by

$$\widetilde{C}_k^\alpha(u,v) := u^{\frac{k}{2}}\widetilde{C}_k^\alpha(\frac{v}{\sqrt{u}}).$$

For instance, $\widetilde{C}_0^\alpha(u,v) = 1$, $\widetilde{C}_1^\alpha(u,v) = 2v$, $\widetilde{C}_2^\alpha(u,v) = 2(\alpha+1)v^2 - u$, etc. Substituting $u = -\Delta_{\mathbb{R}^{n-1}} = -\sum_{j=1}^{n-1}\frac{\partial^2}{\partial x_j^2}$ and $v = \frac{\partial}{\partial x_n}$, we get a differential operator of order $2l$:

$$\widetilde{\mathbb{C}}_{\lambda,\nu} := \operatorname{rest}_{x_n=0} \circ \widetilde{C}_{2l}^{\lambda-\frac{n-1}{2}}(-\Delta_{\mathbb{R}^{n-1}}, \frac{\partial}{\partial x_n}).$$

This closed formula of the differential operator $\widetilde{\mathbb{C}}_{\lambda,\nu}$ was obtained by Juhl [21] (see also [47] for a short proof by the F-method, and [40] for yet another proof by using the residue formula), and the closed formula (2.6) of the symmetry breaking operator $\widetilde{\mathbb{A}}_{\lambda,\nu}$ was obtained by Kobayashi and Speh [52].

The following results answer Question 2.7 (2) and (3); see [52, Theorems 1.8 and 1.9]:

Theorem 2.9. (1) *With notation as above, we have*

$$H(\lambda,\nu) = \begin{cases} \mathbb{C}\widetilde{\mathbb{A}}_{\lambda,\nu} & \text{if } (\lambda,\nu) \in \mathbb{C}^2 \setminus L_{\text{even}} \\ \mathbb{C}\widetilde{\widetilde{\mathbb{A}}}_{\lambda,\nu} \oplus \mathbb{C}\widetilde{\mathbb{C}}_{\lambda,\nu} & \text{if } (\lambda,\nu) \in L_{\text{even}}. \end{cases}$$

(2) *$H(\lambda,\nu)$ contains a nontrivial differential operator if and only if $\nu - \lambda = 0, 2, 4, 6, \dots$. In this case $\widetilde{\mathbb{A}}_{\lambda,\nu}$ is proportional to $\widetilde{\mathbb{C}}_{\lambda,\nu}$, and the proportionality constant vanishes if and only if $(\lambda,\nu) \in L_{\text{even}}$.*

From Theorem 2.9 (2) and Theorem 2.8 (1), we have the following:

Observation 2.10. (1) *Unlike the holomorphic setting in Section 2.1, the localness property fails.*
(2) *Even if an irreducible smooth representation* $\pi^\infty = I(\lambda)^\infty$ *is unitarizable as a representation of* G*, the condition* $\mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty) \neq \{0\}$ *does not imply that the irreducible smooth representation* $\tau^\infty = J(\nu)^\infty$ *is unitarizable as a representation of* G' *(see Section 3.2 for the terminology).*

For $\lambda \in \{n, n+1, n+2, \ldots\}$, $I(\lambda)^\infty$ contains a unique proper infinite-dimensional closed G-submodule. We denote it by $A_{\mathfrak{q}}(\lambda - n)^\infty$, which is the Casselman–Wallach globalization of Zuckerman's derived functor module $A_{\mathfrak{q}}(\lambda - n)$ (see [69, 71]) for some θ-stable parabolic subalgebra $\mathfrak{q}$ of $\mathfrak{g}$. It is unitarizable ([70, 74]) and has nonzero $(\mathfrak{g}, K)$-cohomologies (Vogan–Zuckerman [73]).

By using the explicit formulæ of symmetry breaking operators and certain identities involving these operators, we can identify precisely the images of every subquotient of $I(\lambda)^\infty$ under these operators. In particular, we obtain the following corollary for the branching problem of $A_{\mathfrak{q}}(\lambda)$ modules. We note that in this setting, the restriction $A_{\mathfrak{q}}(\lambda)|_{\mathfrak{g}'}$ is not discretely decomposable as a $(\mathfrak{g}', K')$-module (Definition 4.3).

Corollary 2.11 ([52, Theorem 1.2]). *With notation as above, we have*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(A_{\mathfrak{q}}(i)^\infty|_{G'}, A_{\mathfrak{q}'}(j)^\infty) = \begin{cases} 1 & \textit{if} \quad i \geq j \quad \textit{and} \quad i \equiv j \mod 2, \\ 0 & \textit{if} \quad i < j \quad \textit{and} \quad i \not\equiv j \mod 2. \end{cases}$$

There are some further applications of the explicit formulæ (2.6) (Stage C in the branching problems). For instance, J. Möllers and B. Ørsted recently found an interesting application of the explicit formulæ (2.6) to $L^p - L^q$ estimates of certain boundary-value problems, and to some questions in automorphic forms [58].

3 Preliminary results and basic notation

We review quickly some basic results on (infinite-dimensional) continuous representations of real reductive Lie groups and fix notation. There are no new results in this section.

By a continuous representation π of a Lie group G on a topological vector space V we shall mean that $\pi : G \to \mathrm{GL}_{\mathbb{C}}(V)$ is a group homomorphism from G into the group of invertible endomorphisms of V such that the induced map $G \times V \to V$, $(g, v) \mapsto \pi(g)v$ is continuous. We say π is a (continuous) Hilbert [Banach, Fréchet, $\cdots$] representation if V is a Hilbert [Banach, Fréchet, $\cdots$] space. A continuous Hilbert representation π of G is said to be a unitary representation when all the operators $\pi(g)$ $(g \in G)$ are unitary.

3.1 Decomposition of unitary representations

One of the most distinguished features of *unitary* representations is that they can be built up from the smallest objects, namely, irreducible unitary representations. To be precise, let G be a locally compact group. We denote by $\widehat{G}$ the set of equivalence classes of irreducible unitary representations of G (the *unitary dual*), endowed with the Fell topology.

Fact 3.1 (Mautner–Teleman). *For every unitary representation π of a locally compact group G, there exist a Borel measure $d\mu$ on $\widehat{G}$ and a measurable function $n_\pi : \widehat{G} \to \mathbb{N} \cup \{\infty\}$ such that π is unitarily equivalent to the direct integral of irreducible unitary representations:*

$$\pi \simeq \int_{\widehat{G}}^{\oplus} n_\pi(\sigma)\sigma \, d\mu(\sigma), \tag{3.1}$$

where $n_\pi(\sigma)\sigma$ stands for the multiple of an irreducible unitary representation σ with multiplicity $n_\pi(\sigma)$.

The decomposition (3.1) is unique if G is of type I in the sense of von Neumann algebras, in particular, if G (or G' in later notation) is a real reductive Lie group or a nilpotent Lie group. Then the *multiplicity function* n_π is well-defined up to a measure zero set with respect to $d\mu$. We say that π is *multiplicity-free* if $n_\pi(\sigma) \leq 1$ almost everywhere, or equivalently, if the ring of continuous G-endomorphisms of π is commutative.

The decomposition (3.1) splits into a direct sum of the discrete and continuous parts:

$$\pi \simeq (\pi)_{\mathrm{disc}} \oplus (\pi)_{\mathrm{cont}}, \tag{3.2}$$

where $(\pi)_{\mathrm{disc}}$ is a unitary representation defined on the maximal closed G-invariant subspace that is isomorphic to a discrete Hilbert sum of irreducible unitary representations and $(\pi)_{\mathrm{cont}}$ is its orthogonal complement.

Definition 3.2. We say a unitary representation π is *discretely decomposable* if $\pi = (\pi)_{\mathrm{disc}}$.

3.2 Continuous representations and smooth representations

We would like to treat non-unitary representations as well for branching problems. For this we recall some standard concepts of continuous representations of Lie groups.

Suppose π is a continuous representation of G on a Banach space V. A vector $v \in V$ is said to be *smooth* if the map $G \to V$, $g \mapsto \pi(g)v$ is of C^∞-class. Let V^∞ denote the space of smooth vectors of the representation (π, V). Then V^∞

carries a Fréchet topology with a family of semi-norms $\|v\|_{i_1\cdots i_k} := \|d\pi(X_{i_1})\cdots d\pi(X_{i_k})v\|$, where $\{X_1, \ldots, X_n\}$ is a basis of the Lie algebra $\mathfrak{g}_0$ of G. Then V^∞ is a G-invariant dense subspace of V, and we obtain a continuous Fréchet representation (π^∞, V^∞) of G. Similarly, we can define a representation π^ω on the space V^ω of analytic vectors.

Suppose now that G is a real reductive linear Lie group, K a maximal compact subgroup of G, and $\mathfrak{g}$ the complexification of the Lie algebra $\mathfrak{g}_0$ of G. Let $\mathcal{HC}$ denote the category of Harish-Chandra modules whose objects and morphisms are $(\mathfrak{g}, K)$-modules of finite length and $(\mathfrak{g}, K)$-homomorphisms, respectively.

Let π be a continuous representation of G on a Fréchet space V. Suppose that π is of finite length, namely, there are at most finitely many closed G-invariant subspaces in V. We say π is *admissible* if

$$\dim \operatorname{Hom}_K(\tau, \pi|_K) < \infty$$

for any irreducible finite-dimensional representation τ of K. We denote by V_K the space of K-finite vectors. Then $V_K \subset V^\omega \subset V^\infty$ and the Lie algebra $\mathfrak{g}$ leaves V_K invariant. The resulting $(\mathfrak{g}, K)$-module on V_K is called the underlying $(\mathfrak{g}, K)$-module of π, and will be denoted by π_K.

For any admissible representation π on a Banach space V, the smooth representation (π^∞, V^∞) depends only on the underlying $(\mathfrak{g}, K)$-module. We say (π^∞, V^∞) is an *admissible smooth representation*. By the Casselman–Wallach globalization theory, (π^∞, V^∞) has moderate growth, and there is a canonical equivalence of categories between the category $\mathcal{HC}$ of $(\mathfrak{g}, K)$-modules of finite length and the category of admissible smooth representations of G ([74, Chapter 11]). In particular, the Fréchet representation π^∞ is uniquely determined by its underlying $(\mathfrak{g}, K)$-module. We say π^∞ is the *smooth globalization* of $\pi_K \in \mathcal{HC}$.

For simplicity, by an *irreducible smooth representation*, we shall mean an irreducible admissible smooth representation of G. We denote by $\widehat{G}_{\text{smooth}}$ the set of equivalence classes of irreducible smooth representations of G. Using the category $\mathcal{HC}$ of $(\mathfrak{g}, K)$-modules, we may regard the unitary dual $\widehat{G}$ as a subset of $\widehat{G}_{\text{smooth}}$.

4 Two spaces: $\operatorname{Hom}_{G'}(\tau, \pi|_{G'})$ and $\operatorname{Hom}_{G'}(\pi|_{G'}, \tau)$

Given irreducible continuous representations π of G and τ of a subgroup G', we may consider two settings for branching problems:
Case I. (embedding) continuous G'-homomorphisms from τ to $\pi|_{G'}$;
Case II. (symmetry breaking) continuous G'-homomorphisms from $\pi|_{G'}$ to τ.

We write $\operatorname{Hom}_{G'}(\tau, \pi|_{G'})$ and $\operatorname{Hom}_{G'}(\pi|_{G'}, \tau)$ for the vector spaces of such continuous G'-homomorphisms, respectively. Needless to say, the existence of such G'-intertwining operators depends on the topology of the representation spaces of π and τ.

Cases I and II are related to each other by taking contragredient representations:

$$\mathrm{Hom}_{G'}(\tau, \pi|_{G'}) \subset \mathrm{Hom}_{G'}(\pi^\vee|_{G'}, \tau^\vee),$$
$$\mathrm{Hom}_{G'}(\pi|_{G'}, \tau) \subset \mathrm{Hom}_{G'}(\tau^\vee, \pi^\vee|_{G'}).$$

Thus they are equivalent in the category of unitary representations (see Theorem 4.1 (3)). Furthermore, we shall use a variant of the above duality in analyzing differential symmetry breaking operators (Case II) by means of "discretely decomposable restrictions" of Verma modules (Case I); see the duality (7.3) for the proof of Theorem 7.13 below.

On the other hand, it turns out that Cases I and II are significantly different if we confine ourselves to irreducible smooth representations (see Section 3.2). Such a difference also arises in an analogous problem in the category $\mathcal{HC}$ of Harish-Chandra modules where no topology is specified.

Accordingly, we shall discuss some details for Cases I and II separately, in Sections 5 and 6, respectively.

4.1 K-finite vectors and K'-finite vectors

Let G be a real reductive linear Lie group, and G' a reductive subgroup. We take maximal compact subgroups K and K' of G and G', respectively, such that $K' = K \cap G'$.

We recall that for an admissible representation π of G on a Banach space V, any K-finite vector is contained in V^∞, and the underlying $(\mathfrak{g}, K)$-module π_K is defined on

$$V_K := V_{K\text{-finite}} \qquad (\subset V^\infty).$$

When we regard (π, V) as a representation of the subgroup G' by restriction, we denote by $(V|_{G'})^\infty$ the space of smooth vectors with respect to the G'-action, and write $(\pi|_{G'})^\infty$ for the continuous representation of G' on $(V|_{G'})^\infty$. In contrast to the case $G = G'$, we remark that K'-finite vectors are not necessarily contained in $(V|_{G'})^\infty$ if $G' \subsetneqq G$, because the G'-module $(\pi|_{G'}, V|_{G'})$ is usually not of finite length. Instead, we can define a $(\mathfrak{g}', K')$-module on

$$V_{K'} := V_{K'\text{-finite}} \cap (V|_{G'})^\infty,$$

which we denote simply by $\pi_{K'}$. Obviously we have the following inclusion relations:

$$\begin{array}{ccc} V_K & \subset & V_{K'} \\ \cap & & \cap \\ V^\infty & \subset & (V|_{G'})^\infty \subset V. \end{array} \tag{4.1}$$

None of them coincides in general (e.g., $V_K = V_{K'}$ if and only if π_K is discretely decomposable as $(\mathfrak{g}', K')$-module, as we shall see in Theorem 4.5 below.

We set

$$H_K(\tau,\pi) := \mathrm{Hom}_{\mathfrak{g}',K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'}),$$
$$H_{K'}(\tau,\pi) := \mathrm{Hom}_{\mathfrak{g}',K'}(\tau_{K'}, \pi_{K'}|_{\mathfrak{g}'}).$$

According to the inclusion relation (4.1), for irreducible representations τ of G' we have:

$$H_K(\tau,\pi) \subset H_{K'}(\tau,\pi).$$

In the case where π is a unitary representation of G, the latter captures discrete summands in the branching law of the restriction $\pi|_{G'}$ (see, Theorem 4.1 (3)), whereas the former vanishes even if the latter is nonzero when the continuous part $(\pi|_{G'})_{\mathrm{cont}}$ is not empty (see Theorem 4.5). The spaces of continuous G'-homomorphisms such as $\mathrm{Hom}_{G'}(\tau, \pi|_{G'})$ or $\mathrm{Hom}_{G'}(\tau^\infty, \pi^\infty|_{G'})$ are in-between.

We begin with a general result:

Theorem 4.1. *Suppose that π and τ are admissible irreducible Banach representations of G and G'.*

(1) *We have natural inclusions and an isomorphism:*

$$H_K(\tau,\pi) \hookrightarrow \mathrm{Hom}_{G'}(\tau^\infty, \pi^\infty|_{G'}) \hookrightarrow \mathrm{Hom}_{G'}(\tau^\infty, (\pi|_{G'})^\infty) \xrightarrow{\sim} H_{K'}(\tau,\pi). \quad (4.2)$$

(2) *There are canonical injective homomorphisms:*

$$\mathrm{Hom}_{G'}(\pi|_{G'}, \tau) \hookrightarrow \mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty) \hookrightarrow \mathrm{Hom}_{G'}(\pi^\omega|_{G'}, \tau^\omega) \hookrightarrow \mathrm{Hom}_{\mathfrak{g}',K'}(\pi_K, \tau_{K'}). \quad (4.3)$$

(3) (unitary case) *If τ and π are irreducible unitary representations of G' and G, respectively, then we have natural isomorphisms (where the last isomorphism is conjugate linear):*

$$H_{K'}(\tau,\pi) \xleftarrow{\sim} \mathrm{Hom}_{G'}(\tau^\infty, (\pi|_{G'})^\infty) \xleftarrow{\sim} \mathrm{Hom}_{G'}(\tau, \pi|_{G'}) \simeq \mathrm{Hom}_{G'}(\pi|_{G'}, \tau). \quad (4.4)$$

We write $m_\pi(\tau)$ for the dimension of one of (therefore, any of) the terms in (4.4). Then the discrete part of the restriction $\pi|_{G'}$ (see Definition 3.2) decomposes discretely as

$$(\pi|_{G'})_{\mathrm{disc}} \simeq \sum_{\tau\in\widehat{G'}}{}^{\oplus} m_\pi(\tau)\tau.$$

Remark 4.2. Even if π and τ are irreducible unitary representations of G and G', respectively, the canonical injective homomorphism

$$\mathrm{Hom}_{G'}(\pi|_{G'}, \tau) \hookrightarrow \mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty) \quad (4.5)$$

is not surjective in general.

In fact, we can give an example where the canonical homomorphism (4.5) is not surjective by using the classification of $\mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty)$ for the pair $(G, G') = (O(n+1,1), O(n,1))$ in Section 2.2 as follows. Recall $\mathrm{Hom}_{G'}(I(\lambda)^\infty|_{G'}, J(\nu)^\infty) \neq \{0\}$ for all $(\lambda, \nu) \in \mathbb{C}^2$ with the notation therein. However, for a fixed $\pi \in \widehat{G}$, there exist at most countably many $\tau \in \widehat{G'}$ that occur in the discrete part of the restriction $\pi|_{G'}$, and therefore $\{\tau \in \widehat{G'} : \mathrm{Hom}_{G'}(\pi|_{G'}, \tau) \neq \{0\}\}$ is an infinite set because we have the following bijection:

$$\{\tau \in \widehat{G'} : \mathrm{Hom}_{G'}(\pi|_{G'}, \tau) \neq \{0\}\} \simeq \{\tau \in \widehat{G'} : \mathrm{Hom}_{G'}(\tau, \pi|_{G'}) \neq \{0\}\}.$$

Hence, by taking $\pi^\infty = I(\lambda)^\infty$ for a fixed $\lambda \in \frac{n}{2} + \sqrt{-1}\mathbb{R}$ (unitary axis) or $\lambda \in (0, n)$ (complementary series), we see that the canonical homomorphism (4.5) must be zero when we take τ^∞ to be a representation $I(\nu)^\infty$ for $\nu \in \mathbb{C}$ such that $\nu \notin \frac{n-1}{2} + \sqrt{-1}\mathbb{R}$ and $\nu \notin \mathbb{R}$.

Let us give a proof of Theorem 4.1.

Proof. (1) To see the first inclusion, we prove that any $(\mathfrak{g}', K')$-homomorphism $\iota : \tau_{K'} \to \pi_K|_{\mathfrak{g}'}$ extends to a continuous map $\tau^\infty \to \pi^\infty|_{G'}$. We may assume that ι is nonzero, and therefore, is injective. Since $\iota(\tau_{K'}) \subset \pi_K \subset \pi^\infty$, we can define a Fréchet space W to be the closure of $\iota(\tau_{K'})$ in π^∞, on which G' acts continuously. Its underlying $(\mathfrak{g}', K')$-module is isomorphic to $\iota(\tau_{K'}) \simeq \tau_{K'}$.

Since the continuous representation π^∞ of G is of moderate growth, the Fréchet representation W of the subgroup G' is also of moderate growth. By the Casselman–Wallach globalization theory, there is a G'-homomorphism $\tau^\infty \xrightarrow{\sim} \overline{\iota(\tau_{K'})}\ (= W)$ extending the $(\mathfrak{g}', K')$-isomorphism $\iota : \tau_{K'} \xrightarrow{\sim} \iota(\tau_{K'})$. Hence we have obtained a natural map $\mathrm{Hom}_{\mathfrak{g}', K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'}) \to \mathrm{Hom}_{G'}(\tau^\infty, \pi^\infty|_{G'})$, which is clearly injective because $\tau_{K'}$ is dense in τ^∞.

The second inclusion is obvious.

To see the third inclusion, it suffices to show that any $\iota \in \mathrm{Hom}_{\mathfrak{g}', K'}(\tau_{K'}, \pi_{K'}|_{\mathfrak{g}'})$ extends to a continuous G'-homomorphism from τ^∞ to $(\pi|_{G'})^\infty$. Since $\tau_{K'}$ is an irreducible $(\mathfrak{g}', K')$-module, ι is injective unless ι is zero and $\iota(\tau_{K'})$ is isomorphic to $\tau_{K'}$ as $(\mathfrak{g}', K')$-modules.

Let V be the Banach space on which G acts via π, and W_1 and W_2 the closures of $\iota(\tau_{K'})$ in the Banach space V and the Fréchet space $(V|_{G'})^\infty$, respectively. Then the underlying $(\mathfrak{g}', K')$-modules of W_1 and W_2 are both isomorphic to $\tau_{K'}$. Moreover, $W_2 \subset W_1 \cap (V|_{G'})^\infty$ by definition, and W_2 is closed in $W_1 \cap (V|_{G'})^\infty$ with respect to the Fréchet topology. Since the subspace $\iota(\tau_{K'})$ of W_2 is dense in $W_1 \cap (V|_{G'})^\infty$, we conclude that W_2 coincides with $W_1 \cap (V|_{G'})^\infty$, which is the Casselman–Wallach globalization of the $(\mathfrak{g}', K')$-module $\iota(\tau_{K'}) \simeq \tau_{K'}$. By the uniqueness of the Casselman–Wallach globalization [74, Chapter 11], the $(\mathfrak{g}', K')$-isomorphism $\tau_{K'} \xrightarrow{\sim} \iota(\tau_{K'})$ extends to an isomorphism between Fréchet G'-modules $\tau^\infty \xrightarrow{\sim} W_2 (= W_1 \cap (V|_{G'})^\infty)$.

(2) If $\iota : \pi|_{G'} \to \tau$ is a continuous G'-homomorphism, then

$$\iota(\pi^\infty|_{G'}) \subset \iota((\pi|_{G'})^\infty) \subset \tau^\infty,$$

and thus we have obtained a continuous G'-homomorphism $\iota^\infty : \pi^\infty|_{G'} \to \tau^\infty$ between Fréchet representations. Furthermore $\iota \mapsto \iota^\infty$ is injective because V^∞ is dense in V. This shows the first inclusive relation of the statement (2). The proof for other inclusions are similar.
(3) The last isomorphism in (4.4) is given by taking the adjoint operator. The other isomorphisms are easy to see. The last statement follows from the fact that if $\varphi \in \mathrm{Hom}_{G'}(\tau, \pi|_{G'})$ then φ is a scalar multiple of an *isometric* G'-homomorphism. □

The terms in (4.2) do not coincide in general. In order to clarify when they coincide, we recall from [29] the notion of discrete decomposability of $\mathfrak{g}$-modules.

Definition 4.3. A $(\mathfrak{g}, K)$-module X is said to be *discretely decomposable as a $(\mathfrak{g}', K')$-module* if there is a filtration $\{X_i\}_{i\in\mathbb{N}}$ of $(\mathfrak{g}', K')$-modules such that

- $\bigcup_{i\in\mathbb{N}} X_i = X$ and
- X_i is of finite length as a $(\mathfrak{g}', K')$-module for any $i \in \mathbb{N}$.

The idea was to exclude "hidden continuous spectrum" in an algebraic setting, and discrete decomposability here does not imply complete reducibility. Discrete decomposability is preserved by taking submodules, quotients, and the tensor product with finite-dimensional representations.

Remark 4.4 (see [29, Lemma 1.3]). Suppose that X is a unitarizable $(\mathfrak{g}, K)$-module. Then X is discretely decomposable as a $(\mathfrak{g}', K')$-module if and only if X is isomorphic to an algebraic direct sum of irreducible $(\mathfrak{g}', K')$-modules.

We get much stronger results than Theorem 4.1 in this setting:

Theorem 4.5 (discretely decomposable case). *Assume π is an irreducible admissible representation of G on a Banach space V. Let π_K be the underlying $(\mathfrak{g}, K)$-module. Then the following five conditions on the triple (G, G', π) are equivalent:*

(i) *There exists at least one irreducible $(\mathfrak{g}', K')$-module $\tau_{K'}$ such that* $\mathrm{Hom}_{\mathfrak{g}',K'}(\tau_{K'}, \pi_K) \neq \{0\}$.
(ii) *π_K is discretely decomposable as a $(\mathfrak{g}', K')$-module (see Definition 4.3).*
(iii) *All the terms in (4.2) are the same for any irreducible admissible Banach representation τ of G'.*
(iv) *All the terms in (4.2) are the same for some irreducible admissible Banach representation τ of G'.*
(v) $V_K = V_{K'}$.

Moreover, if (π, V) is a unitary representation, then one of (therefore, any of) the equivalent conditions (i) – (v) implies that the continuous part $(\pi|_{G'})_{\mathrm{cont}}$ of the restriction $\pi|_{G'}$ is empty.

Proof. See [29] for the first statement, and [32] for the second statement. □

4.2 *Some observations on* $\mathrm{Hom}_{G'}(\tau^\infty, \pi^\infty|_{G'})$ *and* $\mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty)$

For a unitary representation (π, V) of G, Fact 3.1 gives an irreducible decomposition of the restriction $\pi|_{G'}$ into irreducible *unitary* representations of G'. However, symmetry breaking operators may exist between unitary and non-unitary representations:

Observation 4.6. *Suppose π is a unitary representation of G, and (τ, W) an irreducible admissible representation of a reductive subgroup G'.*

(1) *If* $\mathrm{Hom}_{G'}(\tau^\infty, \pi^\infty|_{G'}) \neq \{0\}$*, then τ^∞ is unitarizable. Actually, τ occurs as a discrete part of* $(\pi|_{G'})_{\mathrm{disc}}$ *(see (3.2)).*
(2) *It may well happen that* $\mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty) \neq \{0\}$ *even when τ^∞ is not unitarizable.*

In fact, the first assertion is obtained by taking the completion of $\varphi(W^\infty)$ in the Hilbert space V for $\varphi \in \mathrm{Hom}_{G'}(\tau^\infty, \pi^\infty|_{G'})$ as in the proof of Theorem 4.1 (3), where we considered the case $(\pi|_{G'})^\infty$ instead of $\pi^\infty|_{G'}$. Theorem 2.9 gives an example of Observation 4.6 (2).

Here is another example that indicates a large difference between the two spaces, $\mathrm{Hom}_{G'}(\tau^\infty, \pi^\infty|_{G'})$ and $\mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty)$.

Example 4.7. Suppose G is a real simple connected Lie group, and G' is a noncompact closed subgroup of G. Let π be any irreducible unitary representation such that $\dim \pi = \infty$ and $\mathrm{Hom}_G(\pi^\infty, C^\infty(G/G')) \neq \{0\}$. Then by Howe–Moore [20] we have

$$\mathrm{Hom}_{G'}(\mathbf{1}, \pi^\infty|_{G'}) = \{0\} \neq \mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \mathbf{1}).$$

5 Features of the restriction, I : $\mathrm{Hom}_{G'}(\tau, \pi|_{G'})$ (embedding)

In this section, we discuss Case I in Section 4, namely G'-homomorphisms from irreducible G'-modules τ into irreducible G-modules π. We put emphasis on its algebraic analogue in the category $\mathcal{HC}$ of Harish-Chandra modules.

The goals of this section are

(1) (criterion) to find a criterion for the triple (G, G', π) such that

$$\mathrm{Hom}_{\mathfrak{g}', K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'}) \neq \{0\} \quad \text{for some } \tau; \tag{5.1}$$

(2) (classification theory) to classify the pairs (G, G') of reductive groups for which (5.1) occurs for at least one infinite-dimensional $\pi \in \widehat{G}$.

We also discuss recent progress in this direction as a refinement of (2):

(2)′ (classification theory) Classify the triples (G, G', π) for which (5.1) occurs in typical cases (e.g., π_K is Zuckerman's $A_{\mathfrak{q}}(\lambda)$ module, or a minimal representation).

In Section 7 we shall explain two new applications of discretely decomposable restrictions: one is a dimension estimate of differential symmetry breaking operators (Theorem 7.13), and the other is a proof of the "localness property" of symmetry breaking operators (Theorem 7.18); see Observation 2.5 (1).

5.1 Criteria for discrete decomposability of restriction

We review a necessary and sufficient condition for the restriction of Harish-Chandra modules to be discretely decomposable (Definition 4.3), which was established in [28] and [29].

An associated variety $\mathcal{V}_{\mathfrak{g}}(X)$ is a coarse approximation of the $\mathfrak{g}$-modules X, which we recall now from Vogan [72]. We shall use the associated variety for the study of the restrictions of Harish-Chandra modules.

Let $\{U_j(\mathfrak{g})\}_{j \in \mathbb{N}}$ be the standard increasing filtration of the universal enveloping algebra $U(\mathfrak{g})$. Suppose X is a finitely generated $\mathfrak{g}$-module. A filtration $\bigcup_{i \in \mathbb{N}} X_i = X$ is called a *good filtration* if it satisfies the following conditions:

- X_i is finite-dimensional for any $i \in \mathbb{N}$;
- $U_j(\mathfrak{g}) X_i \subset X_{i+j}$ for any $i, j \in \mathbb{N}$;
- There exists n such that $U_j(\mathfrak{g}) X_i = X_{i+j}$ for any $i \geq n$ and $j \in \mathbb{N}$.

The graded algebra $\operatorname{gr} U(\mathfrak{g}) := \bigoplus_{j \in \mathbb{N}} U_j(\mathfrak{g})/U_{j-1}(\mathfrak{g})$ is isomorphic to the symmetric algebra $S(\mathfrak{g})$ by the Poincaré–Birkhoff–Witt theorem and we regard the graded module $\operatorname{gr} X := \bigoplus_{i \in \mathbb{N}} X_i / X_{i-1}$ as an $S(\mathfrak{g})$-module. Define

$$\operatorname{Ann}_{S(\mathfrak{g})}(\operatorname{gr} X) := \{f \in S(\mathfrak{g}) : f v = 0 \text{ for any } v \in \operatorname{gr} X\},$$
$$\mathcal{V}_{\mathfrak{g}}(X) := \{x \in \mathfrak{g}^* : f(x) = 0 \text{ for any } f \in \operatorname{Ann}_{S(\mathfrak{g})}(\operatorname{gr} X)\}.$$

Then $\mathcal{V}_{\mathfrak{g}}(X)$ does not depend on the choice of a good filtration and is called the *associated variety* of X. We denote by $\mathcal{N}(\mathfrak{g}^*)$ the nilpotent variety of the dual space $\mathfrak{g}^*$. We have then the following basic properties of the associated variety [72].

Lemma 5.1. *Let X be a finitely generated $\mathfrak{g}$-module.*

(1) *If X is of finite length, then $\mathcal{V}_{\mathfrak{g}}(X) \subset \mathcal{N}(\mathfrak{g}^*)$.*
(2) *$\mathcal{V}_{\mathfrak{g}}(X) = \{0\}$ if and only if X is finite-dimensional.*
(3) *Let $\mathfrak{h}$ be a Lie subalgebra of $\mathfrak{g}$. Then $\mathcal{V}_{\mathfrak{g}}(X) \subset \mathfrak{h}^{\perp}$ if $\mathfrak{h}$ acts locally finitely on X, where $\mathfrak{h}^{\perp} := \{x \in \mathfrak{g}^* : x|_{\mathfrak{h}} = 0\}$.*

(1) and (3) imply that if X is a $(\mathfrak{g}, K)$-module of finite length, then $\mathcal{V}_{\mathfrak{g}}(X)$ is a $K_{\mathbb{C}}$-stable closed subvariety of $\mathcal{N}(\mathfrak{p}^*)$ because $\mathfrak{k}^{\perp} = \mathfrak{p}^*$.

Dual to the inclusion $\mathfrak{g}' \subset \mathfrak{g}$ of the Lie algebras, we write

$$\mathrm{pr} : \mathfrak{g}^* \to (\mathfrak{g}')^*$$

for the restriction map.

One might guess that irreducible summands of the restriction $\pi|_{G'}$ would be "large" if the irreducible representation π of G is "large". The following theorem shows that such a statement holds if the restriction of the Harish-Chandra module is discretely decomposable (Definition 4.3); however, it is false in general (see Counterexample 5.4 below).

Fact 5.2. *Let X be an irreducible $(\mathfrak{g}, K)$-module.*

(1) *If Y is an irreducible $(\mathfrak{g}', K')$-module such that* $\mathrm{Hom}_{\mathfrak{g}',K'}(Y, X|_{\mathfrak{g}'}) \neq \{0\}$, *then*

$$\mathrm{pr}(\mathcal{V}_{\mathfrak{g}}(X)) \subset \mathcal{V}_{\mathfrak{g}'}(Y).$$

(2) *If $Y^{(j)}$ are irreducible $(\mathfrak{g}', K')$-modules such that* $\mathrm{Hom}_{\mathfrak{g}',K'}(Y^{(j)}, X|_{\mathfrak{g}'}) \neq \{0\}$ $(j = 1, 2)$, *then*

$$\mathcal{V}_{\mathfrak{g}'}(Y_1) = \mathcal{V}_{\mathfrak{g}'}(Y_2).$$

In particular, the Gelfand–Kirillov dimension GK-dim(Y) *of all irreducible $(\mathfrak{g}', K')$-submodules Y of $X|_{\mathfrak{g}'}$ are the same.*

(3) (necessary condition [29, Corollary 3.5]) *If X is discretely decomposable as a $(\mathfrak{g}', K')$-module, then* $\mathrm{pr}(\mathcal{V}_{\mathfrak{g}}(X)) \subset \mathcal{N}((\mathfrak{g}')^*)$, *where $\mathcal{N}((\mathfrak{g}')^*)$ is the nilpotent variety of $(\mathfrak{g}')^*$.*

An analogous statement fails if we replace $\mathrm{Hom}_{\mathfrak{g}',K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'})$ by the space $\mathrm{Hom}_{G'}(\tau, \pi|_{G'})$ of continuous G'-intertwining operators:

False Statement 5.3 *Let π be an irreducible unitary representation of a real reductive Lie group G.*

(1) *If $\tau \in \widehat{G'}$ satisfies* $\mathrm{Hom}_{G'}(\tau, \pi|_{G'}) \neq \{0\}$, *then* $\mathrm{pr}(\mathcal{V}_{\mathfrak{g}}(\pi_K)) \subset \mathcal{V}_{\mathfrak{g}'}(\tau_{K'})$.

(2) *If $\tau^{(j)} \in \widehat{G'}$ satisfy* $\mathrm{Hom}_{G'}(\tau^{(j)}, \pi|_{G'}) \neq \{0\}$ *$(j = 1, 2)$, then* $\mathcal{V}_{\mathfrak{g}'}(\tau^{(1)}_{K'}) = \mathcal{V}_{\mathfrak{g}'}(\tau^{(2)}_{K'})$.

Here are counterexamples to the "False Statement 5.3":

Counterexample 5.4 (1) *There are many triples (G, G', π) such that $\pi \in \widehat{G}$ satisfies $(\pi|_{G'})_{\mathrm{cont}} \neq 0$; see [26, Introduction], [33, Section 3.3], and Theorem 5.14, for instance. In this case,* $\mathrm{pr}(\mathcal{V}_{\mathfrak{g}}(\pi_K)) \not\subset \mathcal{V}_{\mathfrak{g}'}(\tau_{K'})$ *for any $\tau \in \widehat{G'}$ by Fact 5.2 (3).*

(2) *Let* $(G, G') = (G_1 \times G_1, \mathrm{diag}(G_1))$ *with* $G_1 = Sp(n, \mathbb{R})$ $(n \geq 2)$. *Take an irreducible unitary spherical principal series representation* π_1 *induced from the Siegel parabolic subgroup of* G_1, *and set* $\pi = \pi_1 \boxtimes \pi_1$. *Then there exist discrete series representations* $\tau^{(1)}$ *and* $\tau^{(2)}$ *of* G' $(\simeq Sp(n, \mathbb{R}))$, *where* $\tau^{(1)}$ *is a holomorphic discrete series representation and* $\tau^{(2)}$ *is a non-holomorphic discrete series representation, such that*

$$\mathrm{Hom}_{G'}(\tau^{(j)}, \pi) \neq \{0\} \quad (j = 1, 2) \quad \textit{and} \quad \text{GK-dim}(\tau^{(1)}) < \text{GK-dim}(\tau^{(2)}).$$

In fact, it follows from Theorem 5.14 below that $\mathrm{Hom}_{G'}(\tau, \pi) \neq \{0\}$ if and only if τ is a discrete series representation for the reductive symmetric space $Sp(n, \mathbb{R})/GL(n, \mathbb{R})$. Then using the description of discrete series representations [55, 71], we get Counterexample 5.4 (2).

We now turn to an analytic approach to the question of discrete decomposability of the restriction. For simplicity, assume K is connected. We take a maximal torus T of K, and write $\mathfrak{t}_0$ for its Lie algebra. Fix a positive system $\Delta^+(\mathfrak{k}, \mathfrak{t})$ and denote by C_+ $(\subset \sqrt{-1}\mathfrak{t}_0^*)$ the dominant Weyl chamber. We regard $\widehat{T}$ as a subset of $\sqrt{-1}\mathfrak{t}_0^*$, and set $\Lambda_+ := C_+ \cap \widehat{T}$. Then Cartan–Weyl highest weight theory gives a bijection

$$\Lambda_+ \simeq \widehat{K}, \qquad \lambda \mapsto \tau_\lambda.$$

We recall for a subset S of $\mathbb{R}^N$, the asymptotic cone $S\infty$ is the closed cone defined by

$$S\infty := \{y \in \mathbb{R}^N : \text{ there exists a sequence } (y_n, \varepsilon_n) \in S \times \mathbb{R}_{>0} \text{ such that } \lim_{n\to\infty} \varepsilon_n y_n = y \text{ and } \lim_{n\to\infty} \varepsilon_n = 0\}.$$

The asymptotic K-support $AS_K(X)$ of a K-module X is defined by Kashiwara and Vergne [22] as the asymptotic cone of the highest weights of irreducible K-modules occurring in X:

$$AS_K(X) := \mathrm{Supp}_K(X)\infty,$$

where $\mathrm{Supp}_K(X)$ is the K-support of X given by

$$\mathrm{Supp}_K(X) := \{\lambda \in \Lambda_+ : \mathrm{Hom}_K(\tau_\lambda, X) \neq \{0\}\}.$$

For a closed subgroup K' of K, we write $\mathfrak{k}_0'$ for its Lie algebra, and regard $(\mathfrak{k}_0')^\perp = \mathrm{Ker}(\mathrm{pr} : \mathfrak{k}_0^* \to (\mathfrak{k}_0')^*)$ as a subspace of $\mathfrak{k}_0$ via a K-invariant inner product on $\mathfrak{k}_0$. We set

$$C_K(K') := C_+ \cap \sqrt{-1}\,\mathrm{Ad}^*(K)(\mathfrak{k}_0')^\perp.$$

An estimate of the singularity spectrum of the hyperfunction K-character of X yields a criterion of "K'-admissibility" of X for a subgroup K' of K ([28, Theorem 2.8] and [33]):

Fact 5.5. *Let* $G \supset G'$ *be a pair of real reductive linear Lie groups with compatible maximal compact subgroups* $K \supset K'$, *and* X *an irreducible* $(\mathfrak{g}, K)$*-module.*

(1) *The following two conditions on the triple* (G, G', X) *are equivalent:*

(i) *X is K'-admissible, i.e.,* $\dim \operatorname{Hom}_{K'}(\tau, X|_{K'}) < \infty$ *for all* $\tau \in \widehat{K'}$.
(ii) $C_K(K') \cap \operatorname{AS}_K(X) = \{0\}$.

(2) *If one of (therefore either of) (i) and (ii) is satisfied, then X is discretely decomposable as a* $(\mathfrak{g}', K')$*-module.*

5.2 Classification theory of discretely decomposable pairs

We begin with two observations.

First, for a Riemannian symmetric pair, that is, $(G, G') = (G, K)$ where $G' = K' = K$, the restriction $X|_{\mathfrak{g}'}$ is obviously discretely decomposable as a $(\mathfrak{g}', K')$-module for any irreducible $(\mathfrak{g}, K)$-module X, whereas the reductive pair $(G, G') = (\mathrm{SL}(n, \mathbb{C}), \mathrm{SL}(n, \mathbb{R}))$ is an opposite extremal case as the restriction $X|_{\mathfrak{g}'}$ is never discretely decomposable as a $(\mathfrak{g}', K')$-module for any infinite-dimensional irreducible $(\mathfrak{g}, K)$-module X ([29]). There are also intermediate cases such as $(G, G') = (\mathrm{SL}(n, \mathbb{R}), \mathrm{SO}(p, n-p))$ for which the restriction $X|_{\mathfrak{g}'}$ is discretely decomposable for some infinite-dimensional irreducible $(\mathfrak{g}, K)$-modules X and is not for some other X.

Secondly, Harish-Chandra's admissibility theorem [16] asserts that

$$\dim_{\mathbb{C}} \operatorname{Hom}_K(\tau, \pi|_K) < \infty$$

for any $\pi \in \widehat{G}$ and $\tau \in \widehat{K}$.

This may be regarded as a statement for a Riemannian symmetric pair $(G, G') = (G, K)$. Unfortunately, there is a counterexample to an analogous statement for the reductive symmetric pair $(G, G') = (\mathrm{SO}(5, \mathbb{C}), \mathrm{SO}(3, 2))$, namely, we proved in [32] that

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G'}(\tau, \pi|_{G'}) = \infty \quad \text{for some } \pi \in \widehat{G} \text{ and } \tau \in \widehat{G'}.$$

However, it is plausible [32, Conjecture A] to have a generalization of Harish-Chandra's admissibility in the category $\mathcal{HC}$ of Harish-Chandra modules in the following sense:

$$\dim \operatorname{Hom}_{\mathfrak{g}', K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'}) < \infty$$

for any irreducible $(\mathfrak{g}, K)$-module π_K and irreducible $(\mathfrak{g}', K')$-module $\tau_{K'}$.

In view of these two observations, we consider the following conditions (a) – (d) for a pair of real reductive Lie groups (G, G'), and raise a problem:

Problem 5.6. Classify the pairs (G, G') of real reductive Lie groups satisfying the condition (a) below (and also (b), (c) or (d)).

(a) there exist an infinite-dimensional irreducible unitary representation π of G and an irreducible unitary representation τ of G' such that

$$0 < \dim \mathrm{Hom}_{\mathfrak{g}',K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'}) < \infty;$$

(b) there exist an infinite-dimensional irreducible unitary representation π of G and an irreducible unitary representation τ of G' such that

$$0 < \dim \mathrm{Hom}_{\mathfrak{g}',K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'});$$

(c) there exist an infinite-dimensional irreducible $(\mathfrak{g}, K)$-module X and an irreducible $(\mathfrak{g}', K')$-module Y such that

$$0 < \dim \mathrm{Hom}_{\mathfrak{g}',K'}(Y, X|_{\mathfrak{g}'}) < \infty;$$

(d) there exist an infinite-dimensional irreducible $(\mathfrak{g}, K)$-module X and an irreducible $(\mathfrak{g}', K')$-module Y such that

$$0 < \dim \mathrm{Hom}_{\mathfrak{g}',K'}(Y, X|_{\mathfrak{g}'}).$$

Obviously we have the following implications:

$$\begin{array}{ccc} \text{(a)} & \Rightarrow & \text{(b)} \\ \Downarrow & & \Downarrow \\ \text{(c)} & \Rightarrow & \text{(d).} \end{array}$$

The vertical (inverse) implications (c) $\Rightarrow$ (a) and (d) $\Rightarrow$ (b) will mean finite-multiplicity results like Harish-Chandra's admissibility theorem.

For symmetric pairs, Problem 5.6 has been solved in [50, Theorem 5.2]:

Theorem 5.7. *Let (G, G') be a reductive symmetric pair defined by an involutive automorphism σ of a simple Lie group G. Then the following five conditions (a), (b), (c), (d), and*

$$\sigma\beta \neq -\beta \tag{5.2}$$

are equivalent. Here β is the highest noncompact root with respect to a "$(-\sigma)$-compatible" positive system. (See [50] for a precise definition.)

Example 5.8. (1) $\sigma = \theta$ (Cartan involution). Then (5.2) is obviously satisfied because $\theta\beta = \beta$. Needless to say, the conditions (a)–(d) hold when $G' = K$.

(2) The reductive symmetric pairs $(G, G') = (\mathrm{SO}(p_1+p_2, q), \mathrm{SO}(p_1)\times\mathrm{SO}(p_2, q))$, $(\mathrm{SL}(2n, \mathbb{R}), \mathrm{Sp}(n, \mathbb{C}))$, $(\mathrm{SL}(2n, \mathbb{R}), \mathbb{T} \cdot \mathrm{SL}(n, \mathbb{C}))$ satisfy (5.2), and therefore (a)–(d).

The classification of irreducible symmetric pairs (G, G') satisfying one of (therefore all of) (a)–(d) was given in [50]. It turns out that there are fairly many reductive symmetric pairs (G, G') satisfying the five equivalent conditions in Theorem 5.7

when G does not carry a complex Lie group structure, whereas there are a few such pairs (G, G') when G is a complex Lie group. As a flavor of the classification, we present a list in this particular case. For this, we use the following notation, which is slightly different from that used in other parts of this article. Let $G_{\mathbb{C}}$ be a complex simple Lie group, and $G_{\mathbb{R}}$ a real form. Take a maximal compact subgroup $K_{\mathbb{R}}$ of $G_{\mathbb{R}}$, and let $K_{\mathbb{C}}$ be the complexification of $K_{\mathbb{R}}$ in $G_{\mathbb{C}}$. We denote by $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{g}_{\mathbb{R}}$ the Lie algebras of $G_{\mathbb{C}}$, $K_{\mathbb{C}}$, and $G_{\mathbb{R}}$, respectively, and write $\mathfrak{g} = \mathfrak{k} + \mathfrak{p}$ for the complexified Cartan decomposition.

Example 5.9 ([50, Corollary 5.9]). The following five conditions on the pairs $(G_{\mathbb{C}}, G_{\mathbb{R}})$ are equivalent:

(i) $(G_{\mathbb{C}}, K_{\mathbb{C}})$ satisfies (a) (or equivalently, (b), (c), or (d)).
(ii) $(G_{\mathbb{C}}, G_{\mathbb{R}})$ satisfies (a) (or equivalently, (b), (c), or (d)).
(iii) The minimal nilpotent orbit of $\mathfrak{g}$ does not intersect $\mathfrak{g}_{\mathbb{R}}$.
(iv) The minimal nilpotent orbit of $\mathfrak{g}$ does not intersect $\mathfrak{p}$.
(v) The Lie algebras $\mathfrak{g}$, $\mathfrak{k}$, and $\mathfrak{g}_{\mathbb{R}}$ are given in the following table:

$\mathfrak{g}$	$\mathfrak{sl}(2n,\mathbb{C})$	$\mathfrak{so}(m,\mathbb{C})$	$\mathfrak{sp}(p+q,\mathbb{C})$	$\mathfrak{f}_4^{\mathbb{C}}$	$\mathfrak{e}_6^{\mathbb{C}}$
$\mathfrak{k}$	$\mathfrak{sp}(n,\mathbb{C})$	$\mathfrak{so}(m-1,\mathbb{C})$	$\mathfrak{sp}(p,\mathbb{C})+\mathfrak{sp}(q,\mathbb{C})$	$\mathfrak{so}(9,\mathbb{C})$	$\mathfrak{f}_4^{\mathbb{C}}$
$\mathfrak{g}_{\mathbb{R}}$	$\mathfrak{su}^*(2n)$	$\mathfrak{so}(m-1,1)$	$\mathfrak{sp}(p,q)$	$\mathfrak{f}_{4(-20)}$	$\mathfrak{e}_{6(-26)}$

where $m \geq 5$ and $n, p, q \geq 1$.

Remark 5.10. The equivalence (iv) and (v) was announced by Brylinski–Kostant in the context that there is no minimal representation of a Lie group $G_{\mathbb{R}}$ with the Lie algebra $\mathfrak{g}_{\mathbb{R}}$ in the above table (see [7]). The new ingredient here is that this condition on the Lie algebras corresponds to a question of discretely decomposable restrictions of Harish-Chandra modules.

For nonsymmetric pairs, there are a few nontrivial cases where (a) (and therefore (b), (c), and (d)) holds, as follows.

Example 5.11 ([26]). The nonsymmetric pairs $(G, G') = (\mathrm{SO}(4,3), \mathrm{G}_{2(2)})$ and $(\mathrm{SO}(7,\mathbb{C}), \mathrm{G}_2^{\mathbb{C}})$ satisfy (a) (and also (b), (c), and (d)).

Once we classify the pairs (G, G') such that there exists at least one irreducible infinite-dimensional $(\mathfrak{g}, K)$-module X which is discretely decomposable as a $(\mathfrak{g}', K')$-module, then we would like to find all such Xs.

In [49] we carried out this project for $X = A_{\mathfrak{q}}(\lambda)$ by applying the general criterion (Facts 5.2 and 5.5) to reductive symmetric pairs (G, G'). This is a result in Stage A of the branching problem, and we think it will serve as a foundational result for Stage B (explicit branching laws). Here is another example of the classification of the triples (G, G', X) when $G \simeq G' \times G'$, see [50, Theorem 6.1]:

Example 5.12 (tensor product). Let G be a noncompact connected simple Lie group, and let X_j $(j = 1, 2)$ be infinite-dimensional irreducible $(\mathfrak{g}, K)$-modules.

(1) Suppose G is not of Hermitian type. Then the tensor product representation $X_1 \otimes X_2$ is never discretely decomposable as a $(\mathfrak{g}, K)$-module.

(2) Suppose G is of Hermitian type. Then the tensor product representation $X_1 \otimes X_2$ is discretely decomposable as a $(\mathfrak{g}, K)$-module if and only if both X_1 and X_2 are simultaneously highest weight $(\mathfrak{g}, K)$-modules or simultaneously lowest weight $(\mathfrak{g}, K)$-modules.

5.3 Two spaces $\mathrm{Hom}_{G'}(\tau, \pi|_{G'})$ *and* $\mathrm{Hom}_{\mathfrak{g}',K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'})$

There is a canonical injective homomorphism

$$\mathrm{Hom}_{\mathfrak{g}',K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'}) \hookrightarrow \mathrm{Hom}_{G'}(\tau, \pi|_{G'}),$$

however, it is not bijective for $\tau \in \widehat{G'}$ and $\pi \in \widehat{G}$. In fact, we have:

Proposition 5.13. *Suppose that π is an irreducible unitary representation of G. If the restriction $\pi|_{G'}$ contains a continuous spectrum and if an irreducible unitary representation τ of G' appears as an irreducible summand of the restriction $\pi|_{G'}$, then we have*

$$\mathrm{Hom}_{\mathfrak{g}',K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'}) = \{0\} \neq \mathrm{Hom}_{G'}(\tau, \pi|_{G'}).$$

Proof. If $\mathrm{Hom}_{\mathfrak{g}'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'}) = \mathrm{Hom}_{\mathfrak{g}',K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'})$ were nonzero, then the $(\mathfrak{g}, K)$-module π_K would be discretely decomposable as a $(\mathfrak{g}', K')$-module by Theorem 4.5. In turn, the restriction $\pi|_{G'}$ of the unitary representation π would decompose discretely into a Hilbert direct sum of irreducible unitary representations of G' by [32, Theorem 2.7], contradicting the assumption. Hence we conclude $\mathrm{Hom}_{\mathfrak{g}',K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'}) = \{0\}$. □

An example of Proposition 5.13 may be found in [45, Part II] where π is the minimal representation of $G = O(p, q)$ and τ is the unitarization of a Zuckerman derived functor module $A_{\mathfrak{q}}(\lambda)$ for $G' = O(p', q') \times O(p'', q'')$ with $p = p' + p''$ and $q = q' + q''$ ($p', q', p'', q'' > 1$ and $p + q$ even).

Here is another example of Proposition 5.13:

Theorem 5.14. *Let G be a real reductive linear Lie group, and let $\pi = \mathrm{Ind}_P^G(\mathbb{C}_\lambda)$ be a spherical unitary degenerate principal series representation of G induced from a unitary character $\mathbb{C}_\lambda$ of a parabolic subgroup $P = LN$ of G.*

(1) *For any irreducible $(\mathfrak{g}, K)$-module τ_K, we have*

$$\mathrm{Hom}_{\mathfrak{g},K}(\tau_K, \pi_K \otimes \pi_K) = \{0\}.$$

(2) *Suppose now G is a classical group. If N is abelian and P is conjugate to the opposite parabolic subgroup $\overline{P} = L\overline{N}$, then we have a unitary equivalence of the discrete part:*

$$L^2(G/L)_{\mathrm{disc}} \simeq \sum_{\tau \in \widehat{G}}^{\oplus} \dim_{\mathbb{C}} \mathrm{Hom}_G(\tau, \pi \widehat{\otimes} \pi)\, \tau. \tag{5.3}$$

In particular, we have

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(\tau, \pi \widehat{\otimes} \pi) \leq 1$$

for any irreducible unitary representation τ of G. Moreover there exist countably many irreducible unitary representations τ of G such that

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(\tau, \pi \widehat{\otimes} \pi) = 1.$$

A typical example of the setting in Theorem 5.14 (2) is the Siegel parabolic subgroup $P = LN = GL(n, \mathbb{R}) \ltimes \operatorname{Sym}(n, \mathbb{R})$ in $G = Sp(n, \mathbb{R})$.

Proof. (1) This is a direct consequence of Example 5.12.
(2) Take $w_0 \in G$ such that $w_0 L w_0^{-1} = L$ and $w_0 N w_0^{-1} = \overline{N}$. Then the G-orbit through $(w_0 P, eP)$ in $G/P \times G/P$ under the diagonal action is open dense, and therefore Mackey theory gives a unitary equivalence

$$L^2(G/L) \simeq \pi_\lambda \widehat{\otimes} \pi_\lambda \tag{5.4}$$

because $\operatorname{Ad}^*(w_0)\lambda = -\lambda$, see [30] for instance. Since N is abelian, (G, L) forms a symmetric pair (see [64]). Therefore the branching law of the tensor product representation $\pi \widehat{\otimes} \pi$ reduces to the Plancherel formula for the regular representation on the reductive symmetric space G/L, which is known; see [10]. In particular, we have the unitary equivalence (5.3), and the left-hand side of (5.3) is nonzero if and only if $\operatorname{rank} G/L = \operatorname{rank} K/L \cap K$ due to Flensted-Jensen and Matsuki–Oshima [55]. By the description of discrete series representation for G/L by Matsuki–Oshima [55] and Vogan [71], we have the conclusion. $\square$

5.4 Analytic vectors and discrete decomposability

Suppose π is an irreducible unitary representation of G on a Hilbert space V, and G' is a reductive subgroup of G as before. Any G'-invariant closed subspace W in V contains G'-analytic vectors (hence, also G'-smooth vectors) as a dense subspace. However, W may not contain nonzero G-smooth vectors (hence, also G-analytic vectors). In view of Theorem 4.5 in the category $\mathcal{HC}$ of Harish-Chandra modules, we think that this is related to the existence of a continuous spectrum in the branching law of the restriction $\pi|_{G'}$. We formulate a problem related to this delicate point below. As before, π^∞ and τ^∞ denote the space of G-smooth vectors and G'-smooth vectors for representations π and τ of G and G', respectively. An analogous notation is applied to π^ω and τ^ω.

Problem 5.15. Let (π, V) be an irreducible unitary representation of G, and G' a reductive subgroup of G. Are the following four conditions on the triple (G, G', π) equivalent?

(i) There exists an irreducible $(\mathfrak{g}', K')$-module $\tau_{K'}$ such that

$$\mathrm{Hom}_{\mathfrak{g}',K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'}) \neq \{0\}.$$

(ii) There exists an irreducible unitary representation τ of G' such that

$$\mathrm{Hom}_{G'}(\tau^{\omega}, \pi^{\omega}|_{G'}) \neq \{0\}.$$

(iii) There exists an irreducible unitary representation τ of G' such that

$$\mathrm{Hom}_{G'}(\tau^{\infty}, \pi^{\infty}|_{G'}) \neq \{0\}.$$

(iv) The restriction $\pi|_{G'}$ decomposes discretely into a Hilbert direct sum of irreducible unitary representations of G'.

Here are some remarks on Problem 5.15.

Remark 5.16. (1) In general, the implication (i) $\Rightarrow$ (iv) holds ([32, Theorem 2.7]).
(2) If the restriction $\pi|_{K'}$ is K'-admissible, then (i) holds by [29, Proposition 1.6] and (iv) holds by [26, Theorem 1.2].
(3) The implication (iv) $\Rightarrow$ (i) was raised in [32, Conjecture D], and some affirmative results have been announced by Duflo and Vargas in a special setting where π is Harish-Chandra's discrete series representation (cf. [12]). A related result is given in [77].
(4) Even when the unitary representation $\pi|_{G'}$ decomposes discretely (i.e., (iv) in Problem 5.15 holds), it may happen that $V^{\infty} \subsetneqq (V|_{G'})^{\infty}$. The simplest example for this is as follows. Let (π', V') and (π'', V'') be infinite-dimensional unitary representations of noncompact Lie groups G' and G'', respectively. Set $G = G' \times G''$, with G' realized as a subgroup of G as $G' \times \{e\}$, and set $\pi = \pi' \boxtimes \pi''$. Then $V^{\infty} \subsetneqq (V|_{G'})^{\infty}$ because $(V'')^{\infty} \subsetneqq V''$.

6 Features of the restriction, II : $\mathrm{Hom}_{G'}(\pi|_{G'}, \tau)$ (symmetry breaking operators)

In the previous section, we discussed embeddings of irreducible G'-modules τ into irreducible G-modules π (or the analogous problem in the category $\mathcal{HC}$ of Harish-Chandra modules); see Case I in Section 4. In contrast, we consider the opposite order in this section, namely, continuous G'-homomorphisms from irreducible G-modules π to irreducible G'-modules τ, see Case II in Section 4. We highlight the case where π and τ are admissible smooth representations (Casselman–Wallach globalization of modules in the category $\mathcal{HC}$). Then it turns out that the spaces $\mathrm{Hom}_{G'}(\pi^{\infty}|_{G'}, \tau^{\infty})$ or $\mathrm{Hom}_{\mathfrak{g}',K'}(\pi_K|_{\mathfrak{g}'}, \tau_{K'})$ are much larger in general than the spaces $\mathrm{Hom}_{G'}(\tau^{\infty}, \pi^{\infty}|_{G'})$ or $\mathrm{Hom}_{\mathfrak{g}',K'}(\tau_{K'}, \pi_K|_{\mathfrak{g}'})$ considered in Section 5. Thus the primary concern here will be with obtaining an upper estimate for the dimensions of those spaces.

It would make reasonable sense to find branching laws (Stage B) or to construct symmetry breaking operators (Stage C) if we know *a priori* the nature of the multiplicities in branching laws. The task of Stage A of the branching problem is to establish a criterion and to give a classification of desirable settings. In this section, we consider:

Problem 6.1. (1) (finite multiplicities) Find a criterion for when a pair (G, G') of real reductive Lie groups satisfies

$$\dim \mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty) < \infty \quad \text{for any } \pi^\infty \in \widehat{G}_{\mathrm{smooth}} \text{ and } \tau^\infty \in \widehat{G'}_{\mathrm{smooth}}.$$

Classify all such pairs (G, G').

(2) (uniformly bounded multiplicities) Find a criterion for when a pair (G, G') of real reductive Lie groups satisfies

$$\sup_{\pi^\infty \in \widehat{G}_{\mathrm{smooth}}} \sup_{\tau^\infty \in \widehat{G'}_{\mathrm{smooth}}} \dim \mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty) < \infty.$$

Classify all such pairs (G, G').

One may also think of variants of Problem 6.1. For instance, we may refine Problem 6.1 by considering it as a condition on the triple (G, G', π) instead of a condition on the pair (G, G'):

Problem 6.2. (1) Classify the triples (G, G', π^∞) with $G \supset G'$ and $\pi^\infty \in \widehat{G}_{\mathrm{smooth}}$ such that

$$\dim \mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty) < \infty \quad \text{for any } \tau^\infty \in \widehat{G'}_{\mathrm{smooth}}. \tag{6.1}$$

(2) Classify the triples (G, G', π^∞) such that

$$\sup_{\tau^\infty \in \widehat{G'}_{\mathrm{smooth}}} \dim \mathrm{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty) < \infty. \tag{6.2}$$

Problem 6.1 has been solved recently for all reductive symmetric pairs (G, G'); see Sections 6.3 and 6.4. On the other hand, Problem 6.2 has no complete solution even when (G, G') is a reductive symmetric pair. Here are some partial answers to Problem 6.2 (1):

Example 6.3. (1) If (G, G') satisfies (PP) (see the list in Theorem 6.14), then the triple (G, G', π) satisfies (6.1) whenever $\pi^\infty \in \widehat{G}_{\mathrm{smooth}}$.

(2) If π is K'-admissible, then (6.1) is satisfied. A necessary and sufficient condition for the K'-admissibility of $\pi|_{K'}$, Fact 5.5, is easy to check in many cases. In particular, a complete classification of the triples (G, G', π) such that $\pi|_{K'}$ is K'-admissible was recently accomplished in [49] in the setting where $\pi_K = A_{\mathfrak{q}}(\lambda)$ and where (G, G') is a reductive symmetric pair.

We give a conjectural statement concerning Problem 6.2 (2).

Conjecture 6.4. Let (G, G') be a reductive symmetric pair. If π is an irreducible highest weight representation of G or if π is a minimal representation of G, then the uniform boundedness property (6.2) would hold for the triple (G, G', π^∞).

Some evidence was given in [35, Theorems B and D] and in [45, 46].

6.1 Real spherical homogeneous spaces

A complex manifold $X_{\mathbb{C}}$ with an action of a complex reductive group $G_{\mathbb{C}}$ is called *spherical* if a Borel subgroup of $G_{\mathbb{C}}$ has an open orbit in $X_{\mathbb{C}}$. Spherical varieties have been studied extensively in the context of algebraic geometry and finite-dimensional representation theory. In the real setting, in search of a broader framework for global analysis on homogeneous spaces than the usual (e.g., reductive symmetric spaces), we propose the following:

Definition 6.5 ([27]). Let G be a real reductive Lie group. We say a connected smooth manifold X with G-action is *real spherical* if a minimal parabolic subgroup P of G has an open orbit in X, or equivalently $\#(P\backslash X) < \infty$.

The equivalence in Definition 6.5 was proved in [5] by using Kimelfeld [23] and Matsuki [54]; see [48, Remark] and references therein for related earlier results.

Here are some partial results on the classification of real spherical homogeneous spaces.

Example 6.6. (1) If G is compact, then all G-homogeneous spaces are real spherical.
(2) Any semisimple symmetric space G/H is real spherical. The (infinitesimal) classification of semisimple symmetric spaces was accomplished by Berger [3].
(3) G/N is real spherical where N is a maximal unipotent subgroup of G.
(4) For G of real rank one, real spherical homogeneous spaces of G are classified by Kimelfeld [23].
(5) Any real form G/H of a spherical homogeneous space $G_{\mathbb{C}}/H_{\mathbb{C}}$ is real spherical [48, Lemma 4.2]. The latter were classified by Krämer [53], Brion, [6], and Mikityuk [56]. In particular, if G is quasi-split, then the classification problem of real spherical homogeneous spaces G/H reduces to that of the known classification of spherical homogeneous spaces.
(6) The triple product space $(G \times G \times G)/\operatorname{diag} G$ is real spherical if and only if G is locally isomorphic to the direct product of compact Lie groups and some copies of $O(n, 1)$ (Kobayashi [27]).
(7) Real spherical homogeneous spaces of the form $(G \times G')/\operatorname{diag} G'$ for symmetric pairs (G, G') were recently classified. We review this in Theorem 6.14 below.

The second and third examples form the basic geometric settings for analysis on reductive symmetric spaces and Whittaker models. The last two examples play a role in Stage A of the branching problem, as we see in the next subsection.

The significance of this geometric property is that the group G controls the space of functions on X in the sense that the finite-multiplicity property holds for the regular representation of G on $C^\infty(X)$:

Fact 6.7 ([48, Theorems A and C]). *Suppose G is a real reductive linear Lie group, and H is an algebraic reductive subgroup.*

(1) *The homogeneous space G/H is real spherical if and only if*

$$\mathrm{Hom}_G(\pi^\infty, C^\infty(G/H)) \text{ is finite-dimensional for all } \pi^\infty \in \widehat{G}_{\mathrm{smooth}}.$$

(2) *The complexification $G_{\mathbb{C}}/H_{\mathbb{C}}$ is spherical if and only if*

$$\sup_{\pi^\infty \in \widehat{G}_{\mathrm{smooth}}} \dim_{\mathbb{C}} \mathrm{Hom}_G(\pi^\infty, C^\infty(G/H)) < \infty.$$

See [48] for upper and lower estimates of the dimension, and also for the non-reductive case. The proof uses the theory of regular singularities of a system of partial differential equations by taking an appropriate compactification with normal crossing boundaries.

6.2 A geometric estimate of multiplicities : (PP) and (BB)

Suppose that G' is an algebraic reductive subgroup of G. For Stage A in the branching problem for the restriction $G \downarrow G'$, we apply the general theory of Section 6.1 to the homogeneous space $(G \times G')/\operatorname{diag} G'$.

Let P be a minimal parabolic subgroup of G, and P' a minimal parabolic subgroup of G'.

Definition-Lemma 6.8 ([48]) *We say the pair (G, G') satisfies the property* (PP) *if one of the following five equivalent conditions is satisfied:*
(PP1) *$(G \times G')/\operatorname{diag} G'$ is real spherical as a $(G \times G')$-space.*
(PP2) *G/P' is real spherical as a G-space.*
(PP3) *G/P is real spherical as a G'-space.*
(PP4) *G has an open orbit in $G/P \times G/P'$ via the diagonal action.*
(PP5) *$\#(P'\backslash G/P) < \infty$.*

Since the above five equivalent conditions are determined by the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}'$, we also say that the pair $(\mathfrak{g}, \mathfrak{g}')$ of reductive Lie algebras satisfies (PP), where $\mathfrak{g}$ and $\mathfrak{g}'$ are the Lie algebras of the Lie groups G and G', respectively.

Remark 6.9. If the pair $(\mathfrak{g}, \mathfrak{g}')$ satisfies (PP), in particular, (PP5), then there are only finitely many possibilities for $\operatorname{Supp} T$ for symmetry breaking operators $T : C^\infty(G/P, \mathcal{V}) \to C^\infty(G'/P', \mathcal{W})$ (see Definition 7.9 below). This observation has become a guiding principle to formalise a strategy in classifying all symmetry breaking operators used in [52], as we shall discuss in Section 7.2.

Next we consider another property, to be denoted (BB), which is stronger than (PP). Let $G_{\mathbb{C}}$ be a complex Lie group with Lie algebra $\mathfrak{g}_{\mathbb{C}} = \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$, and $G'_{\mathbb{C}}$ a subgroup of $G_{\mathbb{C}}$ with complexified Lie algebra $\mathfrak{g}'_{\mathbb{C}} = \mathfrak{g}' \otimes_{\mathbb{R}} \mathbb{C}$. We do not assume either $G \subset G_{\mathbb{C}}$ or $G' \subset G'_{\mathbb{C}}$. Let $B_{\mathbb{C}}$ and $B'_{\mathbb{C}}$ be Borel subgroups of $G_{\mathbb{C}}$ and $G'_{\mathbb{C}}$, respectively.

Definition-Lemma 6.10 *We say the pair* (G, G') *(or the pair* $(\mathfrak{g}, \mathfrak{g}')$*) satisfies the property* (BB) *if one of the following five equivalent conditions is satisfied:*
(BB1) $(G_{\mathbb{C}} \times G'_{\mathbb{C}})/\operatorname{diag} G'_{\mathbb{C}}$ *is spherical as a* $(G_{\mathbb{C}} \times G'_{\mathbb{C}})$*-space.*
(BB2) $G_{\mathbb{C}}/B'_{\mathbb{C}}$ *is spherical as a* $G_{\mathbb{C}}$*-space.*
(BB3) $G_{\mathbb{C}}/B_{\mathbb{C}}$ *is spherical as a* $G'_{\mathbb{C}}$*-space.*
(BB4) $G_{\mathbb{C}}$ *has an open orbit in* $G_{\mathbb{C}}/B_{\mathbb{C}} \times G_{\mathbb{C}}/B'_{\mathbb{C}}$ *via the diagonal action.*
(BB5) $\#(B'_{\mathbb{C}} \backslash G_{\mathbb{C}}/B_{\mathbb{C}}) < \infty$.

The above five equivalent conditions (BB1) – (BB5) are determined only by the complexified Lie algebras $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}'_{\mathbb{C}}$.

Remark 6.11. (1) (BB) implies (PP).
(2) If both G and G' are quasi-split, then (BB) $\Leftrightarrow$ (PP).

In fact, the first statement follows immediately from [48, Lemmas 4.2 and 5.3], and the second statement is clear.

6.3 Criteria for finiteness/boundedness of multiplicities

In this and the next subsections, we give an answer to Problem 6.1. The following criteria are direct consequences of Fact 6.7 and a careful consideration of the topology of representation spaces, and are proved in [48].

Theorem 6.12. *The following three conditions on a pair of real reductive algebraic groups* $G \supset G'$ *are equivalent:*

(i) (Symmetry breaking) $\operatorname{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty)$ *is finite-dimensional for any pair* $(\pi^\infty, \tau^\infty)$ *of irreducible smooth representations of* G *and* G'.
(ii) (Invariant bilinear form) *There exist at most finitely many linearly independent* G'*-invariant bilinear forms on* $\pi^\infty|_{G'} \widehat{\otimes} \tau^\infty$*, for any* $\pi^\infty \in \widehat{G}_{\text{smooth}}$ *and* $\tau^\infty \in \widehat{G'}_{\text{smooth}}$.
(iii) (Geometry) *The pair* (G, G') *satisfies the condition (PP) (Definition-Lemma 6.8).*

Theorem 6.13. *The following three conditions on a pair of real reductive algebraic groups* $G \supset G'$ *are equivalent:*

(i) (Symmetry breaking) *There exists a constant* C *such that*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty) \leq C$$

for any $\pi^\infty \in \widehat{G}_{\text{smooth}}$ *and* $\tau^\infty \in \widehat{G'}_{\text{smooth}}$.

(ii) (Invariant bilinear form) *There exists a constant C such that*

$$\dim_{\mathbb{C}} \operatorname{Hom}_{G'}(\pi^\infty|_{G'} \widehat{\otimes} \tau^\infty, \mathbb{C}) \le C$$

for any $\pi^\infty \in \widehat{G}_{\text{smooth}}$ and $\tau^\infty \in \widehat{G'}_{\text{smooth}}$.

(iii) (Geometry) *The pair (G, G') satisfies the condition (BB) (Definition-Lemma 6.10).*

6.4 Classification theory of finite-multiplicity branching laws

This section gives a complete list of the reductive symmetric pairs (G, G') such that $\dim \operatorname{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty)$ is finite or bounded for all $\pi^\infty \in \widehat{G}_{\text{smooth}}$ and $\tau^\infty \in \widehat{G'}_{\text{smooth}}$. Owing to the criteria in Theorems 6.12 and 6.13, the classification is reduced to that of (real) spherical homogeneous spaces of the form $(G \times G')/\operatorname{diag} G'$, which was accomplished in [44] by using an idea of “linearization” :

Theorem 6.14. *Suppose (G, G') is a reductive symmetric pair. Then the following two conditions are equivalent:*

(i) $\operatorname{Hom}_{G'}(\pi^\infty|_{G'}, \tau^\infty)$ *is finite-dimensional for any pair $(\pi^\infty, \tau^\infty)$ of admissible smooth representations of G and G'.*

(ii) *The pair $(\mathfrak{g}, \mathfrak{g}')$ of their Lie algebras is isomorphic (up to outer automorphisms) to a direct sum of the following pairs:*

A) Trivial case: $\mathfrak{g} = \mathfrak{g}'$.
B) Abelian case: $\mathfrak{g} = \mathbb{R}$, $\mathfrak{g}' = \{0\}$.
C) Compact case: $\mathfrak{g}$ *is the Lie algebra of a compact simple Lie group.*
D) Riemannian symmetric pair: $\mathfrak{g}'$ *is the Lie algebra of a maximal compact subgroup K of a noncompact simple Lie group G.*
E) Split rank one case ($\operatorname{rank}_{\mathbb{R}} G = 1$):
  E1) $(\mathfrak{o}(p+q, 1), \mathfrak{o}(p) + \mathfrak{o}(q, 1))$ $(p + q \ge 2)$,
  E2) $(\mathfrak{su}(p+q, 1), \mathfrak{s}(\mathfrak{u}(p) + \mathfrak{u}(q, 1)))$ $(p + q \ge 1)$,
  E3) $(\mathfrak{sp}(p+q, 1), \mathfrak{sp}(p) + \mathfrak{sp}(q, 1))$ $(p + q \ge 1)$,
  E4) $(\mathfrak{f}_{4(-20)}, \mathfrak{o}(8, 1))$.
F) Strong Gelfand pairs and their real forms:
  F1) $(\mathfrak{sl}(n+1, \mathbb{C}), \mathfrak{gl}(n, \mathbb{C}))$ $(n \ge 2)$,
  F2) $(\mathfrak{o}(n+1, \mathbb{C}), \mathfrak{o}(n, \mathbb{C}))$ $(n \ge 2)$,
  F3) $(\mathfrak{sl}(n+1, \mathbb{R}), \mathfrak{gl}(n, \mathbb{R}))$ $(n \ge 1)$,
  F4) $(\mathfrak{su}(p+1, q), \mathfrak{u}(p, q))$ $(p + q \ge 1)$,
  F5) $(\mathfrak{o}(p+1, q), \mathfrak{o}(p, q))$ $(p + q \ge 2)$.
G) Group case: $(\mathfrak{g}, \mathfrak{g}') = (\mathfrak{g}_1 + \mathfrak{g}_1, \operatorname{diag} \mathfrak{g}_1)$ *where*
  G1) $\mathfrak{g}_1$ *is the Lie algebra of a compact simple Lie group,*
  G2) $(\mathfrak{o}(n, 1) + \mathfrak{o}(n, 1), \operatorname{diag} \mathfrak{o}(n, 1))$ $(n \ge 2)$.

H) Other cases:
H1) $(\mathfrak{o}(2n,2), \mathfrak{u}(n,1))$ $(n \geq 1)$.
H2) $(\mathfrak{su}^*(2n+2), \mathfrak{su}(2) + \mathfrak{su}^*(2n) + \mathbb{R})$ $(n \geq 1)$.
H3) $(\mathfrak{o}^*(2n+2), \mathfrak{o}(2) + \mathfrak{o}^*(2n))$ $(n \geq 1)$.
H4) $(\mathfrak{sp}(p+1,q), \mathfrak{sp}(p,q) + \mathfrak{sp}(1))$.
H5) $(\mathfrak{e}_{6(-26)}, \mathfrak{so}(9,1) + \mathbb{R})$.

Among the pairs $(\mathfrak{g}, \mathfrak{g}')$ in the list (A)–(H) in Theorem 6.14 describing finite multiplicities, those pairs having uniform bounded multiplicities are classified as follows.

Theorem 6.15. *Suppose* (G, G') *is a reductive symmetric pair. Then the following two conditions are equivalent:*

(i) *There exists a constant* C *such that*

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\pi^{\infty}|_{G'}, \tau^{\infty}) \leq C$$

for any $\pi^{\infty} \in \widehat{G}_{\mathrm{smooth}}$ *and* $\tau^{\infty} \in \widehat{G'}_{\mathrm{smooth}}$.

(ii) *The pair of their Lie algebras* $(\mathfrak{g}, \mathfrak{g}')$ *is isomorphic* (*up to outer automorphisms*) *to a direct sum of the pairs in* (A)*,* (B) *and* (F1) – (F5).

Proof. Theorem 6.14 follows directly from Theorem 6.12 and [44, Theorem 1.3]. Theorem 6.15 follows directly from Theorem 6.13 and [44, Proposition 1.6]. □

Example 6.16. In connection with branching problems, some of the pairs appeared earlier in the literature. For instance,

(F1), (F2) ⋯ finite-dimensional representations (strong Gelfand pairs) [53];
(F2), (F5) ⋯ tempered unitary representations (Gross–Prasad conjecture) [14];
(G2) ⋯ tensor product, trilinear forms [8, 27];
(F1)–(F5) ⋯ multiplicity-free restrictions [2, 68].

7 Construction of symmetry breaking operators

Stage C in the branching problem asks for an explicit construction of intertwining operators. This problem depends on the geometric models of representations of a group G and its subgroup G'. In this section we discuss symmetry breaking operators in two models, i.e., in the setting of real flag manifolds (Sections 7.1–7.3) and in the holomorphic setting (Sections 7.4–7.5).

7.1 Differential operators on different base spaces

We extend the usual notion of differential operators between two vector bundles on the *same* base space to those on *different* base spaces X and Y with a morphism $p : Y \to X$ as follows.

Definition 7.1. Let $\mathcal{V} \to X$ and $\mathcal{W} \to Y$ be two vector bundles, and $p : Y \to X$ a smooth map between the base manifolds. A continuous linear map $T : C^\infty(X, \mathcal{V}) \to C^\infty(Y, \mathcal{W})$ is said to be a *differential operator* if

$$p(\operatorname{Supp}(Tf)) \subset \operatorname{Supp} f \quad \text{for all } f \in C^\infty(X, \mathcal{V}), \tag{7.1}$$

where Supp stands for the support of a section.

The condition (7.1) shows that T is a local operator in the sense that for any open subset U of X, the restriction $(Tf)|_{p^{-1}(U)}$ is determined by the restriction $f|_U$.

Example 7.2. (1) If $X = Y$ and p is the identity map, then the condition (7.1) is equivalent to the condition that T is a differential operator in the usual sense, due to Peetre's theorem [61].

(2) If $p : Y \to X$ is an immersion, then any operator T satisfying (7.1) is locally of the form

$$\sum_{(\alpha,\beta)\in\mathbb{N}^{m+n}} g_{\alpha\beta}(y) \frac{\partial^{|\alpha|+|\beta|}}{\partial y^\alpha \partial z^\beta}\bigg|_{z_1=\cdots=z_n=0} \qquad \text{(finite sum)},$$

where $\{(y_1, \ldots, y_m, z_1, \ldots, z_n)\}$ are local coordinates of X such that Y is given locally by the equation $z_1 = \cdots = z_n = 0$, and $g_{\alpha\beta}(y)$ are matrix-valued functions on Y.

7.2 Distribution kernels for symmetry breaking operators

In this section, we discuss symmetry breaking operators in a geometric setting, where representations are realized in the space of smooth sections for homogeneous vector bundles.

Let G be a Lie group, and $\mathcal{V} \to X$ a homogeneous vector bundle, namely, a G-equivariant vector bundle such that the G-action on the base manifold X is transitive. Likewise, let $\mathcal{W} \to Y$ be a homogeneous vector bundle for a subgroup G'. The main assumption of our setting is that there is a G'-equivariant map $p : Y \to X$. For simplicity, we also assume that p is injective, and do not assume any relationship between $p^*\mathcal{V}$ and $\mathcal{W}$. Then we have continuous representations of G on the

Fréchet space $C^\infty(X, \mathcal{V})$ and of the subgroup G' on $C^\infty(Y, \mathcal{W})$, but it is not obvious if there exists a nonzero continuous G'-homomorphism (symmetry breaking operator)

$$T : C^\infty(X, \mathcal{V}) \to C^\infty(Y, \mathcal{W}).$$

In this setting, a basic problem is:

Problem 7.3. (1) (Stage A) Find an upper and lower estimate of the dimension of the space $\mathrm{Hom}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W}))$ of symmetry breaking operators.

(2) (Stage A) When is $\mathrm{Hom}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W}))$ finite-dimensional for any G-equivariant vector bundle $\mathcal{V} \to X$ and any G'-equivariant vector bundle $\mathcal{W} \to Y$?

(3) (Stage B) Given equivariant vector bundles $\mathcal{V} \to X$ and $\mathcal{W} \to Y$, determine the dimension of $\mathrm{Hom}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W}))$.

(4) (Stage C) Construct explicit elements in $\mathrm{Hom}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W}))$.

Here are some special cases:

Example 7.4. Suppose $G = G'$, X is a (full) real flag manifold G/P where P is a minimal parabolic subgroup of G, and Y is algebraic.

(1) In this setting, Problem 7.3 (1) and (2) were solved in [48]. In particular, a necessary and sufficient condition for Problem 7.3 (2) is that Y is real spherical, by Fact 6.7 (1) (or directly from the original proof of [48, Theorem A]).

(2) Not much is known about precise results for Problem 7.3 (3), even when $G = G'$. On the other hand, Knapp–Stein intertwining operators or Poisson transforms are examples of explicit intertwining operators when Y is a real flag manifold or a symmetric space, respectively, giving a partial solution to Problem 7.3 (4).

Example 7.5. Let G be the conformal group of the standard sphere $X = S^n$, let G' be the subgroup that leaves the totally geodesic submanifold $Y = S^{n-1}$ invariant, and let $\mathcal{V} \to X$, $\mathcal{W} \to Y$ be G-, G'-equivariant line bundles, respectively. Then $\mathcal{V}$ and $\mathcal{W}$ are parametrized by complex numbers λ and ν, respectively, up to signatures. In this setting Problem 7.3 (3) and (4) were solved in [52]. This is essentially the geometric setup for the classification of $\mathrm{Hom}_{O(n,1)}(I(\lambda)^\infty, J(\nu)^\infty)$ which was discussed in Section 2.2.

We return to the general setting. Let H be an algebraic subgroup of G, (λ, V) a finite-dimensional representation of H, and $\mathcal{V} := G \times_H V \to X := G/H$ the associated G-homogeneous bundle. Likewise, let (ν, W) be a finite-dimensional representation of $H' := H \cap G'$, and $\mathcal{W} := G' \times_{H'} W \to Y := G'/H'$ the associated G'-equivariant bundle. Denote by $\mathbb{C}_{2\rho}$ the one-dimensional representation of H defined by $h \mapsto |\det(\mathrm{Ad}(h) : \mathfrak{g}/\mathfrak{h} \to \mathfrak{g}/\mathfrak{h})|^{-1}$. Then the volume density bundle $\Omega_{G/H}$ of G/H is given as a homogeneous bundle $G \times_H \mathbb{C}_{2\rho}$. Let $(\lambda^\vee, V^\vee)$ be the contragredient representation of the finite-dimensional representation (λ, V) of H.

Then the dualizing bundle $\mathcal{V}^* := \mathcal{V}^\vee \otimes \Omega_{G/H}$ is given by $\mathcal{V}^* \simeq G \times_H (V^\vee \otimes \mathbb{C}_{2\rho})$ as a homogeneous vector bundle.

By the Schwartz kernel theorem, any continuous operator $T : C^\infty(X, \mathcal{V}) \to C^\infty(Y, \mathcal{W})$ is given by a distribution kernel $k_T \in \mathcal{D}'(X \times Y, \mathcal{V}^* \boxtimes \mathcal{W})$. We write

$$m : G \times G' \to G, \quad (g, g') \mapsto (g')^{-1} g,$$

for the multiplication map. If T intertwines G'-actions, then k_T is G'-invariant under the diagonal action, and therefore k_T is of the form $m^* K_T$ for some $K_T \in \mathcal{D}'(X, \mathcal{V}^*) \otimes W$. We have shown in [52, Proposition 3.1] the following proposition:

Proposition 7.6. *Suppose X is compact. Then the correspondence $T \mapsto K_T$ induces a bijection:*

$$\mathrm{Hom}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W})) \xrightarrow{\sim} (\mathcal{D}'(X, \mathcal{V}^*) \otimes W)^{\Delta(H')}.$$

Using Proposition 7.6, we can give a solution to Problem 7.3 (2) when X is a real flag manifold:

Theorem 7.7. *Suppose P is a minimal parabolic subgroup of G, $X = G/P$, and $Y = G'/(G' \cap P)$. Then $\mathrm{Hom}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W}))$ is finite-dimensional for any G-equivariant vector bundle $\mathcal{V} \to X$ and any G'-equivariant vector bundle $\mathcal{W} \to Y$ if and only if $G/(G' \cap P)$ is real spherical.*

Proof. We set $\widetilde{Y} := G/(G' \cap P)$ and $\widetilde{\mathcal{W}} := G \times_{(G' \cap P)} W$. Then Proposition 7.6 implies that there is a canonical bijection:

$$\mathrm{Hom}_G(C^\infty(X, \mathcal{V}), C^\infty(\widetilde{Y}, \widetilde{\mathcal{W}})) \xrightarrow{\sim} \mathrm{Hom}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W})).$$

We apply [48, Theorem A] to the left-hand side, and get the desired conclusion for the right-hand side. □

The smaller X is, the more likely it will be that there exists Y satisfying the finiteness condition posed in Problem 7.3 (2). Thus one might be interested in replacing the *full* real flag manifold by a *partial* real flag manifold in Theorem 7.7. By applying the same argument as above to a generalization of [48] to a partial flag manifold in [41, Corollary 6.8], we get

Proposition 7.8. *Suppose P is a (not necessarily minimal) parabolic subgroup of G and $X = G/P$. Then the finiteness condition for symmetry breaking operators in Problem 7.3 (2) holds only if the subgroup $G' \cap P$ has an open orbit in G/P.*

Back to the general setting, we endow the double coset space $H' \backslash G/H$ with the quotient topology via the canonical quotient $G \to H' \backslash G/H$. Owing to Proposition 7.6, we associate a closed subset of $H' \backslash G/H$ to each symmetry breaking operator:

Definition 7.9. Given a continuous symmetry breaking operator $T : C^\infty(X, \mathcal{V}) \to C^\infty(Y, \mathcal{W})$, we define a closed subset $\operatorname{Supp} T$ in the double coset space $H'\backslash G/H$ as the support of $K_T \in \mathcal{D}'(X, \mathcal{V}^*) \otimes W$.

Example 7.10. If $H = P$, a minimal parabolic subgroup of G, and if H' has an open orbit in G/P, then $\#(H'\backslash G/P) < \infty$. In particular, there are only finitely many possibilities for $\operatorname{Supp} T$.

Definition 7.11. Let $T : C^\infty(X, \mathcal{V}) \to C^\infty(Y, \mathcal{W})$ be a continuous symmetry breaking operator.

1) We say T is a *regular* symmetry breaking operator if $\operatorname{Supp} T$ contains an interior point of $H'\backslash G/H$. We say T is *singular* if T is not regular.
2) We say T is a *differential* symmetry breaking operator if $\operatorname{Supp} T$ is a singleton in $H'\backslash G/H$.

Remark 7.12. The terminology "differential symmetry breaking operator" in Definition 7.11 makes reasonable sense. In fact, T is a differential operator in the sense of Definition 7.1 if and only if $\operatorname{Supp} T$ is a singleton in $H'\backslash G/H$ (see [51, Part I, Lemma 2.3]).

The strategy of [52] for the classification of *all* symmetry breaking operators for (G, G') satisfying (PP) is to use the stratification of H'-orbits in G/H by the closure relation. To be more precise, the strategy is:

- to obtain all differential symmetry breaking operators, which corresponds to the singleton in $H'\backslash G/H$, or equivalently, to solve certain branching problems for generalized Verma modules (see Section 7.3 below) via the duality (7.3),
- to construct and classify $\{T \in H(\lambda, \nu) : \operatorname{Supp} T \subset \overline{S}\}$ modulo $\{T \in H(\lambda, \nu) : \operatorname{Supp} T \subset \partial S\}$ for $S \in G'\backslash G/H$ inductively.

The "F-method" [38, 40, 47, 51] gives a conceptual and a practical tool to construct differential symmetry breaking operators in Step 1. The second step may involve analytic questions such as the possibility of an extension of an H'-invariant distribution on an H'-invariant subset of G/H satisfying a differential equation to an H'-invariant distribution solution on the whole of G/H (e.g., [52, Chapter 11, Sect. 4]), and an analytic continuation and residue calculus with respect to some natural parameter (e.g., [52, Chapters 8 and 12]).

We expect that the methods developed in [52] for the classification of symmetry breaking operators for the pair $(G, G') = (\mathrm{O}(n + 1, 1), \mathrm{O}(n, 1))$ would work for some other pairs (G, G') such as those satisfying (PP) (see Theorem 6.14 for the list), or more strongly those satisfying (BB) (see Theorem 6.15 for the list).

7.3 Finiteness criterion for differential symmetry breaking operators

As we have seen in Theorem 7.7 and Proposition 7.8, it is a considerably strong restriction on the G'-manifold Y for the space $\mathrm{Hom}_{G'}(C^\infty(X,\mathcal{V}), C^\infty(Y,\mathcal{W}))$ of symmetry breaking operators to be finite-dimensional, which would be a substantial condition for further study in Stages B and C of the branching problem. On the other hand, if we consider only *differential* symmetry breaking operators, then it turns out that there are much broader settings for which the finite-multiplicity property (or even the multiplicity-free property) holds. The aim of this subsection is to formulate this property.

In order to be precise, we write $\mathrm{Hom}_{G'}(C^\infty(X,\mathcal{V}), C^\infty(Y,\mathcal{W}))$ for the space of continuous symmetry breaking operators, and $\mathrm{Diff}_{G'}(C^\infty(X,\mathcal{V}), C^\infty(Y,\mathcal{W}))$ for that of differential symmetry breaking operators. Clearly we have

$$\mathrm{Diff}_{G'}(C^\infty(X,\mathcal{V}), C^\infty(Y,\mathcal{W})) \subset \mathrm{Hom}_{G'}(C^\infty(X,\mathcal{V}), C^\infty(Y,\mathcal{W})). \tag{7.2}$$

We now consider the problem analogous to Problem 7.3 by replacing the right-hand side of (7.2) with the left-hand side.

For simplicity, we consider the case where $\mathcal{V} \to X$ is a G-equivariant line bundle over a real flag manifold G/P, and write $\mathcal{L}_\lambda \to X$ for the line bundle associated to a one-dimensional representation λ of P. We use the same letter λ to denote the corresponding infinitesimal representation of the Lie algebra $\mathfrak{p}$, and write $\lambda \gg 0$ if $\langle \lambda|_{\mathfrak{j}}, \alpha\rangle \gg 0$ for all $\alpha \in \Delta(\mathfrak{n}^+, \mathfrak{j})$ where $\mathfrak{j}$ is a Cartan subalgebra contained in the Levi part $\mathfrak{l}$ of the parabolic subalgebra $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}^+$.

We say a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}$ is $\mathfrak{g}'$-*compatible* if $\mathfrak{p}$ is defined as the sum of eigenspaces with nonnegative eigenvalues for some hyperbolic element in $\mathfrak{g}'$. Then $\mathfrak{p}' := \mathfrak{p} \cap \mathfrak{g}'$ is a parabolic subalgebra of $\mathfrak{g}'$ and we have compatible Levi decompositions $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}^+$ and $\mathfrak{p}' = (\mathfrak{l} \cap \mathfrak{g}') + (\mathfrak{n}^+ \cap \mathfrak{g}')$. We are ready to state an answer to a question analogous to Problem 7.3 (1) and (2) for *differential* symmetry breaking operators (cf. [40]).

Theorem 7.13 (local operators). *Let G' be a reductive subgroup of a real reductive linear Lie group G, $X = G/P$ and $Y = G'/P'$ where P is a parabolic subgroup of G and $P' = P \cap G'$ such that the parabolic subalgebra $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}^+$ of $\mathfrak{g}$ is $\mathfrak{g}'$-compatible.*

(1) (finite multiplicity) *For any finite-dimensional representations V and W of the parabolic subgroups P and P', respectively, we have*

$$\dim_{\mathbb{C}} \mathrm{Diff}_{G'}(C^\infty(X,\mathcal{V}), C^\infty(Y,\mathcal{W})) < \infty,$$

where $\mathcal{V} = G \times_P V$ and $\mathcal{W} = G' \times_{P'} W$ are equivariant vector bundles over X and Y, respectively.

(2) (uniformly bounded multiplicity) *If* $(\mathfrak{g}, \mathfrak{g}')$ *is a symmetric pair and* $\mathfrak{n}^+$ *is abelian, then for any finite-dimensional representation* V *of* P,

$$C_V := \sup_W \dim_{\mathbb{C}} \mathrm{Diff}_{G'}(C^\infty(X, \mathcal{V}), C^\infty(Y, \mathcal{W})) < \infty.$$

Here W *runs over all finite-dimensional irreducible representations of* P'. *Furthermore,* $C_V = 1$ *if* V *is a one-dimensional representation* λ *of* P *with* $\lambda \gg 0$.

Proof. The classical duality between Verma modules and principal series representations in the case $G = G'$ (e.g., [17]) can be extended to the context of the restriction of reductive groups $G \downarrow G'$, and the following bijection holds (see [51, Part I, Corollary 2.9]):

$$\mathrm{Hom}_{(\mathfrak{g}', P')}(U(\mathfrak{g}') \otimes_{U(\mathfrak{p}')} W^\vee, U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} V^\vee) \simeq \mathrm{Diff}_{G'}(C^\infty(G/P, \mathcal{V}), C^\infty(G'/P', \mathcal{W})). \quad (7.3)$$

Here $(\lambda^\vee, V^\vee)$ denotes the contragredient representation of (λ, V). The right-hand side of (7.3) concerns Case II (symmetry breaking) in Section 4, whereas the left-hand side of (7.3) concerns Case I (embedding) in the BGG category $\mathcal{O}$. An analogous theory of discretely decomposable restriction in the Harish-Chandra category $\mathcal{HC}$ (see Sections 4 and 5) can be developed more easily and explicitly in the BGG category $\mathcal{O}$, which was done in [37]. In particular, the $\mathfrak{g}'$-compatibility is a sufficient condition for the "discrete decomposability" of generalized Verma modules $U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F$ when restricted to the reductive subalgebra $\mathfrak{g}'$. Thus the proof of Theorem 7.13 is reduced to the next proposition.

Proposition 7.14. *Let* $\mathfrak{g}'$ *be a reductive subalgebra of* $\mathfrak{g}$. *Suppose that a parabolic subalgebra* $\mathfrak{p} = \mathfrak{l} + \mathfrak{n}^+$ *is* $\mathfrak{g}'$*-compatible.*
(1) *For any finite-dimensional* $\mathfrak{p}$*-module* F *and* $\mathfrak{p}'$*-module* F',

$$\dim \mathrm{Hom}_{\mathfrak{g}'}(U(\mathfrak{g}') \otimes_{U(\mathfrak{p}')} F', U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} F) < \infty.$$

(2) *If* $(\mathfrak{g}, \mathfrak{g}')$ *is a symmetric pair and* $\mathfrak{n}^+$ *is abelian, then*

$$\sup_{F'} \dim \mathrm{Hom}_{\mathfrak{g}'}(U(\mathfrak{g}') \otimes_{U(\mathfrak{p}')} F', U(\mathfrak{g}) \otimes_{U(\mathfrak{p})} \mathbb{C}_\lambda) = 1$$

for any one-dimensional representation λ *of* $\mathfrak{p}$ *with* $\lambda \ll 0$. *Here the supremum is taken over all finite-dimensional simple* $\mathfrak{p}'$*-modules* F'.

Proof. (1) The proof is parallel to [37, Theorem 3.10] which treated the case where F and F' are simple modules of P and P', respectively.
(2) See [37, Theorem 5.1]. □

Hence Theorem 7.13 is proved. □

Remark 7.15. If we drop the assumption $\lambda \gg 0$ in Theorem 7.13 (2) or $\lambda \ll 0$ in Proposition 7.14 (2), then the multiplicity-free statement may fail. In fact, the computation in Section 2.1 gives a counterexample where $(\mathfrak{g}, \mathfrak{g}') = (\mathfrak{sl}(2,\mathbb{C}) + \mathfrak{sl}(2,\mathbb{C}), \mathrm{diag}(\mathfrak{sl}(2,\mathbb{C})))$; see Remark 2.6 (3).

Remark 7.16. (1) (Stage B) In the setting of Proposition 7.14 (2), Stage B in the branching problem (finding explicit branching laws) have been studied in [35, 37] in the BGG category $\mathcal{O}$ generalizing earlier results by Kostant and Schmid [65].

(2) (Stage C) In the setting of Theorem 7.13 (2), one may wish to find an explicit formula for the unique *differential* symmetry breaking operators. So far, this has been done only in some special cases; see [9, 11] for the Rankin–Cohen bidifferential operator, Juhl [21] in connection with conformal geometry, and [47, 51] using the Fourier transform ("F-method" in [38]).

We end this subsection by applying Theorem 7.13 and Theorem 6.12 to the reductive symmetric pair $(G, G') = (GL(n_1 + n_2, \mathbb{R}), GL(n_1, \mathbb{R}) \times GL(n_2, \mathbb{R}))$, and observe a sharp contrast between differential and continuous symmetry breaking operators, i.e., the left-hand and right-hand sides of (7.2), respectively.

Example 7.17. Let $n = n_1 + n_2$ with $n_1, n_2 \geq 2$. Let P, P' be minimal parabolic subgroups of

$$(G, G') = (GL(n, \mathbb{R}), GL(n_1, \mathbb{R}) \times GL(n_2, \mathbb{R})),$$

respectively, and set $X = G/P$ and $Y = G'/P'$. Then:

(1) For all finite-dimensional representations V of P and W of P',

$$\dim_{\mathbb{C}} \mathrm{Diff}_{G'}(\mathrm{Ind}_P^G(V)^\infty, \mathrm{Ind}_{P'}^{G'}(W)^\infty) < \infty.$$

Furthermore if V is a one-dimensional representation $\mathbb{C}_\lambda$ with $\lambda \gg 0$ in the notation of Theorem 7.13, then the above dimension is 0 or 1.

(2) For some finite-dimensional representations V of P and W of P',

$$\dim_{\mathbb{C}} \mathrm{Hom}_{G'}(\mathrm{Ind}_P^G(V)^\infty, \mathrm{Ind}_{P'}^{G'}(W)^\infty) = \infty.$$

7.4 Localness theorem in the holomorphic setting

In the last example (Example 7.17) and also Theorem 2.9 in Section 2.2, we have seen in the real setting that differential symmetry breaking operators are "very special" among continuous symmetry breaking operators. In this subsection we explain the remarkable phenomenon in the holomorphic framework that any continuous symmetry breaking operator between two representations under certain special geometric settings is given by a differential operator; see Observation 2.5

(1) for the $\mathrm{SL}(2,\mathbb{R})$ case. A general case is formulated in Theorem 7.18 below. The key idea of the proof is to use the theory of discretely decomposable restrictions [26, 28, 29], briefly explained in Section 5. A conjectural statement is given in the next subsection.

Let $G \supset G'$ be real reductive linear Lie groups, $K \supset K'$ their maximal compact subgroups, and $G_{\mathbb{C}} \supset G'_{\mathbb{C}}$ connected complex reductive Lie groups containing $G \supset G'$ as real forms, respectively. The main assumption of this subsection is that $X := G/K$ and $Y := G'/K'$ are Hermitian symmetric spaces. To be more precise, let $Q_{\mathbb{C}}$ and $Q'_{\mathbb{C}}$ be parabolic subgroups of $G_{\mathbb{C}}$ and $G'_{\mathbb{C}}$ with Levi subgroups $K_{\mathbb{C}}$ and $K'_{\mathbb{C}}$, respectively, such that the following commutative diagram consists of holomorphic maps:

$$\begin{array}{ccccc} & Y = G'/K' & \subset & X = G/K & \\ \text{Borel embedding} & \cap & & \cap & \text{Borel embedding} \\ & G'_{\mathbb{C}}/Q'_{\mathbb{C}} & \subset & G_{\mathbb{C}}/Q_{\mathbb{C}}. & \end{array} \tag{7.4}$$

Theorem 7.18 ([51, Part I]). *Let $\mathcal{V} \to X$, $\mathcal{W} \to Y$ be G-equivariant, G'-equivariant holomorphic vector bundles, respectively.*

(1) (localness theorem) *Any G'-homomorphism from $\mathcal{O}(X,\mathcal{V})$ to $\mathcal{O}(Y,\mathcal{W})$ is given by a holomorphic differential operator, in the sense of Definition 7.1, with respect to a holomorphic embedding $Y \hookrightarrow X$.*

We extend $\mathcal{V}$ and $\mathcal{W}$ to holomorphic vector bundles over $G_{\mathbb{C}}/Q_{\mathbb{C}}$ and $G'_{\mathbb{C}}/Q'_{\mathbb{C}}$, respectively.

(2) (extension theorem) *Any differential symmetry breaking operator in (1) defined on Hermitian symmetric spaces extends to a $G'_{\mathbb{C}}$-equivariant holomorphic differential operator $\mathcal{O}(G_{\mathbb{C}}/Q_{\mathbb{C}},\mathcal{V}) \to \mathcal{O}(G'_{\mathbb{C}}/Q'_{\mathbb{C}},\mathcal{W})$ with respect to a holomorphic map between the flag varieties $G'_{\mathbb{C}}/Q'_{\mathbb{C}} \hookrightarrow G_{\mathbb{C}}/Q_{\mathbb{C}}$.*

Remark 7.19. The representation π on the Fréchet space $\mathcal{O}(G/K,\mathcal{V})$ is a maximal globalization of the underlying $(\mathfrak{g},K)$-module π_K in the sense of Schmid [66], and contains some other globalizations having the same underlying $(\mathfrak{g},K)$-module π_K (e.g., the Casselman–Wallach globalization π^∞). One may ask whether an analogous statement holds if we replace $(\pi,\mathcal{O}(G/K,\mathcal{V}))$ and $(\tau,\mathcal{O}(G'/K',\mathcal{W}))$ by other globalizations such as π^∞ and τ^∞. This question was raised by D. Vogan during the conference at MIT in May 2014. We gave an affirmative answer in [51, Part I] by proving that the natural inclusions

$$\mathrm{Hom}_{G'}(\pi,\tau) \subset \mathrm{Hom}_{G'}(\pi^\infty,\tau^\infty) \subset \mathrm{Hom}_{\mathfrak{g}',K'}(\pi_K,\tau_{K'})$$

are actually bijective in our setting.

7.5 Localness conjecture for symmetry breaking operators on cohomologies

It might be natural to ask a generalization of Theorem 7.18 to some other holomorphic settings, from holomorphic sections to Dolbeault cohomologies, and from highest weight modules to $A_{\mathfrak{q}}(\lambda)$ modules.

Problem 7.20. To what extent does the localness and extension theorem hold for symmetry breaking operators between Dolbeault cohomologies?

