Weighted Quasi-Arithmetic Mean on Two-Dimensional Regions and Their Applications

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Abstract. This paper discusses a decision maker's attitude regarding risks, for example risk neutral, risk averse and risk loving in micro-economics by the convexity and concavity of utility functions. Weighted quasi-arithmetic means on two-dimensional regions are introduced, and some conditions on utility functions are discussed to characterize the decision maker's attitude. Risk premiums on two-dimensional regions are given and demonstrated. Some approaches to construct two-dimensional utilities from one-dimensional ones are given, and a lot of examples of weighted quasi-arithmetic means are shown.

1 Introduction

Weighted quasi-arithmetic means are important tools for subjective estimation of data in decision making such as management, artificial intelligence and so on, and they are also strongly related to utility functions in micro-economics (Fishburn [3]). Yoshida [9–11] has studied weighted quasi-arithmetic means of an interval by weighted aggregation operations where Kolmogorov [6] and Nagumo [7] studied the aggregation operators and Aczél [1] developed the theory regarding weighted aggregation. Yoshida [11] has discussed the relations between weighted quasi-arithmetic means on an interval and decision maker's attitude regarding risks.

For a continuous strictly increasing function $\varphi : [a, b] \mapsto (-\infty, \infty)$ as a decision maker's *utility function* and a continuous function $\omega : [a, b] \mapsto (0, \infty)$ as a *weighting function*, a *weighted quasi-arithmetic mean* on a closed interval $[a, b]$ is defined by

$$\varphi^{-1}\left(\int_a^b \varphi(x)\,\omega(x)\,dx \Big/ \int_a^b \omega(x)\,dx \right). \tag{1.1}$$

Equation (1.1) is mathematically a *mean value* given by a real number $\mu (\in [a, b])$ satisfying

$$\varphi(\mu) \int_a^b \omega(x)\,dx = \int_a^b \varphi(x)\,\omega(x)\,dx \tag{1.2}$$

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in the *mean value theorem for integration*. On the other hand, since φ is continuous and strictly increasing, decision maker's risk averse attitude is described as the following condition:

$$\varphi(E(X)) \geq E(\varphi(X)) \tag{1.3}$$

for all real valued random variables X, where $E(\cdot)$ denotes the expectation with some probability measure. Its equivalent representation with density function and normalization is

$$\varphi(\nu) \int_a^b \omega(x)\, dx \geq \int_a^b \varphi(x)\, \omega(x)\, dx \tag{1.4}$$

for all weighting functions ω, where $\nu(\in [a, b])$ is the risk neutral mean defined by

$$\nu = \int_a^b x\, \omega(x)\, dx \Big/ \int_a^b \omega(x)\, dx. \tag{1.5}$$

From (1.2) and (1.4), decision maker's risk averse attitude is