In order to formulate the problem more precisely, we introduce the following assumption on the pair (G, G') of real reductive groups:

$$K \text{ has a normal subgroup of positive dimension which is contained in } K'. \tag{7.5}$$

Here, K and $K' = K \cap G'$ are maximal compact subgroups of G and G', respectively, as usual. We write $K^{(2)}$ for the normal subgroup in (7.5), $\mathfrak{k}_0^{(2)}$ for the corresponding Lie algebra, and $\mathfrak{k}^{(2)}$ for its complexification. Then the assumption (7.5) means that we have direct sum decompositions

$$\mathfrak{k} = \mathfrak{k}^{(1)} \oplus \mathfrak{k}^{(2)}, \qquad \mathfrak{k}' = \mathfrak{k}'^{(1)} \oplus \mathfrak{k}^{(2)}$$

for some ideals $\mathfrak{k}^{(1)}$ of $\mathfrak{k}$ and $\mathfrak{k}'^{(1)}$ of $\mathfrak{k}'$, respectively. The point here is that $\mathfrak{k}^{(2)}$ is common to both $\mathfrak{k}$ and $\mathfrak{k}'$.

We take $H \in \sqrt{-1}\mathfrak{k}_0^{(2)}$, define a θ-stable parabolic subalgebra of $\mathfrak{g}$ by

$$\mathfrak{q} \equiv \mathfrak{q}(H) = \mathfrak{l} + \mathfrak{u}$$

as the sum of eigenspaces of $\mathrm{ad}(H)$ with nonnegative eigenvalues, and set $L := G \cap Q_{\mathbb{C}}$ where $Q_{\mathbb{C}} = N_{G_{\mathbb{C}}}(\mathfrak{q})$ is the parabolic subgroup of $G_{\mathbb{C}}$. Then L is a reductive subgroup of G with complexified Lie algebra $\mathfrak{l}$, and we have an open embedding $X := G/L \subset G_{\mathbb{C}}/Q_{\mathbb{C}}$ through which G/L carries a complex structure. The same element H defines complex manifolds $Y := G'/L' \subset G'_{\mathbb{C}}/Q'_{\mathbb{C}}$ with the obvious notation.

In summary, we have the following geometry that generalizes (7.4):

$$\begin{array}{ccccc} Y = & G'/L' & \subset X = & G/L & \\ \text{open} & \cap & & \cap & \text{open} \\ & G'_{\mathbb{C}}/Q'_{\mathbb{C}} & \subset & G_{\mathbb{C}}/Q_{\mathbb{C}}. & \end{array}$$

It follows from the assumption (7.5) that the compact manifold $K/L \cap K$ coincides with $K'/L' \cap K'$. Let S denote the complex dimension of the complex compact manifolds $K/L \cap K \simeq K'/L' \cap K'$.

Example 7.21. (1) (Hermitian symmetric spaces) Suppose that $K^{(2)}$ is abelian. Then $Y \subset X$ are Hermitian symmetric spaces, $S = 0$, and we obtain the geometric setting of Theorem 7.18.

(2) $(G, G') = (\mathrm{U}(p, q; \mathbb{F}), \mathrm{U}(p'; \mathbb{F}) \times \mathrm{U}(p'', q; \mathbb{F}))$ with $p = p' + p''$ for $\mathbb{F} = \mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, and $K^{(2)} = \mathrm{U}(q; \mathbb{F})$. Then neither G/L nor G'/L' is a Hermitian symmetric space but the assumption (7.5) is satisfied. Thus the conjecture below applies.

For a finite-dimensional holomorphic representation V of $Q_{\mathbb{C}}$, we define a holomorphic vector bundle $G_{\mathbb{C}} \times_{Q_{\mathbb{C}}} V$ over the generalized flag variety $G_{\mathbb{C}}/Q_{\mathbb{C}}$, and write $\mathcal{V} := G \times_L V$ for the G-equivariant holomorphic vector bundle over $X = G/L$ as the restriction $(G_{\mathbb{C}} \times_{Q_{\mathbb{C}}} V)|_{G/L}$. Then the Dolbeault cohomology $H^j_{\bar\partial}(X, \mathcal{V})$ naturally carries a Fréchet topology by the closed range theorem of the $\bar\partial$-operator, and gives the maximal globalization of the underlying $(\mathfrak{g}, K)$-modules, which are isomorphic to Zuckerman's derived functor modules $\mathcal{R}^j_{\mathfrak{q}}(V \otimes \mathbb{C}_{-\rho})$ [69, 75]. Similarly for G', given a finite-dimensional holomorphic representation W of $Q'_{\mathbb{C}}$, we form a G'-equivariant holomorphic vector bundle $\mathcal{W} := G' \times_{L'} W$ over $Y = G'/L'$ and define a continuous representation of G' on the Dolbeault cohomologies $H^j_{\bar\partial}(Y, \mathcal{W})$. In this setting we have the discrete decomposability of the restriction by the general criterion (see Fact 5.5).

Proposition 7.22. *The underlying* $(\mathfrak{g}, K)$*-modules* $H^j_{\bar\partial}(X, \mathcal{V})_K$ *are* K'*-admissible. In particular, they are discretely decomposable as* $(\mathfrak{g}', K')$*-modules.*

Explicit branching laws in some special cases (in particular, when $\dim V = 1$) of Example 7.21 (1) and (2) may be found in [35] and [15, 25], respectively.

We are now ready to formulate a possible extension of the localness and extension theorem for holomorphic functions (Theorem 7.18) to Dolbeault cohomologies that gives geometric realizations of Zuckerman's derived functor modules.

Conjecture 7.23. Suppose we are in the above setting, and let V and W be finite-dimensional representations of $Q_{\mathbb{C}}$ and $Q'_{\mathbb{C}}$, respectively.

(1) (localness theorem) Any continuous G'-homomorphism

$$H^S_{\bar\partial}(X, \mathcal{V}) \to H^S_{\bar\partial}(Y, \mathcal{W})$$

is given by a holomorphic differential operator with respect to a holomorphic embedding $Y \hookrightarrow X$.

(2) (extension theorem) Any such operator in (1) defined on the open subsets $Y \subset X$ of $G'_{\mathbb{C}}/Q'_{\mathbb{C}} \subset G_{\mathbb{C}}/Q_{\mathbb{C}}$, respectively, extends to a $G'_{\mathbb{C}}$-equivariant holomorhic differential operator with respect to a holomorphic map between the flag varieties $G'_{\mathbb{C}}/Q'_{\mathbb{C}} \hookrightarrow G_{\mathbb{C}}/Q_{\mathbb{C}}$.

The key ingredient of the proof of Theorem 7.18 for Hermitian symmetric spaces was the discrete decomposability of the restriction of the representation (Fact 2.2 (2)). Proposition 7.22 is a part of the evidence for Conjecture 7.23 in the general setting.

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Equations for a filtration of sheets and the variety of singular elements of a complex semisimple Lie algebra

Bertram Kostant

To David Vogan whose brilliant mathematical career continues to exhibit an ever-increasing number of contributions to mathematics

Abstract This paper connects results on Amitsur–Levitski identities for simple Lie algebras, ideals in Borel subalgebras, commutative Lie subalgebras in simple Lie algebras, filtration of sheets, and recent work with Nolan Wallach on the variety of singular elements in a complex semisimple Lie algebra.

Key words: standard identities, Borel subalgebras, Amitsur–Levitski, sheets, singular elements

MSC (2010): 16RXX, 16S30, 20B35, 20G07, 22E25

1 Main concepts, basic definitions, and results

Let $\mathfrak{g}$ be a complex semisimple Lie algebra. The two papers, [4] and [5], were written independently of each other; also neither references the other. The paper [4], among other things, deals with equations for a filtration of sheets in $\mathfrak{g}$, denoted by Var $R^k(\mathfrak{g})$. The paper [5], a joint work with Nolan Wallach, deals with equations for the variety of singular elements in $\mathfrak{g}$. One of the main points in this paper, previously overlooked, is that the leading term in the filtration of sheets is Var$R^r(\mathfrak{g})$, where r is given by (27) as noted in Remark 4.4.

To begin with we recall some basic definitions and earlier results.

The Amitsur–Levitski theorem is a famous result. The field $\mathbb{F}$ is denoted subsequently by F. It states that for any field F, any $2n$ elements of the $n \times n$ matrix

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algebra $M(n, F)$ satisfy an identity well known as the standard identity. To discuss the standard identity, let R be an associative ring and for any $k \in \mathbb{Z}$ and $x_i, \ldots, x_k$, in R one defines an alternating sum of products

$$[[x_1, \ldots x_k]] = \sum_{\sigma \in Sym\, k} sg(\sigma)\, x_{\sigma(1)} \cdots x_{\sigma(k)}. \tag{1}$$

Now R satisfies the standard identity of degree k if $[[x_1, \ldots, x_k]] = 0$ for any choice of the $x_i \in R$. Of course R is commutative if and only if it satisfies the standard identity of degree 2.

Now for any $n \in \mathbb{Z}$ and field F, let $M(n, F)$ be the algebra of $n \times n$ matrices over F. The following is the famous Amitsur–Levitski theorem.

Theorem 1.1. *$M(n, F)$ satisfies the standard identity of degree 2n.*

Remark 1.2. By restricting to matrix units, for a proof it suffices to take $F = \mathbb{C}$.

Without any knowledge that it was a known theorem, we came upon Theorem 1.1 in [1] a long time ago from the point of Lie algebra cohomology. In fact the result follows from the fact that if $\mathfrak{g} = M(n, \mathbb{C})$, then the restriction to $\mathfrak{g}$ of the primitive cohomology class of degree $2n + 1$ of $M(n + 1, \mathbb{C})$ to $\mathfrak{g}$ vanishes.

Of course $\mathfrak{g}_1 \subset \mathfrak{g}$ where $\mathfrak{g}_1 = \mathrm{Lie}\,\mathrm{SO}(n, \mathbb{C})$. Assume n is even. One proves that the restriction to $\mathfrak{g}_1$ of the primitive class of degree $2n - 1$ (highest primitive class) of $\mathfrak{g}$ vanishes on $\mathfrak{g}_1$. This leads to a new standard identity, namely

Theorem 1.3.

$$[[x_1, \ldots, x_{2n-2}]] = 0 \tag{2}$$

for any choice of $x_i \in \mathfrak{g}_1$.

Remark 1.4. Theorem 1.3 is immediately evident when $n = 2$.

Theorems 1.1 and 1.3 suggest that standard identities can be viewed as a subject in Lie theory. Theorem 1.5 below offers support for this idea. Let $\mathfrak{r}$ be a complex reductive Lie algebra and let

$$\pi : \mathfrak{r} \to \mathrm{End}\, V \tag{3}$$

be a finite-dimensional complex completely reducible representation. If $w \in \mathfrak{r}$ is nilpotent, then $\pi(w)^k = 0$ for some $k \in \mathbb{Z}$. Let $\varepsilon(\pi)$ be the minimal integer k such that $\pi(w)^k = 0$ for all nilpotent $w \in \mathfrak{r}$. In case π is irreducible, one can easily give a formula for $\varepsilon(\pi)$ in terms of the highest weight. If $\mathfrak{g}$ (resp. $\mathfrak{g}_1$) is given as above and π (resp. π_1) is the defining representation, then $\varepsilon(\pi) = n$ and $\varepsilon(\pi_1) = n - 1$. Consequently the following theorem (see [4]) generalizes Theorems 1.1 and 1.3.

Note that suprisingly the theorem relates the standard identity to the action of the nilcone under π.

Theorem 1.5. *Let* $\mathfrak{r}$ *be a complex reductive Lie algebra and let* π *be as above. Then for any* $x_i \in \mathfrak{r}$, $i = 1, \ldots, 2\varepsilon(\pi)$, *one has*

$$[[\hat{x}_1, \ldots, \hat{x}_{2\varepsilon(\pi)}]] = 0, \tag{4}$$

where $\hat{x}_i = \pi(x_i)$.

2 The nilcone and standard identities

Henceforth $\mathfrak{g}$, until mentioned otherwise, will be an arbitrary reductive complex finite-dimensional Lie algebra. Let $T(\mathfrak{g})$ be the tensor algebra over $\mathfrak{g}$ and let $S(\mathfrak{g}) \subset T(\mathfrak{g})$ (resp. $A(\mathfrak{g}) \subset T(\mathfrak{g})$) be the subspace of symmetric (resp. alternating) tensors in $T(\mathfrak{g})$. The natural grading on $T(\mathfrak{g})$ restricts to a grading on $S(\mathfrak{g})$ and $A(\mathfrak{g})$. In particular, where multiplication is tensor product, one notes

Proposition 2.1. $A^j(\mathfrak{g})$ *is the span of* $[[x_1, \ldots, x_j]]$ *over all choices of* x_i, $i = 1, \ldots, j$, *in* $\mathfrak{g}$.

Now let $U(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}$. Then $U(\mathfrak{g})$ is the quotient algebra of $T(\mathfrak{g})$ so that there is an algebra epimorphism

$$\tau : T(\mathfrak{g}) \to U(\mathfrak{g}).$$

Let $Z = \operatorname{Cent} U(\mathfrak{g})$ and let $E \subset U(\mathfrak{g})$ be the graded subspace spanned by all powers e^j, $j = 1, \ldots,$ where $e \in \mathfrak{g}$ is nilpotent. In [2, Theorem 21] we proved that

$$U(\mathfrak{g}) = Z \otimes E. \tag{5}$$

where tensor product identifies with multiplication.

In [4, Theorem 3.4] we proved

Theorem 2.2. *For any* $k \in \mathbb{Z}$ *one has*

$$\tau(A^{2k}(\mathfrak{g})) \subset E^k. \tag{6}$$

Theorem 1.5 is then an immediate consequence of Theorem 2.2. Indeed, using the notation of Theorem 1.5, let $\pi_U : U(\mathfrak{g}) \to \operatorname{End} V$ be the algebra extension of π to $U(\mathfrak{g})$. One then has

Theorem 2.3. *If* $E^k \subset \operatorname{Ker} \pi_U$, *then*

$$[[\hat{x}_1, \ldots, \hat{x}_{2k}]] = 0 \tag{7}$$

for any $x_i, \ldots, x_{2k}$ *in* $\mathfrak{g}$.

3 Quotients of the tensor algebra

The Poincaré–Birkhoff–Witt theorem says that the restriction $\tau : S(\mathfrak{g}) \to U(\mathfrak{g})$ is a linear isomorphism. Consequently, given any $t \in T(\mathfrak{g})$ there exists a unique element $\overline{t}$ in $S(\mathfrak{g})$ such that

$$\tau(t) = \tau(\overline{t}). \tag{8}$$

Let $A^{even}(\mathfrak{g})$ be the span of alternating tensors of even degree. Restricting to $A^{even}(\mathfrak{g})$, one has a $\mathfrak{g}$-module map

$$\Gamma_T : A^{even}(\mathfrak{g}) \to S(\mathfrak{g})$$

defined so that if $a \in A^{even}(\mathfrak{g})$, then

$$\tau(a) = \tau(\Gamma_T(a)). \tag{9}$$

Now the (commutative) symmetric algebra $P(\mathfrak{g})$ over $\mathfrak{g}$ and exterior algebra $\wedge\mathfrak{g}$ are quotient algebras of $T(\mathfrak{g})$. The restriction of the quotient map clearly induces $\mathfrak{g}$-module isomorphisms

$$\begin{aligned} \tau_S &: S(\mathfrak{g}) \to P(\mathfrak{g}) \\ \tau_A &: A^{even}(\mathfrak{g}) \to \wedge^{even}\mathfrak{g}, \end{aligned} \tag{10}$$

where $\wedge^{even}\mathfrak{g}$ is the commutative subalgebra of $\wedge\mathfrak{g}$ spanned by elements of even degree.

We may complete the commutative diagram defining

$$\Gamma : \wedge^{even}\mathfrak{g} \to P(\mathfrak{g}) \tag{11}$$

so that on $A^{even}(\mathfrak{g})$, one has

$$\tau_S \circ \Gamma_T = \Gamma \circ \tau_A. \tag{12}$$

By (6) one notes that for $k \in \mathbb{Z}$, one has

$$\Gamma : \wedge^{2k}\mathfrak{g} \to P^k(\mathfrak{g}). \tag{13}$$

The Killing form extends to a nonsingular symmetric bilinear form on $P(\mathfrak{g})$ and $\wedge\mathfrak{g}$. This enables us to identify $P(\mathfrak{g})$ with the algebra of polynomial functions on $\mathfrak{g}$ and to identify $\wedge\mathfrak{g}$ with its dual space $\wedge\mathfrak{g}^*$ where $\mathfrak{g}^*$ is the dual space to $\mathfrak{g}$.

Let $R^k(\mathfrak{g})$ be the image of (13), i.e., the image of Γ, so that $R^k(\mathfrak{g})$ is a $\mathfrak{g}$-module of homogeneous polynomial functions of degree k on $\mathfrak{g}$. The significance of $R^k(\mathfrak{g})$ has to do with the dimensions of Ad $\mathfrak{g}$ adjoint (= coadjoint) orbits. Any such orbit is symplectic and hence is even dimensional. For $j \in \mathbb{Z}$, let

$$\mathfrak{g}^{(2j)} = \{x \in \mathfrak{g} \mid \dim [\mathfrak{g}, x] = 2j\}.$$

We recall that a $2j$ $\mathfrak{g}$-sheet is an irreducible component of $\mathfrak{g}^{(2j)}$. Let

$$\mathrm{Var}\, R^k(\mathfrak{g}) = \{x \in \mathfrak{g} \mid p(x) = 0,\ \forall\, p \in R^k(\mathfrak{g})\}.$$

In [4, Proposition 3.2] we prove

Theorem 3.1. *One has*

$$\mathrm{Var}\, R^k(\mathfrak{g}) = \cup_{2j<2k}\, \mathfrak{g}^{(2j)} \tag{14}$$

or that $\mathrm{Var}\, R^k(\mathfrak{g})$ *is the union of all* $2j$ $\mathfrak{g}$*-sheets for* $j < k$.

Let γ be the transpose of Γ. Thus

$$\gamma : P(\mathfrak{g}) \to \wedge^{even}\mathfrak{g}, \tag{15}$$

and one has for $p \in P(\mathfrak{g})$ and $u \in \wedge\mathfrak{g}$,

$$(\gamma(p), u) = (p, \Gamma(u)). \tag{16}$$

One also notes

$$\gamma : P^k(\mathfrak{g}) \to \wedge^{2k}\mathfrak{g}. \tag{17}$$

A proof of Theorem 3.1 depends upon establishing some nice algebraic properties of γ. Since we have, via the Killing form, identified $\mathfrak{g}$ with its dual, $\wedge\mathfrak{g}$ is the underlying space for a standard cochain complex $(\wedge\mathfrak{g}, d)$ where d is the coboundary operator of degree $+1$. In particular, if $x \in \mathfrak{g}$, then $dx \in \wedge^2\mathfrak{g}$. Identifying $\mathfrak{g}$ here with $P^1(\mathfrak{g})$, one has a map

$$P^1(\mathfrak{g}) \to \wedge^2\mathfrak{g}. \tag{18}$$

Theorem 3.2. *The map* (15) *is the homomorphism of commutative algebras extending* (18)*. In particular, for any* $x \in \mathfrak{g}$

$$\gamma(x^k) = (-dx)^k. \tag{19}$$

The connection with Theorem 3.1 follows from

Proposition 3.3. *Let* $x \in \mathfrak{g}$*. Then* $x \in \mathfrak{g}^{(2k)}$ *if and only if* k *is maximal such that* $(dx)^k \neq 0$*, in which case there is a scalar* $c \in \mathbb{C}^\times$ *such that*

$$(dx)^k = c\ w_1 \wedge \cdots \wedge w_{2k} \tag{20}$$

where w_i*,* $i = 1, \ldots, 2k$*, is a basis of* $[x, \mathfrak{g}]$.

For a proof of Theorem 3.2 and Proposition 3.3, see [4, Theorem 1.4 and Proposition 1.3].

We wish to explicitly describe the $\mathfrak{g}$-module $R^k(\mathfrak{g})$. (See [4, §1.2]). Let $J = P(\mathfrak{g})^{\mathfrak{g}}$ so that J is the ring of $\mathrm{Ad}\,\mathfrak{g}$ polynomial invariants. Let $\mathrm{Diff}\, P(\mathfrak{g})$ be the algebra of differential operators on $P(\mathfrak{g})$ with constant coefficients. One then has an algebra isomorphism

$$P(\mathfrak{g}) \to \text{Diff}\, P(\mathfrak{g}), \qquad q \mapsto \partial_q$$

where for $p, q, f \in P(\mathfrak{g})$, one has

$$(\partial_q p, f) = (p, qf) \tag{21}$$

and ∂_x, for $x \in \mathfrak{g}$, is the partial derivative defined by x.

Let $J_+ \subset J$ be the J-ideal of all $p \in J$ with zero constant term and let

$$H = \{q \in P(\mathfrak{g}) \mid \partial_p q = 0 \quad \forall\, p \in J_+\}.$$

H is a graded $\mathfrak{g}$-module whose elements are called harmonic polynomials. Then one knows (see [2, Theorem 11]) that

$$P(\mathfrak{g}) = J \otimes H, \tag{22}$$

where tensor product is realized by polynomial multiplication.

It is immediate from (21) that H is the orthocomplement of the ideal $J_+ P(\mathfrak{g})$ in $P(\mathfrak{g})$. However since γ is an algebra homomorphism one has

$$J_+ P(\mathfrak{g}) \subset \text{Ker}\,\gamma$$

since one easily has that $J_+ \subset \text{Ker}\,\gamma$. Indeed this is clear since

$$\gamma(J_+) \subset d(\wedge\,\mathfrak{g}) \cap (\wedge\,\mathfrak{g})^{\mathfrak{g}} = 0. \tag{23}$$

But then (16) implies

Theorem 3.4. *For any $k \in \mathbb{Z}$ one has*

$$R^k(\mathfrak{g}) \subset H.$$

Let $\text{Sym}(2k, 2)$ be the subgroup of the symmetric group $\text{Sym}(2k)$ defined by $\text{Sym}(2k, 2) = \{\sigma \in \text{Sym}(2k) \mid \sigma$ permutes the set of unordered pairs $(1, 2)$, $(3, 4)$, ..., $((2k-1), 2k)\}$. That is, if $\sigma \in \text{Sym}(2k, 2)$ and $1 \le i \le k$, there exists $1 \le j \le k$ such that as unordered sets

$$(\sigma(2i-1), \sigma(2i)) = ((2j-1), 2j).$$

It is clear that $\text{Sym}(2k, 2)$ is a subgroup of order $2^k \cdot k!$. Let $\Pi(k)$ be a cross-section of the set of left cosets of $\text{Sym}(2k, 2)$ in $\text{Sym}(2k)$ so that one has a disjoint union

$$\text{Sym}(2k) = \cup\, \nu\, \text{Sym}(2k, 2) \tag{24}$$

indexed by $\nu \in \Pi(k)$.

Remark 3.5. One notes that the cardinality of $\Pi(k)$ is $(2k-1)(2k-3)\cdots 1$ and the correspondence

$$\nu \mapsto ((\nu(1), \nu(2)), (\nu(3), \nu(4)), \ldots, (\nu((2k-1)), \nu(2k)))$$

sets up a bijection of $\Pi(k)$ with the set of all partitions of $(1, 2, \ldots, 2k)$ into a union of subsets each of which has two elements. We also observe that $\Pi(k)$ may be chosen — and will be chosen — such that $sg\,\nu = 1$ for all $\nu \in \Pi(k)$. This is clear since the sg character is not trivial on $\mathrm{Sym}(k, 2)$ for $k \geq 1$.

The following is a restatement of the results in [4, §3.2] (see especially [4, (3.25) and (3.29)]).

Theorem 3.6. *For any $k \in \mathbb{Z}$ there exists a nonzero scalar c_k such that for any x_i $i = 1, \ldots, 2k$, in $\mathfrak{g}$,*

$$\Gamma(x_1 \wedge \cdots \wedge x_{2k}) = c_k \sum_{\nu \in \Pi(k)} [x_{\nu(1)}, x_{\nu(2)}] \cdots [x_{\nu(2k-1)}, x_{\nu(2k)}]. \tag{25}$$

Furthermore the homogeneous polynomial of degree k on the right side of (25) is harmonic and $R^k(\mathfrak{g})$ is the span of all such polynomials for an arbitrary choice of the x_i.

4 On the variety of singular elements – joint with Nolan Wallach

Let $\mathfrak{h}$ be a Cartan sublgebra of $\mathfrak{g}$ and let $\ell = \dim \mathfrak{h}$ so $\ell = \operatorname{rank} \mathfrak{g}$. Let Δ be the set of roots of $(\mathfrak{h}, \mathfrak{g})$ and let $\Delta_+ \subset \Delta$ be a choice of positive roots. Let $r = \operatorname{card} \Delta_+$ so that $n = \ell + 2r$ where we fix $n = \dim \mathfrak{g}$. We assume a well ordering is defined on Δ_+. For any $\varphi \in \Delta$ let e_φ be a corresponding root vector. The choices will be normalized only insofar as $(e_\varphi, e_{-\varphi}) = 1$ for all $\varphi \in \Delta$. From Proposition 3.3 one recovers the well-known fact that $\mathfrak{g}^{(2k)} = 0$ for $k > r$ and $\mathfrak{g}^{(2r)}$ is the set of all regular elements in $\mathfrak{g}$. One also notes then that (16) implies $\mathrm{Var}\, R^r(\mathfrak{g})$ reduces to 0 if $k > r$ whereas Theorem 3.1 implies

$$\mathrm{Var}\, R^r(\mathfrak{g}) \text{ is the set of all singular elements in } \mathfrak{g}. \tag{26}$$

The paper [5] is mainly devoted to a study of a special construction of $R^r(\mathfrak{g})$ and a determination of its remarkable $\mathfrak{g}$-module structure.

It is a classic theorem of C. Chevalley that J is a polynomial ring in ℓ homogeneous generators p_i so that we can write

$$J = \mathbb{C}[p_1, \ldots, p_\ell].$$

Let $d_i = \deg p_i$. Then if we put $m_i = d_i - 1$, the m_i are referred to as the exponents of $\mathfrak{g}$, and one knows that

$$\sum_{i=1}^{\ell} m_i = r. \tag{27}$$

Henceforth assume $\mathfrak{g}$ is simple so that the adjoint representation is irreducible. Let $y_j,\ j = 1,\dots,n$, be the basis of $\mathfrak{g}$. One defines an $\ell \times n$ matrix $Q = Q_{ij},\ i = 1,\dots,\ell,\ j = 1,\dots,n$ by putting

$$Q_{ij} = \partial_{y_j} p_i. \tag{28}$$

Let $S_i,\ i = 1,\dots,\ell$, be the span of the entries of Q in the i^{th} row. The following is immediate.

Proposition 4.1. *$S_i \subset P^{m_i}(\mathfrak{g})$. Furthermore S_i is stable under the action of $\mathfrak{g}$ and as a $\mathfrak{g}$-module S_i transforms according to the adjoint representation.*

If V is a $\mathfrak{g}$-module, let V_{ad} be the set of all vectors in V which transform according to the adjoint representation. The equality (24) readily implies $P(\mathfrak{g})_{\mathrm{ad}} = J \otimes H_{\mathrm{ad}}$.

I proved the following result some time ago (See [2, §5.4]. Especially see [2, (5.4.6) and (5.4.7) in §5.4]).

Theorem 4.2. *The multiplicity of the adjoint representation in H_{ad} is ℓ. Furthermore the invariants p_i can be chosen so that $S_i \subset H_{\mathrm{ad}}$ for all i and the $S_i, i = 1,\dots,\ell$, are indeed the ℓ occurrences of the adjoint representation in H_{ad}.*

Clearly there are $\binom{n}{\ell}$ $\ell \times \ell$ minors in the matrix Q. The determinant of any of these minors is an element of $P^r(\mathfrak{g})$ by (27). In [5] we offer a different formulation of $R^r(\mathfrak{g})$ by proving the following.

Theorem 4.3. *The determinant of any $\ell \times \ell$ minor of Q is an element of $R^r(\mathfrak{g})$ and indeed $R^r(\mathfrak{g})$ is the span of the determinants of all these minors.*

Remark 4.4. In effect, Theorem 4.3 achieves the goal in Section 1, namely that

1. $R^r\mathfrak{g}$ arises from the matrix Q, and
2. $\mathrm{Var}R^r(\mathfrak{g})$ is the leading term in the filtration $\mathrm{Var}R^k(\mathfrak{g})$ of sheets in $\mathfrak{g}$.

5 The $\mathfrak{g}$-module structure of $R^r(\mathfrak{g})$

The adjoint action of $\mathfrak{g}$ on $\wedge\mathfrak{g}$ extends to $U(\mathfrak{g})$ so that $\wedge\mathfrak{g}$ is a $U(\mathfrak{g})$-module. If $\mathfrak{s} \subset \mathfrak{g}$ is any subpace and $k = \dim\mathfrak{s}$, let $[\mathfrak{s}] = \wedge^k\mathfrak{s}$ so that $[\mathfrak{s}]$ is a 1-dimensional subspace of $\wedge^k\mathfrak{g}$. Let $M_k \subset \wedge^k\mathfrak{g}$ be the span of all $[\mathfrak{s}]$ where $\mathfrak{s}$ is any k-dimensional commutative Lie subalgebra of $\mathfrak{g}$. If no such subalgebra exists, put $M_k = 0$. It is clear that M_k is a $\mathfrak{g}$-submodule of $\wedge^k\mathfrak{g}$. Let $\mathrm{Cas} \in Z$ be the Casimir element corresponding to the Killing form. The following theorem was proved as [3, Theorem (5)].

Theorem 5.1. *For any $k \in \mathbb{Z}$, let μ_k be the maximal eigenvalue of* Cas *on $\wedge^k\mathfrak{g}$. Then $\mu_k \le k$. Moreover $\mu_k = k$ if and only if $M_k \ne 0$ in which case M_k is the eigenspace for the maximal eigenvalue k.*

Let Φ be a subset of Δ. Write, in increasing order,

$$\Phi = \{\varphi_1, \ldots, \varphi_k\}, \tag{29}$$

where $k = \operatorname{card} \Phi$.

Let

$$e_\Phi = e_{\varphi_1} \wedge \cdots \wedge e_{\varphi_k}$$

so that $e_\Phi \in \wedge^k \mathfrak{g}$ is an ($\mathfrak{h}$) weight vector with weight

$$\langle \Phi \rangle = \sum_{i=1}^{k} \varphi_i .$$

Let $\mathfrak{n}$ be the Lie algebra spanned by e_φ for $\varphi \in \Delta_+$ and let $\mathfrak{b}$ the Borel subalgebra of $\mathfrak{g}$ defined by putting $\mathfrak{b} = \mathfrak{h} + \mathfrak{n}$. Now a subset $\Phi \subset \Delta_+$ will be called an ideal in Δ_+ if the span $\mathfrak{n}_\Phi$ of e_φ for $\varphi \in \Phi$ is an ideal of $\mathfrak{b}$. In such a case $\mathbb{C}e_\Phi$ is stable under the action of $\mathfrak{b}$ and hence if $V_\Phi = U(\mathfrak{g}) \cdot e_\Phi$ then, where $k = \operatorname{card} \Phi$,

$$V_\Phi \subset \wedge^k \mathfrak{g}$$

is an irreducible $\mathfrak{g}$-module of highest weight $\langle \Phi \rangle$ having $\mathbb{C}e_\Phi$ as the highest weight space. We will say Φ is abelian if $\mathfrak{n}_\Phi$ is an abelian ideal of $\mathfrak{b}$. Let

$$\mathcal{A}(k) = \{\Phi \mid \Phi \text{ be an abelian ideal of cardinality } k \text{ in } \Delta_+\}.$$

The following theorem was established in [3]. (See especially [3, Theorems (7) and (8)].)

Theorem 5.2. *If Φ, Ψ are distinct ideals in Δ_+, then V_Φ and V_Ψ are inequivalent (i.e., $\langle \Phi \rangle \neq \langle \Psi \rangle$). Furthermore if $M_k \neq 0$, then*

$$M_k = \oplus_{\Phi \in \mathcal{A}(k)} V_\Phi \tag{30}$$

so that, in particular, M_k is a multiplicity-1 $\mathfrak{g}$-module.

We now focus on the case where $k = \ell$. Clearly $M_\ell \neq 0$ since $\mathfrak{g}^x$ is an abelian subalgebra of dimension ℓ for any regular $x \in \mathfrak{g}$. Let $\mathcal{I}(\ell)$ be the set of all ideals of cardinality ℓ. The following theorem giving the remarkable structure of $R^r(\mathfrak{g})$ as a $\mathfrak{g}$-module is one of the main results in [4].

Theorem 5.3. *One has $\mathcal{I}(\ell) = \mathcal{A}(\ell)$ so that*

$$M_\ell = \oplus_{\Phi \in \mathcal{I}(\ell)} V_\Phi . \tag{31}$$

Moreover as $\mathfrak{g}$-modules one has the equivalence

$$R^r(\mathfrak{g}) \cong M_\ell \tag{32}$$

so that $R^r(\mathfrak{g})$ *is a multiplicity*-1 $\mathfrak{g}$-*module with* card $\mathcal{I}(\ell)$ *irreducible components and* Cas *takes the value* ℓ *on each and every one of the* $\mathcal{I}(\ell)$ *distinct components.*

Example 5.4. If $\mathfrak{g}$ is of type A_ℓ, then the elements of $\mathcal{I}(\ell)$ can be identified with Young diagrams of size ℓ. In this case therefore the number of irreducible components in $R^r(\mathfrak{g})$ is $P(\ell)$ where P here is the classical partition function.

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On conjugacy classes in a reductive group

George Lusztig

Dedicated to David Vogan
on the occasion of his 60th birthday

Abstract Let G be a connected reductive group over an algebraically closed field. We define a decomposition of G into finitely many strata such that each stratum is a union of conjugacy classes of fixed dimension; the strata are indexed purely in terms of the Weyl group and the indexing set is independent of the characteristic.

Key words: Conjugacy class, Springer correspondence, reductive group, Weyl group

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Introduction

0.1 Let $\mathbf{k}$ be an algebraically closed field of characteristic $p \geq 0$ and let G be a connected reductive algebraic group over $\mathbf{k}$. Let W be the Weyl group of G. Let $\mathrm{cl}(W)$ be the set of conjugacy classes of W.

In [St] Steinberg defined the notion of regular element in G (an element whose conjugacy class has dimension as large as possible, that is $\dim(G) - \mathrm{rk}(G)$) and showed that the set of regular elements in G form an open dense subset G_{reg}. The goal of this paper is to define a partition of G into finitely many strata, one of which is G_{reg}. Each stratum of G is a union of conjugacy classes of G of the same dimension. The set of strata is naturally indexed by a set which depends only on W

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as a Coxeter group, not on the underlying root system and not on the ground field $\mathbf{k}$. We give two descriptions of the indexing set above:

(i) one in terms of a class of irreducible representations of W which we call 2-special representations (they are obtained by truncated induction from special representations of certain reflection subgroups of W);
(ii) one in terms of $\mathrm{cl}(W)$ (modulo a certain equivalence relation).

In the case where W is irreducible we give a third description of the indexing set above:

(iii) in terms of the sets of unipotent classes in the various versions of G over $\overline{\mathbf{F}}_r$ for a variable prime number r, glued together according to the set of unipotent classes in the version of G over $\mathbf{C}$.

The definition of strata in the form (i) and (iii) are based on Springer's correspondence (see [Spr] when $p = 0$ or $p \gg 0$ and [L3] for any p) connecting irreducible representations of W with unipotent classes; when W is irreducible, the definition of strata in the form (iii) is related to that in the form (ii) by the results of [L8, L10] connecting $\mathrm{cl}(W)$ with unipotent classes in G.

Since (i),(ii) are two incarnations of our indexing set, they are in canonical bijection with each other. In particular we obtain a canonical map from $\mathrm{cl}(W)$ to the set of irreducible representations of W whose image consists of the 2-special representations (when G is $GL_n(\mathbf{k})$ this is a bijection). We also show that the dimension of a conjugacy class in a stratum of G is independent of the ground field. (This statement makes sense since the parametrization of the strata is independent of the ground field.) In particular, we see that if $n \geq 1$, then the following three conditions on an integer k are equivalent:

- there exists a conjugacy class of dimension k in $\mathrm{SO}_{2n+1}(\mathbf{C})$;
- there exists a conjugacy class of dimension k in $\mathrm{Sp}_{2n}(\mathbf{C})$;
- there exists a conjugacy class of dimension k in $\mathrm{Sp}_{2n}(\overline{\mathbf{F}}_2)$.

The proof shows that the following fourth condition is equivalent to the three conditions above: there exists a unipotent conjugacy class of dimension k in $\mathrm{Sp}_{2n}(\overline{\mathbf{F}}_2)$.

In Section 5 we sketch an alternative approach to the definition of strata which is based on an extension of the ideas in [L8], and Springer's correspondence does not appear in it.

In Section 6 we dicuss extensions of our results to the Lie algebra of G and to the case where G is replaced by a disconnected reductive group. We also define a partition of the set of compact regular semisimple elements in a loop group into strata analogous to the partition of G into strata. Moreover, we give a conjectural description of the strata of G (assuming that $\mathbf{k} = \mathbf{C}$) which is based on an extension of a construction in [KL].

0.2 Notation. For an algebraic group H over $\mathbf{k}$, we denote by H^0 the identity component of H. For a subgroup T of H we denote by $N_H T$ the normalizer of T in H. Let $\mathfrak{g}$ be the Lie algebra of G. For $g \in G$ we denote by $Z_G(g)$ the centralizer of g in G and by g_s (resp. g_u) the semisimple (resp. unipotent) part of g. Let $\mathcal{B}$ be the variety of Borel subgroups of G. Let $\mathcal{B}_g = \{B \in \mathcal{B}; g \in B\}$. Let l be a prime number $\neq p$. For an algebraic variety X over $\mathbf{k}$ we denote by $H^i(X)$ the l-adic cohomology of X in degree i; if X is projective let $H_i(X) = \mathrm{Hom}(H^i(X), \mathbf{Q}_l)$.

For any (finite) Weyl group Γ, we denote by Irr Γ a set of representatives for the isomorphism classes of irreducible representations of Γ over $\mathbf{Q}$. For any $\tau \in \mathrm{Irr} W$ let n_τ be the smallest integer $i \geq 0$ such that τ appears with > 0 multiplicity in the i-th symmetric power of the reflection representation of W; if this multiplicity is 1, we say that τ is *good.*

A *bipartition* is a sequence $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ in $\mathbf{N}$ such that $\lambda_m = 0$ for $m \gg 0$ and $\lambda_1 \geq \lambda_3 \geq \lambda_5 \geq \dots, \lambda_2 \geq \lambda_4 \geq \lambda_6 \geq \dots$. We write $|\lambda| = \lambda_1 + \lambda_2 + \lambda_3 + \dots$. We say that λ is a bipartition of n if $|\lambda| = n$. Let BP^n be the set of bipartitions of n. Let $e, e' \in \mathbf{N}$. We say that a bipartition $(\lambda_1, \lambda_2, \lambda_3, \dots)$ has excess (e, e') if $\lambda_i + e \geq \lambda_{i+1}$ for $i = 1, 3, 5, \dots$ and $\lambda_i + e' \geq \lambda_{i+1}$ for $i = 2, 4, 6, \dots$. Let $BP^n_{e,e'}$ be the set of bipartitions of n which have excess (e, e').

A *partition* is a sequence $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ in $\mathbf{N}$ such that $\lambda_m = 0$ for $m \gg 0$ and $\lambda_1 \geq \lambda_2 \geq \lambda_3 \geq \dots$. Thus a partition is the same as a bipartition of excess $(0, 0)$. On the other hand, a bipartition is the same as an ordered pair of partitions $((\lambda_1, \lambda_3, \lambda_5, \dots), (\lambda_2, \lambda_4, \lambda_6, \dots))$.

Let $\mathcal{P} = \{2, 3, 5, \dots\}$ be the set of prime numbers.

1 The 2-special representations of a Weyl group

1.1 Let V, V^* be finite-dimensional $\mathbf{Q}$-vector spaces with a given perfect bilinear pairing $\langle, \rangle : V \times V^* \to \mathbf{Q}$. Let R (resp. $\check{R}$) be a finite subset of $V - \{0\}$ (resp. $V^* - \{0\}$) with a given bijection $\alpha \leftrightarrow \check{\alpha}$, $R \leftrightarrow \check{R}$, such that $\langle \alpha, \check{\alpha} \rangle = 2$ for any $\alpha \in R$ and $\langle \alpha, \check{\beta} \rangle \in \mathbf{Z}$ for any $\alpha, \beta \in R$; it is assumed that $\beta - \langle \beta, \check{\alpha} \rangle \alpha \in R$, $\check{\beta} - \langle \alpha, \check{\beta} \rangle \check{\alpha} \in \check{R}$ for any $\alpha, \beta \in R$ and that $\alpha \in R \implies \alpha/2 \notin R$. Thus, $(V, V^*, R, \check{R})$ is a reduced root system. Let V_0 (resp. V_0^*) be the $\mathbf{Q}$-subspace of V (resp. V^*) spanned by R (resp. $\check{R}$). Let $\mathrm{rk}(R) = \dim V_0 = \dim V_0^*$. Let W be the (finite) subgroup of $GL(V)$ generated by the reflections $s_\alpha : x \mapsto x - \langle x, \check{\alpha} \rangle \alpha$ in V for various $a \in R$; it may be identified with the subgroup of $GL(V^*)$ generated by the reflections ${}^t s_a : x' \mapsto x' - \langle \alpha, x' \rangle \check{\alpha}$ in V^* for various $\alpha \in R$. For any $e \in V$ let $R_e = \{\alpha \in R; \langle e, \check{\alpha} \rangle \in \mathbf{Z}\}$, $\check{R}_e = \{\check{\alpha}; \alpha \in R_e\}$; note that $(V, V^*, R_e, \check{R}_e)$ is a root system with Weyl group $W_e = \{w \in W; w(e) - e \in \sum_{\alpha \in R} \mathbf{Z}\alpha\}$. Similarly, for any $e' \in V^*$ let $R_{e'} = \{\alpha \in R; \langle \alpha, e' \rangle \in \mathbf{Z}\}$, $\check{R}_{e'} = \{\check{\alpha}; \alpha \in R_{e'}\}$; note that $(V, V^*, R_{e'}, \check{R}_{e'})$ is a root system with Weyl group $W_{e'} = \{w \in W; w(e') - e' \in \sum_{\alpha \in R} \mathbf{Z}\check{\alpha}\}$. For any $(e, e') \in V \times V^*$ let $R_{e,e'} = R_e \cap R_{e'}$, $\check{R}_{e,e'} = \check{R}_e \cap \check{R}_{e'}$. Then $(V, V^*, R_{e,e'}, \check{R}_{e,e'})$ is a root system; let $W_{e,e'}$ be its Weyl group (a subgroup of $W_e \cap W_{e'}$). Note that $W_{0,e'} = W_{e'}$, $W_{e,0} = W_e$, $W_{0,0} = W$. For $E \in \mathrm{Irr}(W_{e,e'})$ let n_E be as in 0.2.

Let $(e_1, e_1') \in V \times V^*$, $(e_2, e_2') \in V \times V^*$ be such that $R_{e_1,e_1'} \subset R_{e_2,e_2'}$ (so that $W_{e_1,e_1'} \subset W_{e_2,e_2'}$). In this case, if $E \in \mathrm{Irr}(W_{e_1,e_1'})$ is good, there is a unique

$E_0 \in \mathrm{Irr}(W_{e_2,e_2'})$ such that E_0 appears in $\mathrm{Ind}_{W_{e_1,e_1'}}^{W_{e_2,e_2'}}(E)$ and $n_{E_0} = n_E$, see [LS1, 3.2]; moreover, E_0 is good. We set $E_0 = j_{W_{e_1,e_1'}}^{W_{e_2,e_2'}}(E)$. Note that if we have also $R_{e_2,e_2'} \subset R_{e_3,e_3'}$ where $(e_3, e_3') \in V \times V^*$, then we have the transitivity property:

(a) $j_{W_{e_1,e_1'}}^{W_{e_3,e_3'}}(E) = j_{W_{e_2,e_2'}}^{W_{e_3,e_3'}}(j_{W_{e_1,e_1'}}^{W_{e_2,e_2'}}(E))$.

Let $\mathcal{S}(W_{e,e'}) \subset \mathrm{Irr}(W_{e,e'})$ be the set of *special* representations of $W_{e,e'}$, see [L1]; note that any $E \in \mathcal{S}(W_{e,e'})$ is good. Hence $j_{W_{e,e'}}^{W}(E) \in \mathrm{Irr}(W)$ is defined. We say that $E_0 \in \mathrm{Irr}(W)$ is 2-*special* if $E_0 = j_{W_{e,e'}}^{W}(E)$ for some $(e, e') \in V \times V^*$ and some $E \in \mathcal{S}(W_{e,e'})$. Let $\mathcal{S}_2(W)$ be the set of all 2-special representations of W (up to isomorphism). From the definition we see that

(b) *$\mathcal{S}_2(W)$ is unchanged when $(V, V^*, R, \check{R})$ is replaced by $(V^*, V, \check{R}, R)$.*

Let $\mathcal{S}_1(W)$ (resp. $'\mathcal{S}_1(W)$) be the set of all $E_0 \in \mathrm{Irr}(W)$ such that $E_0 = j_{W_e}^{W}(E)$ (resp. $E_0 = j_{W_{e'}}^{W}(E)$) for some $e \in V$, $E \in \mathcal{S}(W_e)$ (resp. $e' \in V^*$, $E \in \mathcal{S}(W_{e'})$). The analogue of (b) with $\mathcal{S}_2(W)$ replaced by $\mathcal{S}_1(W)$ is not true in general; instead, if $(V, V^*, R, \check{R})$ is replaced by $(V^*, V, \check{R}, R)$, then $\mathcal{S}_1(W)$ becomes $'\mathcal{S}_1(W)$ and $'\mathcal{S}_1(W)$ becomes $\mathcal{S}_1(W)$.

Now, for any $e' \in V^*$ the subset $\mathcal{S}_1(W_{e'}) \subset \mathrm{Irr}(W_{e'})$ is defined; it consists of all $E' \in \mathrm{Irr}(W_{e'})$ such that $E' = j_{W_{e,e'}}^{We'}(E)$ for some $e \in V$ and some $E \in \mathcal{S}(W_{e,e'})$. Note that any $E' \in \mathcal{S}_1(W_{e'})$ is good. From (a) we see that

(c) *$\mathcal{S}_2(W)$ consists of all $E_0 \in \mathrm{Irr}(W)$ such that $E_0 = j_{W_{e'}}^{W}(E')$ for some $e' \in V^*$ and some $E' \in \mathcal{S}_1(W_{e'})$.*

We say that $e' \in V^*$ (resp. $(e, e') \in V \times V^*$) is *isolated* if $\mathrm{rk}(R_{e'}) = \mathrm{rk}(R)$ (resp. $\mathrm{rk}(R_{e,e'}) = \mathrm{rk}(R)$). We show:

(d) *$\mathcal{S}_2(W)$ consists of all $E_0 \in \mathrm{Irr}(W)$ such that $E_0 = j_{W_{e,e'}}^{W}(E)$ for some isolated $(e, e') \in V \times V^*$ and some $E \in \mathcal{S}(W_{e,e'})$.*

Let $E_0 \in \mathcal{S}_2(W)$. By definition, we can find $(e, e') \in V \times V^*$ and $E \in \mathcal{S}(W_{e,e'})$ such that $E_0 = j_{W_{e,e'}}^{W}(E)$. We can find an isolated $e_1' \in V^*$ such that $R_{e'}$ is rationally closed in $R_{e_1'}$ that is, $R_{e_1'} \cap \sum_{\alpha \in R_{e'}} \mathbf{Q}\alpha = R_{e'}$. Applying the analogous statement to $(V^*, V, \check{R}_{e_1'}, R_{e_1'})$, e, instead of $(V, V^*, R, \check{R})$, e', we can find $e_1 \in V$ such that $\mathrm{rk}(R_{e_1} \cap R_{e_1'}) = \mathrm{rk}(R_{e_1'})$ and $R_e \cap R_{e_1'}$ is rationally closed in $R_{e_1} \cap R_{e_1'}$. It follows that (e_1, e_1') is isolated and $R_e \cap R_{e'}$ is rationally closed in $R_{e_1} \cap R_{e_1'}$; hence $E_1 := j_{W_{e,e'}}^{W_{e_1,e_1'}}(E)$ is in $\mathcal{S}(W_{e_1,e_1'})$, see [L1]. By (a), we have $E_0 = j_{W_{e_1,e_1'}}^{W}(E_1)$. This proves (d).

We have the following variant of (d):

(e) $\mathcal{S}_2(W)$ *consists of all* $E_0 \in \mathrm{Irr}(W)$ *such that* $E_0 = j_{W_{e'}}^W(\widetilde{E})$ *for some isolated* $e' \in V^*$ *and some* $\widetilde{E} \in \mathcal{S}_1(W_{e'})$.

Let $E_0 \in \mathcal{S}_2(W)$. Let E, e, e' be as in (d). We have $E = j_{W_{e'}}^W(\widetilde{E})$ where $\widetilde{E} = j_{W_{e,e'}}^{W_{e'}}(E) \in \mathcal{S}_1(W_{e'})$ and $\mathrm{rk}(R_{e'}) = \mathrm{rk}(R)$. Conversely, if $e' \in V^*$ and $\widetilde{E} \in \mathcal{S}_1(W_{e'})$, then, by (c), $j_{W_{e'}}^W(\widetilde{E}) \in \mathcal{S}_2(W)$ (even without the assumption that $\mathrm{rk}(R_{e'}) = \mathrm{rk}(R)$). This proves (e).

Let $R' \subset R$ be such that (if $\check{R}'$ is the image of R' under $R \leftrightarrow \check{R}$), $(V, V^*, R', \check{R}')$ is a root system (with Weyl group W') and R' is rationally closed in R. Note that $R' = R_e$ for some $e \in V$ and $R' = R_{e'}$ for some $e' \in V^*$. We show:

(f) If $E \in \mathcal{S}_1(W')$, then $j_{W'}^W(E) \in \mathcal{S}_1(W)$.
(g) If $E \in \mathcal{S}_2(W')$, then $j_{W'}^W(E) \in \mathcal{S}_2(W)$.

We prove (f). Let $e' \in V^*$ be such that $R' = R_{e'}$. We have $E = j_{W_{e,e'}}^{W_{e'}}(E')$ for some $e \in V$ and some $E' \in \mathcal{S}(W_{e,e'})$. Hence $j_{W'}^W(E) = j_{W_{e,e'}}^W(E') = j_{W_e}^W(E'')$ where $E'' = j_{W_{e,e'}}^{W_e}(E')$. Now $R_{e,e'}$ is rationally closed in R_e, hence $E'' \in \mathcal{S}(W_e)$, see [L1]. We see that $j_{W'}^W(E) \in \mathcal{S}_1(W)$.

We prove (g). Let $e \in V$ be such that $R' = R_e$. We have $E = j_{W_{e,e'}}^{W_e}(E')$ for some $e' \in V^*$ and some $E' \in \mathcal{S}_1(W_{e,e'})$. Hence $j_{W'}^W(E) = j_{W_{e,e'}}^W(E') = j_{W_{e'}}^W(E'')$ where $E'' = j_{W_{e,e'}}^{W_{e'}}(E')$. Now $R_{e,e'}$ is rationally closed in $R_{e'}$, hence $E'' \in \mathcal{S}(W_{e'})$, see (f). We see that $j_{W'}^W(E) \in \mathcal{S}_2(W)$.

1.2 There are unique direct sum decompositions $V_0 = \oplus_{i \in I} V_i$, $V_0^* = \oplus_{i \in I} V_i^*$ such that $R = \sqcup_{i \in I}(R \cap V_i)$, $\check{R} = \sqcup_{i \in I}(\check{R} \cap V_i)$ and for any $i \in I$, $(V_i, V_i^*, R \cap V_i, \check{R} \cap V_i)$ is an irreducible root system for (with Weyl group W_i); the bijection $R \cap V_i \leftrightarrow \check{R} \cap V_i$ is induced by $R \leftrightarrow \check{R}$). We have canonically $W = \prod_{I \in I} W_i$ and $\mathcal{S}_2(W) = \prod_{i \in I} \mathcal{S}_2(W_i)$ (via external tensor product).

1.3 In this subsection we assume that $(V, V^*, R, \check{R})$ is irreducible. Now W acts naturally on the set of subgroups W' of W of form $W_{e'}$ for various isolated $e' \in V^*$. The types of various W' which appear in this way are well known and are described below in each case.

(a) R of type A_n, $n \geq 0$: W' of type A_n.
(b) R of type B_n, $n \geq 2$: W' of type $B_a \times D_b$ where $a \in \mathbf{N}$, $b \in \mathbf{N} - \{1\}$, $a + b = n$.
(c) R of type C_n, $n \geq 2$: W' of type $C_a \times C_b$ where $a, b \in \mathbf{N}$, $a + b = n$.
(d) R of type D_n, $n \geq 4$: W' of type $D_a \times D_b$ where $a, b \in \mathbf{N} - \{1\}$, $a + b = n$.
(e) R of type E_6: W' of type E_6, A_5A_1, $A_2A_2A_2$.
(f) R of type E_7: W' of type E_7, D_6A_1, A_7, A_5A_2, $A_3A_3A_1$.
(g) R of type E_8: W' of type E_8, E_7A_1, E_6A_2, D_5A_3, A_4A_4, $A_5A_2A_1$, A_7A_1, A_8, D_8.

(h) R of type F_4: W' of type F_4, B_3A_1, A_2A_2, A_3A_1, B_4.
(i) R of type G_2: W' of type G_2, A_2, A_1A_1.

(We use the convention that a Weyl group of type B_n or D_n with $n = 0$ is $\{1\}$.)

1.4 In this subsection we assume that $(V, V^*, R, \check{R})$ is irreducible. Now W acts naturally on the set of subgroups W' of W of form $W_{e,e'}$ for various isolated $(e, e') \in V \times V^*$. The types of various W' which appear in this way are described below in each case. (For type F_4 and G_2 we denote by τ a non-inner involution of W).

(a) R of type A_n: W' of type A_n.
(b) R of type B_n or C_n: W' of type $B_a \times B_b \times D_c \times D_d$ where $a, b \in \mathbf{N}$, $c, d \in \mathbf{N} - \{1\}$, $a + b + c + d = n$.
(c) R of type D_n: W' of type $D_a \times D_b \times D_c \times D_d$ where $a, b, c, d \in \mathbf{N} - \{1\}$, $a + b + c + d = n$.
(d) R of type E_6: W' as in 1.3(e).
(e) R of type E_7: W' as in 1.3(f) and also W' of type $D_4A_1A_1A_1$.
(f) R of type E_8: W' as in 1.3(g) and also W' of type D_6D_2, D_4D_4, $A_3A_3A_1A_1$, $A_2A_2A_2A_2$.
(g) R of type F_4: W' as in 1.3(h), the images under τ of the subgroups W' of type A_3A_1, B_4 in 1.3(h) and also W' of type B_2B_2.
(h) R of type G_2: W' as in 1.3(i) and the image under τ of the subgroup W' of type A_2 in 1.3(i).

1.5 If $R' \subset R$, $\check{R}' \subset \check{R}$ are such that $(V, V^*, R', \check{R}')$ is a root system (with the bijection $R' \leftrightarrow \check{R}'$ being induced by $R \leftrightarrow \check{R}$) then, setting $\overline{R}' = R \cap \sum_{\alpha \in R'} \mathbf{Q}\alpha$, $\overline{\check{R}}' = \check{R} \cap \sum_{\alpha \in R'} \mathbf{Q}\check{\alpha}$, we obtain a root system $(V, V^*, \overline{R}', \overline{\check{R}}')$. We set

$$N_{R'} = \sharp(\sum_{\alpha \in \overline{R}'} \mathbf{Z}\alpha / \sum_{\alpha \in R'} \mathbf{Z}\alpha) \in \mathbf{Z}_{\geq 1}.$$

For any $e' \in V^*$ we set $N_{e'} = N_{R_{e'}}$.

Now let $r \in \mathcal{P}$. Let $\mathcal{S}_2^r(W)$ be the set of all $E_0 \in \mathrm{Irr}(W)$ such that for some isolated $e' \in V^*$ with $N_{e'} = r^k$ for some $k \in \mathbf{N}$ and for some $E \in \mathcal{S}_1(W_{e'})$ we have $E_0 = j_{W_{e'}}^W(E)$. Note that $\mathcal{S}^1(W) \subset \mathcal{S}_2^r(W) \subset \mathcal{S}_2(W)$.

Now assume that $(V, V^*, R, \check{R})$ is irreducible. We show:

(a) If R is of type A_n, $n \geq 0$, then $\mathcal{S}_2^r(W) = \mathcal{S}_2(W) = \mathcal{S}_1(W) = \mathcal{S}(W)$.
(b) If R is of type B_n or C_n, $n \geq 2$, then $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$ if $r \neq 2$ and $\mathcal{S}_2^2(W) = \mathcal{S}_2(W)$.
(c) If R is of type D_n, $n \geq 4$, then $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$ if $r \neq 2$ and $\mathcal{S}_2^2(W) = \mathcal{S}_2(W)$.
(d) If R is of type E_6, then $\mathcal{S}_2^r(W) = \mathcal{S}_2(W) = \mathcal{S}_1(W)$.
(e) If R is of type E_7, then $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$ if $r \neq 2$ and $\mathcal{S}_2^2(W) = \mathcal{S}_2(W)$.
(f) If R is of type E_8, then $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$ if $r \notin \{2, 3\}$ and $\mathcal{S}_2^2(W) \cup \mathcal{S}_2^3(W) = \mathcal{S}_2(W)$.

(g) If R is of type F_4, then $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$ if $r \neq 2$ and $\mathcal{S}_2^2(W) = \mathcal{S}_2(W)$.
(h) If R is of type G_2, then $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$ if $r \neq 3$ and $\mathcal{S}_2^3(W) = \mathcal{S}_2(W)$.

We prove (a). In this case for any isolated $e' \in V^*$ we have $N_{e'} = 1$ and the result follows from 1.1(d),(e), 1.3.

We prove (b), (c). In these cases for any isolated $e' \in V^*$, $N_{e'}$ is a power of 2 (see 1.3) and the equality $\mathcal{S}_2^2(W) = \mathcal{S}_2(W)$ follows from 1.1(e). Moreover, if e' is isolated and $N_{e'}$ is not divisible by 2, then $W_{e'} = W$ so that for $r \neq 2$ we have $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$.

In cases (d), (e), (f) we shall use the fact that for any $e' \in V^*$:

(i) we can find $e \in V$ such that $W_{e'} = W_e$, so that if $E \in \mathcal{S}(W_{e'})$, then $j_{W_{e'}}^W(E) \in \mathcal{S}_1(W)$.

(This property does not always hold in cases (g),(h).)

We prove (d). If $e' \in V^*$ is isolated and $W_{e'} \neq W$, then from 1.3 we see that $W_{e'}$ is of type $A_2A_2A_2$ or A_5A_1 so that $\mathcal{S}_1(W_{e'}) = \mathcal{S}(W_{e'})$; using this and 1.1(e) we see that $\mathcal{S}_2(W) = \mathcal{S}_2^r = \mathcal{S}_1(W)$. (We have used (i).)

We prove (e). If $e' \in V^*$ is isolated and $W_{e'}$ is not of type E_7 (with $N_{e'} = 1$) or D_6A_1 (with $N_{e'} = 2$), then from 1.3 we see that $W_{e'}$ is of type A_7 or A_5A_2 or $A_3A_3A_1$ so that $\mathcal{S}_1(W_{e'}) = \mathcal{S}(W_{e'})$. We see that $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$ if $r \neq 2$ and $\mathcal{S}_2^2(W) = \mathcal{S}_2(W)$. (We have used (i).)

We prove (f). If $e' \in V^*$ is isolated and $W_{e'}$ is not of type E_8 (with $N_{e'} = 1$) or E_7A_1 (with $N_{e'} = 2$) or E_6A_2 (with $N_{e'} = 3$) or D_5A_3 (with $N_{e'} = 4$) or D_8 (with $N_{e'} = 2$), then from 1.3 we see that $W_{e'}$ is of type A_4A_4 or $A_5A_2A_1$ or A_7A_1 or A_8, so that $\mathcal{S}_1(W_{e'}) = \mathcal{S}(W_{e'})$; we see that $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$ if $r \notin \{2,3\}$ and $\mathcal{S}_2^2(W) \cup \mathcal{S}_2^3(W) = \mathcal{S}_2(W)$. (We have used (i).)

We prove (g). If $e' \in V^*$ is isolated and $W_{e'}$ is not of type F_4 (when $N_{e'} = 1$) or B_3A_1 (with $N_{e'}$ a power of 2) or B_4 (with $N_{e'}$ a power of 2), then from 1.3 we see that $W_{e'}$ is of type A_2A_2 (with $N_{e'} = 3$) or A_3A_1 (with $N_{e'}$ a power of 2) so that $\mathcal{S}_1(W_{e'}) = \mathcal{S}(W_{e'})$. Moreover, if $e' \in V^*$ is isolated and $W_{e'}$ is of type A_2A_2, then (i) holds for this e'. We see that $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$ if $r \neq 2$ and $\mathcal{S}_2^2(W) = \mathcal{S}_2(W)$.

We prove (h). If $e' \in V^*$ is isolated and $W_{e'}$ is not of type G_2 (with $N_{e'} = 1$), then from 1.3 we see that $W_{e'}$ is of type A_2 (with $N_{e'} = 3$) or A_1A_1 (when $N_{e'} = 2$) so that $\mathcal{S}_1(W_{e'}) = \mathcal{S}(W_{e'})$. Moreover, if $e' \in V^*$ is isolated and $W_{e'}$ is of type A_1A_1, then (i) holds for this e'. We see that $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$ if $r \neq 3$ and $\mathcal{S}_2^3(W) = \mathcal{S}_2(W)$.

This proves (a)–(h). From (a)–(h) we deduce:

(j) We have $\mathcal{S}_2(W) = \mathcal{S}_2^2(W) \cup \mathcal{S}_2^3(W)$. If $r \in \mathcal{P} - \{2,3\}$, then $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$.

The following result can be verified by computation.

(k) If R is of type E_7, then $\mathcal{S}_2^2(W) - \mathcal{S}_1(W) = \{84_{15}\}$. If R is of type E_8, then $\mathcal{S}_2^2(W) - \mathcal{S}_1(W) = \{1050_{10}, 840_{14}, 168_{24}, 972_{32}\}$ and $\mathcal{S}_2^3(W) - \mathcal{S}_1(W) = \{175_{12}\}$. If R is of type F_4, then $\mathcal{S}_2^2(W) - \mathcal{S}_1(W) = \{9_6, 4_7, 4_8, 2_{16}\}$. If R is of type G_2, then $\mathcal{S}_2^3(W) - \mathcal{S}_1(W) = \{1_3\}$.

(In each case we specify a representation E by a symbol d_n where d is the degree of E and $n = n_E$. For type F_4 and G_2 the specified representations are uniquely determined by the additional condition that they are not in $\mathcal{S}_1(W)$.)

(l) $\mathcal{S}_2^2(W) \cap \mathcal{S}_2^3(W) = \mathcal{S}_1(W)$.

The inclusion $\mathcal{S}_1(W) \subset \mathcal{S}_2^2(W) \cap \mathcal{S}_2^3(W)$ is obvious. The reverse inclusion for R of type $\neq E_8$ follows from the fact that for such R we have either $\mathcal{S}_2^2(W) = \mathcal{S}_1(W)$ or $\mathcal{S}_2^3(W) = \mathcal{S}_1(W)$, see (a)–(h). Thus we can assume that R is of type E_8. In this case the result follows from (k).

1.6 Let $r \in \mathcal{P}$. Let $V_r^* = \{e' \in V^*; N_{e'}/r \notin \mathbf{Z}\}$. Let $\widetilde{\mathcal{S}}_2^r(W)$ be the set of all $E_0 \in \mathrm{Irr}(W)$ such that for some $e' \in V_r^*$ and some $E \in \mathcal{S}_2^r(W_{e'})$ we have $E_0 = j_{W_{e'}}^W(E)$. (Note that any $E \in \mathcal{S}_2^r(W_{e'})$ is good.) Note that $\mathcal{S}_2^r(W) \subset \widetilde{\mathcal{S}}_2^r(W)$ (take $e' = 0$ in the definition of $\widetilde{\mathcal{S}}_2^r(W)$). We show:

(a) $\mathcal{S}_2(W) \subset \widetilde{\mathcal{S}}_2^r(W)$.

We can assume that $(V, V^*, R, \check{R})$ is irreducible. Let $E_0 \in \mathcal{S}_2(W)$. We must show that $E_0 \in \widetilde{\mathcal{S}}_2^r(W)$. By 1.1(e) we can find an isolated $e' \notin V^*$ and $\widetilde{E} \in \mathcal{S}_1(W_{e'})$ such that $E_0 = j_{W_{e'}}^W(\widetilde{E})$. If $N_{e'}/r \notin \mathbf{Z}$ then we have $E_0 \in \widetilde{\mathcal{S}}_2^r(W)$ since $\mathcal{S}_1(W_{e'}) \subset \mathcal{S}_2^r(W_{e'})$. If $N_{e'}$ is a power of r, then from definitions we have $E_0 \in \mathcal{S}_2^r(W)$, hence $E_0 \in \widetilde{\mathcal{S}}_2^r(W)$. Thus we may assume that $N_{e'}$ is not a power of r and is $N_{e'}/r \in \mathbf{Z}$. This forces R to be of type E_8 and $W_{e'}$ to be of type $A_5A_2A_1$ (see 1.3); we then have $N_{e'} = 6$ and $r \in \{2, 3\}$. In particular we must have $\widetilde{E} \in \mathcal{S}(W_{e'})$. If $\widetilde{E}$ is not the sign representation of $W_{e'}$, then we have $\widetilde{E} = j_{W_{e_1'}}^{W_{e'}}(\mathrm{sign})$ for some $e_1' \in V^*$ such that $W_{e_1'}$ is a proper parabolic subgroup of $W_{e'}$. Replacing $W_{e_1'}$ by a W-conjugate we can assume that $W_{e_1'}$ is a proper parabolic subgroup of W so that $j_{W_{e'}}^W(\mathrm{sign}) \in \mathcal{S}(W)$ and in particular, $E_0 \in \widetilde{\mathcal{S}}_2^r(W)$. Thus we can assume that $\widetilde{E}$ is the sign representation of $W_{e'}$. We have $W_{e'} \subset W_{e_2'}$ where $W_{e_2'}$ is of type E_7A_1 and by the definition of $\mathcal{S}_1(W_{e_2'})$ we have

$$\widetilde{E}_2 := j_{W_{e'}}^{W_{e_2'}}(\mathrm{sign}) \in \mathcal{S}_1(W_{e_2'}).$$

If $r = 3$, we have $e_2' \in V_r^*$ hence $E_0 = j_{W_{e_2'}}^W(\widetilde{E}_2) \in \widetilde{\mathcal{S}}_2^r(W)$. We have $W_{e'} \subset W_{e_3'}$ where $W_{e_3'}$ is of type E_6A_2 and by the definition of $\mathcal{S}_1(W_{e_3'})$, we have $\widetilde{E}_3 := j_{W_{e'}}^{W_{e_3'}}(\mathrm{sign}) \in \mathcal{S}_1(W_{e_3'})$. If $r = 2$, we have $e_3' \in V_r^*$ hence $E_0 = j_{W_{e_3'}}^W(\widetilde{E}_3) \in \widetilde{\mathcal{S}}_2^r(W)$. This completes the proof of (a).

We show:

(b) $\widetilde{\mathcal{S}}_2^r(W) \subset \mathcal{S}_2(W)$.

We can assume that $(V, V^*, R, \check{R})$ is irreducible. Let $E_0 \in \widetilde{\mathcal{S}}_2^r(W)$. We must show that $E_0 \in \mathcal{S}_2(W)$. Assume first that $r \notin \{2, 3\}$. Then by results in 1.5 we have

$E \in \mathcal{S}_1(W_{e'})$, hence by 1.1(c) we have $E_0 \in \mathcal{S}_2(W)$. Next we assume that $r = 3$. If $W_{e'} \neq W$, then by results in 1.5 we have $E \in \mathcal{S}_1(W_{e'})$ hence by 1.1(c) we have $E_0 \in \mathcal{S}_2(W)$. Thus we can assume that $W_{e'} = W$ so that $E_0 = E \in \mathcal{S}_2^r(W)$. Since $\mathcal{S}_2^r(W) \subset \mathcal{S}_2(W)$ we see that $E_0 \in \mathcal{S}_2(W)$.

We now assume that $r = 2$. We can find $e' \in V_r^*$ and $E \in \mathcal{S}_2^r(W_{e'})$ such that $E_0 = j_{W_{e'}}^W(E)$. We can find an isolated $e'_1 \in V^*$ such that $N_{e'_1}$ is odd, $R_{e'} \subset R_{e'_1}$ and $R_{e'}$ is rationally closed in $R_{e'_1}$. Let $E' = j_{W_{e'}}^{W_{e'_1}}(E)$. Since $E \in \mathcal{S}_2(W_{e'})$ we have $E' \in \mathcal{S}_2(W_{e'_1})$, see 1.1(g) and $E_0 = j_{W_{e'_1}}^W(E')$. It is then enough to prove the following statement:

(c) If $e' \in V_r^*$ is isolated ($r = 2$) and $E \in \mathcal{S}_2(W_{e'})$, then $E_0 = j_{W_{e'}}^W(E) \in \mathcal{S}_2(W)$.

If $W_{e'} = W$, then $E_0 = E \in \mathcal{S}_2(W)$, as required. If R is of type A_n, B_n, C_n, D_n, then in (c) we have automatically $W_{e'} = W$ hence (c) holds in these cases. Thus we can assume in (c) that R is of exceptional type and $W_{e'} \neq W$. Then $W_{e'}$ is of the following type: $A_2A_2A_2$ (if R is of type E_6); A_5A_2 (if R is of type E_7); A_4A_4 or A_8 or E_6A_2 (if R is of type E_8); A_2A_2, as in 1.3(h) (if R is of type F_4); A_2, as in 1.3(i) (if R is of type G_2). In each case we have $\mathcal{S}_2(W_{e'}) = \mathcal{S}_1(W_{e'})$, see 1.5. Thus $E \in \mathcal{S}_1(W_{e'})$. Using 1.1(e) we see that $E_0 \in \mathcal{S}_2(W)$. This proves (c) hence (b).

Combining (a), (b) we obtain

(d) $\widetilde{\mathcal{S}}_2^r(W) = \mathcal{S}_2(W)$.

In the case where $r = 0$, we set $V_0^* = V^*$, $\mathcal{S}_2^0(W) = \mathcal{S}_1(W)$, $\widetilde{\mathcal{S}}_2^0(W) = \mathcal{S}_2(W)$.

2 The strata of G

2.1 We return to the setup of the introduction. Thus G is a connected reductive algebraic group over $\mathbf{k}$. Let $\mathcal{T}$ be "the" maximal torus of G; let $X = \text{Hom}(\mathcal{T}, \mathbf{k}^*)$, $Y = \text{Hom}(\mathbf{k}^*, \mathcal{T})$, $V = \mathbf{Q} \otimes X$, $V^* = \mathbf{Q} \otimes Y$. We have an obvious perfect bilinear pairing $\langle , \rangle : V \times V^* \to \mathbf{Q}$. Let $R \subset V$ be the set of roots and let $\check{R} \subset V^*$ be the set of corrots. Then $(V, V^*, R, \check{R})$ is as in 1.1. The associated Weyl group W (as in 1.1) that is, the Weyl group of G, can be viewed as an indexing set for the orbits of G acting diagonally on $\mathcal{B} \times \mathcal{B}$; we denote by $\mathcal{O}_w$ the orbit corresponding to $w \in W$. Note that W is naturally a Coxeter group.

Let $g \in G$. Let W_g be the Weyl group of the connected reductive group $H := Z_G(g_s)^0$. We can view W_g as a subgroup of W as follows. Let β be a Borel subgroup of H and let T be a maximal torus of β. We define an isomorphism $b_{T,\beta} : N_H T/T \xrightarrow{\sim} W_g$ by $n'T \mapsto H$-orbit of $(\beta, n'\beta n'^{-1})$. Similarly for any $B \in \mathcal{B}$ such that $T \subset B$ we define an isomorphism $a_{T,B} : N_G T/T \xrightarrow{\sim} W$ by $n'T \mapsto G$-orbit of $(B, n'Bn'^{-1})$. Now assume that $B \in \mathcal{B}$ is such that $B \cap H = \beta$.

We define an embedding $c_{T,\beta,B} : W_g \to W$ as the composition $W_g \xrightarrow{b_{T,\beta}^{-1}} N_H T/T \to N_G T/T \xrightarrow{a_{T,B}} W$ where the middle map is the obvious embedding. If $B' \in \mathcal{B}$ also satisfies $B' \cap H = \beta$, then we have $B' = nBn^{-1}$ for some $n \in N_G T$ and from the definitions we have $c_{T,\beta,B'}(w) = a_{T,B}(nT)c_{T,\beta,B}(w)a_{B,T}(nT)^{-1}$ for any $w \in W_g$. Thus $c_{T,\beta,B}$ depends (up to composition with an inner automorphism of W) only on T, β and we can denote it by $c_{T,\beta}$. Since the set of pairs T, β as above form a homogeneous space for the connected group H, we see that $c_{T,\beta}$ is independent of T, β (up to composition with an inner automorphism of W) hence it does not depend on any choice. We see that there is a well-defined collection $\mathcal{C}$ of embeddings $W_g \to W$ so that any two of them differ only by composition by an inner automorphism of W.

Define $\rho \in \mathrm{Irr}(W_g)$ by the condition that under the Springer correspondence for H, ρ corresponds to the H-conjugacy class of g_u and the trivial local system on it. We choose $f \in \mathcal{C}$; then we can view ρ as an irreducible representation of $f(W_g)$, a subgroup of W such that $f(W_g) = W_{e'}$ for some $e' \in V_p^*$, see 1.6. By [L5, 1.4] we have $\rho \in \mathcal{S}_2^p(f(W_g))$, see 1.5, 1.6. Hence $\widetilde{\rho} := j_{f(W_g)}^W(\rho) \in \widetilde{\mathcal{S}}_2^p(W)$ is well defined. Since $\widetilde{\mathcal{S}}_2^p(W) = \mathcal{S}_2(W)$, see 1.6, we have $\widetilde{\rho} \in \mathcal{S}_2(W)$. This is independent of the choice of f since f is well defined up to composition by an inner automorphism of W.

2.2 Let $g \in G$. Let $d = d_g = \dim \mathcal{B}_g$. The embedding $h_g : \mathcal{B}_g \to \mathcal{B}$ induces a linear map $h_{g*} : H_{2d}(\mathcal{B}_g) \to H_{2d}(\mathcal{B})$. Now $H^{2d}(\mathcal{B}_g), H^{2d}(\mathcal{B})$ carry natural W-actions, see [L3], and this induces natural W-actions on $H_{2d}(\mathcal{B}_g), H_{2d}(\mathcal{B})$ which are compatible with h_{g*}. Hence W acts naturally on the subspace $h_{g*}(H_{2d}(\mathcal{B}_g))$ of $H_{2d}(\mathcal{B})$.

The following result gives an alternative description of the map $g \mapsto \widetilde{\rho}$ (in 2.1) from G to $\mathrm{Irr}W$.

(a) *The W-submodule $h_{g*}(H_{2d}(\mathcal{B}_g))$ of $H_{2d}(\mathcal{B})$ is isomorphic to the W-module $\overline{\mathbf{Q}}_l \otimes \widetilde{\rho}$ where $\rho, \widetilde{\rho}$ are associated to g as in 2.1.*

First, we note that $h_{g*}(H_{2d}(\mathcal{B}_g)) \neq 0$; indeed it is clear that for any irreducible component D of $\mathcal{B}_g$ (necessarily of dimension d), the image of the fundamental class of D under h_{g*} is nonzero (we ignore Tate twists). Let $\mathcal{B}'$ be the variety of Borel subgroups of $Z_G(g_s)^0$. Let $\mathcal{B}'_{g_u} = \{\beta \in \mathcal{B}'; g_u \in \beta\}$. Then $\dim \mathcal{B}' = d$ and W_g (see 2.1) acts naturally on $H_{2d}(\mathcal{B}'_{g_u})$; from the definitions, the W-module $H_{2d}(\mathcal{B}_g)$ is isomorphic to $\mathrm{Ind}_{W_g}^W H_{2d}(\mathcal{B}'_{g_u})$. From the definitions we have $n_\rho = d$ and the W_g-module $H_{2d}(\mathcal{B}'_{g_u})$ is of the form $\oplus_{i \in [1,s]}(\overline{\mathbf{Q}}_l \otimes E_i)^{\oplus c_i}$ where $E_i \in \mathrm{Irr}(W_g)$, $c_i \in \mathbf{N}$ satisfy $E_1 = \rho$, $c_1 = 1$ and $n_{E_i} > d$ for $i > 1$. It follows that the W-module $H_{2d}(\mathcal{B}_g)$ is of the form $\oplus_{i \in [1,s]}(\mathrm{Ind}_{W_g}^W(\overline{\mathbf{Q}}_l \otimes E_i))^{\oplus c_i}$. Now $\mathrm{Ind}_{W_g}^W(\overline{\mathbf{Q}}_l \otimes E_1)$ contains $\overline{\mathbf{Q}}_l \otimes \widetilde{\rho}$ with multiplicity 1 and all its other irreducible constituents are of the form $\overline{\mathbf{Q}}_l \otimes E$ with $n_E > d$; moreover, for $i > 1$, any irreducible constituent E of $\mathrm{Ind}_{W_g}^W(\overline{\mathbf{Q}}_l \otimes E_i)$ satisfies $n_E > d$. Thus the W-module $H_{2d}(\mathcal{B}_g)$ contains $\overline{\mathbf{Q}}_l \otimes \widetilde{\rho}$ with multiplicity 1 and all its other irreducible constituents are of the form $\overline{\mathbf{Q}}_l \otimes E$ with $n_E > d$; these other irreducible constituents are necessarily mapped to 0 by h_{g*} and the irreducible constituent isomorphic to $\overline{\mathbf{Q}}_l \otimes \widetilde{\rho}$ is mapped injectively by h_{g*} since $h_{g*} \neq 0$. It follows that the image of h_{g*} is isomorphic to $\overline{\mathbf{Q}}_l \otimes \widetilde{\rho}$ as a W-module. This proves (a).

2.3 By 2.1, 2.2 we have a well-defined map $\phi : G \to \mathcal{S}_2(W)$, $g \mapsto \widetilde{\rho}$ where $\overline{\mathbf{Q}}_l \otimes \widetilde{\rho} = h_{g*}((H_{2d_g}(\mathcal{B}_g))$ (notation of 2.1, 2.2). The fibres $G_E = \phi^{-1}(E)$ of ϕ $(E \in \mathcal{S}_2(W))$ are called the *strata* of G. They are clearly unions of conjugacy classes of G. Note the strata of G are indexed by the finite set $\mathcal{S}_2(W)$ which depends only on the Weyl group W and not on the underlying root system (see 1.1(b)) or on the characteristic of $\mathbf{k}$.

One can show that any stratum of G is a union of pieces in the partition of G defined in [L3, 3.1]; in particular, it is a constructible subset of G.

2.4 We have the following result.

(a) *Any stratum G_E $(E \in \mathcal{S}_2(W))$ of G is a (non-empty) union of G-conjugacy classes of fixed dimension, namely* $2\dim\mathcal{B} - 2n$ *where $n = n_E$, see 0.2. At most one G-conjugacy class in G_E is unipotent.*

Since $\mathcal{S}_2(W) = \widetilde{\mathcal{S}}_2^p(W)$, see 1.6, we have $E \in \widetilde{\mathcal{S}}_2^p(W)$. Hence there exists $e' \in V_p^*$ and $\rho \in \mathcal{S}_2^p(W_{e'})$ such that $E = j_{W_{e'}}^W(\rho)$. We can find a semisimple element of finite order $s \in G$ such that W_s (viewed as a subgroup of W as in 2.1) is equal to $W_{e'}$. By [L5, 1.4] we can find a unipotent element u in $Z_G(s)^0$ such that ρ is the Springer representation of W_s defined by u and the trivial local system on its $Z_G(s)^0$-conjugacy class. Then $E = \phi(su)$ so that $G_E \neq \emptyset$. Let γ be a G-conjugacy class in G_E. Let $g \in \gamma$. Let ρ (resp. $\widetilde{\rho}$) be the irreducible representation of W_g (resp. W) defined by g_u as in 2.1. Let $n_\rho, n_{\widetilde{\rho}}$ be as in 0.2. By the definition of $\widetilde{\rho}$ we have $n_\rho = n_{\widetilde{\rho}}$. By assumption we have $\widetilde{\rho} = E$, hence $n_{\widetilde{\rho}} = n$ and $n_\rho = n$. By a known property of Springer's representations, n_ρ is equal to the dimension of the variety of Borel subgroups of $Z_G(g_s)^0$ that contain g_u; hence by a result of Steinberg (for $p = 0$) and Spaltenstein [Spa, 10.15] (for any p), n_ρ is equal to

$$(\dim(Z_{Z_G(g_s)^0}(g_u)^0 - \mathrm{rk}(Z_G(g_s)^0))/2 = (\dim(Z_G(g)^0) - \mathrm{rk}(G))/2.$$

It follows that $(\dim(Z_G(g)^0) - \mathrm{rk}(G))/2 = n$ and the desired formula for $\dim\gamma$ follows. Now assume that γ, γ' are two unipotent G-conjugacy classes contained in G_E. Then the Springer representation of W associated to γ is the same as that associated to γ', namely E. By properties of Springer representations, it follows that $\gamma = \gamma'$. This proves (a).

2.5 In this and the next subsection we assume that W is irreducibble. Let $r \in \mathcal{P} \cup \{0\}$. Let G^r be a connected reductive group of the same type as G over an algebraically closed field of characteristic r, whose Weyl group is identified with W. Let $\mathcal{U}^r$ be the set of unipotent classes of G^r. By [L5, 1.4] we have a canonical bijection

$$\psi^r : \mathcal{U}^r \xrightarrow{\sim} \mathcal{S}_2^r(W)$$

which, to a unipotent class γ, associates the Springer representation of W corresponding to γ and the constant local system on γ. We define an embedding $h^r : \mathcal{U}^0 \to \mathcal{U}^r$ as the composition

$$\mathcal{U}^0 \xrightarrow{\psi^0} \mathcal{S}_2^0(W) = \mathcal{S}_1(W) \to \mathcal{S}_2^r(W) \xrightarrow{(\psi^r)^{-1}} \mathcal{U}^r$$

where the unnamed map is the inclusion.

Consider the relation $\cong$ on $\sqcup_{r\in\mathcal{P}}\mathcal{U}^r$ for which $x \in \mathcal{U}^r$, $y \in \mathcal{U}^{r'}$ (where $r, r' \in \mathcal{P}$) satisfy $x \cong y$ if either $r = r'$ and $x = y$ or $r \neq r'$ and $x = h^r(z)$, $y = h^{r'}(z)$ for some $z \in \mathcal{U}^0$. We show that $\cong$ is an equivalence relation. It is enough to show that if $x \in \mathcal{U}^r$, $y \in \mathcal{U}^{r'}$, $u \in \mathcal{U}^{r''}$ are such that $r \neq r'$, $r' \neq r''$ and $x = h^r(z)$, $y = h^{r'}(z)$, $y = h^{r'}(\tilde{z})$, $u = h^{r''}(\tilde{z})$ for some $z \in \mathcal{U}^0, \tilde{z} \in \mathcal{U}^0$, then $x \cong u$. From $h^{r'}(z) = h^{r'}(\tilde{z})$ and the injectivity of $h^{r'}$ we have $z = \tilde{z}$. Thus, if $r \neq r''$, we have $x \cong u$, while if $r = r''$, we have $x = u$. Thus, $\cong$ is indeed an equivalence relation.

Let $\mathcal{U}^*$ be $\sqcup_{r\in\mathcal{P}}\mathcal{U}^r$ modulo the equivalence relation $\cong$. Let $\sqcup_{r\in\mathcal{P}}\mathcal{U}^r \to \mathcal{S}_2(W)$ be the map whose restriction to $\mathcal{U}^r$ is ψ^r followed by the inclusion $\mathcal{S}_2^r(W) \subset \mathcal{S}_2(W)$ (for any r). We show:

(a) *This map induces a bijection* $\psi^* : \mathcal{U}^* \xrightarrow{\sim} \mathcal{S}_2(W)$.

To show that ψ^* is a well-defined map it is enough to verify that if $z \in \mathcal{U}^0$, then for any $r, r' \in \mathcal{P}$, we have $\psi^r h^r(z) = \psi^{r'} h^{r'}(z)$ in $\mathcal{S}_2(W)$; but both sides of the equality to be verified are equal to $\psi^0(z)$. Let $E \in \mathcal{S}_2(W)$. By 1.5(j) there exists $r \in \mathcal{P}$ such that $E \in \mathcal{S}_2^r(W)$, hence $E = \psi^r(x)$ for some $x \in \mathcal{U}^r$. It follows that ψ^* is surjective. We show that ψ^* is injective. It is enough to show that

(b) if $x \in \mathcal{U}^r$, $y \in \mathcal{U}^{r'}$ ($r, r' \in \mathcal{P}$ distinct) satisfy $\psi^r(x) = \psi^{r'}(y)$, then there exists $z \in \mathcal{U}^0$ such that $x = h^r(z)$, $y = h^{r'}(z)$.

If $r \neq \{2, 3\}$, then $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$, hence $\psi^r(x) = \psi^0(z)$ for some $z \in \mathcal{U}^0$. We then have $\psi^{r'}(y) = \psi^0(z)$. It follows that $h^r(z) = x$, $h^{r'}(z) = y$, as required. Similarly, if $r' \neq \{2, 3\}$, then the conclusion of (b) holds. Thus we can assume that $r \in \{2, 3\}$, $r' \in \{2, 3\}$. Since $r \neq r'$ we have $\{r, r'\} = \{2, 3\}$. Hence $\psi^r(x) = \psi^{r'}(y) \in \mathcal{S}_2^2(W) \cap \mathcal{S}_2^3(W) = \mathcal{S}_1(W)$; the last equality follows from 1.5(l). Thus we have $\psi^r(x) = \psi^{r'}(y) = \psi^0(z)$ for some $z \in \mathcal{U}^0$. It follows that $h^r(z) = x$, $h^{r'}(z) = y$, as required.

From (a) we deduce the following:

(c) *The strata of* G *are naturally indexed by the set* $\mathcal{U}^*$.

The proof of (a) shows also that $\mathcal{U}^*$ is equal to $\mathcal{U}^2 \sqcup \mathcal{U}^3$ with the identification of $h^2(z), h^3(z)$ for any $z \in \mathcal{U}^0$.

We can now state the following result.

(d) *Let* $E \in \mathcal{S}_2(W)$. *Then for some* $r \in \mathcal{P}$, *the stratum* G_E^r *contains a unipotent class. In fact,* r *can be assumed to be* 2 *or* 3.

Under (a), E corresponds to an element of $\mathcal{U}^*$ which is the equivalence class of some element $\gamma \in \mathcal{U}^r$ with $r \in \{2,3\}$. Let $g \in G^r$ be an element in the unipotent conjugacy class γ. From the definitions we see that $g \in G^r_E$. This proves (d).

2.6 We show that the set $\mathcal{U}^*$ has a natural partial order. If $\mathcal{S}_2^r(W) = \mathcal{S}_1(W)$ (type A and E_6), we have $\mathcal{U}^* = \mathcal{U}^0$ which has a natural partial order defined by the closure relation of unipotent classes in G^0. If $\mathcal{S}_1(W) \neq \mathcal{S}_2^r(W)$ for a unique $r \in \mathcal{P}$ (type $\neq A, E_6, E_8$), we have $\mathcal{U}^* = \mathcal{U}^r$ which has a natural partial order defined by the closure relation of unipotent classes in G^r. Assume now that G is of type E_8. Then we can identify $\mathcal{U}^2, \mathcal{U}^3$ with subsets of $\mathcal{U}^*$ whose union is $\mathcal{U}^*$ and whose intersection is $\mathcal{U}^0$. Both subsets $\mathcal{U}^2, \mathcal{U}^3$ have natural partial orders defined by the closure relation of unipotent classes in G^2 and G^3. If $\gamma, \gamma' \in \mathcal{U}^*$, we say that $\gamma \leq \gamma'$ if there exists a sequence $\gamma = \gamma_0, \gamma_1, \dots, \gamma_s = \gamma'$ in $\mathcal{U}^*$ such that for any $i \in [1, s]$ there exists $r \in \{2,3\}$ such that

(a) $\gamma_{i-1} \in \mathcal{U}^r, \gamma_i \in \mathcal{U}^r$, $\gamma_{i-1} \leq \gamma_i$ in the partial order of unipotent classes in G^r;

note that if for some i, (a) holds for both $r = 2$ and $r = 3$, then we have $\gamma_{i-1} \in \mathcal{U}^0$, $\gamma_i \in \mathcal{U}^0$, $\gamma_{i-1} \leq \gamma_i$ in the partial order of unipotent classes in G^0. One can show that this partial order on $\mathcal{U}^*$ induces the usual partial orders on the subsets $\mathcal{U}^2$, $\mathcal{U}^3, \mathcal{U}^0$.

2.7 Let W_a be the semidirect product of W with the subgroup of V generated by R (an affine Weyl group); let $'W_a$ be the semidirect product of W with the subgroup of V^* generated by $\check{R}$ (another affine Weyl group). We consider four triples:

(a) $(\mathcal{S}(W), X_0, Z_0)$
(b) $(\mathcal{S}_1(W), X_1, Z_1)$
(c) $('\mathcal{S}_1(W), 'X_1, 'Z_1)$
(d) $(\mathcal{S}_2(W), X_2, Z_2)$

where $X_0, X_1, 'X_1$ is the set of two-sided cells in $W, W_a, 'W_a$ respectively, Z_0 is the set of special unipotent classes in G with $p = 0$, Z_1 is the set of unipotent classes in G with $p = 0$, $'Z_1$ is the set of unipotent classes in the Langlands dual G^* of G with $p = 0$, Z_2 is the set of strata of G with $p = 0$ and X_2 remains to be defined. The three sets in each of these four triples are in canonical bijection with each other (assuming that X_2 has been defined). Moreover, each set in (a) is naturally contained in the corresponding set in (b) and (replacing G by G^*) in the corresponding set in (c); each set in (b) is contained in the corresponding set in (d) and (replacing G by G^*) each set in (c) is contained in the corresponding set in (d).

It remains to define X_2. It seems plausible that the (trigonometric) double affine Hecke algebra **H** associated by Cherednik to W has a natural filtration by two-sided ideals whose successive subquotients can be called two-sided cells and form the desired set X_2. The inclusion of the Hecke algebra of W_a and that of $'W_a$ into **H** should induce the embeddings $X_1 \subset X_2$, $'X_1 \subset X_2$ and X_2 should be in natural bijection with $\mathcal{S}_2(W)$ and with the set of strata of G.

3 Examples

3.1 We write the adjoint group of G as a product $\prod_i G_i$ where each G_i is simple with Weyl group W_i so that $W = \prod_i W_i$. Let $E \in \mathcal{S}_2(W)$. We have $E = \boxtimes_i E_i$ where $E_i \in \mathcal{S}_2(W_i)$. Now G_E is the inverse image of $\prod_i (G_i)_{E_i}$ under the obvious map $G \to \prod_i G_i$.

When E is the sign representation of W, then G_E is the centre of G; when E is the unit representation of W, G_E is the set of elements of G which are regular in the sense of Steinberg [St].

By 2.5(a) and 2.6 applied to G_i, the set $\mathcal{S}_2(W_i)$ has a natural partial order. Since $\mathcal{S}_2(W)$ can be identified as above with $\prod_i \mathcal{S}_2(W_i)$, $\mathcal{S}_2(W)$ is naturally a partially ordered set (a product of partially ordered sets). Hence by 2.3 the set of strata of G is naturally a partially ordered set.

3.2 Assume that $G = GL(V)$ where V is a $\mathbf{k}$-vector space of dimension $n \geq 1$. Let $g \in G$. For any $x \in \mathbf{k}^*$ let V_x be the generalized x-eigenspace of $g : V \to V$ and let $\lambda_1^x \geq \lambda_2^x \geq \lambda_3^x \geq \dots$ be the sequence in $\mathbf{N}$ whose nonzero terms are the sizes of the Jordan blocks of $x^{-1}g : V_x \to V_x$. Let ${}^g\lambda$ be the sequence ${}^g\lambda_1 \geq {}^g\lambda_2 \geq {}^g\lambda_3 \geq \dots$ given by ${}^g\lambda_j = \sum_{x\in\mathbf{k}^*} \lambda_j^x$. Now $g \mapsto {}^g\lambda$ defines a map from G onto the set of partitions of n. From the definitions we see that the fibres of this map are exactly the strata of G. If $g \in G$ and ${}^g\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$, then

$$\dim(\mathcal{B}_g) = \sum_{k\geq 1}(n - (\lambda_1 + \lambda_2 + \dots + \lambda_k)).$$

3.3 Repeating the definition of sheets in a semisimple Lie algebra over $\mathbf{C}$ (see [Bo]), one can define the sheets of G as the maximal irreducible subsets of G which are unions of conjugacy classes of fixed dimension. One can show that if G is as in 3.2, the sheets of G are the same as the strata of G, as described in 3.2. (In this case, the sheets of G, or rather their Lie algebra analogue, are described in [Pe]. They are smooth varieties.) This is not true for a general G (the sheets of G do not usually form a partition of G; the strata of G are not always irreducible). In [Ca] it is shown that if p is 0 or a good prime for G, then any stratum is a union of sheets and that the closure of a stratum is not necessarily a union of strata, even if G is of type A.

3.4 In the next few subsections we will describe explicitly the strata of G when G is a symplectic or special orthogonal group.

Given a partition $\nu = (\nu_1 \geq \nu_2 \geq \dots)$, a *string* of ν is a maximal subsequence $\nu_i, \nu_{i+1}, \dots, \nu_j$ of ν consisting of equal > 0 numbers; the string is said to have an odd origin if i is odd and an even origin if i is even.

For an even $N \in \mathbf{N}$, let Z_N^1 be the set of partitions $\nu = (\nu_1 \geq \nu_2 \geq \dots)$ of N such that any odd number appears an even number of times in ν. We show:

(a) *There is a canonical bijection* $Z_N^1 \leftrightarrow BP_{1,1}^{N/2}$ *(notation of 0.2).*

To $\nu \in Z_N^1$ we associate $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ as follows: each string $2a, 2a, \dots, 2a$ in ν is replaced by $a, a, \dots, a$ of the same length; each string $2a+1, 2a+1, \dots, 2a+1$ (necessarily of even length) in ν is replaced by $a, a+1, a, a+1, \dots, a, a+1$ of the same length. The resulting entries form a bipartition $\lambda \in BP_{1,1}^{N/2}$. Now $\nu \mapsto \lambda$ establishes the bijection (a).

For an even $N \in \mathbf{N}$, let Z_N^2 be the set of partitions $\nu = (\nu_1 \geq \nu_2 \geq \dots)$ of N such that any odd number appears an even number of times in ν and any even > 0 number which appears an even > 0 number of times in ν has an associated label 0 or 1. We show:

(b) *There is a canonical bijection* $Z_N^2 \leftrightarrow BP_{2,2}^{N/2}$ *(notation of 0.2).*

To $\nu \in Z_N^2$ we associate $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ as follows: each string $2a, 2a, \dots, 2a$ of odd length or of even length and label 1 in ν is replaced by $a, a, \dots, a$ of the same length; each string $2a, 2a, \dots, 2a$ of even length and label 0 in ν is replaced by $a-1, a+1, a-1, a+1, \dots, a-1, a+1$ of the same length; each string $2a+1, 2a+1, \dots, 2a+1$ (necessarily of even length) in ν is replaced by $a, a+1, a, a+1, \dots, a, a+1$ of the same length. The resulting entries form a bipartition $\lambda \in BP_{2,2}^{N/2}$. Now $\nu \mapsto \lambda$ establishes the bijection (b).

Assume for example that $N = 6$. The bijection (b) is:

$$(6\dots) \leftrightarrow (3\dots)$$
$$(42\dots) \leftrightarrow (21\dots)$$
$$(411\dots) \leftrightarrow (201\dots)$$
$$(33\dots) \leftrightarrow (12\dots)$$
$$(222\dots) \leftrightarrow (111\dots)$$
$$((22)_1 11\dots) \leftrightarrow (1101\dots)$$
$$((22)_0 110\dots) \leftrightarrow (0201\dots)$$
$$(21111\dots) \leftrightarrow (10101\dots)$$
$$(111111\dots) \leftrightarrow (010101\dots).$$

Here we write $\dots$ instead of $000\dots$. (Compare [LS2, 6.1].)

3.5 Assume that $G = \mathrm{Sp}(V)$ where V is a $\mathbf{k}$-vector space of dimension N with a fixed nondegenerate symplectic form.

Let $g \in G$. For any $x \in \mathbf{k}^*$ let V_x be the generalized x-eigenspace of $g : V \to V$. Let $d_x = \dim V_x$. For any $x \in \mathbf{k}^*$ such that $x^2 \neq 1$ let $\lambda_1^x \geq \lambda_2^x \geq \lambda_3^x \geq \dots$ be the partition of d_x whose nonzero terms are the sizes of the Jordan blocks of $x^{-1}g : V_x \to V_x$.

For $x \in \mathbf{k}^*$ such that $x^2 = 1$, let $\nu^x \in Z_{d_x}^1$ (if $p \neq 2$) and $\nu^x \in Z_{d_x}^2$ (if $p = 2$) be again the partition of d_x whose nonzero terms are the sizes of the Jordan blocks of the unipotent element $x^{-1}g \in \mathrm{Sp}(V_x)$. (When $p = 2$, ν^x should also include a labelling with 0 and 1 associated to $x^{-1}g \in \mathrm{Sp}(V_x)$ as in [L10, 1.4].) Let $\lambda^x = (\lambda^x_1, \lambda^x_2, \lambda^x_3, \dots)$ be the bipartition of $d_x/2$ associated to ν^x by 3.4(a),(b). Thus $\lambda^x \in BP^{d_x/2}_{1,1}$ (if $p \neq 2$), $\lambda^x \in BP^{d_x/2}_{2,2}$ (if $p = 2$). Note that λ^x is the bipartition such that the Springer representation attached to the unipotent element $x^{-1}g \in \mathrm{Sp}(V_x)$ (an irreducible representation of the Weyl group of type $B_{d_x/2}$) is indexed in the standard way by λ^x. Define ${}^g\lambda = ({}^g\lambda_1, {}^g\lambda_2, {}^g\lambda_3, \dots)$ by ${}^g\lambda_j = \sum_x \lambda^x_j$ where x runs over a set of representatives for the orbits of the involution $a \mapsto a^{-1}$ of $\mathbf{k}^*$. Note that ${}^g\lambda \in BP^{N/2}_{2,2}$. Thus we have defined a (surjective) map $g \mapsto {}^g\lambda$, $G \to BP^{N/2}_{2,2}$. From the definitions we see that the fibres of this map are exactly the strata of G.

If $g \in G$ and ${}^g\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$, then

(a) $\dim(\mathcal{B}_g) = \sum_{k\ge 1}((N/2) - (\lambda_1 + \lambda_2 + \dots + \lambda_k))$.

We now consider the case where $N = 4$. In this case we have $\mathcal{S}_2(W) = \mathrm{Irr}(W)$; hence there are five strata. One stratum is the union of all conjugacy classes of dimension 8 (it corresponds to the unit representation); one stratum is the union of all conjugacy classes of dimension 6 (it corresponds to the reflection representation of W). There are two strata which are unions of conjugacy classes of dimension 4 (they correspond to the two one-dimensional representations of W other than unit and sign); if $p = 2$, both these strata are single unipotent classes; if $p \neq 2$, one of these strata is a semisimple class and the other is a unipotent class times the centre of G. The centre of G is a stratum (it corresponds to the sign representation of W).

The results in this subsection show that under the standard identification $\mathrm{Irr}(W) = BP^{N/2}$, we have

(b) $\mathcal{S}_2(W) = BP^{N/2}_{2,2}$.

Under this identification the map $g \mapsto {}^g\lambda$, $G \to BP^{N/2}_{2,2}$ becomes the map $g \mapsto E$ where $g \in G_E$.

3.6 For $N \in \mathbf{N}$, let ${}'Z^1_N$ be the set of partitions $\nu = (\nu_1 \ge \nu_2 \ge \dots)$ such that any even > 0 number appears an even number of times in ν and $\nu_1 + \nu_2 + \dots = N$.

(a) *If N is odd, then there is a canonical bijection ${}'Z^1_N \leftrightarrow BP^{(N-1)/2}_{2,0}$.*

To $\nu \in {}'Z^1_N$ we associate $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ as follows: each string $2a, 2a, \dots, 2a$ of ν (necessarily of even length) is replaced by $a-1, a+1, a-1, a+1, \dots, a-1, a+1$ of the same length (if the string has odd origin) or by $a, a, \dots, a$ of the same length (if the string has even origin); each string $2a+1, 2a+1, \dots, 2a+1$ of ν is replaced by $a, a+1, a, a+1, \dots$ of the same length (if the string has odd origin) or by $a+1, a, a+1, a, \dots$ of the same length (if the string has even origin). The resulting entries form a bipartition $\lambda \in BP^{(N-1)/2}_{2,0}$. Now $\nu \mapsto \lambda$ establishes the bijection (a).

(b) *If N is even, then there is a canonical bijection* $'Z^1_N \leftrightarrow BP^{N/2}_{0,2}$.

To $\nu \in {}'Z^1_N$ we associate $\lambda=(\lambda_1, \lambda_2, \lambda_3, \dots)$ as follows: each string $2a, 2a, \dots, 2a$ of ν (necessarily of even length) is replaced by $a-1, a+1, a-1, a+1, \dots, a-1, a+1$ of the same length (if the string has even origin) or by $a, a, \dots, a$ of the same length (if the string has odd origin); each string $2a+1, 2a+1, \dots, 2a+1$ of ν is replaced by $a, a+1, a, a+1, \dots$ of the same length (if the string has even origin) or by $a+1, a, a+1, a, \dots$ of the same length (if the string has odd origin). The resulting entries form a bipartition $\lambda \in BP^{N/2}_{0,2}$. Now $\nu \mapsto \lambda$ establishes the bijection (b).

3.7 Assume that $p \neq 2$ and that $G = \mathrm{SO}(V)$ where V is a $\mathbf{k}$-vector space of odd dimension $N \geq 1$ with a fixed nondegenerate quadratic form.

Let $g \in G$. For any $x \in \mathbf{k}^*$, let V_x be the generalized x-eigenspace of $g : V \to V$. Let $d_x = \dim V_x$. For any $x \in \mathbf{k}^*$ such that $x^2 \neq 1$ let $\lambda^x_1 \geq \lambda^x_2 \geq \lambda^x_3 \geq \dots$ be the partition of d_x whose nonzero terms are the sizes of the Jordan blocks of $x^{-1}g : V_x \to V_x$.

For $x \in \mathbf{k}^*$ such that $x^2 = 1$ let $\nu^x \in {}'Z^1_{d_x}$ again be the partition of d_x whose nonzero terms are the sizes of the Jordan blocks of the unipotent element $x^{-1}g \in \mathrm{SO}(V_x)$. Let $\lambda^x = (\lambda^x_1, \lambda^x_2, \lambda^x_3, \dots)$ be the bipartition of $d_x/2$ associated to ν^x by 3.6(a) if $x = 1$ and by 3.6(b) if $x = -1$. Thus $\lambda^x \in BP^{(d_x-1)/2}_{2,0}$ if $x = 1$, $\lambda^x \in BP^{d_x/2}_{0,2}$ if $x = -1$. Note that λ^x is the bipartition such that the Springer representation attached to the unipotent element $x^{-1}g \in \mathrm{SO}(V_x)$ (an irreducible representation of the Weyl group of type $B_{(d_x-1)/2}$, if $x = 1$, or of type $D_{d_x/2}$, if $x = -1$) is indexed by λ^x. Define ${}^g\lambda = ({}^g\lambda_1, {}^g\lambda_2, {}^g\lambda_3, \dots)$ by ${}^g\lambda_j = \sum_x \lambda^x_j$ where x runs over a set of representatives for the orbits of the involution $a \mapsto a^{-1}$ of $\mathbf{k}^*$. Note that ${}^g\lambda \in BP^{(N-1)/2}_{2,2}$. Thus we have defined a (surjective) map $g \mapsto {}^g\lambda$, $G \to BP^{(N-1)/2}_{2,2}$. From the definitions we see that the fibres of this map are exactly the strata of G. Under the identification $\mathcal{S}_2(W) = BP^{(N-1)/2}_{2,2}$, see 3.5(b), the map $g \mapsto {}^g\lambda$, $G \to BP^{(N-1)/2}_{2,2}$ becomes the map $g \mapsto E$ where $g \in G_E$.

If $g \in G$ and ${}^g\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$, then

$$\dim(\mathcal{B}_g) = \sum_{k \geq 1}((N-1)/2 - (\lambda_1 + \lambda_2 + \dots + \lambda_k)).$$

3.8 Assume that $p = 2$ and that $G = \mathrm{SO}(V)$ where V is a $\mathbf{k}$-vector space of odd dimension $N \geq 1$ with a given quadratic form, such that the associated symplectic form has radical $\mathbf{r}$ of dimension 1 and the restriction of the quadratic form to $\mathbf{r}$ is nonzero. In this case there is an obvious morphism from G to the symplectic group G' of $V/\mathbf{r}$ which is an isomorphism of abstract groups. From the definitions we see that this morphism maps each stratum of G bijectively onto a stratum of G' (which has been described in 3.5).

3.9 For an even $N \in \mathbf{N}$, let $'Z_N^2$ be the set of partitions with labels $\nu = (\nu_1 \geq \nu_2 \geq \dots)$ in Z_N^2 (see 3.4) such that the number of nonzero entries of ν is even.

(a) *If N is even, then there is a canonical bijection $'Z_N^2 \leftrightarrow BP_{0,4}^{N/2}$.*

To $\nu \in {}'Z_N^2$ we associate $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$ as follows: each string $2a, 2a, 2a, \dots$ of ν of odd length or of even length and label 1 is replaced by $a-1, a+1, a-1, a+1, \dots$ of the same length (if the string has even origin) or $a+1, a-1, a+1, a-1, \dots$ of the same length (if the string has odd origin); each string $2a, 2a, 2a, \dots$ of ν of even length and label 0 is replaced by $a-2, a+2, a-2, a+2, \dots$ of the same length (if the string has even origin) or $a, a, a, a, \dots$ of the same length (if the string has odd origin); each string $2a+1, 2a+1, 2a+1, \dots$ of ν (necessarily of even length) is replaced by $a-1, a+2, a-1, a+2, \dots$ of the same length (if the string has even origin) or $a+1, a, a+1, a, \dots$ of the same length (if the string has odd origin). The resulting entries form a bipartition $\lambda \in BP_{0,4}^{N/2}$. Now $\nu \mapsto \lambda$ establishes the bijection (a).

Assume for example that $N = 8$. The bijection (a) is:

$$
\begin{aligned}
(62\dots) &\leftrightarrow (40\dots)\\
((44)_1\dots) &\leftrightarrow (31\dots)\\
((44)_0\dots) &\leftrightarrow (22\dots)\\
(4211\dots) &\leftrightarrow (3010\dots)\\
(3311\dots) &\leftrightarrow (2110\dots)\\
((2222)_1\dots) &\leftrightarrow (2020\dots)\\
((2222)_0\dots) &\leftrightarrow (1111\dots)\\
((22)_1 1111\dots) &\leftrightarrow (201010\dots)\\
((22)_0 1111\dots) &\leftrightarrow (111010\dots)\\
(11111111\dots) &\leftrightarrow (10101010\dots).
\end{aligned}
$$

Here we write $\dots$ instead of $000\dots$. (Compare [LS2, 6.2].)

3.10 Assume that $G = SO(V)$ where V is a $\mathbf{k}$-vector space of even dimension N with a fixed nondegenerate quadratic form. Let $g \in G$. For any $x \in \mathbf{k}^*$ let V_x be the generalized x-eigenspace of $g : V \to V$. Let $d_x = \dim V_x$. For any $x \in \mathbf{k}^*$ such that $x^2 \neq 1$ let $\lambda_1^x \geq \lambda_2^x \geq \lambda_3^x \geq \dots$ be the partition whose nonzero terms are the sizes of the Jordan blocks of $x^{-1}g : V_x \to V_x$. For $x \in \mathbf{k}^*$ such that $x^2 = 1$ let $\nu^x \in {}'Z_{d_x}^1$ (if $p \neq 2$) and $\nu^x \in {}'Z_{d_x}^2$ (if $p = 2$) be again the partition of d_x whose nonzero terms are the sizes of the Jordan blocks of the unipotent element $x^{-1}g \in SO(V_x)$. (When $p = 2$, ν^x should also include a labelling with 0 and 1 associated to $x^{-1}g$ viewed as an element of $Sp(V_x)$ as in [L10, 1.4].) Let $\lambda^x = (\lambda_1^x, \lambda_2^x, \lambda_3^x, \dots)$ be the bipartition of $d_x/2$ associated to ν^x by 3.6(b), 3.9(a). Thus $\lambda^x \in BP_{0,2}^{d_x/2}$ (if $p \neq 2$), $\lambda^x \in BP_{0,4}^{d_x/2}$ (if $p = 2$). Note that λ^x is the bipartition such that the Springer representation attached to the unipotent element $x^{-1}g \in \mathrm{SO}(V_x)$ (an irreducible representation of the Weyl group of type $D_{d_x/2}$) is indexed by λ^x. Define ${}^g\lambda = ({}^g\lambda_1, {}^g\lambda_2, {}^g\lambda_3, \dots)$ by ${}^g\lambda_j = \sum_x \lambda_j^x$ where x runs over a set of representatives for the orbits of the involution $a \mapsto a^{-1}$ of $\mathbf{k}^*$. Note that ${}^g\lambda \in BP_{0,4}^{N/2}$ and that $g \mapsto {}^g\lambda$ defines a (surjective) map $G \to BP_{0,4}^{N/2}$. From the definitions we see that the fibres of this map are exactly the strata of G (except for the fibre over a bipartition $(\lambda_1, \lambda_2, \lambda_3, \dots)$ with $\lambda_1 = \lambda_2, \lambda_3 = \lambda_4, \dots$ in which case the fibre is a union of two strata). If $g \in G$ and ${}^g\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots)$, then

(a) $\dim(\mathcal{B}_g) = \sum_{k \geq 1}((N/2) - (\lambda_1 + \lambda_2 + \cdots + \lambda_k))$.

Viewing W as a subgroup of index 2 of a Weyl group W' of type B_n, we can associate to any $\lambda \in BP^{N/2}$ one or two irreducible representations of W which appear in the restriction to W of the irreducible representation of W' indexed by λ; the representation(s) of W associated to $\lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4, \dots)$ are the same as those associated to $\iota(\lambda) := (\lambda_2, \lambda_1, \lambda_4, \lambda_3, \dots)$; here $\iota : BP^{N/2} \to BP^{N/2}$ is an involution and the set of orbits is denoted by $BP^{N/2}/\iota$. This gives a surjective map $f : \mathrm{Irr}(W) \to BP^{N/2}/\iota$ whose fibre at the orbit of λ has one element if $\lambda \neq \iota(\lambda)$ and two elements if $\lambda = \iota(\lambda)$. Let $\iota' : \mathrm{Irr}(W) \to \mathrm{Irr}(W)$ be the involution whose orbits are the fibres of f and let $\mathcal{S}_2(W)/\iota'$ be the set of orbits of the restriction of ι' to $\mathcal{S}_2(W)$. The results in this subsection show that f induces a bijection

(b) $\mathcal{S}_2(W)/\iota' \xrightarrow{\sim} BP_{0,4}^{N/2}$.

We have used the fact that the intersection of $BP_{0,4}^{N/2}$ with an orbit of $\iota : BP^{N/2} \to BP^{N/2}$ has at most one element; more precisely,

$$\{\lambda \in BP^{N/2}; \lambda \in BP_{0,4}^{N/2} \text{ and } \iota(\lambda) \in BP_{0,4}^{N/2}\} = \{\lambda \in BP^{N/2}; \lambda = \iota(\lambda)\}.$$

Under the identification (b), the map $g \mapsto {}^g\lambda$, $G \to BP_{0,4}^{N/2}$ becomes the map $g \mapsto E$ (up to the action of ι') where $g \in G_E$.

3.11 Assume that $p \neq 2$ and $n \geq 3$. If $G = \mathrm{SO}_{2n+1,\mathbf{k}}$ then the stratum of minimal dimension > 0 consists of a semisimple class of dimension $2n$; if $G = \mathrm{Sp}_{2n,\mathbf{k}}/\pm 1$ then the stratum of minimal dimension > 0 consists of a unipotent class of dimension $2n$ (that of transvections). The corresponding $E \in \mathrm{Irr}(W)$ is one-dimensional.

3.12 Assume that G is simple of type E_8. In this case G has exactly 75 strata. If $p \neq 2, 3$ then exactly 70 strata contain unipotent elements. If $p = 2$ (resp. $p = 3$) then exactly 74 (resp. 71) strata contain unipotent elements. The unipotent class of dimension 58 is a stratum. If $p \neq 2$, there is a stratum which is a union of a semisimple class and a unipotent class (both of dimension 128); in particular this stratum is disconnected.

4 A map from conjugacy classes in W to 2-special representations of W

4.1 In this subsection we shall define a canonical surjective map

(a) $'\Phi : \mathrm{cl}(W) \to \mathcal{S}_2(W)$.

We preserve the setup of 2.5. We will first define the map (a) assuming that G is simple. In [L8] we have defined for any $r \in \mathcal{P}$ a surjective map $\mathrm{cl}(W) \to \mathcal{U}^r$; we denote this map by Φ^r. Let $C \in \mathrm{cl}(W)$. We define an element $\Phi(C) \in \mathcal{U}^*$ as follows. If $\Phi^r(C) \in h^r(z_r)$ (with $z_r \in \mathcal{U}^0$) for all $r \in \mathcal{P}$, then $z_r = z$ is independent of r (see [L10, 0.4]) and we define $\Phi(C)$ to be the equivalence class of $h^r(z)$ for any $r \in \mathcal{P}$. If $\Phi^r(C) \notin h^r(\mathcal{U}^0)$ for some $r \in \mathcal{P}$, then r is unique. (The only case where r can be possibly not unique is in type E_8 in which case we use the tables in [L10, 2.6].) We then define $\Phi(C)$ to be the equivalence class of $\Phi^r(C)$. Thus we have defined a surjective map $\Phi : \mathrm{cl}(W) \to \mathcal{U}^*$. By composing Φ^r with $\psi^r : \mathcal{U}^r \xrightarrow{\sim} \mathcal{S}_2^r(W)$, see 2.5, and with the inclusion $\mathcal{S}_2^r(W) \subset \mathcal{S}_2(W)$, we obtain a map $'\Phi^r : \mathrm{cl}(W) \to \mathcal{S}_2(W)$. Similarly, by composing Φ with $\psi^* : \mathcal{U}^* \xrightarrow{\sim} \mathcal{S}_2(W)$, see 2.5(a), we obtain a surjective map $'\Phi : \mathrm{cl}(W) \to \mathcal{S}_2(W)$. Note that for $C \in \mathrm{cl}(W)$, $'\Phi(C)$ can be described as follows. If $'\Phi^r(C) \in \mathcal{S}_1(W)$ for all $r \in \mathcal{P}$, then $'\Phi^r(C)$ is independent of r, and we have $'\Phi(C) = {}'\Phi^r(C)$ for any r. If $'\Phi^r(C) \notin \mathcal{S}_1(W)$ for some $r \in \mathcal{P}$, then such r is unique and we have $'\Phi(C) = {}'\Phi^r(C)$.

We return to the general case. We write the adjoint group of G as a product $\prod_i G_i$ where each G_i is simple with Weyl group W_i. We can identify $W = \prod_i W_i$, $\mathrm{cl}(W) = \prod_i \mathrm{cl}(W_i)$, $\mathcal{S}_2(W) = \prod_u \mathcal{S}_2(W_i)$ (via external tensor product). Then $'\Phi_i : \mathrm{cl}(W_i) \to \mathcal{S}_2(W_i)$ is defined as above for each i. We set $'\Phi = \prod_i {}'\Phi_i : \mathrm{cl}(W) \to \mathcal{S}_2(W)$.

For C, C' in $\mathrm{cl}(W)$ we write $C \sim C'$ if $'\Phi(C) = {}'\Phi(C')$. This is an equivalence relation on $\mathrm{cl}(W)$. Let $\underline{\mathrm{cl}}(W)$ be the set of equivalence classes. Note that:

(b) *$'\Phi$ induces a bijection $\underline{\mathrm{cl}}(W) \to \mathcal{S}_2(W)$.*

We see that, via (b),

(c) *the strata of G are naturally indexed by the set $\underline{\mathrm{cl}}(W)$.*

4.2 We preserve the setup of 2.5. Now $'\Phi$ in 4.1(a) is a map between two sets which depend only on W, not on the underlying root system, see 1.1(b). We show that

(a) *$'\Phi$ itself depends only on W, not on the underlying root system.*

We can assume that G is adjoint, simple. We can also assume that G is not of simply laced type. In this case there is a unique $r \in \mathcal{P}$ such that $\mathcal{S}_2(W) = \mathcal{S}_2^r(W)$ so that we have simply $'\Phi = {}'\Phi^r : \mathrm{cl}(W) \to \mathcal{S}_2(W)$. Thus $'\Phi$ is the composition

(b) $\mathrm{cl}(W) \xrightarrow{\Phi^r} \mathcal{U}^r \xrightarrow{\psi^r} \mathcal{S}_2(W)$.

We now use the fact the maps in (b) are compatible with the exceptional isogeny between groups G^2 of type B_n and C_n or of type F_4 and F_4 (resp. between groups G^3 of type G_2 and G_2). This implies (a).

4.3 Assume that G is simple. The map $'\Phi$ in 4.1 is defined in terms of $'\Phi^r$ which is the composition of $\Phi^r : \mathrm{cl}(W) \to \mathcal{U}^r$ (which is described explicitly in each case in [L10]) and $\psi^r : \mathcal{U}^r \leftarrow \mathcal{S}_2^r(W)$ which is given by the Springer correspondence. Therefore $'\Phi$ is explicitly computable. In this subsection we describe this map in the case where W is of classical type.

If W is of type $A_n, n \geq 1$, then $\mathrm{cl}(W)$ can be identified with the set of partitions of n: to a conjugacy class of a permutation of n objects we associate the partition whose nonzero terms are the sizes of the disjoint cycles of which the permutation is a product. We identify $\mathcal{S}_2(W) = \mathrm{Irr}(W)$ with the set of partitions in the standard way (the unit representation corresponds to the partition $(n, 0, 0 \dots)$). With these identifications the map $'\Phi$ is the identity map.

Assume now that W is a Weyl group of type B_n or C_n, $n \geq 2$. Let X be a set with $2n$ elements with a given fixed point free involution τ. We identify W with the group of permutations of X which commute with τ. To any $w \in W$, we can associate an element $\nu \in Z^2_{2n}$ (see 3.4) as follows. The nonzero terms of the partition ν are the sizes of the disjoint cycles of which w is a product. To each string $c, c, \dots, c$ of ν of even length with $c > 0$ even we attach the label 1 if at least one of its terms represents a cycle which commutes with τ; otherwise we attach to it the label 0. This defines a (surjective) map $\mathrm{cl}(W) \to Z^2_{2n}$ which by results of [L10] can be identified with the map $\Phi^2 : \mathrm{cl}(W) \to \mathcal{U}^2$. Composing this with the bijection 3.4(b) we obtain a surjective map $\mathrm{cl}(W) \to BP^n_{2,2}$ or equivalently (see 3.5(b)) $\mathrm{cl}(W) \to \mathcal{S}_2(W)$. This is the same as $'\Phi$.

Next we assume that W is a Weyl group of type D_n, $n \geq 4$. We can identify W with the group of *even* permutations of X (as above) which commute with τ (as above). To any $w \in W$ we associate an element $\nu \in Z^2_{2n}$ as for type B_n above. This element is actually contained in $'Z^2_{2n}$ (see 3.9) since w is an even permutation. This defines a (surjective) map $\mathrm{cl}(W) \to {}'Z^2_{2n}$ which by results of [L10] can be identified with the composition of $\Phi^2 : \mathrm{cl}(W) \to \mathcal{U}^2$ with the obvious map from $\mathcal{U}^2$ to the set of orbits of the conjugation action of the full orthogonal group on $\mathcal{U}^2$. Composing this with the bijection 3.9(a) we obtain a surjective map $\mathrm{cl}(W) \to BP^n_{0,4}$ or equivalently (see 3.10(b)) a surjective map $\mathrm{cl}(W) \to \mathcal{S}_2(W)/\iota'$ (notation of 3.10). This is the same as the composition of $'\Phi$ with the obvious map $\mathcal{S}_2(W) \to \mathcal{S}_2(W)/\iota'$.

4.4 In this and the next five subsections we describe the map $'\Phi : \mathrm{cl}(W) \to \mathcal{S}_2(W)$ in the case where W is of exceptional type. The results will be expressed as diagrams $[a, b, \dots] \mapsto d_n$ where $a, b, \dots$ is the list of conjugacy classes in W (with notation of [C]) which are mapped by $'\Phi$ to an irreducible representation E denoted d_n (here d denotes the degree of E and the index $n = n_E$ as in 0.2). We also mark by $*_r$ those E which are in $\mathcal{S}_2(W) - \mathcal{S}_1(W)$; here r is the unique prime such that $E \in \mathcal{S}_2^r(W)$. Note that the notation d_n does not determine E for types G_2 and F_4; for these types it may happen that there are two E's with same d_n.

Type G_2

$$[G_2] \mapsto 1_0 \qquad [A_1 + \widetilde{A}_1] \mapsto 2_2 \qquad [A_1] \mapsto 1_3$$
$$[A_2] \mapsto 2_1 \qquad [\widetilde{A}_1] \mapsto 1_3, *_3 \qquad [A_0] \mapsto 1_6$$

4.5 Type F_4.

$$[F_4] \mapsto 1_0 \qquad [A_2 + \widetilde{A}_1] \mapsto 4_7$$
$$[B_4] \mapsto 4_1 \qquad [\widetilde{A}_2 + A_1] \mapsto 4_7, *_2$$
$$[F_4(a_1)] \mapsto 9_2 \qquad [B_2] \mapsto 4_8, *_2$$
$$[D_4, B_3] \mapsto 8_3 \qquad [\widetilde{A}_2] \mapsto 8_9$$
$$[C_3 + A_1, C_3] \mapsto 8_3 \qquad [A_2] \mapsto 8_9$$
$$[D_4(a_1)] \mapsto 12_4 \qquad [4A_1, 3A_1, 2A_1 + \widetilde{A}_1, A_1 + \widetilde{A}_1] \mapsto 9_{10}$$
$$[A_3 + \widetilde{A}_1] \mapsto 16_5 \qquad [2A_1] \mapsto 4_{13}$$
$$[A_3] \mapsto 9_6 \qquad [A_1] \mapsto 2_{16}$$
$$[B_2 + A_1] \mapsto 9_6, *_2 \qquad [\widetilde{A}_1] \mapsto 2_{16}, *_2$$
$$[\widetilde{A}_2 + \widetilde{A}_2] \mapsto 6_6 \qquad [A_0] \mapsto 1_{24}$$

4.6 Type E_6.

$$[E_6] \mapsto 1_0 \qquad [D_4] \mapsto 24_6 \qquad [2A_2] \mapsto 24_{12}$$
$$[E_6(a_1)] \mapsto 6_1 \qquad [A_4] \mapsto 81_6 \qquad [A_2 + A_1] \mapsto 64_{13}$$
$$[D_5] \mapsto 20_2 \qquad [D_4(a_1)] \mapsto 80_7 \qquad [A_2] \mapsto 30_{15}$$
$$[E_6(a_2)] \mapsto 30_3 \qquad [A_3 + 2A_1, A_3 + A_1] \mapsto 60_8 \qquad [4A_1, 3A_1] \mapsto 15_{16}$$
$$[A_5 + A_1, A_5] \mapsto 15_4 \qquad [3A_2, 2A_2 + A_1] \mapsto 10_9 \qquad [2A_1] \mapsto 20_{20}$$
$$[D_5(a_1)] \mapsto 64_4 \qquad [A_3] \mapsto 81_{10} \qquad [A_1] \mapsto 6_{25}$$
$$[A_4 + A_1] \mapsto 60_5 \qquad [A_2 + 2A_1] \mapsto 60_{11} \qquad [A_0] \mapsto 1_{36}$$

4.7 Type E_7.

$$[E_7] \mapsto 1_0 \qquad [E_7(a_4)] \mapsto 315_7$$
$$[E_7(a_1)] \mapsto 7_1 \qquad [D_5] \mapsto 189_7$$
$$[E_7(a_2)] \mapsto 27_2 \qquad [E_6(a_2)] \mapsto 405_8$$
$$[E_7(a_3)] \mapsto 56_3 \qquad [D_6(a_2) + A_1, D_6(a_2)] \mapsto 280_8$$
$$[E_6] \mapsto 21_3 \qquad [A_5 + A_2, (A_5 + A_1)'] \mapsto 70_9$$
$$[E_6(a_1)] \mapsto 120_4 \qquad [(A_5 + A_1)'', A_5''] \mapsto 216_9$$
$$[D_6 + A_1, D_6] \mapsto 35_4 \qquad [D_5(a_1) + A_1] \mapsto 378_9$$
$$[A_7] \mapsto 189_5 \qquad [D_5(a_1)] \mapsto 420_{10}$$
$$[A_6] \mapsto 105_6 \qquad [A_4 + A_2] \mapsto 210_{10}$$

$[D_6(a_1)] \mapsto 210_6$

$[D_5 + A_1] \mapsto 168_6$

$[A_4 + A_1] \mapsto 512_{11}$

$[A_5'] \mapsto 105_{12}$

$[D_4 + 3A_1, D_4 + 2A_1, D_4 + A_1] \mapsto 84_{12}$

$[A_4] \mapsto 420_{13}$

$[2A_3 + A_1, A_3 + A_2 + A_1] \mapsto 210_{13}$

$[A_3 + A_2] \mapsto 378_{14}$

$[D_4] \mapsto 105_{15}$

$[D_4(a_1) + A_1] \mapsto 405_{15}$

$[A_3 + A_2] \mapsto 84_{15}, *_2$

$[A_3 + 3A_1, (A_3 + 2A_1)'] \mapsto 216_{16}$

$[D_4(a_1)] \mapsto 315_{16}$

$[(A_3 + 2A_1)'', (A_3 + A_1)''] \mapsto 280_{17}$

$[3A_2, 2A_2 + A_1] \mapsto 70_{18}$

$[(A_3 + A_1)'] \mapsto 189_{20}$

$[A_3] \mapsto 210_{21}$

$[2A_2] \mapsto 168_{21}$

$[A_2 + 3A_1] \mapsto 105_{21}$

$[A_2 + 2A_1] \mapsto 189_{22}$

$[A_2 + A_1] \mapsto 120_{25}$

$[7A_1, 6A_1, 5A_1, (4A_1)'] \mapsto 15_{28}$

$[A_2] \mapsto 56_{30}$

$[(4A_1)'', (3A_1)''] \mapsto 35_{31}$

$[(3A_1)'] \mapsto 21_{36}$

$[2A_1] \mapsto 27_{37}$

$[A_1] \mapsto 7_{46}$

$[A_0] \mapsto 1_{63}$

4.8 Type E_8

$[E_8] \mapsto 1_0$

$[E_8(a_1)] \mapsto 8_1$

$[E_8(a_2)] \mapsto 35_2$

$[E_8(a_4)] \mapsto 112_3$

$[E_7 + A_1, E_7] \mapsto 84_4$

$[E_8(a_5)] \mapsto 210_4$

$[D_8] \mapsto 560_5$

$[E_7(a_1)] \mapsto 567_6$

$[E_8(a_3)] \mapsto 700_6$

$[D_8(a_1), D_7] \mapsto 400_7$

$[E_8(a_7)] \mapsto 1400_7$

$[E_8(a_6)] \mapsto 1400_8$

$[E_7(a_3)] \mapsto 2268_{10}$

$[E_6(a_1) + A_1] \mapsto 4096_{11}$

$[D_8(a_3)] \mapsto 1400_{11}$

$[E_6] \mapsto 525_{12}$

$[D_7(a_2)] \mapsto 4200_{12}$

$[D_6 + 2A_1, D_6 + A_1, D_6] \mapsto 972_{12}$

$[E_6(a_1)] \mapsto 2800_{13}$

$[A_7 + A_1] \mapsto 4536_{13}$

$[A_7'] \mapsto 6075_{14}$

$[A_6 + A_1] \mapsto 2835_{14}$

$[D_5 + A_2] \mapsto 840_{14}, *_2$

$[A_6] \mapsto 4200_{15}$

$[E_7(a_2)+A_1, E_7(a_2)] \mapsto 1344_8$

$[E_6+A_2, E_6+A_1] \mapsto 448_9$

$[D_8(a_2)] \mapsto 3240_9$

$[D_7(a_1)] \mapsto 1050_{10}, *_2$

$[A_7''] \mapsto 175_{12}, *_3$

$[A_8] \mapsto 2240_{10}$

$[D_6(a_1)] \mapsto 5600_{15}$

$[E_8(a_8)] \mapsto 4480_{16}$

$[D_5+2A_1, D_5+A_1] \mapsto 3200_{16}$

$[E_7(a_4)+A_1, E_7(a_4)] \mapsto 7168_{17}$

$[2D_4, D_6(a_2)+A_1, D_6(a_2)] \mapsto 4200_{18}$

$[E_6(a_2)+A_2, E_6(a_2)+A_1] \mapsto 3150_{18}$

$[A_5+A_2+A_1, A_5+A_2, A_5+2A_1, (A_5+A_1)''] \mapsto 2016_{19}$

$[D_5(a_1)+A_3, D_5(a_1)+A_2] \mapsto 1344_{19}$

$[D_5] \mapsto 2100_{20}$

$[2A_4, A_4+A_3] \mapsto 420_{20}$

$[E_6(a_2)] \mapsto 5600_{21}$

$[D_4+A_3] \mapsto 4200_{21}$

$[(A_5+A_1)'] \mapsto 3200_{22}$

$[D_5(a_1)+A_1] \mapsto 6075_{22}$

$[A_4+A_2+A_1] \mapsto 2835_{22}$

$[A_4+A_2] \mapsto 4536_{23}$

$[A_4+2A_1] \mapsto 4200_{24}$

$[D_4+A_2] \mapsto 168_{24}, *_2$

$[D_5(a_1)] \mapsto 2800_{25}$

$[A_4+A_1] \mapsto 4096_{26}$

$[2D_4(a_1), D_4(a_1)+A_3, (2A_3)''] \mapsto 840_{26}$

$[D_4+4A_1, D_4+3A_1, D_4+2A_1, D_4+A_1] \mapsto 700_{28}$

$[D_4(a_1)+A_2] \mapsto 2240_{28}$

$[2A_3+2A_1, A_3+A_2+2A_1, 2A_3+A_1, A_3+A_2+A_1] \mapsto 1400_{29}$

$[A_4] \mapsto 2268_{30}$

$[(2A_3)'] \mapsto 3240_{31}$

$[D_4(a_1)+A_1] \mapsto 1400_{32}$

$[A_3+A_2] \mapsto 972_{32}, *_2$

$[A_3+4A_1, A_3+3A_1, (A_3+2A_1)''] \mapsto 1050_{34}$

$[D_4] \mapsto 525_{36}$

$[4A_2, 3A_2+A_1, 2A_2+2A_1] \mapsto 175_{36}$

$[D_4(a_1)] \mapsto 1400_{37}$

$[(A_3+2A_1)', A_3+A_1] \mapsto 1344_{38}$

$[3A_2, 2A_2+A_1] \mapsto 448_{39}$

$[2A_2] \mapsto 700_{42}$

$[A_2+4A_1, A_2+3A_1] \mapsto 400_{43}$

$[A_3] \mapsto 567_{46}$

$[A_2+2A_1] \mapsto 560_{47}$

$[A_2+A_1] \mapsto 210_{52}$

$[8A_1, 7A_1, 6A_1, 5A_1, (4A_1)''] \mapsto 50_{56}$

$[A_2] \mapsto 112_{63}$

$[(4A_1)', 3A_1] \mapsto 84_{64}$

$[2A_1] \mapsto 35_{74}$

$[A_1] \mapsto 8_{91}$

$[A_0] \mapsto 1_{120}$

4.9 In the tables in 4.4–4.8 the E which are not marked with $*_r$ are in $\mathcal{S}_1(W)$; they are expressed explicitly in the form $j_{W_{e'}}^W(E')$ with $e' \in V^*$, $E' \in \mathcal{S}(W_{e'})$ in the tables of [L6].

We now consider the E in the tables 4.4–4.8 which are marked with $*_r$.

Type G_2:

$1_3 = j_{W'}^W(\text{sign})$ where W' is of type A_2 but not of form $W_{e'}, e' \in V^*$.

Type F_4:

$9_6 = j_{W'}^W(\text{E'})$ where W' is of type B_4 but not of form $W_{e'}, e' \in V^*$ and $\dim E' = 6, n_{E'} = 6$;
$4_7 = j_{W'}^W(\text{sign})$ where W' is of type $A_3 A_1$ but not of form $W_{e'}, e' \in V^*$;
$4_8 = j_{W'}^W(\text{sign})$ where W' is of type $B_2 B_2$;
$2_{12} = j_{W'}^W(\text{sign})$ where W' is of type B_4 but not of form $W_{e'}, e' \in V^*$.

Type E_7:

$84_{15} = j_{W'}^W(\text{sign})$ where W' is of type $D_4 A_1 A_1 A_1$.

Type E_8:

$1050_{10} = j_{W'}^W(\text{E'})$ where W' is of type $D_6 A_1 A_1$ and $\dim E' = 30, n_{E'} = 10$;
$175_{12} = j_{W'}^W(\text{sign})$ where W' is of type $A_2 A_2 A_2 A_2$.
$840_{14} = j_{W'}^W(\text{sign})$ where W' is of type $A_3 A_3 A_1 A_1$.
$168_{24} = j_{W'}^W(\text{sign})$ where W' is of type $D_4 D_4$.
$972_{32} = j_{W'}^W(\text{sign})$ where W' is of type $D_6 A_1 A_1$.

4.10 For any $C \in \text{cl}(W)$ let m_C be the dimension of the 1-eigenspace of an element in C in the reflection representation of W. We have the following result.

(a) *For any $E \in \mathcal{S}_2(W)$, the restriction of $C \mapsto m_C$ to ${}'\Phi^{-1}(E) \subset \text{cl}(W)$ reaches its minimum at a unique element of ${}'\Phi^{-1}(E)$, denoted by C_E.*

We can assume that G is simple. When G is of exceptional type, (a) follows from the tables 4.4-4.8. When G is of classical type, (a) follows from [L10, 0.2].

Note that $E \mapsto C_E$ is a cross section of the surjective map ${}'\Phi : \text{cl}(W) \to \mathcal{S}_2(W)$. It defines a bijection of $\mathcal{S}_2(W)$ with a subset $\text{cl}_0(W)$ of $\text{cl}(W)$.

5 A second approach

5.1 In this section we sketch another approach to defining the strata of G in which Springer representations do not appear. Let $\mathrm{cl}(G)$ be the set of conjugacy classes in G. Let $\underline{l} : W \to \mathbf{N}$ be the length function of the Coxeter group W. For $w \in W$ let

$$G_w = \{g \in G; (B, gBg^{-1}) \in \mathcal{O}_w \text{ for some } B \in \mathcal{B}\}.$$

For $C \in \mathrm{cl}(W)$ let

$$C_{\min} = \{w \in C; \underline{l} : C \to \mathbf{N} \text{ reaches minimum at } w\}$$

and let $G_C = G_w$ where $w \in C_{\min}$.

As pointed out in [L8, 0.2], from [L8, 1.2(a)] and [GP, 8.2.6(b)] it follows that G_C is independent of the choice of w in $C_{\min}$. From [L8] it is known that G_C contains unipotent elements; in particular, $G_C \neq \emptyset$. Clearly, G_C is a union of conjugacy classes. Let

$$\delta_C = \min_{\gamma \in \mathrm{cl}(G); \gamma \subset G_C} \dim \gamma,$$

$$\boxed{G_C} = \bigcup_{\substack{\gamma \in \mathrm{cl}(G); \\ \gamma \subset G_C, \dim \gamma = \delta_C}} \gamma.$$

Then $\boxed{G_C}$ is $\neq \emptyset$, a union of conjugacy classes of fixed dimension, δ_C. We have the following result.

5.2 Theorem Let $C \in \mathrm{cl}(W)$, $E \in \mathcal{S}_2(W)$ be such that $'\Phi(C) = E$, see 4.1. We have $\boxed{G_C} = G_E$.

We can assume that G is almost simple and that $\mathbf{k}$ is an algebraic closure of a finite field. The proof in the case of exceptional groups is reduced in 5.3 to a computer calculation. The proof for classical groups, which is based on combining the techniques of [L8], [L9] and [L12], will be given elsewhere.

5.3 In this subsection we assume that $\mathbf{k}$ is an algebraic closure of a finite field $\mathbf{F}_q$ and that G is simply connected, defined and split over $\mathbf{F}_q$ with Frobenius map $F : G \to G$.

Let γ be an F-stable conjugacy class of G. Let $\gamma' = \{g_s; g \in \gamma\}$, an F-stable semisimple conjugacy class in G. For every $s \in \gamma'$ let $\gamma(s) = \{u \in Z_G(s); u \text{ unipotent}, us \in \gamma\}$, a unipotent conjugacy class of $Z_G(s)$. We fix $s_0 \in \gamma'^F$ and we set $H = Z_G(s_0)$, $\gamma_0 = \gamma(s_0)$. Let W_H be the Weyl group of H. As in 2.1, we can regard W_H as a subgroup of W (the embedding of W_H into W is canonical up to composition with an inner automorphism of W).

By replacing if necessary F by a power of F, we can assume that H contains a maximal torus which is defined and split over $\mathbf{F}_q$. For any F-stable maximal torus

T of G, R^1_T is the virtual representation of G^F defined as in [DL, 1.20] (with $\theta = 1$ and with B omitted from notation). Replacing T, G by T', H where T' is an F-stable maximal torus of H, we obtain a virtual representation $R^1_{T',H}$ of H^F.

For any $z \in W$ we denote by R^1_z the virtual representation R^1_T of G^F where T is an F-stable maximal torus of G of type given by the conjugacy class of z in W. For any $z' \in W_H$ we denote by $R^1_{z',H}$ the virtual representation $R^1_{T',H}$ of H^F where T' is an F-stable maximal torus of H of type given by the conjugacy class of z' in W_H. For $E' \in \mathrm{Irr}W$ we set $R_{E'} = |W|^{-1} \sum_{y \in W} \mathrm{tr}(y, E') R^1_y$. Then for any $z \in W$, we have $R^1_z = \sum_{E' \in \mathrm{Irr}W} \mathrm{tr}(z, E') R_{E'}$.

Let $w \in W$. We show the following:

(a)
$$\begin{aligned} &|\{(g, B) \in \gamma^F \times \mathcal{B}^F; (B, gBg^{-1}) \in \mathcal{O}_w\}| \\ &\quad = |G^F||H^F|^{-1} \sum_{\substack{E \in \mathrm{Irr}W, E' \in \mathrm{Irr}W,\\ E'' \in \mathrm{Irr}W_H, y}} \mathrm{tr}(T_w, E_q)(\rho_E, R_{E'}) \\ &\quad \times (E'|_{W_H} : E'')|Z_{W_H}(y)|^{-1} \mathrm{tr}(y, E'') \sum_{u \in \gamma_0^F} \mathrm{tr}(u, R^1_{y,H}), \end{aligned}$$

where y runs over a set of representatives for the conjugacy classes in W_H and T_w, E_q, ρ_E are as in [L8, 1.2]. Let N be the left-hand side of (a). As in [L8, 1.2(c)] we see that

$$N = \sum_{E \in \mathrm{Irr}W} \mathrm{tr}(T_w, E_q) A_E$$

with

$$A_E = |G^F|^{-1} \sum_{g \in \gamma^F} \sum_T |T^F|(\rho_E, R^1_T)\mathrm{tr}(g, R^1_T),$$

where T runs over all maximal tori of G defined over $\mathbf{F}_q$. We have

$$\begin{aligned} A_E &= |G^F|^{-1} \sum_{s \in \gamma'^F, u \in \gamma(s)^F} \sum_T |T^F|(\rho_E, R^1_T)\mathrm{tr}(su, R^1_T) \\ &= |H^F|^{-1} \sum_{u \in \gamma_0^F} \sum_T |T^F|(\rho_E, R^1_T)\mathrm{tr}(s_0 u, R^1_T). \end{aligned}$$

By [DL, 4.2] we have

$$\mathrm{tr}(s_0 u, R^1_T) = |H^F|^{-1} \sum_{x \in G^F; x^{-1}Tx \subset H} \mathrm{tr}(u, R^1_{x^{-1}Tx, H}),$$

hence

$$A_E = |H^F|^{-2} \sum_{u\in\gamma_0^F} \sum_T |T^F|(\rho_E, R_T^1) \sum_{x\in G^F; x^{-1}Tx\subset H} \mathrm{tr}(u, R^1_{x^{-1}Tx,H})$$
$$= |G^F||H^F|^{-2} \sum_{T'\subset H} |T'^F|(\rho_E, R^1_{T'}) \sum_{u\in\gamma_0^F} \mathrm{tr}(u, R^1_{T',H}),$$

where T' runs over the maximal tori of H defined over $\mathbf{F}_q$. Using the classification of maximal tori of H defined over $\mathbf{F}_q$, we obtain

$$A_E = |G^F||H^F|^{-1}|W_H|^{-1} \sum_{z\in W_H} (\rho_E, R_z^1) \sum_{u\in\gamma_0^F} \mathrm{tr}(u, R^1_{z,H})$$
$$= |G^F||H^F|^{-1}|W_H|^{-1} \sum_{z\in W_H} \sum_{E'\in\mathrm{Irr}W} \mathrm{tr}(z, E')(\rho_E, R_{E'}) \sum_{u\in\gamma_0^F} \mathrm{tr}(u, R^1_{z,H}).$$

This clearly implies (a).

Now assume that G is almost simple of exceptional type and that w has minimal length in its conjugacy class in W. We can also assume that $q-1$ is sufficiently divisible. Then the right-hand side of (a) can be explicitly determined using a computer. Indeed, it is an entry of the product of several large matrices whose entries are explicitly known. In particular the quantities $\mathrm{tr}(T_w, E_q)$ (known from the works of Geck and Geck–Michel, see [GP, 11.5.11]) are available through the CHEVIE package [GH]. The quantities $(\rho_E, R_{E'})$ are coefficients of the nonabelian Fourier transform in [L2, 4.15]. The quantities $(E'|_{W_H} : E'')$ are available from the induction tables in the CHEVIE package. The quantities $\mathrm{tr}(y, E'')$ are available through the CHEVIE package. The quantities $\mathrm{tr}(u, R^1_{y,H})$ are Green functions; I thank Frank Lübeck for providing me with the tables of Green functions for groups of rank ≤ 8 in GAP format. I also thank Gongqin Li for her help with programming in GAP to perform the actual computation using these data.

Thus the number $|\{(g, B) \in \gamma^F \times \mathcal{B}^F; (B, gBg^{-1}) \in \mathcal{O}_w\}|$ is explicitly computable. It turns out that it is a polynomial in q. Note that the set $\{(g, B) \in \gamma \times \mathcal{B}; (B, gBg^{-1}) \in \mathcal{O}_w\}$ is nonempty if and only if this polynomial is non zero. Thus the condition that $\gamma \subset G_w$ can be tested. This can be used to check that Theorem 5.2 holds for exceptional groups.

5.4 If C is the conjugacy class containing the Coxeter elements of W, then $G_C = \boxed{G_C}$ is the union of all conjugacy classes of dimension $\dim G - \mathrm{rk}(G)$, see [St].

6 Variants

6.1 The results in this subsection will be proved elsewhere. In this subsection we assume that G is simple and that G' is a disconnected reductive algebraic group G over $\mathbf{k}$ with identity component G, such that G'/G is cyclic of order r and such that the homomorphism $\epsilon : G'/G \to \mathrm{Aut}(W)$ (the automorphism group of W as a Coxeter group) induced by the conjugation action of G'/G on G is injective. Note that (G, r) must be of type $(A_n, 2)$ $(n \geq 2)$ or $(D_n, 2)$ $(n \geq 4)$ or $(D_4, 3)$ or $(E_6, 2)$. Let D be a connected component of G' other than G. We will give a definition of the strata of D, extending the definition of strata of G. Let $\epsilon_D : W \to W$ be the image of D under ϵ. Let $\mathrm{cl}_D W$ be the set of conjugacy classes in W twisted by ϵ_D (as in [L12, 0.1]). Let $\mathrm{cl}(D)$ be the set of G-conjugacy classes in D. For $w \in W$ let

$$D_w = \{g \in D; (B, gBg^{-1}) \in \mathcal{O}_w \text{ for some } B \in \mathcal{B}\}.$$

For $C \in \mathrm{cl}_D(W)$ let

$$C_{\min} = \{w \in C; \underline{l} : C \to \mathbf{N} \text{ reaches minimum at } w\}.$$

and let $D_C = D_w$ where $w \in C_{\min}$. This is independent of the choice of w in $C_{\min}$. One can show that $D_C \neq \emptyset$. Clearly, D_C is a union of G-conjugacy classes in D. Let

$$\delta_C = \min_{\gamma \in \mathrm{cl}(D); \gamma \subset D_C} \dim \gamma,$$

$$\boxed{D_C} = \bigcup_{\substack{\gamma \in \mathrm{cl}(D);\\ \gamma \subset D_C, \dim \gamma = \delta_C}} \gamma.$$

Then $\boxed{D_C}$ is $\neq \emptyset$, a union of G-conjugacy classes of fixed dimension, δ_C. One can show that $\cup_{C \in \mathrm{cl}_D(W)} \boxed{D_C} = D$; moreover, one can show that if $C, C' \in \mathrm{cl}_D(W)$, then $\boxed{D_C}, \boxed{D_{C'}}$ are either equal or disjoint. (Some partial results in this direction are contained in [L12].) Let $\sim$ be the equivalence relation on $\mathrm{cl}_D(W)$ given by $C \sim C'$ if $\boxed{D_C} = \boxed{D_{C'}}$ and let $\underline{\mathrm{cl}}_D(W)$ be the set of equivalence classes. We see that there is a unique partition of D into pieces (called *strata*) indexed by $\underline{\mathrm{cl}}_D(W)$ such that each stratum is of the form $\boxed{D_C}$ for some $C \in \mathrm{cl}_D(W)$. One can show that the equivalence relation $\sim$ on $\mathrm{cl}_D(W)$ and the function $C \mapsto d_C$ on $\mathrm{cl}_D(W)$ depend only on W and its automorphism ϵ_D; in particular they do not depend on $\mathbf{k}$. When $p = r$, each stratum of D contains a unique unipotent G-conjugacy class in D; this gives a bijection $\underline{\mathrm{cl}}_D(W) \leftrightarrow \mathcal{U}_D^r$ where $\mathcal{U}_D^r$ is the set of unipotent G-conjugacy classes in D (with $p = r$). This bijection coincides with the bijection $\underline{\mathrm{cl}}_D(W) \leftrightarrow \mathcal{U}_D^r$ described explicitly in [L11]. Thus the strata of D can also be indexed by $\mathcal{U}_D^r$. We can also index them by a certain set of irreducible representations of W^{ϵ_D} (the fixed point set of $\epsilon_D : W \to W$) using the bijection [L4, II] between $\mathcal{U}_D^r$ and a set of irreducible representations of W^{ϵ_D} (an extension of the Springer correspondence).

6.2 Assume that G is adjoint. We identify $\mathcal{B}$ with the variety of Borel subalgebras of $\mathfrak{g}$. For any $\xi \in \mathfrak{g}$ let $\mathcal{B}_\xi = \{\mathfrak{b} \in \mathcal{B}; \xi \in \mathfrak{b}\}$ and let $d = \dim \mathcal{B}_\xi$. The subspace of $H_{2d}(\mathcal{B})$ spanned by the images of the fundamental classes of the irreducible components of $\mathcal{B}_\xi$ is an irreducible W-module denoted by τ_ξ. We also denote by τ_ξ

the corresponding W-module over $\mathbf{Q}$. Thus we have a well-defined map $\mathfrak{g} \to \mathrm{Irr} W$, $\xi \mapsto \tau_\xi$. The nonempty fibres of this map are called the *strata* of $\mathfrak{g}$. Each stratum of $\mathfrak{g}$ is a union of adjoint orbits of fixed dimension; exactly one of these orbits is nilpotent. The image of the map $\xi \mapsto \tau_\xi$ is the subset of $\mathrm{Irr}(W)$ denoted by $\mathcal{T}_W^p$ in [L7]; when $p = 0$ this is $\mathcal{S}_1(W)$.

6.3 In this subsection we assume that G is semisimple simply connected. Let K be the field of formal power series $\mathbf{k}((\epsilon))$ and let $\hat{G} = G(K)$. Let $\hat{\mathcal{B}}$ be the set of Iwahori subgroups of $\hat{G}$ viewed as an increasing union of projective algebraic varieties over $\mathbf{k}$. Let $\hat{W}$ be the affine Weyl group associated to $\hat{G}$ viewed as an infinite Coxeter group. Let $G(K)_{rsc}$ be the set of all $g \in G(K)$ that are compact (that is such that $\hat{\mathcal{B}}_g = \{B \in \hat{\mathcal{B}}; g \in B\}$ is nonempty) and regular semisimple. If $g \in G(K)_{rsc}$, then $\hat{\mathcal{B}}_g$ is a union of projective algebraic varieties of fixed dimension $d = d_g$ (see [KL] for a closely related result) hence the homology space $H_{2d}(\hat{\mathcal{B}}_g)$ is well defined and it carries a natural $\hat{W}$-action (see [L13]). Similarly the homology space $H_{2d}(\hat{\mathcal{B}})$ is well-defined and it carries a natural $\hat{W}$-action. The embedding $h_g : \hat{\mathcal{B}}_g \to \hat{\mathcal{B}}$ induces a linear map $h_{g*} : H_{2d}(\hat{\mathcal{B}}_g) \to H_{2d}(\hat{\mathcal{B}})$ which is compatible with the $\hat{W}$-actions. Hence $\hat{W}$ acts naturally on the (finite-dimensional) subspace $E_g := h_{g*}(H_{2d}(\hat{\mathcal{B}}_g))$ of $H_{2d}(\hat{\mathcal{B}})$, but this action is not irreducible in general. Note that E_g is the subspace of $H_{2d}(\hat{\mathcal{B}})$ spanned by the images of the fundamental classes of the irreducible components of $\hat{\mathcal{B}}_g, \overline{\mathbf{Q}}_l$ (we ignore Tate twists), hence is $\neq 0$. For $g, g' \in G(K)_{rsc}$ we say that $g \sim g'$ if $d_g = d_{g'}$ and $E_g = E_{g'}$. This is an equivalence relation on $G(K)_{rsc}$. The equivalence classes for $\sim$ are called the *strata* of $G(K)_{rsc}$. Note that $G(K)_{rsc}$ is a union of countably many strata and each stratum is a union of conjugacy classes of $G(K)$ contained in $G(K)_{rsc}$.

6.4 In this subsection we state a conjectural definition of the strata of G in the case where $\mathbf{k} = \mathbf{C}$ based on an extension of a construction in [KL]. Let K be as in 6.3. Let $g \in G$. Let $\mathfrak{z} \subset \mathfrak{g}$ be the Lie algebra of $Z_G(g_s)$ and let $\xi = \log(g_u) \in \mathfrak{z}$. Let $\mathfrak{p}$ be a parahoric subalgebra of $\mathfrak{g}_K := K \otimes \mathfrak{g}$ with pro-nilradical $\mathfrak{p}_n$ such that $\mathfrak{p} = \mathfrak{z} \oplus \mathfrak{p}_n$ as $\mathbf{C}$-vector spaces. By the last corollary in [KL, §6], there exists a non-empty subset $\mathfrak{U}$ of $\xi + \mathfrak{p}_n$ (open in the power series topology) and $\sigma \in \mathrm{cl}(W)$ such that for any $x \in \mathfrak{U}$, x is regular semisimple in a Cartan subalgebra of $\mathfrak{g}_K$ of type σ (see [KL, §1,§6]). Note that σ does not depend on the choice of $\mathfrak{U}$. We expect that it does not depend on the choice of $\mathfrak{p}$ and that $g \mapsto \sigma$ is a map $G \to \mathrm{cl}(W)$ whose fibres are exactly the strata of G. By the identification 4.1(c) this induces an injective map $\underline{\mathrm{cl}}(W) \to \mathrm{cl}(W)$ whose image is expected to be the subset $\mathrm{cl}_0(W)$ in 4.10 and whose composition with the obvious map $\mathrm{cl}(W) \to \underline{\mathrm{cl}}(W)$ is expected to be the identity map of $\underline{\mathrm{cl}}(W)$.

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Hecke algebras and involutions in Coxeter groups

George Lusztig and David A. Vogan, Jr.

The first author dedicates this paper to the second author and wishes him many productive years

Abstract Let W be a Coxeter group and let M be the free $Z[v, v^{-1}]$-module with basis indexed by the involutions of W. We show how the recent results of Elias and Williamson on Soergel bimodules can be used to give an alternative definition of an action of the Hecke algebra of W on M.

Key words: Involution, Coxeter group, Hecke algebra, Soergel bimodule

MSC (2010): Primary 20G99

Introduction

Let W be a finitely generated Coxeter group with a fixed involutive automorphism $w \mapsto w^*$ which leaves stable the set of simple reflections. An element $w \in W$ is said to be a $*$-twisted involution if $w^{-1} = w^*$. Let $\mathbf{I} = \{w \in W; w^{-1} = w^*\}$ be the set of $*$-twisted involutions of W. Let $\mathcal{A}' = \mathbf{Z}[v, v^{-1}]$ where v is an indeterminate. In [LV] we have defined (geometrically) an action of the Hecke algebra of W (with parameter v^2) on the free $\mathcal{A}'$-module $\mathcal{M}$ with basis $\{a_w; w \in \mathbf{I}\}$, assuming that W is a Weyl group. In [L3] a definition of the Hecke algebra action on $\mathcal{M}$ was given in a purely algebraic way, without assumption on W. The purpose of this paper is

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to give a more conceptual approach to the definition of the Hecke algebra action on $\mathcal{M}$, based on the theory of Soergel bimodules [S] and on the recent results of Elias and Williamson [EW] in that theory.

In this paper we interpret $\mathcal{M}$ as a (modified) Grothendieck group associated to the category of Soergel bimodules corresponding to W and to a 2-periodic functor of this category to itself, defined using $*$ and by switching left and right multiplication in a bimodule. The action of the Hecke algebra appears quite naturally in this interpretation; however, we must find a way to compute explicitly the action of a generator $T_s + 1$ of the Hecke algebra (s is a simple reflection) on a basis element a_w of $\mathcal{M}$ so that we recover the formulas of [LV], [L3]. The formula has four cases depending on whether sw is equal to ws^* or not and on whether the length of sw is smaller or larger than that of w. In each case, $(T_s + 1)a_w$ is a linear combination $c'a_{w'} + c''a_{w''}$ of two basis elements $a_{w'}, a_{w''}$ where one of w', w'' is equal to w, the other is sw or sws^* and the length of w' is smaller than that of w''. We cannot prove the formulas directly. Instead we compute directly the coefficient c' and then observe that if c' is known, then c'' is automatically known from the fact that we have a Hecke algebra action. The computation of c' occupies Sections 4 and 5 (see Theorem 5.2). It has two cases (depending on whether sw' is equal to $w's^*$ or not). The two cases require quite different proofs.

As an application of Theorem 6.2 (which is essentially a corollary of Theorem 5.2) we outline a proof (6.3) of a positivity conjecture ([L3, Conjecture 9.12]) stating that, if $y, w \in \mathbf{I}$ and $\delta \in \{1, -1\}$, then the polynomial $P^{\sigma}_{y,w}$ introduced in [L3] (and earlier in [LV] in the case of Weyl groups) satisfies $(P_{y,w}(u) + \delta P^{\sigma}_{y,w}(u))/2 \in \mathbf{N}[u]$ where $P_{y,w}$ is the polynomial introduced in [KL]. This is a refinement of the statement [EW] that $P_{y,w}(u) \in \mathbf{N}[u]$ which holds for any $y, w \in W$. In Section 7 we show as another application of our results that $\mathcal{M}$ admits a filtration by Hecke algebra submodules whose subquotients are indexed by the two-sided cells of W. Under a boundedness assumption we show that the Hecke algebra acts on such a subquotient by something resembling a W-graph.

Contents

1 2-periodic functors

1.1 In this section we review some results from [L1, §11].

Let k be a field of characteristic zero. Let $\mathfrak{C}$ be a k-linear category, that is a category in which the space of morphisms between any two objects has a given k-vector space structure such that the composition of morphisms is bilinear and such that finite direct sums exist. A functor from one k-linear category to another is said to be k-linear if it respects the k-vector space structures.

Let $\mathcal{K}(\mathfrak{C})$ be the Grothendieck group of $\mathfrak{C}$ that is, the free abelian group generated by symbols $[A]$ for each $A \in \mathfrak{C}$ (up to isomorphism) with relations $[A \oplus B] = [A| + [B]$ for any $A, B \in \mathfrak{C}$. A k-linear functor $M \mapsto M^\sharp$, $\mathfrak{C} \to \mathfrak{C}$ is said to be 2-*periodic* if $M \mapsto (M^\sharp)^\sharp$ is the identity functor $\mathfrak{C} \to \mathfrak{C}$. Assuming that such a functor is given we define a new k-linear category $\mathfrak{C}_\sharp$ as follows. The objects of $\mathfrak{C}_\sharp$ are pairs (A, ϕ) where $A \in \mathfrak{C}$ and $\phi : A^\sharp \to A$ is an isomorphism in $\mathfrak{C}$ such that the composition $(A^\sharp)^\sharp \xrightarrow{\phi^\sharp} A^\sharp \xrightarrow{\phi} A$ is the identity map of A. Let $(A, \phi), (A', \phi')$ be two objects of $\mathfrak{C}_\sharp$. We define a k-linear map $\mathrm{Hom}_{\mathfrak{C}}(A, A') \to \mathrm{Hom}_{\mathfrak{C}}(A, A')$ by $f \mapsto f^! := \phi' f^\sharp \phi^{-1}$. Note that $(f^!)^! = f$. By definition, $\mathrm{Hom}_{\mathfrak{C}_\sharp}((A, \phi), (A', \phi')) = \{f \in \mathrm{Hom}_{\mathfrak{C}}(A, A'); f = f^!\}$, a k-vector space. The direct sum of two objects (A, ϕ), (A', ϕ') is $(A \oplus A', \phi \oplus \phi')$. Clearly, if $(A, \phi) \in \mathfrak{C}_\sharp$, then $(A, -\phi) \in \mathfrak{C}_\sharp$. An object (A, ϕ) of $\mathfrak{C}_\sharp$ is said to be *traceless* if there exists an object B of $\mathfrak{C}$ and an isomorphism $A \cong B \oplus B^\sharp$ under which ϕ corresponds to an isomorphism $B^\sharp \oplus B \xrightarrow{\sim} B \oplus B^\sharp$ which carries the first (resp. second) summand of $B^\sharp \oplus B$ onto the second (resp. first) summand of $B \oplus B^\sharp$.

Let $\mathcal{K}_\sharp(\mathfrak{C})$ be the quotient of $\mathcal{K}(\mathfrak{C}_\sharp)$ by the subgroup $\mathcal{K}^0(\mathfrak{C}_\sharp)$ generated by the elements $[B, \phi]$ where (B, ϕ) is any traceless object of $\mathfrak{C}_\sharp$. We show that

(a) $[A, -\phi] = -[A, \phi]$ for any $(A, \phi) \in \mathfrak{C}_\sharp$.

Indeed, if we define $\phi' : A^\sharp \oplus A \to A \oplus A^\sharp$ by $(x, y) \mapsto (y, x)$ and $\tau : A \oplus A \to A \oplus A^\sharp$ by $(x, y) \mapsto (x + y, \phi^{-1}(x) - \phi^{-1}(y))$, then τ defines an isomorphism of $(A, \phi) \oplus (A, -\phi)$ with the traceless object $(A \oplus A^\sharp, \phi')$.

2 A review of Soergel modules

2.1 In this section we review some results of Soergel [S] and of Elias–Williamson [EW].

Recall that W is a Coxeter group. The canonical set of generators (assumed to be finite) is denoted by S. Let $x \mapsto l(x)$ be the length function on W and let $\leq$ be the Bruhat order on W. Let $\mathfrak{h}$ be a reflection representation of W over the real numbers $\mathbf{R}$, as in [EW]; for any $s \in S$ we fix a linear form $\alpha_s : \mathfrak{h} \to \mathbf{R}$ whose kernel is equal to the fixed point set of $s : \mathfrak{h} \to \mathfrak{h}$. Let R be the algebra of polynomial functions $\mathfrak{h} \to \mathbf{R}$ with the $\mathbf{Z}$-grading in which linear functions $\mathfrak{h} \to \mathbf{R}$ have degree 2. Note that

W acts naturally on R; we write this action as $w : r \mapsto {}^w r$, and for $s \in S$ we set $R^s = \{r \in R; {}^s r = r\}$, a subalgebra of R. Let $R^{>0} = \{r \in R; r(0) = 0\}$. Let $\widehat{R}$ be the completion of R with respect to the maximal ideal $R^{>0}$.

Let $\mathcal{R}$ be the category whose objects are $\mathbf{Z}$-graded (R, R)-bimodules such that the left action of $\mathbf{R}$ is the same as the right action of $\mathbf{R}$, in which for $M, M' \in \mathcal{R}$, $\mathrm{Hom}_{\mathcal{R}}(M, M')$ is the space of homomorphisms of (R, R)-bimodules $M \to M'$ compatible with the $\mathbf{Z}$-gradings. For $M \in \mathcal{R}$ and $n \in \mathbf{Z}$, the shift $M[n]$ is the object of $\mathcal{R}$ equal in degree i to M in degree $i + n$. For M, M' in $\mathcal{R}$ we set $MM' = M \otimes_R M'$; this is naturally an object of $\mathcal{R}$. For M, M' in $\mathcal{R}$ we set

$$M'^M = \oplus_{n \in \mathbf{Z}} \mathrm{Hom}_{\mathcal{R}}(M, M'[n]),$$

viewed as an object of $\mathcal{R}$ with $(rf)(m) = f(rm)$, $(fr)(m) = f(mr)$ for $m \in M, f \in M'^M, r \in R$. For any $M \in \mathcal{R}$ we set $\underline{M} = M/MR^{>0} = M \otimes_R \mathbf{R}$ where $\mathbf{R}$ is identified with $R/R^{>0}$. We view $\underline{M}$ as a $\mathbf{Z}$-graded $\mathbf{R}$-vector space. For any $M \in \mathcal{R}$ we set

$$\widehat{M} = M^\wedge = M \otimes_R \widehat{R},$$

viewed as a $\mathbf{Z}$-graded right $\widehat{R}$-module.

For $s \in S$ let $B_s = R \otimes_{R^s} R[1] \in \mathcal{R}$. More generally, for any $x \in W$, Soergel [S, Bemerkung 6.16] shows that there is an object B_x of $\mathcal{R}$ (unique up to isomorphism) such that B_x is an indecomposable direct summand of $B_{s_1} B_{s_2} \cdots B_{s_q}$ for some/any reduced expression $w = s_1 s_2 \cdots s_q$ ($s_i \in S$) and such that B_x is not a direct summand of $B_{s'_1} B_{s'_2} \cdots B_{s'_p}$ whenever $s'_1, \dots, s'_p \in S, p < q$. Let $\widetilde{C}$ be the full subcategory of $\mathcal{R}$ whose objects are isomorphic to finite direct sums of shifts of objects of the form B_x for various $x \in W$. Let C be the full subcategory of $\mathcal{R}$ whose objects are isomorphic to finite direct sums of objects of the form B_x for various $x \in W$. From [S] it follows that for $M, M' \in \widetilde{C}$ we have $MM' \in \widetilde{C}$.

(In the case where W is a Weyl group of a reductive group G, C can be thought of as the category of semisimple G-equivariant perverse sheaves on the product $\mathcal{B}^2$ of two copies of the flag manifold and $\widetilde{C}$ can be thought of as the category whose objects are complexes of sheaves on $\mathcal{B}^2$ which are (non-canonically) direct sums of objects of semisimple G-equivariant perverse sheaves with shifts. Then $M, M' \mapsto MM'$ corresponds to convolution of complexes of sheaves.)

For any $x \in W$ let R_x be the object of $\mathcal{R}$ such that $R_x = R$ as a left R-module and such that for $m \in R_x, r \in R$ we have $mr = ({}^x r)m$. The following result appears in [S, Bemerkung 6.15]:

(a) *For any $M \in \widetilde{C}$, R_x^M is a finitely generated graded free right R-module; hence* $\dim_{\mathbf{R}} \underline{R_x^M} < \infty$.

Note that for any $i, n \in \mathbf{Z}$ we have $\underline{R_x^{M[n]}}_i = \underline{R_x^M}_{i-n}$.

(In the case where W is a Weyl group of a reductive group G then $\underline{R_x^M}_i$ can be thought of as the dual of a stalk of a cohomology sheaf of a complex of sheaves on $\mathcal{B}^2$ at a point in the G-orbit on $\mathcal{B}^2$ corresponding to x.)

Let $t \in \mathrm{Hom}_{\mathcal{R}}(B_s[-1], R_s) = (R_s^{B_s})_1$ be the unique element such that

$$t(1 \otimes \alpha_s + 1 \otimes \alpha_s) = 0, \qquad t(1 \otimes 1) = 1.$$

The image of t in $\underline{R_s^{B_s}}_1$ is an $\mathbf{R}$-basis of this one-dimensional $\mathbf{R}$-vector space. Hence we have canonically $\underline{R_s^{B_s}}_1 = \mathbf{R}$.

2.2 Let $x \in W$. From [EW] it follows that $\mathrm{Hom}_{\mathcal{R}}(B_x, B_x) = \mathbf{R}$ and from [S, Bemerkung 6.16] it follows that $\dim \underline{R_x^{B_x}}_{l(x)} = 1$. Thus $\underline{R_x^{B_x}}_{l(x)} \otimes_{\mathbf{R}} B_x$ is an object of C isomorphic to B_x and defined up to unique isomorphism (even though B_x was defined only up to non-unique isomorphism). From now on we will use the notation B_x for this new object. It satisfies

$$\underline{R_x^{B_x}}_{l(x)} = \mathbf{R}.$$

When $x = s \in S$, this agrees with the earlier description of B_s.

2.3 Let $x, x' \in W$, $x \neq x'$. From [EW] it follows that $\mathrm{Hom}_{\mathcal{R}}(B_{x'}, B_x) = 0$. This, together with the equality $\mathrm{Hom}_{\mathcal{R}}(B_x, B_x) = \mathbf{R}$, implies that the objects B_x are simple in C. Conversely, it is clear that any simple object of C is isomorphic to some B_x.

2.4 Let $\mathcal{A} = \mathbf{Z}[u, u^{-1}]$ where u is an indeterminate. Let $\mathbf{H}$ be the free $\mathcal{A}$-module with basis $T_w, w \in W$. It is known that there is a unique associative $\mathcal{A}$-algebra structure on $\mathbf{H}$ such that $T_w T_{w'} = T_{ww'}$ whenever $l(ww') = l(w) + l(w')$ and $T_s^2 = u^2 T_1 + (u^2 - 1) T_s$ for $s \in S$. Note that T_1 is a unit element. Let $\{c_w; w \in W\}$ be the $\mathcal{A}$-basis of $\mathbf{H}$ which in [KL] was denoted by $\{C'_w; w \in W\}$. Recall that

(a) $$c_w = u^{-l(w)} \sum_{y \leq w} P_{y,w}(u^2) T_y$$

where $P_{y,w} = 1$ if $y = w$ and $P_{y,w}$ is a polynomial of degree $\leq (l(w) - l(y) - 1)/2$ if $y < w$. We regard $\mathcal{K}(\widetilde{C})$ as an $\mathcal{A}$-module by $u^n[M] = [M[-n]]$ for $M \in \widetilde{C}, n \in \mathbf{Z}$. Note that $\mathcal{K}(\widetilde{C})$ is an associative $\mathcal{A}$-algebra with product defined by $[M][M'] = [MM']$ for $M \in \widetilde{C}$, $M' \in \widetilde{C}$. From [S, Theorems 1.10 and 5.3] we see that

(b) *the assignment* $M \mapsto \sum_{y \in W, i \in \mathbf{Z}} \dim(\underline{R_y^M})_i u^{-i} T_y$ *defines an* $\mathcal{A}$*-algebra isomorphism* $\chi : \mathcal{K}(\widetilde{C}) \xrightarrow{\sim} \mathbf{H}$.

From [EW, Theorem 1.1] it follows that

(c) $$\chi(B_x) = c_x.$$

3 The H-module $\mathcal{M}$

3.1 In this section we preserve the setup of Section 2.

Recall that $w \mapsto w^*$ is an involutive automorphism $W \xrightarrow{\sim} W$ leaving S stable. We can assume that there exists an involutive $\mathbf{R}$-linear map $\mathfrak{h} \to \mathfrak{h}$ (denoted again by $x \mapsto x^*$) which satisfies $(wx)^* = w^*x^*$ for $w \in W, x \in \mathfrak{h}$ and satisfies $\alpha_{s^*} = (\alpha_s)^*$ for $s \in S$. We fix such a linear map. It induces a ring involution $R \to R$ denoted again by $r \mapsto r^*$. For $M \in \mathcal{R}$ let $M^\sharp$ be the object of $\mathcal{R}$ which is equal to M as a graded $\mathbf{R}$-vector space, but left (resp. right) multiplication by $r \in R$ on $M^\sharp$ equals right (resp. left) multiplication by r^* on M. Clearly, $(M^\sharp)^\sharp = M$. If $f : M_1 \to M_2$ is a morphism in $\mathcal{R}$, then f can be also viewed as a morphism $M_1^\sharp \to M_2^\sharp$ in $\mathcal{R}$. Note that $M \mapsto M^\sharp$ is an $\mathbf{R}$-linear, 2-periodic functor $\mathcal{R} \to \mathcal{R}$. Hence $\mathcal{R}_\sharp$ is well-defined; see 1.1.

If $M_1, M_2 \in \mathcal{R}$, then we have an obvious identification $M_1^\sharp M_2^\sharp = (M_2 M_1)^\sharp$ as objects in $\mathcal{R}$; it is given by $x_1 \otimes x_2 \mapsto x_2 \otimes x_1$.

Let $s \in S$. The $\mathbf{R}$-linear isomorphism $\omega_s : B_{s^*}[-1] \xrightarrow{\sim} B_s[-1]$ given by $x \otimes_{R^{s^*}} y \mapsto y^* \otimes_{R^s} x^*$ for $x, y \in R$ can be viewed as an isomorphism $B_{s^*}^\sharp[-1] \xrightarrow{\sim} B_s[-1]$ in $\mathcal{R}$ or as an isomorphism $B_{s^*}^\sharp \xrightarrow{\sim} B_s$ in $\mathcal{R}$.

Now let $x \in W$ and let $s_1 s_2 \cdots s_k$ be a reduced expression for x. Since B_x is an indecomposable direct summand of $B_{s_1} B_{s_2} \cdots B_{s_k}$ (and k is minimal with this property) we see that $B_x^\sharp$ is an indecomposable direct summand of

$$(B_{s_1} B_{s_2} \cdots B_{s_k})^\sharp = B_{s_k}^\sharp \cdots B_{s_2}^\sharp B_{s_1}^\sharp \cong B_{s_k^*} \cdots B_{s_2^*} B_{s_1^*}$$

(and k is minimal with this property), hence by [S, Bemerkung 6.16] we have

(a) $$B_x^\sharp \cong B_{(x^*)^{-1}}.$$

(We use that $s_k^* \cdots s_2^* s_1^*$ is a reduced expression for $(x^*)^{-1}$.) In particular we have $B_x^\sharp \in C$. It follows that $M \in C \implies M^\sharp \in C$ and $M \in \widetilde{C} \implies M^\sharp \in \widetilde{C}$. Note that $M \mapsto M^\sharp$ are $\mathbf{R}$-linear, 2-periodic functors $C \to C$ and $\widetilde{C} \to \widetilde{C}$. Hence $C_\sharp, \widetilde{C}_\sharp$ are defined as in 1.1 and $\mathcal{K}_\sharp(C)$, $\mathcal{K}_\sharp(\widetilde{C})$ are well-defined abelian groups.

3.2 Recall that $\mathbf{I} = \{y \in W; y^{-1} = y^*\}$. Let $x \in W$. We define

$$\mathfrak{f}_x : R_x^\sharp \to R_{(x^*)^{-1}} \quad \text{by} \quad r \mapsto \mathfrak{f}_x(r) = ({}^{x^{-1}}r)^*.$$

This is an isomorphism in $\mathcal{R}$.

Now assume that $x \in \mathbf{I}$; then $\mathfrak{f}_x : R_x^\sharp \to R_x$ is given by $r \mapsto \mathfrak{f}_x(r) = {}^x(r^*)$ and $(R_x, \mathfrak{f}_x) \in \mathcal{R}_\sharp$; thus $(R_x[i], \mathfrak{f}_x[i]) \in \mathcal{R}_\sharp$ for any $i \in \mathbf{Z}$. Hence, if $(M, \phi) \in \widetilde{C}_\sharp$ and $i \in \mathbf{Z}$, then $f \mapsto f^!$, $\mathrm{Hom}_{\mathcal{R}}(M, R_x[i]) \to \mathrm{Hom}_{\mathcal{R}}(M, R_x[i])$ is defined as in 1.1. Taking the direct sum over $i \in \mathbf{Z}$, we obtain a map $f \mapsto f^!$, $R_x^M \to R_x^M$ such that $(f^!)^! = f$. (We always write R_x^M instead of $(R_x)^M$.) From the definitions, for $f \in R_x^M, r \in R$, we have $(fr)^! = r^* f^!$, $(rf)^! = f^! r^*$. Since for $r \in R, b \in R_x$ we have $rb = b({}^{x^{-1}}r)$, we see that $R^{>0}R_x = R_x R^{>0}$ so that $R^{>0}(R_x^M) = (R_x^M)R^{>0}$; we see that $f \mapsto f^!$ induces an $\mathbf{R}$-linear (involutive) map $\underline{R_x^M} \to \underline{R_x^M}$ and (for any i) an $\mathbf{R}$-linear involutive map $\underline{R_x^M}_i \to \underline{R_x^M}_i$ denoted by $\mathcal{Y}_{x,\phi,i}^M$. Let

$$\epsilon_i^x(M,\phi) = \mathrm{tr}_{\mathbf{R}}(\mathcal{Y}^M_{x,\phi,i}, \underline{R^M_x}) \in \mathbf{Z}.$$

We now take $M = B_x$ (still assuming $x \in \mathbf{I}$ so that $(B_x, \phi) \in \widetilde{C}_\sharp$ for some ϕ). Then $\underline{R^{B_x}_x}_{l(x)} = \mathbf{R}$, hence $\epsilon^x_{l(x)}(B_x,\phi) = \pm 1$. We can normalize $\phi : B_x^\sharp \to B_x$ uniquely so that $\epsilon^x_{l(x)}(B_x,\phi) = 1$. We shall denote this normalized ϕ by ϕ_x.

Due to 2.3, we can apply [L1, §11.1.8] to $C, \sharp$; we see that

(a) $\mathcal{K}_\sharp(C)$ is a free abelian group with basis $\{[B_x,\phi_x]; x \in \mathbf{I}\}$.

3.3 It will be convenient to introduce a square root of u. Let $\mathcal{A}' = \mathbf{Z}[v, v^{-1}]$ where v is an indeterminate. We view $\mathcal{A} = \mathbf{Z}[u, u^{-1}]$ as a subring of $\mathcal{A}'$ by setting $u = v^2$. Note that $\mathcal{K}_\sharp(\widetilde{C})$ can be viewed as an $\mathcal{A}'$-module with $v^n[M,\phi] = [M[-n],\phi]$ for $(M,\phi) \in \widetilde{C}_\sharp$, $n \in \mathbf{Z}$. We show:

(a) *The map* $q : \mathcal{A}' \otimes \mathcal{K}_\sharp(C) \to \mathcal{K}_\sharp(\widetilde{C})$, $v^n \otimes [M,\phi] \mapsto [M[-n],\phi]$ *is an isomorphism of* $\mathcal{A}'$*-modules.*

The map q is clearly well-defined. To prove that it is surjective we shall use the functors $M \mapsto \tau_{\le i} M$ from $\widetilde{C}$ to $\widetilde{C}$ (resp. $M \mapsto \mathcal{H}^i M$ from $\widetilde{C}$ to C) defined in [EW, §6.2]. (Here $i \in \mathbf{Z}$.) These define in an obvious way functors $\widetilde{C}_\sharp \to \widetilde{C}_\sharp$ (resp. $\widetilde{C}_\sharp \to C_\sharp$) denoted again by $\tau_{\le i}$ (resp. $\mathcal{H}^i$). Let $(M, \phi) \in \widetilde{C}_\phi$. From the definition we have an exact sequence in $\widetilde{C}$ (with morphisms in $\widetilde{C}_\sharp$)

$$0 \to \tau_{\le i-1} M \xrightarrow{e} \tau_{\le i} M \xrightarrow{e'} \mathcal{H}^i M[-i] \to 0$$

which is split, but the splitting is not necessarily given by morphisms in $\widetilde{C}_\sharp$. Thus there exist morphisms

$$\tau_{\le i-1} M \xleftarrow{f} \tau_{\le i} M \xleftarrow{f'} \mathcal{H}^i M[-i]$$

in $\widetilde{C}$ such that $e'f' = 1$, $fe = 1$, $f'e' + ef = 1$. Now $f^!, f'^!$ are defined as in 1.1 and, since $e^! = e, e'^! = e'$ (notation of 1.1), we have $e'f'^! = 1$, $f^!e = 1$, $f'^!e' + ef^! = 1$; hence setting $\widetilde{f} = (f + f^!)/2$, $\widetilde{f}' = (f' + f'^!)/2$, we have $e'\widetilde{f}' = 1$, $\widetilde{f}e = 1$, $\widetilde{f}'e' + e\widetilde{f} = 1$ and $\widetilde{f}^! = \widetilde{f}$, $\widetilde{f}'^! = \widetilde{f}'$. Thus we obtain a new splitting of the exact sequence above which is given by morphisms in $\widetilde{C}_\sharp$. It follows that

$$(\tau_{\le i} M, \phi) \cong (\tau_{\le i-1} M, \phi) \oplus (\mathcal{H}^i M[-i], \phi)$$

in $\widetilde{C}_\sharp$ (the maps ϕ are induced by $M^\sharp \to M$). Hence $[\tau_{\le i} M, \phi] = [\tau_{\le i-1} M, \phi] + [\mathcal{H}^i M[-i], \phi]$ in $\mathcal{K}_\sharp(\widetilde{C})$. Since $[M,\phi] = [\tau_{\le i} M, \phi]$ for $i \gg 0$ and $0 = [\tau_{\le i} M, \phi]$ for $-i \gg 0$ we deduce that $[M,\phi] = \sum_i [\mathcal{H}^i M[-i], \phi]$. This proves the surjectivity of q.

We define $\mathcal{K}(\widetilde{C}_\sharp) \to \mathcal{A}' \otimes \mathcal{K}(C_\sharp)$ by $[M,\phi] \mapsto \sum_{n \in \mathbf{Z}} v^{-n} [\mathcal{H}^n M, \phi_n]$ where ϕ_n is induced by ϕ. This clearly induces a homomorphism $q' : \mathcal{K}_\sharp(\widetilde{C}) \to \mathcal{A}' \otimes \mathcal{K}_\sharp(C)$ which satisfies $q'q = 1$. It follows that q is injective, completing the proof of (a).

□

3.4 Using 3.2(a) and 3.3(a), we see that

(a) $\mathcal{K}_\sharp(\widetilde{C})$ *is a free* $\mathcal{A}'$*-module with basis* $\{[B_x, \phi_x]; x \in \mathbf{I}\}$,

(notation of 3.2).

3.5 Let $\mathcal{M}$ be the free $\mathcal{A}'$-module with basis $\{a_x; x \in \mathbf{I}\}$. For any $(M, \phi) \in \widetilde{C}_\sharp$ and any $y \in \mathbf{I}$ we set

$$\epsilon^y(M, \phi) = \sum_{i \in \mathbf{Z}} \epsilon_i^y(M, \phi) v^{-i} \in \mathcal{A}'.$$

The homomorphism $\mathcal{K}(\widetilde{C}_\sharp) \to \mathcal{M}$,

$$[M, \phi] \mapsto \sum_{y \in \mathbf{I}} \epsilon^y(M, \phi) a_y$$

clearly factors through an $\mathcal{A}'$-module homomorphism

(a) $$\chi' : \mathcal{K}_\sharp(\widetilde{C}) \to \mathcal{M}.$$

We show that

(b) χ' *is an isomorphism.*

For $x \in \mathbf{I}$ let $\widetilde{A}_x = \chi'([B_x, \phi_x])$. We can write $\widetilde{A}_x = \sum_{y \in \mathbf{I}} f_{y,x} a_y$ where $f_{y,x} \in \mathcal{A}'$ are zero for all but finitely many y. In view of 3.4(a), to prove (b) it is enough to show:

(c) *Let* $y \in \mathbf{I}$. *If* $y \not\leq x$, *then* $f_{y,x} = 0$. *If* $y \leq x$, *then* $f_{y,x} = v^{-l(x)} \widetilde{P}_{y,x}(u)$ *where* $\widetilde{P}_{y,x} = 1$ *if* $y = x$ *and* $\widetilde{P}_{y,x}$ *is a polynomial with integer coefficients of degree* $\leq (l(x) - l(y) - 1)/2$ *if* $y < x$.

Assume that $f_{y,x} \neq 0$. Then for some i we have $\epsilon_i^y(B_x, \phi_x) \neq 0$, hence $\underline{R_y^{B_x}} \neq 0$. Using 2.4(b), (c) we deduce that the coefficient of T_y in c_x is nonzero; thus we have $y \leq x$, as required. Next we assume that $y \leq x$. We have $v^{l(x)} f_{y,x} = \sum_i \epsilon_i^y(B_x, \phi_x) v^{-i+l(x)}$, hence it is enough to show that

$\epsilon_i^y(B_x, \phi_x) \neq 0$ implies $-i + l(x) \in 2\mathbf{N}$ and $-i + l(x) \leq l(x) - l(y)$ with strict inequality unless $x = y$.

Now $\epsilon_i^y(B_x, \phi_x) \neq 0$ implies $\underline{R_y^{B_x}}_i \neq 0$. Hence it is enough to show that

$\underline{R_y^{B_x}}_i \neq 0$ implies $-i + l(x) \in 2\mathbf{N}$ and $-i + l(x) \leq l(x) - l(y)$ with strict inequality unless $x = y$.

By 2.4(a), (b), (c) we have

$$\sum_{j \in \mathbf{Z}} \dim \underline{R_y^{B_x}}_j u^{-j+l(x)} = P_{y,x}(u^2)$$

and the desired result follows from the properties of $P_{y,x}$ (see 2.4(a)). This proves (c) hence also (b). □

Next we note that for $y \in \mathbf{I}$, $y \leq x$ and $\delta \in \{1, -1\}$ the following holds:

(d) $$(P_{y,x}(u) + \delta \widetilde{P}_{y,x}(u))/2 \in \mathbf{N}[u].$$

We have

$$P_{y,x}(u) + \delta \widetilde{P}_{y,x}(u) = \sum_{j \in \mathbf{Z}} \dim \underline{R_y^{B_x}}_j v^{-j+l(x)} + \delta \sum_{j \in \mathbf{Z}} \epsilon_j^y(B_x, \phi_x) v^{-j+l(x)};$$

hence it is enough to show that

$$\dim \underline{R_y^{B_x}}_j + \delta \epsilon_j^y(B_x, \phi_x) \in 2\mathbf{N}.$$

This follows from the fact that for an involutive automorphism τ of a real vector space V, we have $\dim(V) + \delta \mathrm{tr}(\tau, V) \in 2\mathbf{N}$.

3.6 For any $M \in \widetilde{C}$ we define a functor $F_M : \widetilde{C}_\sharp \to \widetilde{C}_\sharp$ by

$$(M', \phi) \mapsto (MM'M^\sharp, \phi')$$

where $\phi' : (MM'M^\sharp)^\sharp = MM'^\sharp M^\sharp \to MM'M^\sharp$ is given by

$$m_1 \otimes m' \otimes m_2 \mapsto m_2 \otimes \phi(m') \otimes m_1.$$

Note that F_M induces an $\mathcal{A}'$-linear map $\mathcal{K}(\widetilde{C}_\sharp) \to \mathcal{K}(\widetilde{C}_\sharp)$ which clearly maps $\mathcal{K}^0(\widetilde{C}_\sharp)$ into itself, hence it induces an $\mathcal{A}'$-linear map $\overline{F}_M : \mathcal{K}_\sharp(\widetilde{C}) \to \mathcal{K}_\sharp(\widetilde{C})$. If $M_1, M_2 \in \widetilde{C}$, we have $F_{M_1 M_2} = F_{M_1} F_{M_2}$, hence $\overline{F}_{M_1 M_2} = \overline{F}_{M_1} \overline{F}_{M_2}$; moreover for any $(M, \phi) \in \widetilde{C}_\sharp$ we have

$$\begin{aligned} F_{M_1 \oplus M_2}(M, \phi) &= ((M_1 \oplus M_2) M (M_1^\sharp \oplus M_2^\sharp), \phi') \\ &= F_{M_1}(M, \phi) \oplus F_{M_2}(M, \phi) \oplus (\widetilde{M}, \widetilde{\phi}) \end{aligned}$$

(for a suitable ϕ') where $\widetilde{M} = M_2 M M_1^\sharp \oplus M_1 M M_2^\sharp$ and $\widetilde{\phi} : \widetilde{M}^\sharp \to \widetilde{M}$ are such that $(\widetilde{M}, \widetilde{\phi})$ is a traceless object of $\widetilde{C}_\sharp$. It follows that $\overline{F}_{M_1 \oplus M_2} = \overline{F}_{M_1} + \overline{F}_{M_2}$. We see that $[M] \mapsto \overline{F}_M$ makes $\mathcal{K}_\sharp(\widetilde{C})$ into a (left) $\mathcal{K}(\widetilde{C})$-module. From the definitions, for any $M \in \widetilde{C}$, $(M', \phi) \in \widetilde{C}_\sharp$, $n \in \mathbf{Z}$, we have $F_{M[n]}(M', \phi) = F_M(M'[2n], \phi)$. Hence for $h \in \mathcal{K}(\widetilde{C})$, $h' \in \mathcal{K}_\sharp(\widetilde{C})$, $n \in \mathbf{Z}$, we have $(u^n h)h' = v^{2n}(hh') = u^n(hh')$. Via the isomorphism $\chi : \mathcal{K}(\widetilde{C}) \xrightarrow{\sim} \mathbf{H}$ in 2.4(b) and the isomorphism $\chi' : \mathcal{K}_\sharp(\widetilde{C}) \xrightarrow{\sim} \mathcal{M}$ in 3.5(a), (b), $\mathcal{M}$ becomes a (left) $\mathbf{H}$-module (with $u \in \mathbf{H}$ acting on $\mathcal{M}$ as multiplication by $u = v^2$).

4 Some exact sequences

4.1 In this section we fix $s \in S$ and we write α instead of α_s so that $\alpha^* = \alpha_{s^*}$. Let $R^{s^*,>0} = R^{s^*} \cap R^{>0}$. Let $\underline{\underline{R}} = R/R^{s^*,>0}R$, a $\mathbf{Z}$-graded $\mathbf{R}$-algebra which can be naturally identified with $\mathbf{R}[\alpha^*]/(\alpha^{*2})$ (it is zero except in degree 0 and 2). Let $\underline{\underline{\mathcal{R}}}$ be the category whose objects are $\mathbf{Z}$-graded right $\underline{\underline{R}}$-modules. For any $M' \in \mathcal{R}$ we write $\underline{\underline{M'}} = M'/M'R^{s^*,>0} = M' \otimes_{R^{s^*}} \mathbf{R}$, where $\mathbf{R} = R^{s^*}/R^{s^*,>0}$ is viewed as a R^{s^*}-algebra in the obvious way. Note that $\underline{\underline{M'}}$ is naturally an object of $\underline{\underline{\mathcal{R}}}$.

4.2 For any $M \in \mathcal{R}$ we write $R.M$ (resp. $M.R$) instead of $R \otimes_{R^s} M \in \mathcal{R}$ (resp. $M \otimes_{R^{s^*}} R \in \mathcal{R}$); for $r \in R, m \in M$ we write

$$r.m \text{ (resp. } m.r) \text{ instead of } r \otimes m \in R.M \text{ (resp. } m \otimes r \in M.R).$$

Note that any element of $R.M$ (resp. $M.R$) can be written uniquely in the form $\sum_{i\in\{0,1\}} \alpha^i.m_i$ (resp. $\sum_{i\in\{0,1\}} m_i\alpha^{*i}$) where $m_i \in M$.

For $M, N \in \mathcal{R}$ let ${}'\mathrm{hom}(M,N)$ (resp. $\mathrm{hom}'(M,N)$) be the set of maps $M \to N$ which are homomorphisms of (R^s, R)-bimodules (resp. (R, R^{s^*})-bimodules) and are compatible with the $\mathbf{Z}$-gradings; let

$$\begin{aligned} {}'\mathrm{hom}^\bullet(M,N) &= \oplus_{i\in\mathbf{Z}}{}'\mathrm{hom}(M,N[i]),\\ \mathrm{hom}'^\bullet(M,N) &= \oplus_{i\in\mathbf{Z}}\mathrm{hom}'(M,N[i]). \end{aligned}$$

The statements (i)–(ii) below are easily verified.

(i) *There is a unique group isomorphism*

$${}'\mathrm{hom}^\bullet(M,N) \xrightarrow{\sim} N^{R.M} \quad (\text{resp.}\ \mathrm{hom}'^\bullet(M,N) \xrightarrow{\sim} N^{M.R}),$$

$f \mapsto F$, such that for $m \in M$ we have

$$F(1.m) = f(m), \quad F(\alpha.m) = \alpha f(m),$$

(resp. $F(m.1) = f(m)$, $F(m.\alpha^*) = f(m)\alpha^*$); this is in fact an isomorphism in $\mathcal{R}$, provided that ${}'\mathrm{hom}^\bullet(M,N)$ (resp. $\mathrm{hom}'^\bullet(M,N)$) is viewed as an object of $\mathcal{R}$ with $(rf)(m) = r(f(m))$, $(fr)(m) = (f(m))r$ for $m \in M, r \in R$ and $f \in {}'\mathrm{hom}^\bullet(M,N)$ (resp. $f \in \mathrm{hom}'^\bullet(M,N)$).

(ii) *The map*

$$f \mapsto G, \quad G(m) = \alpha.f(m) + 1.f(\alpha m)$$

(resp. $G(m) = f(m).\alpha^* + f(m\alpha^*).1$) is an isomorphism

$${}''\mathrm{hom}^\bullet(M,N[-2]) \xrightarrow{\sim} (R.N)^M \ (\text{resp. } \mathrm{hom}''^\bullet(M,N[-2]) \to (N.R)^M$$

in $\mathcal{R}$, provided that ${}'\mathrm{hom}^\bullet(M,N)$ (resp. $\mathrm{hom}'^\bullet(M,N)$) is viewed as an object of $\mathcal{R}$ with $(rf)(m) = f(rm)$, $(fr)(m) = f(mr)$ for $m \in M, r \in R$ and $f \in {}'\mathrm{hom}^\bullet(M,N)$ (resp. $f \in \mathrm{hom}'^\bullet(M,N)$).

Combining (i), (ii) we see that

(iii) *The map* $F \mapsto G$,

$$G(m) = \alpha.F(1.m) + 1.F(1.\alpha m)$$

(resp. $G(m) = F(m.1).\alpha^* + F(m\alpha^*.1).1$*) is an isomorphism*

$$(N[-2])^{R.M} \xrightarrow{\sim} (R.N)^M \text{ (resp. } (N[-2])^{M.R} \xrightarrow{\sim} (N.R)^M)$$

of (R^s, R)*-bimodules (resp. of* (R, R^{s^*})*-bimodules).*

(We use that the two (R, R)-bimodule structures on ${}'\hom^\bullet(M, N)$ described in 4.2(i), (ii) restrict to the same (R^s, R)-bimodule structure and that the two (R, R)-bimodule structures on $\hom'^\bullet(M, N)$ described in 4.2(i), (ii) restrict to the same (R, R^{s^*})-bimodule structure.)

4.3 For any $M' \in \mathcal{R}$ we write

$$R.M'.R \text{ instead of } R \otimes_{R^s} M' \otimes_{R^{s^*}} R \in \mathcal{R};$$

for r, r' in R and $m' \in M'$ we write $r.m'.r'$ instead of $r \otimes m' \otimes r' \in R.M'.R$. Note that any element $\xi \in R.M'.R$ can be written uniquely in the form

$$\sum_{i,j \in \{0,1\}} \alpha^i.\xi_{ij}.a^{*j},$$

where $\xi_{ij} \in M'$.

For $M, N \in \mathcal{R}$ let $\hom(M, N)$ be the set of maps $M \to N$ which are homomorphisms of (R^s, R^{s^*})-bimodules and which are compatible with the $\mathbf{Z}$-gradings; let $\hom^\bullet(M, N) = \oplus_{i \in \mathbf{Z}} \hom(M, N[i])$.

We view $\hom^\bullet(M, N)$ as an object of $\mathcal{R}$ in two ways: for $f \in \hom^\bullet(M, N), r \in R, m \in M$ we set either

(a) $$(rf)(m) = r(f(m)), \quad (fr)(m) = (f(m))r;$$

(b) $$\text{or } (rf)(m) = f(rm), \quad (fr)(m) = f(mr).$$

The statements (i), (ii) below are easily verified.

(i) *There is a unique group isomorphism* $\hom^\bullet(M, N) \xrightarrow{\sim} N^{R.M.R}$ *in* $\mathcal{R}$, $f \mapsto F$ *such that for any* $m \in M$, *we have* $F(1.m.1) = f(m)$, $F(\alpha.m.1) = \alpha f(m)$, $F(1.m.\alpha^*) = f(m)\alpha^*$, $F(\alpha.m.\alpha^*) = \alpha f(m)\alpha^*$; *this is in fact an isomorphism in* $\mathcal{R}$ *provided that* $\hom^\bullet(M, N)$ *is viewed as an object of* $\mathcal{R}$ *as in (a).*

(ii) *The map* $f \mapsto G$,

$$G(m) = 1.f(\alpha m\alpha^*).1 + \alpha.f(m\alpha^*).1 + 1.f(\alpha m).\alpha^* + \alpha.f(m).\alpha^*$$

is an isomorphism $\hom^\bullet(M, N[-4]) \xrightarrow{\sim} (R.N.R)^M$ *in* $\mathcal{R}$ *provided that* $\hom^\bullet(M, N)$ *is viewed as an object of* $\mathcal{R}$ *as in (b).*

Combining (i), (ii) we see that

(iii) *The map* $F \mapsto G$,

$$G(m) = \alpha.F(1.m.1).\alpha^* + \alpha.F(1.m\alpha^*.1).1 + 1.F(1.\alpha m.1).\alpha^* + 1.F(1.\alpha m\alpha^*.1).1$$

is an isomorphism $(N[-4])^{R.M.R} \xrightarrow{\sim} (R.N.R)^M$ *of* (R^s, R^{s^*})*-bimodules.*

(We use that the two (R, R)-bimodule structures on $\hom^\bullet(M, N)$ described in (a), (b) restrict to the same (R^s, R^{s^*})-bimodule structure.)

4.4 Let $M \in \widetilde{C}$ and let $\omega \in W$. We define an exact sequence

(a) $$0 \to R^M_\omega \xrightarrow{c} R^{R.M}_\omega \xrightarrow{d} R^M_{s\omega}[2]$$

as follows. We identify $R^{R.M}_\omega = {}'\hom^\bullet(M, R_\omega)$ as objects of $\mathcal{R}$ as in 4.2(i); then c is the obvious inclusion $R^M_\omega \subset {}'\hom^\bullet(M, R_\omega)$ and $d : {}'\hom^\bullet(M, R_\omega) \to R^M_{s\omega}[2]$ is given by $f \mapsto f'$, where $f'(m) = {}^s(f(\alpha m) - \alpha f(m))$. (The kernel of d is clearly R^M_ω.) Now (a) induces sequences

(b) $$0 \to \underline{R^M_\omega} \to \underline{R^{R.M}_\omega} \to \underline{R^M_{s\omega}[2]} \to 0,$$

(c) $$0 \to \underline{\underline{R^M_\omega}} \to \underline{\underline{R^{R.M}_\omega}} \to \underline{\underline{R^M_{s\omega}[2]}} \to 0.$$

We state the following result.

(d) *If* $l(\omega) < l(s\omega)$*, then the sequences (b), (c) are exact.*

For (b) this is implicit in the proof in [S, Proposition 5.7, Corollary 5.16] of the fact that, under the assumption of (d) the alternating sum of dimensions of the terms of (b) is zero (in each degree). The statement for (c) can be reduced to that for (b) as follows. The $\underline{\underline{R}}$-modules in (c) are free of finite rank (we use 2.1(a)) and the kernel and cokernel of right multiplication by α^* in these $\underline{\underline{R}}$-modules form sequences which can be identified with the sequence (b) which are already known to be exact; it follows that the sequence (c) is exact. □

Next we define an exact sequence

(e) $$0 \to R^M_\omega \xrightarrow{c'} R^{M.R}_\omega \xrightarrow{d'} R^M_{\omega s^*}[2]$$

as follows. We identify $R^{M.R}_\omega = \hom'^\bullet(M, R_\omega)$ as objects of $\mathcal{R}$ as in Property 4.2(i); then c' is the obvious inclusion $R^M_\omega \subset \hom'^\bullet(M, R_\omega)$ and $d' : \hom'^\bullet(M, R_\omega) \to R^M_{\omega s^*}[2]$ is given by $f \mapsto f'$, where $f'(m) = f(m\alpha^*) - f(m)\alpha^*$ (the product $f(m)\alpha^*$ is computed in the right R-module structure of R_ω). (The kernel of d' is clearly R^M_ω.) Now (e) induces sequences

(f) $$0 \to \underline{R^M_\omega} \to \underline{R^{M.R}_\omega} \to \underline{R^M_{\omega s^*}[2]} \to 0,$$

(g) $$0 \to \underline{\underline{R^M_\omega}} \to \underline{\underline{R^{M.R}_\omega}} \to \underline{\underline{R^M_{\omega s^*}}}[2] \to 0.$$

We now state the following result.

(h) *If $l(\omega) < l(\omega s^*)$, then the sequences (f), (g) are exact.*

For any $x \in W$ we have an isomorphism

(i) $$R^M_x \xrightarrow{\sim} R^{M^\sharp}_{x^{-1}}$$

as $\mathbf{R}$-vector spaces (not in $\mathcal{R}$) given by $f \mapsto \widetilde{f}$ where $(\widetilde{f})(m) = {}^{x^{-1}}(f(m))$ for any $m \in M$ (we identify $M, M^\sharp$ as sets). It carries $R^M_x R^{>0}$ onto $R^{M^\sharp}_{x^{-1}} R^{>0}$, hence it induces an isomorphism $\underline{R^M_x} \xrightarrow{\sim} \underline{R^{M^\sharp}_{x^{-1}}}$ of graded $\mathbf{R}$-vector spaces. Applying an isomorphism like (i) to each term of the sequence (f) we get a sequence

$$0 \to \underline{R^{M^\sharp}_{\omega^{-1}}} \to \underline{R^{R\otimes_{R_{s^*}} M^\sharp}_{\omega^{-1}}} \to \underline{R^{M^\sharp}_{s^*\omega^{-1}}[2]} \to 0;$$

(we use that $(M.R)^\sharp = R\otimes_{R_{s^*}} M^\sharp$). This sequence is a special case of the sequence (b) (with M, ω, s replaced by $M^\sharp, \omega^{-1}, s^*$); hence, by (d), it is exact (we use that $l(\omega) < l(s^*\omega^{-1})$). It follows that the sequence (f) is exact. From this we deduce the exactness of (g) in the same way as we deduced the exactness of (c) from that of (b). □

4.5 Let $w \in W$. We set
$$N = R_w.$$

For $r \in R$ and $b \in N$, we write $b \circ r$ for the element of N given by the right R-module structure on N. We define some subsets of $R.N.R$ as follows:

$$\begin{aligned}
Y &= \{1.\alpha b.1 + \alpha.b.1 + 1.\alpha b'.\alpha^* + \alpha.b'.\alpha^*; b, b' \in N\},\\
Y' &= \{1.b' \circ \alpha^*.1 + \alpha.b \circ \alpha^*.1 + 1.b'.\alpha^* + \alpha.b.\alpha^*; b, b' \in N\},\\
V &= \{1.\alpha b \circ \alpha^*.1 + \alpha.b \circ \alpha^*.1 + 1.\alpha b.\alpha^* + \alpha.b.\alpha^*; b \in N\} = Y \cap Y',\\
Z &= \{1.(\alpha b + b' \circ \alpha^* - \alpha b'' \circ \alpha^*).1 + \alpha.b.1 + 1.b'.\alpha^* + \alpha.b''.\alpha^*; b, b', b'' \in N\}\\
&= Y + Y'.
\end{aligned}$$

It is easy to verify that Y, Y' are subobjects of $R.N.R$ in $\mathcal{R}$. Hence V, Z are subobjects of $R.N.R$ in $\mathcal{R}$.

By a straightforward computation we see that (a)–(d) below hold:

(a) *the map* $\tau_1 : V \to N[-4]$, $1.\alpha b \circ \alpha^*.1 + \alpha.b \circ \alpha^*.1 + 1.\alpha b.\alpha^* + \alpha.b.\alpha^* \mapsto b$ *is an isomorphism in* $\mathcal{R}$;

(b) *the map* $Y \to R_{ws^*}[-2]$, $1.\alpha b.1 + \alpha.b.1 + 1.\alpha b'.\alpha^* + \alpha.b'.\alpha^* \mapsto b - b' \circ \alpha^*$, *induces an isomorphism* $\tau_2 : Y/V \xrightarrow{\sim} R_{ws^*}[-2]$ *in* $\mathcal{R}$;

(c) *the map* $Y' \to R_{sw}[-2]$, $1.b' \circ \alpha^*.1 + \alpha.b \circ \alpha^*.1 + 1.b'.\alpha^* + \alpha.b.\alpha^* \mapsto {}^s(b' - \alpha b)$, *induces an isomorphism* $\tau_3 : Y/V' \xrightarrow{\sim} R_{sw}[-2]$ *in* $\mathcal{R}$;

(d) *the map* $R.N.R \to R_{sws^*}$,

$$1.b_0.1 + \alpha.b_1.1 + 1.b_2.\alpha^* + \alpha.b_3.\alpha^* \mapsto {}^s(-b_0 + \alpha b_1 + b_2 \circ \alpha^* - \alpha b_3 \circ \alpha^*),$$

induces an isomorphism $\tau_4 : R.N.R/Z \xrightarrow{\sim} R_{sws^*}$.

The proof is a straightforward computation.

In the remainder of this section we fix $M \in \widetilde{C}$.

Lemma 4.6. *Assume that* $l(w) < l(ws^*)$. *The obvious sequence*

(a)
$$0 \to \underline{\underline{V^M}} \to \underline{\underline{Y^M}} \to \underline{\underline{(Y/V)^M}} \to 0$$

is exact.

We can identify $N.R[-2] = Y$ (as objects of $\mathcal{R}$) by $b.r \mapsto \alpha.b.r + 1.\alpha b.r$ (for $b \in N, r \in R$). We can identify $N[-4] = V$ via τ_1 in 4.5(a) and $R_{ws^*}[-2] = Y/V$ via τ_2 in 4.5(b) Then (a) becomes a sequence

$$0 \to \underline{\underline{(N[-4])^M}} \to \underline{\underline{(N.R[-2])^M}} \to \underline{\underline{(R_{ws^*}[-2])^M}} \to 0.$$

By 4.2(iii) we can identify $\underline{\underline{(N.R[-2])^M}} = \underline{\underline{(N[-4])^{M.R}}}$ (as **R**-vector spaces). The previous sequence becomes a sequence

$$0 \to \underline{\underline{N^{M[4]}}} \to \underline{\underline{N^{M[4].R}}} \to \underline{\underline{(R_{ws^*})^{M[4]}[2]}} \to 0.$$

This is of the type appearing in 4.4(c) with M replaced by $M[4]$ hence is exact by 4.4(d). □

Lemma 4.7. *Assume that* $l(sw) < l(sws^*)$. *The obvious sequence*

(a)
$$0 \to \underline{\underline{(Z/Y)^M}} \to \underline{\underline{(R.N.R/Y)^M}} \to \underline{\underline{(R.N.R/Z)^M}} \to 0$$

is exact.

Consider the exact sequence $0 \to N[-2] \xrightarrow{c} R.N \xrightarrow{c'} R_{sw} \to 0$ in which c is $b \mapsto \alpha.b + \alpha b.1$ and c' maps $r'.b$ to $r'^s b$ (where $r' \in R, b \in N$). Applying $\otimes_{R^{s^*}} R$ we obtain an exact sequence $0 \to N.R[-2] \to R.N.R \to R_{sw}.R \to 0$. Here we identify $N.R[-2] = Y$ as in the proof of Lemma 4.6 and we obtain an exact sequence $0 \to Y \to R.N.R \to R_{sw}.R \to 0$ in $\mathcal{R}$. Hence we obtain an identification $R.N.R/Y = R_{sw}.R$ under which $r'.b.r \in R.N.R/Y$ corresponds to

$r'^s b.r \in R_{sw}.R$. We identify $Z/Y = (Y + Y')/Y = Y'/V = R_{sw}[-2]$ via the isomorphism τ_3 in 4.5(c) and $R.N.R/Z = R_{sws^*}$ via the isomorphism τ_4 in 4.5(d). Then (a) becomes

$$0 \to \underline{\underline{(R_{sw}[-2])^M}} \to \underline{\underline{(R_{sw}.R)^M}} \to \underline{\underline{R^M_{sws^*}}} \to 0.$$

By 4.2(iii) we can identify $(R_{sw}.R)^M = (R_{sw}[-2])^{M.R}$. The previous sequence becomes

$$0 \to \underline{\underline{(R_{sw}[-2])^M}} \to \underline{\underline{(R_{sw}[-2])^{M.R}}} \to \underline{\underline{R^M_{sws^*}}} \to 0.$$

This sequence is (up to shift) of the type appearing in 4.4(g) (with ω replaced by sw) hence is exact by 4.4(h). □

Lemma 4.8. *Assume that $l(w) < l(sw)$. The obvious sequence*

(a) $$0 \to \underline{\underline{Y^M}} \to \underline{\underline{(R.N.R)^M}} \to \underline{\underline{(R.N.R/Y)^M}} \to 0$$

is exact.

We identify $Y = N.R[-2]$ as in the proof of Lemma 4.6 and $R.N.R/Y = R_{sw}.R$ as in the proof of Lemma 4.7. Then (a) becomes the sequence

$$0 \to \underline{\underline{(N.R[-2])^M}} \to \underline{\underline{(R.N.R)^M}} \to \underline{\underline{(R_{sw}.R)^M}} \to 0.$$

By 4.2(iii) and 4.3(iii) we can identify

$$(N.R[-2])^M = (N[-4])^{M.R}, \quad (R.N.R)^M = (N[-4])^{R.M.R},$$

$$(R_{sw}.R)^M = (R_{sw}[-2])^{M.R}$$

and the previous sequence becomes

$$0 \to \underline{\underline{(N[-4])^{M.R}}} \to \underline{\underline{(N[-4])^{R.M.R}}} \to \underline{\underline{(R_{sw}[-2])^{M.R}}} \to 0.$$

This sequence is of the type appearing in 4.4(c) with M replaced by $M.R[4]$, hence is exact by 4.4(d). □

4.9 With M, N as in 4.5, we set $P = \hom^\bullet(M, N)$ regarded as an object of $\mathcal{R}$ as in 4.3(a). We define subsets $\mathcal{V}, \mathcal{Y}, \mathcal{Y}', \mathcal{Z}$ of P as follows:

$$\mathcal{V} = \{f \in P; f(\alpha m) = \alpha f(m), f(m\alpha^*) = f(m) \circ \alpha^* \text{ for all } m \in M\};$$
$$\mathcal{Y} = \{f \in P; f(\alpha m) = \alpha f(m) \text{ for all } m \in M\};$$
$$\mathcal{Y}' = \{f \in P; f(m\alpha^*) = f(m) \circ \alpha^* \text{ for all } m \in M\};$$
$$\mathcal{Z} = \{f \in P; f(\alpha m\alpha^*) - \alpha f(m\alpha^*) - f(\alpha m) \circ \alpha^* + \alpha f(m) \circ \alpha^* = 0 \text{ for all } m \in M\}.$$

Note that $\mathcal{V}, \mathcal{Y}, \mathcal{Y}', \mathcal{Z}$ are subobjects of P in $\mathcal{R}$. Under the bijection

$$P \leftrightarrow (R.N.R)^M[4]$$

in 4.3(ii), $\mathcal{V}, \mathcal{Y}, \mathcal{Y}', \mathcal{Z}$ correspond respectively to the subsets V^M, Y^M, Y'^M, Z^M of $(R.N.R)^M$. Thus we have natural bijections $\mathcal{V} \leftrightarrow V^M$, $\mathcal{Y} \leftrightarrow Y^M$, $\mathcal{Y}' \leftrightarrow Y'^M$, $\mathcal{Z} \leftrightarrow Z^M$ as (R^s, R^{s^*})-bimodules. From the definitions it is clear that

(a) $$\mathcal{V} = N^M$$

as objects of $\mathcal{R}$. Since $P \cong N^{R.M.R}$ as objects of $\mathcal{R}$, we see from 2.1(a) that P is a finitely generated right R-module. Since R is a Noetherian ring, it follows that $\mathcal{V}, \mathcal{Y}, \mathcal{Y}', \mathcal{Z}$ (which are subobjects of P) are also finitely generated right R-modules.

Lemma 4.10. *Assume that $l(w) < l(ws^*)$. The map (in $\mathcal{R}$)*

(a) $$\mathcal{Y} \to (R_{ws^*}[2])^M, \quad f \mapsto f', \text{ where } f'(m) = f(m\alpha^*) - f(m) \circ \alpha^*,$$

induces an isomorphism $\underline{\mathcal{Y}/\mathcal{V}} \xrightarrow{\sim} \underline{(R_{ws^*}[2])^M}$ and an isomorphism $(\mathcal{Y}/\mathcal{V})^\wedge \xrightarrow{\sim} ((R_{ws^*}[2])^M)^\wedge$ (notation of 2.1).

The map (a) is clearly a well-defined morphism in $\mathcal{R}$ and its kernel is clearly equal to $\mathcal{V}$. Thus we have an exact sequence $0 \to \mathcal{V} \to \mathcal{Y} \to (R_{ws^*}[2])^M$ (in $\mathcal{R}$). Using 4.9(a) and the identification $\mathcal{Y} = N^{M.R}$ (see 4.2(i)) this exact sequence becomes an exact sequence $0 \to N^M \to N^{M.R} \to (R_{ws^*}[2])^M$ (in $\mathcal{R}$) which induces the exact sequence $0 \to \underline{N^M} \to \underline{N^{M.R}} \to \underline{(R_{ws^*}[2])^M} \to 0$ (a special case of 4.4(c), (d)) that is an exact sequence $0 \to \underline{\mathcal{V}} \to \underline{\mathcal{Y}} \to \underline{(R_{ws^*}[2])^M} \to 0$. Applying $\otimes_R \mathbf{R}$ to the exact sequence $0 \to \mathcal{V} \to \mathcal{Y} \to \mathcal{Y}/\mathcal{V} \to 0$ we deduce an exact sequence $\underline{\mathcal{V}} \to \underline{\mathcal{Y}} \to \underline{\mathcal{Y}/\mathcal{V}} \to 0$. It follows that both $\underline{(R_{ws^*}[2])^M}$ and $\underline{\mathcal{Y}/\mathcal{V}}$ can be identified with the cokernel of the map $\underline{\mathcal{V}} \to \underline{\mathcal{Y}}$. Thus, $\underline{\mathcal{Y}/\mathcal{V}} \xrightarrow{\sim} \underline{(R_{ws^*}[2])^M}$. Now the injective homomorphism $\mathcal{Y}/\mathcal{V} \to (R_{ws^*}[2])^M$ induces an injective homomorphism $(\mathcal{Y}/\mathcal{V})^\wedge \to ((R_{ws^*}[2])^M)^\wedge$ which becomes surjective after applying $\otimes_{\widehat{R}} \mathbf{R}$; hence, by the Nakayama lemma, it is surjective before applying $\otimes_{\widehat{R}} \mathbf{R}$. $\square$

Lemma 4.11. *Assume that $l(w) < l(sw)$. The map (in $\mathcal{R}$)*

(a) $$\mathcal{Y}' \to (R_{sw}[2])^M, \quad f \mapsto f', \text{ where } f'(m) = {}^s(f(\alpha m) - \alpha f(m)),$$

induces an isomorphism $\underline{\mathcal{Y}'/\mathcal{V}} \xrightarrow{\sim} \underline{(R_{sw}[2])^M}$ and an isomorphism $(\mathcal{Y}'/\mathcal{V})^\wedge \xrightarrow{\sim} ((R_{sw}[2])^M)^\wedge$.

The proof is almost a repetition of that of Lemma 4.10. The map (a) is clearly a well-defined morphism in $\mathcal{R}$ and its kernel is clearly equal to $\mathcal{V}$. Thus we have an exact sequence $0 \to \mathcal{V} \to \mathcal{Y}' \to (R_{sw}[2])^M$ (in $\mathcal{R}$). Using 4.9(a) and the identification $\mathcal{Y}' = N^{R.M}$ (see 4.2(i)) this exact sequence becomes an exact sequence $0 \to N^M \to N^{R.M} \to (R_{sw}[2])^M$ (in $\mathcal{R}$) which induces the exact sequence $0 \to \underline{N^M} \to \underline{N^{R.M}} \to \underline{(R_{sw}[2])^M} \to 0$ (a special case of 4.4(b), (d)) that is an exact sequence $0 \to \underline{\mathcal{V}} \to \underline{\mathcal{Y}'} \to \underline{(R_{sw}[2])^M} \to 0$. Applying $\otimes_R \mathbf{R}$ to the exact sequence $0 \to \mathcal{V} \to \mathcal{Y}' \to \mathcal{Y}'/\mathcal{V} \to 0$, we deduce an exact sequence $\underline{\mathcal{V}} \to \underline{\mathcal{Y}'} \to \underline{\mathcal{Y}'/\mathcal{V}} \to 0$. It follows that both $\underline{(R_{sw}[2])^M}$ and $\underline{\mathcal{Y}'/\mathcal{V}}$ can be identified with the cokernel of the map $\underline{\mathcal{V}} \to \underline{\mathcal{Y}'}$. Thus, $\underline{\mathcal{Y}'/\mathcal{V}} \xrightarrow{\sim} \underline{(R_{sw}[2])^M}$. Now the injective homomorphism $\mathcal{Y}'/\mathcal{V} \to (R_{sw}[2])^M$ induces an injective homomorphism $(\mathcal{Y}'/\mathcal{V})^\wedge \to ((R_{sw}[2])^M)^\wedge$ which becomes surjective after applying $\otimes_{\widehat{R}} \mathbf{R}$; hence, by the Nakayama lemma, it is surjective before applying $\otimes_{\widehat{R}} \mathbf{R}$. □

Lemma 4.12. *Assume that $l(w) < l(sw)$. Let $P' = \hom'^\bullet(M, R_{sw})$; we view P' as an object of $\mathcal{R}$ as in 4.2(i). The map (in $\mathcal{R}$)*

(a) $$P \to P'[2], \quad f \mapsto f', \quad f'(m) = {}^s(f(\alpha m) - \alpha f(m)),$$

induces an isomorphism $\underline{P/\mathcal{Y}} \xrightarrow{\sim} \underline{P'[2]}$ (hence, using $P' = R_{sw}^{M.R}$, see 4.2(i)) an isomorphism $\underline{P/\mathcal{Y}} \xrightarrow{\sim} \underline{(R_{sw}[2])^{M.R}}$; it also induces an isomorphism $(P/\mathcal{Y})^\wedge \xrightarrow{\sim} ((R_{sw}[2])^{M.R})^\wedge$.

The map (a) is clearly a well-defined morphism in $\mathcal{R}$ and its kernel is clearly equal to $\mathcal{Y}$. Thus we have an exact sequence $0 \to \mathcal{Y} \to P \to P'[2]$ in $\mathcal{R}$. By 4.2(i) we can identify $\mathcal{Y} = N^{M.R}$ and our exact sequence becomes the exact sequence $0 \to N^{M.R} \to N^{R.M.R} \to (R_{sw}[2])^{M.R}$ in $\mathcal{R}$ which induces an exact sequence $0 \to \underline{N^{M.R}} \to \underline{N^{R.M.R}} \to \underline{(R_{sw}[2])^{M.R}} \to 0$ (a special case of 4.4(b), (d) with M replaced by $M.R$). Thus we have an exact sequence $0 \to \underline{\mathcal{Y}} \to \underline{P} \to \underline{P'[2]} \to 0$. Applying $\otimes_R \widehat{R}$ to the exact sequences

$$0 \to \mathcal{Y} \to P \to P/\mathcal{Y} \to 0, \qquad 0 \to \mathcal{Y} \to P \to P'[2],$$

we obtain exact sequences

$$0 \to \widehat{\mathcal{Y}} \to \widehat{P} \to \widehat{P/\mathcal{Y}} \to 0, \qquad 0 \to \widehat{\mathcal{Y}} \to \widehat{P} \to \widehat{P}'[2].$$

From the surjectivity of $\underline{P} \to \underline{P'[2]}$ and the Nakayama lemma it follows that $\widehat{P} \to \widehat{P}'[2]$ in the last exact sequence is surjective. Hence the obvious map $\widehat{P/\mathcal{Y}} \to \widehat{P}'[2]$ is an isomorphism (both sides can be identified with coker$(\widehat{\mathcal{Y}} \to \widehat{P})$). Applying $\otimes_{\widehat{R}} \mathbf{R}$ we deduce that the obvious map $\underline{P/\mathcal{Y}} \to \underline{P'[2]}$ is an isomorphism. Thus, $\underline{P/\mathcal{Y}} \xrightarrow{\sim} \underline{(R_{sw}[2])^{M.R}}$. Now the injective homomorphism $P/\mathcal{Y} \to (R_{sw}[2])^M$ induces an injective homomorphism $(P/\mathcal{Y})^\wedge \to ((R_{sw}[2])^M)^\wedge$ which becomes surjective after applying $\otimes_{\widehat{R}} \mathbf{R}$; hence, by the Nakayama lemma, it is surjective before applying $\otimes_{\widehat{R}} \mathbf{R}$. □

Lemma 4.13. *Assume that $l(w) < l(sw) < l(sws^*)$. Then the map (in $\mathcal{R}$) $P \to (R_{sws^*}[4])^M$,*

(a) $$f \mapsto f', \quad f'(m) = {}^s(f(\alpha m\alpha^*) - \alpha f(m\alpha^*) - f(\alpha m) \circ \alpha^* + \alpha f(m) \circ \alpha^*),$$

induces an isomorphism $\underline{P/\mathcal{Z}} \xrightarrow{\sim} \underline{(R_{sws^*}[4])^M}$ *and an isomorphism* $(P/\mathcal{Z})^\wedge \xrightarrow{\sim} ((R_{sws^*}[4])^M)^\wedge$.

The map (in $\mathcal{R}$*)* $\mathcal{Z} \to (R_{sw}[2])^M$,

$$\text{(b)} \qquad f \mapsto f', \quad f'(m) = {}^s(f(\alpha m) - \alpha f(m))$$

induces an isomorphism $\underline{\mathcal{Z}/\mathcal{Y}} \xrightarrow{\sim} \underline{(R_{sw}[2])^M}$ *and an isomorphism* $(\mathcal{Z}/\mathcal{Y})^\wedge \xrightarrow{\sim} ((R_{sw}[2])^M)^\wedge$.

The map (a) is clearly a well-defined morphism in $\mathcal{R}$ and its kernel is clearly equal to $\mathcal{Z}$. Thus we have an exact sequence $0 \to \mathcal{Z}/\mathcal{Y} \to P/\mathcal{Y} \to (R_{sws^*}[4])^M$. Applying $\otimes_R \widehat{R}$ gives again an exact sequence

$$\text{(c)} \qquad 0 \to \widehat{\mathcal{Z}/\mathcal{Y}} \to \widehat{P/\mathcal{Y}} \to ((R_{sws^*}[4])^M)^\wedge.$$

From 4.4(f), (h) we have an exact sequence

$$\text{(d)} \qquad 0 \to \underline{(R_{sw}[2])^M} \to \underline{(R_{sw}[2])^{M.R}} \to \underline{(R_{sws^*}[4])^M} \to 0.$$

Hence $\underline{(R_{sw}[2])^{M.R}} \to \underline{(R_{sws^*}[4])^M}$ is surjective, that is (using Lemma 4.12) $\underline{P/\mathcal{Y}} \to \underline{R^M_{sws^*}}$ is surjective. Using this and the Nakayama lemma, we see that $\widehat{P/\mathcal{Y}} \to (R^M_{sws^*})^\wedge$ is surjective. This is just the last map in (c); thus, (c) becomes an exact sequence

$$0 \to \widehat{\mathcal{Z}/\mathcal{Y}} \to \widehat{P/\mathcal{Y}} \to ((R_{sws^*}[4])^M)^\wedge \to 0.$$

This exact sequence of $\widehat{R}$-modules splits since, according to 2.1(a), the $\widehat{R}$-module $((R_{sws^*}[4])^M)^\wedge$ is free. Hence, applying $\otimes_{\widehat{R}}\mathbf{R}$ gives an exact sequence

$$\text{(e)} \qquad 0 \to \underline{\mathcal{Z}/\mathcal{Y}} \to \underline{P/\mathcal{Y}} \to \underline{(R_{sws^*}[4])^M} \to 0.$$

From the obvious exact sequence $0 \to \mathcal{Z}/\mathcal{Y} \to P/\mathcal{Y} \to P/\mathcal{Z} \to 0$ we deduce an exact sequence $\underline{\mathcal{Z}/\mathcal{Y}} \to \underline{P/\mathcal{Y}} \to \underline{P/\mathcal{Z}} \to 0$. Using this and (d), we see that both $\underline{P/\mathcal{Z}}$ and $\underline{(R_{sws^*}[4])^M}$ can be identified with the cokernel of the map $\underline{\mathcal{Z}/\mathcal{Y}} \to \underline{P/\mathcal{Y}}$. Using (d) and (e), where we identify $\underline{(R_{sw}[2])^{M.R}} = \underline{P/\mathcal{Y}}$ (see Lemma 4.12), we see that both $\underline{\mathcal{Z}/\mathcal{Y}}$ and $\underline{(R_{sw}[2])^M}$ can be identified with the kernel of the map $\underline{P/\mathcal{Y}} \to \underline{P/\mathcal{Z}}$. Thus, we have $\underline{P/\mathcal{Z}} \xrightarrow{\sim} \underline{(R_{sws^*}[4])^M}$ and $\underline{\mathcal{Z}/\mathcal{Y}} \xrightarrow{\sim} \underline{(R_{sw}[2])^M}$. Now, the injective homomorphism $P/\mathcal{Z} \to (R_{sws^*}[4])^M$ (resp. $\mathcal{Z}/\mathcal{Y} \to (R_{sw}[2])^M$) induces an injective homomorphism $(P/\mathcal{Z})^\wedge \to ((R_{sws^*}[4])^M)^\wedge$ (resp. $(\mathcal{Z}/\mathcal{Y})^\wedge \to ((R_{sw}[2])^M)^\wedge$) which becomes surjective after applying $\otimes_{\widehat{R}}\mathbf{R}$; hence, by the Nakayama lemma, it is surjective before applying $\otimes_{\widehat{R}}\mathbf{R}$. □

Lemma 4.14. *Assume that* $l(w) < l(sw) = l(ws^*)$. *The obvious sequence*

$$0 \to \underline{\mathcal{V}} \to \underline{P} \to \underline{P/\mathcal{V}} \to 0$$

is exact.

From the exact sequence $0 \to \mathcal{Y}/\mathcal{V} \to P/\mathcal{V} \to P/\mathcal{Y} \to 0$ we deduce an exact sequence $0 \to (\mathcal{Y}/\mathcal{V})^\wedge \to (P/\mathcal{V})^\wedge \to (P/\mathcal{Y})^\wedge \to 0$ in which $(\mathcal{Y}/\mathcal{V})^\wedge$ is a free $\widehat{R}$-module (by Lemma 4.10 and 2.1(a)) and $(P/\mathcal{Y})^\wedge$ is a free $\widehat{R}$-module (by Lemma 4.12 and 2.1(a)). It follows that

(a) $(P/\mathcal{V})^\wedge$ is a free $\widehat{R}$-module.

From the obvious exact sequence $0 \to \mathcal{V} \to P \to P/\mathcal{V} \to 0$ we deduce an exact sequence $0 \to \widehat{\mathcal{V}} \to \widehat{P} \to (P/\mathcal{V})^\wedge \to 0$ which is split, due to (a). It follows that it remains exact after applying $\otimes_{\widehat{R}}\mathbf{R}$. ☐

Lemma 4.15. *Assume that* $l(w) < l(sw) < l(sws^*)$. *The obvious sequence*

$$0 \to \underline{\mathcal{Z}/\mathcal{V}} \to \underline{P/\mathcal{V}} \to \underline{P/\mathcal{Z}} \to 0$$

is exact.

From the obvious exact sequence $0 \to \mathcal{Z}/\mathcal{V} \to P/\mathcal{V} \to P/\mathcal{Z} \to 0$ we deduce an exact sequence $0 \to (\mathcal{Z}/\mathcal{V})^\wedge \to (P/\mathcal{V})^\wedge \to (P/\mathcal{Z})^\wedge \to 0$ which is split, since the $\widehat{R}$-module $(P/\mathcal{Z})^\wedge$ is free, by Lemma 4.13 and 2.1(a). It follows that it remains exact after applying $\otimes_{\widehat{R}}\mathbf{R}$. ☐

Lemma 4.16. *Assume that* $l(w) < l(sw) < l(sws^*)$. *The sum of the obvious homomorphisms* $\underline{\mathcal{Y}/\mathcal{V}} \xrightarrow{c} \underline{\mathcal{Z}/\mathcal{V}}$ *and* $\underline{\mathcal{Y}'/\mathcal{V}} \xrightarrow{c'} \underline{\mathcal{Z}/\mathcal{V}}$ *is an isomorphism*

$$\underline{\mathcal{Y}/\mathcal{V}} \oplus \underline{\mathcal{Y}'/\mathcal{V}} \xrightarrow{\sim} \underline{\mathcal{Z}/\mathcal{V}}.$$

From the obvious exact sequence $0 \to \mathcal{Y}/\mathcal{V} \to \mathcal{Z}/\mathcal{V} \to \mathcal{Z}/\mathcal{Y} \to 0$ we deduce an exact sequence $0 \to (\mathcal{Y}/\mathcal{V})^\wedge \to (\mathcal{Z}/\mathcal{V})^\wedge \to (\mathcal{Z}/\mathcal{Y})^\wedge \to 0$ which is split, since the $\widehat{R}$-module $(\mathcal{Z}/\mathcal{Y})^\wedge$ is free, by Lemma 4.13 and 2.1(a). It follows that after applying $\otimes_{\widehat{R}}\mathbf{R}$ we get an exact sequence

$$0 \to \underline{\mathcal{Y}/\mathcal{V}} \xrightarrow{c} \underline{\mathcal{Z}/\mathcal{V}} \xrightarrow{d} \underline{\mathcal{Z}/\mathcal{Y}} \to 0.$$

We consider the composition $dc' : \underline{\mathcal{Y}'/\mathcal{V}} \to \underline{\mathcal{Z}/\mathcal{Y}}$. By Lemma 4.11 we can identify $\underline{\mathcal{Y}'/\mathcal{V}} = \underline{(R_{sw}[2])^M}$ and by Lemma 4.13 we can identify $\underline{\mathcal{Z}/\mathcal{Y}} = \underline{(R_{sw}[2])^M}$. Under these identifications the map dc' becomes the identity map of $\underline{(R_{sw}[2])^M}$. In particular, dc' is an isomorphism. This implies immediately the statement of the lemma. ☐

5 Trace computations

5.1 To simplify notation, for $x \in W, r \in R$ we shall write ${}^x r^*$ instead of ${}^x(r^*)$. Recall that if $x \in \mathbf{I}$, then $r \mapsto {}^x r^*$ is an involution $R \to R$ (as an $\mathbf{R}$-vector space) denoted by $\mathfrak{f}_x$ in 3.2.

In this section we fix $(M, \phi) \in \widetilde{\mathcal{C}}_\sharp$, $s \in \mathbf{I}$ and $w \in \mathbf{I}$ such that $l(w) < l(sw)$; we have automatically $l(w) < l(ws^*)$. As in 4.5 we set $N = R_w$ and define the notation $b \circ r$ for $b \in N, r \in R$ as in 4.5. In the case where $sw = ws^*$, we set $N' = R_{sw}$. In the case where $sw \neq ws^*$, we set $N'' = R_{sws^*}$.

For $b \in N, r \in R$ we have

$$\mathfrak{f}_w(b \circ r) = r^* \mathfrak{f}_w(b), \quad \mathfrak{f}_w(rb) = \mathfrak{f}_w(b) \circ r^*.$$

The involution $f \mapsto f^!$, $N^M \to N^M$, given by $f^!(m) = \mathfrak{f}_w(f(\phi(m)))$, induces an involution

$$\Theta : \underline{N^M} \to \underline{N^M}.$$

In the case where $sw = ws^*$, we have $sw \in \mathbf{I}$ and the involution $f \mapsto f^!$, $N'^M \to N'^M$, given by $f^!(m) = \mathfrak{f}_{sw}(f(\phi(m)))$, induces an involution

$$\Theta' : \underline{N'^M} \to \underline{N'^M}.$$

In the case where $sw \neq ws^*$, we have $sws^* \in \mathbf{I}$ and the involution $f \mapsto f^!$, $N''^M \to N''^M$, given by $f^!(m) = \mathfrak{f}_{sws^*}(f(\phi(m)))$, induces an involution

$$\Theta'' : \underline{N''^M} \to \underline{N''^M}.$$

Now Θ (or Θ' or Θ'', if defined) induces a degree-preserving involution of $\underline{N^M}$ (or $\underline{N'^M}$, or $\underline{N''^M}$) denoted again by Θ (or Θ' or Θ'').

By 3.6 we have $(R.M.R, \phi') \in \widetilde{\mathcal{C}}_\sharp$ where

$$\phi' : R.M.R \to R.M.R$$

is the $\mathbf{R}$-linear map such that $r_1.m.r_2 \mapsto r_2^*.\phi(m').r_1^*$ for $r_1, r_2 \in R$, $m \in M$. (Recall that $R.M.R \in \widetilde{\mathcal{C}}$ is defined in 4.3.) We have $\phi'^2 = 1$. Let $\Psi : N^{R.M.R} \to N^{R.M.R}$ be the $\mathbf{R}$-linear involution such that for any $F \in N^{R.M.R}$ and any $r_1, r_2 \in R$, $m \in M$, we have

$$\Psi(F)(r_1.m.r_2) = \mathfrak{f}_w(F(\phi'(r_1.m.r_2))) = \mathfrak{f}_w(F(r_2^*.\phi(m).r_1^*)).$$

(This is a special case of the definition of $f \mapsto f^!$ in 1.1.) It induces a degree-preserving involution of $\underline{N^{R.M.R}}$ denoted again by Ψ.

We now state the main result of this section. (In this section all traces are taken over $\mathbf{R}$.)

Theorem 5.2. *Recall that $w \in \mathbf{I}$, $l(w) < l(sw)$. Let $i \in \mathbf{Z}$. If $sw \neq ws^*$, then*

(a) $$\operatorname{tr}(\Psi, \underline{N^{R.M.R}}_i) = \operatorname{tr}(\Theta, \underline{N^M}_i) + \operatorname{tr}(\Theta'', \underline{N''^M}_{i+4}).$$

If $sw = ws^*$, then

(b) $$\mathrm{tr}(\Psi, \underline{N^{R.M.R}}_i) = \mathrm{tr}(\Theta, \underline{N^M}_i) + \mathrm{tr}(\Theta, \underline{N^M}_{i+2}) - \mathrm{tr}(\Theta', \underline{N'^M}_{i+2}) + \mathrm{tr}(\Theta', \underline{N'^M}_{i+4}).$$

Note that the following identities (with ϕ' as in 5.1) are equivalent to the theorem.

(c) $$\epsilon^w(R.M.R, \phi') = \epsilon^w(M, \phi) + \epsilon^{sws^*}(M, \phi)v^4 \text{ if } sw \neq ws^*,$$

(d) $$\epsilon^w(R.M.R, \phi') = \epsilon^w(M, \phi)(v^2 + 1) + \epsilon^{sw}(M, \phi)(v^4 - v^2) \text{ if } sw = ws^*.$$

The proof will occupy the remainder of this section.

5.3 We identify $N^{R.M.R}$ with $P = \hom^\bullet(M, N)$ (as objects of $\mathcal{R}$) as in 4.3(i). Then Ψ becomes an involution of P denoted again by Ψ. It is given by $f \mapsto f^!$ where $f^!(m) = \mathfrak{f}_w(f(\phi(m)))$. This induces a degree-preserving involution of $\underline{P}$ denoted again by Ψ. For any i we clearly have

(a) $$\mathrm{tr}(\Psi, \underline{N^{R.M.R}}_i) = \mathrm{tr}(\Psi, \underline{P}_i).$$

5.4 In this subsection we assume that $sw \neq ws^*$ so that $l(w) < l(sw) < l(sws^*)$ and $sws^* \in \mathbf{I}$.

Let $\mathcal{V}, \mathcal{Y}, \mathcal{Y}', \mathcal{Z}$ be the subobjects of P defined in 4.9. From the definition we see that $\Psi : P \to P$ preserves $\mathcal{V}$ and $\mathcal{Z}$; it interchanges $\mathcal{Y}$ and $\mathcal{Y}'$. Now for any $\xi \in P$, we have $\Psi(\xi R^{>0}) = R^{>0}\Psi(\xi) = \Psi(\xi)R^{>0}$. (We use that $R^{>0}b = b \circ R^{>0}$ for $b \in N$.) It follows that $\mathcal{V}R^{>0}, \mathcal{Z}R^{>0}$ are preserved by Ψ and $\mathcal{Y}R^{>0}, \mathcal{Y}'R^{>0}$ are interchanged by Ψ. Hence Ψ induces involutions of $\underline{\mathcal{V}}_i, \underline{P/\mathcal{Z}}_i, \underline{\mathcal{Z}/\mathcal{V}}_i$ (denoted again by Ψ) and the two summands $\underline{\mathcal{Y}/\mathcal{V}}_i, \underline{\mathcal{Y}'/\mathcal{V}}_i$ of $\underline{\mathcal{Z}/\mathcal{V}}_i$ (see Lemma 4.16) are interchanged by $\Psi : \underline{\mathcal{Z}/\mathcal{V}}_i \to \underline{\mathcal{Z}/\mathcal{V}}_i$. Hence we have $\mathrm{tr}(\Psi, \underline{\mathcal{Z}/\mathcal{V}}_i) = 0$ and (using Lemmas 4.14 and 4.15) we have

(a) $$\mathrm{tr}(\Psi, \underline{P}_i) = \mathrm{tr}(\Psi, \underline{\mathcal{V}}_i) + \mathrm{tr}(\Psi, \underline{P/\mathcal{V}}_i) = \mathrm{tr}(\Psi, \underline{\mathcal{V}}_i) + \mathrm{tr}(\Psi, \underline{P/\mathcal{Z}}_i).$$

We now show that the map (say τ), $P \to R_{sws^*}[4]$ in 4.13(a) satisfies

(b) $$\tau(\Psi(f)) = \Theta''(\tau(f))$$

for any $f \in P$. For $m \in M$ we have

$$\tau(\Psi(f))(m) = {}^s(\mathfrak{f}_w(f(\phi(\alpha m\alpha^*))) - \alpha\mathfrak{f}_w(f(\phi(m\alpha^*))) - \mathfrak{f}_w(f(\phi(\alpha m))) \circ \alpha^* + \alpha\mathfrak{f}_w(f(\phi(m))) \circ \alpha^*),$$

and

$$\begin{aligned}\Theta''(\tau(f))(m) &= \mathfrak{f}_{sws^*}(\tau(f)(\phi(m))) \\ &= \mathfrak{f}_{sws^*}({}^s(f(\alpha\phi(m)\alpha^*) - \alpha f(\phi(m)\alpha^*) \\ &\quad - f(\alpha\phi(m)) \circ \alpha^* + \alpha f(\phi(m)) \circ \alpha^*)).\end{aligned}$$

It is enough to show that for any $m' \in M$, we have

$$^{s}(\mathfrak{f}_w(f(m'))) = \mathfrak{f}_{sws^*}(^{s}(f(m')))$$

or that

$$^{s}(^{w}(f(m')^*)) = {}^{sws^*}(^{s^*}(f(m')^*)).$$

This is clear; (b) is proved. Using (b) and Lemma 4.13 we deduce that

(c) $$\mathrm{tr}(\Psi, \underline{P/\mathcal{Z}}_i) = \mathrm{tr}(\Theta'', \underline{N''^M}_{i+4}).$$

We clearly have $\mathcal{V} = N^M$ and $\mathrm{tr}(\Psi, \underline{\mathcal{V}}_i) = \mathrm{tr}(\Theta, \underline{N^M}_i)$. Introducing this and (c) into (a) and using 5.3(a) we obtain 5.2(a) and (equivalently) 5.2(c).

5.5 For the remainder of this section we assume that $sw = ws^*$ so that $sw \in \mathbf{I}$. Note that we have $^{w}\alpha^* = \alpha$, hence $b \circ \alpha^* = \alpha b$ for $b \in N$.

In this case the involution $\Psi : P \to P$ preserves $PR^{s^*,>0}$ (more precisely, we have $\Psi(fR^{s^*,>0}) = \Psi(f)R^{s^*,>0}$ for any $f \in P$), hence it induces an involution of $\underline{\underline{P}}$ denoted again by Ψ. (We use that $^{w}(R^{s^*}) = R^s$, hence $^{w}(R^{s^*} \cap R^{>0}) = R^s \cap R^{>0}$). Moreover the involution Ψ of $\underline{\underline{P}}$ is $\underline{\underline{R}}$-linear. (We use that $^{w}\alpha^* = \alpha$.)

Let $\Phi : (R.N.R)^M \to (R.N.R)^M$ be the $\mathbf{R}$-linear involution which corresponds to $\Psi : P \to P$ under the bijection $P[-4] \xrightarrow{\sim} (R.N.R)^M$ in 4.3(ii). Since that bijection is compatible with the (R^s, R^{s^*})-bimodule structures, it follows that Φ preserves the subset $(R.N.R)^M R^{s^*,>0}$; more precisely we have

(a) $$\Phi(\xi R^{s^*,>0}) = \Phi(\xi)R^{s^*,>0} \text{ for any } \xi \in (R.N.R)^M,$$

hence Φ induces an $\mathbf{R}$-linear involution of $\underline{\underline{(R.N.R)^M}}$ (which is not necessarily $\underline{\underline{R}}$-linear). For any i we have from the definition:

(b) $$\mathrm{tr}(\Phi, \underline{\underline{(R.N.R)^M}}_i) = \mathrm{tr}(\Psi, \underline{\underline{P}}_{i-4}).$$

Note that $\underline{\underline{P}}$ is a free right $\mathbf{R}[\alpha^*]/(\alpha^{*2})$-module. Hence we have exact an sequence of $\mathbf{R}$-vector spaces

$$0 \to \underline{P}_{i-6} \xrightarrow{c} \underline{\underline{P}}_{i-4} \xrightarrow{d} \underline{P}_{i-4} \to 0,$$

where c is induced by right multiplication by α^* and we have $d\Theta = \Theta d, c\Theta = \Theta c$. It follows that we have

$$\mathrm{tr}(\Psi, \underline{\underline{P}}_{i-4}) = \mathrm{tr}(\Psi, \underline{P}_{i-4}) + \mathrm{tr}(\Theta, \underline{P}_{i-6}).$$

Introducing this in (b), we obtain

(c) $$\mathrm{tr}(\Phi, \underline{\underline{(R.N.R)^M}}_i) = \mathrm{tr}(\Psi, \underline{P}_{i-4}) + \mathrm{tr}(\Psi, \underline{P}_{i-6}).$$

5.6 We define a map $\xi \mapsto \check{\xi}$, $R.N.R \to R.N.R$ by

(a) $$1.b_0.1 + \alpha.b_1.1 + 1.b_2.\alpha^* + \alpha.b_3.\alpha^* \mapsto 1.\mathfrak{f}_w(b_0).1 + \alpha.\mathfrak{f}_w(b_2).1 + 1.{}^w\mathfrak{f}_w(b_1).\alpha^* + \alpha.\mathfrak{f}_w(b_3).\alpha^*$$

where $b_i \in N$. Then $\xi \mapsto \check{\xi}$ is an involution of the **R**-vector space $R.N.R$ such that $(r_1.b.r_2)^\vee = r_2^*.\mathfrak{f}_w(b).r_1^*$ for $r_1, r_2 \in R, b \in N$. Hence in the (R, R)-bimodule structure of $R.N.R$ we have $(r\xi)^\vee = \check{\xi}r^*$, $(\xi r)^\vee = r^*\check{\xi}$ for $r \in R$, $\xi \in R.N.R$. Thus $(R.N.R, \xi \mapsto \check{\xi}) \in \mathcal{R}_\sharp$.

From the definitions we see that $\Phi : (R.N.R)^M \to (R.N.R)^M$ is given explicitly by $G \mapsto G^!$, where for any $G \in (R.N.R)^M$ and any $m \in M$ we have

(b) $$G^!(m) = (G(\phi(m)))^\vee.$$

(This is a special case of the definition of $f \mapsto f^!$ in 1.1.)

5.7 Let V, Y be the subsets of $R.N.R$ defined as in 4.5. (They are subobjects in $\mathcal{R}$.) In addition to the subsets V, Y we shall need the following subsets of $R.N.R$:

$$X = \{1.\alpha b' \circ \alpha^*.1 + \alpha.b.1 + 1.b.\alpha^* + \alpha.b'.\alpha^*; b, b' \in N\},$$
$$U = \{1.(\alpha b - b' \circ \alpha^* + \alpha b'' \circ \alpha^*).1 + \alpha.b.1 + 1.b'.\alpha^* + \alpha.b''.\alpha^*; b, b', b'' \in N\}$$
$$= X + Y.$$

Note that $X \cap Y = V$. Using our assumptions on w, it is easy to verify that X is a subobject of $R.N.R$ in $\mathcal{R}$, hence $U = X + Y$ is a subobject of $R.N.R$ in $\mathcal{R}$.

By a straightforward computation we see that (a), (b) below hold:

(a) the map $X \to N[-2]$, $1.\alpha b' \circ \alpha^*.1 + \alpha.b.1 + 1.b.\alpha^* + \alpha.b'.\alpha^* \mapsto {}^s(b - \alpha b')$ induces an isomorphism $X/V \xrightarrow{\sim} N[-2]$ in $\mathcal{R}$;

(b) the map $R.N.R \to N'$,

$$1.b_0.1 + \alpha.b_1.1 + 1.b_2.\alpha^* + \alpha.b_3.\alpha^* \mapsto {}^s(-b_0 + \alpha b_1 - \alpha b_2 + \alpha^2 b_3),$$

induces an isomorphism $R.N.R/U \xrightarrow{\sim} N'$ in $\mathcal{R}$.

Lemma 5.8. *The obvious sequence*

(a) $$0 \to \underline{\underline{(U/Y)^M}} \to \underline{\underline{(R.N.R/Y)^M}} \to \underline{\underline{(R.N.R/U)^M}} \to 0$$

is exact.

Note that $N'.R = N.R$. Indeed, it is enough to show that $N = N'$ as (R, R^{s^*})-bimodules. It is also enough to show that if $r \in R^{s^*}$, then ${}^w r = {}^{ws^*} r$; this follows from ${}^{s^*} r = r$. We identify $R.N.R/Y = R_{sw}.R = N'.R$ (hence $R.N.R/Y = N.R$) as in the proof of Lemma 4.7, $U/Y = (X + Y)/Y = X/V = N[-2]$ as in 5.7(a), and $R.N.R/U = N'$ as in 5.7(b). Then (a) becomes a sequence

$$0 \to \underline{\underline{(N[-2])^M}} \to \underline{\underline{(N.R)^M}} \to \underline{\underline{N'^M}} \to 0$$

or equivalently, a sequence

$$0 \to \underline{\underline{(N[-2])^M}} \to \underline{\underline{(N[-2])^{M.R}}} \to \underline{\underline{N'^M}} \to 0$$

which is exact by 4.4(g), (h). □

5.9 We write $N^0 = R^s$, $N^1 = \alpha R^s = R^s \circ \alpha^*$ so that $N = N^0 \oplus N^1$ as an (R^s, R^{s^*})-bimodule. For $b \in N$ we can write uniquely $b = b^0 + b^1$ where $b^i \in N^i$. For $i \in \{0, 1\}$ let

$$(R.N.R)^i = \{1.b_0.1 + \alpha.b_1.1 + 1.b_2.\alpha^* + \alpha.b_3.\alpha^* \in R.N.R; \quad b_j \in N^i \text{ for } j = 0, 1, 2, 3\}.$$

Using the fact that $\alpha^2 \in R^s$ we see that $(R.N.R)^i$ is a subobject of $R.N.R$ in $\mathcal{R}$. Thus, we have $R.N.R = (R.N.R)^0 \oplus (R.N.R)^1$ as objects of $\mathcal{R}$. For $i \in \{0, 1\}$ we set

$$X^i = X \cap (R.N.R)^i = \{1.\alpha b' \circ \alpha^*.1 + \alpha.b.1 + 1.b.\alpha^* + \alpha.b'.\alpha^*; b, b' \in N^i\},$$

$$(R.N.R/X)^i = (R.N.R)^i / X^i.$$

Then X^i is a subobject of X in $\mathcal{R}$, $(R.N.R/X)^i$ is a subobject of $R.N.R/X$ in $\mathcal{R}$ and we have

(a) $$X = X^0 \oplus X^1, \quad R.N.R/X = (R.N.R/X)^0 \oplus (R.N.R/X)^1$$

as objects of $\mathcal{R}$. We have

(b) $$X = V \oplus X^0,$$

(c) $$R.N.R/X = U/X \oplus (R.N.R/X)^0.$$

We prove (b). We must show that for any $b, b' \in N$ there are unique $\beta \in N^0, \beta' \in N^0, b'' \in N$ such that

$$1.\alpha b' \circ \alpha^*.1 + \alpha.b.1 + 1.b.\alpha^* + \alpha.b'.\alpha^* = 1.\alpha\beta' \circ \alpha^*.1 + \alpha.\beta.1 + 1.\beta.\alpha^* + \alpha.\beta'.\alpha^* + 1.\alpha b'' \circ \alpha^*.1 + \alpha.\alpha b''.1 + \alpha.b''.\alpha^* + 1.\alpha b''.\alpha^*$$

or equivalently, $b = \beta + \alpha b'', b' = \beta' + b''$. Setting $b'' = b - \beta'$ we see that we must show that there are unique $\beta \in N^0, \beta' \in N^0$ such that $b - \alpha b' = \beta - \alpha\beta'$. This is obvious.

We prove (c). It is enough to show that

(i) $R.N.R = U + (R.N.R)^0$,
(ii) $U \cap ((R.N.R)^0 + X^1) = X$.

For (i) we must show that given $b_1, b_2, b_3, b_4 \in N$, there exist $b, b', b'' \in N$ and $\beta_1, \beta_2, \beta_3, \beta_4 \in N^0$ such that

$$b_1 = b + \beta_1, b_2 = b' + \beta_2, b_3 = b'' + \beta_3, b_4 = \alpha b - b' \circ \alpha^* + \alpha b'' \circ \alpha^* + \beta_4.$$

Setting $\beta_2 = \beta_3 = 0$, $b = b_1 - \beta_1, b' = b_2, b'' = b_3$, we see that it is enough to show that there exist $\beta_1, \beta_4 \in N^0$ such that

$$b_4 - \alpha b_1 + \alpha b_2 - \alpha^2 b_3 = \beta_4 - \alpha\beta_1.$$

This is obvious.

For (ii) we must show that given $b, b', b'' \in N$ and $\beta, \beta' \in N^1$ such that

$$\begin{aligned} 1.(\alpha b - b' \circ \alpha^* + \alpha b'' \circ \alpha^*).1 + \alpha.b.1 + 1.b'.\alpha^* + \alpha.b''.\alpha^* \\ - (1.\alpha\beta' \circ \alpha^*.1 + \alpha.\beta.1 + 1.\beta.\alpha^* + \alpha.\beta'.\alpha^*) \in (R.N.R)^0, \end{aligned}$$

we have $b = b'$. Our assumption implies $b^1 = \beta$, $b'^1 = \beta$, $b''^1 = \beta'$, $(\alpha b - \alpha b' + \alpha^2 b'')^1 = \alpha^2\beta'$ (that is $(b - b' + \alpha b'')^0 = \alpha\beta'$). Thus, $(b - b')^1 = 0$ and $(b - b')^0 = 0$, so that $b - b' = 0$. This proves (c). $\square$

Now (a), (b) yield isomorphisms (in $\mathcal{R}$)

$$X^0 \to X/V, \quad X^1 \xrightarrow{\sim} V;$$

the first one is induced by the identity map, the second one is the restriction to X^1 of the first projection $X = V \oplus X^0 \to V$. Moreover, (a), (c) yield isomorphisms (in $\mathcal{R}$)

$$(R.N.R/X)^0 \to R.N.R/U, \quad (R.N.R/X)^1 \to U/X;$$

the first one is induced by the identity map, the second one is the restriction to $(R.N.R)^1$ of the first projection $R.N.R/X = U/X \oplus (R.N.R/X)^0 \to U/X$.

Lemma 5.10. *The obvious sequence*

(a) $$0 \to \underline{\underline{X^M}} \to \underline{\underline{(R.N.R)^M}} \to \underline{\underline{(R.N.R/X)^M}} \to 0$$

is exact.

Consider the obvious commutative diagram with exact horizontal and vertical lines

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & V & \longrightarrow & X & \longrightarrow & U/Y & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Y & \longrightarrow & R.N.R & \longrightarrow & R.N.R/Y & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & Y/V & \longrightarrow & R.N.R/X & \longrightarrow & R.N.R/U & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ & & 0 & & 0 & & 0 & & \end{array}$$

Here the non-middle horizontal maps are split as exact sequences in $\mathcal{R}$. Indeed by the results in Section 5.9 they can be identified with the obvious split exact sequences

$$0 \to X^1 \to X^0 \oplus X^1 \to X^0 \to 0,$$

$$0 \to (R.N.R/X)^1 \to (R.N.R/X)^0 \oplus (R.N.R/X)^1 \to (R.N.R/X)^0 \to 0.$$

From this we deduce the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \underline{\underline{V^M}} & \longrightarrow & \underline{\underline{X^M}} & \longrightarrow & \underline{\underline{(U/Y)^M}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \underline{\underline{Y^M}} & \longrightarrow & \underline{\underline{(R.N.R)^M}} & \longrightarrow & \underline{\underline{(R.N.R/Y)^M}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \underline{\underline{(Y/V)^M}} & \longrightarrow & \underline{\underline{(R.N.R/X)^M}} & \longrightarrow & \underline{\underline{(R.N.R/U)^M}} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

in which the middle horizontal line and the non-middle vertical lines are exact sequences (see Lemmas 4.8, 4.6 and 5.8) and in which the non-middle horizontal lines are (split) exact sequences. This implies, by diagram chasing, that the middle vertical line is an exact sequence. □

5.11 From 5.6(a) we see that X, $(R.N.R)^0$, $(R.N.R)^1$ are stable under the involution $\xi \mapsto \check{\xi}$ of $R.N.R$. Hence that involution induces involutions on X, on $(R.N.R)/X$, on X^i and on $(R.N.R/X)^i$ (for $i = 0, 1$) which in turn induce (by formulas like 5.6(b)) involutions on X^M, on $((R.N.R)/X)^M$, on $(X^i)^M$ and on $((R.N.R/X)^i)^M$ (for $i = 0, 1$) which are denoted again by Φ. Using 5.5(a) we see that each of these involutions preserve the image of right multiplication by $R^{s^*,>0}$, hence we have induced involutions on $\underline{\underline{X^M}}$, on $\underline{\underline{((R.N.R)/X)^M}}$, on $\underline{\underline{(X^i)^M}}$ and on $\underline{\underline{((R.N.R/X)^i)^M}}$ (for $i = 0, 1$) which are denoted again by Φ.

Using the definitions we see that the exact sequence 5.10(a) is compatible with the involutions Φ on each of its terms. Using the definitions we also see that the obvious direct sum decompositions

$$\underline{\underline{X^M}} = \underline{\underline{(X^0)^M}} \oplus \underline{\underline{(X^1)^M}},$$

$$\underline{\underline{(R.N.R/X)^M}} = \underline{\underline{((R.N.R/X)^0)^M}} \oplus \underline{\underline{((R.N.R/X)^1)^M}}$$

are compatible with the involutions Φ on each of their terms. It follows that for $i \in \mathbf{Z}$ we have

$$\text{(a)}\quad \mathrm{tr}(\Phi, \underline{\underline{(R.N.R)^M}}_i) = \mathrm{tr}(\Phi, \underline{\underline{X^M}}_i) + \mathrm{tr}(\Phi, \underline{(R.N.R/X)^M}_i)$$
$$= \mathrm{tr}(\Phi, \underline{\underline{(X^0)^M}}_i) + \mathrm{tr}(\Phi, \underline{\underline{(X^1)^M}}_i)$$
$$+ \mathrm{tr}(\Phi, \underline{\underline{((R.N.R/X)^0)^M}}_i) + \mathrm{tr}(\Phi, \underline{\underline{((R.N.R/X)^1)^M}}_i).$$

5.12 By a straightforward computation we see that the maps t_i in (a)–(d) below are isomorphisms in $\mathcal{R}$:

(a) $t_1 : X^1 \to N[-4]$, given by

$$1.\alpha\beta' \circ \alpha^*.1 + \alpha.\beta.1 + 1.\beta.\alpha^* + \alpha.\beta'.\alpha^* \mapsto \alpha^{-1}\beta + \beta$$

with $\beta, \beta' \in N^1$;
(b) $t_1 : X^0 \to N[-2]$, given by

$$1.\alpha\beta' \circ \alpha^*.1 + \alpha.\beta.1 + 1.\beta.\alpha^* + \alpha.\beta'.\alpha^* \mapsto \beta + \alpha\beta'$$

with $\beta, \beta' \in N^0$;
(c) $t_3 : (R.N.R/X)^1 \to N'[-2]$ induced by

$$1.\beta_0.1 + \alpha.\beta_1.1 + 1.\beta_2.\alpha^* + \alpha.\beta_3.\alpha^* \mapsto \beta_1 - \beta_2 + \alpha^{-1}\beta_0 - \alpha\beta_3$$

with $\beta_0, \beta_1, \beta_2, \beta_3 \in N^1$;
(d) $t_4 : (R.N.R/X)^0 \to N'$ induced by

$$1.\beta_0.1 + \alpha.\beta_1.1 + 1.\beta_2.\alpha^* + \alpha.\beta_3.\alpha^* \mapsto \alpha\beta_1 - \alpha\beta_2 + \beta_0 - \alpha^2\beta_3$$

with $\beta_0, \beta_1, \beta_2, \beta_3 \in N^0$.

5.13 The identities (a)–(d) below express a connection between $\xi \mapsto \check{\xi}$ and the isomorphisms t_j in 5.12:

(a) if $\xi \in X^1$, then $t_1(\check{\xi}) = {}^w(t_1(\xi))^*$;
(b) if $\xi \in X^0$, then $t_2(\check{\xi}) = {}^w(t_2(\xi))^*$;
(c) if $\xi \in (R.N.R/X)^1$, then $t_3(\check{\xi}) = {}^{sw}(t_3(\xi))^*$;
(d) if $\xi \in (R.N.R/X)^0$, then $t_4(\check{\xi}) = {}^{sw}(t_4(\xi))^*$.

Here $t_i(\xi)$ is viewed as an element of R and the shift is ignored.
We prove (a). Let $\xi = 1.\alpha\beta' \circ \alpha^*.1 + \alpha.\beta.1 + 1.\beta.\alpha^* + \alpha.\beta'.\alpha^* \in X^1$ be as in Section 5.9. Then

$$\check{\xi} = 1.\alpha^w\beta'^* \circ \alpha^*.1 + \alpha.{}^w\beta^*.1 + 1.{}^w\beta^*.\alpha^* + \alpha.{}^w\beta'^*.\alpha^*$$

and we must show that $\alpha^{-1w}\beta^* + {}^wb'^* = {}^w(\alpha^{-1}\beta + \beta')^*$. This follows from the equality ${}^w\alpha^* = \alpha$.

We prove (b). In this case we must show that ${}^w\beta^* + \alpha^w b'^* = {}^w(\beta + \alpha\beta')^*$ for $\beta, \beta' \in N^0$. This again follows from the equality ${}^w\alpha^* = \alpha$.

We prove (c). Let $\xi = 1.\beta_0.1 + \alpha.\beta_1.1 + 1.\beta_2.\alpha^* + \alpha.\beta_3\alpha^*$ be as in Section 5.9. Then $\check{\xi} = 1.{}^w\beta_0^*.1 + \alpha.{}^w\beta_2^*.1 + 1.{}^w\beta_1^*.\alpha^* + \alpha.{}^w\beta_3^*.\alpha^*$ and we must show that

$${}^w\beta_2^* - {}^w\beta_1^* + \alpha^{-1w}\beta_0^* - \alpha^w\beta_3^* = {}^{sw}(\beta_1 - \beta_2 + \alpha^{-1}\beta_0 - \alpha\beta_3)^*.$$

Since $\beta_i \in N^1$ we have ${}^{s^*}\beta_i^* = -b_i^*$; we have also ${}^{s^*}\alpha^* = -\alpha^*$ and $sw = ws^*$. Thus

$$\begin{aligned}{}^{sw}(\beta_1 - \beta_2 + \alpha^{-1}\beta_0 - \alpha\beta_3)^* &= {}^w({}^{s^*}\beta_1^* - {}^{s^*}\beta_2^* - (\alpha^*)^{-1s^*}\beta_0^* + \alpha^{*s^*}\beta_3^*)\\ &= {}^w(-\beta_1^* + \beta_2^* + (\alpha^*)^{-1}\beta_0^* - \alpha^*\beta_3^*),\end{aligned}$$

as desired.

We prove (d). In this case we must show that

$$\alpha^w\beta_2^* - \alpha^w\beta_1^* - {}^w\beta_0^* + \alpha^{2w}\beta_3^* = {}^{sw}(\alpha\beta_1 - \alpha\beta_2 - \beta_0 + \alpha^2\beta_3)^*$$

for $\beta_i \in N_0$. We have ${}^{s^*}\beta_i^* = b_i^*$; we have also ${}^{s^*}\alpha^* = -\alpha^*$ and $sw = ws^*$. Thus

$$\begin{aligned}&{}^{sw}(\alpha\beta_1 - \alpha\beta_2 - \beta_0 + \alpha^2\beta_3)^*\\ &\qquad = {}^w(-\alpha^{*s^*}\beta_1^* + \alpha^{*s^*}\beta_2^* - {}^{s^*}\beta_0^* + \alpha^{*2s^*}\beta_3^*)\\ &\qquad = {}^w(-\alpha^*\beta_1^* + \alpha^*\beta_2^* - \beta_0^* - \alpha^{*2}\beta_3^*),\end{aligned}$$

as desired. □

5.14 The involution $f \mapsto f^!$, $N^M \to N^M$, given by $f^!(m) = \mathfrak{f}_w(f(\phi(m)))$, induces an involution $\Theta : \underline{N^M} \to \underline{N^M}$ (see 5.1) and also an involution $\underline{\underline{N^M}} \to \underline{\underline{N^M}}$ denoted again by Θ. The involution $f \mapsto f^!$, $N'^M \to N'^M$, given by $f^!(m) = \mathfrak{f}_{sw}(f(\phi(m))$, induces an involution $\Theta' : \underline{N'^M} \to \underline{N'^M}$ (see 5.1) and also an involution $\underline{\underline{N'^M}} \to \underline{\underline{N'^M}}$ denoted again by Θ'. Using that ${}^{w^{-1}}\alpha = \alpha^*$ and ${}^{(sw)^{-1}}\alpha = -\alpha^*$, we see that:

(a) *$\Theta : \underline{\underline{N^M}} \to \underline{\underline{N^M}}$ is $\mathbf{R}[\alpha^*]/(\alpha^{*2})$-linear; $\Theta' : \underline{\underline{N'^M}} \to \underline{\underline{N'^M}}$ is only $\mathbf{R}$-linear and satisfies $\Theta'(f\alpha^*) = -\Theta'(f)\alpha^*$ for $f \in \underline{\underline{N'^M}}$.*

Note that $\underline{\underline{N^M}}$, $\underline{\underline{N'^M}}$ are free right $\mathbf{R}[\alpha^*]/(\alpha^{*2})$-modules. Hence for any i we have exact sequences of $\mathbf{R}$-vector spaces

$$0 \to \underline{N^M}_{i-2} \xrightarrow{c} \underline{\underline{N^M}}_i \xrightarrow{d} \underline{N^M}_i \to 0,$$

$$0 \to \underline{N'^M}_{i-2} \xrightarrow{c'} \underline{\underline{N'^M}}_i \xrightarrow{d'} \underline{N'^M}_i \to 0,$$

where c, c' are induced by right multiplication by α^* and we have $d\Theta = \Theta d$, $d'\Theta' = \Theta' d$, $c\Theta = \Theta c$, $c'\Theta' = -\Theta' c'$. (We use (a).) It follows that for any $i \in \mathbf{Z}$ we have

(b) $$\mathrm{tr}(\Theta, \underline{\underline{N^M}}_i) = \mathrm{tr}(\Theta, \underline{N^M}_i) + \mathrm{tr}(\Theta, \underline{N^M}_{i-2}),$$

(c) $$\mathrm{tr}(\Theta', \underline{\underline{N'^M}}_i) = \mathrm{tr}(\Theta, \underline{N'^M}_i) - \mathrm{tr}(\Theta, \underline{N'^M}_{i-2}).$$

Using the isomorphisms in 5.12 and the identities in 5.13, we see that for any $i \in \mathbf{Z}$ we have

$$\begin{aligned}
\mathrm{tr}(\Phi, \underline{\underline{(X^0)^M}}_i) &= \mathrm{tr}(\Theta, \underline{\underline{N^M}}_{i-4}),\\
\mathrm{tr}(\Phi, \underline{\underline{(X^1)^M}}_i) &= \mathrm{tr}(\Theta, \underline{\underline{N^M}}_{i-2}),\\
\mathrm{tr}(\Phi, \underline{\underline{((R.N.R/X)^0)^M}}_i) &= \mathrm{tr}(\Theta', \underline{\underline{N'^M}}_{i-2}),\\
\mathrm{tr}(\Phi, \underline{\underline{((R.N.R/X)^1)^M}}_i) &= \mathrm{tr}(\Theta', \underline{\underline{N'^M}}_i).
\end{aligned}$$

Introducing this into 5.11(a) we deduce

$$\begin{aligned}
\mathrm{tr}(\Phi, \underline{\underline{(R.N.R)^M}}_i) &= \mathrm{tr}(\Theta, \underline{\underline{N^M}}_{i-4}) + \mathrm{tr}(\Theta, \underline{\underline{N^M}}_{i-2})\\
&\quad + \mathrm{tr}(\Theta', \underline{\underline{N'^M}}_{i-2}) + \mathrm{tr}(\Theta', \underline{\underline{N'^M}}_i),
\end{aligned}$$

from which (taking into account (b), (c) and 5.5(c)) we deduce

$$\begin{aligned}
&\mathrm{tr}(\Psi, \underline{P}_{i-4}) + \mathrm{tr}(\Psi, \underline{P}_{i-6})\\
&\quad = \mathrm{tr}(\Theta, \underline{N^M}_{i-4}) + \mathrm{tr}(\Theta, \underline{N^M}_{i-6}) + \mathrm{tr}(\Theta, \underline{N^M}_{i-2}) + \mathrm{tr}(\Theta, \underline{N^M}_{i-4})\\
&\quad + \mathrm{tr}(\Theta', \underline{N'^M}_{i-2}) - \mathrm{tr}(\Theta', \underline{N'^M}_{i-4}) + \mathrm{tr}(\Theta', \underline{N'^M}_i) - \mathrm{tr}(\Theta', \underline{N'^M}_{i-2}).
\end{aligned}$$

We multiply this equality by v^{-i} and sum over all i. We get

$$\begin{aligned}
&\sum_i \mathrm{tr}(\Psi, \underline{P}_{i-4})v^{-i} + \sum_i \mathrm{tr}(\Psi, \underline{P}_{i-6})v^{-i}\\
&\quad = \sum_i \mathrm{tr}(\Theta, \underline{N^M}_{i-4})v^{-i} + \sum_i \mathrm{tr}(\Theta, \underline{N^M}_{i-6})v^{-i} + \sum_i \mathrm{tr}(\Theta, \underline{N^M}_{i-2})v^{-i}\\
&\quad + \sum_i \mathrm{tr}(\Theta, \underline{N^M}_{i-4})v^{-i} + \sum_i \mathrm{tr}(\Theta', \underline{N'^M}_{i-2})v^{-i} - \sum_i \mathrm{tr}(\Theta', \underline{N'^M}_{i-4})v^{-i}\\
&\quad + \sum_i \mathrm{tr}(\Theta', \underline{N'^M}_i)v^{-i} - \sum_i \mathrm{tr}(\Theta', \underline{N'^M}_{i-2})v^{-i},
\end{aligned}$$

that is (using also 5.3(a)):

$$\begin{aligned}
&\epsilon^w(R.M.R, \phi')v^{-4} + \epsilon^w(R.M.R, \phi')v^{-6}\\
&\qquad = \epsilon^w(M, \phi)v^{-4} + \epsilon^w(M, \phi)v^{-6} + \epsilon^w(M, \phi)v^{-2} + \epsilon^w(M, \phi)v^{-4}\\
&\qquad + \epsilon^{sw}(M, \phi)v^{-2} - \epsilon^{sw}(M, \phi)v^{-4} + \epsilon^{sw}(M, \phi) - \epsilon^{sw}(M, \phi)v^{-2},
\end{aligned}$$

where ϕ' is as in 5.1. We divide both sides by $v^{-4} + v^{-6}$; we obtain

$$\epsilon^w(R.M.R, \phi') = \epsilon^w(M, \phi) + \epsilon^w(M, \phi)v^2 - \epsilon^{sw}(M, \phi)v^2 + \epsilon^{sw}(M, \phi)v^4.$$

This proves 5.2(d) and (equivalently 5.2(b)). Theorem 5.2 is proved.

6 Applications

6.1 Theorem 6.2 below describes the action of $T_s + 1 \in \mathbf{H}$ in the $\mathbf{H}$-module $\mathcal{M}$ (see 3.5, 3.6) for a fixed $s \in S$. We set

$$\mathbf{I}' = \{z \in \mathbf{I}, l(z) < l(sz)\}, \quad \mathbf{I}'' = \{z \in \mathbf{I}, l(z) > l(sz)\}.$$

$$\mathbf{I}_e = \{z \in \mathbf{I}; sz = zs^*\}, \quad \mathbf{I}_n = \{z \in \mathbf{I}; sz \neq zs^*\},$$

$$\mathbf{I}'_e = \mathbf{I}' \cap \mathbf{I}_e, \quad \mathbf{I}'_n = \mathbf{I}' \cap \mathbf{I}_n, \quad \mathbf{I}''_e = \mathbf{I}'' \cap \mathbf{I}_e, \quad \mathbf{I}''_n = \mathbf{I}'' \cap \mathbf{I}_n.$$

We denote by $w \mapsto \widetilde{w}$ the involution of $\mathbf{I}$ given by $w \mapsto sw$ if $w \in \mathbf{I}_e$ and $w \mapsto sws^*$ if $w \in \mathbf{I}_n$.

Theorem 6.2. *In the* $\mathbf{H}$*-module* $\mathcal{M}$ *the following identities hold for any* $z \in \mathbf{I}$*:*

$$(T_s + 1)a_z = \begin{cases} (u+1)(a_z + a_{\widetilde{z}}) & \text{if } z \in \mathbf{I}'_e, \\ (u^2 - u)(a_z + a_{\widetilde{z}}) & \text{if } z \in \mathbf{I}''_e, \\ a_z + a_{\widetilde{z}} & \text{if } z \in \mathbf{I}'_n, \\ u^2(a_z + a_{\widetilde{z}}) & \text{if } z \in \mathbf{I}''_n. \end{cases}$$

(Recall that $u = v^2$.) We define a map $\mathbf{I} \to \mathbf{I}'$, $z \mapsto \widehat{z}$ by $z \mapsto z$ if $z \in \mathbf{I}'$ and $z \mapsto \widetilde{z}$ if $z \in \mathbf{I}''$. For any $z \in \mathbf{I}$ we set

$$(T_s + 1)a_z = \sum_{y \in \mathbf{I}} c_{y,z} a_y$$

where $c_{y,z} \in \mathcal{A}'$. The following equality (for any $(M, \phi) \in \widetilde{C}_\sharp$) is a reformulation of Theorem 5.2:

$$\sum_{y \in \mathbf{I}'} \sum_{z \in \mathbf{I}} \epsilon^z(M, \phi) c_{y,z} a_y = \sum_{y \in \mathbf{I}'_n} (\epsilon^{\widetilde{y}}(M, \phi)v^4 + \epsilon^y(M, \phi))a_y$$
$$+ \sum_{y \in \mathbf{I}'_e} (\epsilon^{\widetilde{y}}(M, \phi)v^4 - \epsilon^{\widetilde{y}}(M, \phi)v^2 + \epsilon^y(M, \phi)v^2 + \epsilon^y(M, \phi))a_y.$$

Taking $(M, \phi) = (B_x, \phi_x)$ (see 3.2) we see that for any $x \in \mathbf{I}$, we have

$$\sum_{z \in \mathbf{I}} \epsilon^z(B_x, \phi_x) c_{y,z} = \begin{cases} \epsilon^{\widetilde{y}}(B_x, \phi_x)v^4 + \epsilon^y(B_x, \phi_x) & \text{if } y \in \mathbf{I}'_n \\ \epsilon^{\widetilde{y}}(B_x, \phi_x)(v^4 - v^2) + \epsilon^y(B_x, \phi_x)(v^2 + 1) & \text{if } y \in \mathbf{I}'_e. \end{cases}$$

Since the functions $z \mapsto [x \mapsto \epsilon^z(B_x, \phi_x)]$ from $\mathbf{I}$ to the set of maps $\mathbf{I} \to \mathcal{A}'$ are linearly independent (see 3.4, 3.5), we deduce that for $y \in \mathbf{I}'_n$, $z \in \mathbf{I}$ we have

$$c_{y,z} = v^4 \text{ if } y = \widetilde{z}; c_{y,z} = 1 \text{ if } y = z; c_{y,z} = 0 \text{ if } z \notin \{y, \widetilde{y}\};$$

and for $y \in \mathbf{I}'_e$, $z \in \mathbf{I}$ we have

$$c_{y,z} = v^4 - v^2 \text{ if } y = \widetilde{z}; c_{y,z} = v^2 + 1 \text{ if } y = z; c_{y,z} = 0 \text{ if } z \notin \{y, \widetilde{y}\}.$$

Thus for any $z \in \mathbf{I}$ we have

(a) $$(T_s + 1)a_z = r_z a_{\widehat{z}} + \sum_{y \in \mathbf{I}''} c_{y,z} a_y$$

where $r_z = v^4$ if $z \in \mathbf{I}''_n$, $r_z = 1$ if $z \in \mathbf{I}'_n$, $r_z = v^4 - v^2$ if $z \in \mathbf{I}''_e$, $r_z = v^2 + 1$ if $z \in \mathbf{I}'_e$.

We apply (T_s+1) to both sides of (a) and we use that $(T_s+1)^2 = (u^2+1)(T_s+1)$ in $\mathbf{H}$. We obtain

$$\begin{aligned}(u^2 + 1)r_z a_{\widehat{z}} &+ \sum_{y \in \mathbf{I}''} (u^2 + 1)c_{y,z} a_y \\ &= r_z r_{\widehat{z}} a_{\widehat{z}} + r_z \sum_{y \in \mathbf{I}''} c_{y,\widehat{z}} a_y + \sum_{y \in \mathbf{I}''} r_y c_{y,z} a_{\widehat{y}} + \sum_{y \in \mathbf{I}'', y' \in \mathbf{I}''} c_{y,z} c_{y',y} a_{y'}\end{aligned}$$

for any $z \in \mathbf{I}$. Taking the coefficients of a_y with $y \in \mathbf{I}'$ in the two sides of the last equality we obtain

$$(u^2 + 1)r_z \delta_{\widehat{z},y} = r_z r_{\widehat{z}} \delta_{\widehat{z},y} + r_{\widehat{y}} c_{\widehat{y},z}.$$

We see that if $y \in \mathbf{I}'$, then $c_{\widehat{y},z} = 0$ unless $y = \widehat{z}$ in which case we have $c_{\widehat{y},z} = r_{\widehat{y}}^{-1} r_z((u^2 + 1) - r_{\widehat{z}})$. The theorem follows. □

6.3 By Theorem 6.2, the $\mathbf{H}$-module $\mathcal{M}$ is identified with the $\mathbf{H}$-module denoted in [L3, §0.3] by $\underline{M}$ in such a way that to $a_y \in \mathcal{M}$ corresponds to $a_y \in \underline{M}$ in [L3]. The duality functor $M \mapsto D(M)$ [S, proof of Proposition 5.9] can be used to define a $\mathbf{Z}$-linear map $^-: \mathcal{K}_\sharp(\widetilde{C}) \to \mathcal{K}_\sharp(\widetilde{C})$ which satisfies

- $\overline{v^n \xi} = v^{-n}\overline{\xi}$ for $\xi \in \mathcal{K}_\sharp(\widetilde{C})$, $n \in \mathbf{Z}$,
- $\overline{[B_x, \phi_x]} = [B_x, \phi_x]$ for any $x \in \mathbf{I}$ and
- $\overline{u^{-1}(T_s + 1)\xi} = u^{-1}(T_s + 1)\overline{\xi}$ for any $s \in S$ and any $\xi \in \mathcal{K}_\sharp(\widetilde{C})$.

It follows that the operator $^-: \mathcal{K}_\sharp(\widetilde{C}) \to \mathcal{K}_\sharp(\widetilde{C})$ corresponds under the bijection χ' in 3.5(a), (b) to the operator $^-: \mathcal{M} \to \mathcal{M}$ given by [L3, Theorem 0.2]. It follows that for $x \in \mathbf{I}$, $\widetilde{A}_x = v^{-l(x)} \sum_{y \in \mathbf{I}; y \le w} \widetilde{P}_{y,x}(u) a_y \in \mathcal{M}$ (see 3.5) is fixed by the operator $^-: \mathcal{M} \to \mathcal{M}$ in [L3, Theorem 0.2] where $\widetilde{P}_{y,x}$ are as in 3.5(c). Using [L3, Theorem 0.4], it follows that for $x \in \mathbf{I}$ we have $\widetilde{A}_x = A_x$ (notation of [L3, Theorem 0.4]) and that for $y \in \mathbf{I}$, $y \le x$, $\widetilde{P}_{y,x}$ coincides with the polynomial $P^\sigma_{x,y}$

introduced in [L3, Theorem 0.4]. Using now 3.5(d), we see that for $y \in \mathbf{I}$, $y \leq x$ and $\delta \in \{1, -1\}$, the following holds:

(a) $$(P_{y,x}(u) + \delta P^{\sigma}_{y,x}(u))/2 \in \mathbf{N}[u].$$

This proves Conjecture 9.12 in [L3]. (In the case where W is a Weyl group this was already known from [LV].)

6.4 For $x, y \in W$ we have $c_x c_y = \sum_{z \in W} h_{x,y,z}(u) c_z$ where $h_{x,y,z}(u) \in \mathbf{N}[u, u^{-1}]$. Hence for $z, w \in W$, we have $c_z c_w c_{(z^*)^{-1}} = \sum_{w' \in W} \widetilde{h}_{z,w,w'}(u) c_w$ where

$$\widetilde{h}_{z,w,w'}(u) = \sum_{z' \in W} h_{z,w,z'}(u) h_{z',z^{*-1},w'}(u) = \sum_{z' \in W} h_{w,z^{*-1},z'}(u) h_{z,z',w'}(u).$$

For $z \in W$, $w \in \mathbf{I}$, we write

$$c_z A_w = \sum_{w' \in \mathbf{I}} b_{z,w,w'}(v) A_{w'}$$

where $b_{z,w,w'}(v) \in \mathcal{A}'$. For $z \in W$, $w, w' \in \mathbf{I}$ and $\delta \in \{1, -1\}$ the following holds:

(a) $$(\widetilde{h}_{z,w,w'}(u) + \delta b_{z,w,w'}(u))/2 \in \mathbf{N}[u, u^{-1}].$$

The proof is analogous to that of 6.3(a). (In the case where W is a Weyl group and $* = 1$ this was stated in [LV, §5.1].)

7 The H-module $\mathcal{M}_c$

7.1 Let $\leq_L, \leq_{LR}$ be the preorders on W defined as in [KL, after Theorem 1.3]; let $\sim_L, \sim_{LR}$ be the associated equivalence relations on W. In this section we fix an equivalence class c for $\sim_{LR}$ that is, a two-sided cell of W. For $w \in W$ we write $w \leq_{LR} c$ if $w \leq_{LR} w'$ for some $w' \in c$; we write $w <_{LR} c$ if $w \leq_{LR} c$ and $w \notin c$. Let $\mathcal{M}_{\leq c}$ (resp. $\mathcal{M}_{<c}$) be the $\mathcal{A}'$-submodule of $\mathcal{M}$ generated by the elements A_x with $x \in \mathbf{I}$ such that $x \leq_{LR} c$ (resp. $x <_{LR} c$). We show:

(a) *$\mathcal{M}_{\leq c}$ is an* **H**-*submodule of* $\mathcal{M}$.

With the notation in 6.4 it is enough to show that if $z \in W$ and $w, w' \in \mathbf{I}$ satisfy $b_{z,w,w'}(v) \neq 0$ and $w \leq_{LR} c$, then $w' \leq_{LR} c$. Using 6.4(a) we have $\widetilde{h}_{z,w,w'}(u) \neq 0$, hence $\sum_{z' \in W} h_{z,w,z'}(u) h_{z',z^{-1},w'}(u) \neq 0$. It follows that for some $z' \in W$, we have $h_{z,w,z'}(u) \neq 0$ and $h_{z',z^{-1},w'}(u) \neq 0$, hence $w' \leq_{LR} z' \leq_{LR} w$ and $w' \leq_{LR} w$ so that $w' \leq_{LR} c$, as required. □

A similar proof shows:

(b) *$\mathcal{M}_{<c}$ is an* **H**-*submodule of* $\mathcal{M}$.

We now define $\mathcal{M}_c = \mathcal{M}_{\leq c}/\mathcal{M}_{<c}$. From (a), (b) we see that $\mathcal{M}_c$ inherits an $\mathbf{H}$-module structure from $\mathcal{M}_{\leq c}$. For $x \in \mathbf{I} \cap c$ we denote the image of $A_x \in \mathcal{M}_{\leq c}$ in $\mathcal{M}_c$ again by A_x. Note that $\{A_x; x \in \mathbf{I} \cap c\}$ is an $\mathcal{A}'$-basis of $\mathcal{M}_c$.

7.2 In the remainder of this paper we assume that (W, l) satisfies the boundedness property in [L2, §13.2]. (This holds automatically when W is finite or an affine Weyl group, and it probably holds in general.) Then the function $\mathbf{a} : W \to \mathbf{N}$ is defined as in [L2, §13.6].

We recall the following properties:

(i) if z, z' in W satisfy $z \sim_{LR} z'$, then $\mathbf{a}(z) = \mathbf{a}(z')$;
(ii) if z, z' in W satisfy $z \leq_L z'$ and $\mathbf{a}(z) = \mathbf{a}(z')$, then $z \sim_L z'$.

(See [L2, Conjectures 14.2, P4, P9] and [L2, Ch.15]; the assumptions 15.1(a), (b) in [L2, Ch.15] are satisfied by [EW].)

In this subsection we fix $s \in S$. For $w \in W$ we set $\epsilon_w = (-1)^{l(w)}$. Let $y, w \in \mathbf{I}$. As in [LV, §4.1], [L3, §6.1], we set

(a) $$v^{-l(w)+l(y)} P^{\sigma}_{y,w}(v) = \delta_{y,w} + \mu'_{y,w} v^{-1} + \mu''_{y,w} v^{-2} \mod v^{-3}\mathbf{Z}[v^{-1}],$$

where $\mu'_{y,w} \in \mathbf{Z}$, $\mu''_{y,w} \in \mathbf{Z}$. (When $y \not\leq w$ we set $P^{\sigma}_{y,w} = 0$.)

As in [LV, §4.3], [L3, §6.2], for any $y, w \in \mathbf{I}$ such that $sy < y < sw > w$, $\epsilon_y = \epsilon_w$, we define $\mathcal{M}^s_{y,w} \in \mathcal{A}'$ by

$$\mathcal{M}^s_{y,w} = \mu''_{y,w} - \sum_{x \in \mathbf{I}; y<x<w, sx<x} \mu'_{y,x}\mu'_{x,w} - \delta_{sw,ws^*}\mu'_{y,sw} + \mu'_{sy,w}\delta_{sy,ys^*}.$$

We have the following result.

Let $w \in \mathbf{I} \cap c$. In the $\mathbf{H}$-module $\mathcal{M}_c$ we have the following identities:

(b) *if $sw < w$, then $c_s A_w = (u + u^{-1}) A_w$;*
(c) *if $sw > w$, then $c_s A_w = \Xi + \sum_{z \in \mathbf{I} \cap c; sz<z<sw, \epsilon_z=\epsilon_w} \mathcal{M}^s_{z,w} A_z$,*

where Ξ is given by

$$\Xi = \begin{cases} A_{sws^*} & \text{if } sw \neq ws^* > w \text{ and } sws^* \in c, \\ 0 & \text{otherwise.} \end{cases}$$

To prove (b), (c) we make use of the formula for $c_s A_w$ given in [LV, Theorem 4.4] (for Weyl groups) and [L3, Theorem 6.3] in the general case and show that all terms of that formula which involve $(v + v^{-1})$ belong to $\mathcal{M}_{<c}$ and can therefore be neglected. It is enough to prove the following statements:

(d) If $sw = ws^* > w$, then $sw <_{LR} c$.
(e) If $sw > w$ and $z \in \mathbf{I}$, $\epsilon_z = -\epsilon_w$, $sz < z < sw$, $\mu'_{z,w} \neq 0$, then $z <_{LR} c$.

We prove (d). Since $sw > w$ we have $sw \leq_L w$. If $sw \sim_L w$, then by [KL, (2.4)], for any $t \in S$ such that $(sw)t < sw$, we have $wt < w$; in particular, since $sws^* = w < sw$ we have $ws^* < w$, a contradiction. Thus, we have $sw \not\sim_L w$.

From $sw \leq_L w$, $sw \not\sim_L w$, we deduce that $sw \not\sim_{LR} w$. (If $sw \sim_{LR} w$, then $\mathbf{a}(sw) = \mathbf{a}(w)$, see (i); from $sw \leq_L w, \mathbf{a}(sw) = \mathbf{a}(w)$ we deduce $sw \sim_L w$ by (ii), a contradiction.) Now (d) follows.

We prove (e). Since $\mu'_{z,w} \neq 0$, the coefficient of $v^{l(w)-l(z)-1}$ in $P^{\sigma}_{z,w}(v)$ is $\neq 0$. Using 6.3(a) we deduce that the coefficient of $v^{l(w)-l(z)-1}$ in $P_{z,w}(v)$ is $\neq 0$. Since $sz < z < sw > w$, the last coefficient is known to be equal to $h_{s,w,z}$ (an integer), see [KL]. Thus we have $h_{s,w,z} \neq 0$ so that $z \leq_L w$. If $z \sim_L w$, then by [KL, (2.4)], for any $t \in S$ such that $zt < z$ we have $wt < w$; but from $sz < z$, $z \in \mathbf{I}$, we deduce $zs^* < z$, hence $ws^* < w$. From $ws^* < w$ and $w \in \mathbf{I}$ we deduce $sw < w$, a contradiction. Thus we have $z \not\sim_L w$. From $z \leq_L w$, $z \not\sim_L w$, we deduce that $z \not\sim_{LR} w$. (If $z \sim_{LR} w$, then $\mathbf{a}(z) = \mathbf{a}(w)$ by (i); from $z \leq_L w, \mathbf{a}(z) = \mathbf{a}(w)$ we deduce $z \sim_L w$ by (ii), a contradiction.) Now (e) follows.

This completes the proof of (b) and (c). □

7.3 For $\delta \in \{1, -1\}$ let $\mathcal{M}^{\delta}_c$ be the $\mathcal{A}'$-submodule of $\mathcal{M}_c$ generated by

$$\{A_x; x \in \mathbf{I} \cap c, \epsilon_x = \delta\}.$$

From 7.2(b), (c) we see that $\mathcal{M}^{\delta}_c$ is an $\mathbf{H}$-submodule of $\mathcal{M}$. Clearly, we have $\mathcal{M}_c = \mathcal{M}^1_c \oplus \mathcal{M}^{-1}_c$ as $\mathbf{H}$-modules.

7.4 The formulas 7.2(b), (c) for the action of c_s in the basis $\{A_x; x \in \mathbf{I} \cap c\}$ of $\mathcal{M}_c$ are similar to those in a W-graph (see [KL]) since the coefficients in the right-hand side of 7.2(c) are integer constants. (Unlike the case of W-graph these integer constants can in principle depend on s, although we do not know an example when they do.) Note that the action of left multiplication by c_s in the basis $\{A_x; x \in \mathbf{I}\}$ of $\mathcal{M}$ is not given by a W-graph, due to the appearence of terms involving $v + v^{-1}$.

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Comparing and characterizing some constructions of canonical bases from Coxeter systems

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Dedicated to David Vogan on the occasion of his 60th birthday

Abstract The Iwahori–Hecke algebra $\mathcal{H}$ of a Coxeter system (W, S) has a "standard basis" indexed by the elements of W and a "bar involution" given by a certain antilinear map. Together, these form an example of what Webster calls a pre-canonical structure, relative to which the well-known Kazhdan–Lusztig basis of $\mathcal{H}$ is a canonical basis. Lusztig and Vogan defined a representation of a modified Iwahori–Hecke algebra on the free $\mathbb{Z}[v, v^{-1}]$-module generated by the set of twisted involutions in W, and showed that this module has a unique pre-canonical structure compatible with the $\mathcal{H}$-module structure, which admits its own canonical basis which can be viewed as a generalization of the Kazhdan–Lusztig basis. One can modify the definition of Lusztig and Vogan's module to obtain other pre-canonical structures, each of which admits a unique canonical basis indexed by twisted involutions. We classify all of the pre-canonical structures which arise in this manner, and explain the relationships between their resulting canonical bases. Some of these canonical bases are equivalent in a trivial fashion to Lusztig and Vogan's construction, while others appear to be unrelated. Along the way, we also clarify the differences between Webster's notion of a canonical basis and the related concepts of an IC basis and a P-kernel.

Key words: Canonical bases, pre-canonical structures, twisted involutions, Kazhdan–Lusztig basis, Iwahori–Hecke algebras, Coxeter groups

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1 Introduction

Let (W, S) be a Coxeter system and write $\mathcal{H}$ for its associated Iwahori–Hecke algebra. This algebra has a "standard basis" indexed by the elements of W, whose structure constants have a simple inductive formula. The Kazhdan–Lusztig basis of $\mathcal{H}$ is the unique basis which is invariant under a certain antilinear map $\mathcal{H} \to \mathcal{H}$, referred to as the "bar involution," and whose elements are each unitriangular linear combinations of standard basis elements with respect to the Bruhat order. The standard basis and bar involution of $\mathcal{H}$ are an example of what Webster [22] calls a *pre-canonical structure*, relative to which the Kazhdan–Lusztig basis is a *canonical basis*. This terminology, whose precise definition we review in Section 2.1, is useful for organizing several similar constructions attached to Coxeter systems. Webster's idea of a canonical basis is closely related to Du's notion of an *IC basis* [4] and also to Stanley's notion of a *P-kernel* [19], and in Section 2.2 we discuss the relationship between these three concepts.

In [13, 14, 15], Lusztig and Vogan study a representation of a modified Iwahori–Hecke algebra $\mathcal{H}_2$ on the free $\mathbb{Z}[v, v^{-1}]$-module generated by the set of twisted involutions $\mathbf{I} = \mathbf{I}(W, S)$ in a Coxeter group. (See Section 2.4 for the definition of this set; though we mean something more general, in this introduction one can simply take $\mathbf{I} = \{w \in W : w^2 = 1\}$.) They show that this module has a unique pre-canonical structure which is compatible with the action of $\mathcal{H}_2$, and that this structure admits a canonical basis, of which the Kazhdan–Lusztig basis can be viewed as a special case.

The definition of Lusztig and Vogan's $\mathcal{H}_2$-representation has a particularly simple form, and gives an example of a *generic* $(\mathcal{H}_2, \mathbf{I})$*-structure* as defined in Section 3.4. It turns out that there are a number of slight modifications one can make to this definition which produce other $\mathcal{H}_2$-module structures on the free $\mathbb{Z}[v, v^{-1}]$-algebra generated by $\mathbf{I}$; some (but not all) of these modules likewise possess a unique pre-canonical structure compatible with the action of $\mathcal{H}_2$; in each such case there is a unique associated canonical basis. We review Lusztig and Vogan's results in Section 4.1, and derive from them a family of analogous theorems (along the lines just described) in Section 4.2. In Section 4.3 we present another variation of these results, in which the role of the modified Iwahori–Hecke algebra $\mathcal{H}_2$ is replaced by the usual algebra $\mathcal{H}$. These constructions give three canonical bases indexed by the twisted involutions in a Coxeter group; these bases all can be seen as generalizations of the Kazhdan–Lusztig basis of $\mathcal{H}$, but, somewhat unexpectedly, they do not appear to be related to each other in any simple way.

In Sections 3.3 and 3.4 we describe a precise sense in which these three bases account for all canonical bases on this space. Specifically, we define in Section 3.1 a category whose objects are pre-canonical structures on free $\mathbb{Z}[v, v^{-1}]$-modules. Our definition of morphisms in this category has the following appealing properties:

(i) Canonical bases arising from isomorphic pre-canonical structures are always related in a simple way; in particular, their coefficients (when written as sums of standard basis elements) are equal up to a change of sign or the variable substitution $v \mapsto -v$; see Corollary 3.9.

(ii) Assume the free $\mathbb{Z}[v, v^{-1}]$-module generated by W has a pre-canonical structure in which the natural basis W is standard. If this structure satisfies a natural compatibility condition with an $\mathcal{H}$-representation on the ambient space, then it is isomorphic to the pre-canonical structure on $\mathcal{H}$ itself, and so it has a unique canonical basis which can be identified in the sense of (i) with the Kazhdan–Lusztig basis; see Theorem 3.12.

With respect to these definitions, our main results are as follows. Suppose we are given a pre-canonical structure on the free $\mathbb{Z}[v, v^{-1}]$-module generated by the set of twisted involutions in W, in which the natural basis $\mathbf{I}$ is the standard one. We prove that

(1) If the structure is compatible with any representation of $\mathcal{H}$ of a certain natural form, then it is isomorphic to the pre-canonical structure we define in Section 4.3; see Theorem 3.16.
(2) If the structure is compatible (in a certain natural sense) with a representation of the modified Iwahori–Hecke algebra $\mathcal{H}_2$, then it is isomorphic to one of four pre-canonical structures: the one Lusztig and Vogan define in [13, 14], the one we define in Section 4.2, or one of two non-isomorphic structures derived from the one given in Section 4.3; see Theorem 3.20.

These results provide some formal justification for considering the pre-canonical structures described in Sections 4.1, 4.2, and 4.3 to be particularly natural objects. Lusztig and Vogan have given two interpretations of the first structure, in terms of the geometry of an associated algebraic group when W is a Weyl group [14] and in terms of the theory of Soergel bimodules for general W [15]. It remains an open problem to give similar interpretations of the two other pre-canonical structures.

2 Preliminaries

2.1 Canonical bases

Throughout we let $\mathcal{A} = \mathbb{Z}[v, v^{-1}]$ denote the ring of Laurent polynomials with integer coefficients in a single indeterminant. We write $f \mapsto \overline{f}$ for the ring involution of $\mathcal{A}$ with $v \mapsto v^{-1}$ and say that a map $\varphi : U \to V$ between $\mathcal{A}$-modules is *$\mathcal{A}$-antilinear* if $\varphi(fu) = \overline{f} \cdot \varphi(u)$ for $f \in \mathcal{A}$ and $u \in U$. Let V be a free $\mathcal{A}$-module.

Definition 2.1. A *(balanced) pre-canonical structure* on V consists of

- a "bar involution" ψ given by an $\mathcal{A}$-antilinear map $V \to V$ with $\psi^2 = 1$.
- a "standard basis" $\{a_c\}$ with partially ordered index set $(C, \leq)$ such that

$$\psi(a_c) \in a_c + \sum_{c' < c} \mathcal{A} \cdot a_{c'}.$$

This is equivalent to Webster's definition of a *balanced pre-canonical structure* [22, Definition 1.5]. In this work we will only consider pre-canonical structures which are balanced in this sense, and from this point on we drop the adjective "balanced" and just refer to "pre-canonical structures." The reader should note, however, that in [22] a pre-canonical structure refers to a slightly more general construction which includes Definition 2.1 as a special case.

Assume V has a pre-canonical structure $(\psi, \{a_c\})$; we then have this accompanying notion.

Definition 2.2. A set of vectors $\{b_c\}$ in V indexed by $(C, \leq)$ is a *canonical basis* if

(C1) each vector b_c in the basis is invariant under ψ.
(C2) each vector b_c in the basis is in the set $b_c = a_c + \sum_{c'<c} v^{-1}\mathbb{Z}[v^{-1}] \cdot a_{c'}$.

This definition of a canonical basis is slightly different from the one which Webster gives [22, Definition 1.7], but is equivalent when the pre-canonical structure on V is balanced (which we assume everywhere in this work) by [22, Lemma 1.8].

Example 2.3. We view the ring $\mathcal{A}$ itself as possessing the pre-canonical structure in which the bar involution is the map $f \mapsto \overline{f}$ and the standard basis is the singleton set $\{1\}$. This structure admits a canonical basis, which is again just $\{1\}$.

The following crucial property of a canonical basis appears in the introduction of [22]; its elementary proof is an instructive exercise.

Proposition 2.4 (Webster [22]). *A pre-canonical structure admits at most one canonical basis.*

It is usually difficult to describe elements of a canonical basis explicitly. However, one can often at least guarantee that a canonical basis exists. Continue to assume V is a free $\mathcal{A}$-module with a pre-canonical structure $(\psi, \{a_c\})$ whose standard basis is indexed by $(C, \leq)$.

Theorem 2.5 (Du [4]). *If all lower intervals $(-\infty, x] = \{c \in C : c \leq x\}$ in the partially ordered index set $(C, \leq)$ are finite, then the pre-canonical structure on V admits a canonical basis.*

Proof. The result is equivalent to [4, Theorem 1.2 and Remark 1.2.1(1)]. One can also adapt the argument Lusztig gives in [13, Section 4.9], which proves the existence of a canonical basis in one particular pre-canonical structure but makes sense in greater generality. □

Webster lists several examples of pre-canonical structures from representation theory in the introduction of [22]. Pre-canonical structures, such as in these examples, arise naturally from graded categorifications, by which we broadly mean isomorphisms

$$V \xrightarrow{\sim} [\mathscr{C}] \tag{2.1}$$

where $\mathscr{C}$ is an additive category with $\mathbb{Z}$-graded objects, and $[\mathscr{C}]$ denotes its split Grothendieck group: this is the $\mathcal{A}$-module generated by the symbols $[C]$ for objects $C \in \mathscr{C}$, subject to the relations $[A]+[B] = [C]$ whenever $A \oplus B \cong C$ and $v^n[C] = [C(n)]$ where $C(n)$ is the object $C \in \mathscr{C}$ with its grading shifted down by n. The bar involution of a pre-canonical structure on V should then correspond via (2.1) to a duality functor on $\mathscr{C}$, and elements of the standard basis should arise as some set of easily located objects in $\mathscr{C}$, each of which contains a unique indecomposable summand not found in smaller objects. A canonical basis in turn should correspond to a representative set of indecomposable objects which are self-dual with respect to some choice of grading shift.

Example 2.6. The pre-canonical structure on $V = \mathcal{A}$ comes from the categorification taking $\mathscr{C}$ to be the category of finitely generated $\mathbb{Z}$-graded free R-modules (with R any commutative ring), with morphisms given by grading preserving R-linear maps. For this category, there is a unique ring isomorphism $\mathcal{A} \xrightarrow{\sim} [\mathscr{C}]$ identifying $1 \in \mathcal{A}$ with $[\mathbb{1}] \in [\mathscr{C}]$, where $\mathbb{1}$ denotes the graded R-module whose nth component is R when $n = 0$ and is 0 otherwise. The bar involution $f \mapsto \overline{f}$ on $\mathcal{A}$ is the decategorification of the duality functor $M \mapsto \mathrm{Hom}(M, \mathbb{1})$ where $\mathrm{Hom}(M, \mathbb{1})$ denotes the graded R-module whose nth component is the set of grading preserving R-linear maps $M \to \mathbb{1}(n)$.

In general, confronted with some natural pre-canonical structure, it is an interesting problem (which in the present work we do not address) to identify a categorification which can explain the existence and special properties of an associated canonical basis.

2.2 Comparison with IC bases and P-kernels

Webster's definition of canonical bases is similar to two concepts appearing earlier in the literature: *IC bases* as formalized by Du in [4] and *P-kernels* as introduced by Stanley in [19]. We review this terminology here, and explain how one may view canonical bases as special cases of IC bases, and P-kernels as special cases of pre-canonical structures. We remind the reader that for us, all pre-canonical structures (as specified by Definition 2.1) are what Webster [22, Definition 1.5] calls *balanced pre-canonical structures.*

To begin, we recall the following definition from [4], studied elsewhere, for example, in [3, 6].

Definition 2.7. Let V be a free $\mathcal{A}$-module with

- a "bar involution" ψ given by an $\mathcal{A}$-antilinear map $V \to V$ with $\psi^2 = 1$.
- a "standard basis" $\{a_c\}$ with index set C.

A set of vectors $\{b_c\}$ of V is an *IC basis* relative to $(\psi, \{a_c\})$ if it is the *unique* basis such that

$$\psi(b_c) = b_c \qquad \text{and} \qquad b_c \in a_c + \sum_{c' \in C} v^{-1}\mathbb{Z}[v^{-1}] \cdot a_{c'} \qquad \text{for each } c \in C.$$

Remark 2.8. In [3, 4, 6], this definition is formulated slightly differently. There, one begins with a bar involution ψ, a basis $\{m_c\}_{c\in C}$ of V, and a function $r : C \to \mathbb{Z}$. An IC basis of V is then defined exactly as above relative to ψ and the standard basis $\{a_c\}$ given by setting $a_c = v^{-r(c)}m_c$. One passes to our definition by assuming $r = 0$; there is clearly no loss of generality in this reduction.

The initial data in the definition of an IC basis is more general than a pre-canonical structure in two aspects: there is no condition on the action of the bar involution on the standard basis, and the index set C is no longer required to be partially ordered. When the initial data $(\psi, \{a_c\})$ is a pre-canonical structure, the notions of a canonical basis and an IC basis are equivalent:

Proposition 2.9. *Let V be a free $\mathcal{A}$-module with a pre-canonical structure $(\psi, \{a_c\})$. Relative to $(\psi, \{a_c\})$, a set of vectors $\{b_c\}$ in V is a canonical basis if and only if it is an IC basis.*

Proof. Suppose $\{b_c\}$ is an IC basis relative to $(\psi, \{a_c\})$. Let $f_{x,y} \in v^{-1}\mathbb{Z}[v^{-1}]$ for $x, y \in C$ be the polynomials such that $b_y = a_y + \sum_{x\in C} f_{x,y}a_x$. To show that $\{b_c\}$ is a canonical basis, we must check that $f_{x,y} = 0$ whenever $x \not< y$. This follows since if $y \in C$ is fixed and $x \in C$ is maximal among all elements $x \not< y$ with $f_{x,y} \neq 0$, then the equality $b_y = \psi(b_y)$ together with the unitriangular formula for ψ implies that $f_{x,y} = \overline{f_{x,y}}$, which is impossible for a nonzero element of $v^{-1}\mathbb{Z}[v^{-1}]$.

Now suppose conversely that $\{b_c\}$ is a canonical basis. This basis automatically has both desired properties of an IC basis, so it remains only to show that it is the unique basis with these properties. This follows from Proposition 2.4, since the argument in the previous paragraph shows that any other basis $\{b'_c\}$ with the desired properties of an IC basis is a canonical basis. □

Stanley first introduced in [19] the concept of a P-kernel for any locally finite poset P, which Brenti studied subseqently in [2, 3]. To define P-kernels we must review some terminology for partially ordered sets; [20, Chapter 3] serves as the standard reference for this material.

Let P be a partially ordered set (i.e., a poset) and let $\mathrm{Int}(P) = \{(x, y) \in P^2 : x \le y\}$. Assume the poset P is *locally finite*, i.e., that $\{t \in P : x \le t \le y\}$ is finite for all $x, y \in P$. Let R be a commutative ring and let q be an indeterminate. The *incidence algebra* $I(P; R[q])$ is the set of functions $f : \mathrm{Int}(P) \to R[q]$, with sums and scalar multiplication given pointwise and products given by

$$(fg)(x, y) = \sum_{x\le t\le y} f(x,t)g(t, y) \qquad \text{for } f, g : \mathrm{Int}(P) \to R[q].$$

This algebra has a unit given by the function $\delta_P : \mathrm{Int}(P) \to R[q]$ with $\delta_P(x, y) = \delta_{x,y}$ for $x, y \in P$. A function $f : \mathrm{Int}(P) \to R[q]$ is invertible if and only if $f(x, x)$ is a unit in $R[q]$ for all $x \in P$. We adopt the convention of setting $f(x, y) = 0$ whenever $f : \mathrm{Int}(P) \to R[q]$ and $x, y \in P$ are elements such that $x \not\leq y$.

Finally, let $r : P \to \mathbb{Z}$ be a function such that $r(x) < r(y)$ if $x < y$, and define $r(x, y) = r(y) - r(x)$ for $x, y \in P$. Relative to the initial data (P, R, q, r), we have the following definition, which can be found as [19, Definition 6.2] or in [3, Section 2].

Definition 2.10. An element $K \in I(P; R[q])$ is a P*-kernel* if

(1) $K(x, x) = 1$ for all $x \in P$.
(2) There exists an invertible $f \in I(P; R[q])$ such that

$$(Kf)(x, y) = q^{r(x,y)}\overline{f(x, y)} \qquad \text{for } x, y \in P.$$

An invertible element $f \in I(P)$ satisfying (2) is called K*-totally acceptable*.

Brenti proves the following result as [2, Theorem 6.2]. This statement strengthens an earlier result [19, Corollary 6.7] due to Stanley.

Theorem 2.11 (Brenti [2]). *Suppose $K \in I(P; R[q])$ is a P-kernel. If P is locally finite, then there exists a unique K-totally acceptable element $\gamma \in I(P; R[q])$ such that*

$$\gamma(x, x) = 1 \qquad \textit{and} \qquad \deg_q(\gamma(x, y)) < \tfrac{1}{2} r(x, y)$$

for all $x, y \in P$ with $x < y$. Call γ the KLS-function *of K. (Here "KLS" abbreviates "Kazhdan–Lusztig–Stanley.")*

Returning to our earlier convention, we let C be an index set with a partial order $\leq$. Assume the hypothesis of Theorem 2.5 (i.e., that all lower intervals in C are finite) and let V be the free $\mathcal{A}$-module with a basis given by the symbols a_c for $c \in C$. To translate the language of P-kernels into pre-canonical structures, assume $P = (C, \leq)$ and $R = \mathbb{Z}$ and $q = v^2$. Given a P-kernel K, we may then define $\psi_K : V \to V$ as the $\mathcal{A}$-antilinear map with

$$\psi_K(a_y) = \sum_{x \in C} v^{r(x,y)} \cdot \overline{K(x, y)} \cdot a_x \qquad \text{for } y \in C.$$

Note that our assumption that C has finite lower intervals ensures that the sum on the right side of this formula is well-defined.

In [2], Brenti proves that P-kernels are equivalent to IC bases of a special form. It turns out that this special form is essentially the requirement that the initial data $(\psi, \{a_c\})$ of an IC basis form a pre-canonical structure. Brenti's results thus translate via Proposition 2.9 into the following statement relating P-kernels and canonical bases.

Theorem 2.12 (Brenti [2]). *Assume $P = (C, \leq)$ and $R = \mathbb{Z}$ and $q = v^2$.*

(a) The map $K \mapsto \psi_K$ is a bijection from the set of P-kernels to the set of maps ψ such that $(\psi, \{a_c\})$ is a pre-canonical structure on V with the property that

$$\psi(a_y) \in \mathbb{Z}[v^{-2}]\text{-span}\{v^{r(x,y)}a_x : x \in C\} \qquad \text{for } y \in C.$$

(b) If γ is the KLS-function of a P-kernel K and $\{b_c\}$ is the canonical basis of V relative to the pre-canonical structure $(\psi_K, \{a_c\})$, then

$$b_y = \sum_{x \in C} v^{-r(x,y)} \cdot \gamma(x,y) \cdot a_x \qquad \text{for } y \in C.$$

Remark 2.13. Note that part (b) is only a meaningful statement if the KLS-function γ and the canonical basis $\{b_c\}$ both exist, but this follows from Theorems 2.5 and 2.11 since we assume all lower intervals in $P = (C, \leq)$ are finite. Observe that since $\gamma(x,y) \in \mathbb{Z}[v^2]$ for all $x, y \in C$, this result shows that not all canonical bases correspond to KLS-functions of P-kernels.

Proof. The definition of ψ_K makes sense for any $K \in I(P, R[q])$, and part (a) is equivalent to the statement that $(\psi_K, \{a_c\})$ is a pre-canonical structure if and only if K is a P-kernel. Clearly $(\psi_K, \{a_c\})$ is a pre-canonical structure if and only if $K(x,x) = 1$ for all $x \in P$ and $\psi_K^2 = 1$. The assertion that these two properties hold if and only if K is a P-kernel is precisely [2, Proposition 3.1], since the map ι defined in part (ii) of that result is just $\psi_{K^{-1}}$ (with $m_c = v^{r(c)}a_c$). Part (b) is equivalent to [2, Theorem 3.2] by Proposition 2.9. □

2.3 Pre-canonical module structures

In this short section we introduce a useful variant of Definition 2.1. Suppose $\mathcal{B}$ is an $\mathcal{A}$-algebra with a pre-canonical structure; write $\overline{b}$ for the image of $b \in \mathcal{B}$ under the corresponding bar involution. (For us, all algebras are unital and associative.) Let V be a $\mathcal{B}$-module which is free as an $\mathcal{A}$-module.

Definition 2.14. A *pre-canonical $\mathcal{B}$-module structure* on V is a pre-canonical structure whose bar involution $\psi : V \to V$ commutes with the bar involution of $\mathcal{B}$ in the sense that

$$\psi(bx) = \overline{b} \cdot \psi(x) \qquad \text{for all } b \in \mathcal{B} \text{ and } x \in V.$$

Observe that a pre-canonical structure is thus the same thing as a pre-canonical $\mathcal{A}$-module structure. The additional compatibility condition satisfied by a pre-canonical $\mathcal{B}$-module structure can be useful for proving uniqueness statements. In particular, we have the following lemma.

Lemma 2.15. *Suppose V has a basis $\{a_c\}$ with partially ordered index set $(C, \leq)$. If V is generated as a $\mathcal{B}$-module by the minimal elements of the basis $\{a_c\}$, then there exists at most one pre-canonical $\mathcal{B}$-module structure on V in which $\{a_c\}$ serves as the "standard basis."*

Proof. Suppose ψ and ψ' are two $\mathcal{A}$-antilinear maps $V \to V$ which, together with $\{a_c\}$, give V a pre-canonical $\mathcal{B}$-module structure. Let $U \subset V$ be the set of elements on which ψ and ψ' agree. Then U is a $\mathcal{B}$-submodule which contains the minimal elements of the basis $\{a_c\}$. Since these elements generate V, we have $U = V$, so $\psi = \psi'$. □

2.4 Twisted involutions

We review here the definition of the set of twisted involutions attached to a Coxeter system. This set has many interesting combinatorial properties; see [7, 8, 9, 10, 11, 18]. Let (W, S) be any Coxeter system. Write $\ell : W \to \mathbb{N}$ for the associated length function and $\leq$ for the Bruhat order. We denote by $\mathrm{Aut}(W, S)$ the group of automorphisms $\theta : W \to W$ such that $\theta(S) = S$, and define

$$W^+ = \{(x, \theta) : x \in W \text{ and } \theta \in \mathrm{Aut}(W, S)\}.$$

We extend the length function and Bruhat order to W^+ by setting $\ell(x, \theta) = \ell(x)$ and by setting $(x, \theta) \leq (x', \theta')$ if and only if $\theta = \theta'$ and $x \leq x'$. The set W^+ has the structure of a group, in which multiplication of elements is given by

$$(x, \alpha)(y, \beta) = (x \cdot \alpha(y), \alpha\beta).$$

We view $W \subset W^+$ as a subgroup by identifying $x \in W$ with the pair $(x, 1)$. Likewise, we view $\mathrm{Aut}(W, S) \subset W^+$ as a subgroup by identifying $\theta \in \mathrm{Aut}(W, S)$ with the pair $(1, \theta)$. With respect to these inclusions, W^+ is a semidirect product $W \rtimes \mathrm{Aut}(W, S)$.

Definition 2.16. The set of *twisted involutions* of a Coxeter system (W, S) is

$$\mathbf{I} = \mathbf{I}(W, S) = \{w \in W^+ : w = w^{-1}\}.$$

A pair $(x, \theta) \in W^+$ belongs to $\mathbf{I}$ if and only if $\theta = \theta^{-1}$ and $\theta(x) = x^{-1}$. In this situation, often in the literature the element $x \in W$ is referred to as a twisted involution, relative to the automorphism θ. We have defined twisted involutions slightly more generally as ordinary involutions of the extended group W^+, since all of the results we will state are true relative to any choice of automorphism θ.

If $s \in S$ and $w = (x, \theta) \in \mathbf{I}$, then $sws = (s \cdot x \cdot \theta(s), \theta)$ is also a twisted involution. The latter may be equal to w; in particular, $sws = w$ if and only if $sw = ws$, in which case $sw \in \mathbf{I}$. Let $s \ltimes w$ denote whichever of sws or sw is in $\mathbf{I} \setminus \{w\}$; i.e., define

$$s \ltimes w = \begin{cases} sws & \text{if } sw \neq ws \\ sw & \text{if } sw = ws \end{cases} \qquad \text{for } s \in S \text{ and } w \in \mathbf{I}. \tag{2.2}$$

While $s \ltimes (s \ltimes w) = w$, the operation $\ltimes$ does not extend to an action of W of $\mathbf{I}$.

The restriction of the Bruhat order on W to $\mathbf{I}$ forms a poset with many special properties. Concerning this, we will just need the following result, which rephrases [7, Theorem 4.8].

Theorem 2.17 (Hultman [7]). *The poset* $(\mathbf{I}, \leq)$ *is graded, and its rank function* $\rho : \mathbf{I} \to \mathbb{N}$ *satisfies*

$$\rho(s \ltimes w) = \rho(w) - 1 \qquad \Leftrightarrow \qquad \ell(s \ltimes w) < \ell(w) \qquad \Leftrightarrow \qquad \ell(sw) = \ell(w) - 1$$

for all $s \in S$ *and* $w \in \mathbf{I}$.

We reserve the notation ρ in all later sections to denote the rank function of $(\mathbf{I}, \leq)$. Note that $\rho(1) = 0$, and so one can compute $\rho(w)$ inductively using the equivalent identities in the theorem. As with ℓ, there are explicit formulas for ρ when W is a classical Weyl group; see [10, 11].

2.5 Kazhdan–Lusztig basis

In this final preliminary section we recall briefly the definition of the Kazhdan–Lusztig basis of the Iwahori–Hecke algebra of a Coxeter system. As references for this material, we mention [1, 12, 21]. Continue to let (W, S) be a Coxeter system with length function $\ell : W \to \mathbb{N}$ and Bruhat order $\leq$. We write $\mathcal{H} = \mathcal{H}(W, S)$ to denote the free $\mathcal{A}$-module with a basis given the symbols H_w for $w \in W$. There is a unique $\mathcal{A}$-algebra structure on $\mathcal{H}$ such that

$$H_s H_w = \begin{cases} H_{sw} & \text{if } sw > w \\ H_{sw} + (v - v^{-1}) H_w & \text{if } sw < w \end{cases} \qquad \text{for } s \in S \text{ and } w \in W.$$

The *Iwahori–Hecke algebra* of (W, S) is $\mathcal{H}$ equipped with this structure.

The unit of $\mathcal{H}$ is the basis element H_1, which often we write as 1 or simply omit. Observe that $H_s^{-1} = H_s + (v^{-1} - v)$ and that $H_w = H_{s_1} \cdots H_{s_k}$ whenever $w = s_1 \cdots s_k$ is a reduced expression. Hence every basis element H_w for $w \in W$ is invertible. We denote by $H \mapsto \overline{H}$ the $\mathcal{A}$-antilinear map $\mathcal{H} \to \mathcal{H}$ with $\overline{H_w} = (H_{w^{-1}})^{-1}$ for $w \in W$. One checks that this map is a ring involution, and we have the following result from Kazhdan and Lusztig's seminal work [12].

Theorem 2.18 (Kazhdan and Lusztig [12]). *Define*

- *the "bar involution" of* $\mathcal{H}$ *to be the map* $H \mapsto \overline{H}$.
- *the "standard basis" of* $\mathcal{H}$ *to be* $\{H_w\}$ *with partially ordered index set* $(W, \leq)$.

This is a pre-canonical structure on $\mathcal{H}$ *and it admits a canonical basis* $\{\underline{H}_w\}$.

The canonical basis $\{\underline{H}_w\}$ is the *Kazhdan–Lusztig basis* of $\mathcal{H}$. It is a simple exercise to show for $s \in S$ that $\underline{H}_s = H_s + v^{-1}$. Define $h_{y,w} \in \mathbb{Z}[v^{-1}]$ for $y, w \in W$ such that $\underline{H}_w = \sum_{y \in W} h_{y,w} H_y$. We note the following well-known property of these polynomials.

Proposition 2.19 (Kazhdan and Lusztig [12]). *If $y \leq w$, then*

$$v^{\ell(w)-\ell(y)}h_{y,w} \in 1 + v^2\mathbb{Z}[v^2].$$

Remark 2.20. Define $q = v^2$ and $P_{y,w} = v^{\ell(w)-\ell(y)}h_{y,w}$ for $y, w \in W$. The polynomials $P_{y,w} \in \mathbb{Z}[q]$ are usually called the *Kazhdan–Lusztig polynomials* of the Coxeter system (W, S).

The Kazhdan–Lusztig basis has several remarkable positivity properties; for example, it is now known from work of Elias and Williamson [5] that for all $x, y \in W$ one has $h_{x,y} \in \mathbb{N}[v^{-1}]$ and $\underline{H}_x\underline{H}_y \in \mathbb{N}[v, v^{-1}]\text{-span}\{\underline{H}_z : z \in W\}$. Available proofs of such phenomena make extensive use of the interpretation of the Iwahori–Hecke algebra $\mathcal{H}$ as the split Grothendieck of an appropriate category (in [5], the category of Soergel bimodules). This is an important motivation for the problem of constructing categorifications which give rise to pre-canonical structures of interest.

3 Characterizations

The results of Kazhdan and Lusztig in the previous section give us a canonical basis for the free $\mathcal{A}$-module generated by any Coxeter group W. In turn, recent results of Lusztig and Vogan [13, 14, 15] construct a canonical basis of the free $\mathcal{A}$-module generated by the set of twisted involutions in W. In this section our goal, broadly speaking, is to characterize the ways one can modify such constructions to get other canonical bases, and to explain how such bases differ from each other.

3.1 Morphisms for pre-canonical structures

To this end, our first task is to describe what it means for two pre-canonical structures to be the same. This amounts to defining what should comprise a morphism between pre-canonical structures on free $\mathcal{A}$-modules. In this pursuit we are guided by the principle that if a morphism exists from one pre-canonical structure to another, and if the first structure admits a canonical basis, then the second structure should admit a canonical basis which can be described explicitly in terms of the first basis.

The following is a natural but rigid notion of (iso)morphism compatible with this philosophy. Suppose V and V' are free $\mathcal{A}$-modules with respective pre-canonical structures $(\psi, \{a_c\})$ and $(\psi', \{a'_c\})$. We say that an $\mathcal{A}$-linear map $\varphi : V \to V'$ is a *strong isomorphism* of pre-canonical structures if φ restricts to an order-preserving bijection $\{a_c\} \to \{a'_c\}$ between standard bases and φ commutes with bar involutions in the sense that $\varphi \circ \psi = \psi' \circ \varphi$. Under these conditions, φ is necessarily invertible as an $\mathcal{A}$-linear map. The inverse and composition of strong isomorphisms of pre-canonical structures are again strong isomorphisms of pre-canonical structures.

Moreover, if $\varphi : V \to V'$ is a strong isomorphism of pre-canonical structures and V admits a canonical basis $\{b_c\}$, then $\{\varphi(b_c)\}$ is a canonical basis of V'.

There are other situations in which we would like to consider two pre-canonical structures to be "the same" besides when they are strongly isomorphic. We illustrate this as follows. Continue to let V be a free $\mathcal{A}$-module with a pre-canonical structure $(\psi, \{a_c\})$ whose standard basis is indexed by $(C, \leq)$. Suppose for each index $c \in C$ we have an element $d_c \in \mathcal{A}$. Let $u_c = d_c a_c$ and consider the set of rescaled basis elements $\{u_c\}$, likewise indexed by $(C, \leq)$. These elements are linearly independent if and only if each $d_c \neq 0$, so assume this condition holds and define $U = \mathcal{A}\text{-span}\{u_c : c \in C\}$. One naturally asks when $(\psi, \{u_c\})$ is a pre-canonical structure on the submodule $U \subset V$. Since we have

$$\psi(u_c) \in \frac{\overline{d_c}}{d_c} \cdot u_c + \sum_{c' < c} \mathcal{A} \cdot \frac{\overline{d_c}}{d_{c'}} \cdot u_{c'}$$

it follows that $(\psi, \{u_c\})$ is a pre-canonical structure on U at least when (i) each $d_c = \overline{d_c}$ and (ii) $d_c = q_{c',c} d_{c'}$ for some $q_{c',c} \in \mathcal{A}$ whenever $c' < c$ in C. Moreover, the first of these sufficient conditions is also necessary. Note that if (i) and (ii) hold, then $q_{c',c} = \overline{q_{c',c}}$ and so $q_{c',c} \in \mathbb{Z}[v + v^{-1}]$ since $\mathbb{Z}[v + v^{-1}]$ is the set of bar invariant elements of $\mathcal{A}$.

Assume conditions (i) and (ii) hold and further that V admits a canonical basis $\{b_c\}$ with respect to the pre-canonical structure $(\psi, \{a_c\})$. If U also has a canonical basis, then one asks how it is related to the basis $\{a_c\}$; in particular, when does some rescaling of $\{b_c\}$ give a canonical basis for U? By condition (C2) in Definition 2.2, it follows that the only possible such basis would be given by $\{d_c b_c\}$. Since

$$d_c b_c \in u_c + \sum_{c' < c} v^{-1}\mathbb{Z}[v^{-1}] \cdot q_{c',c} \cdot u_{c'},$$

it follows that $\{d_c b_c\}$ is a canonical basis for U at least when $q_{c',c} \in \mathbb{Z}[v^{-1}]$. Since $\mathbb{Z} = \mathbb{Z}[v^{-1}] \cap \mathbb{Z}[v + v^{-1}]$, we may summarize this discussion with the following lemma.

Lemma 3.1. *For each index $c \in C$ let $d_c \in \mathcal{A}$ and define*

$$u_c = d_c a_c \qquad \textit{and} \qquad U = \mathcal{A}\text{-span}\{u_c : c \in C\}.$$

Suppose the following conditions hold:

(i) $d_c \in \mathbb{Z}[v + v^{-1}]$ and $d_c \neq 0$ for all $c \in C$.
(ii) $d_c / d_{c'} \in \mathbb{Z}$ whenever $c' < c$.

Then $(\psi, \{u_c\})$ is a pre-canonical structure on U. If $\{b_c\}$ is a canonical basis of V, then $\{d_c b_c\}$ is a canonical basis of U.

Morphisms between pre-canonical structures should at least include strong isomorphisms and also the $\mathcal{A}$-linear maps $D : V \to V'$ given by $D(a_c) = d_c a_c$ when the conditions hold in the preceding proposition. There is a third kind of map which should form a morphism; in particular, it is natural to consider the map Φ given by (3.1) to be a morphism between the pre-canonical structures on $\mathcal{H}$ and $\mathcal{H}_2$, as we will see in the following lemma.

Let ϵ be a ring endomorphism of $\mathcal{A}$. Such a map is $\mathbb{Z}$-linear and completely determined by its value at $v \in \mathcal{A}$, which must be a unit, since $\epsilon(v)\epsilon(v^{-1}) = \epsilon(vv^{-1}) = \epsilon(1) = 1$. It follows that $\epsilon(v) = \pm v^n$ for some $n \in \mathbb{Z}$. Call n the *degree* of the endomorphism ϵ. We say that a map $\varphi : M \to N$ between $\mathcal{A}$-modules is ϵ*-linear* if $\varphi(fm) = \epsilon(f)\varphi(m)$ for $f \in \mathcal{A}$ and $m \in M$.

Lemma 3.2. *Let ϵ be a ring endomorphism of $\mathcal{A}$ and write $\tau : V \to V$ and $\phi : V \to V$ for the respective ϵ-linear and $\mathcal{A}$-antilinear maps with*

$$\tau(a_c) = a_c \qquad \text{and} \qquad \phi(a_c) = \tau \circ \psi(a_c) \qquad \text{for } c \in C.$$

Then $(\phi, \{a_c\})$ is another pre-canonical structure on V. If $\{b_c\}$ is a canonical basis of V relative to $\{\psi, \{a_c\})$ and ϵ has positive degree, then $\{\tau(b_c)\}$ is a canonical basis of V relative to $(\phi, \{a_c\})$.

Proof. That $(\phi, \{a_c\})$ is a pre-canonical structure is clear from the definitions, and checking that $\{\tau(b_c)\}$ is a canonical basis relative to this structure is straightforward. □

Motivated by the preceding lemmas, we adopt the following definition. Let V and V' be free $\mathcal{A}$-modules with pre-canonical structures $(\psi, \{a_c\})$ and $(\psi', \{a'_c\})$. Assume the standard bases $\{a_c\}$ and $\{a'_c\}$ have the same partially ordered index set $(C, \leq)$.

Definition 3.3. A map $\varphi : V \to V'$ is a *morphism* of pre-canonical structures if

(i) The map φ is ϵ-linear for a positive degree ring endomorphism $\epsilon : \mathcal{A} \to \mathcal{A}$.
(ii) There are nonzero polynomials $d_c \in \mathcal{A}$ for $c \in C$ with $d_c/d_{c'} \in \mathbb{Z}$ whenever $c' < c$, such that if $D : V \to V$ is the $\mathcal{A}$-linear map with $D(a_c) = d_c a_c$ for $c \in C$, then $\psi' \circ \varphi = \varphi \circ \psi^D$, where we define $\psi^D = D^{-1} \circ \psi \circ D$.

Remark 3.4. The polynomials d_c in condition (ii) belong to $\mathbb{Z}[v + v^{-1}]$ since the coefficients of a_c in $\varphi^{-1} \circ \psi' \circ \varphi(a_c)$ and in $\psi^D(a_c)$, which must be equal, are 1 and $\overline{d_c}/d_c$ respectively. This observation and the fact that $d_c/d_{c'} \in \mathbb{Z}$ whenever $c' < c$ in C ensure that ψ^D is a well-defined map $V \to V$, even though D^{-1} may not be.

If $\varphi : V \to V'$ is a morphism of pre-canonical structures, then we call a map $D : V \to V$ of the form in condition (ii) of Definition 3.3 a *scaling factor* of φ. If $V' \subset V$ and φ is equal to one of its scaling factors, then we call φ a *scaling morphism*. We define the *degree* of any morphism φ to be the degree of the ring endomorphism ϵ in condition (i). If $V = V'$ and $\{a_c\} = \{a'_c\}$ and the identity is a scaling factor of φ, then we call φ a *parametric morphism*.

In the rest of this section we describe some properties of morphisms in this sense. We fix some notation. Let V and V' and V'' be free $\mathcal{A}$-modules with pre-canonical structures $(\{a_c\}, \psi)$ and $(\{a'_c\}, \psi')$ and $(\{a''_c\}, \psi'')$. Assume the standard bases of these structures all have the same partially ordered index set $(C, \leq)$, and suppose $\varphi : V \to V'$ and $\varphi' : V' \to V''$ are morphisms of pre-canonical structures.

Proposition 3.5. *The composition $V \xrightarrow{\varphi} V' \xrightarrow{\varphi'} V''$ is a morphism of pre-canonical structures. The collection of pre-canonical structures on free $\mathcal{A}$-modules forms a category.*

The proposition follows in an elementary way from the definitions; we omit its proof.

Proposition 3.6. *Every morphism of pre-canonical structures is equal to some composition $\iota \circ \sigma \circ \tau$ where ι is a strong isomorphism, σ is a scaling morphism, and τ is a parametric morphism.*

Proof. Let ϵ be the $\mathcal{A}$-endomorphism of positive degree such that φ is ϵ-linear. Define $\tau : V \to V$ and $\phi : V \to V$, relative to $(\psi, \{a_c\})$ and ϵ, as in Lemma 3.2. Then $(\psi, \{a_c\})$ and $(\phi, \{a_c\})$ are both pre-canonical structures on V and $\tau : V \to V$ is a parametric morphism from the first to the second.

Let D be a scaling factor of φ so that $D(a_c) = d_c a_c$ for some $d_c \in \mathbb{Z}[v + v^{-1}]$ for each $c \in C$. Let $d'_c = \epsilon(d_c)$ and write $\sigma : V \to V$ for the $\mathcal{A}$-linear map with $\sigma(a_c) = d'_c a_c$. Define $u_c = d'_c a_c$ and $U = \mathcal{A}\text{-span}\{u_c : c \in C\}$ as in Lemma 3.1. Then $(\phi, \{u_c\})$ is a pre-canonical structure on U and the map $\sigma : V \to U$ is a scaling morphism from $(\phi, \{a_c\})$ to $(\phi, \{u_c\})$.

Finally, define $\iota : U \to V'$ as the $\mathcal{A}$-linear map with $\iota(u_c) = a'_c$ for $c \in C$. This is a strong isomorphism, since for any $c \in C$ we have

$$\iota \circ \phi(u_c) = d'_c \cdot \iota \circ \tau \circ \psi(a_c) = \varphi \circ \psi^D(a_c) = \psi' \circ \varphi(a_c) = \psi'(a'_c) = \psi' \circ \iota(u_c).$$

As both $\iota \circ \psi$ and $\psi' \circ \iota$ are $\mathcal{A}$-antilinear, this identity shows that the two maps are equal. The composition $\iota \circ \sigma \circ \tau$ agrees with φ at each basis element a_c, and both maps are ϵ-linear, so they are equal. □

Proposition 3.7. *Suppose the pre-canonical structure on V admits a canonical basis $\{b_c\}$. Then the pre-canonical structure on V' also admits a canonical basis $\{b'_c\}$. If D is a scaling factor of φ and $\beta : V \to V$ is the $\mathcal{A}$-linear map with $\beta(a_c) = b_c$ for each $c \in C$, then the composition*

$$\varphi \circ D^{-1} \circ \beta \circ D \circ \beta^{-1}$$

is a well-defined map $V \to V'$ which restricts to an order-preserving bijection $\{b_c\} \to \{b'_c\}$.

Proof. Let $b'_c = \varphi \circ D^{-1} \circ \beta \circ D \circ \beta^{-1}(b_c)$. It suffices to check that this element satisfies the defining conditions of a canonical basis. This is a simple exercise which is left to the reader. □

Proposition 3.8. *A morphism of pre-canonical structures is an isomorphism (that is, there exists a morphism of pre-canonical structures which is its left and right inverse) if and only if it has degree 1 and it has a scaling factor whose eigenvalues are each ± 1.*

Proof. If φ has degree 1 and a scaling factor D whose eigenvalues are each ± 1, then $D = D^{-1}$ and φ is an ϵ-linear bijection (where $\epsilon = \epsilon^{-1}$ is a ring involution of $\mathcal{A}$) and it follows that the inverse map φ^{-1} is well-defined and a morphism of pre-canonical structures with scaling factor $\varphi \circ D \circ \varphi^{-1}$. Hence in this case φ is an isomorphism of pre-canonical structures. Suppose conversely that D is a scaling factor for φ and that φ^{-1} exists and is a morphism with scaling factor D'. Then φ must have degree 1 since otherwise φ is not invertible. To show that φ has some scaling factor all of whose eigenvalues are ± 1, let $D'' = \varphi \circ D \circ \varphi^{-1}$. Then

$$\psi' = \varphi \circ (\varphi^{-1} \circ \psi' \circ \varphi) \circ \varphi^{-1} = D''^{-1} \circ (\varphi \circ \psi \circ \varphi^{-1}) \circ D'' = (D'D'')^{-1} \circ \psi' \circ (D'D'').$$

For each $c \in C$ let d_c and d'_c be the elements of $\mathbb{Z}[v + v^{-1}]$ such that $D(a_c) = d_c a_c$ and $D'(a'_c) = d'_c a'_c$. Now, write $\sim$ for the minimal equivalence relation on C such that $c \sim c'$ whenever $c, c' \in C$ are such that the coefficient $f_{c',c}$ of $a'_{c'}$ in $\psi'(a'_c)$ is nonzero. The equation above implies

$$f_{c',c} = d_c / d_{c'} \cdot d'_c / d'_{c'} \cdot f_{c',c},$$

so since $d_c/d_{c'}$ and $d'_c/d'_{c'}$ are both integers, these quotients must each be ± 1. Hence if K is an equivalence class under $\sim$, then $d_c/d_{c'} \in \{\pm 1\}$ for any $c, c' \in K$. For each such equivalence class K, choose an arbitrary $c \in K$ and let $d_K = d_c$. Now let $E : V \to V$ be the $\mathcal{A}$-linear map with $E(a_c) = d_K a_c$ where K is the equivalence class of $c \in C$. We claim that

$$\psi = E^{-1} \circ \psi \circ E.$$

This follows since if the coefficient of $a_{c'}$ in $\psi(a_c)$ is some polynomial $f \in \mathcal{A}$, then the coefficient of $a_{c'}$ in $E^{-1} \circ \psi \circ E(a_c)$ is $d_K / d_{K'} \cdot f$ where K and K' are the equivalence classes of c and c'. If $f = 0$, then these coefficients are both zero, and if $f \neq 0$, then the coefficient of $a'_{c'}$ in $\psi'(a'_c)$ is also nonzero, so $K = K'$ and our coefficients are again equal. From this claim, we conclude that $E^{-1} D$ is another scaling factor of φ. The eigenvalues of this scaling factor are each ± 1 since if K is the equivalence class of $c \in C$, then $d_c / d_K \in \{\pm 1\}$. □

The following corollary shows that the structure constants of canonical bases arising from isomorphic pre-canonical structures differ only by a factor of ± 1 or the substitution $v \mapsto -v$.

Corollary 3.9. *Suppose the pre-canonical structures on V and V' are isomorphic and admit canonical bases $\{b_c\}$ and $\{b'_c\}$. Define $f_{x,y}(t), g_{x,y}(t) \in \mathbb{Z}[t]$ such that*

$$b_y = \sum_{x \le y} f_{x,y}(v^{-1}) a_x \qquad \text{and} \qquad b'_y = \sum_{x \le y} g_{x,y}(v^{-1}) a'_x.$$

Then for each $x, y \in C$ there are $\varepsilon_i \in \{\pm 1\}$ such that $f_{x,y}(t) = \varepsilon_1 \cdot g_{x,y}(\varepsilon_2 t)$.

Proof. Let $\varphi : V \to V'$ be an isomorphism of pre-canonical structures. By the previous proposition, φ has a scaling factor D whose eigenvalues are all ± 1, and φ

is ϵ-linear where $\epsilon \in \mathrm{End}(\mathcal{A})$ is either the identity or the ring homomorphism with $v \mapsto -v$. Given these considerations, the corollary follows from Proposition 3.7. □

3.2 Generic structures on group elements

In this and the two sections which follow, we consider Hecke algebra modules of a certain generic form. We are interested in classifying such generic structures, saying which structures admit compatible pre-canonical structures, and identifying when such pre-canonical structures are isomorphic in the sense of Definition 3.3. The solutions to these problems will recover some constructions already studied in the literature, but will also reveal other structures not previously examined. The unexpected existence of these "extra" solutions is the primary motivation for our results.

In this section, the type of generic module structure which we study is a natural generalization of the regular representation of a Hecke algebra. Our results here are useful mostly for comparison with the theorems in the next sections. The proofs in this section are only sketched, since they are just simpler versions of the arguments we use to establish the results in Sections 3.3 and 3.4.

If X is a set, then we write $\mathcal{A}X$ for the free $\mathcal{A}$-module generated by X, and let $\mathrm{End}(\mathcal{A}X)$ denote the $\mathcal{A}$-module of $\mathcal{A}$-linear maps $\mathcal{A}X \to \mathcal{A}X$. A *representation* of $\mathcal{H}$ in some $\mathcal{A}$-module $\mathcal{M}$ is an $\mathcal{A}$-algebra homomorphism $\mathcal{H} \to \mathrm{End}(\mathcal{M})$. Now consider a 2×2 matrix $\gamma = (\gamma_{ij})$ with entries in $\mathcal{A}$. Given a Coxeter system (W, S), we let $\rho_\gamma : \{H_s : s \in S\} \to \mathrm{End}(\mathcal{A}W)$ denote the map with

$$\rho_\gamma(H_s)(w) = \begin{cases} \gamma_{11} \cdot sw + \gamma_{12} \cdot w & \text{if } sw > w \\ \gamma_{21} \cdot sw + \gamma_{22} \cdot w & \text{if } sw < w \end{cases} \qquad \text{for } s \in S \text{ and } w \in W.$$

Definition 3.10. The matrix γ is an $(\mathcal{H}, W)$-*structure* if for every Coxeter system (W, S), the map ρ_γ extends to a representation of $\mathcal{H} = \mathcal{H}(W, S)$ in $\mathcal{A}W$.

An $(\mathcal{H}, W)$-structure $\gamma = (\gamma_{ij})$ is *trivial* if $\gamma_{11} = \gamma_{21} = 0$ and $\gamma_{12} = \gamma_{22} \in \{v, -v^{-1}\}$. Such a structure defines an $\mathcal{H}$-representation which decomposes as a direct sum of irreducible submodules given by free $\mathcal{A}$-modules of rank one. The definition of $\mathcal{H}$ affords an obvious example of a nontrivial $(\mathcal{H}, W)$-structure: namely, the matrix γ with $\gamma_{11} = \gamma_{21} = 1$ and $\gamma_{12} = 0$ and $\gamma_{22} = v - v^{-1}$.

Theorem 3.11. *Every nontrivial* $(\mathcal{H}, W)$*-structure is equal to*

$$\begin{bmatrix} \alpha & 0 \\ \alpha^{-1} & v - v^{-1} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \alpha & v - v^{-1} \\ \alpha^{-1} & 0 \end{bmatrix}$$

for some unit α *in* $\mathcal{A}$*. All nontrivial* $(\mathcal{H}, W)$*-structures define isomorphic* $\mathcal{H}$*-representations.*

Recall that the units in the ring $\mathcal{A}$ are the monomials of the form $\pm v^n$ for $n \in \mathbb{Z}$.

Proof (sketch). The given matrices are $(\mathcal{H}, W)$-structures, since those on the left (respectively, right) describe the action of H_s for $s \in S$ on the basis $\{\alpha^{-\ell(w)} H_w : w \in W\}$ (respectively, $\{\alpha^{-\ell(w)} \overline{H_w} : w \in W\}$) of $\mathcal{H}$. The corresponding $\mathcal{H}$-representations are evidently all isomorphic to the regular representation of $\mathcal{H}$ on itself. That there are no other nontrivial $(\mathcal{H}, W)$-structures follows by a simpler version of the argument used in the proof of Theorem 3.15. □

An $(\mathcal{H}, W)$-structure γ defines an $\mathcal{H}$-module structure on $\mathcal{A}W$ for every Coxeter system (W, S). We say that γ is *pre-canonical* if each of these $\mathcal{H}$-modules has a pre-canonical $\mathcal{H}$-module structure in which W, partially ordered by the Bruhat order, is the "standard basis." It follows from the preceding theorem and Lemma 2.15 that if γ is nontrivial and pre-canonical, then there is a unique bar involution $\psi : \mathcal{A}W \to \mathcal{A}W$ such that (ψ, W) is a pre-canonical $\mathcal{H}$-module structure. By Theorem 2.5, this pre-canonical structure always admits a canonical basis.

Theorem 3.12. *Exactly 4 nontrivial $(\mathcal{H}, W)$-structures are pre-canonical. The 4 associated pre-canonical structures on $\mathcal{A}W$ are all isomorphic (in the sense of Definition 3.3) to the pre-canonical structure on $\mathcal{H}$ given in Theorem 2.18.*

Proof (sketch). The proof is similar to that of Theorem 3.16. Let γ be a nontrivial, pre-canonical $(\mathcal{H}, W)$-structure. Then γ must be one of the two matrices in Theorem 3.11 for some unit $\alpha \in \mathcal{A}$. One first argues that $\alpha = \overline{\alpha}$ so that $\alpha \in \{\pm 1\}$. Next, one observes that γ remains pre-canonical if α is replaced with $-\alpha$, and that the pre-canonical structures associated to these two $(\mathcal{H}, W)$-structures are always isomorphic. One may therefore assume $\alpha = 1$. It remains to prove that if γ is the right-hand matrix in Theorem 3.11 then its associated pre-canonical structure is isomorphic to the pre-canonical structure on $\mathcal{H}$ given in Theorem 2.18. This can be deduced from [12, Lemma 2.1(i)], after noting that the $\mathcal{A}$-linear map with $w \mapsto \overline{H_w}$ defines an isomorphism between $\mathcal{A}W$ viewed as an $\mathcal{H}$-module via γ and $\mathcal{H}$ viewed as a left module over itself. □

3.3 Generic structures on twisted involutions

In this section we introduce a second kind of generic structure, which concerns Hecke algebra modules on the space of twisted involutions in a Coxeter group. Quite nontrivial results of Lusztig and Vogan [13, 14, 15] provide interesting examples of this type of generic structure, which is what motivates their study. Our results here depend on Lusztig and Vogan's work, which we review in Section 4. For this reason, we defer most proofs to Section 5.

Consider a 4×2 matrix $\gamma = (\gamma_{ij})$ with entries in $\mathcal{A}$. Given a Coxeter system (W, S), writing $\mathbf{I} = \mathbf{I}(W, S)$, we let $\rho_\gamma : \{H_s : s \in S\} \to \mathrm{End}(\mathcal{A}\mathbf{I})$ denote the map with

$$\rho_\gamma(H_s)(w) = \begin{cases} \gamma_{11}\cdot sws + \gamma_{12}\cdot w & \text{if } s \ltimes w = sws > w \\ \gamma_{21}\cdot sws + \gamma_{22}\cdot w & \text{if } s \ltimes w = sws < w \\ \gamma_{31}\cdot sw + \gamma_{32}\cdot w & \text{if } s \ltimes w = sw > w \\ \gamma_{41}\cdot sw + \gamma_{42}\cdot w & \text{if } s \ltimes w = sw < w \end{cases}$$

for $s \in S$ and $w \in \mathbf{I}$.

Definition 3.13. The matrix γ is an $(\mathcal{H}, \mathbf{I})$-*structure* if for every Coxeter system (W, S), the map ρ_γ extends to a representation of $\mathcal{H} = \mathcal{H}(W, S)$ in $\mathcal{A}\mathbf{I} = \mathcal{A}\mathbf{I}(W, S)$.

It would make sense to view ρ_γ as a map $\{H_s : s \in S\} \to \mathrm{End}(\mathcal{A}W)$ by the same formula. However, combining some computations with the analysis in Section 5.1, one can show that ρ_γ only extends to a representation of $\mathcal{H}$ in $\mathcal{A}W$ for every Coxeter system (W, S) when γ is *trivial*, where we say that $\gamma = (\gamma_{ij})$ is trivial if $\gamma_{11} = \gamma_{21} = \gamma_{31} = \gamma_{41} = 0$ and $\gamma_{12} = \gamma_{22} = \gamma_{32} = \gamma_{42} \in \{v, -v^{-1}\}$. Before we can classify the nontrivial $(\mathcal{H}, \mathbf{I})$-structures, we need to describe the following basic notation of equivalence between structures:

Lemma 3.14. *Let* $A, B, C, D, E, F, G, H \in \mathcal{A}$ *and suppose* $\alpha, \beta \in \mathbb{Q}(v) - \{0\}$ *such that* $A\alpha^{-1}$ *and* $C\alpha$ *and* $E\beta^{-1}$ *and* $G\beta$ *all belong to* $\mathcal{A}$. *Let*

$$\gamma = \begin{bmatrix} A & B \\ C & D \\ E & F \\ G & H \end{bmatrix} \qquad \textit{and} \qquad \gamma[\alpha, \beta] = \begin{bmatrix} A\alpha^{-1} & B \\ C\alpha & D \\ E\beta^{-1} & F \\ G\beta & H \end{bmatrix}.$$

If γ *is a* $(\mathcal{H}, \mathbf{I})$*-structure, then so is* $\gamma[\alpha, \beta]$. *In this case, we say that* γ *and* $\gamma[\alpha, \beta]$ *are* diagonally equivalent. *If* $\alpha, \beta \in \mathcal{A}$, *then* γ *and* $\gamma[\alpha, \beta]$ *define isomorphic representations of* $\mathcal{H}$.

Proof. Assume γ is an $(\mathcal{H}, \mathbf{I})$-structure. The $\mathcal{H}$-representation ρ_γ extends to a representation in the larger $\mathcal{A}$-module $\mathbb{Q}(v)\mathbf{I}$ by linearity. Define $T : \mathbb{Q}(v)\mathbf{I} \to \mathbb{Q}(v)\mathbf{I}$ as the $\mathbb{Q}(v)$-linear map with

$$T(w) = \alpha^{\ell(w)-\rho(w)} \cdot \beta^{2\rho(w)-\ell(w)} \cdot w \qquad \text{for } w \in \mathbf{I}$$

where on the right $\rho : \mathbf{I} \to \mathbb{N}$ is defined as in Theorem 2.17. Then $\gamma[\alpha, \beta]$ is an $(\mathcal{H}, \mathbf{I})$-structure since $\rho_{\gamma[\alpha,\beta]}(H) = T^{-1} \circ \rho_\gamma(H) \circ T$ for all $H \in \mathcal{H}$. □

Let $u = v - v^{-1}$ and define four 4×2 matrices as follows:

$$\Gamma = \begin{bmatrix} 1 & 0 \\ 1 & u \\ 1 & 1 \\ u & u-1 \end{bmatrix}, \quad \Gamma' = \begin{bmatrix} 1 & u \\ 1 & 0 \\ 1 & u-1 \\ u & 1 \end{bmatrix}, \quad \Gamma'' = \begin{bmatrix} 1 & 0 \\ 1 & u \\ 1 & -1 \\ -u & u+1 \end{bmatrix}, \quad \Gamma''' = \begin{bmatrix} 1 & u \\ 1 & 0 \\ 1 & u+1 \\ -u & -1 \end{bmatrix}.$$

The proof of the following theorem will be given in Section 5.1.

Theorem 3.15. *Each of Γ, Γ', Γ'', and Γ''' is an $(\mathcal{H}, \mathbf{I})$-structure and every nontrivial $(\mathcal{H}, \mathbf{I})$-structure is diagonally equivalent to one of these.*

An $(\mathcal{H}, \mathbf{I})$-structure γ defines an $\mathcal{H}$-module structure on $\mathcal{A}\mathbf{I}$ for every Coxeter system (W, S). Analogous to our definition for $(\mathcal{H}, W)$-structures, we say that γ is *pre-canonical* if each of these $\mathcal{H}$-modules has a pre-canonical $\mathcal{H}$-module structure in which $\mathbf{I}$, partially ordered by the Bruhat order, is the "standard basis." We have the same remark as concerned pre-canonical $(\mathcal{H}, W)$-structures: by the preceding theorem and Lemma 2.15, if γ is a nontrivial, pre-canonical $(\mathcal{H}, \mathbf{I})$-structure, then for each choice of Coxeter system (W, S) there is a unique bar involution $\psi : \mathcal{A}\mathbf{I} \to \mathcal{A}\mathbf{I}$ such that $(\psi, \mathbf{I})$ is a pre-canonical $\mathcal{H}$-module structure, and this structure always admits a canonical basis. We have this analogue of Theorem 3.12, whose proof will be given in Section 5.1.

Theorem 3.16. *Exactly 16 nontrivial $(\mathcal{H}, \mathbf{I})$-structures are pre-canonical; in particular, each of Γ, Γ', Γ'', and Γ''' is pre-canonical. However, the 16 associated pre-canonical structures on $\mathcal{A}\mathbf{I}$ are all isomorphic (in the sense of Definition 3.3).*

3.4 Generic structures for a modified Iwahori–Hecke algebra

Let $\mathcal{H}_2$ be the free $\mathcal{A}$-module with a basis given the symbols K_w for $w \in W$, with the unique $\mathcal{A}$-algebra structure such that

$$K_s K_w = \begin{cases} K_{sw} & \text{if } sw > w \\ K_{sw} + (v^2 - v^{-2}) K_w & \text{if } sw < w \end{cases} \qquad \text{for } s \in S \text{ and } w \in W.$$

We call this the *Iwahori–Hecke algebra of* (W, S) *with parameter* v^2. We again denote by $K \mapsto \overline{K}$ the $\mathcal{A}$-antilinear map $\mathcal{H}_2 \to \mathcal{H}_2$ with $\overline{K_w} = K_{w^{-1}}^{-1}$ for $w \in W$. This "bar involution" together with the "standard basis" $\{K_w\}$ indexed by $(W, \leq)$ forms a pre-canonical structure on $\mathcal{H}_2$, which admits a canonical basis $\{\underline{K}_w\}$. The $\mathbb{Z}$-linear map

$$\Phi : \mathcal{H} \to \mathcal{H}_2 \tag{3.1}$$

with $\Phi(v^n H_w) = v^{2n} K_w$ is an injective ring homomorphism and $\underline{K}_w = \Phi(\underline{H}_w)$ for all $w \in W$.

We adapt the definition of an $(\mathcal{H}, \mathbf{I})$-structure to the modified Iwahori–Hecke algebra $\mathcal{H}_2$ in the following natural way. Consider a 4×2 matrix $\gamma = (\gamma_{ij})$ with entries in $\mathcal{A}$. Define $\rho_{\gamma,2} : \{K_s : s \in S\} \to \mathrm{End}(\mathcal{A}\mathbf{I})$ again by the formula (3.3) except with H_s replaced by K_s; that is, let $\rho_{\gamma,2}$ be the composition of ρ_γ with the obvious bijection $\{K_s : s \in S\} \to \{H_s : s \in S\}$.

Definition 3.17. The matrix γ is an $(\mathcal{H}_2, \mathbf{I})$-*structure* if for every Coxeter system (W, S), the map $\rho_{\gamma,2}$ extends to a representation of $\mathcal{H}_2 = \mathcal{H}_2(W, S)$ in $\mathcal{A}\mathbf{I} = \mathcal{A}\mathbf{I}(W, S)$.

Despite the formal similarity of this definition to Definition 3.13, there are at least two good reasons to consider $(\mathcal{H}_2, \mathbf{I})$-structures in addition to $(\mathcal{H}, \mathbf{I})$-structures. First, the generic structures which have so far been uncovered "in nature," through the work of Lusztig and Vogan [13, 14, 15], are in fact $(\mathcal{H}_2, \mathbf{I})$-structures, and we will actually deduce the existence of the $(\mathcal{H}, \mathbf{I})$-structures in the previous section from the existence of such $(\mathcal{H}_2, \mathbf{I})$-structures. Second, we will find that a more complicated and interesting classification applies to "pre-canonical" $(\mathcal{H}_2, \mathbf{I})$-structures, which does not follow directly from the theorems in Section 3.3.

Given a matrix γ over $\mathcal{A}$, define $[\gamma]_2$ by applying the ring endomorphism of $\mathcal{A}$ with $v \mapsto v^2$ to the entries of γ. The following observation motivates this notation.

Lemma 3.18. *If γ is an $(\mathcal{H}, \mathbf{I})$-structure, then $[\gamma]_2$ is an $(\mathcal{H}_2, \mathbf{I})$-structure.*

As before, we say that an $(\mathcal{H}_2, \mathbf{I})$ structure γ is *trivial* if $\gamma_{11} = \gamma_{21} = \gamma_{31} = \gamma_{41} = 0$ and $\gamma_{12} = \gamma_{22} = \gamma_{32} = \gamma_{42} \in \{v^2, -v^{-2}\}$. Lemma 3.14 holds *mutatis mutandis* with "$(\mathcal{H}, \mathbf{I})$-structure" replaced by "$(\mathcal{H}_2, \mathbf{I})$-structure" and "$\mathcal{H}$" replaced by "$\mathcal{H}_2$." Define two $(\mathcal{H}_2, \mathbf{I})$-structures to be *diagonally equivalent* as in that result. The classification of $(\mathcal{H}_2, \mathbf{I})$-structures up to diagonal equivalence is no different than for $(\mathcal{H}, \mathbf{I})$-structures:

Theorem 3.19. *Let Γ, Γ', Γ'', and Γ''' be the $(\mathcal{H}, \mathbf{I})$-structures defined before Theorem 3.15. Then every nontrivial $(\mathcal{H}_2, \mathbf{I})$-structure is diagonally equivalent to $[\Gamma]_2$, $[\Gamma']_2$, $[\Gamma'']_2$, or $[\Gamma''']_2$.*

The proof of this result will be sketched in Section 5.1. Define an $(\mathcal{H}_2, \mathbf{I})$-structure γ to be *pre-canonical* exactly as for $(\mathcal{H}, \mathbf{I})$-structures: namely, say that γ is pre-canonical if, for every Coxeter system (W, S), there exists a pre-canonical $\mathcal{H}_2$-module structure on $\mathcal{A}\mathbf{I}$ (relative to the $\mathcal{H}_2$-module structure defined by γ) in which $\mathbf{I}$, partially ordered by the Bruhat order, is the "standard basis." Just like for $(\mathcal{H}, W)$-structures and $(\mathcal{H}, \mathbf{I})$-structures, if an $(\mathcal{H}_2, \mathbf{I})$-structure is nontrivial and pre-canonical, then by Lemma 2.15 it associates a unique pre-canonical $\mathcal{H}_2$-structure to $\mathcal{A}\mathbf{I}$ for each Coxeter system (W, S).

To classify the pre-canonical $(\mathcal{H}_2, \mathbf{I})$-structures, we define Δ and Δ' as the matrices

$$\Delta = \begin{bmatrix} 1 & 0 \\ 1 & v^2 - v^{-2} \\ v + v^{-1} & 1 \\ v - v^{-1} & v^2 - 1 - v^{-2} \end{bmatrix} \qquad \text{and} \qquad \Delta' = \begin{bmatrix} 1 & 0 \\ 1 & v^2 - v^{-2} \\ v^{-1} + v & -1 \\ v^{-1} - v & v^2 + 1 - v^{-2} \end{bmatrix}.$$

In addition, let

$$\Delta'' = [\Gamma]_2 \qquad \text{and} \qquad \Delta''' = [\Gamma'']_2.$$

Observe that Δ and Δ'' (respectively, Δ' and Δ''') are diagonally equivalent, which is how we deduce that Δ and Δ' are $(\mathcal{H}_2, \mathbf{I})$-structures. Note, however, the $\mathcal{H}_2$-module structures defined by the pairs Δ and Δ'' (respectively, Δ' and Δ''') are technically not isomorphic, although they would be if all of our algebras and modules were defined over the field $\mathbb{Q}(v)$ instead of the ring $\mathcal{A}$. We have this analogue of Theorems 3.12 and 3.16.

Theorem 3.20. *Exactly 32 nontrivial* $(\mathcal{H}_2, \mathbf{I})$*-structures are pre-canonical; in particular, each of* Δ, Δ', Δ'', *and* Δ''' *is pre-canonical. The 32 associated pre-canonical structures on* $\mathcal{A}\mathbf{I}$ *are each isomorphic (in the sense of Definition 3.3) to one of the structures arising from* Δ, Δ', Δ'', *or* Δ'''.

The proof of this theorem appears at the end of Section 5.1.

Remark 3.21. The pre-canonical structures on $\mathcal{A}\mathbf{I}$ defined by the $(\mathcal{H}, \mathbf{I})$-structures Γ and Γ'' are isomorphic by Theorem 3.16, and one might expect this to imply that the pre-canonical structures defined by $\Delta'' = [\Gamma]_2$ and $\Delta''' = [\Gamma'']_2$ are likewise isomorphic. The reason this does not follow is that the latter structures admit canonical bases $\{b_w\}$ and $\{b'_w\}$ of the form $b_w = \sum_{y \leq w} f_{y,w}(v^{-2})y$ and $b'_w = \sum_{y \leq w} f_{y,w}(-v^{-2})y$ for some polynomials $f_{y,w}(t) \in \mathbb{Z}[t]$. Corollary 3.9 shows that such canonical bases cannot arise from isomorphic pre-canonical structures, provided $f_{y,w}(t)$ are sufficiently complicated polynomials.

It will follow from the discussion in Sections 4.3 and 4.2 (or more concretely, from small computations) that the pre-canonical structures which Δ, Δ', Δ'', and Δ''' associate to $\mathcal{A}\mathbf{I}$ are generally not isomorphic. Thus, we are left with the interesting question of explaining where these four structures come from. The structure Δ is what has appeared naturally from geometric considerations in the work of Lusztig and Vogan [13, 14, 15], and one can account for Δ'' and Δ''' as the two distinct "extensions" of the unique isomorphism class of pre-canonical $(\mathcal{H}, \mathbf{I})$-structures. The origins of the remaining pre-canonical $(\mathcal{H}_2, \mathbf{I})$-structure Δ' remains more mysterious.

4 Existence proofs

The results in Sections 3.3 and 3.4 assert, in one direction, that certain generic structures exist, or equivalently, that certain formulas define $\mathcal{H}$- or $\mathcal{H}_2$-modules structures on $\mathcal{A}\mathbf{I}$ (and sometimes also admit compatible pre-canonical structures) for all Coxeter systems (W, S). In this section we prove some existence statements of this type, which we require for the proofs of Theorems 3.15, 3.16, 3.19, and 3.20 given in Section 5.1.

4.1 A canonical basis for twisted involutions

Our starting point is the following result of Lusztig and Vogan, first proved in [14] in the case that W is a Weyl group or affine Weyl group, then extended in [13] to arbitrary Coxeter systems by elementary methods. Lusztig and Vogan's preprint [15] provides another proof of this result, using the machinery of Soergel bimodules developed by Elias and Williamson in [5].

Theorem 4.1 (Lusztig and Vogan [14, 15]; Lusztig [13]). *There is a unique $\mathcal{H}$-module*

$$\mathcal{L} = \mathcal{L}(W, S)$$

which, as an $\mathcal{A}$-module, is free with a basis given by the symbols L_w for $w \in \mathbf{I}$, and which satisfies

$$K_s L_w = \begin{cases} L_{sws} & \text{if } sw \neq ws > w \\ L_{sws} + (v^2 - v^{-2})L_w & \text{if } sw \neq ws < w \\ (v + v^{-1})L_{sw} + L_w & \text{if } sw = ws > w \\ (v - v^{-1})L_{sw} + (v^2 - 1 - v^{-2})L_w & \text{if } sw = ws < w \end{cases}$$

for $s \in S$ and $w \in \mathbf{I}$.

Proof. This is [13, Theorem 0.1], where $v^2 = u$ and $K_s = u^{-1}T_s$ and $L_w = a'_w = v^{-\ell(w)}a_w$. □

The preceding theorem shows that the matrix Δ in Section 3.4 is an $(\mathcal{H}_2, \mathbf{I})$-structure. The following result, which combines [13, Theorem 0.2, Theorem 0.4, and Proposition 4.4], shows that this $(\mathcal{H}_2, \mathbf{I})$-structure is pre-canonical. Here, for $x \in W$ we write $\mathrm{sgn}(x) = (-1)^{\ell(x)}$.

Theorem 4.2 (Lusztig and Vogan [14]; Lusztig [13]). *Define*

- *the "bar involution" of $\mathcal{L}$ to be the $\mathcal{A}$-antilinear map $\mathcal{L} \to \mathcal{L}$, denoted $L \mapsto \overline{L}$, with*
$$\overline{L_{(x,\theta)}} = \mathrm{sgn}(x) \cdot \overline{K_x} \cdot L_{(x^{-1},\theta)} \qquad \text{for } (x, \theta) \in \mathbf{I}.$$
- *the "standard basis" of $\mathcal{L}$ to be $\{L_w\}$ with the partially ordered index set $(\mathbf{I}, \leq)$.*

This is a pre-canonical $\mathcal{H}_2$-module structure on $\mathcal{L}$, and it admits a canonical basis $\{\underline{L}_w\}$.

Observe, by Lemma 2.15, that the pre-canonical $\mathcal{H}_2$-module structure thus defined on $\mathcal{L}$ is the unique one in which $\{L_w\}$ serves as the "standard basis." Following the convention in [13], we define $\pi_{y,w} \in \mathbb{Z}[v^{-1}]$ for $y, w \in \mathbf{I}$ such that $\underline{L}_w = \sum_{y \in \mathbf{I}} \pi_{y,w} L_y$. Note that $\pi_{y,w} = \delta_{y,w}$ if $y \not< w$. We note the following degree bound from [13, Section 4.9(c)].

Proposition 4.3 (Lusztig [13]). *If $y, w \in \mathbf{I}$ such that $y \leq w$, then*

$$v^{\ell(w)-\ell(y)}\pi_{y,w} \in 1 + v^2\mathbb{Z}[v^2].$$

Remark 4.4. The polynomials $v^{\ell(w)-\ell(y)}\pi_{y,w}$ are denoted $P^{\sigma}_{y,w}$ in [13, 14, 16, 17]. Lusztig proves an inductive formula [13, Theorem 6.3] for the action of $\underline{K}_s = K_s + v^{-2} \in \mathcal{H}_2$ on $\underline{L}_w$ which can be used to compute these polynomials; see [17, Section 2.1].

While the polynomials $\pi_{y,w}$ may have negative coefficients, they still possess certain positivity properties. Recall that $h_{y,w} \in \mathbb{N}[v^{-1}]$ are the polynomials such that $\underline{H}_w = \sum_{y \in W} h_{y,w} H_y$. Given $y, w \in W$ and $\theta, \theta' \in \mathrm{Aut}(W, S)$, define $h_{(y,\theta),(w,\theta')}$ to be $h_{y,w}$ if $\theta = \theta'$ and zero otherwise. Lusztig [13, Theorem 9.10] has shown that

$$\tfrac{1}{2}\left(h_{y,w} \pm \pi_{y,w}\right) \in \mathbb{Z}[v^{-1}] \qquad \text{for all } y, w \in \mathbf{I}$$

and has conjectured that these polynomials actually belong to $\mathbb{N}[v^{-1}]$. Lusztig and Vogan provide a geometric proof of this conjecture when W is a Weyl group (see [14, Section 3.2]) and outline a proof for arbitrary Coxeter systems in [15]. The canonical basis $\{\underline{L}_w\}$ conjecturally displays some other positivity properties, which are considered in detail in [16, 17].

4.2 Another pre-canonical $\mathcal{H}_2$-module structure

Here we deduce from the results in the previous section that the matrix Δ' in Section 3.20 is a pre-canonical $(\mathcal{H}_2, \mathbf{I})$-structure. The pre-canonical $\mathcal{H}_2$-module structure on $\mathcal{A}\mathbf{I}$ associated to this generic structure admits a canonical basis which is not related in any obvious way to the basis $\{\underline{L}_w\}$ in the previous section, although it has similar properties. It is an open problem to find an interpretation of this new canonical basis along the lines of [14, 15].

First we have this analogue of Theorem 4.1, showing that Δ' is in fact an $(\mathcal{H}_2, \mathbf{I})$-structure.

Theorem 4.5. *There is a unique $\mathcal{H}_2$-module*

$$\mathcal{L}' = \mathcal{L}'(W, S)$$

which, as an $\mathcal{A}$-module, is free with a basis given by the symbols L'_w for $w \in \mathbf{I}$, and which satisfies

$$K_s L'_w = \begin{cases} L'_{sws} & \text{if } sw \neq ws > w \\ L'_{sws} + (v^2 - v^{-2})L'_w & \text{if } sw \neq ws < w \\ (v^{-1} + v)L'_{sw} - L'_w & \text{if } sw = ws > w \\ (v^{-1} - v)L'_{sw} + (v^2 + 1 - v^{-2})L'_w & \text{if } sw = ws < w \end{cases}$$

for $s \in S$ and $w \in \mathbf{I}$.

Proof. Define $f^z_{x,y} \in \mathcal{A}$ for $x \in W$ and $y, z \in W$ such that

$$(-1)^{\rho(y)} K_x L_y = \sum_{z \in \mathbf{I}} (-1)^{\rho(z)} f^z_{x,y} L_z.$$

It is a straightforward exercise to check, using the well-known relations defining $\mathcal{H}_2$ (see, e.g., [13, Section 2.1]), that there is a unique $\mathcal{H}_2$-module structure on $\mathcal{L}'$ in which $K_x L'_y = \mathrm{sgn}(x) \sum_{z\in\mathbf{I}} \overline{f^z_{x,y}} L'_z$ for $x \in W$ and $y \in \mathbf{I}$. In this $\mathcal{H}_2$-module structure, the generators K_s for $s \in S$ act on the basis elements L'_w according to the given formula. □

We have this analogue of Theorem 4.2, which shows that Δ' is pre-canonical.

Theorem 4.6. *Define*

- *the "bar involution" of $\mathcal{L}'$ as the $\mathcal{A}$-antilinear map $\mathcal{L}' \to \mathcal{L}'$, denoted $L' \mapsto \overline{L'}$, with*

$$\overline{L'_{(x,\theta)}} = \overline{K_x} \cdot L'_{(x^{-1},\theta)} \qquad \textit{for } (x,\theta) \in \mathbf{I}.$$

- *the "standard basis" of $\mathcal{L}'$ to be $\{L'_w\}$ with the partially ordered index set $(\mathbf{I}, \leq)$.*

This is a pre-canonical $\mathcal{H}_2$-module structure on $\mathcal{L}'$, and it admits a canonical basis $\{\underline{L}'_w\}$.

By Lemma 2.15, this is the unique pre-canonical $\mathcal{H}_2$-module structure on $\mathcal{L}'$ in which $\{L'_w\}$ is the "standard basis."

Proof. Define $r_{y,w} \in \mathcal{A}$ for $y, w \in I$ such that $\overline{L_w} = \sum_{y\in\mathbf{I}} r_{y,w} L_y$ and let $f^z_{x,y}$ be as in the proof of Theorem 4.5. Let $L \mapsto \widetilde{L}$ be the $\mathcal{A}$-antilinear map with $\widetilde{L'_w} = \sum_{y\in\mathbf{I}} (-1)^{\rho(w)-\rho(y)} \cdot \overline{r_{y,w}} \cdot L'_y$ for $w \in \mathbf{I}$. We claim that $\widetilde{L} = \overline{L}$ for all $L \in \mathcal{L}'$. To prove this, we note that if $w = (x,\theta) \in \mathbf{I}$, then

$$K_{x^{-1}} \widetilde{L'_w} = \mathrm{sgn}(x) \sum_{y\in\mathbf{I}} \sum_{z\in\mathbf{I}} (-1)^{\rho(z)-\rho(y)} \cdot \overline{r_{y,w} \cdot f^z_{x^{-1},y}} \cdot L'_z,$$

while

$$\begin{aligned} L_w &= \mathrm{sgn}(x) K_{x^{-1}} \overline{L_w} \\ &= \mathrm{sgn}(x) \sum_{y\in\mathbf{I}} \sum_{z\in\mathbf{I}} (-1)^{\rho(z)-\rho(y)} \cdot r_{y,w} \cdot f^z_{x^{-1},y} \cdot L_z. \end{aligned}$$

We deduce that $K_{x^{-1}} \widetilde{L'_w} = L'_w = K_{x^{-1}} \overline{L'_w}$ since the right side of the first equation is the image of the right side of the second under the $\mathcal{A}$-antilinear map $\mathcal{L} \to \mathcal{L}'$ with $L_z \mapsto L'_z$ for $z \in \mathbf{I}$. Since $K_{x^{-1}}$ is invertible this shows that $\widetilde{L'_w} = \overline{L'_w}$ for $w \in \mathbf{I}$ which suffices to prove our claim.

Given the claim, it follows from Theorem 4.2 that the bar involution and standard basis of $\mathcal{L}'$ form a pre-canonical structure, and it is easy to show that the identity $\overline{K_s L_w} = \overline{K_s} \cdot \overline{L_w}$ implies $\overline{K_s L'_w} = \overline{K_s} \cdot \overline{L'_w}$ for $s \in S$ and $w \in \mathbf{I}$. Hence the bar involution and standard basis of $\mathcal{L}'$ form a pre-canonical $\mathcal{H}_2$-module structure, which admits a canonical basis $\{\underline{L}'_w\}$ by Theorem 2.5. □

We spend the rest of this section establishing a few properties of the canonical basis $\{\underline{L}'_w\}$. Define $\pi'_{y,w} \in \mathbb{Z}[v^{-1}]$ for $y, w \in \mathbf{I}$ as the polynomials such that

$\underline{L}'_w = \sum_{y\in\mathbf{I}} \pi'_{y,w} L'_y$. We introduce some notation to state a recurrence for computing these polynomials. First, for $y, w \in \mathbf{I}$ let

$$\mu'(y,w) = (\text{the coefficient of } v^{-1} \text{ in } \pi'_{y,w}),$$
$$\mu''(y,w) = (\text{the coefficient of } v^{-2} \text{ in } \pi'_{y,w}) + (v+v^{-1})\mu'(y,w).$$

Next, for $s \in S$ and $y, w \in \mathbf{I}$, define

$$\mu'(s,y,w) = \delta_{sy<y}\cdot\mu''(y,w) + \delta_{sy,ys}\cdot(\ell(y)-\ell(sy))\cdot\mu'(sy,w) - \sum_{\substack{y<z<w\\ sz<z}} \mu'(y,z)\mu'(z,w).$$

Here $\delta_{sy<y}$ is 1 if $sy < y$ and 0 otherwise. In what follows, recall that $\underline{K}_s = K_s + v^{-2}$ for $s \in S$.

Proposition 4.7. *Let $w \in \mathbf{I}$ and $s \in S$ such that $w < sw$.*

(a) If $sw \neq ws$, then $\underline{K}_s\underline{L}'_w = \underline{L}'_{sws} + \sum_{y<sws} \mu'(s,y,w)\underline{L}'_y$.
(b) If $sw = ws$, then

$$\underline{K}_s\underline{L}'_w = (v+v^{-1})\underline{L}'_{sw} - \underline{L}'_w + \sum_{y<sw} (\mu'(s,y,w) - \mu'(y,sw))\underline{L}'_y.$$

Remark 4.8. Lusztig [13, Theorem 6.3(c)] shows that the canonical basis $\{\underline{L}_w\} \subset \mathcal{L}$ in the previous section is such that $\underline{K}_s\underline{L}_w = (v^2+v^{-2})\underline{L}_w$ if $s \in S$ and $w \in \mathbf{I}$ and $sw < w$. This property has no simple analogue for the canonical basis $\{\underline{L}'_w\} \subset \mathcal{L}'$.

Proof. Each part of the proposition follows by showing that the difference between the two sides of the desired equality both (i) is an element of the set $\sum_{y<s\ltimes w} v^{-1}\mathbb{Z}[v^{-1}]\cdot L'_y$ and (ii) is invariant under the bar operator of $\mathcal{L}'$. Since the only such element with these two properties is 0, the given identities must hold. The observation (ii) follows in either case from Theorem 4.6, while showing that property (i) holds is a straightforward exercise from Theorem 4.5. □

Write $f \equiv g \pmod 2$ if $f, g \in \mathcal{A}$ are such that $f - g \in 2\mathcal{A}$, and define $\pi_{y,w}$ and $h_{y,w}$ for $y, w \in \mathbf{I}$ as in the previous section. We note the following relationship between $\pi'_{y,w}$, $\pi_{y,w}$, and $h_{y,w}$.

Proposition 4.9. *For all $y, w \in \mathbf{I}$ it holds that $\pi'_{y,w} \equiv \pi_{y,w} \equiv h_{y,w} \pmod 2$.*

Proof. The second congruence is [13, Theorem 9.10]. For $F \in \mathcal{L}$ and $G \in \mathcal{L}'$, we write $F \equiv G \pmod 2$ if $F = \sum_{y\in\mathbf{I}} f_y L_y$ and $G = \sum_{y\in\mathbf{I}} g_y L'_y$ for some polynomials $f_y, g_y \in \mathcal{A}$ with $f_y \equiv g_y \pmod 2$ for all $y \in \mathbf{I}$. To prove the first congruence we must show that $\underline{L}_w \equiv \underline{L}'_w \pmod 2$ for all $w \in \mathbf{I}$. This automatically holds if $\rho(w) = 0$. Let $w \in \mathbf{I}$ and $s \in S$ be such that $w < sw$ and assume $\underline{L}_y \equiv \underline{L}'_y \pmod 2$ if $y < s \ltimes w$. It suffices to show under this hypothesis that

$$\underline{L}_{s\ltimes w} \equiv \underline{L}'_{s\ltimes w} \pmod 2. \tag{4.1}$$

Towards this end, define $\mu(y,w) \in \mathbb{Z}$ for $y, w \in \mathbf{I}$ as the coefficient of v^{-1} in $\pi_{y,w}$, and let

$$X_{s,w} = \begin{cases} \underline{L}_{sws} & \text{if } sw \neq ws \\ (v+v^{-1})\underline{L}_{sw} - \sum_{y<sw} \mu(y,sw)\underline{L}_y & \text{if } sw = ws \end{cases}$$

and

$$X'_{s,w} = \begin{cases} \underline{L}'_{sws} & \text{if } sw \neq ws \\ (v+v^{-1})\underline{L}'_{sw} - \sum_{y<sw} \mu'(y,sw)\underline{L}'_y & \text{if } sw = ws. \end{cases}$$

We claim that to prove the congruence (4.1) it is enough show that $X_{s,w} \equiv X'_{s,w} \pmod 2$. This is obvious if $sw \neq ws$ so assume $sw = ws$ and $X_{s,w} \equiv X'_{s,w} \pmod 2$. We must check that $\pi'_{y,sw} \equiv \pi_{y,sw} \pmod 2$ for all $y \leq sw$; to this end we argue by induction on $\rho(sw) - \rho(y)$. By definition $\pi'_{sw,sw} = \pi_{sw,sw} = 1$. Fix $y < sw$ and suppose $\pi'_{z,sw} \equiv \pi_{z,sw} \pmod 2$ for $y < z \leq sw$. The congruence $X_{s,w} \equiv X'_{s,w} \pmod 2$ implies

$$(v+v^{-1})\pi_{y,sw} - \sum_{y\leq z<sw} \mu(z,sw)\pi_{y,z} \equiv (v+v^{-1})\pi'_{y,sw} - \sum_{y\leq z<sw} \mu'(z,sw)\pi'_{y,z} \pmod 2.$$

By hypothesis, the terms indexed by $z > y$ in the sums on either side of this congruence cancel, and we obtain

$$(v+v^{-1})\pi_{y,sw} - \mu(y,sw) \equiv (v+v^{-1})\pi'_{y,sw} - \mu'(y,sw) \pmod 2.$$

It is an elementary exercise, noting that $\pi_{y,sw}$ and $\pi'_{y,sw}$ both belong to $v^{-1}\mathbb{Z}[v^{-1}]$, to show that this congruence implies $\pi_{y,sw} \equiv \pi'_{y,sw} \pmod 2$, and so we conclude by induction that (4.1) holds. This proves our claim.

We now argue that $X_{s,w} \equiv X'_{s,w} \pmod 2$. For this we observe that there are unique polynomials $a_{s,y,w}, a'_{s,y,w} \in \mathcal{A}$ such that

$$X_{s,w} = \underline{K}_s\underline{L}_w - \sum_{y<s\ltimes w} a_{s,y,w}\underline{L}_y \qquad \text{and} \qquad X'_{s,w} = \underline{K}_s\underline{L}'_w - \sum_{y<s\ltimes w} a'_{s,y,w}\underline{L}'_y.$$

Indeed, the polynomials $a'_{s,y,w}$ are given by Proposition 4.7, and an entirely analogous statement decomposing the product $\underline{K}_s\underline{L}_w$ gives the polynomials $a_{s,y,w}$. It is not difficult to show, by deriving a formula for $a_{s,y,w}$ similar to the one for $\mu'(s,y,w)$, that the hypothesis $\underline{L}_y \equiv \underline{L}'_y \pmod 2$ for $y < s \ltimes w$ implies $a_{s,y,w} \equiv a'_{s,y,w} \pmod 2$. Hence to prove $X_{s,w} \equiv X'_{s,w} \pmod 2$ we need only check that $\underline{K}_s\underline{L}_w \equiv \underline{K}_s\underline{L}'_w \pmod 2$. As we assume $\underline{L}_w \equiv \underline{L}'_w \pmod 2$, this follows by comparing Theorems 4.1 and 4.5, which shows more generally that $\underline{K}_s F \equiv \underline{K}_s G \pmod 2$ whenever $F \in \mathcal{L}$ and $G \in \mathcal{L}'$ such that $F \equiv G \pmod 2$. □

The polynomials $\pi'_{y,w}$ also satisfy the same degree bound as $\pi_{y,w}$ and $h_{y,w}$.

Proposition 4.10. *If* $y, w \in \mathbf{I}$ *such that* $y \leq w$, *then* $v^{\ell(w)-\ell(y)}\pi'_{y,w} \in 1 + v^2\mathbb{Z}[v^2]$.

Proof. The proposition holds if $\rho(w) = 0$ since then $\pi'_{y,w} = \delta_{y,w}$. Let $w \in \mathbf{I}$ and $s \in S$ such that $w < sw$ and assume $v^{\ell(z)-\ell(y)}\pi'_{y,z} \in 1 + v^2\mathbb{Z}[v^2]$ for all $y \leq z < s \ltimes w$. It suffices to show under this hypothesis that

$$v^{\ell(s\ltimes w)-\ell(y)}\pi'_{y,s\ltimes w} \in 1 + v^2\mathbb{Z}[v^2] \qquad \text{for all } y \in \mathbf{I} \text{ with } y \leq s \ltimes w. \tag{4.2}$$

To this end, define $X'_{s,w}$ as in the proof of Proposition 4.9 and let $p_y \in \mathcal{A}$ for $y \in \mathbf{I}$ be such that $X'_{s,w} = \sum_{y \leq s\ltimes w} p_y L'_y$. We claim that to prove (4.2) it is enough to show that

$$v^{\ell(w)-\ell(y)+2}p_y \in 1 + v^2\mathbb{Z}[v^2] \qquad \text{for all } y \in \mathbf{I} \text{ with } y \leq s \ltimes w. \tag{4.3}$$

This follows when $sw \neq ws$ as then $\ell(s \ltimes w) = \ell(w) + 2$ and $p_y = \pi'_{y,s\ltimes w}$. Alternatively, suppose that $sw = ws$ and (4.3) holds. We then have

$$p_y = (v + v^{-1})\pi'_{y,sw} - \mu'(y, sw) - \sum_{y<z<sw} \mu'(z, sw)\pi'_{y,z}. \tag{4.4}$$

To deduce (4.2), we argue by induction on $\ell(sw)-\ell(y)$. If $y = sw$, then the desired containment holds automatically. Let $y < sw$ and suppose $v^{\ell(sw)-\ell(z)}\pi'_{z,sw} \in 1 + v^2\mathbb{Z}[v^2]$ for $y < z \leq sw$. Then $\mu'(z, sw)$ is nonzero for $z > y$ only if $\ell(w) - \ell(z)$ is even, so if we multiply both sides of (4.4) by $v^{\ell(w)-\ell(y)+2}$, then it follows from (4.3) via our inductive hypothesis that

$$(v^2 + 1)v^{\ell(sw)-\ell(y)}\pi'_{y,sw} - v^{\ell(sw)-\ell(y)+1}\mu'(y, sw) \in 1 + v^2\mathbb{Z}[v^2].$$

Since we always have $\pi'_{y,sw} \in v^{-1}\mathbb{Z}[v^{-1}]$ and $\mu'(y, sw) \in \mathbb{Z}$, this containment can only hold if $\mu'(y, sw) = 0$ whenever $\ell(sw) - \ell(y)$ is even. We deduce from this that in fact

$$(v^2 + 1)v^{\ell(sw)-\ell(y)}\pi'_{y,sw} \in 1 + v^2\mathbb{Z}[v^2],$$

and it is easy to see that this implies $v^{\ell(sw)-\ell(y)}\pi'_{y,sw} \in 1 + v^2\mathbb{Z}[v^2]$, which is what we needed to show. We conclude by induction that (4.3) implies (4.2).

We now argue that (4.3) holds. Fix $y \leq s \ltimes w$. Proposition 4.7 then implies

$$p_y = (a + \delta_{sw,ws}) \cdot \pi'_{y,w} + b \cdot \pi'_{s\ltimes y,w} - \Sigma$$

where

$$(a, b) = \begin{cases} (v^{-2},\ 1) & \text{if } sy \neq ys > y \\ (v^2,\ 1) & \text{if } sy \neq ys < y \\ (v^{-2} - 1,\ v^{-1} - v) & \text{if } sy = ys > y \\ (v^2 + 1,\ v^{-1} + v) & \text{if } sy = ys < y \end{cases}$$

and

$$\Sigma = \sum_{z < s \ltimes w} \mu'(s, z, w)\pi'_{y,z}.$$

Since we assume that $v^{\ell(z')-\ell(z)}\pi'_{z,z'} \in 1 + v^2\mathbb{Z}[v^2]$ for $z \le z' \le w$, inspecting our definition shows that $\mu'(s, z, w)$ is an integer when $\ell(w) - \ell(z)$ is even and an integer multiple of $v + v^{-1}$ when $\ell(w) - \ell(z)$ is odd. Consequently, it follows that

$$v^{\ell(w)-\ell(y)+2}\Sigma \in v^2\mathbb{Z}[v^2].$$

In turn, since $y \le s \ltimes w$, [9, Lemma 2.7] implies that $s \ltimes y \le w$ if $sy < y$ and that $y \le w$ if $sy > y$. Using this fact and the hypothesis stated in the second sentence of this proof, one checks that

$$v^{\ell(w)-\ell(y)+2}\left((a + \delta_{sw,ws}) \cdot \pi'_{y,w} + b \cdot \pi'_{s\ltimes y,w}\right) \in 1 + v^2\mathbb{Z}[v^2].$$

Combining these observations, we conclude that (4.3) holds. □

Despite these results, there does not appear to be any simple relationship between the polynomials $\pi_{y,w}$ and $\pi'_{y,w}$, and it is unclear what positivity properties the latter polynomials possess, if any. In general, $\pi'_{y,w}$ may have both positive and negative coefficients. The combination of Propositions 2.19, 4.3, 4.9, and 4.10 shows that

$$\tfrac{1}{2}\left(h_{y,w} \pm \pi'_{y,w}\right) \qquad \text{and} \qquad \tfrac{1}{2}\left(\pi_{y,w} \pm \pi'_{y,w}\right) \tag{4.5}$$

are polynomials in v^{-1} with integer coefficients, which become polynomials in v^2 when multiplied by $v^{\ell(w)-\ell(y)}$. Unlike the analogous polynomials $\frac{1}{2}\left(h_{y,w} \pm \pi_{y,w}\right)$ discussed at the end of the previous section (which conjecturally belong to $\mathbb{N}[v^{-1}]$), the four polynomials in (4.5) can each have both positive and negative coefficients.

4.3 A third canonical basis for twisted involutions

We finally prove here that the matrix Γ from Section 3.3 is a pre-canonical $(\mathcal{H}, \mathbf{I})$-structure. This provides us with another a canonical basis indexed by the twisted involutions in a Coxeter group, but not related in any transparent way to our other bases $\{\underline{L}_w\}$ and $\{\underline{L}'_w\}$. It is an open problem to find an interpretation of this third basis.

Theorem 4.11. *There is a unique $\mathcal{H}$-module*

$$\mathcal{I} = \mathcal{I}(W, S)$$

which, as an $\mathcal{A}$-module, is free with a basis given by the symbols I_w for $w \in \mathbf{I}$, and which satisfies

$$H_s I_w = \begin{cases} I_{sws} & \text{if } s \ltimes w = sws > w \\ I_{sws} + (v - v^{-1}) I_w & \text{if } s \ltimes w = sws < w \\ I_{sw} + I_w & \text{if } s \ltimes w = sw > w \\ (v - v^{-1}) I_{sw} + (v - 1 - v^{-1}) I_w & \text{if } s \ltimes w = sw < w \end{cases}$$

for $s \in S$ *and* $w \in \mathbf{I}$.

Proof. Define $J_w = (v + v^{-1})^{2\rho(w) - \ell(w)} L_w \in \mathcal{L}$ and let

$$\mathcal{J} = \mathbb{Z}[v^2, v^{-2}]\text{-span}\{J_w : w \in \mathbf{I}\}.$$

Define $\phi : \mathcal{I} \to \mathcal{J}$ as the $\mathbb{Z}$-linear bijection with $v^n I_w \mapsto v^{2n} J_w$ for $w \in \mathbf{I}$. With $\Phi : \mathcal{H} \to \mathcal{H}_2$ the ring homomorphism (3.1), the multiplication formula $HI = \phi^{-1}(\Phi(H)\phi(I))$ for $H \in \mathcal{H}$ and $I \in \mathcal{I}$ makes $\mathcal{I}$ into an $\mathcal{H}$-module, and one checks that relative to this structure, the action of H_s on I_w is described by precisely the given formula. This $\mathcal{H}$-module structure is unique since the elements H_s for $s \in S$ generate $\mathcal{H}$ as an $\mathcal{A}$-algebra. □

Theorem is equivalent to the assertion that Γ defined before Theorem 3.15 is an $(\mathcal{H}, \mathbf{I})$-structure. In turn we have this analogue of Theorems 4.2 and 4.6 showing that Γ is pre-canonical.

Theorem 4.12. *Define*

- *the "bar involution" of $\mathcal{I}$ as the $\mathcal{A}$-antilinear map $\mathcal{I} \to \mathcal{I}$, denoted $I \mapsto \overline{I}$, with*

$$\overline{I_{(x,\theta)}} = \operatorname{sgn}(x) \cdot \overline{H_x} \cdot I_{(x^{-1},\theta)} \qquad \text{for } (x, \theta) \in \mathbf{I}.$$

- *the "standard basis" of $\mathcal{I}$ to be $\{I_w\}$ with the partially ordered index set* $(\mathbf{I}, \leq)$.

This is a pre-canonical $\mathcal{H}$-module structure on $\mathcal{I}$ and it admits a canonical basis $\{\underline{I}_w\}$.

Again by Lemma 2.15, this is the unique pre-canonical $\mathcal{H}$-module structure on $\mathcal{I}$ in which $\{I_w\}$ serves as the "standard basis."

Proof. Define $\mathcal{J}$ and $\Phi : \mathcal{H} \to \mathcal{H}_2$ and $\phi : \mathcal{I} \to \mathcal{J}$ as in the proof of Theorem 4.11. The bar involution given in Theorem 4.2 for $\mathcal{L}$ restricts to an $\mathcal{A}$-antilinear map $\mathcal{J} \to \mathcal{J}$. Denote this restricted map by ψ', and write $\psi : I \mapsto \overline{I}$ for the bar involution of $\mathcal{I}$. Since $\Phi(\overline{H_x}) = \overline{K_x}$ for all $x \in W$, it follows that $\psi = \phi^{-1} \circ \psi' \circ \phi$, and from this identity, the claim that $(\psi, \{I_w\})$ is a pre-canonical $\mathcal{H}$-module structure on $\mathcal{I}$ follows as a consequence of Theorem 4.2. Given this, we conclude that a canonical basis $\{\underline{I}_w\}$ exists by Theorem 2.5. □

Remark 4.13. Suppose (W', S') is a Coxeter system such that $W = W' \times W'$ and $S = S' \sqcup S'$. Let $\theta \in \operatorname{Aut}(W, S)$ be the automorphism with $\theta(x, y) = (y, w)$. There is then an injective $\mathcal{A}$-module homomorphism $\mathcal{H}(W', S') \to \mathcal{I}(W, S)$ with

$$H_w \mapsto I_{((w,w^{-1}),\theta)} \qquad \text{which also maps} \qquad \underline{H}_w \mapsto \underline{I}_{((w,w^{-1}),\theta)} \qquad \text{for } w \in W'.$$

Via this map, one may view the canonical basis of $\mathcal{I}$ as a generalization of the Kazhdan–Lusztig basis of $\mathcal{H}$. The canonical bases of $\mathcal{L}$ and $\mathcal{L}'$ generalize the canonical basis of $\mathcal{H}_2$ in an entirely analogous fashion.

Define $\iota_{y,w} \in \mathbb{Z}[v^{-1}]$ for $y, w \in \mathbf{I}$ such that $\underline{I}_w = \sum_{y\in\mathbf{I}} \iota_{y,w} I_y$ and let

$$\nu(s, y, w) = \begin{cases} \text{the coefficient of } v^{-1} \text{ in } \iota_{y,w} & \text{if } sy < y \\ \text{the coefficient of } v^{-1} \text{ in } \iota_{sy,w} & \text{if } sy = ys > y \\ 0 & \text{otherwise} \end{cases}$$

for $s \in S$ and $y, w \in \mathbf{I}$. Recall that $\underline{H}_s = H_s + v^{-1}$ for $s \in S$.

Proposition 4.14. *If $s \in S$ and $w \in \mathbf{I}$ such that $w < sw$, then*

$$\underline{H}_s \underline{I}_w = \underline{I}_{s\ltimes w} + \delta_{sw,ws} \underline{I}_w + \sum_{y<w} \nu(s, y, w) \underline{I}_y.$$

Remark 4.15. Unlike the canonical basis $\{\underline{L}_w\}$ (see the remark after Proposition 4.7), there is no simple formula for $\underline{H}_s \underline{I}_w$ when $s \in S$ such that $sw < w$.

Proof. The difference between the two sides of the desired identity is invariant under the bar involution of $\mathcal{I}$ and is also an element of $\sum_{y<s\ltimes w} v^{-1}\mathbb{Z}[v^{-1}] \cdot I_y$, as is straightforward to check from the definition of $\nu(s, y, w)$ and Theorem 4.11. The only such element in $\mathcal{I}$ is 0. □

We note one more result. Recall the definition of $\rho : \mathbf{I} \to \mathbb{N}$ from Theorem 2.17.

Proposition 4.16. *If $y, w \in \mathbf{I}$ such that $y \le w$, then $v^{\rho(w)-\rho(y)} \iota_{y,w} \in 1 + v\mathbb{Z}[v]$.*

Proof. The proof is by induction on $\rho(w)$ using Proposition 4.14. We omit the details, which are similar to and somewhat simpler than those in the proof of Proposition 4.10. □

Computations indicate that there is no obvious relationship between the polynomials $\iota_{y,w}$ and the other polynomials $h_{y,w}, \pi_{y,w}, \pi'_{y,w} \in \mathbb{Z}[v^{-1}]$ we have seen so far. For example, suppose $|S| = 2$ so that (W, S) is a dihedral Coxeter system. Then the values of $v^{\ell(w)-\ell(y)} h_{y,w}$ (for $y, w \in W$) and $v^{\ell(w)-\ell(y)} \pi_{y,w}$ (for $y, w \in \mathbf{I}$) are all 0 or 1; see [17, Theorem 4.3]. However, the polynomials $v^{\rho(w)-\rho(y)} \iota_{y,w}$ for $y, w \in \mathbf{I}$ can achieve any of the values 0, 1, $1+v$, $1-v$, or $1-v^2$. The polynomials $\iota_{y,w}$ may thus have negative coefficients, and do not in general satisfy any parity condition analogous to Proposition 4.9.

This means that the pre-canonical structure on $\mathcal{I}$ does not arise from a P-kernel, since by the preceding proposition it is not in the image of the bijection in Theorem 2.12 for any choice of function $r : \mathbf{I} \to \mathbb{Z}$. By contrast, it follows from [12, §2], [13, Proposition 4.4(b)], and the proof of Theorem 4.6, respectively, that the pre-canonical structures on $\mathcal{H}$, $\mathcal{L}$, and $\mathcal{L}'$ are all in the image of this bijection relative to the function $r = \ell$, and so correspond to P-kernels.

5 Uniqueness proofs

In this section we at last give the proofs to the main results in Sections 3.3 and 3.4. Throughout, we recall our earlier definitions of $(\mathcal{H}, \mathbf{I})$- and $(\mathcal{H}_2, \mathbf{I})$-structures, and what it means for such structures to be pre-canonical.

5.1 Proofs for results on generic structures

We first prove Theorem 3.15, classifying all nontrivial $(\mathcal{H}, \mathbf{I})$-structures, after stating two lemmas. Denote by Θ the $\mathcal{A}$-algebra automorphism of $\mathcal{H}$ with $\Theta(H_s) = -H_s + v - v^{-1}$ for $s \in S$. Observe that more generally $\Theta(H_w) = \mathrm{sgn}(w) \cdot \overline{H_w}$ for $w \in W$.

Lemma 5.1. *The involution of the set of* 4×2 *matrices with entries in* $\mathcal{A}$ *given by the map*

$$\Theta : \begin{bmatrix} A & B \\ C & D \\ E & F \\ G & H \end{bmatrix} \mapsto \begin{bmatrix} -A & v + v^{-1} - B \\ -C & v + v^{-1} - D \\ -E & v + v^{-1} - F \\ -G & v + v^{-1} - H \end{bmatrix}$$

restricts to an involution of the set of $(\mathcal{H}, \mathbf{I})$*-structures.*

Proof. Observe that if γ is an $(\mathcal{H}, \mathbf{I})$-structure, then $\rho_{\Theta(\gamma)}$ is the $\mathcal{H}$-representation $\rho_\gamma \circ \Theta$. □

The next lemma is more technical. Fix a choice of parameters A, B, C, D, E, F, G, $H \in \mathcal{A}$ and define γ as in Lemma 3.14.

Lemma 5.2. *If* γ *is an* $(\mathcal{H}, \mathbf{I})$*-structure, then the following properties hold:*

(a) $(B - v)(B + v^{-1}) = (D - v)(D + v^{-1}) = -AC$.
(b) $(F - v)(F + v^{-1}) = (H - v)(H + v^{-1}) = -EG$.
(c) If A *or* C *is nonzero, then* $B + D = v - v^{-1}$ *and* $D - H \in \{\pm 1\}$.
(d) If E *or* G *is nonzero, then* $F + H = v - v^{-1}$ *and* $B - F \in \{\pm 1\}$.
(e) If A, C, E, G *are all nonzero, then* $B \in \{0, v - v^{-1}\}$.

Proof. In this proof we abbreviate by letting $\rho = \rho_\gamma$. Suppose $s, t \in S$ are such that st has order 3. Since ρ defines a representation of $\mathcal{H}$, we have $(\rho(H_s) - v)(\rho(H_s) + v^{-1})w = 0$ for all $w \in \mathbf{I}$. Expanding the left side of this identity for the elements $w \in \{1, s, t, sts\} \subset W \cap \mathbf{I}$ yields the equations in parts (a) and (b), and also the identities

$$X(B + D + v^{-1} - v) = 0 \qquad \text{and} \qquad Y(F + H + v^{-1} - v) = 0$$

for $X \in \{A, C\}$ and $Y \in \{E, G\}$. It follows that if A or C is nonzero, then $B + D = v - v^{-1}$ and that if E or G is nonzero, then $F + H = v - v^{-1}$.

We also have $\rho(H_s)\rho(H_t)\rho(H_s)w = \rho(H_t)\rho(H_s)\rho(H_t)w$ for all $w \in \mathbf{I}$. Expanding both sides of this identity for $w \in \{1, s, t, sts\} \subset W \cap \mathbf{I}$ and then comparing coefficients yields the identities

$$X(D^2 + (B - D)H - EG) = 0 \quad \text{and} \quad Y(F^2 + B(H - F) - AC) = 0 \quad (5.1)$$

again for $X \in \{A, C\}$ and $Y \in \{E, G\}$. Assume A or C is nonzero, so that we can take X to be nonzero. Then $B - D = v - v^{-1} - 2D$ and $-EG = (H - v)(H + v^{-1})$. Substituting these identities into the first equation in (5.1) and dividing both sides by X produces the equation

$$D^2 + (v - v^{-1} - 2D)H + (H - v)(H + v^{-1}) = 0.$$

The left-hand sides simplifies to the expression $(D - H)^2 - 1$, and thus $D - H \in \{\pm 1\}$. This establishes part (c). In a similar way one finds that if E or G is nonzero, then $B - F \in \{\pm 1\}$, which establishes part (d).

To prove part (e), suppose now that $s, t \in S$ are such that st has order 4. Then $(\rho(H_s)\rho(H_t))^2 w = (\rho(H_t)\rho(H_s))^2 w$ for all $w \in \mathbf{I}$. Expanding both sides of this equation for $w = 1$ and comparing the coefficients of sts yields the identity $AE(DF + BH - EG) = 0$. Assume A, C, E, G are all nonzero. Then, after dividing both sides by AE and applying the substitutions $D = v - v^{-1} - B$ and $H = v - v^{-1} - F$ and $-EG = (F - v)(F + v^{-1})$, our previous identity becomes

$$(v - v^{-1} - B)B + (B - F)^2 - 1 = 0.$$

Since $(B - F)^2 - 1 = 0$ by part (d), either $B = 0$ or $B = v - v^{-1}$, as claimed. □

Proof (of Theorem 3.15). We first show that $\Gamma, \Gamma', \Gamma'', \Gamma'''$ are all $(\mathcal{H}, \mathbf{I})$-structures. The matrices Γ and Γ''' are $(\mathcal{H}, \mathbf{I})$-structures since the corresponding representations just describe the action of $\mathcal{H}$ on the respective bases $\{I_w\}$ and $\{\overline{I_w}\}$ of $\mathcal{I}$, as defined in Theorem 4.11. The matrices Γ' and Γ'' are $(\mathcal{H}, \mathbf{I})$-structures by Lemmas 3.14 and 5.1, since $\Gamma' = \Theta(\Gamma)[-1, -1]$ and $\Gamma'' = \Theta(\Gamma''')[-1, -1]$.

Fix a choice of parameters $A, B, C, D, E, F, G, H \in \mathcal{A}$ and define the 4×2 matrix γ as in Lemma 3.14. Assume γ is an $(\mathcal{H}, \mathbf{I})$-structure. We show that γ is diagonally equivalent to Γ, Γ', Γ'', or Γ'''. There are four cases to consider:

- Suppose $AC = EG = 0$. Then $B, D, F, H \in \{-v^{-1}, v\}$ by Lemma 5.2, and by Lemma 3.14 we may assume that $A, C, E, G \in \{0, 1\}$. There are 144 choices of parameters satisfying these conditions. With the aid of the computer algebra system MAGMA, we have checked that the only matrices γ of this form that are $(\mathcal{H}, \mathbf{I})$-structures are the two trivial ones. (For this calculation, it suffices just to consider finite Coxeter systems of rank three.)
- Suppose $AC \neq 0$ and $EG = 0$. By Lemma 3.14 we may then assume that $E, G \in \{0, 1\}$. By the second and third parts of Lemma 5.2, it follows that $F, H \in \{-v^{-1}, v\}$ and $D \in \{H \pm 1\}$ and $B = v - v^{-1} - D$. By Lemma 3.14 and the first part of Lemma 5.2, finally, we may assume that $A = 1$ and $C = -(D - v)(D + v^{-1}) \neq 0$. This leaves 8 possible choices of parameters,

and we have checked (again with the help of a computer) that for each of the resulting matrices γ, there are finite Coxeter systems (W, S) for which ρ_γ fails to define an $\mathcal{H}(W, S)$-representation. Hence it cannot occur that $AC \neq 0$ and $EG = 0$.

- It follows by similar considerations that it cannot happen that $AC = 0$ and $EG \neq 0$.
- Finally suppose $AC \neq 0$ and $EG \neq 0$ so that A, C, E, G are all nonzero. By Lemma 5.2 we then have $B \in \{0, v - v^{-1}\}$ and $D = v - v^{-1} - B$ and $F \in \{B \pm 1\}$ and $H = v - v^{-1} - F$ and $AC = 1$ and $EG \in \{\pm(v - v^{-1})\}$; more specifically, Lemma 5.2 implies that $EG = v - v^{-1}$ when $B = 0 = F - 1$ or $B = v - v^{-1} = F - 1$ while in all other cases $EG = v^{-1} - v$. There are thus four choices for the quadruple (B, D, F, H) and it is easy to see by Lemma 3.14 that in each case γ is diagonally equivalent to one of Γ, Γ', Γ'', or Γ'''.

This completes the proof of the theorem. □

The property of an $(\mathcal{H}, \mathbf{I})$-structure being pre-canonical is preserved under the operations in Lemmas 3.14 and 5.1, in the following precise sense.

Lemma 5.3. *If γ is a nontrivial, pre-canonical $(\mathcal{H}, \mathbf{I})$-structure, then $\gamma[-1, -1]$ is also, and the (unique) associated pre-canonical structures on $\mathcal{A}\mathbf{I}$ are isomorphic via the identity map, which has as a scaling factor the $\mathcal{A}$-linear map $\mathcal{A}\mathbf{I} \to \mathcal{A}\mathbf{I}$ with $w \mapsto (-1)^{\rho(w)} w$ for $w \in \mathbf{I}$.*

Proof. Let γ be a nontrivial, pre-canonical $(\mathcal{H}, \mathbf{I})$-structure, and let $\gamma' = \gamma[-1, -1]$. Let $(\psi, \mathbf{I})$ be the unique pre-canonical structure on $\mathcal{A}\mathbf{I}$ such that $\psi\left(\rho_\gamma\left(\overline{H}\right) I\right) = \rho_\gamma\left(\overline{H}\right) \psi(I)$ for $H \in \mathcal{H}$ and $I \in \mathcal{A}\mathbf{I}$. Let $\psi' = D^{-1} \circ \psi \circ D$ where $D : \mathcal{A}I \to \mathcal{A}I$ is the $\mathcal{A}$-linear map with $D(w) = (-1)^{\rho(w)} w$ for $w \in \mathbf{I}$. Since $\rho_{\gamma'}(H) = D^{-1} \circ \rho_\gamma(H) \circ D$ for $H \in \mathcal{H}$, it follows that $(\psi', \mathbf{I})$ is a pre-canonical structure on $\mathcal{A}\mathbf{I}$ such that

$$\psi'\left(\rho_{\gamma'}\left(\overline{H}\right) I\right) = \rho_{\gamma'}\left(\overline{H}\right) \psi'(I) \qquad \text{for } H \in \mathcal{H} \text{ and } I \in \mathcal{A}\mathbf{I}.$$

Thus γ' is pre-canonical. Moreover, the identity map $\mathcal{A}\mathbf{I} \to \mathcal{A}\mathbf{I}$ is evidently an isomorphism between the pre-canonical structures $(\psi, \mathbf{I})$ and $(\psi', \mathbf{I})$, with D as a scaling factor. □

Lemma 5.4. *If γ is a nontrivial, pre-canonical $(\mathcal{H}, \mathbf{I})$-structure, then so is $\Theta(\gamma)$, and the (unique) associated pre-canonical structures on $\mathcal{A}\mathbf{I}$ are strongly isomorphic via the identity map.*

Proof. Let γ be a nontrivial, pre-canonical $(\mathcal{H}, \mathbf{I})$-structure, and define $\gamma' = \Theta(\gamma)$. Let $(\psi, \mathbf{I})$ be the unique pre-canonical structure on $\mathcal{A}\mathbf{I}$ such that $\psi\left(\rho_\gamma\left(\overline{H}\right) I\right) = \rho_\gamma\left(\overline{H}\right) \psi(I)$ for $H \in \mathcal{H}$ and $I \in \mathcal{A}\mathbf{I}$. Then it also holds that $\psi\left(\rho_{\gamma'}\left(\overline{H}\right) I\right) = \rho_{\gamma'}\left(\overline{H}\right) \psi(I)$ for $H \in \mathcal{H}$ and $I \in \mathcal{A}\mathbf{I}$ since $\rho_{\gamma'}(H) = \rho(\Theta(H))$ and $\overline{\Theta(H)} = \Theta\left(\overline{H}\right)$. Thus γ' is also pre-canonical and its associated pre-canonical structure is strongly isomorphic to the one associated to γ. □

Before we can prove Theorem 3.16, we require an additional lemma. For this, let

$$\mathcal{I}, \quad \mathcal{I}', \quad \mathcal{I}'', \quad \text{and} \quad \mathcal{I}'''$$

be the free $\mathcal{A}$-modules with a basis given by the symbols I_w, I'_w, I''_w, and I'''_w respectively for $w \in \mathbf{I}$. View these as $\mathcal{H}$-modules relative to the $(\mathcal{H}, \mathbf{I})$-structure Γ, Γ', Γ'', and Γ''' respectively. Of course, $\mathcal{I}$ defined in this way is the same thing as $\mathcal{I}$ defined by Theorem 4.11. In addition, let ϵ denote the ring endomorphism of $\mathcal{A}$ with $\epsilon(v) = -v$.

Lemma 5.5. *There are unique pre-canonical $\mathcal{H}$-module structures on $\mathcal{I}$, $\mathcal{I}'$, $\mathcal{I}''$, $\mathcal{I}'''$, respectively, in which $\{I_w\}$, $\{I'_w\}$, $\{I''_w\}$, $\{I'''_w\}$ indexed by $(\mathbf{I}, \leq)$ are the "standard bases." Moreover, these pre-canonical structures are all isomorphic; the following maps are isomorphisms:*

(a) The $\mathcal{A}$-linear map $\mathcal{I} \to \mathcal{I}'$ with $I_w \mapsto I'_w$ for $w \in \mathbf{I}$.
(b) The $\mathcal{A}$-linear map $\mathcal{I}'' \to \mathcal{I}'''$ with $I''_w \mapsto I'''_w$ for $w \in \mathbf{I}$.
(c) The ϵ-linear map $\mathcal{I} \to \mathcal{I}'''$ with $I_w \mapsto I'''_w$ for $w \in \mathbf{I}$.

Finally, the morphisms in (a), (b), (c) have as respective scaling factors the $\mathcal{A}$-linear maps with

$$I_w \mapsto (-1)^{\rho(w)} I_w \quad \textit{and} \quad I''_w \mapsto (-1)^{\rho(w)} I''_w \quad \textit{and} \quad I_w \mapsto I_w \quad \textit{for } w \in \mathbf{I}.$$

Remark 5.6. The "bar involution" of $\mathcal{I}$ in the pre-canonical structure mentioned in this result is the one defined before Theorem 4.11. One can show, though we omit the details here, that the "bar involutions" of $\mathcal{I}'$, $\mathcal{I}''$, and $\mathcal{I}'''$ are the respective $\mathcal{A}$-antilinear maps with

$$I'_{(x,\theta)} \mapsto H_x \cdot I'_{(x^{-1},\theta)} \quad \text{and} \quad I''_{(x,\theta)} \mapsto \overline{H_x} \cdot I''_{(x^{-1},\theta)}$$
$$\text{and} \quad I'''_{(x,\theta)} \mapsto \operatorname{sgn}(x) \cdot H_x \cdot I'''_{(x^{-1},\theta)}$$

for twisted involutions $(x, \theta) \in \mathbf{I}$.

Proof. The uniqueness of the pre-canonical $\mathcal{H}$-module structures is clear from Lemma 2.15. From Theorem 4.12 we already have a bar involution $I \mapsto \overline{I}$ on $\mathcal{I}$ which forms a pre-canonical $\mathcal{H}$-module structure with $\{I_w\}$ as the standard basis. Define $r_{y,w} \in \mathcal{A}$ for $y, w \in \mathbf{I}$ such that $\overline{I_w} = \sum_{y \in \mathbf{I}} r_{y,w} I_y$. In addition, for $x \in W$ and $y, z \in \mathbf{I}$, let $f^x_{y,z} \in \mathcal{A}$ be such that $H_x I_y = \sum_{z \in \mathbf{I}} f^x_{y,z} I_z$.

Let $\mathcal{J}$ be the free $\mathcal{A}$-module with a basis given by the symbols J_w for $w \in \mathbf{I}$. View this as an $\mathcal{H}$-module relative to the $(\mathcal{H}, \mathbf{I})$-structure $\gamma = \Gamma''[-1, -1] = \Theta\ (\Gamma''')$, and define $J \mapsto \overline{J}$ as the $\mathcal{A}$-antilinear map $\mathcal{J} \to \mathcal{J}$ with $\overline{J_w} = \sum_{y \in \mathbf{I}} \epsilon(r_{y,w}) J_y$ for $w \in \mathbf{I}$. It is immediate that this bar involution forms a pre-canoncal structure on $\mathcal{J}$ with $\{J_w\}$ as the standard basis. Since $\overline{H_s J_y} = -\sum_{z \in \mathbf{I}} \epsilon(f^s_{y,z}) J_z$ for all $s \in S$ and $y \in \mathbf{I}$, it follows moreover that $\overline{H_s J_y} = \overline{H_s} \cdot \overline{J_y}$, which suffices to show that $\overline{H} \cdot \overline{J} = \overline{HJ}$ for all $H \in \mathcal{H}$ and $J \in \mathcal{J}$. We thus have a pre-canonical $\mathcal{H}$-module structure on $\mathcal{J}$. It is clear that the ϵ-linear map $\mathcal{I} \to \mathcal{J}$

with $I_w \mapsto J_w$ is an isomorphism of the pre-canonical structures on $\mathcal{I}$ and $\mathcal{J}$, which has the identity map as a scaling factor.

One deduces the remaining assertions in the lemma from the existence of these isomorphic pre-canonical structures on $\mathcal{I}$ and $\mathcal{J}$, using Lemmas 5.3 and 5.4 and the fact that

$$\Gamma' = \Theta(\Gamma)[-1,-1] \quad \text{and} \quad \Gamma'' = \gamma[-1,-1] \quad \text{and} \quad \Gamma''' = \Theta(\gamma). \qquad \square$$

Proof (of Theorem 3.16). Let γ be a nontrivial $(\mathcal{H}, \mathbf{I})$-structure which is pre-canonical, and write $\psi : \mathcal{A}\mathbf{I} \to \mathcal{A}\mathbf{I}$ for the associated bar involution. We claim that γ_{11} and γ_{31} must then belong to $\mathbb{Z}[v + v^{-1}]$. To see this, let $\theta \in \mathrm{Aut}(W, S)$ be an involution and let $s \in S$. If $s \neq \theta(s)$, then $w = (s \cdot \theta(s), \theta) \in \mathbf{I}$ and we have

$$\begin{aligned}\overline{\gamma_{11}} \cdot \psi(w) + \overline{\gamma_{12}} \cdot \theta &= \psi(\gamma(H_s)\theta) \\ &= \gamma(H_s + v^{-1} - v)\theta = \gamma_{11} \cdot w + (\gamma_{12} + v^{-1} - v) \cdot \theta.\end{aligned}$$

On the other hand, if $s = \theta(s)$, then $w = (s, \theta) \in \mathbf{I}$ and we have

$$\begin{aligned}\overline{\gamma_{31}} \cdot \psi(w) + \overline{\gamma_{32}} \cdot \theta &= \psi(\gamma(H_s)\theta) \\ &= \gamma(H_s + v^{-1} - v)\theta = \gamma_{31} \cdot w + (\gamma_{32} + v^{-1} - v) \cdot \theta.\end{aligned}$$

These equations, compared with the unitriangular property of the bar involution, imply that $\overline{\gamma_{11}} = \gamma_{11}$ and $\overline{\gamma_{31}} = \gamma_{31}$; hence these two parameters must belong to $\mathbb{Z}[v + v^{-1}]$ as claimed. Since Theorem 3.15 implies that

$$\gamma_{11} \cdot \gamma_{21} = 1 \qquad \text{and} \qquad \gamma_{31} \cdot \gamma_{41} \in \{\pm(v - v^{-1})\},$$

it necessarily follows that $\gamma_{11}, \gamma_{31} \in \{\pm 1\}$. From Theorem 3.15 we conclude that for some $\varepsilon_i \in \{\pm 1\}$ we have $\gamma[\varepsilon_1, \varepsilon_2] \in \{\Gamma, \Gamma', \Gamma'', \Gamma'''\}$. Thus γ must be one of 16 different $(\mathcal{H}, \mathbf{I})$-structures. It is a simple exercise to show that γ is pre-canonical if and only if $\gamma[\varepsilon_1, \varepsilon_2]$ is pre-canonical; moreover, the associated pre-canonical structures are isomorphic. Hence, by Lemma 5.5 we conclude that all 16 possibilities for γ are pre-canonical, and that the associated pre-canonical structures are all isomorphic to the one in Theorem 4.12. $\square$

Finally, we return to Theorems 3.19 and 3.20. These results follow by arguments similar to the ones just given, and so we only sketch the main ideas to their proofs.

Proof (Sketch of Theorem 3.19's proof). The result follows by nearly the same argument as in the proof Theorem 3.15, using three lemmas analogous to Lemmas 3.14, 5.1, and 5.2, *mutatis mutandis*. We omit the details. $\square$

Proof (Sketch of Theorem 3.20's proof). One deduces that at most 32 nontrivial $(\mathcal{H}_2, \mathbf{I})$-structures are pre-canonical exactly as in the proof of Theorem 3.16: first argue that any such structure γ has $\overline{\gamma_{11}} = \gamma_{11}$ and $\overline{\gamma_{31}} = \gamma_{31}$, and then appeal to Theorem 3.19. The claim that these $(\mathcal{H}_2, \mathbf{I})$-structures are in fact all pre-canonical, along with the second sentence in the theorem, follows from Lemmas 5.3 and 5.4, which hold *mutatis mutandis* with "$(\mathcal{H}, \mathbf{I})$-structure" replaced by $(\mathcal{H}_2, \mathbf{I})$-structure" and Θ replaced by a slightly different involution on 4×2 matrices. $\square$

5.2 Application to inversion formulas

In this last section, we use the lemmas in the previous section to prove an inversion formula for the canonical bases introduced in Section 4. Let V be a free $\mathcal{A}$-module of finite rank, with a pre-canonical structure $(\psi, \{a_c\})$, the standard basis indexed by $(C, \leq)$. Define V^* as the set of $\mathcal{A}$-linear maps $V \to \mathcal{A}$. This is naturally a free $\mathcal{A}$-module: a basis is given by the $\mathcal{A}$-linear maps $a_c^* : V \to V$ for $c \in C$ defined by

$$a_c^*(a_{c'}) = \delta_{c,c'} \qquad \text{for } c' \in C.$$

Define $\psi^* : V^* \to V^*$ as the $\mathcal{A}$-antilinear map such that

$$\psi^*(f)(v) = \overline{f \circ \psi(v)} \qquad \text{for } f \in V^* \text{ and } v \in V.$$

Also let $\leq^{\text{op}}$ denote the partial order on C with $c \leq^{\text{op}} c'$ if and only if $c' \leq c$. The following appears in a slightly more general form as [22, Proposition 7.1].

Proposition 5.7 (Webster [22]). *The "bar involution" ψ^* and "standard basis" $\{a_c^*\}$, indexed by the partially ordered set $(C, \leq^{\text{op}})$, form a pre-canonical structure on V^*. If V has a canonical basis $\{b_c\}$, then the dual basis $\{b_c^*\}$ of V^* is canonical relative to $(\psi^*, \{a_c^*\})$.*

Let $\mathcal{B}$ denote a free $\mathcal{A}$-algebra with a pre-canonical structure; write $\overline{b}$ for the image of $b \in \mathcal{B}$ under the corresponding bar involution. Suppose V is a $\mathcal{B}$-module and $(\psi, \{a_c\})$ is a pre-canonical $\mathcal{B}$-module structure. Assume $\mathcal{B}$ has a distinguished $\mathcal{A}$-algebra antiautomorphism $b \mapsto b^\dagger$. We may then view V^* as a $\mathcal{B}$-module by defining bf for $b \in \mathcal{B}$ and $f \in V^*$ to be the map with the formula

$$(bf)(v) = f(b^\dagger v) \qquad \text{for } v \in V. \tag{5.2}$$

Proposition 5.8. *Suppose the maps $b \mapsto b^\dagger$ and $b \mapsto \overline{b}$ commute. Then the pre-canonical structure $(\psi^*, \{a_c^*\})$ on V^* is a pre-canonical $\mathcal{B}$-module structure.*

Proof. One just needs to check that if $b \in \mathcal{B}$ and $f \in V^*$, then $\psi^*(bf) = \overline{b} \cdot \psi^*(f)$, and this is straightforward from the commutativity hypothesis in the proposition. □

Assume (W, S) is a finite Coxeter system, so that W has a longest element w_0. Recall that since the longest element is unique, we have $w_0 = w_0^{-1} = \theta(w_0)$ for all $\theta \in \text{Aut}(W, S)$. Write θ_0 for the inner automorphism of W given by $w \mapsto w_0 w w_0$. This map is an automorphism of the poset $(W, \leq)$ and in particular is length-preserving [1, Proposition 2.3.4(ii)]; thus it belongs to $\text{Aut}(W, S)$. In fact, θ_0 lies in the center of $\text{Aut}(W, S)$. Let $w_0^+ = (w_0, \theta_0) \in W^+$. Observe that w_0^+ is a central involution in W^+, and so if $w = (x, \theta) \in \mathbf{I}$, then $ww_0^+ = (xw_0, \theta\theta_0) \in \mathbf{I}$.

We may use the results in the previous sections to prove an inversion formula for the structure constants of the canonical bases of $\mathcal{L}$, $\mathcal{L}'$, and $\mathcal{I}$ given in Theorems 4.2, 4.6, and 4.12.

Theorem 5.9. *Let $F \in \{\pi, \pi', \iota\}$. Then*

$$\sum_{w\in\mathbf{I}}(-1)^{\rho(x)+\rho(w)} \cdot F_{x,w} \cdot F_{yw_0^+,ww_0^+} = \delta_{x,y} \qquad \textit{for } x, y \in \mathbf{I}.$$

Lusztig proves the version of this statement with $F = \pi$ as [13, Theorem 7.7].

Proof. We only consider the case $F = \iota$, as the argument in the other cases is similar. There is a unique antiautomorphism $H \mapsto H^\dagger$ of $\mathcal{H}$ with $H_w \mapsto H_{w^{-1}}$ for $w \in W$. We make $\mathcal{I}^*$ into an $\mathcal{H}$-module relative to this antiautomorphism via the formula (5.2). Let $s \in S$ and $w \in \mathbf{I}$. Since w_0^+ is central, we have $sw = ws$ if and only if $sww_0^+ = ww_0^+s$. Since $x \le y$ if and only if $yw_0 \le xw_0$ for any $x, y \in W$ (see [1, Proposition 2.3.4(i)]), it follows that $sw < w$ if and only if $sww_0^+ > sww_0^+$, and also that $\rho(xw_0^+) - \rho(yw_0^+) = \rho(y) - \rho(x)$ for $x, y \in \mathbf{I}$. Given these facts it is straightforward to check that if $\mathcal{I}'$ is the $\mathcal{H}$-module defined before Lemma 5.5, then the $\mathcal{A}$-linear map $\varphi : \mathcal{I}' \to \mathcal{I}^*$ with $\varphi(I'_w) = I^*_{ww_0^+}$ for $w \in \mathbf{I}$ is an isomorphism of $\mathcal{H}$-modules.

We have a pre-canonical $\mathcal{H}$-module structure on $\mathcal{I}'$ from Lemma 5.5. Likewise, since the maps $H \mapsto H^\dagger$ and $H \mapsto \overline{H}$ commute, we have a pre-canonical $\mathcal{H}$-module structure on $\mathcal{I}^*$ from Proposition 5.8. Write ψ^* for the bar involution of $\mathcal{I}^*$ in this structure. Then $(\varphi^{-1} \circ \psi^* \circ \varphi, \{I'_w\})$ is another pre-canonical $\mathcal{H}$-module structure on $\mathcal{I}'$, so the uniqueness assertion in Lemma 5.5 implies that $\varphi^{-1} \circ \psi^* \circ \varphi$ is equal to the bar involution $I \mapsto \overline{I}$ on $\mathcal{I}'$, and thus φ is a strong isomorphism between the pre-canonical structures on $\mathcal{I}'$ and $\mathcal{I}^*$. Composing φ with the map in Lemma 5.5(a), it follows that the $\mathcal{A}$-linear map $\mathcal{I} \to \mathcal{I}^*$ with $I_w \mapsto I^*_{ww_0^+}$ is an isomorphism of pre-canonical structures (though not of $\mathcal{H}$-modules), having as a scaling factor the $\mathcal{A}$-linear map $D : \mathcal{I} \to \mathcal{I}$ with $D(I_w) = (-1)^{\rho(w)} I_w$ for $w \in \mathbf{I}$.

From Proposition 3.7, we deduce that elements of the canonical basis $\{\underline{I}^*_w\}$ of $\mathcal{I}^*$ have the form $\underline{I}^*_y = I^*_y + \sum_{w>y}(-1)^{\rho(y)-\rho(w)}\iota_{yw_0^+,ww_0^+} \cdot I^*_w$. Since $\underline{I}^*_y(\underline{I}_x) = \delta_{x,y}$ for $x, y \in \mathbf{I}$ by Proposition 5.7, we deduce that $MN = 1$ where M and N are the $(\mathbf{I} \times \mathbf{I})$-indexed matrices with $M_{y,w} = (-1)^{\rho(y)-\rho(w)}\iota_{ww_0^+,yw_0^+}$ and $N_{w,x} = \iota_{w,x}$. Since M and N are finite square matrices, $MN = 1$ implies $NM = 1$; the desired inversion formula is equivalent to the second equality. □

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Upper semicontinuity of KLV polynomials for certain blocks of Harish-Chandra modules

William M. McGovern

To David Vogan on his 60th birthday

Abstract We show that the coefficients of Kazhdan–Lusztig–Vogan polynomials attached to certain blocks of Harish-Chandra modules satisfy a monotonicity property relative to the closure order on K-orbits in the flag variety.

Key words: Kazhdan–Lusztig–Vogan polynomial, Harish-Chandra module, upper semicontinuity

MSC (2010): Primary 22E46; Secondary 20G05

Let G be a complex connected reductive group with Lie algebra $\mathfrak{g}$ and Borel subgroup B. Recall that the flag variety G/B decomposes into finitely many B-orbits $\mathcal{O}_w$, which are indexed by elements w of the Weyl group W of G. The closure $\overline{\mathcal{O}_w}$ of an orbit $\mathcal{O}_w$ is called a Schubert variety. Kazhdan and Lusztig introduced polynomials $P_{v,w}$ in one variable q, indexed by pairs v, w of elements in W in [KL79], which, as they later showed, measure the singularities of Schubert varieties [KL80]. More precisely, they showed that the coefficient c_i of q^i in $P_{v,w}$ satisfies

$$c_i = \dim IH_x^{2i}(\overline{\mathcal{O}_w}; \overline{\mathbb{Q}}_p)$$

for any $x \in \mathcal{O}_v$, where the right side denotes the local $2i$th intersection cohomology group with values in the constant sheaf $\mathbb{Q}_p$, p is a prime, and w_0 is the longest

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element of W [KL80]. A fundamental result of Irving, first proved in [I88], using results of [GJ81], asserts that the singularities of $\overline{\mathcal{O}_w}$ increase as one goes down; more precisely, if $c_i^{v,w}$ denotes the coefficient of q_i in $P_{v,w}$, then $c_i^{v,w} \geq c_i^{v',w}$ whenever $v \leq v' \leq w$ in the Bruhat order on W. (Following Li and Yong [LY11], we call this property *upper semicontinuity*.) Irving's proof uses representation theory; later Braden and MacPherson gave a geometric argument in [BM01]. Braden and MacPherson's proof applies to many stratified varieties with a torus action (not just Schubert varieties), but not to closures of orbits of a symmetric subgroup on G/B. The purpose of this note is to establish the corresponding inequality in some cases for coefficients of Kazhdan–Lusztig–Vogan (KLV) polynomials attached to such orbit closures.

So let θ be an involutive automorphism of G and K the fixed points of θ acting on G. If $\mathcal{O}, \mathcal{O}'$ are two K-orbits in G/B with $\overline{\mathcal{O}} \subset \overline{\mathcal{O}'}$ and if γ, γ' are K-equivariant sheaves of one-dimensional $\overline{\mathbb{Q}}_p$ vector spaces on $\mathcal{O}, \mathcal{O}'$, respectively (more briefly, one-dimensional sheaves), then Lusztig and Vogan have constructed a polynomial $P_{\gamma,\gamma'}$ such that the coefficient d_i of q^i in $P_{\gamma,\gamma'}$ equals the dimension of the local $2i$th intersection cohomology sheaf of the Deligne–Goresky–MacPherson extension of γ' to the closure of $\mathcal{O}'$, supported at a point in $\mathcal{O}$ [LV83, V83]. If γ and γ' are trivial, then we write $P_{\mathcal{O},\mathcal{O}'}$ instead of $P_{\gamma,\gamma'}$. We then ask under what conditions, given three orbits $\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3$ with $\overline{\mathcal{O}_1} \subset \overline{\mathcal{O}_2} \subset \overline{\mathcal{O}_3}$, do we have

$$d_i^{\mathcal{O}_1,\mathcal{O}_3} \geq d_i^{\mathcal{O}_2,\mathcal{O}_3}$$

for all i, where the terms denote the coefficients of q^i in the polynomials corresponding to the pair of orbits in the superscripts. In general, this fails, for example if $G = \mathrm{SL}(3)$ (as we will observe in an example below). The problem stems from the existence of nontrivial sheaves γ, even though the inequality is stated for trivial γ only. Now we can state our result.

Theorem 1 *With notation as above, assume that all K-orbits $\mathcal{O}$ admit only the trivial sheaf (equivalently, all orbits $\mathcal{O}$ are simply connected, or (as is well known) all Cartan subgroups of the real form G_0 of G corresponding to K are connected). If $\overline{\mathcal{O}_1} \subseteq \overline{\mathcal{O}_2} \subseteq \overline{\mathcal{O}_3}$, then $d_i^{\mathcal{O}_1,\mathcal{O}_3} \geq d_i^{\mathcal{O}_2,\mathcal{O}_3}$.*

Proof. The proof follows Irving's proof in [I88] for Schubert varieties closely, supplemented by basic facts on block equivalence for Harish-Chandra modules from [V83] and [V81]. We begin by observing that the hypothesis on G and K implies that none of the roots of $\mathfrak{g}$ relative to any K-orbit (or to the corresponding θ-stable Cartan subgroup of G) are of type II, whence the recursion formulas of [V83, 6.14,(a,b,c,e)] imply that the constant term of $P_{\mathcal{O},\mathcal{O}'}$ equals 1 whenever $\overline{\mathcal{O}} \subseteq \overline{\mathcal{O}'}$. Moreover, applying the circle action of [V83, §5], we find that whenever any orbit lying in the image of a simple reflection applied to a given orbit has closure containing that of the orbit, then this image is single-valued and the simple root in question is either complex or noncompact imaginary type I. As every orbit can be obtained from a closed orbit by repeated application of simple reflections raising it in the closure order, it follows that there is a single block $\mathcal{B}$ containing all simple $(\mathfrak{g}, K)$ modules of trivial infinitesimal character.

Now we appeal to "IC4 duality". Vogan has shown in [V82] that there is another (possibly disconnected) complex group G' with symmetric subgroup K' and Lie algebra $\mathfrak{g}'$ and a block $\mathcal{B}'$ of $(\mathfrak{g}', K')$-modules, which we may take to have trivial infinitesimal character, such that there is a bijection D between the set S of K-orbits in G/B and a set S' of one-dimensional sheaves γ over K'-orbits in G'/B' that is order-reversing on the underlying orbits relative to the closure order. (The sheaves in S' parametrize the irreducible modules in $\mathcal{B}'$.) In particular, since there is a unique maximal (open) K-orbit in G/B, there is a unique orbit minimal among the orbits corresponding to the sheaves in S'. Moreover, the values of the KLV polynomials for $\mathcal{B}$ at 1 count the multiplicities of composition factors in standard $(\mathfrak{g}', K')$-modules for $\mathcal{B}'$; both the irreducible and the standard modules in $\mathcal{B}'$ are indexed by elements of S'.

Casian and Collingwood have refined this result along the lines of the Gabber–Joseph refinement of the Kazhdan–Lusztig conjecture for Verma modules: they showed that the coefficients of KLV polynomials count multiplicities of composition factors in the layers of the weight filtrations of standard modules in $\mathcal{B}'$ [CC89, 1.12], with standard modules indexed by sheaves on lower orbits occurring further down than those indexed by sheaves on higher orbits. (Here the standard modules in $\mathcal{B}'$ are normalized to have unique irreducible quotients, not unique irreducible submodules.) In particular, all standard modules in $\mathcal{B}'$ have a single copy of the unique simple standard module in this block at the lowest layer of the weight filtration. Now the proofs of Theorem 2.6.3 and Lemma 2.6.5 of [CI92] carry over to show that the simple standard module is the unique simple subquotient of any standard module in $\mathcal{B}'$ of largest GK dimension and is the socle of that module. It then follows from [C89] that all standard modules in $\mathcal{B}'$ are rigid, so that their socle and weight filtrations coincide.

We now show inductively that whenever γ, γ' are two elements of S' with $\gamma = D(\mathcal{O}_1), \gamma' = D(\mathcal{O}_2)$ and $\overline{\mathcal{O}_2} \subseteq \overline{\mathcal{O}_1}$, then the standard module X_γ indexed by γ embeds in $X_{\gamma'}$. (This condition on γ, γ' is *not* equivalent to requiring that $\gamma \le \gamma'$ in the Bruhat order.) We have just shown that this embedding holds if X_γ is simple. In general we assume inductively that this result holds for all pairs γ_1, γ_2 with either $\gamma_2 < \gamma'$ in the closure order, or else $\gamma_2 = \gamma', \gamma_1 < \gamma$. As in the proof of Proposition 6.14 of [V83], locate a simple reflection $s = s_\alpha$ such that either γ, γ', or both may be realized as the single-valued image of s applied via the circle action of W to an appropriate element of S. For definiteness assume that $\gamma = s \circ \gamma_1$ while α is real for γ' and does not satisfy the parity condition, so that $s \circ \gamma'$ is empty; the other cases are similar.

Applying the wall-crossing operator corresponding to s to the standard module X_{γ_1}, we get a module having X_{γ_1} as a submodule with quotient specified by either Proposition 8.2.7 or Proposition 8.4.5 of [V81], depending on whether α is complex or noncompact imaginary type I for γ_1. Applying the same operator to $X_{\gamma'}$, we get a module having $X_{\gamma'}$ as a submodule whose quotient is again X'_γ, by Proposition 8.4.9 of [V81]. Hence the embedding of X_{γ_1} into $X_{\gamma'}$ induces an embedding of the image $T_\alpha(X_{\gamma_1})$ of X_{γ_1} under the wall-crossing operator T_α into $T_\alpha(X_{\gamma'})$, which induces a map from the quotient of $T_\alpha(X_{\gamma_1})$ by its submodule isomorphic to X_{γ_1}

to the corresponding quotient of $T_\alpha(X_{\gamma'})$, which is $X_{\gamma'}$. This induced map between the quotients is injective when restricted to the socle of its domain, since there is exactly one copy of the unique irreducible standard module in $\mathcal{B}'$ inside X_{γ_1}, at the lowest level of the socle filtration, and exactly two copies of the irreducible standard module in $T_\alpha(X_{\gamma_1})$, one at the lowest and the other at the next-to-lowest level of the socle filtration. Hence the induced map is injective, and we get an embedding of X_γ into $X_{\gamma'}$, as desired.

We now conclude the proof in exactly the same way that Irving did in the Schubert variety case [I88, Corollary 4]: $d_i^{\mathcal{O}_1,\mathcal{O}_3}$ counts the multiplicity of a suitable composition factor in the $2i$th level of the socle filtration of a suitable standard module X for $(\mathfrak{g}', K')$, while $d_i^{\mathcal{O}_2,\mathcal{O}_3}$ counts the multiplicity of the same composition factor in the $2i$th level of the socle filtration of a submodule of X. The desired inequality follows at once from the definition of the socle filtration. Notice that the Bruhat order of [V83] coincides with the order by containment of closures for K-orbits in G/B, in contrast to the situation for K'-orbits in G'/B', thanks to [RS90, 7.11,(vii)]. □

We remark that the theorem extends to KLV polynomials for principal blocks (containing the trivial representation) of $(\mathfrak{g}, K)$-modules even if not all Cartan subgroups of the real form G_0 of G are connected, provided that: there is only a single conjugacy class of disconnected Cartan subgroups, the groups in this class have only two components, and all modules attached to trivial sheaves on orbits lie in the principal block. This covers the cases

$$G = \mathrm{GL}(2p), \quad K = \mathrm{GL}(p) \times \mathrm{GL}(p);$$
$$G = \mathrm{SO}(2n), \quad K = \mathrm{GL}(n);$$
$$G = E_7, \quad K = E_6 \times \mathbb{C}.$$

On the other hand, the case $G = G_2$, $K = \mathrm{SL}(2) \times \mathrm{SL}(2)$ does not work: there is only one conjugacy class of disconnected Cartan subgroups, but the groups in it have four components. Here the theorem holds for the nonprincipal block, which has only one simple module, but fails for the principal one.

A more subtle failure occurs for $G = \mathrm{SL}(3)$, $K = \mathrm{SO}(3)$: here there is only one conjugacy class of disconnected Cartan subgroups and the groups in it have only two components, but not all modules attached to trivial sheaves on orbits lie in the same block. There is a nontrivial sheaf γ attached to the open orbit $\mathcal{O}_1$ in G/B and a lower orbit $\mathcal{O}_2$ (admitting only the trivial sheaf), such that the KLV polynomial attached to the pair $((\mathcal{O}_1, \gamma), \mathcal{O}_2)$ is 0, but the one attached to $((\mathcal{O}_1, \gamma), (\mathcal{O}_1, \gamma))$ is 1.

We further remark that Collingwood and Irving have explored the properties of Harish-Chandra modules in the block $\mathcal{B}$ (rather than its dual block $\mathcal{B}'$), in the special case where the real form G_0 has only one conjugacy class of Cartan subgroups. Here the standard modules do not satisfy inclusion relations corresponding to inclusion of orbit closures, but many other familiar properties of modules in category $\mathcal{O}$ do carry over [CI92].

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Hodge theory and unitary representations

Wilfried Schmid and Kari Vilonen

Dedicated to David Vogan, on the occasion of his sixtieth birthday

Abstract We describe our conjecture about the irreducible unitary representations of reductive Lie groups, in the special case of $\mathrm{SL}(2,\mathbb{R})$.

Key words: Representation theory, Hodge theory

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In our paper [4] we formulated a conjecture on unitary representations of reductive Lie groups. We are currently working towards a proof; the technical difficulties are formidable. It has been suggested that an explicit description in the case of $\mathrm{SL}(2,\mathbb{R})$ would be helpful. The unitary representations of $\mathrm{SL}(2,\mathbb{R})$ have been worked out in great detail, of course, but even in this special case our construction of the inner product in terms of the $\mathcal{D}$-module realization is not obvious.

We begin with a quick summary of our conjecture in the general case of a reductive, linear, connected Lie group $G_\mathbb{R}$, with maximal compact subgroup $K_\mathbb{R}$. We let G and K denote the complexifications. The complex group G contains a unique compact real form $U_\mathbb{R}$ such that $U_\mathbb{R} \cap G_\mathbb{R} = K_\mathbb{R}$. Then $U_\mathbb{R}$ acts transitively on the flag variety X of G, and K acts with finitely many orbits. The points of X correspond to Borel subalgebras $\mathfrak{b}$ of the Lie algebra $\mathfrak{g}$ of G. The quotients

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$\mathfrak{h} = \mathfrak{b}/[\mathfrak{b},\mathfrak{b}]$ constitute the fibers of a flat vector bundle. Since X is simply connected, we can think of $\mathfrak{h}$ as a fixed vector space. This is the "universal Cartan", and is acted on by the "universal Weyl group" W. Its dual $\mathfrak{h}^*$ contains the "universal root system" Φ and the system of positive roots Φ^+, chosen so that $[\mathfrak{b},\mathfrak{b}]$ becomes the direct sum of the negative root spaces; $\mathfrak{h}^*$ also contains the "universal weight lattice" Λ. Further notation: lower case Fraktur letters, such as $\mathfrak{g}_{\mathbb{R}}$, $\mathfrak{k}_{\mathbb{R}}$, $\mathfrak{g}$, refer to the Lie algebras of $G_{\mathbb{R}}$, $K_{\mathbb{R}}$, G, etc.

Via the Harish Chandra isomorphism, $\mathfrak{h}^*$ parameterizes the characters χ_λ of the center of the universal enveloping algebra, with $\chi_\lambda = \chi_\mu$ if and only if $\mu = w\lambda$ for some $w \in W$. We shall say that a Harish Chandra module M has a real infinitesimal character if it is of the form χ_λ with $\lambda \in \mathbb{R} \otimes_{\mathbb{Z}} \Lambda$. David Vogan, many years ago, pointed out that to understand the irreducible unitary representations of $G_{\mathbb{R}}$ it suffices to treat the case of real infinitesimal character [3]. Let then M_λ be an irreducible Harish Chandra module with real infinitesimal character χ_λ. Since λ is determined only up to the Weil group action, we may and shall assume that λ is dominant, i.e.,

$$(\alpha, \lambda) \geq 0 \quad \text{for all } \alpha \in \Phi^+. \tag{1}$$

To determine whether or not M_λ underlies an irreducible unitary representation, one needs to know if it carries a nonzero $\mathfrak{g}_{\mathbb{R}}$-invariant hermitian form $(\ ,\ )_{\mathfrak{g}_{\mathbb{R}}}$ — this question has a simple answer, see below — and, when that is the case, if $(\ ,\ )_{\mathfrak{g}_{\mathbb{R}}}$ has a definite sign.

Vogan and his coworkers [5] made the important observation that the condition of having a real infinitesimal character ensures the existence of a nonzero $\mathfrak{u}_{\mathbb{R}}$-invariant hermitian form $(\ ,\ )_{\mathfrak{u}_{\mathbb{R}}}$. If both types of hermitian forms exist, they are explicitly related: the Cartan involution $\theta : \mathfrak{g} \to \mathfrak{g}$ then acts also on the Harish Chandra module M_λ, and after suitable rescaling of the hermitian forms,

$$(v_1, v_2)_{\mathfrak{u}_{\mathbb{R}}} = (\theta v_1, v_2)_{\mathfrak{g}_{\mathbb{R}}} . \tag{2}$$

The $\mathfrak{u}_{\mathbb{R}}$-invariant form is easier to deal with, both computationally and from a geometric point of view. At the same time the action of θ on M_λ can be described quite concretely. Thus, if one understands the hermitian form $(\ ,\ )_{\mathfrak{u}_{\mathbb{R}}}$, one can decide if M_λ is unitarizable.

As usual we write ρ for the half sum of the positive roots. We let $\mathcal{D}$ denote the sheaf of linear differential operators, with algebraic coefficients, on the flag variety X; here X is equipped with the Zariski topology. The sheaf of algebras $\mathcal{D}$ can be twisted by G-equivariant line bundles, and more generally, by any $\lambda \in \mathfrak{h}^*$. It is convenient to parameterize the twists so that $\mathcal{D}_\lambda$, for $\lambda \in \Lambda + \rho$, acts on sections of the G-equivariant line bundle $\mathcal{L}_{\lambda-\rho} \to X$ with Chern class $\lambda - \rho \in \Lambda \cong H^2(X, \mathbb{Z})$; for arbitrary $\lambda \in \mathfrak{h}$, one then defines $\mathcal{D}_\lambda$ by a process of analytic continuation. The sheaves $\mathcal{D}_\lambda$ are G-equivariant, in the Zariski sense locally isomorphic to $\mathcal{D}$, and every $\zeta \in \mathfrak{g}$ acts as an infinitesimal automorphism and thus defines a global section of $\mathcal{D}_\lambda$. Note that $\mathcal{D}_\rho = \mathcal{D}$, and that $\mathcal{D}_{-\rho}$ acts on sections $\mathcal{L}_{-2\rho}$ = canonical bundle of X.

The Beilinson-Bernstein construction[1] realizes the irreducible Harish Chandra module M_λ, with λ real and dominant as in (1), as the space of global sections

$$M_\lambda \;=\; H^0(X, \mathcal{M}_\lambda) \tag{3}$$

of an irreducible, K-equivariant sheaf of $\mathcal{D}_\lambda$-modules $\mathcal{M}_\lambda$ — or for short, an irreducible, K-equivariant $\mathcal{D}_\lambda$-module. Then $\mathfrak{g}$ acts on $M_\lambda = H^0(X, \mathcal{M}_\lambda)$ via the inclusion $\mathfrak{g} \hookrightarrow H^0(X, \mathcal{D}_\lambda)$. The correspondence between the Harish Chandra module M_λ and the "Harish Chandra sheaf" $\mathcal{M}_\lambda$ extends functorially to all Harish Chandra modules of finite length, with infinitesimal character χ_λ. Irreducible Harish Chandra sheaves $\mathcal{M}_\lambda$ are easy to describe: they arise from direct images, in the category of $\mathcal{D}_\lambda$-modules, under the embedding $j : Q \hookrightarrow X$ of a K-orbit Q in X, applied to a K-equivariant "twisted local system" $\mathbb{C}_{Q,\lambda}$ on Q, with twist $\lambda - \rho$. A formal, general definition of $\mathbb{C}_{Q,\lambda}$ would lead too far; in the special case of $G_{\mathbb{R}} = \mathrm{SL}(2,\mathbb{R})$ we describe it implicitly in (12) below, where its generating section will be denoted by $\sigma_0^{\frac{\lambda-1}{2}}$. In any case, the tensor product $\mathcal{O}_Q \otimes_{\mathbb{C}} \mathbb{C}_{Q,\lambda}$ has the structure of a $\mathcal{D}_{Q,\lambda}$-module on the K-orbit Q, and the direct image $j_*(\mathcal{O}_Q \otimes_{\mathbb{C}} \mathbb{C}_{Q,\lambda})$ that of a Harish Chandra sheaf: a K-equivariant $\mathcal{D}_\lambda$-module on X. In general the direct image is not irreducible, but it contains a unique irreducible $\mathcal{D}_\lambda$-submodule[2], and

$$\mathcal{M}_\lambda \;=\; \text{unique irreducible submodule of } \; j_*(\mathcal{O}_Q \otimes_{\mathbb{C}} \mathbb{C}_{Q,\lambda})\,. \tag{4}$$

The realization (4) of irreducible Harish Chandra sheaves is unique. It almost sets up a bijection between irreducible Harish Chandra modules M_λ, with the parameter λ of the infinitesimal character as in (1), and K-equivariant, irreducible local systems $\mathbb{C}_{Q,\lambda}$, with twist $\lambda - \rho$, on K-orbits $Q \subset X$ — the qualifier "almost" is necessary because when λ is singular, certain irreducible Harish Chandra sheaves have no nonzero global sections. This phenomenon explains why the classification of irreducible Harish Chandra modules with *regular infinitesimal character* looks simpler than that of irreducible Harish Chandra modules with *singular infinitesimal character*.

We shall not attempt to summarize Saito's theory of mixed Hodge modules here. Rather, we shall state the relevant facts, which apply to all members of the category of "geometrically constructible" Harish Chandra sheaves $\mathcal{M}_\lambda$; this includes in particular the sheaves obtained by the standard $\mathcal{D}$-module operations applied to $\mathcal{D}_\lambda$-modules of the type $j_*(\mathcal{O}_Q \otimes_{\mathbb{C}} \mathbb{C}_{Q,\lambda})$ and their $\mathcal{D}_\lambda$-subsheaves. A mild generalization of Saito's theory[3] puts three additional structures on each object $\mathcal{M}_\lambda$. First, the *weight filtration*, a functorial, finite increasing filtration

$$0 \subset W_0\mathcal{M}_\lambda \subset W_1\mathcal{M}_\lambda \subset \cdots \subset W_k\mathcal{M}_\lambda \subset \cdots \subset W_n\mathcal{M}_\lambda \;=\; \mathcal{M}_\lambda$$

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[1] A more detailed summary of the Beilinson-Bernstein construction of Harish Chandra modules can be found in [2].

[2] Since we assumed $G_{\mathbb{R}}$, and hence also K, to be connected, any $\mathcal{D}_\lambda$-subsheaf of a Harish Chandra sheaf is automatically K-equivariant and is therefore also a Harish Chandra sheaf.

[3] Without the assumption of an underlying rational structure, which Saito requires.

by $\mathcal{D}_\lambda$-subsheaves, with completely reducible quotients $W_k\mathcal{M}_\lambda / W_{k-1}\mathcal{M}_\lambda$ which are themselves objects in the category of Harish Chandra sheaves. Secondly, the *Hodge filtration*, a typically infinite, increasing filtration

$$0 \subset F_a\mathcal{M}_\lambda \subset \cdots \subset F_p\mathcal{M}_\lambda \subset F_{p+1}\mathcal{M}_\lambda \subset \cdots \subset \mathcal{M}_\lambda = \cup_{p\geq a} F_p\mathcal{M}_\lambda$$

by $\mathcal{O}_X$-coherent, K-equivariant, $\mathcal{O}_X$-submodules. This is a good filtration in the sense of $\mathcal{D}$-module theory: let $(\mathcal{D}_\lambda)_d \subset \mathcal{D}_\lambda$ denote the $\mathcal{O}_X$-subsheaf of differential operators of degree at most d; then

$$(\mathcal{D}_\lambda)_d\, F_p\mathcal{M}_\lambda \subseteq F_{p+d}\mathcal{M}_\lambda\,, \quad \text{with equality holding if } p \gg 0\,.$$

The third ingredient, the *polarization* on any irreducible Harish Chandra sheaf $\mathcal{M}_\lambda$, is a nontrivial $\mathcal{D}_\lambda \times \overline{\mathcal{D}}_\lambda$-bilinear pairing

$$P : \mathcal{M}_\lambda \times \overline{\mathcal{M}}_\lambda \longrightarrow \mathcal{C}^{-\infty}(X_\mathbb{R})\,. \tag{5}$$

Here $\mathcal{C}^{-\infty}(X_\mathbb{R})$ refers to the sheaf of distributions on X, considered as a C^∞ manifold, and $\overline{\mathcal{M}}_\lambda$ is the complex conjugate of $\mathcal{M}_\lambda$, viewed as a $\overline{\mathcal{D}}_\lambda$-module on X, equipped with the complex conjugate algebraic structure.

Morphisms in the category of mixed Hodge modules preserve both filtrations strictly: if $T : \mathcal{M} \to \mathcal{N}$ is a morphism, then $T(F_p\mathcal{M}) = (T\,\mathcal{M}) \cap (F_p\mathcal{N})$, and analogously for the weight filtration. We should also mention Saito's normalization of the indexing of the two filtrations. Going back to (4), the Hodge filtration on the sheaf $\mathcal{O}_Q \otimes_\mathbb{C} \mathbb{C}_{Q,\lambda}$ on Q is trivial, in the sense that $F_0(\mathcal{O}_Q \otimes_\mathbb{C} \mathbb{C}_{Q,\lambda}) = \mathcal{O}_Q \otimes_\mathbb{C} \mathbb{C}_{Q,\lambda}$ and $F_{-1}(\mathcal{O}_Q \otimes_\mathbb{C} \mathbb{C}_{Q,\lambda}) = 0$. As a sheaf on Q it is irreducible and has weight equal to $\dim Q$. The process of direct image shifts the lowest index of the Hodge filtration to $a = \operatorname{codim} Q$, and puts the weights into degrees $\geq \dim Q$, with the irreducible subsheaf $\mathcal{M}_\lambda$ having weight equal to $\dim Q$.

The polarization leads to a geometric description of the $\mathfrak{u}_\mathbb{R}$-invariant hermitian form on any irreducible Harish Chandra module M_λ. Let ω denote the — unique, up to scaling — $U_\mathbb{R}$-invariant measure on $X_\mathbb{R}$. Like any smooth measure on a compact C^∞ manifold it can be integrated against any distribution. When M_λ is realized as the space of global sections of the corresponding Harish Chandra sheaf $\mathcal{M}_\lambda$ as in (3), then

$$(s_1, s_2)_{\mathfrak{u}_\mathbb{R}} = \int_{X_\mathbb{R}} P(s_1, \bar{s}_2)\,\omega \quad \text{for } s_1, s_2 \in H^0(X, \mathcal{M}_\lambda)\,,$$

does indeed define a $\mathfrak{u}_\mathbb{R}$-invariant hermitian form, as can be checked readily [4]. The Cartan involution θ acts on X and on the set of K-orbits in X. If θ fixes a particular K-orbit Q, then it acts on the twisted local systems on Q, and if it fixes also the twisted local system $\mathbb{C}_{Q,\lambda}$ as in (4), then it acts on the sections of the direct image $j_*(\mathcal{O}_Q \otimes_\mathbb{C} \mathbb{C}_{Q,\lambda})$ and of its unique irreducible subsheaf $\mathcal{M}_\lambda$. In this sense, the action of θ on the Harish Chandra module M_λ, which relates $(\ ,\ )_{\mathfrak{u}_\mathbb{R}}$ to $(\ ,\ )_{\mathfrak{g}_\mathbb{R}}$ as in (2), is visible geometrically.

Via the global section functor the Hodge and weight filtrations induce filtrations on M_λ, the space of global sections of the Harish Chandra sheaf $\mathcal{M}_\lambda$, wether or not the latter is irreducible:

$$\begin{aligned} &0 \subset W_0 M_\lambda \subset W_1 M_\lambda \subset \cdots \subset W_k M_\lambda \subset \cdots \subset W_n M_\lambda = M_\lambda\,, \\ &0 \subset F_a M_\lambda \subset \cdots \subset F_p M_\lambda \subset F_{p+1} M_\lambda \subset \cdots \subset M_\lambda = \cup_{p \geq a} F_p M_\lambda\,. \end{aligned}$$

The $W_k M_\lambda$ are Harish Chandra submodules of M_λ, and the $F_p M_\lambda$ are finite dimensional, K-invariant subspaces. In the irreducible case the weight filtration collapses as was mentioned earlier, M_λ in (3,4) has weight dim Q, and the lowest index in the Hodge filtration is $a = \operatorname{codim} Q$. We can now state our conjecture. It asserts that if M_λ is irreducible, the $\mathfrak{u}_\mathbb{R}$-invariant hermitian form is nondegenerate on each $F_p M_\lambda$, and

$$(-1)^{p-a}\,(s,s)_{\mathfrak{u}_\mathbb{R}} \;>\; 0 \;\text{ for all nonzero }\; s \in F_p M_\lambda \cap (F_{p-1} M_\lambda)^\perp\,.$$

Whenever M_λ also admits a $\mathfrak{g}_\mathbb{R}$-invariant hermitian form, it would be related to a $\mathfrak{u}_\mathbb{R}$-invariant one via (2), and the resulting hermitian form $(\ ,\ )_{\mathfrak{g}_\mathbb{R}}$ would then have a definite sign if and only if M_λ is unitarizable.

The significance of the conjecture is discussed in [4]. While it does not amount to a description of the unitary dual of $G_\mathbb{R}$ in terms of representation parameters, it puts the study of the irreducible unitary representation into a functorial context.

We now turn to the example of $\mathrm{SL}(2,\mathbb{R})$. It is conjugate to $\mathrm{SU}(1,1)$ under an inner automorphism of $\mathrm{SL}(2,\mathbb{C})$, and various formulas have a simpler appearance for $\mathrm{SU}(1,1)$. Thus we suppose $G = \mathrm{SL}(2,\mathbb{C})$,

$$\begin{aligned} &G_\mathbb{R} = \mathrm{SU}(1,1) = \left\{ \begin{pmatrix} \alpha & \beta \\ -\bar\beta & \bar\alpha \end{pmatrix} \,\middle|\, \alpha,\,\beta \in \mathbb{C}\,,\; |\alpha|^2 - |\beta|^2 = 1 \right\}, \\ &K_\mathbb{R} = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \bar\alpha \end{pmatrix} \,\middle|\, \alpha \in \mathbb{C}\,,\; |\alpha| = 1 \right\}, \quad K = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} \,\middle|\, \alpha \in \mathbb{C}^* \right\}, \end{aligned} \tag{6}$$

and $U_\mathbb{R} = \mathrm{SU}(2)$. These groups act on the flag variety of G,

$$X = \mathbb{P}^1 = \mathbb{C} \cup \{\infty\}\,, \tag{7}$$

by linear fractional transformations, and K acts with three orbits, namely $\{0\}$, $\{\infty\}$, and $\mathbb{C}^*$. In the notation of (6,7), K acts on $\mathbb{C}^*$ by α^2, so $\mathbb{C}^*$ admits two irreducible K-equivariant local systems, corresponding to the trivial and the nontrivial character of the (component group of the) generic isotropy group $\{\pm 1\}$. This is true both in the scalar, i.e., non-twisted, and twisted case. Since K is connected, the two point orbits admit only the trivial irreducible K-equivariant local system. The Cartan involution,

$$\theta = \text{conjugation by } \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}\,,$$

is inner; it preserves each of the three orbits and the K-equivariant local systems on them. Thus all irreducible Harish Chandra modules with real infinitesimal character admit both $\mathfrak{u}_{\mathbb{R}}$- and $\mathfrak{g}_{\mathbb{R}}$-invariant hermitian forms.

The dual $\mathfrak{h}^*$ of the universal Cartan can be identified with $\mathbb{C}$ so that $\Lambda \cong \mathbb{Z}$, $\Phi \cong \{\pm 2\}$, and $\rho \cong 1$. With this identification an infinitesimal character χ_λ is real in the earlier sense if and only if $\lambda \in \mathbb{R}$, and $\lambda \in \mathbb{R}$ is dominant if and only if $\lambda \geq 0$. The standard SL_2-triple

$$e_+ \;=\; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad e_- \;=\; \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h \;=\; \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

spans $\mathfrak{g}$ over $\mathbb{C}$, with $\mathfrak{k}$ spanned by h, and satisfies the conjugation relations $\overline{e_+} = e_-$, $\overline{h} = -h$. The elements of this triple operate on the sheaf of algebraic functions on $\mathbb{C} \cup \{\infty\}$ by infinitesimal left translation. One computes readily that via this action,

$$e_+ \;\cong\; -\frac{d}{dz}, \quad e_- \;\cong\; z^2 \frac{d}{dz}, \quad h \;\cong\; -2z \frac{d}{dz}. \tag{8}$$

For example, e_+ acts on $f(z)$ by the derivative with respect to t, at the origin, of $f(\exp(-t\, e_+)z) = f(z - t)$, resulting in the formula $(e_+ f)(z) = -z \frac{df}{dz}(z)$; the other cases are treated similarly.

The G-equivariant line bundle $\mathcal{L}_2$ coincides with the tangent bundle of $\mathbb{P}^1$, so we can identify (8) with a basis of the space of global sections of $\mathcal{L}_2$. However, for notational reasons, we choose the new symbols σ_2, σ_0, σ_{-2}, corresponding to e_+, h, e_-, in that order. Then σ_2 vanishes to the second order at ∞, σ_{-2} vanishes to the second order at 0, and σ_0 has first-order zeros at both 0 and ∞, and these are the only zeroes in each case. Moreover,

$$\begin{aligned} &e_+ \sigma_2 \;=\; 0\,, \quad h\,\sigma_2 \;=\; 2\sigma_2\,, \quad e_- \sigma_2 \;=\; -\sigma_0 \;=\; \tfrac{1}{2}\, z\, \sigma_2\,, \\ &e_+ \sigma_0 \;=\; -2\sigma_2 \;=\; -z^{-1}\sigma_0\,, \quad h\,\sigma_0 \;=\; 0\,, \quad e_- \sigma_0 \;=\; 2\,\sigma_{-2} \;=\; -z\sigma_0\,, \\ &e_+ \sigma_{-2} \;=\; \sigma_0 \;=\; -\tfrac{1}{2}\, z^{-1}\sigma_{-2}\,, \quad h\,\sigma_{-2} \;=\; -2\sigma_{-2}\,, \quad e_- \sigma_{-2} \;=\; 0\,, \end{aligned} \tag{9}$$

as can be read off from (8). The $U_{\mathbb{R}}$-invariant measure on $\mathbb{P}^1$ is

$$\omega \;=\; (1 + |z|^2)^{-2}\, dz\, d\bar{z}\,. \tag{10}$$

The coefficient of $dz\, d\bar{z}$ in this formula can be interpreted as the squared length of $\frac{d}{dz}$ with respect to the $U_{\mathbb{R}}$-invariant hermitian metric, or equivalently, the squared length of σ_2 relative to the $U_{\mathbb{R}}$-invariant hermitian metric on the line bundle $\mathcal{L}_2$. Thus

$$\|\sigma_2\| \;=\; \frac{1}{1 + |z|^2}\,, \quad \|\sigma_0\| \;=\; \frac{2\,|z|}{1 + |z|^2}\,, \quad \|\sigma_{-2}\| \;=\; \frac{|z|^2}{1 + |z|^2} \tag{11}$$

describes the length, as measured by the $U_{\mathbb{R}}$-invariant metric on $\mathcal{L}_2$, of the three sections σ_2, σ_0, σ_{-2}.

As was mentioned already, there exist two irreducible K-invariant local systems, with twist $\lambda-\rho$, on the K-orbit $\mathbb{C}^*$, corresponding to the trivial and the nontrivial character of the generic isotropy subgroup $\{\pm 1\}$ of K. The corresponding Harish Chandra sheaves can be realized as

$$\begin{aligned} \mathcal{M}_{\mathbb{C}^*,\lambda,\text{even}} &= \{\, f(z)\,\sigma_0^{\frac{\lambda-1}{2}} \mid f\in\mathbb{C}(z)\,\}\,, \\ \mathcal{M}_{\mathbb{C}^*,\lambda,\text{odd}} &= \{\, f(z)\,z^{1/2}\,\sigma_0^{\frac{\lambda-1}{2}} \mid f\in\mathbb{C}(z)\,\}\,. \end{aligned} \tag{12}$$

These are Zariski-locally defined algebraic functions, multiplied by the "section" $\sigma_0^{\frac{\lambda-1}{2}}$ of the formal power $\mathcal{L}_2^{\frac{\lambda-1}{2}}$, either on $\mathbb{C}^*$ (in the even case), where the section is well defined, or its twofold cover (in the odd case). As such they are naturally $\mathcal{D}_\lambda$-modules on $\mathbb{C}^*$, and then, via the direct image functor corresponding to the open embedding $\mathbb{C}^*\subset\mathbb{C}\cup\{\infty\}$, on all of $\mathbb{C}\cup\{\infty\}$. How $\mathcal{D}_\lambda$ acts is not so relevant for us, but the action of $\mathfrak{g}\subset\Gamma\mathcal{D}_\lambda$ is. That action is given by the product rule, with $\mathfrak{g}$ acting on $f(z)$ or $z^{1/2}f(z)$ according to the formulas (8), and on the formal powers of σ_0 according to (9). Since σ_0 has first order zeroes at 0 and ∞,

$$\begin{aligned} f(z)\,\sigma_0^{\frac{\lambda-1}{2}} &\sim f(z)\,z^{\frac{\lambda-1}{2}} \quad \text{near the origin, and} \\ f(z)\,\sigma_0^{\frac{\lambda-1}{2}} &\sim f(z)\,z^{-\frac{\lambda-1}{2}} \quad \text{near } \infty\,. \end{aligned} \tag{13}$$

In particular, $\mathcal{M}_{\mathbb{C}^*,\lambda,\text{even}}$ is reducible if and only if λ is an odd integer, whereas $\mathcal{M}_{\mathbb{C}^*,\lambda,\text{odd}}$ reduces if and only if λ is even.

We recall that the sheaves (12), when restricted to $\mathbb{C}^*$, are irreducible and have weight one, which is the dimension of $\mathbb{C}^*$. That remains correct for these sheaves on all of $\mathbb{C}\cup\{\infty\}$ when they are irreducible:

$$\begin{aligned} W_0\,\mathcal{M}_{\mathbb{C}^*,\lambda,\text{even}} = 0 \quad &\text{and} \quad W_1\,\mathcal{M}_{\mathbb{C}^*,\lambda,\text{even}} = \mathcal{M}_{\mathbb{C}^*,\lambda,\text{even}} \quad \text{if } \lambda\notin 2\mathbb{Z}+1\,, \\ W_0\,\mathcal{M}_{\mathbb{C}^*,\lambda,\text{odd}} = 0 \quad &\text{and} \quad W_1\,\mathcal{M}_{\mathbb{C}^*,\lambda,\text{odd}} = \mathcal{M}_{\mathbb{C}^*,\lambda,\text{odd}} \quad \text{if } \lambda\notin 2\mathbb{Z}\,. \end{aligned}$$

In the reducible case,

$$\begin{aligned} W_0\,\mathcal{M}_{\mathbb{C}^*,2m+1,\text{even}} = 0\,, \quad W_1\,\mathcal{M}_{\mathbb{C}^*,2m+1,\text{even}} &= \mathcal{O}_{\mathbb{P}^1}(\mathcal{L}_{2m})\,, \\ \text{and } W_2\,\mathcal{M}_{\mathbb{C}^*,2m+1,\text{even}} &= \mathcal{M}_{\mathbb{C}^*,2m+1,\text{even}}\,; \\ W_0\,\mathcal{M}_{\mathbb{C}^*,2m,\text{odd}} = 0\,, \quad W_1\,\mathcal{M}_{\mathbb{C}^*,2m,\text{odd}} &= \mathcal{O}_{\mathbb{P}^1}(\mathcal{L}_{2m-1})\,, \\ \text{and } W_2\,\mathcal{M}_{\mathbb{C}^*,2m,\text{odd}} &= \mathcal{M}_{\mathbb{C}^*,2m,\text{odd}}\,. \end{aligned}$$

To justify these descriptions of the weight filtrations one should notice that σ_0^m can be viewed as a section of $\mathcal{L}_2^m = \mathcal{L}_{2m}$, and $z^{1/2}\sigma_0^{1/2}$ as a meromorphic section of $\mathcal{L}_1$. The quotients $\mathrm{gr}_{W,2}\,\mathcal{M}_{\mathbb{C}^*,2m+1,\text{even}}$ and $\mathrm{gr}_{W,2}\,\mathcal{M}_{\mathbb{C}^*,2m,\text{odd}}$ are Harish Chandra sheaves supported on $\{0,\infty\}$. We shall discuss these later.

The Hodge filtration for the sheaves (12) starts at level $a=0$, since that is the codimension. In general the Hodge filtration of the direct image under an open embedding is governed by the — not necessarily integral — order of poles. The case of $\mathbb{C}^* \hookrightarrow \mathbb{C}$, and analogously for $\mathbb{C}^* \hookrightarrow \mathbb{C}^* \cup \{\infty\}$, is especially simple: poles of order ≤ 1 have Hodge level 0, those of order ≤ 2 have Hodge level 1, and so forth. Thus, in view of (12,13), for $n \in \mathbb{Z}$ and $p \geq 0$,

$$\begin{aligned} z^n\, \sigma_0^{\frac{\lambda-1}{2}} \in F_p\, \mathcal{M}_{\mathbb{C}^*,\lambda,\mathrm{even}} &\iff -\tfrac{\lambda+1}{2} - p \leq n \leq \tfrac{\lambda+1}{2} + p\,, \\ z^{n+1/2} \sigma_0^{\frac{\lambda-1}{2}} \in F_p\, \mathcal{M}_{\mathbb{C}^*,\lambda,\mathrm{odd}} &\iff -\tfrac{\lambda+1}{2} - p \leq n + \tfrac{1}{2} \leq \tfrac{\lambda+1}{2} + p\,, \end{aligned} \tag{14}$$

for all $n \in \mathbb{Z}$ and $p \geq 0$. In the reducible case, there is a connection between the induced Hodge filtrations on the quotient sheaves and the intrinsic Hodge filtrations on the quotients; this too will be described later.

We now turn to the polarizations of the sheaves $\mathcal{M}_{\mathbb{C}^*,2n+1,\mathrm{even}}$, $\mathcal{M}_{\mathbb{C}^*,2n+1,\mathrm{odd}}$ and the resulting hermitian forms on their spaces of global sections. On the K-orbit $\mathbb{C}^*$ these sheaves are always irreducible, and the only possible hermitian pairing of the type (5) on $\mathbb{C}^*$ is, up to scaling,

$$P\left(f(z)\, \sigma_0^{\frac{\lambda-1}{2}}\,,\ \overline{g(z)\, \sigma_0^{\frac{\lambda-1}{2}}} \right) \;=\; f(z)\, \overline{g(z)}\, \|\sigma_0\|^{\lambda-1}\,,$$

which is a real analytic function, and thus distribution, on $\mathbb{C}^*$. This is correct in both cases, if we take $f, g \in \mathbb{C}(z)$ in the case of $\mathcal{M}_{\mathbb{C}^*,2n+1,\mathrm{even}}$, and $f, g \in z^{1/2}\mathbb{C}(z)$ in the case of $\mathcal{M}_{\mathbb{C}^*,2n+1,\mathrm{odd}}$; (11) makes the last factor on the right explicit. The two sheaves were defined as the $\mathcal{D}_\lambda$-module direct image under the open embedding $\mathbb{C}^* \hookrightarrow \mathbb{C} \cup \{\infty\}$ which, as always in the case of open embeddings, coincides with the $\mathcal{O}$-module direct image. The general theory ensures that

$$f(z)\, \overline{g(z)}\, \|\sigma_0\|^{\lambda-1} \tag{15}$$

makes sense as global distribution on $\mathbb{C} \cup \{\infty\}$, for all global sections $f(z)\, \sigma_0^{\frac{\lambda-1}{2}}$, $g(z)\, \sigma_0^{\frac{\lambda-1}{2}}$, provided the sheaf in question is irreducible.

To see how this works out in the current setting, applied to the spaces of global sections $M_{\mathbb{C}^*,2n+1,\mathrm{even}}$, $M_{\mathbb{C}^*,2n+1,\mathrm{odd}}$ of the two sheaves, we note that

$$\begin{aligned} M_{\mathbb{C}^*,\lambda,\mathrm{even}} \quad &\text{has basis} \quad \{\, z^n\, \sigma_0^{\frac{\lambda-1}{2}} \mid n \in \mathbb{Z} \,\}, \quad \text{and} \\ M_{\mathbb{C}^*,\lambda,\mathrm{odd}} \quad &\text{has basis} \quad \{\, z^n\, \sigma_0^{\frac{\lambda-1}{2}} \mid n \in \mathbb{Z} + 1/2 \,\}. \end{aligned} \tag{16}$$

For reasons of radial symmetry we only need to consider the integral of the expression (15) over $\mathbb{C} \cup \{\infty\}$ against the $U_{\mathbb{R}}$-invariant measure ω when f and g are the same basis element. With the convention of (16), with n denoting either an integer or a true half integer, and using (10,11), we find that

$$
\begin{aligned}
(z^n\,\sigma_0^{\frac{\lambda-1}{2}}, z^n\,\sigma_0^{\frac{\lambda-1}{2}})_{\mathfrak{u}_{\mathbb{R}}} \;&=\; \int_{\mathbb{C}\cup\{\infty\}} |z|^{2n}\,\|\sigma_0\|^{\lambda-1}\,\omega \\
&= \int_{\mathbb{C}\cup\{\infty\}} \frac{4\,|z|^{2n+\lambda-1}}{(1+|z|^2)^{\lambda+1}}\,dz\,d\bar{z} \;=\; 8\,\pi \int_0^\infty \frac{r^{2n+\lambda}\,dr}{(1+r^2)^{\lambda+1}} \\
&= 4\,\pi \int_0^\infty \frac{u^{n+(\lambda-1)/2}\,du}{(1+u)^{\lambda+1}}\,.
\end{aligned}
\tag{17}
$$

This integral converges if and only if $-(\lambda+1)/2 < n < (\lambda+1)/2$. Since the integrand is positive, the integral has a strictly positive value in the range of convergence.

To continue the integral meromorphically beyond the range of convergence, one uses integration by parts. Formally, for $s, t \in \mathbb{R}$,

$$
s \int_0^\infty \frac{u^{s-1}\,du}{(1+u)^t} \;=\; t \int_0^\infty \frac{u^s\,du}{(1+u)^{t+1}}\,.
$$

Applying this identity with $s = n + (\lambda+1)/2$ and $t = \lambda + 1$, one finds that the integral changes sign and becomes strictly negative for $-(\lambda+3)/2 < n < -(\lambda+1)/2$. The same argument, with $-n$ substituted for n, shows that the integral is also strictly negative for $(\lambda+1)/2 < n < (\lambda+3)/2$. Then the pattern continues: when n is negative and decreased by one, or if n is positive and increased by one, the integral changes sign. Poles occur when λ is an odd integer in the even case, or an even integer in the odd case, i.e., exactly when the module becomes reducible. That is what must happen, of course; the polarization is well defined only in the irreducible range. The preceding discussion can be summarized succinctly in terms of the Hodge filtration: with ϵ referring to either the even or the odd parity, in the irreducible range,

$$
\begin{aligned}
&s \in F_0\,M_{\mathbb{C}^*,\lambda,\epsilon} \implies \text{the integral defining } (s,s)_{\mathfrak{u}_{\mathbb{R}}} \text{ converges} \\
&\qquad s \in F_p\,M_{\mathbb{C}^*,\lambda,\epsilon} \cap (F_p\,M_{\mathbb{C}^*,\lambda,\epsilon})^\perp,\;\; s \neq 0 \implies (-1)^p\,(s,s)_{\mathfrak{u}_{\mathbb{R}}} > 0\,;
\end{aligned}
$$

cf. (14). The second statement is the assertion of our conjecture in the case of the open K-orbit $\mathbb{C}^*$

The change in sign is directly related to a change in the weight filtration. Let $\lambda_0 > 0$ be a reduction point — i.e., a positive odd or even integer, depending on whether the parity is even or odd. In terms of the basis (16), with the same convention of letting n refer to an integer or true half integer, depending on the parity,

$$
W_1\,M_{\mathbb{C}^*,\,\lambda_0,\epsilon} \quad \text{has basis} \quad \{\, z^n\,\sigma_0^{\frac{\lambda-1}{2}} \;\mid\; -(\lambda-1) \le 2n \le \lambda - 1 \,\}\,.
$$

Thus, as the parameter λ crosses the reduction point λ_0 going from left to right, the sign of $(z^n\,\sigma_0, z^n\,\sigma_0)_{\mathfrak{u}_{\mathbb{R}}}$ remains the same if and only if $z^n\,\sigma_0 \in W_1\,M_{\mathbb{C}^*,\,\lambda_0,\epsilon}$. This is one instance of a general fact. As the parameter for a family of induced representations crosses a reduction point, the sign changes of (,) are governed by the weight filtration; that is the assertion of the Jantzen conjecture proved by

Beilinson–Bernstein [1]. The jumps of the Hodge filtration at the reduction point line up with the weight filtration to produce exactly the sign changes predicted by our conjecture.

When $\lambda = m > 0$ is a positive integer and ϵ denotes the opposite parity, i.e., the parity of $m+1$, the Harish Chandra module $W_1\, M_{\mathbb{C}^*,m,\epsilon}$ has dimension m, and the integral (17) converges for all basis elements: this is the usual description of the positive definite $U_{\mathbb{R}}$-invariant inner product on the irreducible m-dimensional representation. For $\lambda = 0$ and ϵ odd, $W_1\, M_{\mathbb{C}^*,m,\epsilon}$ reduces to zero. The corresponding sheaf $W_1\, \mathcal{M}_{\mathbb{C}^*,\,0,\mathrm{odd}}$ is the one and only irreducible Harish Chandra sheaf for $G_{\mathbb{R}} = \mathrm{SU}(1,1)$ without nonzero sections.

The two singleton orbits $\{0\}$, $\{\infty\}$ are related by an outer automorphism of $G_{\mathbb{R}} = \mathrm{SU}(1,1)$. Thus it is only necessary to discuss the orbit $\{0\}$. Since K fixes the origin, it must act on the geometric fiber of any $\mathcal{D}_\lambda$-module supported at $\{0\}$, and that forces an integral twisting parameter:

$$\lambda \;=\; m \;\in\; \mathbb{Z}_{\geq 0}\,.$$

In the untwisted case, the only irreducible $\mathcal{D}$-module supported at the origin in $\mathbb{C}$ is the one generated by the "holomorphic delta function",

$$\mathcal{D}_{\mathbb{C}}\, \delta_0 \;=\; \mathbb{C}[z, z^{-1}]/\mathbb{C}[z]\,.$$

Thus $\delta_0 \cong z^{-1}$, and the SL_2-triple (8) acts according to the formulas

$$h\, \delta_0 \;=\; 2\delta_0\,, \quad e_-\, \delta_0 \;=\; 0\,, \tag{18}$$

with e_+ acting freely, by normal differentiation. The section σ_2 of $\mathcal{L}_2$ in (8) is nonzero at the origin and is K-invariant, and this leads to a description of the sheaf $\mathcal{M}_{\{0\},m}$, or equivalently, to its space of global sections $M_{\{0\},m}$,

$$M_{\{0\},m} \quad \text{has basis} \quad \{\, (\tfrac{d^n}{dz^n}\, \delta_0)\, \sigma_2^{(m-1)/2} \;\mid\; n \geq 0\,\}\,;$$

here $\sigma_2^{(m-1)/2}$ can be viewed as a section of $\mathcal{L}_{m-1}$, a section that is regular and nonzero except at ∞. The SL_2-triple e_+, h, e_- acts by the product rule, on $\frac{d^n}{dz^n}\, \delta_0$ according to (8) and (18), and on $\sigma_2^{(m-1)/2}$ according to (9). In particular,

$$\begin{aligned} e_+ \left((\tfrac{d^n}{dz^n}\, \delta_0)\, \sigma_2^{(m-1)/2} \right) \;&=\; -\,(\tfrac{d^{n+1}}{dz^{n+1}}\, \delta_0)\, \sigma_2^{(m-1)/2}\,, \\ h \left((\tfrac{d^n}{dz^n}\, \delta_0)\, \sigma_2^{(m-1)/2} \right) \;&=\; (2n+m+1)(\tfrac{d^{n+1}}{dz^{n+1}}\, \delta_0)\, \sigma_2^{(m-1)/2}\,, \\ e_- \left((\tfrac{d^n}{dz^n}\, \delta_0)\, \sigma_2^{(m-1)/2} \right) \;&=\; n(n+1)(\tfrac{d^{n-1}}{dz^{n-1}}\, \delta_0)\, \sigma_2^{(m-1)/2}\,, \end{aligned}$$

as can be checked readily.

The inclusion $\{0\} \hookrightarrow \mathbb{C} \cup \{\infty\}$ is a very special case of a closed embedding. In general the $\mathcal{D}$-module direct image of an irreducible module under a closed embedding remains irreducible, so the weight filtration collapses. The effect of closed embeddings on the Hodge filtration also has a simple description: the Hodge index is increased by the order of normal derivative. In the case of $M_{\{0\},m}$ this means that

$$\begin{aligned} W_0\, M_{\{0\},m} \;&=\; M_{\{0\},m}\,, \\ F_p\, M_{\{0\},m} \quad &\text{has basis} \quad \{\, (\tfrac{d^n}{dz^n}\,\delta_0)\,\sigma_2^{(m-1)/2} \;\mid\; 0 \le n \le p-1\,\}\,, \end{aligned}$$

because the weight equals the dimension of the support, and the Hodge filtration starts at the codimension of the support.

The polarization pairs δ_0 and $\overline{\delta_0}$ into $\delta_{\mathbb{R},0}$, the delta function in the usual sense on $\mathbb{C} \cong \mathbb{R}^2$. It also pairs $\sigma_2^{(m-1)/2}$ and its complex conjugate into $\|\sigma_2^{(m-1)/2}\|^2 = \|\sigma_2\|^{m-1} = (1+|z|^2)^{-m+1}$, as follows from (11). Thus

$$\begin{aligned} &(\,(\tfrac{d^k}{dz^k}\,\delta_0)\sigma_2^{(m-1)/2}\,,\,\overline{(\tfrac{d^\ell}{dz^\ell}\,\delta_0)\sigma_2^{(m-1)/2}}\,)_{\mathfrak{u}_\mathbb{R}} = \int_{\mathbb{C}\cup\{\infty\}} \tfrac{d^k}{dz^k}\,\tfrac{d^\ell}{d\bar z^\ell}\,\|\sigma_2^{(m-1)/2}\|^2\,\delta_{\mathbb{R},0}\;\omega \\ &= \int_{\mathbb{C}\cup\{\infty\}} \tfrac{d^k}{dz^k}\,\tfrac{d^\ell}{d\bar z^\ell}\,(1+|z|^2)^{-m-1}\,\delta_{\mathbb{R},0}\;dz\,d\bar z \;=\; \tfrac{d^k}{dz^k}\,\tfrac{d^\ell}{d\bar z^\ell}\,(1+|z|^2)^{-m-1}\Big|_{z=0} \end{aligned}$$

vanishes unless $k=\ell$, in which case

$$(\,(\tfrac{d^k}{dz^k}\,\delta_0)\sigma_2^{(m-1)/2}\,,\,\overline{(\tfrac{d^k}{dz^k}\,\delta_0)\sigma_2^{(m-1)/2}}\,)_{\mathfrak{u}_\mathbb{R}} \;=\; (-1)^k\,k!\,\prod\nolimits_{j=1}^{k}\,(m+j)\,.$$

That, of course, is consistent with our conjecture in this particular instance.

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On elliptic factors in real endoscopic transfer I

Diana Shelstad

To David Vogan, for his 60th birthday

Abstract This paper is concerned with the structure of packets of representations and some refinements that are helpful in endoscopic transfer for real groups. It includes results on the structure and transfer of packets of limits of discrete series representations. It also reinterprets the Adams–Johnson transfer of certain non-tempered representations via spectral analogues of the Langlands–Shelstad factors, thereby providing structure and transfer compatible with the associated transfer of orbital integrals. The results come from two simple tools introduced here. The first concerns a family of splittings of the algebraic group G under consideration; such a splitting is based on a fundamental maximal torus of G rather than a maximally split maximal torus. The second concerns a family of Levi groups attached to the dual data of a Langlands or an Arthur parameter for the group G. The introduced splittings provide explicit realizations of these Levi groups. The tools also apply to maps on stable conjugacy classes associated with the transfer of orbital integrals. In particular, they allow for a simpler version of the definitions of Kottwitz–Shelstad for twisted endoscopic transfer in certain critical cases. The paper prepares for spectral factors in twisted endoscopic transfer that are compatible in a certain sense with the standard factors discussed here. This compatibility is needed for Arthur's global theory. The twisted factors themselves will be defined in a separate paper.

Key words: Endoscopic transfer, twisted endoscopic transfer, transfer factor, real reductive group, orbital integral, spectral transfer

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1 Introduction

Our main purpose is to continue a study of the coefficients appearing in the spectral identities of endoscopic transfer for real groups. The coefficients carry information about the structure of packets of irreducible representations, and in the global theory of endoscopy this structure plays a central role in determining if certain irreducible representations are automorphic or not; see [Ar13].

Here we will consider both the standard and the more general twisted versions of endoscopic transfer. We focus on the *fundamental case* where the endoscopic group and the ambient group share, in a certain precise sense, fundamental maximal tori; see Section 3.3. It includes the case where the ambient group G is cuspidal and the endoscopic group H_1 is elliptic. We call this the cuspidal-elliptic setting; see Section 3.4. Then $G(\mathbb{R})$ and $H_1(\mathbb{R})$ share fundamental Cartan subgroups that are elliptic, i.e., compact modulo the centers of $G(\mathbb{R})$ and $H_1(\mathbb{R})$ respectively. Thus there is a discrete series of representations for each of $G(\mathbb{R})$ and $H_1(\mathbb{R})$ [HC75], along with limits of discrete series representations (see [KZ82]).

Endoscopic transfer begins with the matching of orbital integrals, the so-called geometric side. In the standard version we use the transfer factors of Langlands–Shelstad ([LS87], see also [Sh14]) for the geometric side. Factors with a parallel definition appear in the tempered dual spectral transfer, i.e., as coefficients in the dual spectral identities for tempered irreducible representations [Sh10, Sh08b]. Properties of these spectral factors simplify the related harmonic analysis; for example, inversion of the identities becomes a short exercise (see [Sh08b]).

In preparation for generalizing (in [ShII]) the definition of spectral factors to the twisted setting of Kottwitz–Shelstad [KS99] we will establish three refinements. First, we make use of an alternative simpler description of limits of discrete series packets in terms of elliptic data, i.e., data attached to an elliptic Cartan subgroup (see Remark 5.6), to simplify transfer and structure in that setting.

Second, introducing the nontempered spectrum to our picture, we reinterpret the transfer of Adams–Johnson in terms of data attached directly to the associated Arthur parameters. Here we will consider only parameters that are elliptic in the sense of Arthur. Our new factors are related very simply to the tempered factors already defined, and we check that they do provide the transfer that is precisely dual (i.e., there are no extraneous constants) to that of orbital integrals with the Langlands–Shelstad factors. The inversion properties of our spectral transfer are more delicate than for the tempered case ([AJ87], [Ar89] explain why this must be the case) and will be described in [ShII].

For the third refinement we turn to twisted transfer and the underlying definitions of [KS99]. The transfer of orbital integrals is based on an abstract norm correspondence (γ_1, δ) for suitably regular points γ_1 in endoscopic $H_1(\mathbb{R})$ and δ in $G(\mathbb{R})$. For the fundamental part of the correspondence that concerns us here, we will see that we may limit the twisting automorphism to a family for which the norm correspondence is well-behaved. The standard spectral factors generalize readily for this family [ShII] and we have the standard-twisted compatibility needed in Arthur's global theory [Ar13].

To obtain these refinements we introduce two simple tools. The first involves fundamental splittings. These are particular splittings based on fundamental maximal tori and exist for any G. They work well with both elliptic data and Whittaker data; here Vogan's characterization of generic representations plays a critical role. See Sections 2.3, 6.1. The second tool involves a family of Levi groups in G. First we attach to a Langlands or Arthur parameter a family of L-groups and then we use fundamental splittings to identify their real duals as a family of (nonstandard) Levi groups in G and its inner forms; see Sections 5.2, 6.1.

We begin the paper with fundamental splittings and their properties. The main result is Lemma 2.5. In Part 3 we review the norm correspondence and see, in particular, that the fundamental part of geometric transfer is nonempty if and only if the twisting automorphism preserves a fundamental splitting up to a further twist by an element of $G(\mathbb{R})$. This may be expressed precisely in terms of the norm correspondence itself or in terms of the nonvanishing of geometric transfer factors of [KS99]; see Theorem 3.12, Corollary 3.13. We prove a spectral analogue, but not until Part 9 where we also finish the discussion of Part 4 on certain properties of endoscopic data that will be used in the definition of twisted spectral factors.

The rest of the paper concerns standard transfer and the first two refinements. In Part 5 we turn to the Langlands and Arthur parameters attached to the representations of interest to us here. In the cuspidal-elliptic setting these are the s-elliptic Langlands parameters and the elliptic u-regular Arthur parameters of Sections 5.5–5.7. Given a parameter, we generate data for the various attached packets of representations by means of pairs (G, η), where η is an inner twist of G to a given quasisplit form G^*.

For the limits of discrete series representations attached to an s-elliptic Langlands parameter, we reformulate some well-known properties in terms of our attached Levi groups. For example, the critical Lemma 6.1 characterizes the pairs (G, η) for which we obtain a well-defined (i.e., nonzero) representation. Then Lemma 6.2 gives a description of the packet that allows us to attach an elliptic invariant to each member; see Section 6.4. Lemmas 6.4 and 6.5 describe the application to endoscopic transfer.

Part 6 has further results on limits of discrete series representations that we will apply in various places. For example, as in Section 6.7, every s-elliptic parameter factors through a totally degenerate parameter for an attached Levi group. We will check in [ShII] that this gives a simple characterization of those pairs (G, η) for which the distribution character of the attached representation is elliptic.

The representations attached to elliptic u-regular Arthur parameters are the derived functor modules of Vogan and Zuckerman [Vo84] from the main setting in [AJ87]; we allow without harm a nontrivial split component in the center of G. In the case of regular infinitesimal character they are discussed in [Ar89, Ko90]; see the last paragraph of Section 7.1. Generalizing some familiar L-group constructions we attach directly to the Arthur parameter the following: elliptic data, a family of Levi groups, and an s-elliptic Langlands parameter with the same infinitesimal character and central behavior. We will use these again in [ShII] in the twisted setting. In the present paper we pursue only the case of regular infinitesimal character so that the attached Langlands parameter is elliptic. Lemma 7.5 describes the Arthur packet by means of pairs (G, η). In Part 8 we introduce spectral transfer factors for the Arthur packet by tethering the packet to the elliptic Langlands packet via *relative* factors with good transitivity properties [Sh10]. We then verify in Section 8.3 that the corresponding absolute factors are correct, in the sense already mentioned, for endoscopic transfer with Langlands–Shelstad factors on the geometric side.

To finish this brief sketch we refer to Sections 3.5, 4.1, 4.2, 6.4 and 6.5 where there are further remarks on the properties of transfer factors that are crucial for our approach to work.

Note This paper is an expanded version of part of the preprint "On spectral transfer factors in real twisted endoscopy" posted on the author's website, May 2011.

2 Automorphisms and inner forms

Here we introduce notation to be used throughout the paper, along with definitions and properties related to fundamental splittings. We finish with an application to the *inner forms of a quasisplit pair*.

2.1 Quasi-split pairs and inner forms

By a quasisplit pair we mean a pair (G^*, θ^*), where G^* is a connected, reductive algebraic group defined and quasisplit over $\mathbb{R}$, and θ^* is an $\mathbb{R}$-automorphism of G^* that preserves an $\mathbb{R}$-splitting $\mathrm{spl}^* = (B^*, T^*, \{X_\alpha\})$ of G^*. We assume that the restriction of θ^* to the identity component of the center of G^* is semisimple or, equivalently, that θ^* has finite order.

Recall from [KS99, Appendix B] that (G, θ, η) is defined to be an inner form of (G^*, θ^*) if G is connected, reductive and defined over $\mathbb{R}$, η is an isomorphism from G to G^* that is an inner twist, θ is an $\mathbb{R}$-automorphism of G, and θ coincides with the transport of θ^* to G via η up to an inner automorphism. Notice that if θ^* is the identity, then θ must be an inner automorphism of G defined over $\mathbb{R}$, i.e., θ must act on G as an element of $G_{\mathrm{ad}}(\mathbb{R})$.

Let (G, θ, η) be an inner form of (G^*, θ^*). By the *inner class* of (θ, η) we will mean the set of all pairs (θ', η') where (G, θ', η') is an inner form of (G^*, θ^*) such that

(i) $\eta' \circ \eta^{-1}$ is inner and
(ii) the automorphism $\theta' \circ \theta^{-1}$ of G, which is inner and acts as an element of $G_{\mathrm{ad}}(\mathbb{R})$ by (i), is induced by an element of $G(\mathbb{R})$, i.e., acts as an element of the image of $G(\mathbb{R})$ in $G_{\mathrm{ad}}(\mathbb{R})$ under the natural projection.

We will see that replacing (θ, η) by a member of its inner class has no effect on our final results.

Let (G, θ, η) be an inner form of (G^*, θ^*). Then we choose $u(\sigma) \in G^*_{\text{sc}}$ such that

$$\eta \circ \sigma(\eta)^{-1} = \text{Int}(u(\sigma)). \tag{2.1}$$

Here and throughout the paper we use σ to denote the nontrivial element of $\Gamma = \text{Gal}(\mathbb{C}/\mathbb{R})$. The action of σ on a Γ-set X will be denoted by σ_X or by σ itself when X is evident.

2.2 Fundamental splittings

While $\mathbb{R}$-splittings exist only for quasisplit groups, *fundamental splittings* may be constructed for any connected, reductive G defined over $\mathbb{R}$. We recall the definition (see [Sh12]).

Consider a pair (B, T), where T is a maximal torus in G defined over $\mathbb{R}$ and B is a Borel subgroup of G containing T. We call (B, T) a *fundamental pair* if

(i) T is fundamental, i.e., T is minimally $\mathbb{R}$-split or, equivalently, T has no roots fixed by σ_T and
(ii) the set of (simple) roots of T in B is preserved by $-\sigma_T$.

The existence of fundamental pairs is noted in [Ko86, Section 10.4].

Lemma 2.1. *The set of all fundamental pairs for G forms a single stable conjugacy class in the sense that another fundamental pair (B', T') is conjugate to (B, T) by an element g of G for which* $\text{Int}(g) : T \to T'$ *is defined over* $\mathbb{R}$.

Proof. Observe that (B', T') is conjugate to (B, T) under some element g of G, and then $g^{-1}\sigma(g)$ acts as an element of the Weyl group of T preserving the roots of B, i.e., as the identity element. □

To prescribe a fundamental splitting we start with a fundamental pair (B, T) and pick an $\mathfrak{sl}_2$-triple $\{X_\alpha, H_\alpha, X_{-\alpha}\}$ for each simple root α of T in B. Here we identify the Lie algebra of T with $X_*(T) \otimes \mathbb{C}$ and require H_α be the element identified with the coroot $\alpha^\vee$ of α; $X_\alpha, X_{-\alpha}$ are to be root vectors for $\alpha, -\alpha$ respectively. There is an attached splitting $\text{spl} = (B, T, \{X_\alpha\})$ for G. Conversely, each splitting for G determines uniquely a collection of $\mathfrak{sl}_2$-triples of the above form. We call spl *fundamental* if the Galois action satisfies: $\sigma X_\alpha = X_{\sigma_T \alpha}$ in the case $\sigma_T\alpha \neq -\alpha$, and $\sigma X_\alpha = \varepsilon_\alpha X_{-\alpha}$ in the case $\sigma_T \alpha = -\alpha$, where $\varepsilon_\alpha = \pm 1$.

If $\sigma_T \alpha = -\alpha$, then such a triple $\{H_\alpha, X_\alpha, X_{-\alpha}\}$ determines an $\mathbb{R}$-homomorphism from a real form of SL(2) into G; examples are written in [Sh79a]. The isomorphism class of that real form, split or anisotropic, is uniquely determined by α. If the real form is split, then $\varepsilon_\alpha = 1$ and α is called *noncompact*. If the real form is anisotropic, then $\varepsilon_\alpha = -1$ and α is *compact*.

Lemma 2.2. *Each fundamental pair* (B, T) *extends to a fundamental splitting* $\text{spl} = (B, T, \{X_\alpha\})$. *Moreover, two fundamental splittings extending* (B, T) *are conjugate under* $T_{\text{sc}}(\mathbb{R})$.

Proof. For each simple root α of T in B, pick an $\mathfrak{sl}_2$-triple $\{X_\alpha, H_\alpha, X_{-\alpha}\}$. If $\sigma_T\alpha \neq -\alpha$, then we may arrange that $\sigma X_\alpha = X_{\sigma_T\alpha}$ and $\sigma X_{-\alpha} = X_{-\sigma_T\alpha}$ since $\alpha, \sigma_T\alpha$ are distinct. If $\sigma_T\alpha = -\alpha$, then calculation shows that $\sigma X_\alpha = \lambda X_{-\alpha}$ and $\sigma X_{-\alpha} = \lambda^{-1}X_\alpha$, where λ is real. Then we can adjust the choice of X_α and $X_{-\alpha}$ to arrange that $\sigma X_\alpha = \varepsilon_\alpha X_{-\alpha}$ and $\sigma X_{-\alpha} = \varepsilon_\alpha X_\alpha$, where $\varepsilon_\alpha = \pm 1$.

Suppose we have two such splittings. Then they are conjugate under $T_{\mathrm{ad}}(\mathbb{R})$ since if X_α is replaced by $\mathrm{Int}(t)X_\alpha$ for each B-simple root α, where $t \in T_{\mathrm{sc}}$, then our requirements on the action of σ imply that $\alpha(\sigma(t)t^{-1}) = 1$. Because $T_{\mathrm{sc}}, T_{\mathrm{ad}}$ are fundamental, $T_{\mathrm{sc}}(\mathbb{R})$ and $T_{\mathrm{ad}}(\mathbb{R})$ are connected; see [Ko86, Section 10] and [Sh12, Section 6]. Then the projection $T_{\mathrm{sc}}(\mathbb{R}) \to T_{\mathrm{ad}}(\mathbb{R})$ is surjective, and the desired conjugation exists. □

Corollary 2.3. *Each fundamental splitting* $\mathrm{spl}' = (B', T', \{X'_\alpha\})$ *is conjugate to* spl *by an element g of G for which* $\mathrm{Int}(g) : T \to T'$ *is defined over* $\mathbb{R}$.

2.3 Fundamental splittings of Whittaker type

We return to a quasisplit pair (G^*, θ^*). Recall that θ^* preserves the $\mathbb{R}$-splitting $\mathrm{spl}^* = (B^*, T^*, \{X_\alpha\})$ of G^*. *From now on we will typically use the same notation* $\{X_\alpha\}$ *for the root vectors in any splitting.* We will say a fundamental pair (B, T), or a fundamental splitting $\mathrm{spl}_f = (B, T, \{X_\alpha\})$ *of* G^*, is *of Whittaker type* if all imaginary simple roots of (B, T) are noncompact. We use this terminology because of Vogan's classification theorem [Vo78, Corollary 5.8, Theorem 6.2] for representations with Whittaker model, i.e., for *generic* representations. It is not difficult to check directly that a group G has a fundamental pair of Whittaker type if and only if G is quasisplit over $\mathbb{R}$, although this characterization is naturally part of Vogan's classification.

Lemma 2.4.

(i) There exists a fundamental pair of Whittaker type preserved by θ^* *and*
(ii) each fundamental pair of Whittaker type preserved by θ^* *has an extension to a fundamental splitting* $\mathrm{spl}_{\mathrm{Wh}}$ *of* G^* *preserved by* θ^*.

Proof. (i) We use Steinberg's structure theorems as described in [KS12, Section 3] and [KS99, Section 1.3]. First, attach to spl^* an $\mathbb{R}$-splitting for $(G^*_{\mathrm{sc}})^{\theta^*_{\mathrm{sc}}}$. We may then find h in $(G^*_{\mathrm{sc}})^{\theta^*_{\mathrm{sc}}}$ conjugating the pair determined by this $\mathbb{R}$-splitting to a fundamental pair in $(G^*_{\mathrm{sc}})^{\theta^*_{\mathrm{sc}}}$ of Whittaker type; such a pair exists since $(G^*_{\mathrm{sc}})^{\theta^*_{\mathrm{sc}}}$ is quasisplit. This pair determines uniquely a pair (B, T) for G^* preserved by θ^*. Then (B, T) is fundamental because T can have no real roots; see the proof of Lemma 3.8 below. An examination of root vectors shows further that (B, T) is of Whittaker type.

(ii) Now attach to any fundamental (B, T) of Whittaker type a fundamental pair in $(G^*_{\mathrm{sc}})^{\theta^*_{\mathrm{sc}}}$ also of Whittaker type, and define h in $(G^*_{\mathrm{sc}})^{\theta^*_{\mathrm{sc}}}$ as in (i). Extend (B, T) to a fundamental splitting $\mathrm{spl}_f = (B, T, \{X_\alpha\})$ for G^*. There is $t \in T^*_{\mathrm{sc}}$ such that th transports spl^* to spl_f. Then

$$\theta_f = \mathrm{Int}(th) \circ \theta^* \circ \mathrm{Int}(th)^{-1} = \mathrm{Int}(t\theta^*_{\mathrm{sc}}(t)^{-1}) \circ \theta^* \tag{2.2}$$

preserves spl_f and coincides with θ^* on T. A calculation on root vectors shows that $\sigma(\theta_f) = \theta_f$. For this, note that the Whittaker property of (B, T) implies that $\sigma X_\alpha = X_{-\alpha}$, for each imaginary root vector X_α in spl_f. Thus θ_f is defined over $\mathbb{R}$. Then $\mathrm{Int}(t\theta^*_{\mathrm{sc}}(t)^{-1})$ lies in $T_{\mathrm{ad}}(\mathbb{R})$. Since $T_{\mathrm{sc}}(\mathbb{R}) \to T_{\mathrm{ad}}(\mathbb{R})$ is surjective, we may take $t\theta^*_{\mathrm{sc}}(t)^{-1}$ in $V(\mathbb{R}) = T_{\mathrm{sc}}(\mathbb{R}) \cap V$, where $V = [1 - \theta^*_{\mathrm{sc}}](T_{\mathrm{sc}})$. Now we claim that for fundamental T, the kernel of $H^1(\Gamma, (T_{\mathrm{sc}})^{\theta^*_{\mathrm{sc}}}) \to H^1(\Gamma, T_{\mathrm{sc}})$ is trivial. From the Tate–Nakayama isomorphisms it is enough to show the kernel of $H^{-1}(\Gamma, [X_*(T_{\mathrm{sc}})]^{\theta^*_{\mathrm{sc}}}) \to H^{-1}(\Gamma, X_*(T_{\mathrm{sc}}))$ is trivial. This is immediate since both $-\sigma_T$ and θ^*_{sc} preserve a base for the coroot lattice $X_*(T_{\mathrm{sc}})$. Triviality of the kernel implies that $V(\mathbb{R})$ is connected. Thus we may assume $t \in T_{\mathrm{sc}}(\mathbb{R})$. Then $\theta^* = \mathrm{Int}(t^{-1}) \circ \theta_f \circ \mathrm{Int}(t)$ preserves the splitting $\mathrm{Int}(t^{-1})(\mathrm{spl}_f)$ which is fundamental and of Whittaker type. □

2.4 An application

We continue with an inner form (G, θ, η) of the quasisplit pair (G^*, θ^*). Following [KS99, Chapter 3] we say an element δ of $G(\mathbb{R})$ is θ-semisimple if $\mathrm{Int}(\delta) \circ \theta$ preserves a pair (B, T). We will say that the θ-semisimple element δ of $G(\mathbb{R})$ is *θ-fundamental* if $\mathrm{Int}(\delta) \circ \theta$ preserves a fundamental pair (B, T).

Recall that G is *cuspidal* if and only if a fundamental maximal torus T is *elliptic*, i.e., T is anisotropic modulo the center Z_G of G. In a setting where G is assumed cuspidal, we will use the term *θ-elliptic* interchangeably with *θ-fundamental*. For strongly θ-regular θ-semisimple elements there is another definition of θ-ellipticity (which does not require G to be cuspidal) in [KS99, Introduction]. We observe that a strongly θ-regular θ-semisimple element δ of cuspidal $G(\mathbb{R})$ is θ-elliptic in our present sense if and only if it is θ-elliptic in the sense of [KS99]; see Lemma 3.8(i). *In the general setting we will use exclusively the term θ-fundamental.* The strongly θ-regular θ-semisimple elements of $G(\mathbb{R})$ that are θ-elliptic in the sense of [KS99] are θ-fundamental; this is another consequence of the observation about real roots in the proof of Lemma 3.8(i).

Following Lemma 2.4, we choose a fundamental splitting $\mathrm{spl}_{\mathrm{Wh}}$ of G^* of Whittaker type preserved by θ^*.

Lemma 2.5.

(i) There exists a θ-fundamental element in $G(\mathbb{R})$ if and only if there is (θ_f, η_f) in the inner class of (θ, η) such that θ_f preserves a fundamental splitting for G.

(ii) If such (θ_f, η_f) exists and θ_f preserves the fundamental splitting spl_G, then we may further assume η_f transports spl_G to $\mathrm{spl}_{\mathrm{Wh}}$ and θ_f to θ^*.

Proof. Assume that there exists a θ-fundamental element in $G(\mathbb{R})$. Then we may multiply θ by an element of $\mathrm{Int}(G(\mathbb{R}))$ to obtain an $\mathbb{R}$-automorphism θ' preserving a fundamental pair. Now apply Lemma 2.2 to extend this pair to a fundamental splitting spl_G. Since θ' carries spl_G to another fundamental splitting, the lemma also shows that a further multiplication by an element of $\mathrm{Int}(G(\mathbb{R}))$ provides an $\mathbb{R}$-automorphism θ_f which preserves spl_G. We choose $\eta_f : G \to G^*$ in the inner class of η carrying spl_G to $\mathrm{spl}_{\mathrm{Wh}}$. Then $\eta_f \circ \theta_f \circ \eta_f^{-1}$ and θ^* are automorphisms of G^* which preserve $\mathrm{spl}_{\mathrm{Wh}}$ and differ by an inner automorphism. Hence they coincide. The converse assertion in (i) is immediate, and so the lemma is proved. □

For θ_f as in (ii) of the lemma, write $\eta_f \circ \sigma(\eta_f)^{-1}$ as $\mathrm{Int}(u_f(\sigma))$. Then, applying σ to the equation

$$\eta_f \circ \theta_f \circ \eta_f^{-1} = \theta^*, \tag{2.3}$$

we see that $\mathrm{Int}(u_f(\sigma))$ lies in the torus $(T_{\mathrm{ad}})^{\theta^*_{\mathrm{ad}}}$. Since $(T_{\mathrm{sc}})^{\theta^*_{\mathrm{sc}}} \to (T_{\mathrm{ad}})^{\theta^*_{\mathrm{ad}}}$ is surjective (both are connected; see [KS99, Section 1.1]) we may now assume

$$u_f(\sigma) \in (T_{\mathrm{sc}})^{\theta^*_{\mathrm{sc}}}. \tag{2.4}$$

3 Norms and the fundamental case

Here we include notation and review, and show that the norm correspondence is well-behaved in the fundamental case.

3.1 Endoscopic data

We now consider as quasisplit data, a triple (G^*, θ^*, a), where (G^*, θ^*) is a quasisplit pair as above, and a is a 1-cocycle of the Weil group $W_{\mathbb{R}}$ of $\mathbb{C}/\mathbb{R}$ in the center of the connected Langlands dual group $G^\vee$. Then ϖ will denote the character on $G^*(\mathbb{R})$, or on the real points of an inner form of G^*, attached to a. As always, and without harm, we provide an explicit transition of data between G^* and its Langlands dual ${}^LG = G^\vee \rtimes W_{\mathbb{R}}$ by the choice of $\mathbb{R}$-splitting $\mathrm{spl}^* = (B^*, T^*, \{X_\alpha\})$ of G^* preserved by θ^* and dual Γ-splitting $\mathrm{spl}^\vee = (\mathcal{B}, \mathcal{T}, \{X_{\alpha^\vee}\})$ for $G^\vee$. The action of $W_{\mathbb{R}}$ on $G^\vee$ factors through $W_{\mathbb{R}} \to \Gamma$. Then $\theta^\vee$ is the Γ-automorphism of $G^\vee$ that preserves $\mathrm{spl}^\vee$ and is dual to θ^* as automorphism of the dual based root data. We write ${}^L\theta_a$ for the extension

$$g \times w \to a(w).\theta^\vee(g) \times w$$

of $\theta^\vee$ to an automorphism of ${}^LG$.

We assume $\mathfrak{e}_z$ is a supplemented set of endoscopic data (SED) for (G^*, θ^*, a) and its inner forms. The SED consists of a set $\mathfrak{e} = (H, \mathcal{H}, s)$ of endoscopic data for (G^*, θ^*, a) and a z-pair (H_1, ξ_1) for $\mathfrak{e}$ in the sense of [KS99], although we avoid the additional choice a' from Section 2.1 of [KS99] by adjusting s. There is no harm in assuming $\mathfrak{e}_z$ is bounded in the sense of [Sh14, Section 2]. Recall that H_1 is what we call the endoscopic group defined by the SED; Z_1 will denote the kernel of the z-extension $H_1 \to H$. We remark that SEDs exist for (G^*, θ^*, a) precisely when there are Langlands parameters preserved by ${}^L\theta_a$; see Section 9.1.

As noted in the introduction, we will be concerned mainly with the *fundamental case* for which we make an ad hoc definition in Section 3.3, and more particularly with the *cuspidal-elliptic setting* of Section 3.4.

3.2 Norm correspondence

A norm correspondence for $G(\mathbb{R})$ and an endoscopic group $H_1(\mathbb{R})$ is defined via maps on (twisted) conjugacy classes [KS99, Chapter 3, Section 5.4]. In general, the correspondence is not uniquely determined by (θ, η) and there are examples where it is empty on all or much of the *very regular set* defined in the paragraph following (3.1) below. In preparation for the fundamental case to be introduced in Section 3.3, we review two simpler settings indicated as I, II.

(I) Assume that θ preserves a fundamental splitting or, more precisely, that (θ, η) is of the form (θ_f, η_f) from (ii) in Lemma 2.5.

The equation (2.3) allows us to attach a unique norm correspondence to (θ, η). To begin, there is no need to choose the datum g_θ of [KS99, Chapter 3]; in the formulas there, set $g_\theta = 1$. To compute the cochain z_σ of Lemma 3.1.A of [KS99], write $u(\sigma)$ from (2.1) above as $u_1(\sigma) \cdot z(\sigma)$, where $u_1(\sigma) \in (T_{\mathrm{sc}})^{\theta^*_{\mathrm{sc}}}$ as in (2.4) and $z(\sigma)$ is central in G^*_{sc}. Thus $z_\sigma = (1-\theta^*_{\mathrm{sc}})\, z(\sigma)$. Then, by (2) of Lemma 3.1.A in [KS99], η determines uniquely a Γ-equivariant bijective map from the set $\mathrm{Cl}_{\theta\text{-ss}}(G, \theta)$ of θ-twisted conjugacy classes of θ-semisimple elements in $G(\mathbb{C})$ to the corresponding set $\mathrm{Cl}_{\theta^*\text{-ss}}(G^*, \theta^*)$ for (G^*, θ^*). This map provides the first step in defining the norm correspondence. By restriction, we obtain a Γ-equivariant bijective map from the set $\mathrm{Cl}_{\mathrm{str}\,\theta\text{-reg}}(G, \theta)$ of θ-twisted conjugacy classes of strongly θ-regular elements in $G(\mathbb{C})$ to the corresponding set $\mathrm{Cl}_{\mathrm{str}\,\theta^*\text{-reg}}(G^*, \theta^*)$ for (G^*, θ^*).

For the second step, the endoscopic datum $\mathfrak{e}$ provides a unique Γ-equivariant surjective map from the set $\mathrm{Cl}_{\mathrm{ss}}(H)$ of semisimple conjugacy classes in $H(\mathbb{C})$ to $\mathrm{Cl}_{\theta^*\text{-ss}}(G^*, \theta^*)$. The inverse image of $\mathrm{Cl}_{\mathrm{str}\,\theta^*\text{-reg}}(G^*, \theta^*)$ is, by definition, the set $\mathrm{Cl}_{\mathrm{str}\,G\text{-reg}}(H)$ of strongly G-regular conjugacy classes in $H(\mathbb{C})$; see [KS99, Lemma 3.3.C].

Third, the z-extension $H_1 \to H$ provides a Γ-equivariant surjective map from $\mathrm{Cl}_{\mathrm{ss}}(H_1)$ to $\mathrm{Cl}_{\mathrm{ss}}(H)$, and then by restriction, a Γ-equivariant surjective map from $\mathrm{Cl}_{\mathrm{str}\,G\text{-reg}}(H_1)$ to $\mathrm{Cl}_{\mathrm{str}\,G\text{-reg}}(H)$.

In summary, we have established the following diagram with all arrows Γ-equivariant.

$$
\begin{array}{ccccc}
\mathrm{Cl}_{\mathrm{str}\,\theta\text{-reg}}(G,\theta) & & & & \\
 & \searrow & & & \\
 & & \mathrm{Cl}_{\mathrm{str}\,\theta^*\text{-reg}}(G^*,\theta^*) & & \\
 & & & \nwarrow & \\
 & & \uparrow & & \mathrm{Cl}_{\mathrm{str}\,G\text{-reg}}(H_1) \\
 & & & \swarrow & \\
 & & \mathrm{Cl}_{\mathrm{str}\,G\text{-reg}}(H) & &
\end{array}
\tag{3.1}
$$

Turning now to real points, by the *very regular set* in $H_1(\mathbb{R}) \times G(\mathbb{R})$, we mean the set of all pairs (γ_1, δ) where $\gamma_1 \in H_1(\mathbb{R})$ is strongly G-regular and $\delta \in G(\mathbb{R})$ is strongly θ-regular. Restricting to the real points of the classes in (3.1), we obtain maps from stable θ-twisted conjugacy classes of strongly θ-regular elements in $G(\mathbb{R})$ to stable θ^*-twisted conjugacy classes of strongly θ^*-regular elements in $G^*(\mathbb{R})$, from stable conjugacy classes of strongly G-regular elements in $H(\mathbb{R})$ to stable θ^*-twisted conjugacy classes of strongly θ^*-regular elements in $G^*(\mathbb{R})$, and from the set of stable classes of strongly regular elements in $H_1(\mathbb{R})$ to the stable conjugacy classes of strongly regular elements in $H(\mathbb{R})$. Because $H_1 \to H$ is a z-extension, the last map is surjective and remains surjective when we replace"strongly regular" by "strongly G-regular". As in [KS99, Section 3.3], we now define a *norm correspondence on the very regular set*: $\delta \in G(\mathbb{R})$ has norm γ_1 in $H_1(\mathbb{R})$, i.e., (γ_1, δ) lies in the norm correspondence, if and only if the images of the respective stable classes of γ_1, δ have the same image among the stable θ^*-twisted conjugacy classes of strongly θ^*-regular elements in $G^*(\mathbb{R})$.

To attach data to the norm correspondence as in [KS99, Section 4.4], consider strongly θ-regular $\delta \in G(\mathbb{R})$. Then unraveling the definition of the last paragraph shows that δ has norm γ_1 in $H_1(\mathbb{R})$ if and only if there exist a θ^*-stable pair (B, T) in G^* with T defined over $\mathbb{R}$ and elements g in G^*_{sc}, δ^* in T such that

$$\delta^* = g.\eta(\delta).\theta^*(g)^{-1}, \tag{3.2}$$

and the image γ of δ^* under some admissible $T \to T_{\theta^*} \to T_H$ coincides with the image of γ_1 under $H_1 \to H$. See [KS99, Section 3.3]. This is summarized in the following diagram, where N denotes the projection $T \to T_{\theta^*}$ to coinvariants.

$$
\begin{array}{ccccc}
G(\mathbb{R}) \ni \delta \longrightarrow & \delta^* \in T & & & \\
 & \Big\downarrow & & & \gamma_1 \in H_1(\mathbb{R}) \\
 & & & \swarrow & \\
 & N\delta^* \in T_{\theta^*}(\mathbb{R}) & \longrightarrow & \gamma \in T_H(\mathbb{R}) &
\end{array}
$$

As in [KS99, Section 3.3], we say a maximal torus T in G^* is θ^**-admissible* if there exists a θ^*-stable pair (B, T) in G^*. Also, T_H and its inverse image T_1 in H_1 are θ^**-norm groups for* T if there exists an admissible $T \to T_{\theta^*} \to T_H$. Then write T_1 for the inverse image of T_H in H_1. Every maximal torus over $\mathbb{R}$ in H_1 is a θ^*-norm group for some θ^*-admissible maximal torus T in G^* [KS99, Lemma 3.3.B].

In regard to (3.2) we note the following for use in calculations.

Remark 3.1. Suppose (B_δ, T_δ) is preserved by $\mathrm{Int}(\delta) \circ \theta$, where δ is a strongly θ-regular element of $G(\mathbb{R})$ as above. Then T_δ is the centralizer in G of the abelian reductive subgroup $\mathrm{Cent}_\theta(\delta, G)$ of G. We may arrange that $\mathrm{Int}(g) \circ \eta$ carries (B_δ, T_δ) to (B, T) and $\mathrm{Cent}_\theta(\delta, G)$ to T^{θ^*} with the restriction of $\mathrm{Int}(g) \circ \eta$ to T_δ defined over $\mathbb{R}$.

(II) Replace θ in (I) by θ' of the form $\mathrm{Int}(g_\mathbb{R}) \circ \theta$, $g_\mathbb{R} \in G(\mathbb{R})$.

The norm correspondence is no longer canonical but there is a quick and transparent definition in this case. Namely, the map $\delta \to \delta \cdot g_\mathbb{R}$ carries strongly θ'-regular elements in $G(\mathbb{R})$ to strongly θ-regular elements in $G(\mathbb{R})$, providing a bijection between the stable classes of strongly θ'-regular elements and the stable classes of strongly θ-regular elements. We then extend the definition of the norm correspondence on stable classes to this case in the obvious way. This norm for θ' depends on our choice of $g_\mathbb{R}$ and so we use the terminology $g_\mathbb{R}$-norm. The dependence is that $g_\mathbb{R}$ may be replaced by $z_\mathbb{R} g_\mathbb{R}$, with $z_\mathbb{R} \in Z_G(\mathbb{R})$. Then a strongly G-regular γ_1 in the endoscopic group $H_1(\mathbb{R})$ is a $g_\mathbb{R}$-norm of δ if and only if γ_1 is a $z_\mathbb{R} g_\mathbb{R}$-norm of $\delta z_\mathbb{R}$. See Section 4.2 for the role of $z_\mathbb{R}$ in transfer statements.

3.3 Fundamental case

Let (G^*, θ^*) be a quasisplit pair. A fundamental maximal torus T_1 in H_1 is a θ^*-norm group for some θ^*-admissible maximal torus T in G^*; see Section 3.2. It will be convenient to call $(G^*, \theta^*, \mathfrak{e}_z)$ *fundamental* if we may choose T to be fundamental; see Remark 3.4 below.

In general, write $\mathrm{StrReg}(G^*, \theta^*)$ for the set of all strongly θ^*-regular elements in $G^*(\mathbb{R})$ and $\mathrm{StrReg}(G^*, \theta^*)_f$ for the subset of θ^*-fundamental elements as defined in Section 2.4.

Lemma 3.2. $\mathrm{StrReg}(G^*, \theta^*)_f$ *is nonempty and a union of stable θ^*-twisted conjugacy classes.*

Proof. There is a fundamental pair (B, T) in G^* preserved by θ^*; see the proof of Lemma 3.8 below. Then $T(\mathbb{R})$ contains (many) elements in $\mathrm{StrReg}(G^*, \theta^*)_f$. The rest is immediate from definitions. □

Write $\mathrm{StrReg}_{G^*}(H_1)$ for the set of all strongly G^*-regular elements in $H_1(\mathbb{R})$ and $\mathrm{StrReg}_{G^*}(H_1)_f$ for the subset of elements γ_1 such that the maximal torus $\mathrm{Cent}(\gamma_1, H_1)$ is fundamental in H_1. We call (γ_1, δ) in the very regular set, i.e., in $\mathrm{StrReg}_{G^*}(H_1) \times \mathrm{StrReg}(G^*, \theta^*)$, a *related pair* if it lies in the uniquely defined norm correspondence for (G^*, θ^*), i.e., if γ_1 is a norm of δ.

Lemma 3.3.

(i) $(G^*, \theta^*, \mathfrak{e}_z)$ *is fundamental if and only if*

$$\mathrm{StrReg}_{G^*}(H_1)_f \times \mathrm{StrReg}(G^*, \theta^*)_f$$

contains a related pair.

Now assume $(G^*, \theta^*, \mathfrak{e}_z)$ *is fundamental. Then*

(ii) each δ *in* $\mathrm{StrReg}(G^*, \theta^*)_f$ *has a norm* γ_1 *in* $H_1(\mathbb{R})$ *and* $\gamma_1 \in \mathrm{StrReg}_{G^*}(H_1)_f$,

(iii) if $\gamma_1 \in \mathrm{StrReg}_{G^*}(H_1)_f$ *is a norm of strongly* θ*-regular* δ *in* $G^*(\mathbb{R})$*, then* $\delta \in \mathrm{StrReg}(G^*, \theta^*)_f$.

Proof. (i) Assume that $(G^*, \theta^*, \mathfrak{e}_z)$ is fundamental and choose an admissible $T \to T_{\theta^*} \to T_H$ with both T, T_H fundamental. This provides related pairs in $\mathrm{StrReg}_{G^*}(H_1)_f \times \mathrm{StrReg}(G^*, \theta^*)_f$. Conversely, a related pair in $\mathrm{StrReg}_{G^*}(H_1)_f \times \mathrm{StrReg}(G^*, \theta^*)_f$ provides an admissible $T \to T_{\theta^*} \to T_H$ with both T, T_H fundamental, and $(G^*, \theta^*, \mathfrak{e}_z)$ is fundamental. To check (ii), we may replace δ with a twisted conjugate by an element of $G^*(\mathbb{R})$ and assume that $T_\delta = T$. The result then follows easily; see [KS99, Lemma 4.4.A]. For (iii), suppose (γ_1, δ) is a related pair with attached T_H fundamental. Then Remark 3.1 implies that the stable class of attached θ^*-admissible T is uniquely determined by γ_1, and (iii) follows. □

Remark 3.4. The argument for (iii) shows that $(G^*, \theta^*, \mathfrak{e}_z)$ is *fundamental if and only if* every θ^*-admissible maximal torus T in G^* with a fundamental maximal torus in H_1 as θ^*-norm group is fundamental.

Now consider an inner form (G, θ, η), and define $\mathrm{StrReg}(G, \theta)_f$ in the same way as $\mathrm{StrReg}(G^*, \theta^*)_f$. In general, we modify $\mathrm{StrReg}_{G^*}(H_1)_f$ slightly as in Section 5.4 of [KS99]. Namely, we replace $H_1(\mathbb{R})$ by a suitable coset $H_1(\mathbb{R})^\dagger$ of $H_1(\mathbb{R})$ in $H_1(\mathbb{C})$. Then we define a subset $\mathrm{StrReg}_{G^*}(H_1)_f^\dagger$ of this coset $H_1(\mathbb{R})^\dagger$ which may be empty. For (θ, η) as in (ii) of the next lemma we take, as we may, $H_1(\mathbb{R})^\dagger = H_1(\mathbb{R})$.

Lemma 3.5. *Assume that* $(G^*, \theta^*, \mathfrak{e}_z)$ *is fundamental. Then the following are equivalent for an inner form* (G, θ, η) *of* (G^*, θ^*)*:*

(i) there exists a θ*-fundamental element in* $G(\mathbb{R})$,

(ii) there is (θ_f, η_f) *in the inner class of* (θ, η) *such that* θ_f *preserves a fundamental splitting for* G,

(iii) there exists a related pair in $\mathrm{StrReg}_{G^*}(H_1)_f^\dagger \times \mathrm{StrReg}(G, \theta)_f$.

Proof. We have proved (i) ⇒ (ii) in Lemma 2.5. For (ii) ⇒ (iii) we may further assume that $\theta = \theta_f$ and η transports θ to θ^*. Then the assertion follows easily. (iii) ⇒ (i) is immediate. □

Lemma 3.6. *Assume any one of the equivalent conditions from Lemma 3.5 is satisfied. Then:*

(i) each δ *in* $\mathrm{StrReg}(G,\theta)_f$ *has a norm* γ_1 *in* $H_1(\mathbb{R})$ *and moreover* γ_1 *lies in* $\mathrm{StrReg}_{G^*}(H_1)_f$,

(ii) if $\gamma_1 \in \mathrm{StrReg}_{G^*}(H_1)_f$ *is a norm of strongly* θ*-regular* δ *in* $G(\mathbb{R})$*, then* δ *lies in* $\mathrm{StrReg}(G,\theta)_f$.

Proof. We may assume that θ preserves fundamental splitting spl_G, that θ^* preserves fundamental splitting $\mathrm{spl}_{\mathrm{Wh}}$ of Whittaker type, and that η transports θ to θ^*. Recall that $\mathrm{Int}(\delta)\circ\theta$ preserves the fundamental pair (B_δ, T_δ). Extend the pair to a fundamental splitting spl_δ. Then there is t_δ in $(T_\delta)_{\mathrm{sc}}(\mathbb{R})$ such that $\mathrm{Int}(t_\delta\delta)\circ\theta$ preserves spl_δ. Here, as usual, we have used the same notation t_δ for the image of t_δ in $(T_\delta)(\mathbb{R})$ under $G_{\mathrm{sc}} \to G$. We now choose g in G_{sc} such that $\mathrm{Int}(g)$ carries spl_δ to spl_G. Let T_G be the elliptic maximal torus specified by spl_G. Then $g_\sigma = g\sigma(g)^{-1}$ lies in $(T_G)_{\mathrm{sc}}$, $t_G = gt_\delta^{-1}g^{-1}$ lies in $(T_G)_{\mathrm{sc}}(\mathbb{R})$ and $\delta_G = g\delta\theta(g)^{-1}$ is of the form zt_G, where z is central. Also

$$\sigma(z)^{-1}z = \sigma(\delta_G)^{-1}\delta_G = (1-\theta)g_\sigma, \tag{3.3}$$

so that $N_\theta(z)$ lies in $(T_G)_\theta(\mathbb{R})$. Now apply the twist η which carries spl_G to $\mathrm{spl}_{\mathrm{Wh}}$. Then (i), (ii) follow; see Lemma 4.4.A of [KS99]. □

Example 3.7. For general (G^*,θ^*), consider a basic SED $\mathfrak{e}_z$, i.e., assume that $s = 1$. Then an argument along the same lines as that for Lemma 3.3 shows that $(G^*,\theta^*,\mathfrak{e}_z)$ is fundamental.

3.4 Cuspidal-elliptic setting

By the cuspidal-elliptic setting we mean that G^*, or equivalently an inner form of G^*, is cuspidal and that the endoscopic datum $\mathfrak{e}$ is *elliptic* in the sense that the identity component of the Γ-invariants in the center of $H^\vee$ lies in the center of $G^\vee$ [KS99]. We then call H_1 an *elliptic endoscopic group.*

Lemma 3.8.

(i) Assume G^* *is cuspidal. Then* $(G^*)^{\theta^*}$ *is cuspidal and there exists an elliptic* θ^**-admissible maximal torus* T *in* G^*.

(ii) Assume also that $\mathfrak{e}$ *is elliptic. Then* H_1 *is cuspidal and each elliptic* T_1 *in* H_1 *is a* θ^**-norm group for each elliptic* θ^**-admissible* T *in* G^*.

Proof. There is no harm, for both (i) and (ii), in assuming that G^* is semisimple and simply-connected, so that $I = (G^*)^{\theta^*}$ is connected (as well as reductive) as algebraic group. Consider a pair (B^1, T^1), where T^1 is a fundamental maximal torus defined over $\mathbb{R}$ in I and B^1 is any Borel subgroup of I containing T^1. Set $T = \mathrm{Cent}(T^1, G^*)$ and $B = \mathrm{Norm}(B^1, G^*)$, so that (B,T) is a θ^*-stable pair for G^*. Then T must be fundamental, for otherwise T would have a real root and then a multiple of the restriction of this root to $T^1 = T^{\theta^*}$ would provide us with a real root for T^1 in I; no such root exists since T^1 is fundamental. (i) then follows. For (ii), let T_H be a fundamental maximal torus in H. Then there is some admissible isomorphism $T_H \to T_{\theta^*}$ associated to a θ^*-admissible T. Attach to H the standard endoscopic group J for I as in Section 4.2 of [KS99]. Then T^1 is (isomorphic to) a fundamental maximal torus in J, and moreover J is elliptic because H is. Thus, by (ii) in the case of standard endoscopy, T^1 is anisotropic modulo Z_I. Since T is then anisotropic modulo Z_{G^*} as in (i), T_H is anisotropic modulo Z_H, and (ii) follows. □

Corollary 3.9. $(G^*, \theta^*, \mathfrak{e}_z)$ *is fundamental in the sense of Section 3.3.*

Consider an inner form (G, θ, η). We write sr-ell(G, θ) for the set of all θ-elliptic strongly θ-regular elements in $G(\mathbb{R})$ and sGr-ell$(H_1)^\dagger$ for the set of all strongly G-regular elliptic elements in $H_1(\mathbb{R})^\dagger$.

Corollary 3.10. *The following are equivalent:*

(i) there exists a θ-elliptic element in $G(\mathbb{R})$,
(ii) there is (θ_f, η_f) in the inner class of (θ, η) such that θ_f preserves a fundamental splitting for G,
(iii) there exists a related pair in sGr-ell$(H_1)^\dagger \times$ sr-ell(G, θ).

Proof. By Lemma 3.8, this is a special case of Lemma 3.5. □

Corollary 3.11. *Assume any one of the conditions of Corollary 3.10 is satisfied. Then:*

(i) each δ in sr-ell(G, θ) *has a norm γ_1 in $H_1(\mathbb{R})$ and $\gamma_1 \in$* sGr-ell(H_1),
(ii) if $\gamma_1 \in$ sGr-ell(H_1) *is a norm of strongly θ-regular δ in $G(\mathbb{R})$, then $\delta \in$* sr-ell(G, θ).

Proof. By Lemma 3.8, this is a special case of Lemma 3.3. □

3.5 Consequences for geometric transfer factors

We conclude by summarizing some of the results of Section 3.3 and 3.4 in terms of the transfer factor Δ of [KS99] (see also [KS12, Sh14]) for the matching of orbital integrals, i.e., for geometric twisted transfer [Sh12]. The factor Δ is defined on the very regular set of Section 3.2. By construction, $\Delta(\gamma_1, \delta) \neq 0$ if and only if (γ_1, δ) is a related pair, i.e., γ_1 is a norm of δ. We consider

(i) transfer for quasisplit data (G^*, θ^*) with SED $\mathfrak{e}_z$ and
(ii) transfer for an inner form of the quasisplit data in (i) when $\mathfrak{e}_z$ is fundamental.

Then Lemmas 3.3, 3.6 imply that:

Theorem 3.12.

(i) There exists fundamental γ_1 and θ^*-fundamental δ such that $\Delta(\gamma_1, \delta) \neq 0$ if and only if $(G^*, \theta^*, \mathfrak{e}_z)$ is fundamental.
(ii) Assume $(G^*, \theta^*, \mathfrak{e}_z)$ is fundamental and that (G, θ, η) is an inner form. Then there exist fundamental γ_1 and θ-fundamental δ such that $\Delta(\gamma_1, \delta) \neq 0$ if and only if there exists a θ-fundamental element in $G(\mathbb{R})$.

From this and Lemma 3.8 we conclude:

Corollary 3.13. *In the cuspidal-elliptic setting:*

(i) there exist elliptic γ_1 and θ^*-elliptic δ such that $\Delta(\gamma_1, \delta) \neq 0$ and
(ii) for an inner form (G, θ, η), there exist elliptic γ_1 and θ-elliptic δ such that $\Delta(\gamma_1, \delta) \neq 0$ if and only if there exists a θ-elliptic element in $G(\mathbb{R})$.

We will return to the results of Sections 3.3 and 3.4 in [ShII].

4 Formulating spectral factors

We turn now to some remarks on transfer statements in the setting from Lemma 2.5. We have checked that this setting captures all nontrivial geometric transfer on the fundamental very regular set. There is an analogous statement for the spectral side which we will introduce now but make precise and verify later; see Part 9. We will limit our discussion in the present part to the cuspidal-elliptic setting as the general fundamental case follows quickly.

4.1 Transfer statements

For the main case I, we consider an inner form (G, θ, η) of (G^*, θ^*) for which

(i) the transport of spl_{Wh} to G by η is fundamental and
(ii) θ is the transport of θ^* to G by η.

We have assumed for convenience that G is cuspidal and the endoscopic datum $\mathfrak{e}$ is elliptic. Also for convenience, we will discuss transfer for the tempered rather than the essentially tempered spectrum.

First recall geometric transfer. Test functions are Harish-Chandra Schwartz functions; we consider functions $f \in \mathcal{C}(G(\mathbb{R}), \theta)$ and $f_1 \in \mathcal{C}(H_1(\mathbb{R}), \varpi_1)$ [Sh12, Section 1]. We may also use $C_c^\infty(G(\mathbb{R}), \theta)$ and $C_c^\infty(H_1(\mathbb{R}), \varpi_1)$ by Bouaziz's Theorem (see [Sh12, Section 2]), as we will need in Section 8.2 for the generally nontempered transfer of Adams–Johnson. Measures and integrals will be defined and normalized as in [Sh12]. To be more careful, we should use test measures in place of test functions throughout, in order to have the transfer depend only on the normalization of transfer factors. However, this will be ignored here; see instead the note [Sh].

Theorem 2.1 of [Sh12] shows that for all $f \in \mathcal{C}(G(\mathbb{R}), \theta)$ there exists $f_1 \in \mathcal{C}(H_1(\mathbb{R}), \varpi_1)$ such that

$$SO(\gamma_1, f_1) = \sum_{\delta,\, \theta\text{-}conj} \Delta(\gamma_1, \delta)\ O^{\theta,\varpi}(\delta, f) \tag{4.1}$$

for all strongly G-regular γ_1 in $H_1(\mathbb{R})$. Here $O^{\theta,\varpi}$ denotes a (θ, ϖ)-twisted orbital integral and SO denotes a standard (untwisted) stable orbital integral. We write $f_1 \in \mathrm{Trans}_{\theta,\varpi}(f)$.

Suppose π_1 is a tempered irreducible admissible representation of $H_1(\mathbb{R})$ and Π_1 is its packet. We will assume, usually without further mention, that $\pi_1(Z_1(\mathbb{R}))$ acts by the character ϖ_1; recall Z_1 is the central torus $\mathrm{Ker}(H_1 \to H)$. Let $\mathrm{St}\text{-}\mathrm{Tr}\,\pi_1$ be the stable tempered distribution

$$f_1 \to \sum_{\pi_1' \in \Pi_1} \mathrm{Trace}\, \pi_1'(f_1).$$

Because $f_1 \in \mathcal{C}(H_1(\mathbb{R}), \varpi_1)$ we have taken $\pi_1(f_1)$ as the operator

$$\int_{H_1(\mathbb{R})/Z_1(\mathbb{R})} f_1(h_1)\pi_1(h_1)\, \frac{dh_1}{dz_1}.$$

Following the case of standard endoscopic transfer we may consider the linear form $f \to \mathrm{St}\text{-}\mathrm{Tr}\,\pi_1(f_1)$ on $\mathcal{C}(G(\mathbb{R}), \theta)$, where f_1 is attached to f as in (4.1). For the present discussion we restrict the form to $C_c^\infty(G(\mathbb{R}), \theta)$. It is well-defined by Lemma 5.3 of [Sh79a], and results of Waldspurger [Wa14] (see also [Me13]) show that it is a linear combination of twisted traces of representations of $G(\mathbb{R})$. Our purpose is different. We want to describe certain coefficients closely related to the geometric factors and then later establish that they are correct for such a spectral transfer. Our interest in the spectral transfer statement (4.3) below is in certain constraints it places on our factors. With these constraints in mind we will verify various lemmas before making our definitions. For example, Lemma 9.2 will be the spectral analogue of Corollary 3.10, namely that our present assumption on (G, θ, η) captures all nonempty twistpackets of discrete series representations for inner forms of (G^*, θ^*). Other results require more effort and for these we will introduce further tools.

Let π be a tempered irreducible admissible representation of $G(\mathbb{R})$ and Π denote its packet. We use the same notation for a representation and its isomorphism class; we may also work with unitary representations and unitary isomorphisms. For a related pair of Langlands parameters (see Part 9) we consider the corresponding packets Π_1 for $H_1(\mathbb{R})$ and Π for $G(\mathbb{R})$. The construction of endoscopic data ensures that the packet Π is preserved under the map

$$\pi \to \varpi^{-1} \otimes (\pi \circ \theta).$$

This last property is a simple condition on the Langlands parameter of Π; whenever it is satisfied we call the attached packet (θ, ϖ)-*stable*. Thus we may define a twisted trace on $\oplus_{\pi' \in \Pi} \pi'$. Only those π' fixed by the map will contribute nontrivially. We then define $\Pi^{\theta,\varpi}$ to be the subset of Π consisting of such π' and call $\Pi^{\theta,\varpi}$ a *twistpacket* for (θ, ϖ).

Suppose π belongs to the twistpacket $\Pi^{\theta,\varpi}$ and that the unitary operator $\pi(\theta, \varpi)$ on the space of π interwines $\pi \circ \theta$ and $\varpi \otimes \pi$ or, more precisely, that

$$\pi(\theta(g)) \circ \pi(\theta, \varpi) = \varpi(g).(\pi(\theta, \varpi) \circ \pi(g)), \tag{4.2}$$

for $g \in G(\mathbb{R})$. Then by the twisted trace of π, we mean the linear form

$$f \to \operatorname{Trace} \pi(f) \circ \pi(\theta, \varpi).$$

Note that we have not fixed a normalization of the operator $\pi(\theta, \varpi)$. Also, if f is replaced by $g \to f(xg\theta(x)^{-1})$, then $\operatorname{Trace} \pi(f)\ \pi(\theta, \varpi)$ is multiplied by $\varpi(x)$, for $x \in G(\mathbb{R})$.

Spectral transfer factors will be nonzero complex coefficients $\Delta(\pi_1, \pi)$ such that

$$\text{St-}\operatorname{Trace} \pi_1(f_1) = \textstyle\sum_{\pi \in \Pi^{\theta,\varpi}} \Delta(\pi_1, \pi)\ \operatorname{Trace} \pi(f)\ \pi(\theta, \varpi). \tag{4.3}$$

The factors $\Delta(\pi_1, \pi)$ depend on how we normalize the geometric factors $\Delta(\gamma_1, \delta)$ that prescribe the correspondence (f, f_1). Following the method for standard transfer, we will introduce a geometric-spectral compatibility factor. For standard transfer this factor was canonical. In the twisted case there is a new dependence: the choice of normalization for the operators $\pi(\theta, \varpi)$, $\pi \in \Pi^{\theta,\varpi}$. We may multiply $\pi(\theta, \varpi)$ by a nonzero complex number λ (of absolute value one since we have required unitarity). In standard endoscopy, the term Δ_{II} in $\Delta(\pi_1, \pi)$ comes from the explicit local representation of $f \to \operatorname{Trace} \pi(f)$ around the identity. In the twisted case, we consider a similar twisted term for $f \to \operatorname{Trace} \pi(f)\pi(\theta, \varpi)$ around a certain point, in general not the identity element. We will then see that multiplying $\pi(\theta, \varpi)$ by λ has the effect of dividing Δ_{II} by λ. No other term in $\Delta(\pi_1, \pi)$ will depend on $\pi(\theta, \varpi)$ and so

$$\Delta(\pi_1, \pi)\ \operatorname{Trace} \pi(f)\ \pi(\theta, \varpi)$$

will be independent of the choice for $\pi(\theta, \varpi)$. Then the (geometric-spectral) compatibility factor $\Delta(\pi_1, \pi; \gamma_1, \delta)$ will depend on $\pi(\theta, \varpi)$ but the quotient

$$\Delta(\pi_1, \pi)/\Delta(\pi_1, \pi; \gamma_1, \delta)$$

will not. We conclude that we may define geometric-spectral compatibility as in the standard case [Sh10, Section 12].

4.2 Additional twist by an element of $G(\mathbb{R})$

We now consider the setting II where we twist an automorphism θ as in I by an element $g_{\mathbb{R}}$ of $G(\mathbb{R})$. This yields no new twistpackets but it will be useful to have a precise formulation for transfer with the twisted automorphism.

Denote by $\Delta_{g_{\mathbb{R}}}$ the geometric transfer factors defined using $g_{\mathbb{R}}$-norms. Suppose we replace $g_{\mathbb{R}}$ by $z_{\mathbb{R}}g_{\mathbb{R}}$, where $z_{\mathbb{R}}$ lies in the center of $G(\mathbb{R})$. Then the relative factors

$$\Delta_{g_{\mathbb{R}}}(\gamma_1, \delta; \gamma_1', \delta')$$

and

$$\Delta_{z_{\mathbb{R}}g_{\mathbb{R}}}(\gamma_1, \delta z_{\mathbb{R}}; \gamma_1', \delta' z_{\mathbb{R}})$$

coincide. Indeed we see quickly from the definitions that the only difference between the two is that the element $z_{\mathbb{R}}$ is inserted in the element D constructed for Δ_{III} (see p. 33 of [KS99]) where it clearly has no effect. This property of the relative factors allows us to normalize absolute factors so that

$$\Delta_{z_{\mathbb{R}}g_{\mathbb{R}}}(\gamma_1, \delta z_{\mathbb{R}}) = \Delta_{g_{\mathbb{R}}}(\gamma_1, \delta)$$

for all very regular related pairs (γ_1, δ) for the $g_{\mathbb{R}}$-norm.

The choice of $z_{\mathbb{R}}$ affects the correspondence on test functions. If $f_1 \in \text{Trans}(f)$ for $g_{\mathbb{R}}$-norms, then clearly $f_1 \in \text{Trans}(f_{z_{\mathbb{R}}})$ for $z_{\mathbb{R}}g_{\mathbb{R}}$-norms, where $f_{z_{\mathbb{R}}}$ denotes the translate of f by $(z_{\mathbb{R}})^{-1}$. The extended version of Lemma 5.1.C at the bottom of p. 53 of [KS99] applies also to $g_{\mathbb{R}}$-norms since it is easily rewritten as a statement about relative factors. Thus, if $z_1 \in Z_{H_1}(\mathbb{R})$ has image in $Z_H(\mathbb{R})$ equal to the image of $z_{\mathbb{R}}$ under N, i.e., if $(z_1, z_{\mathbb{R}})$ belongs to the group $C(\mathbb{R})$ from (5.1) of [KS99], then there is quasicharacter ϖ_C on $C(\mathbb{R})$ such that

$$\Delta_{g_{\mathbb{R}}}(z_1\gamma_1, \delta z_{\mathbb{R}}) = \varpi_C(z_1, z_{\mathbb{R}})^{-1}\Delta_{g_{\mathbb{R}}}(\gamma_1, \delta).$$

A calculation with (4.1) now shows that

$$\varpi_C(z_1, z_{\mathbb{R}}).(f_1)_{z_1} \in \text{Trans}(f)$$

for $z_{\mathbb{R}}g_{\mathbb{R}}$-norms.

In Lemma 9.5 we will prove that the central characters $\varpi_{\pi_1}, \varpi_\pi$ for a related pair (π_1, π) have the property that

$$\varpi_{\pi_1}(z_1).\varpi_\pi(z)^{-1} = \varpi_C(z_1, z_{\mathbb{R}}) \tag{4.4}$$

for all $(z_1, z_{\mathbb{R}})$ in $C(\mathbb{R})$. This and (4.2) imply that if the spectral factors $\Delta_{g_{\mathbb{R}}}(\pi_1, \pi)$ and $\Delta_{z_{\mathbb{R}}g_{\mathbb{R}}}(\pi_1, \pi)$ are compatible with geometric $\Delta_{g_{\mathbb{R}}}$ and $\Delta_{z_{\mathbb{R}}g_{\mathbb{R}}}$ respectively, then

$$\Delta_{g_{\mathbb{R}}}(\pi_1, \pi) = \Delta_{z_{\mathbb{R}}g_{\mathbb{R}}}(\pi_1, \pi)$$

for all pairs (π_1, π) as in Section 4.1. Here if $\pi(\theta, \varpi)$ is used in the definition on the left, then $\varpi_\pi(z_\mathbb{R}).\pi(\theta, \varpi)$ is to be used on the right. Our conclusion is then that the spectral factors will be independent of the choice for $g_\mathbb{R}$.

5 Packets and parameters I

Next we review briefly Langlands parameters and Arthur parameters for real groups [La89, Ar89]. We make a construction in Section 5.2 that attaches a *c-Levi group* to a parameter. We will show in subsequent sections how this group provides useful additional information about the parameters we are concerned with. Twisting will be ignored until Part 9.

5.1 Langlands parameters, Arthur parameters

Consider a homomorphism of the form

$$\psi = (\varphi, \rho) : W_\mathbb{R} \times \mathrm{SL}(2, \mathbb{C}) \to {}^L G,$$

where $\varphi : W_\mathbb{R} \to {}^L G$ is an essentially bounded admissible homomorphism and ρ is a continuous homomorphism of $\mathrm{SL}(2, \mathbb{C})$ into $G^\vee$. The conditions on φ mean that $\varphi(w) = \varphi_0(w) \times w$, $w \in W_\mathbb{R}$, where φ_0 is a continuous 1-cocycle of $W_\mathbb{R}$ in $G^\vee$ and $\varphi_0(W_\mathbb{R})$ is a group of semisimple elements in $G^\vee$ that is bounded mod center in the sense that the image of $\varphi_0(W_\mathbb{R})$ in the adjoint form $G^\vee_{\mathrm{ad}}$ under the natural projection $G^\vee \to G^\vee_{\mathrm{ad}}$ is bounded.

An element g of $G^\vee$ acts on the set of such φ by conjugation: $\varphi \to \mathrm{Int}(g) \circ \varphi$. The $G^\vee$-orbits are the essentially bounded Langlands parameters for G^*; see [La89]. Similarly, $G^\vee$ acts on the set of such ψ and the orbits are the Arthur parameters for G^*; see [Ar89].

When we replace G^* by an inner twist (G, η) in our considerations, we will often limit our attention to Langlands parameters which are *relevant to* (G, η) in the usual sense that the image of a representative is contained only in parabolic subgroups of ${}^L G$ relevant to (G, η) [La89]. The essentially bounded Langlands parameters relevant to (G, η) parametrize the essentially tempered packets of irreducible admissible representations of $G(\mathbb{R})$ [La89].

Notation. Occasionally we distinguish between a homomorphism φ, ψ and its $G^\vee$-orbit $\boldsymbol{\varphi}, \boldsymbol{\psi}$ respectively, but much of the time we use the symbols φ or ψ for both.

Let $\psi = (\varphi, \rho)$ be a Arthur parameter and let $S = S_\psi$ denote the centralizer in $G^\vee$ of the image of ψ. Recall that Arthur calls ψ *elliptic* if the identity component of S is central in $G^\vee$ and that this is equivalent to requiring that the image of ψ be contained in no proper parabolic subgroup of ${}^L G$ [Ar89].

For calculations with the Weil group $W_{\mathbb{R}}$ we fix an element w_σ of $W_{\mathbb{R}}$ such that w_σ maps to σ under $W_{\mathbb{R}} \to \Gamma$ and $(w_\sigma)^2 = -1$.

5.2 c-Levi group attached to a parameter

Let $\psi = (\varphi, \rho)$ be an Arthur parameter. Set

$$M_\psi^\vee = \mathrm{Cent}(\varphi(\mathbb{C}^\times), G^\vee).$$

Then $M_\psi^\vee = M^\vee$ is a connected reductive subgroup of $G^\vee$. Because $\varphi(\mathbb{C}^\times)$ is a torus, $M^\vee$ is *Levi* in the sense that there is a parabolic subgroup of $G^\vee$ with $M^\vee$ as a Levi subgroup. Notice that $\varphi(W_{\mathbb{R}})$ normalizes $M^\vee$. We define $\mathcal{M}$ to be the subgroup of ${}^L G$ generated by $M^\vee$ and $\varphi(W_{\mathbb{R}})$. Then $\mathcal{M}$ is a split extension of $W_{\mathbb{R}}$ by $M^\vee$. Notice that $\mathcal{M}$ contains S_ψ.

While $\mathcal{M}$ is typically not endoscopic, i.e., it is not the group $\mathcal{H}$ in some set $(H, \mathcal{H}, s)$ of standard endoscopic data for G, we may extract an L-action on $M^\vee$ in the same way as for the endoscopic case. For this, recall the fixed splitting $\mathrm{spl}^\vee = (\mathcal{B}, \mathcal{T}, \{X_{\alpha^\vee}\})$ for $G^\vee$. There is no harm in assuming that $\varphi_0(\mathbb{C}^\times)$ lies in $\mathcal{T}$ and that $\varphi_0(w_\sigma)$ normalizes $\mathcal{T}$ [La89]. Then $\mathcal{T} \subset M^\vee$ and a simple root $\alpha^\vee$ for $M^\vee \cap \mathcal{B}$ is also simple for $\mathcal{B}$. We then use the same root vector $X_{\alpha^\vee}$. Write $\mathrm{spl}_M^\vee$ for this splitting $(M^\vee \cap \mathcal{B}, \mathcal{T}, \{X_{\alpha^\vee}\})$ for $M^\vee$. To define an L-action on $M^\vee$ we need only to specify an automorphism σ_M of $M^\vee$ that preserves $\mathrm{spl}_M^\vee$ and has order at most two. Since $\mathrm{Int}\,\varphi(w_\sigma)$ preserves $M^\vee$ and has order at most two as automorphism of $\mathcal{T}$, it is clear that there is a unique such σ_M of the form $\mathrm{Int}[m_\sigma \cdot \varphi(w_\sigma)]$, with $m_\sigma \in M^\vee$. For the L-action itself, $W_{\mathbb{R}}$ acts through $W_{\mathbb{R}} \to \Gamma$; in particular, w_σ acts as σ_M and $\mathbb{C}^\times$ acts trivially.

Write ${}^L M_\psi = {}^L M$ for the corresponding L-group $M_\psi^\vee \rtimes W_{\mathbb{R}}$ and M_ψ for a group defined and quasisplit over $\mathbb{R}$ that is dual to ${}^L M_\psi$. In Section 5.4 we will describe explicitly the L-isomophisms $\xi_M : {}^L M_\psi \to \mathcal{M}$ in the critical case, where we have the property • of the next section. In Section 6.2 we will define an embedding over $\mathbb{R}$ of the quasisplit group M_ψ in the quasisplit form G^* in that case (and the general case follows quickly). For now, however, the following observation will be sufficient: in the same sense and by the same arguments as for an endoscopic group (see [KS99, Lemma 3.3.B]), the group M_ψ shares various maximal tori over $\mathbb{R}$ with an inner form G of G^* and all maximal tori over $\mathbb{R}$ in M_ψ are shared with G^*.

The groups $\mathcal{M}$ and M_ψ attached to Arthur parameter $\psi = (\varphi, \rho)$ depend only on φ. We may make the same definitions for any Langlands parameter φ and then we use the notation M_φ. We call the group M_φ a *c-Levi group* of G^*. We will define c-Levi groups in an inner form via inner twists; see Section 6.2. In Section 7.4 we will see M_ψ in a more familiar setting, namely as a Levi subgroup defined over $\mathbb{R}$ of a parabolic subgroup preserved by a Cartan involution.

The group M_φ also appears indirectly in the dual version of the Knapp–Zuckerman decomposition of unitary principal series (see [Sh82, Sections 4, 5]), as we will recall briefly in Part 6. For certain Arthur parameters ψ, a family of inner forms of M_ψ is introduced in [Ko90, Section 9].

5.3 A property for Arthur parameters

Let $\psi = (\varphi, \rho)$ be an Arthur parameter. As above, we choose a representative ψ such that $\varphi_0(\mathbb{C}^\times)$ lies in $\mathcal{T}$ and $\varphi_0(w_\sigma)$ normalizes $\mathcal{T}$. Consider the property:

- there is an element of $\mathcal{M} \cap (G^\vee \times w_\sigma)$ that normalizes $\mathcal{T}$ and acts as -1 on all roots of $G^\vee$.

Notice that • is true if and only if both G^*, M_ψ are cuspidal and share an elliptic maximal torus T, i.e., a maximal torus that is anisotropic modulo the center of G^*. Here we may replace G^* by an inner form (G, η) if we wish; we then write T_G in place of T and assume harmlessly that η maps T_G to T over $\mathbb{R}$.

5.4 L-isomorphisms for attached c-Levi group

We describe next the L-isomorphisms $\xi_M : {}^L M \to \mathcal{M}$ for the case that • is true. There will be no harm in working with standard χ-data and we do so as it returns us to a familiar setting. In particular, Lemma 5.1 below is well known; much of it is stated in [AJ87], [Ar89] without details of proof. We give a quick proof based on some explicit calculations we will need. These calculations also pinpoint dependence on the critical Lemma 3.2 in [La89].

The element described in • may be written as $n \times w_\sigma$, where $n \in G^\vee$ normalizes $\mathcal{T}$ and represents the longest element of the Weyl group of $\mathcal{T}$ in $G^\vee$. Since $n \times w_\sigma \in \mathcal{M}$, our construction of ${}^L M = M^\vee \rtimes W_\mathbb{R}$ yields $n_M \times w_\sigma$ in the group ${}^L M$ normalizing $\mathcal{T}$ and acting as -1 on the roots of $M^\vee$. Then $n_M \in M^\vee$ normalizes $\mathcal{T}$ and represents the longest element of the Weyl group of $\mathcal{T}$ in $M^\vee$. Because $M^\vee$ is Levi, we may multiply n by an element of $\mathcal{T} \cap (M^\vee)_{\mathrm{der}} \subseteq \mathcal{T} \cap (G^\vee)_{\mathrm{der}}$ to obtain n' such that the action of $n' \times w_\sigma \in {}^L G$ on the entire group $M^\vee$ coincides with that of $n_M \times w_\sigma \in {}^L M$. Notice that

$$n'\sigma(n') = n\sigma(n), \tag{5.1}$$

where σ denotes the action of $1 \times w_\sigma \in {}^L G$ on $G^\vee$, and that the action of $1 \times w_\sigma \in {}^L M$ on $M^\vee$ is given by conjugation by

$$(n_M)^{-1} n' \times w_\sigma \in \mathcal{M}. \tag{5.2}$$

Returning to the construction of an L-isomorphism $\xi_M : {}^L M \to \mathcal{M}$, we require ξ_M to act as the identity on $M^\vee$. It remains to define ξ_M on $W_\mathbb{R}$. There is no harm in assuming that the element n above, and thus also n', belongs to $G^\vee_{\text{der}}$, and that n_M belongs to $M^\vee_{\text{der}}$. Then set

$$\xi_M(w_\sigma) = (n_M)^{-1} n' \times w_\sigma.$$

Let ι denote one-half the sum of the coroots for $\mathcal{T}$ in $\mathcal{B}$, and let ι_M be the corresponding term for the coroots in $M^\vee \cap \mathcal{B}$. Notice that because $M^\vee$ is Levi, we have

$$\langle \iota - \iota_M, \alpha^\vee \rangle = 0 \tag{5.3}$$

for all roots $\alpha^\vee$ of $\mathcal{T}$ in $M^\vee$. This, together with $\bullet$, implies that σ_M acts on $\iota - \iota_M$ as -1. For $z \in \mathbb{C}^\times$, define the element $\xi_M(z)$ of $\mathcal{T} \times z$ by

$$\xi_M(z) = (z/\overline{z})^{\iota - \iota_M} \times z.$$

Lemma 5.1.

(i) The map ξ_M extends to a well-defined homomorphism $\xi_M : W_\mathbb{R} \to \mathcal{M}$ and thence to an L-isomophism $\xi_M : {}^L M \to \mathcal{M}$.
(ii) An L-isomophism $\xi'_M : {}^L M \to \mathcal{M}$ extending the identity on $M^\vee$ is of the form $\xi'_M = a \otimes \xi_M$, where a is a 1-cocycle of $W_\mathbb{R}$ in the center $Z_{M^\vee}$ of $M^\vee$.

Proof. Following (5.1) and (5.2) in our construction above, we see that

$$[(n_M)^{-1} n' \times w_\sigma]^2$$

may be rewritten as

$$(n_M \sigma_M(n_M))^{-1}.n\sigma(n) \times (-1).$$

By [La89, Lemma 3.2], this is

$$(-1)^{-2\iota_M}(-1)^{2\iota} \times (-1),$$

and (i) then follows. Also, (ii) is immediate. □

5.5 u-regular Arthur parameters

We continue with an Arthur parameter $\psi = (\varphi, \rho)$. Notice that the image of ρ lies in $M^\vee$. From now on we will limit our attention to *u-regular* Arthur parameters. By this we mean a parameter ψ for which the image of ρ contains a regular unipotent element of $M^\vee$. Then ρ maps regular unipotent elements of $\mathrm{SL}(2, \mathbb{C})$ to regular unipotent elements of $M^\vee$. We include the case that $M^\vee$ is abelian. Then ρ is trivial so that $\psi = (\varphi, \text{triv})$, where φ is a Langlands parameter that is regular in the sense of [Sh10, Section 2]. Representations in the attached essentially tempered packet have regular infinitesimal character. On the other hand, if $M^\vee$ is nonabelian, then there are *u-regular* Arthur parameters where representations in the attached Arthur packet have singular infinitesimal character; see Lemma 7.4.

Observe that for a u-regular Arthur parameter ψ, the centralizer S_ψ of the image of ψ in $G^\vee$ consists exactly of the σ_M-invariants in $Z_{M^\vee}$.

Lemma 5.2. *A u-regular Arthur parameter ψ is elliptic if and only if • is true.*

Proof. There is no harm in assuming G is simply-connected and semisimple, so that $Z_{G^\vee}$ is trivial. Then a nontrivial torus in the σ_M-invariants of $Z_{M^\vee}$ determines a nontrivial $\mathbb{R}$-split torus in a fundamental maximal torus of M, and conversely. □

In the next lemma we assume ψ is elliptic since we have yet to describe ξ_M in general (see [ShII]). By construction, we have $\varphi(W_\mathbb{R})$ contained in $\mathcal{M}$, and so we may factor φ through $\xi_M : {}^L M \to \mathcal{M}$. Define the Langlands parameter φ_M by $\varphi = \xi_M \circ \varphi_M$. Set ${}^L Z_M = Z_{M^\vee} \rtimes W_\mathbb{R} \subseteq {}^L M$.

Lemma 5.3. *The Langlands parameter φ_M factors through ${}^L Z_M$.*

Proof. By (5.3), $\varphi_M(\mathbb{C}^\times)$ lies in ${}^L Z_M$. Because $1\times w_\sigma \in {}^L M$ preserves the splitting $\mathrm{spl}_M^\vee$ of $M^\vee$ we may adjust ψ by an element of $M^\vee$ to arrange also that $\rho(\mathrm{SL}(2,\mathbb{C}))$ contains a regular unipotent element of $M^\vee$ that is fixed by $1\times w_\sigma$. Writing $\varphi_M(w_\sigma)$ as $m(w_\sigma)\times w_\sigma$, we have then that the semisimple element $m(w_\sigma)$ commutes with a regular unipotent element and hence is central in $M^\vee$. □

Remark 5.4. By (ii) of Lemma 5.1 we may replace ξ_M by φ itself. Then φ factors through the trivial parameter $w \to 1 \times w$; this factoring is used for the parameters in [Ko90, Section 9].

5.6 Langlands parameters for discrete series

Discrete series parameters are defined in [La89, Section 3]. They are precisely the Langlands parameters $\boldsymbol{\varphi}$ that are elliptic as Arthur parameters, i.e., such that $\psi = (\varphi, \mathrm{triv})$ is elliptic, and then M_φ is just the elliptic torus T of Section 5.3.

We recall that there is a representative φ for $\boldsymbol{\varphi}$ such that

$$\varphi_0(z) = z^\mu \bar{z}^{\sigma_T \mu},\ z \in \mathbb{C}^\times, \tag{5.4}$$

and

$$\varphi_0(w_\sigma) = e^{2\pi i \lambda} n. \tag{5.5}$$

Here n is the element of the derived group of $G^\vee$ constructed from $\mathrm{spl}^\vee$ to represent the longest element of the Weyl group of $\mathcal{T}$ (see [LS87, Section 2.6]). Then $\varphi(w_\sigma)$ acts on $\mathcal{T}$ as σ_T. Also $\mu, \lambda \in X_*(\mathcal{T}) \otimes \mathbb{C}$ and

$$\frac{1}{2}(\mu - \sigma_T \mu) - \iota \equiv \lambda + \sigma_T \lambda \mod X_*(\mathcal{T}), \tag{5.6}$$

where ι is one-half the sum of the coroots for $\mathcal{T}$ in the Borel subgroup $\mathcal{B}$ provided by spl$^\vee$. Notice that the congruence implies that

$$\langle \mu, \alpha^\vee \rangle \in \mathbb{Z}$$

for all roots $\alpha^\vee$ of $\mathcal{T}$ in $G^\vee$. We require μ to be strictly dominant for spl$^\vee$. Thus μ is determined uniquely by $\boldsymbol{\varphi}$, while λ is determined uniquely modulo $\mathcal{K}$, where

$$\mathcal{K} = X_*(\mathcal{T}) + \{\nu \in X_*(\mathcal{T}) \otimes \mathbb{C} : \sigma_T \nu = -\nu\}.$$

Finally, the representative $\varphi(\mu, \lambda)$ for $\boldsymbol{\varphi}$ is determined uniquely up to the action of $\mathcal{T}$.

5.7 Langlands parameters for limits of discrete series

As in [Sh82], we may use (5.4) and (5.5) above to construct a Langlands parameter $\boldsymbol{\varphi}$ with representative $\varphi = \varphi(\mu, \lambda)$, where (μ, λ) satisfy (5.6) but the strict dominance condition on μ is relaxed to dominance.

Notice that if $\mu_0 \in X_*(\mathcal{T})$ is dominant, then we obtain another such parameter $\varphi(\mu + \mu_0, \lambda + \frac{1}{2}\mu_0)$. This will provide a translation by the rational character μ_0 in the character data of Parts 6 and 7.

Also, because $\langle \mu + \sigma_T \mu, \alpha^\vee \rangle = 0$ for all roots $\alpha^\vee$ of $\mathcal{T}$ in $G^\vee$, the image of $\varphi_0 = \varphi_0(\mu, \lambda)$ is bounded mod center. Thus $\boldsymbol{\varphi}$ is essentially bounded.

Remark 5.5. We will now use the term **s-elliptic** for any Langlands parameter $\boldsymbol{\varphi}$ with representative of the form $\varphi = \varphi(\mu, \lambda)$. By construction, $\bullet$ is true, so that the attached c-Levi group M_φ is cuspidal and shares an elliptic maximal torus T with G.

Remark 5.6. Langlands' definition of the packet attached to relevant $\boldsymbol{\varphi}$ involves components of principal series representations [La89]. Theorem 4.3.2 of [Sh82] shows that $\varphi(\mu, \lambda)$ may be used directly to identify these components as limits of discrete series characters, nondegenerate or not.

6 Packets of limits of discrete series

We pause for a more detailed analysis of the packet of tempered irreducible representations attached to an s-elliptic parameter $\boldsymbol{\varphi}$ with representative $\varphi = \varphi(\mu, \lambda)$. Namely, we expand on Remark 5.6 using the attached c-Levi group M_φ and the strong generic base-point property from [Sh08b, Section 11] which is based on Vogan's classification of generic representations.

6.1 Character data, Whittaker data

Let $\mathcal{C}$ denote the closed Weyl chamber in $X_*(\mathcal{T}) \otimes \mathbb{C}$ dominant for $\text{spl}^\vee$. Recall that $\mu \in \mathcal{C}$. We choose an inner automorphism of G^* carrying $\text{spl}^* = (\text{spl}^\vee)^\vee$ to a fundamental splitting $\text{spl}_{\text{Wh}} = (B, T, \{X_\alpha\})$ for G^* of Whittaker type. We thus have a transport to T of $(\mu, \lambda, \mathcal{C})$. Then $(\mu, \lambda, \mathcal{C})$ becomes character data for a generic discrete series or limit of discrete series representation π^* of $G^*(\mathbb{R})$.

To fix Whittaker data for G^*, by which we mean a $G^*(\mathbb{R})$-conjugacy class of the pairs (B, λ) of [KS99, Section 5.3], we choose an additive character $\psi_\mathbb{R}$ for $\mathbb{R}$ and use the conjugacy class of the pair determined by $\psi_\mathbb{R}$ and spl^*. We may adjust the fundamental splitting spl_{Wh} to arrange that π^* is generic for the chosen Whittaker data. We then say that the splitting is *aligned* with the data. This determines spl_{Wh} uniquely up to $G^*(\mathbb{R})$-conjugacy.

Let (G, η) be an inner twist and let spl_f be a fundamental splitting for G. We use a twist η' in the inner class of η to transport spl_f to spl_{Wh}. This provides us with a further transport of $(\mu, \lambda, \mathcal{C})$ to the maximal torus specified by spl_f. The transported triple serves as character data $(\mu_\pi, \lambda_\pi, \mathcal{C}_\pi)$ for a discrete series or limit of discrete series representation π of $G(\mathbb{R})$ which is determined uniquely by the $G(\mathbb{R})$-conjugacy class of spl_f. In the case of limits of discrete series *we must now allow* $\pi = 0$, i.e., that the distribution attached to the character data is zero. We write $\text{spl}_f = \text{spl}_\pi = (B_\pi, T_\pi, \{X_\alpha\})$, $\eta' = \eta_\pi$ and the character data for π as $(\mu_\pi, \lambda_\pi, \mathcal{C}_\pi)$.

By Lemma 2.1 and the theorem cited in Remark 5.6, as spl_f varies we generate the packet of essentially tempered representations attached to $\varphi = \varphi(\mu, \lambda)$ and possibly some zeros. By the same theorem we obtain all zeros if and only if φ is irrelevant to G.

6.2 Characterizing nonzero limits

Our first concern will be to detect when $\pi = 0$. There is a well-known characterization, in terms of roots, for a limit of discrete series to be nonzero; see [KZ82, Theorem 1.1 (b)]. The precise statement is noted in the next proof. We want a characterization in terms of the c-Levi group M_φ.

Consider the subgroup M^* generated by T and the root vectors X_α from spl_{Wh} for which α is the transport to T of the coroot of a simple root of $\mathcal{T}$ in $M^\vee \cap \mathcal{B}$. Because spl_{Wh} is of Whittaker type, i.e., the simple roots are all noncompact, the group M^* is quasisplit over $\mathbb{R}$. Moreover, we may identify the L-group of M^* with the L-group ${}^L M_\varphi$ constructed in Section 5.2. For this, we reverse the construction of Section 6.1 to determine an $\mathbb{R}$-splitting spl^*_M for M^* from the fundamental splitting $\text{spl}_{Wh,M}$ attached to spl_{Wh} and the additive character $\psi_\mathbb{R}$. Then spl^*_M is unique up to $M^*(\mathbb{R})$-conjugacy. Each such spl^*_M determines a unique isomorphism from ${}^L(M^*)$ to ${}^L M_\varphi$. In summary, M^* provides a concrete realization of the quasisplit group M_φ.

By definition, η_π carries spl_π to $\mathrm{spl}_{\mathrm{Wh}}$. Let $M_\pi = \eta_\pi^{-1}(M^*)$. Then $\eta_\pi : M_\pi \to M$ is an inner twist in the inner class of η that carries T_π to T^* over $\mathbb{R}$. We say that M_π is an *elliptic c-Levi group for* (G, η).

Lemma 6.1. π *is nonzero if and only if the inner twist* $\eta_\pi : M_\pi \to M^*$ *is an* $\mathbb{R}$*-isomorphism.*

Proof. The characterization cited (in sufficient generality for our setting) is that $\pi \neq 0$ if and only if all $\mathcal{C}_\pi$-simple roots α such that $\langle \mu_\pi, \alpha^\vee \rangle = 0$ are noncompact. In other words, $\pi \neq 0$ if and only if the splitting of M_π determined by spl_π is of Whittaker type. Let $\eta_\pi \sigma(\eta_\pi)^{-1} = \mathrm{Int}(u_\pi(\sigma))$, where $u_\pi(\sigma) \in T_{\mathrm{sc}}$. Then because $\mathrm{spl}_{\mathrm{Wh}}$ is of Whittaker type and spl_π is fundamental, a calculation with root vectors shows that this is the same as requiring that $\alpha(u_\pi(\sigma)) = 1$ for all B^*-simple roots α of T in M^*, i.e., that $u_\pi(\sigma)$ lies in the center of $M^*_{(sc)}$. Here $M^*_{(sc)}$ denotes the inverse image of M^* in G^*_{sc} under the natural projection $G^*_{\mathrm{sc}} \to G^*$. The lemma then follows. □

6.3 Generating packets

Fix an inner form (G, η) and assume s-elliptic $\varphi = \varphi(\mu, \lambda)$ is relevant to (G, η). We consider the packet Π of representations of $G(\mathbb{R})$ attached to φ.

Let $\pi \in \Pi$ and assume $\pi \neq 0$. Then:

Lemma 6.2.

(i) $\mathrm{spl}_{\pi'}$ *yields character data for nonzero* π' *in* Π *if and only if*

$$\eta_{\pi'} = \mathrm{Int}(g_*) \circ \eta_\pi \circ \mathrm{Int}(g),$$

where $g \in G(\mathbb{R})$ *and where* $g_* \in G^*_{\mathrm{sc}}$ *normalizes* T *and is such that the restriction of* $\mathrm{Int}(g_*)$ *to* M^* *is defined over* $\mathbb{R}$*.*

(ii) Further, $\pi' = \pi$ *if and only if* $\eta_{\pi'}$ *is of the form* $\eta_\pi \circ \mathrm{Int}(g)$*, where* $g \in G(\mathbb{R})$*.*

Proof. The second assertion is just a restatement of a well-known property of limits of discrete series; see [Sh82, Section 4] or [KZ82, Theorem 1.1(c)]. The first assertion follows from Lemma 6.1. □

6.4 An elliptic invariant

We return now to using the notation π *only for nonzero* representations. Again fix an inner form (G, η) for which s-elliptic $\varphi = \varphi(\mu, \lambda)$ is relevant and consider π in the attached packet Π of representations of $G(\mathbb{R})$. Recall $u_\eta(\sigma), u_\pi(\sigma) \in G^*_{\mathrm{sc}}$; we have $\eta\sigma(\eta)^{-1} = \mathrm{Int}(u_\eta(\sigma))$ and $\eta_\pi\sigma(\eta_\pi)^{-1} = \mathrm{Int}(u_\pi(\sigma))$. Since η_π is in the inner class of η, we write $\eta_\pi = \mathrm{Int}(x_\pi) \circ \eta$ and

$$u_\pi(\sigma) = x_\pi[u_\eta(\sigma)]\sigma(x_\pi)^{-1},$$

where $x_\pi \in G^*_{\mathrm{sc}}$.

If we may choose $u_\eta(\sigma)$, and thence $u_\pi(\sigma)$, to be a cocycle, then we will say that (G,η) is of quasisplit type because (G,η) then occurs as a component of an extended group of quasisplit type. An extended group (introduced by Kottwitz) consists of several pairs (G,η), (G',η'), ... and conditions on the twists η, η' ensure the property (6.3) below; see [Sh08b] for a review and examples. There is a quasisplit component if and only the coboundaries in (6.3) are trivial. Then we say the extended group is of quasisplit type. The quasisplit component, if it exists, is unique [Sh08b].

For pairs (G,η), (G',η') in the same extended group, relative factors for tempered spectral transfer are defined in [Sh08b] (the relative geometric factors were introduced by Kottwitz). When the extended group is of quasisplit type, our already chosen Whittaker data provides a unique normalization Δ_{Wh} of the absolute transfer factors for each component (G,η); see [KS99]. The spectral factors Δ_{Wh} possess the strong base-point property [Sh08b]. In particular, we have the formula (6.4) for discrete series representations. For a general extended group, the results of Kaletha [Ka13] provide a natural normalization for the absolute factors. The setting, and in particular the definition of extended group, is modified with additional structure. For our purposes it is convenient to work with the minimal extended groups of the present setting, and we will allow any normalization of the absolute factors that possesses geometric-spectral compatibility in the sense of [Sh10, Sh08b]. The extended groups will play a more central role when we come to finer structure on packets in [ShII].

We begin our definition of elliptic invariants with the case that (G,η) is of quasisplit type. In this setting we define an absolute invariant $\mathrm{inv}(\pi)$ in $H^1(\Gamma,T)$. Recall that T is the elliptic maximal torus in G^* specified by $\mathrm{spl}_{\mathrm{Wh}}$. First we have by Lemma 6.2 that $u_\pi(\sigma)$ lies in the center $Z_{M^*_{(sc)}}$ of $M^*_{(sc)}$ and so defines an element of $H^1(\Gamma, Z_{M^*_{(sc)}})$. It depends only on π, i.e., only on the $G(\mathbb{R})$-conjugacy class of spl_π. Now $\mathrm{inv}(\pi)$ is defined to be the image of this class under

$$H^1(\Gamma, Z_{M^*_{(sc)}}) \to H^1(\Gamma, Z_{M^*}) \to H^1(\Gamma, T) \tag{6.1}$$

given by the composition of the obvious map $Z_{M^*_{(sc)}} \to Z_{M^*}$ and inclusion $Z_{M^*} \to T$. From the diagram

$$\begin{array}{ccc} Z_{M^*_{(sc)}} & \longrightarrow & T_{\mathrm{sc}} \\ \downarrow & & \downarrow \\ Z_{M^*} & \longrightarrow & T \end{array} \tag{6.2}$$

we conclude that $\mathrm{inv}(\pi)$ lies in the image $\mathcal{E}(T)$ of $H^1(\Gamma, T_{\mathrm{sc}})$ in $H^1(\Gamma, T)$.

We will also make use of the following.

Lemma 6.3. *Suppose that T is a fundamental maximal torus in a connected reductive group G over $\mathbb{R}$. Then $H^1(\Gamma, Z_G) \to H^1(\Gamma, T)$ is injective.*

Proof. Because T is fundamental, both $T_{\mathrm{sc}}(\mathbb{R})$, $T_{\mathrm{ad}}(\mathbb{R})$ are connected and hence $T_{\mathrm{sc}}(\mathbb{R}) \to T_{\mathrm{ad}}(\mathbb{R})$ is surjective; see the proof of Lemma 2.2 for references. A calculation then shows that the kernel of $H^1(\Gamma, Z_G) \to H^1(\Gamma, T)$ is trivial. □

In general, we define a relative invariant $\mathrm{inv}(\pi, \pi')$ when (G, η), (G', η') are components of the same extended group and π, π' belong to packets Π, Π' for $G(\mathbb{R}), G'(\mathbb{R})$ attached to relevant s-elliptic parameters $\varphi = \varphi(\mu, \lambda), \varphi' = \varphi(\mu', \lambda')$ respectively. We follow the method introduced in [LS87, LS90]; see also [KS99, Section 4.4]. First, recall that

$$\partial u_\pi = \partial u_\eta = \partial u_{\eta'} = \partial u_{\pi'} \tag{6.3}$$

takes values in $Z_{G^*_{\mathrm{sc}}}$ as subgroup of T, the elliptic maximal torus in G^* specified by $\mathrm{spl}_{\mathrm{Wh}}$. As in the references, set

$$U_{\mathrm{sc}} = U(T_{\mathrm{sc}}, T_{\mathrm{sc}}) = T_{\mathrm{sc}} \times T_{\mathrm{sc}} \big/ \{(z^{-1}, z) : z \in Z_{G^*_{\mathrm{sc}}}\}$$

and

$$U = U(T, T) = T \times T \big/ \{(z^{-1}, z) : z \in Z_{G^*}\}.$$

Also consider

$$U(Z_{M^*_{(sc)}}) = Z_{M^*_{(sc)}} \times Z_{M^*_{(sc)}} \big/ \{(z^{-1}, z) : z \in Z_{G^*_{\mathrm{sc}}}\}$$

and

$$U(Z_{M^*}) = Z_{M^*} \times Z_{M^*} \big/ \{(z^{-1}, z) : z \in Z_{G^*}\}.$$

Then we replace (6.1) above by

$$H^1(\Gamma, U(Z_{M^*_{(sc)}})) \to H^1(\Gamma, U(Z_{M^*})) \to H^1(\Gamma, U),$$

and use the cocycle that is the image in $U(Z_{M^*_{(sc)}})$ of the pair $(u_\pi(\sigma)^{-1}, u_{\pi'}(\sigma))$ in $Z_{M^*_{(sc)}} \times Z_{M^*_{(sc)}}$. Then we obtain $\mathrm{inv}(\pi, \pi')$ in the image of $H^1(\Gamma, U_{\mathrm{sc}})$ in $H^1(\Gamma, U)$. If the extended group is of quasisplit type, then $\mathrm{inv}(\pi, \pi')$ is the image of $(\mathrm{inv}(\pi)^{-1}, \mathrm{inv}(\pi'))$ under the evident homomorphism $H^1(\Gamma, T) \times H^1(\Gamma, T) \to H^1(\Gamma, U)$.

6.5 Application to endoscopic transfer

We consider standard endoscopic transfer in the cuspidal elliptic setting. If φ_1 is an s-elliptic Langlands parameter for H_1, then its transport φ to G is also s-elliptic; see [Sh08a, Section 11] for how to transport attached character data from the z-extension H_1. If φ_1 is elliptic, then φ may, of course, fail to be elliptic, but φ is at least s-elliptic and moreover the associated triples of nonzero character data are nondegenerate. It is then straightforward to define spectral transfer factors via the Zuckerman translation principle; this is recalled in Section 14 of [Sh10].

For a general s-elliptic related pair (φ_1, φ), however, neither side of the spectral transfer statement has support on the regular elliptic set. Then we have defined the associated transfer factors via an L-group version of the Knapp–Zuckerman (nondegenerate) decomposition of unitary principal series; see [Sh10, Sh08b]. In that form, the factors display desired structure on the packet; see [Sh08b, Section 11].

Our purpose now is to note a simpler description, based on the elliptic invariant of Section 6.4, of the transfer factors for a general s-elliptic related pair. Whittaker data for G^* has been fixed. First, we transport a Γ_T-invariant s_T in the complex dual $T^\vee$ of T to an element s of the maximal torus $\mathcal{T}$ in $G^\vee$ via the method of Section 6.1. To the pair (s, φ) we attach the endoscopic data $\mathfrak{e}(s)$ of Section 7 of [Sh08b], now writing $\mathfrak{e}_z(s)$ since it is already supplemented, as well as the related pair of parameters (φ^s, φ). The attached endoscopic group will be denoted $H^{(s)}$. When φ is singular we have used a different representative, say φ', to display the structure on the packet via Knapp–Zuckerman theory. The conjugacy of φ and φ' under $G^\vee$ determines a canonical isomorphism of the attached abelian groups $\mathbb{S}_\varphi$ and $\mathbb{S}_{\varphi'}$ (see Section 6.6 or 6.7). To examine the effect of this isomorphism on transfer, see [Sh08b, Section 2] for passage to isomorphic endoscopic data and [Sh08b, Section 11] for related results. For present needs, the results of Sections 6.6–6.8 will be sufficient.

Recall from Section 6.4 that our Whittaker data also determine absolute transfer factors Δ_{Wh} for any inner form (G, η) of quasisplit type. We use π^s to denote a representation in the packet for $H^{(s)}(\mathbb{R})$ attached to φ^s; the choice within the packet will not matter. Finally, $\langle -, - \rangle_{\mathrm{tn}}$ will be the Tate–Nakayama pairing between $H^1(\Gamma, T)$ and $\pi_0((T^\vee)^\Gamma)$, and the image of s_T in $\pi_0((T^\vee)^\Gamma)$ will again be written s_T.

Lemma 6.4. *Suppose* (G, η) *is of quasisplit type and* $\varphi(\mu, \lambda)$ *is an s-elliptic parameter relevant to* (G, η). *Then*

$$\Delta_{\mathrm{Wh}}(\pi_s, \pi) = \langle \mathrm{inv}(\pi), s_T \rangle_{\mathrm{tn}} \tag{6.4}$$

for each limit of discrete series representation π *of* $G(\mathbb{R})$ *attached to* $\varphi(\mu, \lambda)$.

Proof. Although not necessary, we reduce easily to the case that the derived group of G is simply-connected as this allows us to refer directly to the first half of the argument for the proof of Theorem 11.5 in [Sh08b]. There a totally degenerate parameter as in Section 6.6 was needed; now we apply the coherent continuation argument to any relevant s-elliptic φ, so obtaining the transfer identity in the middle of p. 400. The formula (6.4) then follows from its truth in the case φ is elliptic. □

Returning to the notation of Section 6.4, recall from [LS90] that we identify $(U_{\mathrm{sc}})^\vee$ with $\mathcal{T}_{\mathrm{sc}} \times \mathcal{T}_{\mathrm{sc}} / \{(z, z) : z \in Z_{G^\vee_{\mathrm{sc}}}\}$ and define s_U as there. Now $\langle \_, \_ \rangle_{\mathrm{tn}}$ will denote the Tate–Nakayama pairing for U. The following requires a minor variant of the last proof but it will be convenient to have a separate statement.

Lemma 6.5. *Suppose* $(G, \eta), (G', \eta')$ *are components of an extended group and that* Δ *is an absolute transfer factor for the extended group. Then*

$$\Delta(\pi_s, \pi) \big/ \Delta(\pi_s, \pi') = \left\langle \mathrm{inv}(\pi, \pi'), s_U \right\rangle_{\mathrm{tn}}$$

for all limits of discrete series representations π, π' *of* $G(\mathbb{R}), G'(\mathbb{R})$.

Remark 6.6. We use transfer factors Δ for the classic version of the Langlands correspondence for real groups. See [Sh14] for (simple) transition to the alternate factors Δ_D.

6.6 Example: totally degenerate parameters

First, the notion of *totally degenerate* character data of Carayol and Knapp [CK07] extends to reductive groups, and since our data are generated by a Langlands parameter we consider the parameter instead. We call an s-elliptic parameter $\varphi = \varphi(\mu, \lambda)$ *totally degenerate* if $\langle \mu, \alpha^\vee \rangle = 0$ for all roots $\alpha^\vee$ of $\mathcal{T}$ in $G^\vee$; see [Sh08b, Section 12].

This definition implies that a totally degenerate parameter is relevant to (G, η), i.e., there is a packet for $G(\mathbb{R})$ attached to the parameter, if and only if G is quasisplit. Thus we may as well assume that $G = G^*$ and $\eta = \mathrm{id}$.

Further, an examination of the congruences for μ, λ shows that totally degenerate parameters exist only for certain cuspidal quasisplit groups. For example, if G_{der} is simply-connected, then such (μ, λ) do exist: they are the data for an extension of the rational character ι on T_{der}, regarded as character on $T_{\mathrm{der}}(\mathbb{R})$, to a continuous quasicharacter on $T(\mathbb{R})$; see [Sh08b]. Then an elliptic endoscopic group for G also has totally degenerate parameters [Sh08b]. So also does each cuspidal standard or c-Levi group X for G because X_{der} is also simply-connected. A z-extension G_z of any cuspidal quasisplit group G has totally degenerate characters for the same reason.

Suppose now that $\varphi = \varphi(\mu, \lambda)$ represents a totally degenerate parameter for $G = G^*$. The congruences for μ, λ further show that the parameter $\boldsymbol{\varphi}$ is uniquely determined by G up to multiplication by element of $H^1(W_{\mathbb{R}}, Z_{G^\vee})$, and hence that the attached packet is uniquely determined up to twisting by a quasicharacter on $G(\mathbb{R})$.

To describe the packet Π attached to totally degenerate φ in terms of the elliptic character data provided by (μ, λ) and the Whittaker data, we may proceed as follows. Recall the fixed $\mathbb{R}$-splitting $\mathrm{spl}^* = (B^*, T^*, \{X_\alpha\})$ for G. There is another representative $\overline{\varphi}$ for $\boldsymbol{\varphi}$ attached to the maximally split maximal torus T^*. We obtain it by applying a sequence of dual Cartan transforms to $\varphi = \varphi(\mu, \lambda)$; the sequence is prescribed by a suitable set of strongly orthogonal roots and the transforms are defined as in the proof of Lemma 4.3.5 in [Sh82]. Write $\overline{\varphi} = \varphi(\mu, \overline{\lambda})$ relative to T^*. These data determine an essentially unitary minimal principal series representation for $G(\mathbb{R})$. By definition of the Langlands correspondence, Π consists of the components of this representation. By Vogan's classification of generic representations [Vo78], these components include generic π^* with attached fundamental splitting $\mathrm{spl}_{\mathrm{Wh}}$. Then Lemma 6.2 shows that we obtain the other components by applying $\mathrm{Int}(g_*)$ to $\mathrm{spl}_{\mathrm{Wh}}$, where $g_* \in G_{\mathrm{sc}}$ and the automorphism $\mathrm{Int}(g_*)$ of G is defined over $\mathbb{R}$. Each such element g_* determines an element of $H^1(\Gamma, Z_{\mathrm{sc}})$, where Z_{sc} denotes the center of G_{sc}. Conversely, each element of $H^1(\Gamma, Z_{\mathrm{sc}})$ has trivial image in $H^1(\Gamma, G_{\mathrm{sc}})$ [Sh08b, Lemma 12.3] and so determines g_* such that $\mathrm{Int}(g_*)$ is defined over $\mathbb{R}$. Finally, two elements of $H^1(\Gamma, Z_{\mathrm{sc}})$ determine the same component if and only if they differ by an element of $\mathrm{Ker}[H^1(\Gamma, Z_{\mathrm{sc}}) \to H^1(\Gamma, Z)]$, where Z denotes the center of G, so that we have bijections

$$\Pi \leftrightarrow G_{\mathrm{ad}}(\mathbb{R}) / \mathrm{Int}(G(\mathbb{R})) \leftrightarrow \mathrm{Image}[H^1(\Gamma, Z_{\mathrm{sc}}) \to H^1(\Gamma, Z)]. \tag{6.5}$$

If we map the image of π in $H^1(\Gamma, Z)$ to $H^1(\Gamma, T)$ under the injective $H^1(\Gamma, Z) \to H^1(\Gamma, T)$, then we recover the elliptic invariant $\mathrm{inv}(\pi)$ defined in Section 6.4.

The group $S_{\overline{\varphi}} = \mathrm{Cent}(\overline{\varphi}(W_{\mathbb{R}}), G^\vee)$ consists of the fixed points in $G^\vee$ for the action of $\sigma \in \Gamma$ by $\overline{\sigma} = \mathrm{Int}(\overline{\varphi}(w_\sigma))$. Thus

$$\mathbb{S}_{\overline{\varphi}} := S_{\overline{\varphi}} / [(Z_{G^\vee})^\Gamma . S^0_{\overline{\varphi}}] = (G^\vee)^{\overline{\Gamma}} / [(Z_{G^\vee})^\Gamma . ((G^\vee)^{\overline{\Gamma}})^0].$$

Notice that $\mathbb{S}_{\overline{\varphi}}$ is isomorphic to Langlands' R-group $R_{\overline{\varphi}}$ for $\boldsymbol{\varphi}$ in this setting; see [Sh82, Section 5.3]. Combining this with the pairing obtained via nondegenerate Knapp–Zuckerman theory (see [Sh08b, Sh10]), we have that Π determines a perfect pairing of

$$\mathrm{Image}[H^1(\Gamma, Z_{\mathrm{sc}}) \to H^1(\Gamma, Z)] \tag{6.6}$$

with

$$(G^\vee)^{\overline{\Gamma}} / (Z_{G^\vee})^\Gamma . ((G^\vee)^{\overline{\Gamma}})^0.$$

In particular, if G is semisimple and simply-connected, then our pairing for the unique totally degenerate packet for $G(\mathbb{R})$ exhibits a perfect pairing of $H^1(\Gamma, Z)$ with $\pi_0[(G^\vee)^{\overline{\Gamma}}]$.

6.7 General limits: factoring parameters

We return to general s-elliptic $\varphi = \varphi(\mu, \lambda) : W_{\mathbb{R}} \to {}^L G$. Since the image of φ lies in $\mathcal{M}$, we factor φ through ${}^L M$, and write $\varphi = \xi_M \circ \varphi_M$, where φ_M is the s-elliptic parameter $\varphi(\mu_M, \lambda_M)$ for M^*, with

$$\mu_M = \mu - (\iota - \iota_M), \ \lambda_M = \lambda.$$

Clearly φ_M is totally degenerate. In summary:

Lemma 6.7. *An s-elliptic parameter φ determines a well-defined totally degenerate parameter for the c-Levi group attached to φ.*

Turning to packets, we start with the quasisplit form G^* and generic π^* whose character data is the transport of $(\mu, \lambda, \mathcal{C})$ to T provided by $\mathrm{spl}_{\mathrm{Wh}}$. Our realization of M_φ as M^* in Section 6.2 determines a fundamental splitting $\mathrm{spl}_{Wh,M}$ and chamber $\mathcal{C}_M$ for T. We use the same notation for the inverse transport of this chamber to $M^\vee$. The transport by $\mathrm{spl}_{Wh,M}$ of dual data $(\mu_M, \lambda_M, \mathcal{C}_M)$ attached to φ_M determines a totally degenerate limit of discrete series representation π_M^* of $M^*(\mathbb{R})$. By construction, π_M^* is generic relative to the Whittaker data attached to $\psi_\mathbb{R}$ and the $\mathbb{R}$-splitting $\mathrm{spl}_M^* = (\overline{B}_M, \overline{T}_M, \{X_\alpha\})$ for M^* from Section 6.1.

Consider now general (G, η) for which $\varphi(\mu, \lambda)$ is relevant. Let Π be the attached packet and consider $\pi \in \Pi$. Recall that $\eta_\pi : M_\pi \to M^*$ is an $\mathbb{R}$-isomorphism. Define the representation π_M of $M_\pi(\mathbb{R})$ by transport: $\pi_M = \pi_M^* \circ \eta_\pi$. Then π_M lies in the totally degenerate packet Π_{M_π} of representations of $M_\pi(\mathbb{R})$ attached to φ_M.

We return to the elliptic invariants of Section 6.4 and consider the subgroup

$$\mathrm{Image}(H^1(\Gamma, Z_{M^*_{\mathrm{sc}}}) \to H^1(\Gamma, T))$$

of

$$\mathrm{Image}(H^1(\Gamma, Z_{M^*_{(sc)}}) \to H^1(\Gamma, T)).$$

From (6.5) and Lemma 6.3, we have an isomorphism of this subgroup with

$$M_{\mathrm{ad}}(\mathbb{R}) / \mathrm{Int}(M(\mathbb{R})).$$

On the other hand, notice that $S_\varphi = \mathrm{Cent}(\varphi(W_\mathbb{R}), G^\vee)$ is contained in $M^\vee$ and hence

$$S_\varphi = S_{\varphi_M} = \mathrm{Cent}(\varphi_M(W_\mathbb{R}), M^\vee)$$

which is the group of fixed points of $M^\vee$ under either of the automorphisms $\mathrm{Int}(\varphi(w_\sigma))$, $\mathrm{Int}(\varphi_M(w_\sigma))$; we arranged in Section 5.4 that these automorphisms act the same way on $M^\vee$. Again write $\mathbb{S}_\varphi$ for the quotient $S_\varphi / (Z_{G^\vee})^\Gamma S_\varphi^0$. Then

$$\mathbb{S}_{\varphi_M} = S_\varphi / (Z_{M^\vee})^\Gamma S_\varphi^0$$

and since $(Z_{M^\vee})^\Gamma \cap ((M^\vee)^\Gamma)^0$ is contained in $(Z_{G^\vee})^\Gamma$ we have an exact sequence

$$1 \to (Z_{M^\vee})^\Gamma / (Z_{G^\vee})^\Gamma \to \mathbb{S}_\varphi \to \mathbb{S}_{\varphi_M} \to 1.$$

6.8 General limits: companion standard Levi group

We continue with the packet Π of the last section. It consists of the components of several essentially tempered principal series representations of $G(\mathbb{R})$. To describe them, we return to the representative $\overline{\varphi}_M = \varphi(\mu_M, \overline{\lambda}_M)$ for $\boldsymbol{\varphi}_M$ in Section 6.6 and set $\overline{\varphi} = \xi_M \circ \overline{\varphi}_M$. Then $\overline{\varphi}$ also represents $\boldsymbol{\varphi}$.

We may replace $\mathrm{spl}_{\mathrm{Wh}}$ by a $G^*(\mathbb{R})$-conjugate and then M^* by its conjugate relative to the same element to arrange that the maximal torus $\overline{T}_M$ in M^* provided by spl^*_M is a standard maximal torus in G^*. We then drop the subscript M in notation for this torus. Here by *standard* we mean that the maximal split torus $\overline{S}$ in $\overline{T}$ is contained in T^* provided by $\mathrm{spl}^* = (\mathrm{spl}^\vee)^\vee$. Let $\overline{M}$ be the standard Levi group $\mathrm{Cent}(\overline{S}, G^*)$. Then ${}^L\overline{M}$ will denote the dual standard Levi group in ${}^LG$, naturally embedded by inclusion.

Lemma 6.8. *The image of $\overline{\varphi}$ lies in ${}^L\overline{M}$ and defines an elliptic parameter for $\overline{M}$.*

Proof. We return to the notation of Section 5.4. We have arranged that $\sigma_{\overline{T}} = \sigma_M$ on $\mathcal{T}$. Then the element $n_{\overline{M}} \times w_\sigma$ in ${}^LG$ coincides with $\xi_M(w_\sigma)$ up to an element of $\mathcal{T} \cap G^\vee_{\mathrm{der}}$. It follows that $\overline{\varphi}(w_\sigma) \in {}^L\overline{M}$ and then that $\overline{\varphi}(W_\mathbb{R}) \subseteq {}^L\overline{M}$. Since $\sigma_M \alpha^\vee = \sigma_{\overline{T}} \alpha^\vee = -\alpha^\vee$ for each root $\alpha^\vee$ of $\mathcal{T}$ in $\overline{M}^\vee$, it is clear that $\overline{\varphi}$ is s-elliptic as Langlands parameter for $\overline{M}$. If we write $\overline{\varphi} = \varphi(\mu, \overline{\lambda})$ relative to $\overline{M}$, then μ is $\overline{M}$-regular for otherwise $\overline{T}$ would have an imaginary root in M^*. □

We continue with $\overline{\varphi} = \varphi(\mu, \overline{\lambda})$ and consider the quasisplit form G^*. The Whittaker data Wh for G^* determines, by restriction, Whittaker data $\mathrm{Wh}_{\overline{M}}$ for $\overline{M}$. We choose a corresponding fundamental splitting $\mathrm{spl}_{\mathrm{Wh}_{\overline{M}}} = (B_{\overline{M}}, \overline{T}, \{X_\alpha\})$ of Whittaker type for $\overline{M}$, and then transport $(\mu, \overline{\lambda})$ to discrete series character data on $\overline{T}$. Via unitary parabolic induction, each discrete series representation in the packet for $\overline{M}(\mathbb{R})$ attached to $\overline{\varphi}$ determines an essentially tempered principal series representation of $G^*(\mathbb{R})$. Then Π consists of the irreducible components of all principal series representations so obtained. Consider next an inner form (G, η) for which φ is relevant. Recall that for $\pi \in \Pi$, η_π transports elliptic character data for π to that for π^*. By (6.5) and Lemma 6.2 we may choose π so that $\pi_M = \pi^*_M \circ \eta_\pi$ is isomorphic to π^*_M. We then adjust our discussion for G^* to describe the packet for $G(\mathbb{R})$; we will not need details here.

From definitions (recalled in [Sh82, Section 5.3]) it is clear that Langlands' version of the R-group is unchanged by passage from ${}^LM$ to ${}^LG$:

$$R_{\overline{\varphi}} = R_{\overline{\varphi}_M}.$$

Also there is a surjective homomorphism $\mathbb{S}_{\overline{\varphi}} \to R_{\overline{\varphi}}$ with kernel that may be identified with the dual of $\mathcal{E}(\overline{T})$ (see [Sh82, Sections 5.3, 5.4]). Because φ_M is totally degenerate we have that $\mathbb{S}_{\overline{\varphi}_M} \to R_{\overline{\varphi}_M} = R_{\overline{\varphi}}$ is an isomorphism. Then by the discussion around (6.6) we have a perfect pairing of $R_{\overline{\varphi}}$ with

$$\mathrm{Image}(H^1(\Gamma, Z_{M^*_{\mathrm{sc}}}) \to H^1(\Gamma, T)) \simeq M_{\mathrm{ad}}(\mathbb{R}) / \mathrm{Int}(M(\mathbb{R})).$$

7 Packets and parameters II

7.1 Data for elliptic u-regular parameters

Suppose that $\psi = (\varphi, \rho)$ is an elliptic u-regular Arthur parameter. Continuing from Section 5.5, we may assume that φ takes the following form:

$$\varphi(z) = z^{\mu}\overline{z}^{\sigma_M \mu} \times z$$

for $z \in \mathbb{C}^{\times}$, and

$$\varphi(w_\sigma) = e^{2\pi i\lambda}.\xi_M(w_\sigma).$$

Here $\mu, \lambda \in X_*(\mathcal{T}) \otimes \mathbb{C}$ and

$$\langle \mu, \alpha^\vee \rangle = 0, \ \langle \lambda, \alpha^\vee \rangle \in \mathbb{Z} \tag{7.1}$$

for all roots $\alpha^\vee$ of $\mathcal{T}$ in $M^\vee$. The element μ is uniquely determined by the $\mathcal{T}$-conjugacy class of the representative φ, and λ is determined uniquely modulo

$$\mathcal{K}_M = X_*(\mathcal{T}) + \{\nu \in X_*(\mathcal{T}) \otimes \mathbb{C} : \sigma_M \nu = -\nu\}.$$

We will use the notation $\varphi = \varphi[\mu, \lambda]$. Notice that in the case $M_\psi = T$, where φ is elliptic, we return to the pair (μ, λ) from Section 5.6.

From our construction of ξ_M and the equation $\varphi(w_\sigma)^2 = \varphi(-1)$, we have immediately the following congruence:

$$\frac{1}{2}(\mu - \sigma_M \mu) - (\iota - \iota_M) \equiv \lambda + \sigma_M \lambda \mod X_*(\mathcal{T}). \tag{7.2}$$

The properties (7.1) allow us to replace σ_M by σ_T in (7.2) and then to rewrite the congruence as

$$\frac{1}{2}[\mu + \iota_M - \sigma_T(\mu + \iota_M)] - \iota \equiv \lambda + \sigma_T \lambda \mod X_*(\mathcal{T}). \tag{7.3}$$

For the second component ρ of ψ we turn to Section 5.5 and the u-regularity property. With first component φ prescribed as above, we may assume that $\rho : \mathrm{SL}(2, \mathbb{C}) \to M^\vee$ is in standard form with cocharacter $2\iota_M$. Then

$$\rho(\mathrm{diag}(|w|^{1/2}, |w|^{-1/2})) = (z\overline{z})^{\iota_M}, \ w \in W_{\mathbb{R}},$$

where $w = z$ or zw_σ, $z \in \mathbb{C}^\times$, as in [Ar89]. We write $\rho = \rho(\iota_M)$.

We observe that (7.3) implies that $\mu + \iota_M \in X_*(\mathcal{T}) \otimes \mathbb{C}$ is integral, i.e.,

$$\langle \mu + \iota_M, \alpha^\vee \rangle \in \mathbb{Z}, \tag{7.4}$$

for all roots of $\mathcal{T}$ in $G^\vee$.

Remark 7.1. μ is at least half-integral; μ is integral if the derived group of G is simply-connected since ι_M is integral in that case.

Recall that $\mathcal{B}$ denotes the Borel subgroup that is part of $\mathrm{spl}^\vee$.

Lemma 7.2. *Let $\boldsymbol{\psi}$ be an elliptic u-regular Arthur parameter. Then there exists a representative $\psi = (\varphi, \rho)$ for $\boldsymbol{\psi}$, where $\varphi = \varphi[\mu, \lambda]$ and $\rho = \rho(\iota_M)$, with both μ and $\mu + \iota_M$ $\mathcal{B}$-dominant.*

Proof. First, we observe that it is sufficient to arrange that $\mu + \iota_M$ is dominant. Since $M^\vee$ is Levi we have that $\langle \iota_M, \alpha^\vee \rangle \leq 0$ for each $\mathcal{B}$-simple $\alpha^\vee$ that is not a root of $\mathcal{T}$ in $M^\vee$. Then dominance of $\mu + \iota_M$ implies $\langle \mu, \alpha^\vee \rangle \geq 0$ for all such $\alpha^\vee$ and so by (7.1), μ is dominant.

Second, suppose we pick $\varphi = \varphi[\mu, \lambda]$, $\rho = \rho(\iota_M)$ as in the paragraphs above. There is ω in the Weyl group of $\mathcal{T}$ in $G^\vee$ such that $\omega(\mu + \iota_M)$ is $\mathcal{B}$-dominant. Let $x \in G^\vee$ normalize $\mathcal{T}$ and act on $\mathcal{T}$ as ω. Set $M_\omega^\vee = xM^\vee x^{-1}$ and $\psi_x = \mathrm{Int}(x) \circ \psi$. If $\alpha^\vee$ is a $\mathcal{B}$-positive root of in $\mathcal{T}$ in $M_\omega^\vee$, then $\omega^{-1}\alpha^\vee$ is a root in $M^\vee$ and so

$$\left\langle \iota_M, \omega^{-1}\alpha^\vee \right\rangle = \left\langle \mu + \iota_M, \omega^{-1}\alpha^\vee \right\rangle = \left\langle \omega(\mu + \iota_M), \alpha^\vee \right\rangle \geq 0.$$

Then $\omega^{-1}\alpha^\vee$ must be $\mathcal{B}$-positive. It now follows that $\iota_{M_\omega} = \omega\iota_M$. Then after multiplying x by an element of $\mathcal{T}$, we may replace ψ by ψ_x in our constructions to complete the proof. ◻

From now on we choose representative $\psi = (\varphi, \rho)$ as in Lemma 7.3.

From (7.3) we conclude that:

Lemma 7.3. $\mu + \iota_M, \lambda$ *are data for an s-elliptic Langlands parameter*

$$\widehat{\varphi} = \varphi(\mu + \iota_M, \lambda).$$

Finally, we set

$$\mu_M = \mu - (\iota - \iota_M),\ \lambda_M = \lambda. \tag{7.5}$$

As one of the ingredients [La89] of the Langlands correspondence for M^*, the parameter $\varphi_M : W_{\mathbb{R}} \to {}^L Z_M$ from Section 5.5 defines a quasicharacter χ_{M^*} on $M^*(\mathbb{R})$. Because of (5.3) and (7.1) the restriction of χ_{M^*} to each Cartan subgroup in $M^*(\mathbb{R})$ takes the form

$$\Lambda(\mu_M, \lambda_M) \tag{7.6}$$

in the Langlands correspondence for real tori [La89]; see [Sh81, Section 9] for a discussion and [Sh10, Section 7] for notation. Further, for an inner twist $\eta : M_\eta \to M^*$ we may replace M^* by $M_\eta = \eta^{-1}(M^*)$. Then the new quasicharacter χ_η on $M_\eta(\mathbb{R})$ depends only on the inner class of η.

On the other hand, $\widehat{\varphi}$ factors through the discrete series parameter

$$\widehat{\varphi}_M = \varphi(\mu_M + \iota_M, \lambda_M)$$

for M^*.

From (7.5) and (7.4) we see that μ_M is integral for $G^\vee$. Clearly:

Lemma 7.4.

(i) $\mu + \iota_M$ *is regular, i.e.,* $\widehat{\varphi}$ *is elliptic, if and only if* μ_M *is* $\mathcal{B}$*-dominant.*
(ii) $\mu + \iota_M$ *is singular if and only if* $\langle \mu_M, \alpha^\vee \rangle = -1$ *for some* $\mathcal{B}$*-simple root* $\alpha^\vee$ *of* $\mathcal{T}$.

Assume that $\mu + \iota_M$ is regular and G has anisotropic center; this is the setting of [Ar89, Section 5], [Ko90, Section 9]. Here we recover the same parameters, but now with data for use in canonical transfer factors; see, for example, Section 8.2. Our parameter $\widehat{\varphi}$ coincides with the discrete series parameter constructed slightly differently in [Ko90, Section 9].

7.2 Character data and elliptic u-regular parameters

We combine the setting of Section 7.1 with that of Section 6.1. Thus G^* is cuspidal, and we have fixed Whittaker data for G^* together with an aligned fundamental splitting $\mathrm{spl}_{\mathrm{Wh}} = (B_{\mathrm{Wh}}, T, \{X_\alpha\})$ for G^*. We now transport the data $(\mu + \iota_M, \lambda, \mathcal{C})$ for $\mathcal{T} \subseteq G^\vee$ of Section 7.1 to data for $T \subseteq G^*$, by the means described in Section 6.1. Recall that M^* is the subgroup of G^* generated by T and the root vectors $\{X_\alpha\}$, for $\alpha^\vee$ a simple root of $\mathcal{T}$ in $M^\vee \cap \mathcal{B}$. We use the same notation for the transported data, except that now we write ι_{M^*} for the transport of ι_M, i.e., for one-half the sum of the roots of T in $B_{\mathrm{Wh}} \cap M^*$.

7.3 Elliptic u-regular data: attached s-elliptic packet

We start with the case that $\mu + \iota_M$ is regular. Consider an inner form (G, η). Replacing η by a member of its inner class if necessary, we assume that the transport $\mathrm{spl}_\eta = \eta^{-1}(\mathrm{spl}_{\mathrm{Wh}})$ of $\mathrm{spl}_{\mathrm{Wh}}$ to G is a fundamental splitting. As in Part 6, to each $G(\mathbb{R})$-conjugacy class of fundamental splittings for G is attached to a discrete series representation $\widehat{\pi}$ of $G(\mathbb{R})$ in the packet $\widehat{\Pi}_G$ for $\widehat{\varphi}$, and conversely. Again write $\mathrm{spl}_{\widehat{\pi}}$ for a representative of this conjugacy class and $\eta_{\widehat{\pi}} = \mathrm{Int}(x_{\widehat{\pi}}) \circ \eta$ for the inner twist carrying $\mathrm{spl}_{\widehat{\pi}}$ to $\mathrm{spl}_{\mathrm{Wh}}$. Set $M_{\widehat{\pi}} = \eta_{\widehat{\pi}}^{-1}(M^*)$. By definition, $\eta_{\widehat{\pi}}$ transports character data $(\mu_{\widehat{\pi}} + \iota_{M_{\widehat{\pi}}}, \lambda_{\widehat{\pi}}, \mathcal{C}_{\widehat{\pi}})$ for $\widehat{\pi}$ to the data $(\mu + \iota_{M^*}, \lambda, \mathcal{C})$ for the elliptic torus T in G^* that is part of $\mathrm{spl}_{\mathrm{Wh}}$. Recall that the latter triple serves as character data for the Wh-generic discrete series representation of $G^*(\mathbb{R})$ in the packet $\widehat{\Pi}_{G^*}$ attached to $\widehat{\varphi}$.

Now allow $\mu + \iota_M$ to be singular. Then we assume that $\widehat{\varphi} = \varphi(\mu + \iota_M, \lambda)$ is relevant to G so that $\widehat{\Pi}_G$ is nonempty. As in Section 6.2, there is attached to $\widehat{\varphi}$ a c-Levi group which we will call E^*. Notice that $E^* \cap M^* = T$. Each $G(\mathbb{R})$-conjugacy class of fundamental splittings of G again has a representative $\mathrm{spl}_{\widehat{\pi}}$, but now $\widehat{\pi}$

(or, more precisely, the attached distribution character) may be zero. We obtain precisely the members $\widehat{\pi}$ of $\widehat{\Pi}_G$ by requiring that $\eta_{\widehat{\pi}} : E_{\widehat{\pi}} \to E^*$ be defined over $\mathbb{R}$ (Lemma 6.1).

7.4 Elliptic u-regular data: attached Arthur packet

For the rest of Parts 7 and 8 we will limit our attention to the case that $\mu + \iota_M$ is regular as we will need it to structure our arguments for the singular case. For convenience we could also require the center of G to be anisotropic, but the general case requires no extra notation and so we will at least write it here. Finally, there is the matter of how we treat z-extensions. We will continue to use the construction needed for the twisted case (see Section 3.1) but defer checking that the Adams–Johnson results may be extended in this manner until we come to the general twisted case.

Consider an inner form (G, η), where $\mathrm{spl}_\eta = (B_\eta, T_\eta, \{X_\alpha\})$ is fundamental and η carries spl_η to $\mathrm{spl}_{\mathrm{Wh}}$. We may fix a Cartan involution c on G of the form $\mathrm{Int}(t_\eta)$, where $t_\eta \in T_\eta(\mathbb{R})$ and $(t_\eta)^2$ is central in G. Then B_η, M_η together generate a c-stable parabolic subgroup P_η of G with M_η as Levi subgroup defined over $\mathbb{R}$. We have the quasicharacter $\chi = \chi_\eta$ on $M_\eta(\mathbb{R})$ described in Section 7.1. Because of (7.6), it is clear that χ_η is unitary modulo the center of $G(\mathbb{R})$. As usual, we will identify a representation with its (appropriate) isomorphism class. Define $\pi(\eta)$ to be the irreducible essentially unitary representation of $G(\mathbb{R})$ attached to χ_η by the method of [Vo84, Theorems 1.2, 1.3]; see Lemma 2.10 of [AJ87].

Suppose we replace η by $\eta^\dagger$ within its inner class and that $\eta^\dagger$ also carries a fundamental splitting of G to $\mathrm{spl}_{\mathrm{Wh}}$. It is convenient to write $\eta^\dagger$ in the form

$$\eta^\dagger = \mathrm{Int}(m^*) \circ \eta \circ \mathrm{Int}(g),$$

where $m^* \in M^*_{\mathrm{sc}}$ and $g \in G_{\mathrm{sc}}$. Let $m = \eta_{\mathrm{sc}}^{-1}(m^*)$. Then we insist also that $\mathrm{Int}(m)$ transports spl_η to another fundamental splitting of G. Since we are concerned with splittings only up to $G(\mathbb{R})$-conjugacy there is no harm in considering only those $\eta^\dagger$ for which $T_{\eta^\dagger} = T_\eta$, and requiring that both $\mathrm{Int}(m)$ and $\mathrm{Int}(g)$ preserve T_η.

We define $\pi(\eta^\dagger)$ by replacing $B_\eta, M_\eta, \chi_\eta$ from the definition of $\pi(\eta)$ with $B_{\eta^\dagger}, M_{\eta^\dagger}, \chi_{\eta^\dagger}$. Then:

Lemma 7.5.

(i) $\pi(\eta^\dagger)$ lies in same Arthur packet Π_G prescribed by Adams–Johnson (enlarged packet in their terminology) as $\pi(\eta)$, and all members of the packet are so obtained.

(ii) $\pi(\eta^\dagger) = \pi(\eta)$ if and only if $\eta^\dagger$ is of the form $\mathrm{Int}(m^*) \circ \eta \circ \mathrm{Int}(g)$, *where $m^* \in M^*_{\mathrm{sc}}$ and $g \in G(\mathbb{R})$.*

Let ω_M, ω_G be the elements of the complex Weyl group $\Omega(G, T_\eta)$ of T_η in G defined by the restrictions of $\mathrm{Int}(m)$, $\mathrm{Int}(g)$ to T_η. Then (ii) says $\pi(\eta^\dagger) = \pi(\eta)$ if and only if we may arrange that ω_G lies in the subgroup $\Omega_{\mathbb{R}}(G, T_\eta)$ of $\Omega(G, T_\eta)$ consisting of those elements that are realized in $G(\mathbb{R})$.

Proof. To compare explicitly with Lemma 2.10 of [AJ87], first note that since we do not assume that the center of G is anisotropic, our elliptic data have an extra component, namely λ as above. The "λ, ρ" of [AJ87] are our μ_M, ι. Note that (7.5) says that

$$\mu_M + \iota = \mu + \iota_M.$$

We further have the alternative short definition after Remark 7.4 for the quasicharacter χ_η, but it is clear from calculations of Section 2 of [AJ87] (or see [Sh79b, Lemma 9.2], [Sh81, Section 9]) that we obtain the same character when we require the center of G to be anisotropic. The claim (i) now follows. More accurately, we have adapted the definitions of Adams–Johnson to the case where there is no restriction on the center for G while retaining (i) of their Lemma 2.10. Because $\mu + \iota_M$ is regular, the claim (ii) follows easily from the character formulas that we will recall in Section 8.3. □

8 Standard factors for elliptic u-regular packets

Here by standard factors we mean the spectral transfer factors for standard endoscopy. We introduce these, with a two-fold purpose, for the elliptic u-regular Arthur packets Π_G of the last section. First, we will check that the Adams–Johnson transfer can be recast in terms of these factors and thereby made compatible with the transfer of orbital integrals using the canonical factors of [LS87]. Second, we will write the factors in a way that allows quick generalization to the twisted setting [ShII].

8.1 Canonical relative factor: setting

We continue with the setting at the end of Part 7. In summary, $\psi = (\varphi, \rho)$ is an elliptic u-regular Arthur parameter with $\varphi = \varphi[\mu, \lambda]$ and $\rho = \rho(\iota_M)$ as in Lemma 7.2. We assume $\mu + \iota_{M^*}$ is regular as well as dominant. Then $\widehat{\varphi}$ is the attached elliptic parameter $\varphi(\mu + \iota_{M^*}, \lambda)$.

To introduce elliptic endoscopic groups as in Section 6.5, we turn to the Γ-invariants in the maximal torus $\mathcal{T}$ from spl$^\vee$. We use the elliptic action of Γ, so that σ acts by Int$\,\widehat{\varphi}(w_\sigma)$. Consider the elliptic SED $\mathfrak{e}_z(s)$ as in Section 6.5, using the notation $(H, \mathcal{H}, s)$ for the endoscopic data and $H^{(s)}$ for the endoscopic group. It will be sufficient for our purposes in [ShII] to consider the case that the Γ-invariant s lies in the center $Z_{M^\vee}$ of $M^\vee$. This is the same as requiring that $H^\vee := \mathrm{Cent}(s, G^\vee)^0$ contain $M^\vee$. Thus we place ourselves in the setting of Adams–Johnson; see [AJ87, 2.16].

Because ψ is elliptic, the subgroup $\mathcal{M} = \mathcal{M}_\psi$ of ${}^L G$ may be generated by $M^\vee$ and either $\varphi(W_\mathbb{R})$ or $\widehat{\varphi}(W_\mathbb{R})$. Thus $\mathcal{H}$, generated by $H^\vee$ and $\widehat{\varphi}(W_\mathbb{R})$, contains $\mathcal{M}$. Since the endoscopic group $H^{(s)}$ is a z-extension of the endoscopic datum H, we will need to thicken $\mathcal{M}$.

Recall that $(Z_{M^\vee})^\Gamma = S_\psi := \mathrm{Cent}(\mathrm{Image}(\psi), G^\vee)$. Thus the image of ψ lies in $\mathcal{M} \subseteq \mathcal{H}$. As a component of the SED $\mathfrak{e}_z(s)$, we have $\xi^{(s)} : \mathcal{H} \to {}^L H^{(s)}$ and thus an elliptic u-regular Arthur parameter for $H^{(s)}$ represented by $\psi^{(s)} = \xi^{(s)} \circ \psi$. Now we attach $\mathcal{M}^{(s)}$ to $\psi^{(s)}$ in the same way we attached $\mathcal{M}$ to ψ. Then $\mathcal{M}^{(s)}$ is what we mean by the thickened version of $\mathcal{M}$. We will thicken various other subgroups when needed, again using the super- or subscript (s) to indicate this.

We now describe transfer factors attached to the pair $(\psi^{(s)}, \psi)$; see [Sh10, Section 9], [Sh08b, Sections 7, 11] for the tempered analogue. Recall $\psi = (\varphi, \rho)$. Then we write $\psi^{(s)}$ as $(\varphi^{(s)}, \rho)$.

First, $\varphi^{(s)} = \varphi[\mu^{(s)}, \lambda^{(s)}]$ and

$$\mu^{(s)} = \mu - \mu^*,\ \lambda^{(s)} = \lambda - \lambda^*. \tag{8.1}$$

The pair (μ^*, λ^*) is from [Sh81]; it is typically nontrivial and is critical for a well-defined transfer of orbital integrals. Here we need its construction for general standard transfer with z-extensions; see Section 11 of [Sh08a]. Also see Section 9.3 below for a detailed contruction in the general twisted case. The formula (8.1) follows from combining the construction with that in Lemma 7.2.

Second, the component ρ of $\psi^{(s)}$ may be written again as $\rho(\iota_M)$. For this we recall the splittings involved in our constructions: we have $\mathrm{spl}^\vee = (\mathcal{B}, \mathcal{T}, \{X_{\alpha^\vee}\})$ for $G^\vee$ with attached $\mathrm{spl}^\vee_M = (\mathcal{B} \cap M^\vee, \mathcal{T}, \{X_{\alpha^\vee}\})$ for $M^\vee$, along with $\mathrm{spl}^\vee_H = (\mathcal{B} \cap H^\vee, \mathcal{T}, \{X_{\alpha^\vee}\})$ for $H^\vee$ and thickened $\mathrm{spl}^\vee_{(s)} = (\mathcal{B}^{(s)}, \mathcal{T}^{(s)}, \{X_{\alpha^\vee}\})$ for $(H^{(s)})^\vee$. Then ι_M is one-half the sum of the coroots of $\mathcal{T}$ in $\mathcal{B} \cap M^\vee = \mathcal{B} \cap \mathcal{M}$. Each such coroot is naturally identified as a coroot of $\mathcal{T}^{(s)}$ in $\mathcal{B}^{(s)} \cap \mathcal{M}^{(s)}$ and conversely, which justifies our use of ι_M for ρ as component of $\psi^{(s)}$.

The c-Levi group $M^{(s)}$ in $H^{(s)}$ is the analogue for $\psi^{(s)}$ of the c-Levi group M^* in G^* attached to ψ. There is a c-Levi group M_H in H such that $M^{(s)} \to M_H$ is a z-extension with kernel Z_1, i.e., with same kernel as the z-extension $H^{(s)} \to H$ provided by the SED $\mathfrak{e}_z(s)$.

Our next step is to define an $\mathbb{R}$-isomorphism $M_H \to M^*$ uniquely up to composition with an element of $\mathrm{Int}[M^*(\mathbb{R})]$, and thence a surjective homomorphism $M_H^{(s)} \to M^*$ with kernel Z_1. For this, recall that in the construction of M^* at the beginning of Section 6.2 we also determined an $\mathbb{R}$-splitting for M^* uniquely up to $M^*(\mathbb{R})$-conjugacy. The same is then true for M_H. There is a unique $\mathbb{R}$-isomorphism $M_H \to M^*$ transporting the latter splitting to the former. We may further assume the isomorphism carries chosen elliptic maximal torus T_H in H to chosen T in G^*; recall that each torus is part of an appropriate fundamental splitting of Whittaker type. If $T^{(s)}$ is the inverse image of T_H in $M_H^{(s)}$, then we have now have a well-defined transport to T of our various data attached to $T^{(s)}$.

8.2 Canonical relative factor: definition

We now define a relative transfer factor in preparation for a nontempered supplement for Section 6.5. Thus (G, η) is an inner form of G^* such that η transports fundamental splitting spl_η, of G to $\mathrm{spl}_{\mathrm{Wh}}$. Let $\pi \in \Pi_G$ (Arthur packet for $G(\mathbb{R})$ attached to ψ) and let $\widehat{\pi} \in \widehat{\Pi}_G$ (discrete series packet for $G(\mathbb{R})$ attached to ψ). Also let $\pi_s \in \Pi_{H^{(s)}}$ and $\widehat{\pi}_s \in \widehat{\Pi}_{H^{(s)}}$ (packets for $H^{(s)}(\mathbb{R})$ attached to $\psi^{(s)}$). Then our first concern will be a relative factor $\Delta(\pi_s, \pi; \widehat{\pi}_s, \widehat{\pi})$.

Attach the cochain $x_{\widehat{\pi}}(\sigma) \in T_{\mathrm{sc}}$ to the discrete series representation $\widehat{\pi}$ as in (6.4)); recall that $\eta_{\widehat{\pi}} = \mathrm{Int}(x_{\widehat{\pi}}) \circ \eta$ and $x_{\widehat{\pi}}(\sigma) = x_{\widehat{\pi}}.u_\eta(\sigma).\sigma(x_{\widehat{\pi}})^{-1}$. Again we write $\mathcal{E}(T)$ for the image of $H^1(\Gamma, T_{\mathrm{sc}})$ in $H^1(\Gamma, T)$ under the homomorphism induced by $T_{\mathrm{sc}} \to T$. Then if (G, η) is a component of an extended group of quasisplit type, so that $u_\eta(\sigma)$ is a cocycle, we map the class of $x_{\widehat{\pi}}(\sigma)$ in $H^1(\Gamma, T_{\mathrm{sc}})$ to $H^1(\Gamma, T)$ to obtain the element $\mathrm{inv}(\widehat{\pi})$ of $\mathcal{E}(T)$.

Turning to π in the Arthur packet for $G(\mathbb{R})$, we pick a twist $\eta^\dagger$ such that $\pi = \pi(\eta^\dagger)$ as in Section 7.4. We write $\eta^\dagger$ as $\mathrm{Int}(x^\dagger) \circ \eta$ and form the cochain $x^\dagger(\sigma) = x^\dagger \cdot u_\eta(\sigma) \cdot \sigma(x^\dagger)^{-1}$. Recall the torus U_{sc} from Section 6.4. The image in U_{sc} of the cochain $(x^\dagger(\sigma)^{-1}, x_{\widehat{\pi}}(\sigma))$ in $T_{\mathrm{sc}} \times T_{\mathrm{sc}}$ is a cocycle whose class in $H^1(\Gamma, U_{\mathrm{sc}})$ we denote by $\mathbf{x}_{\mathrm{sc}}(\eta^\dagger, \widehat{\pi})$. Then $\mathbf{x}(\eta^\dagger, \widehat{\pi})$ is the image of this class in $H^1(\Gamma, U)$. Recall s_U from (6.4) and that in the present setting we assume that the Γ-invariant s lies in the center of $M^\vee$.

Lemma 8.1. $\langle \mathbf{x}(\eta^\dagger, \widehat{\pi}),\ s_U \rangle_{\mathrm{tn}}$ *depends only on* $\pi, \widehat{\pi}$.

Then we define

$$\mathrm{pair}_{(s)}(\pi, \widehat{\pi}) := \Big\langle \mathbf{x}(\eta^\dagger, \widehat{\pi}),\ s_U \Big\rangle_{\mathrm{tn}}.$$

Before proving Lemma 8.1 we examine $x^\dagger(\sigma)$ in the case that (G, η) is a component of an extended group of quasisplit type. Then $x^\dagger(\sigma)$ is a cocycle and so defines an element $\mathbf{x}(\eta^\dagger)$ of $\mathcal{E}(T)$. We have $T \subseteq M^* \subseteq G^*$. Then $\mathcal{E}_{M^*}(T)$ is the image of $H^1(\Gamma, T_{M^*_{\mathrm{sc}}}) \to H^1(\Gamma, T)$. It is a subgroup of $\mathcal{E}(T)$.

Lemma 8.2. *The image of* $\mathbf{x}(\eta^\dagger)$ *in* $\mathcal{E}(T)/\mathcal{E}_{M^*}(T)$ *depends only on* π.

Proof. There is no harm in replacing $x^\dagger(\sigma)$ by its inverse. The twist $\eta^\dagger$ may be replaced only by $\mathrm{Int}(m^*) \circ \eta \circ \mathrm{Int}(g)$, where m^*, g are as specified in Section 7.4. Then $x^\dagger(\sigma)^{-1}$ is replaced by $\sigma(m^*)(m^*)^{-1} \cdot m^* x^\dagger(\sigma)^{-1} \cdot (m^*)^{-1}$. Our assumptions on m^* ensure that $\sigma(m^*)(m^*)^{-1}$ is a cocycle in T_{sc}; its class then has image in $\mathcal{E}_{M^*}(T)$. Finally, the $\mathbb{R}$-automorphism $\mathrm{Int}(m^*) : T_{\mathrm{sc}} \to T_{\mathrm{sc}}$ induces a homomorphism $H^1(\Gamma, T_{\mathrm{sc}}) \to H^1(\Gamma, T_{\mathrm{sc}})$. Passing to T, we may then define a homomorphism $\mathcal{E}(T) \to \mathcal{E}(T)/\mathcal{E}_{M^*}(T)$. From the Tate–Nakayama isomorphism of $H^1(\Gamma, T_{\mathrm{sc}})$ with $H^{-1}(\Gamma, X_*(T_{\mathrm{sc}}))$, we see that the homomorphism coincides with the natural projection, and the lemma follows. □

Now define

$$\mathrm{inv}(\pi) := \mathbf{x}(\eta^{\dagger}) \cdot \mathcal{E}_{M^*}(T)$$

Because s is a Γ-invariant in the center of $M^{\vee}$, we have that

$$\langle \mathcal{E}_{M^*}(T),\ s_T \rangle_{\mathrm{tn}} = 1,$$

and so the Tate–Nakayama pairing for T determines a well-defined sign we will write as

$$\langle \mathrm{inv}(\pi),\ s_T \rangle\,.$$

We may view $\langle \_, \_ \rangle$ as a pairing between $\mathcal{E}(T)/\mathcal{E}_{M^*}(T)$ and $(Z_{M^{\vee}})^{\Gamma}$ or, better, between $\mathcal{E}(T)/\mathcal{E}_{M^*}(T)$ and $(Z_{M^{\vee}})^{\Gamma}/(Z_{G^{\vee}})^{\Gamma}$. In the latter case we identify s_T with its image in $(Z_{M^{\vee}})^{\Gamma}/(Z_{G^{\vee}})^{\Gamma}$ without change in notation. We will say more about the pairing in [ShII].

Notice that Lemma 8.1 is now proved in this setting, i.e., for an extended group of quasisplit type, because

$$\left\langle \mathbf{x}(\eta^{\dagger}, \widehat{\pi}),\ s_U \right\rangle_{\mathrm{tn}} = \mathrm{pair}_{(s)}(\pi, \widehat{\pi}) = \langle \mathrm{inv}(\pi), s_T \rangle^{-1} . \langle \mathrm{inv}(\widehat{\pi}), s_T \rangle_{\mathrm{tn}}\,. \tag{8.2}$$

Proof (of Lemma 8.1). A factoring via the method for the proof of Lemma 8.2, but now in U_{sc} instead of T_{sc}, may be applied to the cocycle defining $\mathbf{x}_{\mathrm{sc}}(\eta^{\dagger}, \widehat{\pi})$. Then we follow closely the rest of the argument to complete the proof. □

Next, we recall the sign

$$\varepsilon(G) := (-1)^{q(G)\,-\,q(G^*)},$$

where $2q(G)$ is the rank of the symmetric space attached to G_{sc}. It is well-defined in general and appears in the tempered character identities for transfer from the inner form (G, η) to G^*; see [Sh79a, Theorem 6.3]. This sign is recast by Kottwitz in [Ko83, p.295] in terms of Galois cohomology. Notice that the choice of inner twist does not matter; see [Ko83, p.292]. In our present setting we have $\pi = \pi(\eta^{\dagger})$. Let $M_{\eta^{\dagger}} = (\eta^{\dagger})^{-1}(M^*)$. Then it is clear from either definition that $\varepsilon(M_{\eta^{\dagger}})$ is independent of the various choices for $\eta^{\dagger}$ and so we write it as $\varepsilon_M(\pi)$.

We conclude then that the relative factor

$$\Delta(\pi_s, \pi; \widehat{\pi}_s, \widehat{\pi}) := \varepsilon_M(\pi).\mathrm{pair}_{(s)}(\pi, \widehat{\pi}) \tag{8.3}$$

is well-defined, i.e., depends only on s, π and $\widehat{\pi}$. This factor and others similarly defined have useful transitivity properties (see [LS87, Section 4.1], [Sh10, Section 4]). We will ignore them for now except to remark that if the discrete series representation $\widehat{\pi}$ has the property that $\eta_{\widehat{\pi}}$ serves as $\eta^{\dagger}$, then

$$\Delta(\pi_s, \pi; \widehat{\pi}_s, \widehat{\pi}) = \varepsilon_M(\pi). \tag{8.4}$$

To define an absolute factor $\Delta(\pi_s, \pi)$, assume that we have absolute geometric factors and absolute spectral factors for the essentially tempered spectrum that are compatible in the sense of [Sh10, Section 12]. This notion of compatibility is defined via another canonical relative factor, and compatible factors are easily shown to exist for all inner forms (G, η); see [Sh10, Section 4]. We then set

$$\Delta(\pi_s, \pi) := \Delta(\pi_s, \pi; \widehat{\pi}_s, \widehat{\pi}).\Delta(\widehat{\pi}_s, \widehat{\pi}). \tag{8.5}$$

In particular if M^* is a torus, so that (π_s, π) is a related pair of discrete series representations, we return the original constructions for the (essentially) tempered spectrum; see [Sh10, Section 9].

Consider an extended group of quasisplit type and use the Whittaker normalization Δ_{Wh} of absolute factors attached to our choice of Whittaker data [KS99, Section 5.3]. Then (8.5), (8.3), (8.2) and the strong base-point property of Whittaker normalization [Sh08b, Theorem 11.5] (recall Section 6.5) imply:

Lemma 8.3.

$$\Delta_{\mathrm{Wh}}(\pi_s, \pi) = \varepsilon_M(\pi).\,\langle \mathrm{inv}(\pi),\ s_T \rangle\,.$$

8.3 Application to the transfer of Adams–Johnson

Continuing in the same setting, we write the correspondence of test functions (more precisely, test measures) as $(f, f^{(s)})$. Then

$$SO(\gamma, f^{(s)}) = \sum_{\delta,\mathrm{conj}} \Delta(\gamma, \delta)\ O(\delta, f) \tag{8.6}$$

for all strongly G-regular γ in $H^{(s)}(\mathbb{R})$ and

$$\text{St-}\,\mathrm{Trace}\,\widehat{\pi}_s(f^{(s)}) = \textstyle\sum_{\widehat{\pi}} \Delta(\widehat{\pi}_s, \widehat{\pi})\ \mathrm{Trace}\,\widehat{\pi}(f). \tag{8.7}$$

Now to consider the pair (π_s, π), we observe that the Adams–Johnson stable combination [AJ87, Theorem 2.13] agrees with

$$\text{St-}\,\mathrm{Trace}\,\pi_s(f^{(s)}) := \textstyle\sum_{\pi'_s \in \Pi_{H^{(s)}}} \varepsilon_M(\pi'_s)\ \mathrm{Trace}\,\pi'_s(f^{(s)}),$$

up to the sign $(-1)^{\gamma(M^*)}$ defined in [AJ87, 2.12].

Next we claim the following transfer for (π_s, π) :

$$\text{St-}\,\mathrm{Trace}\,\pi_s(f^{(s)}) = \textstyle\sum_{\pi \in \Pi_G} \Delta(\pi_s, \pi)\ \mathrm{Trace}\,\pi(f). \tag{8.8}$$

Here $(f, f^{(s)})$ is any pair of test functions related by the geometric transfer (8.6) and $\Delta(\pi_s, \pi)$ is given by (8.3), (8.5) (or by (8.9) below).

Suppose G has anisotropic center, so that we may apply the main transfer theorem of Adams–Johnson directly. We recast the geometric transfer of [AJ87, Section 2] as the correspondence $(f, f^{(s)})$ above; see [LS90, Theorem 2.6.A].

Also, because we must work with C_c^∞-functions, we have applied Bouaziz's Theorem as in [Sh12, Sections 1, 2]. From [AJ87, Theorem 2.21], we then have that the transfer (8.8) is true for some choice of the coefficients, say $\Delta'(\pi_s, \pi)$. With a little more effort we may show that our choice of $\Delta(\pi_s, \pi)$ is correct up to a constant, but we will not need that. Instead, we turn to the transfer (8.7) in the case of the discrete series pairs $(\widehat{\pi}_s, \widehat{\pi})$ from Sections 7.3 and 8.2.

For each pair (π_s, π), where $\pi = \pi(\eta^\dagger)$, we consider all pairs $(\widehat{\pi}_s, \widehat{\pi})$ such that $\eta_{\widehat{\pi}}$ serves as $\eta^\dagger$. From (8.4) and (8.5) we have that

$$\Delta(\pi_s, \pi) = \varepsilon_M(\pi).\Delta(\widehat{\pi}_s, \widehat{\pi}). \tag{8.9}$$

Now we choose $(f, f^{(s)})$ with support within the very regular elliptic set (see Section 3.4). We follow the comparison in [Ko90, Section 9] of the Vogan–Zuckerman character formula for π on the regular elliptic set with the Harish-Chandra formulas for the discrete series characters $\widehat{\pi}$ attached to π. From this we deduce that

$$\operatorname{Trace} \pi(f) = (-1)^{q(M_{\eta^\dagger})} \textstyle\sum_{\widehat{\pi}} \operatorname{Trace} \widehat{\pi}(f) \tag{8.10}$$

for our particular pairs $(f, f^{(s)})$. Multiply across (8.7) by $(-1)^{q(M^*)}$. From that identity, together with (8.9) and (8.10), we then have that

$$\textstyle\sum_{\pi \in \Pi_G} [\Delta(\pi_s, \pi) - \Delta'(\pi_s, \pi)] \operatorname{Trace} \pi(f) = 0$$

for all f supported in the strongly regular elliptic set. It now follows that the coefficients $\Delta(\pi_s, \pi) - \Delta'(\pi_s, \pi)$ are all zero; we could also argue this directly with the transfer of characters as functions. We conclude then that our choice of the constants $\Delta(\pi_s, \pi)$ in (8.8) is correct.

9 Parameters and twistpackets

We now return to the general twisted setting of Section 3.1 and finish the proof of various assertions made earlier.

9.1 Twistpackets

Attached to the triple (G^*, θ^*, a) is the automorphism ${}^L\theta_a$ of ${}^LG$. We are interested in Langlands parameters $\boldsymbol{\varphi}$ preserved by ${}^L\theta_a$, i.e., those $\boldsymbol{\varphi}$ for which

$$S_\varphi^{tw} := \{s \in G^\vee : {}^L\theta_a \circ \varphi = \operatorname{Int}(s) \circ \varphi\}$$

is nonempty, for some, and hence any, representative φ. Then we may construct supplemented endoscopic data for (G^*, θ^*, a) following the last paragraphs of [KS99, Chapter 2]; see Section 3.1.

Let (G, θ, η) be an inner form of (G^*, θ^*). It follows quickly from the Langlands classification, at least in the essentially tempered case, that the L-packet Π for $G(\mathbb{R})$ attached to φ is stable under the operation $\pi \to \varpi^{-1} \otimes (\pi \circ \theta)$. As in Section 4.1, we then say Π is (θ, ϖ)-stable. Conversely, the parameter for a (θ, ϖ)-stable packet is preserved by ${}^L\theta_a$. In general, this operation on a (θ, ϖ)-stable packet Π need have no fixed points, i.e., the twistpacket $\Pi^{\theta,\varpi}$ introduced in Section 4.1 may be empty. We examine this further for φ elliptic.

Suppose $\boldsymbol{\varphi}$ is elliptic and preserved by ${}^L\theta_a$. We use the standard representative $\varphi = \varphi(\mu, \lambda)$ from Section 5.6 and define data (μ_a, λ_a) for the cocycle a in the usual manner: $a(z) = z^{\mu_a}\overline{z}^{\sigma\mu_a}$ for $z \in \mathbb{C}^\times$, and $a(w_\sigma) = e^{2\pi i \lambda_a}$. First, we observe that because φ is regular, each element s of S_φ^{tw} must normalize $\mathcal{T}$. Then because $\theta^\vee$ preserves $\mathrm{spl}^\vee$, s lies in $\mathcal{T}$, so that $S_\varphi^{tw} \subseteq \mathcal{T}$. We conclude then that

$$\theta^\vee \mu = \mu + \mu_a \; and \; \theta^\vee \lambda \equiv \lambda + \lambda_a \bmod \mathcal{K}_f, \tag{9.1}$$

where

$$\mathcal{K}_f = X_*(\mathcal{T}) + [1 - \varphi(w_\sigma)]X_*(\mathcal{T}) \otimes \mathbb{C}.$$

Returning to Section 6.1, we now assume the chosen Whittaker data for G^* is θ^*-stable (see [KS99, Section 5.3]). We have a uniquely defined transport of $(\mu, \lambda, \mathcal{C})$ to character data for the generic discrete series representation π^* attached to φ. Then (9.1) implies that $\pi^* \circ \theta^* \approx \varpi \otimes \pi^*$, or we could argue this directly from Whittaker properties.

9.2 Nonempty fundamental twistpackets

Since the general fundamental case requires only a trivial modification, we continue with the elliptic setting of Section 9.1. We have attached a fundamental splitting spl_π to a discrete series representation π of $G(\mathbb{R})$ in Section 6.1. It is unique up to $G(\mathbb{R})$-conjugacy.

Lemma 9.1. *Suppose that π is a discrete series representation of $G(\mathbb{R})$ such that $\pi \circ \theta \approx \varpi \otimes \pi$. Then there exists $\delta_\pi \in G(\mathbb{R})$ such that $\mathrm{Int}(\delta_\pi) \circ \theta$ preserves spl_π. If spl_π is replaced by another fundamental splitting $\mathrm{Int}(x).\mathrm{spl}_\pi$, where $x \in G(\mathbb{R})$, then δ_π is replaced by an element δ'_π of the form $zx\delta_\pi\theta(x)^{-1}$, where $z \in Z_G(\mathbb{R})$.*

Proof. Since θ transports spl_π to a fundamental splitting for $\pi \circ \theta$ and we may use spl_π as splitting for $\varpi \otimes \pi$, the existence of δ_π is clear. Now, with spl_π fixed, δ_π may be replaced only by an element of $Z_G(\mathbb{R})\delta_\pi$. Next replace spl_π by $\mathrm{Int}(x).\mathrm{spl}_\pi$, where $x \in G(\mathbb{R})$. Then

$$\mathrm{Int}(x) \circ (\mathrm{Int}(\delta_\pi) \circ \theta) \circ \mathrm{Int}(x)^{-1} = \mathrm{Int}(x\delta_\pi\theta(x)^{-1}) \circ \theta$$

preserves $\mathrm{Int}(x).\mathrm{spl}_\pi$, and the lemma follows. □

Lemma 9.2. *If there exist nonempty twistpackets of discrete series (or fundamental series) representations of $G(\mathbb{R})$, then there is (θ_f, η_f) in the inner class of (θ, η) such that θ_f preserves a fundamental splitting for G. The converse is also true in the case that there exists an elliptic (fundamental) Langlands parameter preserved by ${}^L\theta_a$.*

Proof. A nonempty twistpacket provides us with a θ-fundamental element δ_π, and so Lemma 2.5 applies. For the converse, we may assume that θ is as in (ii) of Lemma 2.5. We then apply the remarks of Section 9.1 using the transport of data to G provided by the inner twist η. □

We see then that, as on the geometric side, to capture the elliptic (fundamental) contribution we may assume that, up to a twist by an element of $G(\mathbb{R})$, θ is the transport of θ^* to G by an inner twist η which also carries $\mathrm{spl}_{\mathrm{Wh}}$ to a fundamental splitting for G, i.e., that we are in the setting I of Section 3.2. We will need further information about the element δ_π of Lemma 9.1.

Lemma 9.3.

(i) $\delta_\pi \in G(\mathbb{R})$ has a norm γ_π in $H_1(\mathbb{R})$.
(ii) γ_π lies in $Z_{H_1}(\mathbb{R})$ and its image in $Z_H(\mathbb{R})$ under the projection $H_1 \to H$ is determined uniquely by δ_π.
(iii) If δ_π is replaced by $\delta'_\pi = zx\delta_\pi\theta(x)^{-1}$, where $z \in Z_G(\mathbb{R})$ and $x \in G(\mathbb{R})$, then γ_π is replaced by an element $\gamma'_\pi = z_1\gamma_\pi$, where $(z_1, z) \in C(\mathbb{R})$.

We will explain what we mean by (i) in the proof. The group $C(\mathbb{R})$ is from Section 5.1 of [KS99]; it was recalled in Section 4.2.

Proof. We may as well assume that we are in the setting I of Section 3.2 since the modifications for a further twist by an element of $G(\mathbb{R})$ are immediate. Suppose $\mathrm{Int}(x_\pi) \circ \eta$ carries spl_π to $\mathrm{spl}_{\mathrm{Wh}}$. Then a calculation shows that the element $\delta^*_\pi = x_\pi\eta(\delta_\pi)\theta^*(x_\pi)^{-1}$ has the property that $\mathrm{Int}(\delta^*_\pi) \circ \theta^*$ preserves $\mathrm{spl}_{\mathrm{Wh}}$. Because θ^* also preserves $\mathrm{spl}_{\mathrm{Wh}}$, we conclude that δ^*_π lies in Z_{G^*}. Further, we calculate that $\sigma(\delta^*_\pi)^{-1}.\delta^*_\pi \in (1 - \theta^*)T$. As in (5.1) of [KS99], we regard the coinvariants $(Z_{G^*})_{\theta^*}$ of Z_{G^*} as a subgroup of the coinvariants T_{θ^*} of T. Then under the projection $N : T \to T_{\theta^*}$, δ^*_π maps into $(Z_{G^*})_{\theta^*}$. Since $N(\sigma(\delta^*_\pi)^{-1}.\delta^*_\pi) = 1$, we have that $N(\delta^*_\pi) \in (Z_{G^*})_{\theta^*}(\mathbb{R})$. We identify $(Z_{G^*})_{\theta^*}(\mathbb{R})$ as a subgroup of $Z_H(\mathbb{R})$ and then as a subgroup of $T_H(\mathbb{R})$. Let γ_π be an element of $Z_{H_1}(\mathbb{R})$ whose image under $p : H_1 \to H$ coincides with the image of $N(\delta^*_\pi)$ in $Z_H(\mathbb{R})$. Then γ_π is a T_1-norm of δ_π in the sense of Section 6 of [Sh12]. In general, γ_π is determined up to stable conjugacy by δ_π [Sh12]. Since $\gamma_\pi \in Z_{H_1}(\mathbb{R})$, it is uniquely determined by δ_π. The rest is immediate. □

9.3 Elliptic related pairs of parameters

Let $\mathfrak{e}_z$ be a supplemented set of endoscopic data for (G^*, θ^*, a) as in Section 3.1. We may define *related pairs* of essentially tempered parameters (φ_1, φ) as in Section 2 of [Sh10] for the standard case. The arguments there, and accompanying definitions, apply word-for-word apart from the shift in notation to ϖ_1 for the character on the central subgroup $Z_1(\mathbb{R})$ of $H_1(\mathbb{R})$.

We return to the cuspidal-elliptic setting of Section 3.4 since the general fundamental case follows quickly from this. If an elliptic parameter φ_1 for the endoscopic group H_1 satisfies the stronger requirement of G-regularity [Sh10], then there is an elliptic parameter φ for G^* providing us with a related pair (φ_1, φ). Here it is assumed that φ_1 factors, in the sense of [Sh10, Section 2], through the group $\mathcal{H}$ included in the chosen SED.

We now recall explicit data attached to such pairs (φ_1, φ). To the $\theta^\vee$-stable Γ-splitting $\mathrm{spl}_{G^\vee} = (\mathcal{B}, \mathcal{T}, \{X\})$ of $G^\vee$ we attach a Γ-splitting $\mathrm{spl}^{\theta^\vee}_{G^\vee}$ for the identity component of $(G^\vee)^{\theta^\vee}$ in the standard manner (see, for example, p. 61 of [KS99]). We adjust the endoscopic datum $\mathfrak{e} = (H, \mathcal{H}, s)$ within its isomorphism class so that $s \in \mathcal{T}$, and then fix a Γ-splitting $\mathrm{spl}_{H^\vee} = (\mathcal{B}_{H^\vee}, \mathcal{T}_{H^\vee}, \{Y\})$ for $H^\vee$, where $\mathcal{B}_{H^\vee} = \mathcal{B} \cap H^\vee$ and $\mathcal{T}_{H^\vee} = \mathcal{T} \cap H^\vee = (\mathcal{T}^{\theta^\vee})^0$. Embed $H^\vee$ in $H_1^\vee$ and extend $\mathrm{spl}_{H^\vee}$ to $\mathrm{spl}_{H_1^\vee} = (\mathcal{B}_1, \mathcal{T}_1, \{Y\})$ by taking $\mathcal{B}_1 = \mathrm{Norm}(\mathcal{B}_{H^\vee}, H_1^\vee)$ and $\mathcal{T}_1 = \mathrm{Cent}(\mathcal{T}_{H^\vee}, H_1^\vee)$. None of these choices will matter for transfer factors. Nor will the choice of χ-data (this choice does matter for the construction of geometric Δ_{II} and Δ_{III}). We will thus define all Langlands data μ_1, λ_1, and so on, for packets in familiar terms [La89]; this amounts to the choice of χ-data such that $\chi_{(\alpha^\vee)_{\mathrm{res}}} = (z/\overline{z})^{1/2}$, where $(\alpha^\vee)_{\mathrm{res}}$ denotes the restriction to $(\mathcal{T}^{\theta^\vee})^0$ of a root $\alpha^\vee$ of $\mathcal{T}$ in $\mathcal{B}$.

We follow the approach of Section 11 of [Sh08a] for standard endoscopy. To $\mathrm{spl}_{H^\vee}$, we attach the representative $\varphi_1 = \varphi(\mu_1, \lambda_1)$ as in Section 5.6. Now consider an elliptic parameter $\varphi = \varphi(\mu, \lambda)$ for G^*. We alter the construction slightly. To fix an element of $G^\vee_{\mathrm{der}} \rtimes W_{\mathbb{R}}$ acting on $\mathcal{T} \cap G^\vee_{\mathrm{der}}$ as $t \rightarrow t^{-1}$, we may use either $n_G \times w_\sigma$ defined relative to $\mathrm{spl}_{G^\vee}$ or $n_{G,\theta} \times w_\sigma$ defined relative to $\mathrm{spl}^{\theta^\vee}_{G^\vee}$. It is more convenient to choose the latter. Thus $\varphi = \varphi(\mu, \lambda)$ will mean that

$$\varphi(w_\sigma) = e^{2\pi i \lambda}.n_{G,\theta} \times w_\sigma.$$

We may also drop the dominance requirement on μ. We do require that μ_1 be $\mathcal{B}_1$-dominant. While G-regularity of φ_1 requires that μ be regular when $\varphi(\mu_1, \lambda_1)$ and $\varphi(\mu, \lambda)$ are related, $\mathcal{B}_1$-dominance of μ_1 does not ensure that μ is $\mathcal{B}$-dominant. That case, however, is the only one that will matter to us (in general, an extra sign is needed in transfer factors, see Sections 7, 9 of [Sh10]). Thus we call φ_1 *well-positioned relative to* φ if μ is $\mathcal{B}$-dominant, and make that our assumption throughout. Given φ we can always find such φ_1 and it is unique up to $\mathcal{T}$-conjugacy. It is not difficult to check that this notion is independent of the choices made in its formulation; again see [Sh10].

Finally, we determine the conditions on (μ_1, λ_1) and (μ, λ) for $\varphi_1(\mu_1, \lambda_1)$ and $\varphi(\mu, \lambda)$ to be related. First, for $w \in W_{\mathbb{R}}$, pick $u(w) \in \mathcal{H}$ projecting to w, as follows. For $z \in \mathbb{C}^\times$, $u(z)$ is to act trivially and $u(zw_\sigma)$ is to act on $\mathcal{T}_H$ and $\mathcal{T}_1$ as $n_H \times w_\sigma \in {}^L H$. Since ξ_1 (part of the chosen SED) embeds $\mathcal{H}$ in ${}^L H_1$, we may define

$$\xi_1(u(zw_\sigma)) = t_{\xi_1}(zw_\sigma).n_H \times zw_\sigma$$

and

$$\xi_1(u(z)) = t_{\xi_1}(z) \times z,$$

where each $t_{\xi_1}(w)$ lies in $\mathcal{T}_1$. On the other hand, in ${}^LG$ we have that $u(w_\sigma)$ acts as $n_{G,\theta} \times z w_\sigma$. Write

$$u(w) = t(w) \cdot u'(w),$$

where

$$u'(z) = 1 \times z,\ u'(zw_\sigma) = n_{G,\theta} \times w_\sigma$$

for $z \in \mathbb{C}^\times$. Then $t(w) \in \mathcal{T}$. Let

$$\mathcal{T}_2 = (\mathcal{T}_1 \times \mathcal{T}) / \mathcal{T}_H,$$

where $\mathcal{T}_H$ is embedded by $t \to (t^{-1}, t)$. On $\mathcal{T}_2$ we use the elliptic action σ_2 of Γ inflated to $W_\mathbb{R}$: σ_2 acts as $n_H \times w_\sigma$ on the first component and as $n_{G,\theta} \times w_\sigma$ on the second. Let $t_2(w)$ denote the image in $\mathcal{T}_2$ of $(t_{\xi_1}(w)^{-1}, t(w)) \in \mathcal{T}_1 \times \mathcal{T}$. Then we define

$$(\mu^*, \lambda^*) \in (X_*(\mathcal{T}_2) \otimes \mathbb{C})^2$$

by

$$t_2(z) = z^{\mu^*}.\overline{z}^{\sigma_2 \mu^*} \times z$$

for $z \in \mathbb{C}^\times$, and

$$t_2(w_\sigma) = e^{2\pi i \lambda^*} \times w_\sigma.$$

Notice that we have constructed (μ^*, λ^*) independently of φ_1, φ. The cochain $t_2(w)$ is *not* the cocycle $a_T(w)$ of (4.4) in [KS99]; $a_T(w)$ requires a ρ-shift (ι-shift in our notation) to be applied to the datum μ^*. See Section 11 of [Sh08a] for the case of standard endoscopy, where the torus $\mathcal{T}_2$ collapses to $\mathcal{T}_1$.

Recall that $\varphi_1(W_\mathbb{R})$ is assumed to lie in $\xi_1(\mathcal{H})$. Identify μ_1, μ with their images in $X_*(\mathcal{T}_2) \otimes \mathbb{C}$ under the componentwise embeddings. We may write

$$\varphi_1(w) = t_H(w) \cdot \xi_1(u(w))$$

and

$$\varphi(w) = t_H(w) \cdot t(w) \cdot u'(w),$$

where $t_H(w) \in \mathcal{T}_H$. We now conclude that:

Lemma 9.4. *An elliptic pair $(\varphi_1(\mu_1, \lambda_1), \varphi(\mu, \lambda))$ as above is related if and only if*

$$\mu_1 + \mu^* = \mu \textit{ and } \lambda_1 + \lambda^* \equiv \lambda \bmod \mathcal{K}_f.$$

9.4 An application

We finish with a proof of the formula (4.4) from our discussion on properties required of spectral factors.

Lemma 9.5.

$$\varpi_C((z_1, z)) = \varpi_{\pi_1}(z_1).\varpi_\pi(z)^{-1},$$

for all $(z_1, z) \in C(\mathbb{R})$.

Proof. There is no harm in arguing in G^* since ϖ_{π_1}, ϖ_π may be calculated there. Thus we embed Z_{G^*} in fundamental T and write $z \in Z_{G^*}(\mathbb{R})$ in the form $z = \exp Y \cdot \exp i\pi\lambda^\vee$, where Y lies in the Lie algebra $\mathfrak{z}_{G^*}(\mathbb{R})$ viewed as a subspace of $X_*(T) \otimes \mathbb{C}$ and $\lambda^\vee \in X_*(T)$ is σ_T-invariant. Then it follows easily from the Langlands parametrization that

$$\varpi_\pi(z) = e^{<\mu, Y>} e^{2i\pi <\lambda, \lambda^\vee>}.$$

Here we have, as usual, identified $X^*(T) \otimes \mathbb{C}$ with $X_*(\mathcal{T}) \otimes \mathbb{C}$. Similarly, for $z_1 = \exp Y_1 \cdot \exp i\pi\lambda_1^\vee$, with $Y_1 \in \mathfrak{z}_{H_1}(\mathbb{R}) \subset X_*(T_1) \otimes \mathbb{C}$ and $\lambda_1^\vee \in X_*(T)^{\sigma_{T_1}}$, we have

$$\varpi_{\pi_1}(z_1) = e^{<\mu_1, Y_1>} e^{2i\pi <\lambda_1, \lambda_1^\vee>}.$$

Now identify the torus $\mathcal{T}_2$ from Section 9.2 as the dual of the torus T_2. Then if z, z_1 have the same image in $Z_H(\mathbb{R})$, i.e., if $(z_1, z) \in C(\mathbb{R})$, it follows from our remarks in Section 9.2 that

$$\varpi_{\pi_1}(z_1) \cdot \varpi_\pi(z)^{-1} = e^{- <\mu^*, Y_2>} e^{-2i\pi <\lambda^*, \lambda_2^\vee>},$$

where $Y_2 = (Y_1, Y)$ and $\lambda_2^\vee = (\lambda_1^\vee, \lambda^\vee)$. On the other hand, it is clear from the definitions of ϖ_C and (μ^*, λ^*), and from the relation of the cochain $t_2(w)$ to the cocycle $a_T(w)$ of p. 45 of [KS99] (see Section 9.2), that this last expression is the same as $\varpi_C((z_1, z))$. □

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On the Gelfand–Kirillov dimension of a discrete series representation

Nolan R. Wallach

To David Vogan with admiration

Abstract Lower bounds to the Gelfand–Kirillov dimension of discrete series are given for semisimple Lie groups with finite center by showing that the K-finite vectors are torsion free with respect to enveloping algebras of certain unipotent subgroups. In particular we prove two folk theorems about the Gelfand–Kirillov dimension. The first is that the holomorphic (or anti-holomorphic) discrete series are the "smallest" and representations with Whittaker models for minimal parabolic subgroups are the "largest" (a more precise result in the quasi-split case is due to Kostant). We also show that if G is quaternionic and not of type A or C, then the quaternionic discrete series is the "smallest".

Key words: Gelfand–Kirillov dimension, representation, Whittaker models

MSC (2010): Primary 22E30; Secondary 22E46

1 Introduction

In this short paper we give proofs of several folk theorems that we (and probably everyone else in the field) have believed for many years. One is that a representation of a real reductive group with a Whittaker model for a minimal parabolic must be of maximal Gelfand–Kirillov (GK) dimension (equal to the dimension of the unipotent radical of the parabolic subgroup). In the quasi-split case this is a theorem of Kostant [K], which uses results of Vogan [V]. The other is the "well-known" folk theorem: the holomorphic (or conjugate holomorphic) discrete series

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is the smallest (in the sense of GK-dimension) discrete series of a group that admits such representations. In addition, we will prove an analogous theorem for quaternionic real forms not of type C. That is, the GK-dimension of a discrete series is at least that of a quaternionic discrete series if the representation is not holomorphic or antiholomorphic. This implies that for quaternionic real forms, not of type A, the minimal GK-dimension of a discrete series is $1 + \frac{d}{2}$ with d the real dimension of the corresponding symmetric space.

The proofs that we give are, perhaps, more interesting than the results. They are based on a simple consequence, proved in [W1], of the vanishing theorem of Kostant [K] and Lynch [L] which guaranties that representations with "nice" generalized Whittaker models are torsion free with respect to the enveloping algebra of a corresponding nilpotent Lie algebra. Thus representations with a Whittaker model with respect to a minimal parabolic subgroup are torsion free with respect to the enveloping algebra of the unipotent radical of the parabolic. We also give a general conjecture.

This paper is a slight expansion of a manuscript written by the author in 2010 and then totally forgotten. No doubt the conjecture can be proved using the all-knowing ATLAS.

2 A class of representations of a nilpotent Lie algebra

We recall a result from [W1]. Let $\mathfrak{n}$ denote a nilpotent Lie algebra over $\mathbb{C}$ with universal enveloping algebra $U(\mathfrak{n})$. We consider $U(\mathfrak{n})^*$ as an $\mathfrak{n}$-module under the action $xf(n) = f(nx)$ for $f \in U(\mathfrak{n})^*$ and $n, x \in U(\mathfrak{n})$. We set

$$T(\mathfrak{n}) = \{f \in U(\mathfrak{n})^* \mid f(U(\mathfrak{n})\mathfrak{n}^k) = 0 \text{ for some } k\}.$$

Proposition 2.1. *Let M be an $\mathfrak{n}$-module such that*

$$H^1(\mathfrak{n}, M) = 0$$

and such that the elements of $\mathfrak{n}$ act locally nilpotently (that is, if $v \in M$ there exists k depending on v such that $\mathfrak{n}^k v = 0$). Then M is isomorphic with the tensor product module $T(\mathfrak{n}) \otimes V$ with $\mathfrak{n}$ acting trivially on V.

Corollary 2.2. *Let V be an $\mathfrak{n}$-module satisfying the two conditions:*

1. *The canonical pairing between V and $V_{\mathfrak{n}}^* = \{\lambda \in V^* \mid \mathfrak{n}^k\lambda = 0 \text{ for some } k\}$ is nondegenerate.*
2. *$H^1(\mathfrak{n}, V_{\mathfrak{n}}^*) = 0$.*

Then V is torsion free as a $U(\mathfrak{n})$-module.

Proof. The proposition above implies (see [W1]) that as an $\mathfrak{n}$-module

$$V_{\mathfrak{n}}^* \cong \lim_{k\to\infty} (U(\mathfrak{n})/\mathfrak{n}^k U(\mathfrak{n}))^* \otimes W$$

with $\mathfrak{n}$ acting trivially on W. The nondegeneracy of the pairing and the finite dimensionality of $U(\mathfrak{n})/\mathfrak{n}^k U(\mathfrak{n})$ imply that V is isomorphic with a submodule of

$$\lim_{\infty \longleftarrow k} U(\mathfrak{n})/\mathfrak{n}^k U(\mathfrak{n}) \otimes W^*$$

which is torsion free. □

3 Application to certain $(\mathfrak{g}, K)$-modules

Let G be a real reductive group and set $\mathfrak{g} = \mathrm{Lie}(G)$. Fix K to be a maximal compact subgroup. Let $P_o = M_o A_o N_o$ be a minimal parabolic subgroup with given Langlands decomposition. Let $\mathfrak{n}$ be a Lie subalgebra of $\mathfrak{n}_o = \mathrm{Lie}(N_o)$. Then we will say that $\mathfrak{n}$ is *nice* if there exists a parabolic subalgebra $\mathfrak{p}$ of $\mathfrak{g}_{\mathbb{C}}$ (here subscript $\mathbb{C}$ stands for complexification) such that $\mathfrak{n}_{\mathbb{C}}$ is the nilradical of $\mathfrak{p}$ and $\mathfrak{p}$ is nice in the sense of [BW] (that is, $\mathfrak{g}_{\mathbb{C}}$ has a $\mathbb{Z}$-grading as a Lie algebra, $\mathfrak{g}_{\mathbb{C}} = \oplus_{j \in \mathbb{Z}} \mathfrak{g}_j$, such that $\mathfrak{n}_{\mathbb{C}} = \oplus_{j>0} \mathfrak{g}_j$, $[\mathfrak{n}_{\mathbb{C}}, \mathfrak{n}_{\mathbb{C}}] = \oplus_{j>1} \mathfrak{g}_j$ and there is a Richardson element for $\mathfrak{p}$ in $\mathfrak{g}_1$). We note that this does not imply that $\mathfrak{n}$ is the nilradical of a parabolic subalgebra of $\mathfrak{g}$.

We will now explain an implication of this condition. Let $\lambda : \mathfrak{n}_{\mathbb{C}} \to \mathbb{C}$ be a homomorphism such that $\{X \in \mathfrak{p} \mid \lambda \circ \mathrm{ad} X = 0\}$ is of minimal dimension (i.e., λ is *generic*). We use the notation $\mathbb{C}_\lambda$ for the $\mathfrak{n}$-module $\mathbb{C}$ with $\mathfrak{n}$ acting by λ. If M is a $\mathfrak{g}$-module and λ is a homomorphism of $\mathfrak{n}_{\mathbb{C}}$ to $\mathbb{C}$, then we set

$$M_\lambda^* = \{f \in M^* \mid (X - \lambda(X))^k f = 0 \text{ for some } k \text{ and all } X \in \mathfrak{n}\}.$$

Then M_λ^* is a $\mathfrak{g}$-submodule of M^*. The condition of niceness implies that if λ is generic, then by a slight extension of the theorem in [L] we have

$$H^1(\mathfrak{n}, M_\lambda^* \otimes \mathbb{C}_{-\lambda}) = 0.$$

The result of the previous section implies:

Theorem 3.1. *Let M be an irreducible $(\mathfrak{g}, K)$-module. Let $\mathfrak{n}$ be a nice subalgebra of $\mathfrak{n}_o$ and let λ be a generic homomorphism of $\mathfrak{n}$ to $\mathbb{C}$. Assume that there exists $f \in M^*$ such that $f(Xv) = -\lambda(X) f(v)$ for $v \in M$, $X \in \mathfrak{n}$. Then M is torsion free as a $U(\mathfrak{n}_{\mathbb{C}})$-module.*

Proof. We note that the canonical pairing of M with M^* induces a nonsingular $\mathfrak{n}$-invariant pairing between $M \otimes \mathbb{C}_\lambda$ and $M_\lambda^* \otimes \mathbb{C}_{-\lambda}$. Thus Corollary 2.2 implies that $M \otimes \mathbb{C}_\lambda$ is torsion free as a $U(\mathfrak{n}_{\mathbb{C}})$-module. We now show that if V is torsion free as a $U(\mathfrak{n}_{\mathbb{C}})$-module and if μ is a homomorphism of $\mathfrak{n}$ to $\mathbb{C}$, then $V \otimes \mathbb{C}_\mu$ is torsion free as a $U(\mathfrak{n}_{\mathbb{C}})$-module. This will complete the proof of the theorem.

Obviously we may assume $\mu \neq 0$. We observe that $\mathfrak{n}_{\mathbb{C}} = \ker \mu \oplus \mathbb{C} X$ with $\mu(X) = 1$. Suppose that $n \in U(\mathfrak{n}_{\mathbb{C}})$ and $n(v \otimes 1) = 0$. Using the Poincaré–Birkhoff–Witt theorem we can write $n = \sum_{k=0}^m X^k n_k$ with $n_k \in U(\ker\ \mu)$. Thus

$n(v \otimes 1) = (\sum_{k,l} \binom{k}{l} X^{k-l} n_k v) \otimes 1$. So $(\sum_{k,l} \binom{k}{l} X^{k-l} n_k)v = 0$. We assume that $n \neq 0$. Hence we may assume that $n_m \neq 0$. The highest power of X appearing in $\sum_{k,l} \binom{k}{l} X^{k-l} n_k$ is X^m with coefficient n_m. Hence we see that $\sum \binom{k}{l} X^{k-l} n_k \neq 0$. Since V is torsion free, this implies $v = 0$. □

We will use the notation $\mathrm{Wh}_\lambda(M)$ for the space

$$\{f \in M^* \mid f(Xv) = -\lambda(X) f(v) \text{ for } v \in M, X \in \mathfrak{n}\}.$$

As usual, we define the Gelfand–Kirillov (GK) dimension of a finitely generated $U(\mathfrak{g})$-module, V, as follows: Let W be a finite-dimensional generating set of V. Then if $U^j(\mathfrak{g}) \subset U^{j+1}(\mathfrak{g})$ is the canonical filtration, the GK-dimension of V is

$$\lim_{j \to \infty} \frac{\log \dim(U^j(\mathfrak{g})W)}{\log j}.$$

Corollary 3.2. *Under the hypotheses of Theorem 3.1, the GK-dimension of M is at least* $\dim \mathfrak{n}$.

Proof. Since M is irreducible as a $(\mathfrak{g}, K)$-module, it is admissible and generated as a $U(\mathfrak{g})$-module by one K-type, $V(\gamma)$, $\gamma \in \hat{K}$. Let $v \in V(\gamma)$, $v \neq 0$. Then $\dim U^j(\mathfrak{g})V(\gamma) \geq \dim U^j(\mathfrak{n})v = \dim U^j(\mathfrak{n})$. The Poincaré–Birkhoff–Witt theorem implies that $\dim U^j(\mathfrak{n}) = \binom{d+j}{d}$ with $d = \dim \mathfrak{n}$. □

Corollary 3.3. *Under the hypothesis of Theorem 3.1, if $\mathfrak{n} = \mathfrak{n}_o$, then the GK-dimension of M is* $\dim \mathfrak{n}_o$ *(the maximal possible).*

Proof. The above result implies that the GK-dimension is at least $\dim \mathfrak{n}_o$. Since an admissible finitely generated $(\mathfrak{g}, K)$-module is finitely generated as a $U((\mathfrak{n}_o)_{\mathbb{C}})$-module, $\dim \mathfrak{n}_o$ is also an upper bound. □

We also have the following consequence (using Theorem 16 of [W2]).

Corollary 3.4. *Let G be a finite covering of the quaternionic real form of a simple Lie group not of type C over $\mathbb{C}$. Then if (π, H) is a quaternionic discrete series representation and if P is a parabolic subgroup of G with unipotent radical N of Heisenberg type (up to conjugation it is unique), then the underlying $(\mathfrak{g}, K)$-module H_K is torsion free as a $U(\mathrm{Lie}(N)_{\mathbb{C}})$-module, and hence the GK-dimension of H_K is at least* $\dim N = \frac{1}{2}\dim G/K + 1$.

Remark 3.5. Using the results in [GW] one can show that a quaternionic discrete series has GK-dimension equal to $\dim N$.

4 Nice abelian ideals in the Hermitian symmetric case

Let G be a connected simple Lie group over $\mathbb{R}$ with finite center, let K be a maximal compact subgroup, and let B denote the Killing form on $\mathfrak{g}_{\mathbb{C}} = \mathrm{Lie}(G)_{\mathbb{C}}$, normalized so that the square of the length of a long root is 2. We assume that G/K has a G-invariant complex structure. This occurs if and only if K has a one-dimensional center. We retain the notation of the previous section. Let $H \in \mathfrak{k}_{\mathbb{C}} = \mathrm{Lie}(K)_{\mathbb{C}}$ be such that $\mathfrak{k}_{\mathbb{C}} = \ker \mathrm{ad} H$ and $\mathrm{ad} H$ has eigenvalues 0 and ± 1. We set

$$V^{\pm} = \{X \in \mathfrak{g}_{\mathbb{C}} \mid [H, X] = \pm X\}.$$

Then $\mathfrak{k}_{\mathbb{C}} \oplus V^+$ is a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$. Let $\mathfrak{h}$ denote a Cartan subalgebra of $\mathfrak{k}_{\mathbb{C}}$ that is the complexification of a maximal abelian subspace of $\mathrm{Lie}(K)$.

We choose a system of positive roots Φ^+ for the root system Φ of $(\mathfrak{g}_{\mathbb{C}}, \mathfrak{h})$ such that if $\alpha(H) = 1$, then $\alpha \in \Phi^+$. We recall Harish-Chandra's construction of a maximal set of strongly orthogonal roots. We choose a linear order on the span over $\mathbb{R}$ of the roots compatible with Φ^+. Let γ_1 be the lowest element of Φ^+ such that $\alpha(H) = 1$. This is the unique simple root in $\Phi_0 = \{\alpha \in \Phi \mid \alpha(H) = 1\}$. We set $\Phi_1 = \{\alpha \in \Phi_0 - \{\gamma_1\} \mid \alpha \pm \gamma_1 \notin \Phi\}$. Assume that we have found $\gamma_1, \ldots, \gamma_j$ and Φ_{j+1}. If Φ_{j+1} is empty, then $\gamma_1, \ldots, \gamma_j$ is the desired ordered set. Otherwise, let γ_{j+1} be the lowest element of Φ_{j+1} and let $\Phi_{j+2} = \{\alpha \in \Phi_{j+1} - \{\gamma_{j+1}\} \mid \alpha \pm \gamma_{j+1} \notin \Phi\}$. The procedure terminates after ℓ steps and yields the ordered set of Harish-Chandra's strongly orthogonal roots. If $\lambda \in \mathfrak{h}^*$, then we define $H_\lambda \in \mathfrak{h}$ by $\lambda(h) = B(H_\lambda, h)$ for $h \in \mathfrak{h}$. We set $\mathfrak{h}^- = \sum_{i=1}^{\ell} \mathbb{C} H_{\gamma_i}$.

For each $\alpha \in \Phi_o$, we set $\mathfrak{s}_\alpha$ equal to the span of H_α and the α and $-\alpha$ root spaces. Then $\mathfrak{s}_\alpha$ is a Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$ that is isomorphic with $\mathrm{Lie}(\mathrm{SL}(2, \mathbb{C}))$. We set $\mathfrak{s}_\alpha^{\mathbb{R}} = \mathfrak{s}_\alpha \cap \mathfrak{g}$. Then $\mathfrak{s}_\alpha^{\mathbb{R}}$ is isomorphic with $\mathrm{Lie}(\mathrm{SL}(2, \mathbb{R}))$. Let S_α (resp. $S_\alpha^{\mathbb{R}}$) be the connected subgroup of the inner automorphisms $\mathrm{Int}(\mathfrak{g}_{\mathbb{C}})$ of $\mathfrak{g}_{\mathbb{C}}$ (resp. G) with Lie algebra $\mathfrak{s}_\alpha$ (resp. $\mathfrak{s}_\alpha^{\mathbb{R}}$). We set $\mathfrak{s}_i, \mathfrak{s}_i^{\mathbb{R}}, S_i, S_i^{\mathbb{R}}$ to be the objects defined as above that correspond to $\alpha = \gamma_i$. Let $\sigma_i : \mathfrak{s}_i \to \mathrm{Lie}(\mathrm{SL}(2, \mathbb{C}))$ be such that $H_{\gamma_i} \longmapsto H = \begin{bmatrix} 0 & i \\ -i & 0 \end{bmatrix}$, $\mathfrak{s}_i \cap V^+ \to \mathbb{C} \begin{bmatrix} 1 & -i \\ -i & -1 \end{bmatrix}$, and $\mathfrak{s}_i \cap V^- \to \mathbb{C} \begin{bmatrix} 1 & i \\ i & -1 \end{bmatrix}$. Let C_i be the element of $S_i \subset \mathrm{Aut}(\mathfrak{g}_{\mathbb{C}})$ corresponding to

$$c = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}.$$

Then

$$\mathrm{Ad}(c) H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

We set $h_i = \mathrm{Ad}(C_i) H_{\gamma_i}$ and $\mathfrak{a} = \sum_{i=1}^{\ell} \mathbb{R} h_i$. The general theory implies that $\mathfrak{a}$ is $\mathrm{Lie}(A)$ with A the center of a Levi factor for a minimal parabolic subgroup over $\mathbb{R}$. Set $C = C_1 \cdots C_\ell$. Let $h = \sum_{i=1}^{\ell} h_i$.

If $H = \frac{1}{2} \sum_{i=1}^{\ell} H_{\gamma_i}$, then G/K is said to be of *tube type*. The following two propositions are easily derived from the general theory of Hermitian symmetric spaces.

Proposition 4.1. *If G/K is of tube type and if $M = \{g \in G \mid \mathrm{Ad}(g) h = h\}$ and $\mathfrak{n} = \{X \in \mathfrak{g} \mid \mathrm{ad}(h) X = X\}$, then $\mathfrak{n}$ is abelian and* $\mathrm{Ad}(C) V^+ = \mathfrak{n}_{\mathbb{C}}$.

Proposition 4.2. *Assume that G/K is not of tube type. Then the eigenvalues of* ad$h$ *on* $\mathfrak{g}$ *are* $0, \pm 1, \pm 2$. *Let* $M = \{g \in G \mid \mathrm{Ad}(g)h = h\}$. *Let* $\mathfrak{n}_1$ *and* $\mathfrak{n}_2$ *denote respectively the eigenspaces for* ad$h$ *with eigenvalues* 1 *and* 2. *Furthermore,* $\mathfrak{n} = \mathrm{Ad}(C)V^+ \cap \mathfrak{n}_1 \oplus \mathfrak{n}_2$ *is a real form of* $\mathrm{Ad}(C)V^+$.

We can thus combine the above two propositions to prove the following.

Theorem 4.3. *If G is simple and G/K has a G-invariant complex structure and if $P_o = M_o A_o N_o$ is a minimal parabolic with given Langlands decomposition, then* $\mathrm{Lie}(N_o)$ *has a nice abelian ideal of dimension equal to*

$$\frac{1}{2}\dim_{\mathbb{R}} G/K = \dim_{\mathbb{C}} V^+.$$

Proof. We may take $\mathrm{Lie}(A_o) = \mathfrak{a}$ and take P_o to be a minimal parabolic subgroup containing $M \exp(\mathfrak{n}_2 \oplus \mathfrak{n}_1)$ (in the case of tube type $\mathfrak{n}_1 = 0$). Then $\mathrm{Ad}(C)(\mathfrak{k}_{\mathbb{C}} \oplus V^+)$ is a parabolic subalgebra whose nilradical is the complexification of the Lie algebra $\mathfrak{n}$ as in the propositions above. Since all parabolic subalgebras with commutative nilradical are nice in the sense of [BW], the theorem follows. □

5 Applications to the discrete series

In this section we consider G to be a real reductive group with compact center that admits an irreducible square integrable representation. Fix $P_o = M_o A_o N_o$ to be a minimal parabolic subgroup of G.

Theorem 5.1. *Let $\mathfrak{n}$ be an abelian subalgebra of* $\mathrm{Lie}(N_o)$ *and let N be the corresponding connected subgroup of N_o. Let (π, H) be a square integrable representation of G and let H^∞ be its space of C^∞ vectors. Then there exists an open non-empty set of characters χ of N such that*

$$\mathrm{Wh}_\chi(H^\infty) = \{f \in (H^\infty)' \mid f(\pi(n)\phi) = \chi(n)^{-1} f(\phi), n \in N, \phi \in H^\infty\} \neq 0.$$

(Here $(H^\infty)'$ is the continuous dual space of H^∞ endowed with the usual Fréchet topology.)

Proof. Let $v \in H_K$ and $\phi \in H^\infty$. Then the matrix coefficient defined by $c_{\phi,v}(g) = \langle \pi(g)\phi, v\rangle$ is an element of $\mathcal{C}(G)$ (see the appendix) and the map $T_v : H^\infty \to \mathcal{C}(G)$, $T_v(\phi) = c_{\phi,v}$ is a continuous map of Fréchet spaces. We fix $v \neq 0$. Since $\mathfrak{n}$ is abelian, the observations in the appendix imply that the integrals

$$F(\phi)(\chi) = \int_N \chi(n) c_{\phi,v}(n)\, dn$$

are absolutely convergent for $\chi \in \widehat{N}$. The Plancherel theorem for the Fourier transform implies that $F(\phi)$ defines a continuous function on $\widehat{N}$ that is nonzero if and only if $T_v(\phi) \neq 0$. Since π is irreducible there exists ϕ such that $T_v(\phi) \neq 0$, hence the set of χ such that there exists ϕ for which $F(\phi)(\chi) \neq 0$ must have non-empty interior. The theorem now follows from the results in the appendix, and the fact that $\phi \longmapsto F(\phi)(\chi)$ defines an element of $\mathrm{Wh}_\chi(H^\infty)$. □

Corollary 5.2. *If* $\mathfrak{n}$ *is a nice abelian subalgebra of* $\mathrm{Lie}(N_o)$ *and if* (π, H) *is an irreducible square integrable representation of* G*, then the action of* $U(\mathfrak{n}_{\mathbb{C}})$ *on the* K*-finite* C^∞ *vectors* H_K *is torsion free.*

Proof. Since $\mathfrak{n}$ is nice, the set of linear maps $\lambda : \mathfrak{n} \to i\mathbb{R}$ that are generic is open and dense in $i\mathfrak{n}^*$. Thus the set of differentials shown to exist in the previous theorem must have a non-empty intersection with these generic functionals. Now Corollary 2.2 implies this corollary. □

Corollary 5.3. *If* $\mathfrak{n}$ *is a nice abelian subalgebra of* $\mathrm{Lie}(N_o)$ *and if* (π, H) *is an irreducible square integrable representation of* G*, then* $GK\dim(H_K) \geq \dim \mathfrak{n}$.

Theorem 3.1 combined with this implies:

Corollary 5.4. *Assume that* G/K *has an invariant complex structure. Let* (π, H) *be an irreducible square integrable representation of* G*. Then*

$$GK\dim(H_K) \geq \frac{1}{2}\dim_{\mathbb{R}} G/K.$$

Conjecture 5.5. Let G be simple with finite center and let K be a maximal compact subgroup. If there exists an irreducible square integrable representation π of GK-dimension equal to $\frac{1}{2}\dim G/K$, then

1. G/K admits a G-invariant complex structure and
2. π is either holomorphic or antiholomorphic.

We will prove this conjecture for quaternionic real forms of type other than C.

Malcev [M] classified the maximal abelian Lie subalgebras of the simple Lie algebras over $\mathbb{C}$. The result has an interesting overlap with the theorems above. He showed that if the Lie algebra is not of type B_4, D_4 or G_2, then there is up to inner conjugacy exactly one such algebra (and hence of necessity, using the Borel fixed point theorem, an abelian ideal in a Borel subalgebra). For the case A_n, $n \geq 1$, the maximal dimension is $\left\lfloor \frac{(n+1)^2}{4} \right\rfloor$ (i.e., floor) which is half of the real dimension of the symmetric space $\mathrm{SU}(\lfloor \frac{n}{2} \rfloor, \lceil \frac{n}{2} \rceil)/\mathrm{S}(\mathrm{U}(\lfloor \frac{n}{2} \rfloor) \times \mathrm{U}(\lceil \frac{n}{2} \rceil))$. In the case C_n the maximal dimension is $\frac{n(n+1)}{2}$, which is exactly half of the real dimension of the associated Hermitian symmetric space. In the case of D_n ($n \geq 4$), the maximal dimension is $\frac{n(n-1)}{2}$, which is half of the real dimension of the symmetric space $\mathrm{SO}^*(2n)/\mathrm{U}(n)$ (this space is of tube type if and only if n is even). For E_6 and E_7 the dimensions are 16 and 27, which are also one half of the dimensions of the corresponding Hermitian symmetric space. Thus, in these cases we see that the conjugacy class of maximal abelian subalgebras contains an element defined over $\mathbb{R}$ for a (unique for all D_n) Hermitian symmetric real form. For B_n the only Hermitian real form is (locally) $SO(2n-1,2)$ and half of the real dimension is $2n-1$. If $n=3$ or 4, then Malcev gave the maximal dimension to be 5 and 7 respectively, which fits with the other Hermitian symmetric cases. However, if $n>4$, he gives $\frac{n(n-1)}{2}+1$. For the other simple Lie algebras, we record that for E_8, F_4 and G_2 Malcev gave dimensions respectively of 36, 9 and 3.

6 Quaternionic real forms

Let G be simple with finite center, not of type C, and equal to the quaternionic real form of its complexification. In this section we will show that if (π,H) is an irreducible square integrable representation of G that is not holomorphic or anti-holomorphic (this is a condition only in type A), then H has a generalized Whittaker model for the Heisenberg parabolic subgroup.

Let G be as above and let K be a maximal compact subgroup. Let $P=MAN$ be a parabolic subgroup of G such that N is of Heisenberg type (that is a Heisenberg parabolic). Since G is not of type C (see [W2]) there exists a group homomorphism ϕ of a group locally isomorphic with $SU(2,1)$ into G with the following properties.

- $d\phi$ is injective and if G_1 is the image of ϕ, then $G_1\cap K$ is a maximal compact subgroup of G_1.
- $G_1\cap P$ is a Heisenberg parabolic for G_1 and $N\cap G_1$ contains the center of N.
- $SU(2,1)$ contains a subgroup locally isomorphic with $SU(1,1)$ such that if L is its image under ϕ, then $L\cap N$ is the center of N and $L\cap K$ is maximal compact in L.

Our key result in this context is

Theorem 6.1. *Assume that G is quaternionic and not of type C. Let (π,H) be a square integrable representation of G such that if U is the unipotent radical of a proper parabolic subgroup of L and if $v,w\in H_K$ and $g\in G$, then $\int_U\langle\pi(gu)v,w\rangle\,du=0$ (du a choice of invariant measure). Then G must be of type A and (π,H) is a holomorphic or antiholomorphic discrete series for G.*

Proof. Let $v,w\in H_K$. Then the function $f(g)=c_{v,w}(g)=\langle\pi(g)v,w\rangle$ defines a right and left K-finite element of $\mathcal{C}(G)$. Thus $L_g f_{|L}$ is an element of $\mathcal{C}(L)$, where $L_g f(x)=f(g^{-1}x)$. Thus, Harish-Chandra's characterization of the span of $L\cap K$-finite linear combinations of matrix coefficients of discrete series (cf. [W4, 13.4.2 (2), p.241]) implies that, in particular, the function $f_{|L}$ is a finite linear combination of matrix coefficients of discrete series for L since f is right and left $K\cap L$-finite. Now $K\cap L$ is a one-dimensional compact torus, so we will identify it with $S^1=\{z\in\mathbb{C}\mid |z|=1\}$. We write $u(z)$ for the element of $K\cap L$ corresponding to z.We assume that $u(z)$ acts on v by z^{k_o}. Since G is quaternionic,

$\mathfrak{g} = \mathrm{Lie}(G)_{\mathbb{C}}$ decomposes as $\mathfrak{g}_{-2} \oplus \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2$ with $\mathrm{Ad}(u(z))$ acting as z^j on $\mathfrak{g}_j$. We note that the discrete series of L consists of holomorphic and antiholomorphic representations, that is, all characters of $L \cap K$ that occur are of the form $u(z) \mapsto z^k$ in the holomorphic case with $k < 0$ or $k > 0$ in the antiholomorphic case and there are infinitely many characters. Thus, if $k_o > 0$ (resp. $k_o < 0$), then $f_{|L}$ is in the span of matrix coefficients of antiholomorphic (resp. holomorphic) discrete series. Since we can replace our parametrization of $L \cap K$ with the inverses of the parameters, we may assume that $k_o < 0$. We see that if $g \in U(\mathfrak{g})$ is such that $\mathrm{Ad}(u(z))g = z^j g$, then all of the characters that occur in $U(\mathrm{Lie}(L)_{\mathbb{C}}) \cdot c_{gv,w|_L}$ (here L is acting by the right regular action) are bounded above by j. This implies that all of the weights of $L \cap K$ occurring in H_K are negative. Let $v \in H_K$ be of weight k_o (which is maximal). If $\mathfrak{u} = \mathfrak{g}_1 \oplus \mathfrak{g}_2$, then $\mathfrak{u}v = 0$. We note that $\mathfrak{g}_0 \subset \mathrm{Lie}(K)_{\mathbb{C}}$, hence $\dim U(\mathfrak{g}_0)v < \infty$. Let M be a nonzero irreducible $\mathfrak{g}_0$-submodule of $U(\mathfrak{g}_0)v$. Then we note that $\mathfrak{u}M = 0$, thus M is a $\mathfrak{q} = \mathfrak{g}_0 \oplus \mathfrak{u}$ module. By irreducibility we have a surjective $U(\mathfrak{g})$-module homomorphism

$$U(\mathfrak{g}) \otimes_{U(\mathfrak{q})} M \to H_K$$

via $g \otimes m \longmapsto gm$. This implies that H_K is a square integrable highest weight representation. So G/K is Hermitian symmetric and (π, H) is a holomorphic (or antiholomorphic) discrete series. □

Theorem 6.2. *Assume that G is quaternionic and not of type C. Let (π, H) be an irreducible representation of G and if G is of type A, then assume that (π, H) is not holomorphic or antiholomorphic. Then the set of unitary (one-dimensional) characters χ of N (the unipotent radical of a Heisenberg parabolic) such that* $\mathrm{Wh}_\chi(H^\infty) \neq 0$ *has non-empty interior.*

Proof. Let L be as in the proof of the previous theorem. The theorem implies that there exist $v \in H_K$ and $w \in H^\infty$ such that

$$\int_{L \cap N} \langle \pi(n)v, w \rangle \, dn \neq 0.$$

Fix such a w. We note that if $x \in N$, then Fubini's theorem implies that if $z \in H_K$, then

$$f_z(x) = \int_{L \cap N} \langle \pi(xn)z, w \rangle \, du$$

defines a function in $L^1(N/N \cap L)$. Since $N/N \cap L$ is abelian we can use the methods of the previous section to complete the proof. □

Corollary 6.3. *Conjecture 5.5 is true if G is quaternionic and not of type C.*

Appendix : the convergence of some integrals

In this section we will prove some convergence results used in this paper. Let G be a real reductive group with compact center and let K be a maximal compact subgroup and let $P_o = M_o A_o N_o$ be a minimal parabolic subalgebra. Let Ξ be the Harish-Chandra basic spherical function (cf. [W3, 4.5]). Recall that a norm on G is a continuous function, $\|...\|$ from G to $\mathbb{R}_{\geq 1}$ satisfying

1. $\|g\| = \|g^{-1}\|$,
2. $\|gh\| \leq \|g\| \, \|h\|$,
3. for each $r \geq 1$, $\{g \in G \mid \|g\| \leq r\}$ is compact.

In [W3, 7.2.1] we defined a norm such that $\|k_1 g k_2\| = \|g\|$, $k_1, k_2 \in K$ and proved that there exist positive constants C_1, C_2, q_1, q_2 such that

$$C_1 \|g\|^{-1} (1 + \log \|g\|)^{-q_1} \leq \Xi(g) \leq C_2 \|g\|^{-1} (1 + \log \|g\|)^{q_2}. \qquad (*)$$

Let $\mathcal{C}(G)$ denote Harish-Chandra's Schwartz space. That is the space of C^∞ functions f on G such that if $x, y \in U(\mathrm{Lie}(G))$ and $r > 0$, then

$$\sup_{g \in G} \frac{|R_x L_y f(g)|(1 + \log \|g\|)^r}{\Xi(g)} < \infty.$$

Here L and R are respectively the left and right regular representations of $U(\mathrm{Lie}(G))$. Let $p_{x,y,r}(f)$ denote the sup in the above inequality. Then following Harish-Chandra, $\mathcal{C}(G)$ is endowed with the topology defined by these semi-norms. Thus we have

$$|R_x L_y f(g)| \leq p_{x,y,r}(f) \Xi(g)(1 + \log \|g\|)^{-r}.$$

We note that $(*)$ implies that if we define for $f \in \mathcal{C}(G)$

$$q_{x,y.r}(f) = \sup_{g \in G} |R_x L_y f(g) \, \|g\| \, |(1 + \log \|g\|)^r,$$

then the semi-norms $q_{x,y,r}$ define the same (Fréchet) topology on $\mathcal{C}(G)$. The point of this appendix is to prove

Proposition A.1 *Let $\mathfrak{n}$ be a subalgebra of N_o and let N be the connected subgroup of N_o corresponding to $\mathfrak{n}$. Then if dn denotes a choice of Haar measure on N,*

$$q(f) = \int_N |f(n)| dn$$

defines a continuous semi-norm on $\mathcal{C}(G)$.

Proof. We note that if $r = \dim N_o/N$, then there exist $X_1, \ldots, X_r \in \mathrm{Lie}(N_o)$ such that $\sum_{i=1}^r \mathbb{R}X_i + \mathfrak{n} = \mathrm{Lie}(N_0)$ and if we set $\sum_{i \geq s}^r \mathbb{R}X_i + \mathfrak{n} = \mathfrak{n}_{s-1}$ (so $\mathfrak{n}_r = \mathfrak{n}$), then $[X_{i-1}, \mathfrak{n}_{i-1}] \subset \mathfrak{n}_{i-1}$. This implies that we have a diffeomorphism

$$\mathbb{R}^r \times N \to N_o, \quad (x, n) \mapsto \exp(x_1 X_1) \cdots \exp(x_r X_r) n.$$

Relative to this diffeomorphism $dx_1 \cdots dx_r dn$ is Haar measure, which we denote dn_o, on N_o. Harish-Chandra has shown that there exists $r > 0$ such that

$$\int_{N_o} \Xi(n_o)(1 + \log \|n_o\|)^{-r} dg < \infty.$$

Thus, in the notation above we have

$$\int_{N_o} \|n_o\|^{-1} (1 + \log \|n_o\|)^{-q_1 - r} dn_o < \infty.$$

Fubini's theorem implies that there exists $v = \exp(x_1 X_1) \cdots \exp(x_r X_r)$ such that

$$\int_{N} \|vn\|^{-1} (1 + \log \|vn\|)^{-q_1 - r} dn < \infty.$$

Using the properties of $\|\ldots\|$ above, we have

$$\|vn\| \le \|v\| \, \|n\| \, ;$$

thus since $\|g\| \ge 1$ for all $g \in G$,

$$1 + \log \|vn\| \le 1 + \log \|v\| + \log \|n\| \le (1 + \log \|v\|)(1 + \log \|n\|).$$

Putting this together we have that if $s \ge 0$,

$$\|v\| \, (1 + \log \|v\|)^s \, \|n\| \, (1 + \log \|n\|)^s \ge \|vn\| \, (1 + \log \|vn\|)^s.$$

We have

$$\|n\|^{-1} (1 + \log \|n\|)^{-s} \le \|v\| \, (1 + \log \|v\|)^s \, \|vn\|^{-1} (1 + \log \|vn\|)^{-s}.$$

This implies the proposition since

$$|f(n)| \le q_{1,1,s} \|n\|^{-1} (1 + \log \|n\|)^{-s}.$$

□

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A reducible characteristic variety in type A

Geordie Williamson

Dedicated to David Vogan on the occasion of his 60^{th} birthday

Abstract We show that simple highest weight modules for $\mathfrak{sl}_{12}(\mathbb{C})$ may have reducible characteristic variety. This answers a question of Borho–Brylinski and Joseph from 1984. The relevant singularity under Beilinson–Bernstein localization is the (in)famous Kashiwara–Saito singularity. We sketch the rather indirect route via the p-canonical basis, W-graphs and decomposition numbers for perverse sheaves that led us to examine this singularity.

Key words: Characteristic variety; characteristic cycle; irreducibility; Schubert varieties; Kazhdan–Lusztig cells.

MSC (2010): Primary 22E47; Secondary 55N33, 14F10, 14M15.

1 Introduction

Let $G \supset B \supset T$ denote respectively a complex reductive group, a Borel subgroup and maximal torus. Let W denote its Weyl group, $X = G/B$ the flag variety and T^*X its cotangent bundle. Given $x \in W$ we denote by C_x the corresponding Schubert cell and $T^*_x X \subset T^*X$ its conormal bundle. Let D_X denote the sheaf of algebraic differential operators on X and by $\mathcal{L}_y$ the IC extension of the trivial local system on C_y. We can write the characteristic cycle of $\mathcal{L}_y$ as

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$$CC(\mathcal{L}_y) = \sum_{x \in W} m_{x,y} [\overline{T_x^* X}].$$

We have $m_{x,y} \in \mathbb{Z}_{\geq 0}$ and $m_{x,y} = 0$ unless $x \leq y$ in the Bruhat order. The calculation of the multiplicities $m_{x,y}$ is an important and difficult problem. The question we address in this note is:

Question 1.1. (See [BB85, Conjecture 4.5] and [Jos84, §10.2]) Suppose that $G = \mathrm{SL}_n(\mathbb{C})$. Is $m_{x,y} = 0$ if $x \neq y$ and x and y lie in the same two-sided Kazhdan–Lusztig cell?

This question is equivalent to asking whether the characteristic variety of a simple highest weight module for $\mathfrak{sl}_n(\mathbb{C})$ is irreducible [BB85, Proposition 6.9]. (A sketch: if $\pi : T^*(G/B) \to \mathfrak{sl}_n(\mathbb{C})^*$ denotes the moment map, then the characteristic variety of the global sections of $\mathcal{L}_y$ (a simple highest weight module) agrees with the image of the characteristic variety of $\mathcal{L}_y$ under π [BB85, Corollary 1.5]. The condition on two-sided cells occurs because if $x <_{LR} y$ ($\leq_{LR}$ denotes the Kazhdan–Lusztig two-sided cell preorder), then $\pi(\overline{T_x^* G/B})$ has strictly smaller dimension than $\pi(\overline{T_y^* G/B})$ and hence cannot contribute a reducible component, because characteristic varieties of simple modules are equidimensional [Gab82].) It is known that reducible characteristic varieties occur in other types (e.g., B_2, B_3, C_3) thanks to calculations of Kashiwara and Tanisaki [KT84] and Tanisaki [Tan88].

Kazhdan and Lusztig conjectured (still for $G = \mathrm{SL}_n(\mathbb{C})$) that the characteristic varieties of all $\mathcal{L}_y$ are irreducible [KL80a] (that is, that $m_{x,y} = 0$ if $x \neq y$). Of course this would imply an affirmative answer to the above question. However Kashiwara and Saito [KS97] showed that their conjecture was true if $n < 8$ but false for $n \geq 8$. They discovered a singularity (the *Kashiwara–Saito singularity*) which occurs as a normal slice to a Schubert variety in the flag variety of $\mathrm{SL}_8(\mathbb{C})$, and for which the characteristic variety is reducible. In their example, x and y do not lie in the same two-sided cell, and hence do not provide an example of a reducible characteristic variety of a highest weight module.

In this note we give two permutations $x \leq y$ in S_{12} which lie in the same right cell and such that a normal slice to the Schubert variety corresponding to y along the Schubert cell corresponding to x is isomorphic to the Kashiwara–Saito singularity. This implies that $m_{x,y} \neq 0$, and hence that Question 1.1 has a negative answer.

1.1 Structure of the paper

In §2 we discuss the p-canonical basis and prove a result relating characteristic cycle multiplicities and the p-canonical basis. This result is a simple consequence of an observation of Vilonen and the author [VW12]. We then discuss how positivity properties of the p-canonical basis and computer code of Howlett–Nguyen allows one to narrow the search for potential counterexamples. (Indeed, with 12! = 479 001 600

Schubert varieties in the flag variety of GL_{12}, the challenge is in the finding rather than in the verifying!) In §3 we give the singularity in the GL_{12} flag variety and perform a straightforward calculation to obtain the Kashiwara–Saito singularity.

1.2 Comments on the literature

In [Mel93] a proof is proposed for the irreducibility of characteristic varieties in type A. As we have already remarked, this would imply that Question 1.1 has a positive answer. The faulty step in [Mel93] occurs in the proof of [Mel93, Proposition 3.2] where it is tacitly assumed that [Jos84, 9.12] extends to characteristic varieties; this is false, as our example in §3.5 shows. A statement equivalent to Melnikov's claim is made in the remark on page 54 of [BB85].

2 Motivation from modular representation theory

In this section we sketch the route which led us to consider the singularity in §3.5. We have tried to provide enough details and references that a motivated reader could adapt these techniques to find other interesting (counter)examples. Most of the ideas are already contained in [Wil12], which has more detail than the discussion below.

2.1 The *p*-canonical basis

Let G, B, T be as in the introduction. Let W denote the Weyl group, S its simple reflections, $\leq$ its Bruhat order and ℓ its length function. Consider the flag variety G/B with its stratification by B-orbits (the Schubert stratification):

$$G/B = \bigsqcup_{w \in W} C_w.$$

Fix a field $\Bbbk$ of characteristic $p \geq 0$ and let $D^b_{(B)}(G/B; \Bbbk)$ denote the bounded derived category of constructible sheaves on G/B which are constructible with respect to the Schubert stratification. For $w \in W$ denote by $\mathbf{IC}(w; \Bbbk)$ the intersection cohomology sheaf and $\mathcal{E}(w; \Bbbk)$ the parity sheaf (for the constant pariversity) [JMW09, Wil12] corresponding to $\overline{C_w}$. We will drop the $\Bbbk$ from the notation if it is clear from the context. If $\Bbbk$ is of characteristic 0, then $\mathcal{E}(w; \Bbbk) = \mathbf{IC}(w; \Bbbk)$.

Let $\mathcal{H}$ denote the Hecke algebra of (W, S). It is a free $\mathbb{Z}[v^{\pm 1}]$-module with basis $\{H_w \mid w \in W\}$ and multiplication determined by

$$H_s H_w = \begin{cases} H_{sw} & \text{if } \ell(sw) > \ell(w), \\ (v^{-1} - v)H_w + H_{sw} & \text{if } \ell(sw) < \ell(w). \end{cases}$$

Let $\{\underline{H}_w\}$ denote the Kazhdan–Lusztig or "canonical" basis of $\mathcal{H}$. We use the normalizations of [Soe97]. For example $\underline{H}_s = H_s + vH_{\text{id}}$.

Given a finite-dimensional $\mathbb{Z}$-graded vector space $V = \bigoplus V^i$, let

$$\text{ch}V = \sum_{i \in \mathbb{Z}} \dim V^{-i} v^i \in \mathbb{Z}[v^{\pm 1}]$$

denote its Poincaré polynomial. Given $\mathcal{F} \in D^b_{(B)}(G/B; \Bbbk)$ define

$$\text{ch}\,\mathcal{F} = \sum_{x \in W} \text{ch}\, H^*(\mathcal{F}_x) v^{-\ell(x)} H_x \in \mathcal{H}$$

where $\mathcal{F}_x$ denotes the stalk of $\mathcal{F}$ at the point $xB/B \in C_x \subset G/B$. It is a classical theorem of Kazhdan and Lusztig [KL80b] (see also [Spr82]) that if $\Bbbk$ is of characteristic zero, then

$$\text{ch}\,\mathbf{IC}(w; \Bbbk) = \underline{H}_w. \tag{2.1}$$

For any $w \in W$ we define

$$^p\underline{H}_w := \text{ch}\,\mathcal{E}(w; \Bbbk).$$

(One can show that $^p\underline{H}_w$ only depends on the characteristic p of $\Bbbk$, which explains the notation.) We call the $\{^p\underline{H}_w\}$ the *p-canonical basis* for reasons which the following proposition should make clear:

Proposition 2.1.

(i) $^p\underline{H}_w = H_w + \sum_{x<w} {}^p h_{x,w} H_x$ *with* $^p h_{x,w} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$ *(hence* $\{^p\underline{H}_w \mid w \in W\}$ *is a basis),*

(ii) $^p\underline{H}_w = \sum {}^p m_{x,w} \underline{H}_x$ *for self-dual* $^p m_{x,w} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$,

(iii) if $^p m_{x,w}$ *are as in (ii), then* $^p m_{x,w} = 0$ *unless* $\mathcal{L}(x) \supset \mathcal{L}(w)$ *and* $\mathcal{R}(x) \supset \mathcal{R}(w)$ *where* $\mathcal{L}$ *and* $\mathcal{R}$ *denote left and right descent sets,*

(iv) $^p\underline{H}_x {}^p\underline{H}_y = \sum {}^p\mu^z_{xy} {}^p\underline{H}_z$ *for self-dual* $^p\mu^z_{x,y} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}]$,

(v) for $p \gg 0$, $^p\underline{H}_w = {}^0\underline{H}_w = \underline{H}_w$.

Proof (Sketch of proof). By definition, the parity sheaf $\mathcal{E}(w)$ is supported on $\overline{C_w}$ and its restriction to C_w is isomorphic to a shifted constant sheaf. (i) now follows easily from the definition of ch.

Each $\mathcal{E}(w; \mathbb{F}_p)$ admits a lift $\mathcal{E}(w; \mathbb{Z}_p)$, a parity sheaf with coefficients in $\mathbb{Z}_p$. Then $\mathcal{E}(w, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ is a parity sheaf with coefficients in $\mathbb{Q}_p$, and is hence isomorphic to a direct sum of intersection cohomology complexes. (ii) now follows from (2.1) and the fact that $\mathcal{E}(w; \mathbb{F}_p)$, $\mathcal{E}(w, \mathbb{Z}_p)$ and $\mathcal{E}(w, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ all have the same character (see [Wil12, Theorem 3.10]).

For fixed w, the parity sheaf $\mathcal{E}(w; \mathbb{F}_p)$ may be obtained via pull-back from the partial flag variety G/P where $P \supset B$ is the parabolic subgroup determined by

$\mathcal{R}(w) \subset S$ (see [JMW09, Proposition 4.10]). Hence $\mathcal{R}(x) \supset \mathcal{R}(w)$ as claimed. The statement for left descent sets follows because ${}^p m_{x,w} = {}^p m_{x^{-1},w^{-1}}$ by [Wil12, §3 eq. (4)].

Each parity sheaf admits a lift to the B-equivariant derived category $D^b_B(G/B, \Bbbk)$ where there is a convolution formalism categorifying the multiplication in the Hecke algebra. (iv) then follows because the convolution of two parity sheaves is isomorphic to a direct sum of shifts of parity sheaves [JMW09, Theorem 4.8].

Finally (v) follows from (2.1) and [JMW09, Proposition 2.41] which asserts that $\mathcal{E}(w; \mathbb{F}_p) = \mathbf{IC}(w; \mathbb{F}_p)$ for all but finitely many primes p. □

Warning 2.2. The p-canonical basis depends on the root system of G, not just on its Weyl group. (For example the 2-canonical basis differs in types B_3 and C_3.) Hence one should think about the p-canonical basis as a basis of the Hecke algebra attached to a root system or Cartan matrix rather than a Coxeter system. Kashiwara and Saito observed the same phenomenon for characteristic cycles [KT84, Example 5.4].

2.2 The p-canonical basis and decomposition numbers

We briefly recall the notion of a decomposition number for perverse sheaves. An excellent reference is [Jut09].

Let X denote a complex variety, $Z \subset X$ a locally closed smooth subvariety, and $\mathcal{L}$ a local system of free $\mathbb{Z}$-modules on X. One may consider the intersection cohomology extension[1] $\mathbf{IC}(\overline{Z}; \mathcal{L})$. It is a perverse sheaf with $\mathbb{Z}$-coefficients on X. One has

$$\mathbf{IC}(\overline{Z}; \mathcal{L}) \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbf{IC}(\overline{Z}; \mathcal{L} \otimes_{\mathbb{Z}} \mathbb{Q}),$$

and so $\mathbf{IC}(\overline{Z}; \mathcal{L})$ can be thought of as a $\mathbb{Z}$-form of $\mathbf{IC}(\overline{Z}; \mathcal{L} \otimes \mathbb{Q})$. In general,

$$\mathbf{IC}(\overline{Z}; \mathcal{L}) \otimes^L_{\mathbb{Z}} \mathbb{F}_p \in D^b_c(X; \mathbb{F}_p)$$

is perverse but no longer simple. The decomposition matrix encodes the Jordan–Hölder multiplicities of the simple perverse sheaves occurring in $\mathbf{IC}(\overline{Z}; \mathcal{L}) \otimes^L_{\mathbb{Z}} \mathbb{F}_p$.

In this paper we will be concerned with the flag variety together with its Schubert stratification, as in §2.1. In this case all the strata are simply connected and the decomposition matrix takes the form $(d_{y,x})_{y,x \in W}$ where

$$d_{y,x} := [\mathbf{IC}(\overline{C_y}; \mathbb{Z}) \otimes_{\mathbb{Z}} \mathbb{F}_p : \mathbf{IC}(\overline{C_x}; \mathbb{F}_p)].$$

The relation between the characters of the parity sheaves (i.e., the p-canonical basis) and the decomposition matrix is subtle. For example, recent papers of Achar

[1] For the perversity p, see [Jut09].

and Riche [AR14a, AR14b] prove that knowledge of the p-canonical basis gives (a q-refinement of) the decomposition matrix for perverse sheaves on the Langlands dual flag variety.

Here we will be concerned with a much more limited but simpler relationship. Roughly it says that the first time the p-canonical basis differs from the canonical basis corresponds to the first nontrivial decomposition number (see Proposition 2.1 and above for notation):

Proposition 2.3. *Fix $y \in W$ and suppose that $x < y$ is maximal in the Bruhat order such that ${}^p m_{x,y} \neq 0$. If ${}^p m_{x,y} \in \mathbb{Z}$, then $d_{y,x} = {}^p m_{x,y}$.*

Proof. Fix x and y as in the proposition. Set

$$X = \bigsqcup_{z \geq x} BzB/B, \qquad Z = BxB \subset X, \qquad U = X \setminus U$$

and denote by i (resp. j) the closed (resp. open) embedding of Z (resp. U) into X. Note that X is open in G/B.

For $\Bbbk \in \{\mathbb{F}_p, \mathbb{Z}_p, \mathbb{Q}_p\}$ let $\mathbf{IC}_\Bbbk$ (resp. $\mathcal{E}_\Bbbk$) denote the intersection cohomology (resp. parity) sheaf corresponding to the stratum $ByB/B \subset X$. We have $\mathcal{E}_{\mathbb{Q}_p} \cong \mathbf{IC}_{\mathbb{Q}_p}$ and our assumptions guarantee that $\mathcal{E}$ is perverse with

$$\mathbf{IC}_{\Bbbk|U} \cong \mathcal{E}_{\Bbbk|U}.$$

Hence we need to examine the difference between $\mathbf{IC}_{\mathbb{F}_p}$ and $\mathcal{E}_{\mathbb{F}_p}$ over the closed stratum Z.

Our main tool will be [JMW09, Lemma 2.18] which gives a bijection between isomorphism classes of extensions of a fixed $\mathcal{F}$ on U to X, and isomorphism classes of distinguished triangles on Z of the form

$$A \to i^* j_* \mathcal{F} \to B \overset{[1]}{\to} .$$

If $\mathcal{F}'$ is such an extension, then A and B are given by

$$i^* \mathcal{F}' \cong A \quad \text{and} \quad i^! \mathcal{F}' \cong B[-1]. \tag{2.2}$$

Let us examine the triangle corresponding to the extension $\mathcal{E}_{\mathbb{Z}_p}$ of $\mathbf{IC}_{\mathbb{Z}_p|U}$. It has the form

$$A \to i^* j_* (\mathbf{IC}_{\mathbb{Z}_p|U}) \to B \overset{[1]}{\to} . \tag{2.3}$$

Because Z is contractible we can view (2.3) as a distinguished triangle of $\mathbb{Z}_p$-modules. By (2.2) and the fact that $\mathcal{E}$ is a parity sheaf, we deduce:

(1) $H^m(A)$ and $H^m(B)$ are free $\mathbb{Z}_p$-modules;
(2) $H^m(A) = 0$ if $m - \ell(y)$ is odd, and $H^m(B) = 0$ if $m - \ell(y)$ is even.

The assumptions of the proposition and (2.2) guarantee that

(3) $H^m(A)$ vanishes for $m > -\ell(x)$ and $H^m(B)$ vanishes for $m < -\ell(x) - 1$;
(4) $H^{-\ell(x)}(A)$ is free of rank ${}^p m_{x,y}$ (in particular $\ell(y) - \ell(x)$ is even).

Because $\mathbb{Z}_p$ is hereditary, each of the terms in (2.3) is isomorphic to its cohomology. Hence we can turn the triangle and rewrite it as

$$H^*(B)[-1] \to H^*(A) \to H^*(i^* j_* \mathcal{F}) \xrightarrow{[1]} .$$

By (3) above the only nonzero map component of the first map is

$$\alpha : H^{-\ell(x)-1}(B) \to H^{-\ell(x)}(A).$$

Because $\mathcal{E}$ is indecomposable, α does not map any summand of $H^{-\ell(x)-1}(B)$ isomorphically onto a summand of $H^{-\ell(x)}(A)$ by [JMW09, Lemma 2.21]. In other words, $\alpha \otimes_{\mathbb{Z}_p} \mathbb{F}_p = 0$. On the other hand, we have

$$\mathcal{E} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p \cong \mathbf{IC}_{\mathbb{Q}_p} \oplus \mathbf{IC}(Z)^{\oplus({}^p m_{x,y})},$$

and hence α is an isomorphism over $\mathbb{Q}_p$. In other words, $\ker \alpha = 0$ and the domain and codomain of α are free of the same rank.

By the long exact sequence of cohomology we deduce that:

$$H^m(i^* j_* \mathbf{IC}_{\mathbb{F}_p|U}) = \begin{cases} H^m(A) \otimes \mathbb{F}_p & \text{if } m < \ell(x) - 1, \\ H^{-\ell(x)}(A) \otimes \mathbb{F}_p & \text{if } m = -\ell(x) - 1 \text{ or } m = -\ell(x), \\ H^m(B) \otimes \mathbb{F}_p & \text{if } m > -\ell(x). \end{cases}$$

$$H^m(i^* j_* \mathbf{IC}_{\mathbb{Q}_p|U}) = \begin{cases} H^m(A) \otimes \mathbb{F}_p & \text{if } m < \ell(x) - 1, \\ 0 & \text{if } m = -\ell(x) - 1 \text{ or } m = -\ell(x), \\ H^m(B) \otimes \mathbb{F}_p & \text{if } m > -\ell(x). \end{cases}$$

By the Deligne construction [BBD82, Proposition 2.1.11], we have

$$i^* \mathbf{IC}_{\Bbbk} = i^* \tau_{<-\ell(x)} j_* \mathbf{IC}_{\Bbbk} = \tau_{<-\ell(x)} i^* j_* \mathbf{IC}_{\Bbbk}$$

where $\tau_{<m}$ denotes truncation. Hence if χ denotes the Euler characteristic at any point in Z, we have

$$\chi(\mathbf{IC}_{\mathbb{F}_p}) = \chi(\mathbf{IC}_{\mathbb{Q}_p}) - (-1)^{-\ell(x)}({}^p m_{x,y}).$$

Now we are done: if we write

$$[\mathbf{IC}_{\mathbb{Z}_p} \otimes^L_{\mathbb{Z}_p} \mathbb{F}_p] = [\mathbf{IC}_{\mathbb{F}_p}] + a[\mathbf{IC}(Z)]$$

in the Grothendieck group of $\mathbb{F}_p$-perverse sheaves on X, then taking Euler characteristics over Z yields $a = {}^p m_{x,y}$ as claimed. □

As in the introduction we write $m_{x,y}$ for the the characteristic cycle multiplicities. The following is an immediate consequence of the previous proposition, and [VW12, Theorem 2.1].

Corollary 2.4. *Suppose that $x < y$ are as in the previous proposition. Then*

$$m_{x,y} \geq {}^p m_{x,y}.$$

2.3 Searching for a counterexample

Consider the following variant of Question 1.1 (with notation as in Proposition 2.1):

Question 2.5. Suppose that $G = \mathrm{SL}_n(\mathbb{C})$ and let p be a prime. Is ${}^p m_{x,y} = 0$ if $x \neq y$ and x and y lie in the same two-sided cell?

It will become clear below that a positive answer to Question 1.1 implies a positive answer to Question 2.5. Question 2.5 is also important for modular representation theory, with connections to Lusztig's conjecture around the Steinberg weight [Soe00], amongst other things.

One can show (using Soergel calculus [EW13] or Schubert calculus [HW14]) that the counterexample in §3 also gives a counterexample to Question 2.5. We found the examples by pursuing a naive idea, which is the main theme of [Wil12]: the p-canonical basis has remarkable positivity properties (summarized in Proposition 2.1) and these positivity properties are enough to rule out many potential counterexamples.

Asume that W is an arbitrary Weyl group. For any left cell $C \subset W$ we can consider the corresponding cell module

$$M_C = \bigoplus_{x \in C} \mathbb{Z}[v^{\pm 1}] M_x := \bigoplus_{x \leq_L C} \mathbb{Z}[v^{\pm 1}] \underline{H}_x \Big/ \Big(\bigoplus_{x <_L C} \mathbb{Z}[v^{\pm 1}] \underline{H}_x \Big).$$

The $\mathcal{H}$-module structure in the basis $\{M_x\}$ is encoded in the W-graph of C. Fix a prime p and assume that the p-canonical basis satisfies

$$\text{for all } y \in C \text{ if } {}^p m_{x,y} \neq 0, \text{ then } x \leq_L y. \tag{2.4}$$

Then we may define ${}^p M_y$ as the image of ${}^p \underline{H}_y$ in M_C and obtain in this way a p-canonical basis for the cell module M_C. By Proposition 2.1 it satisfies the following properties:

(1) (positive upper-triangularity) we have

$${}^p M_y = M_y + \sum_{C \ni x < y} {}^p m_{x,y} M_x \text{ with } {}^p m_{x,y} \in \mathbb{Z}_{\geq 0}[v^{\pm 1}] \text{ self-dual;}$$

(2) (positive structure constants) for any $x \in W$,

$${}^p \underline{H}_x \cdot {}^p M_y \in \bigoplus_{z \in C} \mathbb{Z}_{\geq 0}[v^{\pm 1}]({}^p M_z).$$

Example 2.6. Suppose that W is of type B_2 with simple reflections s, t. Consider the left cell $C = \{s, ts, sts\}$. The W-graph is

$$\{s\} \text{———} \{t\} \text{———} \{s\}.$$

In this case there are two possible bases for M_C satisfying (1) and (2). The first is the Kazhdan–Lusztig basis $\{M_x\}$. The second is the basis $\{M'_x\}$ with $M'_x = M_x$ for $x \in \{s, ts\}$ and $M'_{sts} := M_{sts} + M_s$. In this case M' agrees with the image of the 2-canonical basis for B_2 (for an appropriate choice of long and short root).

Now assume that W is of type A_{n-1}. In this case two-sided cells are parametrized by partitions λ of n. Also, all left cells in a fixed two-sided cell have isomorphic W-graphs and hence afford isomorphic (based) representations of the Hecke algebra $\mathcal{H}$.

Lemma 2.7. *Let λ be a partition of n and $E_\lambda \subset W$ the corresponding two-sided cell. Then there exists a left cell $C \subset E_\lambda$ satisfying (2.4).*

Proof (Sketch of proof). Let w_λ denote the longest element of the standard parabolic subgroup $W_\lambda \subset W$ determined by λ. Then $w_\lambda \in E_\lambda$. We claim that the left cell C containing w_λ satisfies (2.4). First, ${}^p\underline{H}_{w_\lambda} = \underline{H}_{w_\lambda}$ by (i) and (iii) of Proposition 2.1 and a simple induction then shows that

$$ {}^p\underline{H}_y = \sum_{x \le_L w_\lambda} {}^p m_{x,y} \underline{H}_x. $$

for all $y \in C$. Hence (2.4) holds. □

It follows that any left cell representation in type A admits a p-canonical basis satisfying the above positive conditions. One can apply computer searches in order to isolate potential counterexamples and then use Soergel calculus [EW13] or Schubert calculus [HW14] to check whether one has indeed found a counterexample.

In order to implement this approach one needs the W-graphs of the left cell representations in type A. These are provided by the wonderful code of Howlett and Nguyen [HN13] for magma [BCP97].

Remark 2.8.

1. Using the recent results of Achar–Riche [AR14a, AR14b] one can show that if there is a counterexample for a left cell corresponding to λ, then there is also a counterexample for the left cell corresponding to the transposed partition λ^t. This allows one to roughly halve the number of left cells which one needs to consider. Experimentally, the above positivity properties are more restrictive in left cells corresponding to partitions "near the top" of the dominance order. (For example for S_4 there is only one solution for the left cell corresponding to the partition $(3, 1)$, whereas there are two for the partition $(2, 1, 1)$.)

2. Lusztig has given a beautiful description of the J-ring for a fixed two-sided cell in S_n as a (based) matrix ring. Using this result, one can show that if the p-canonical basis is trivial (i.e., equal to the image of the Kazhdan–Lusztig basis) in a fixed left cell, then Question 2.5 has a positive answer for that two-sided cell.
3. The above methods yielded another counterexample to Question 2.5, this time in GL_{13}:

$$x = 12132156543765438798765ba98c,$$
$$y = 121321546543765438798765aba9876cba98$$

Here we write x and y as words in the simple transpositions $1, \dots 9, a, b, c$ of S_{13}. Yoshihisa Saito has informed me that in this case one again obtains the Kashiwara–Saito singularity as a normal slice!

3 Two realisations of the Kashiwara–Saito singularity

3.1 Notation

Fix a positive integer $n \geq 1$.

Let S_n denote the symmetric group, which we regard as permutations of the set $\{1, \dots, n\}$. We view S_n as a Coxeter group with Coxeter generators the simple transpositions $s_i = (i, i+1)$ for $1 \leq i \leq n-1$. We write ℓ for the length function on S_n and $\leq$ for the Bruhat order.

We will usually write permutations in "string notation," i.e., we write $x = x_1 x_2 \dots x_n$ to mean that x is the permutation in S_n which sends $1 \mapsto x_1$, $2 \mapsto x_2$, etc. To avoid confusion when using string notation we extend our alphabet of digits $1, \dots, 9$ by the letters $a, b \dots$ with $a = 10$, $b = 11 \dots$.

Let $G = \mathrm{GL}_n(\mathbb{C})$ denote the general linear group of invertible complex matrices. Given $x = x_1 x_2 \dots x_n \in S_n$, we will denote by $\dot{x}$ the corresponding permutation matrix. That is $\dot{x}(e_i) = e_{x_i}$ if $e_1, e_2, \dots, e_n$ denotes the standard basis of $\mathbb{C}^n$.

Let $B \subset G$ denote the Borel subgroup of upper triangular matrices. Let G/B denote the flag variety. Given $y \in S_n$ we denote by

$$C_y = B\dot{y}B/B \subset G/B$$

its Schubert cell and by Z_y the corresponding Schubert variety

$$Z_y := \overline{B\dot{y}B/B} = \overline{C_y}.$$

3.2 Equations for slices to Schubert cells

We recall how to explicitly write down equations for slices to Schubert cells in Schubert varieties. Everything here can be checked reasonably easily by hand with the (possible) exception of the fact that the equations (3.1) are complete.

Let $U_-, U \subset \mathrm{GL}_n(\mathbb{C})$ the subgroups of unipotent lower and upper triangular matrices respectively. The natural map $U_- \to G/B$ is an open immersion, giving a coordinate patch isomorphic to $\mathbb{A}^{\binom{n}{2}}$ around the basepoint $B \in G/B$. Hence for any $x \in S_n$ the natural map $\pi : \dot{x}U_- \to G/B$ gives a coordinate patch around $\dot{x}B \in G/B$. For a permutation $x = x_1 \dots x_n$ we have

$$\dot{x}U_- = \{g = (g_{i,j}) \in \mathrm{GL}_n(\mathbb{C}) \mid g_{x_i,i} = 1 \text{ and } g_{x_i,j} = 0 \text{ for } j > i\}.$$

For $y \in S_n$ the inverse image $\pi^{-1}(Z_y) \subset \dot{x}U_-$ is given by the equations (see [Ful92], [WY08, §3.2] and [WY12, §2.2]):

$$\mathrm{rank}((g_{i,j})_{\substack{a \le i \le n \\ 1 \le j \le b}}) \le \mathrm{rank}((\dot{y}_{i,j})_{\substack{a \le i \le n \\ 1 \le j \le b}}) \quad \text{for all} \quad 1 \le a, b \le n. \tag{3.1}$$

We have

$$\pi^{-1}(C_x) := \{g \in \dot{x}U_- \mid g_{i,j} = 0 \text{ for } i > x_j\}.$$

Hence if we set

$$N_x = \{g \in \dot{x}U_- \mid g_{i,j} = 0 \text{ for } i < x_j\},$$

then N_x is a normal slice to C_x in $\dot{x}U_-$. Hence the singularity of Z_y along C_x is given by $N_x \cap \pi^{-1}(Z_y)$ which is given by intersecting the linear equations describing N_x with the equations (3.1).

Example 3.1. Perhaps an example will help decipher the notation. Consider $n = 4$ and let $x = 2143$ and $y = 4231$. We have

$$N_x = \left\{ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ a & b & 0 & 1 \\ c & d & 1 & 0 \end{pmatrix} \middle| \; a, b, c, d \in \mathbb{C} \right\}$$

and the rank conditions (3.1) reduce in this case to the single equation $ad - bc = 0$.

3.3 The Kashiwara–Saito singularity

Let $M_2(\mathbb{C})$ denote the space of 2×2-complex matrices with coefficients in $\mathbb{C}$. Consider the space S of matrices $M_i \in M_2(\mathbb{C})$ for $i \in \mathbb{Z}/4\mathbb{Z}$ satisfying the two conditions:

$$\det M_i = 0 \text{ for } i \in \mathbb{Z}/4\mathbb{Z},$$

$$M_i M_{i+1} = 0 \text{ for } i \in \mathbb{Z}/4\mathbb{Z}.$$

Clearly S is an affine variety. One can show that it is irreducible of dimension 8. We call S (or more precisely the singularity of S at $0 := (0,0,0,0) \in S$) the *Kashiwara–Saito* singularity. In [KS97] it is shown that the conormal bundle to 0 is a component of the characteristic cycle of the intersection cohomology D-module on S. In particular, the characteristic cycle is reducible.

3.4 Realisation in $\mathbf{GL_8}$

Now let $n = 8$ and consider the permutations

$$u := 21654387, \quad v := 62845173.$$

Then u is the maximal element in the standard parabolic subgroup $\langle s_1, s_3, s_4, s_5, s_7\rangle$ of length $\ell(u) = 8$. We have $u \le v$ and $\ell(v) = 16$.

The following is stated without proof in [KS97, §8.3]. We give the proof here because it is a good warm-up for the calculation in GL_{12} which we need to perform next.

Proposition 3.2. *The singularity of Z_v along C_u is isomorphic to S.*

Proof. If $J := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, we have (as a matrix of block 2×2-matrices):

$$N_u := \left\{ \begin{pmatrix} J & 0 & 0 & 0 \\ A_1 & 0 & J & 0 \\ A_2 & J & 0 & 0 \\ A_0 & A_3 & A_4 & J \end{pmatrix} \;\middle|\; A_i \in M_2(\mathbb{C}) \right\}.$$

Now

$$\dot{v} = \left(\begin{array}{cc|cc|cc|cc} 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$$

and after some checking, one sees that the rank conditions (3.1) reduce to the following equations:

$$A_0 = 0,$$
$$\operatorname{rank}\begin{pmatrix} A_2 & J \\ 0 & A_3 \end{pmatrix} \leq 2, \tag{3.2}$$
$$\operatorname{rank}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \leq 1, \tag{3.3}$$
$$\operatorname{rank}\begin{pmatrix} A_3 & A_4 \end{pmatrix} \leq 1, \tag{3.4}$$
$$\operatorname{rank}\begin{pmatrix} A_1 & 0 & J \\ A_2 & J & 0 \\ 0 & A_3 & A_4 \end{pmatrix} \leq 4. \tag{3.5}$$

Now

$$\begin{pmatrix} A_2 & J \\ 0 & A_3 \end{pmatrix}\begin{pmatrix} I & 0 \\ -JA_2 & J \end{pmatrix} = \begin{pmatrix} 0 & I \\ -A_3JA_2 & A_3J \end{pmatrix},$$

and so (3.2) is equivalent to $A_3JA_2 = 0$. Similarly, one may show that together (3.2) and (3.5) are equivalent to the conditions

$$A_3JA_2 = 0 \text{ and } A_4JA_1 = 0.$$

If we let $K := \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then (3.3) is equivalent to the conditions:

$$A_2KA_1^t = 0 \text{ and } \det A_1 = \det A_2 = 0.$$

Similarly, (3.4) is equivalent to the conditions

$$A_4^tKA_3 = 0 \text{ and } \det A_3 = \det A_4 = 0$$

Hence if we set

$$A_1' := A_1^tJ, \quad A_2' := A_2K, \quad A_3' := A_3J, \quad A_4' := A_4^tK,$$

then the relations become

$$\det A_i' = 0 \text{ for } i \in \{1, 2, 3, 4\},$$
$$A_2'A_1' = A_3'A_2' = A_4'A_3' = A_1'A_4' = 0.$$

This is clearly isomorphic to the Kashiwara–Saito singularity. □

3.5 Realisation in GL_{12}

We will see how to realise the Kashiwara–Saito singularity as a normal slice in Z_y to a Schubert cell C_x. This time x and y belong to the same right cell.

Now let $n = 12$. Consider the permutations in S_{12}:

$$x = 438721a965cb, \quad y = 4387a2c691b5.$$

(Remember that we use string notation and $a = 10$, $b = 11, c = 12$.) The Robinson–Schensted P and Q symbols of x are (following the conventions of [Ari00]):

$$P(x) = \begin{matrix} 1\,5\,9\,b \\ 2\,6\,a\,c \\ 3\,7 \\ 4\,8 \end{matrix} \quad \text{and} \quad Q(x) = \begin{matrix} 1\,3\,7\,b \\ 2\,4\,8\,c \\ 5\,9 \\ 6\,a \end{matrix}.$$

The P and Q symbols of y are

$$P(y) = \begin{matrix} 1\,5\,9\,b \\ 2\,6\,a\,c \\ 3\,7 \\ 4\,8 \end{matrix} \quad \text{and} \quad Q(y) = \begin{matrix} 1\,3\,5\,7 \\ 2\,4\,9\,b \\ 6\,8 \\ a\,c \end{matrix}.$$

In particular, we conclude that x and y are in the same two-sided cell (even the same right cell).

Reduced expressions for x and y are given by

$$x = s_b s_5 s_6 s_7 s_8 s_9 s_5 s_6 s_7 s_8 s_7 s_1 s_2 s_3 s_4 s_5 s_1 s_2 s_3 s_4 s_3 s_1$$
$$y = s_5 s_6 s_7 s_8 s_9 s_a s_b s_a s_1 s_2 s_3 s_4 s_5 s_6 s_7 s_8 s_9 s_7 s_8 s_4 s_5 s_6 s_7 s_1 s_2 s_3 s_4 s_5 s_3 s_1.$$

We have $x \le y$, $\ell(x) = 22$ and $\ell(y) = 30$.

Recall the Kashiwara–Saito singularity S from the previous section.

Proposition 3.3. *The singularity of Z_y along C_x is isomorphic to S.*

Proof. The normal slice N_x to C_x inside the full flag variety is given by the space of matrices

$$\begin{pmatrix} 0 & 0 & J & 0 & 0 & 0 \\ J & 0 & 0 & 0 & 0 & 0 \\ B_1 & 0 & A_1 & 0 & J & 0 \\ B_2 & J & 0 & 0 & 0 & 0 \\ B_3 & B_5 & A_2 & J & 0 & 0 \\ B_4 & B_6 & B_7 & A_3 & A_4 & J \end{pmatrix}$$

where $J := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ as above and the A_i, B_i are in $M_2(\mathbb{C})$. Now, after some checking one sees that the rank conditions (3.1) give that the intersection of N_x and Z_y is cut out by the equations:

$$B_i = 0,$$
$$\operatorname{rank}\begin{pmatrix} A_3 & A_4 \end{pmatrix} \le 1,$$
$$\operatorname{rank}\begin{pmatrix} 0 & A_1 \\ J & 0 \\ 0 & A_2 \end{pmatrix} \le 3 \quad \Leftrightarrow \quad \operatorname{rank}\begin{pmatrix} A_1 \\ A_2 \end{pmatrix} \le 1,$$
$$\operatorname{rank}\begin{pmatrix} A_2 & J \\ 0 & A_3 \end{pmatrix} \le 2,$$
$$\operatorname{rank}\begin{pmatrix} 0 & A_1 & 0 & J \\ J & 0 & 0 & 0 \\ 0 & A_2 & J & 0 \\ 0 & 0 & A_3 & A_4 \end{pmatrix} \le 6 \quad \Leftrightarrow \quad \operatorname{rank}\begin{pmatrix} A_1 & 0 & J \\ A_2 & J & 0 \\ 0 & A_3 & A_4 \end{pmatrix} \le 4.$$

Looking at the proof of Proposition 3.2 it is now clear that $N \cap Z_y \cong S$, the Kashiwara–Saito singularity. □

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During a visit to MIT last year David Vogan asked me whether the results of [VW12] could produce new examples of reducible characteristic cycles, and asked about Question 1.1. It is a pleasure to dedicate this paper to David, thank him for his many wonderful contributions to Lie theory and to wish him a happy birthday!

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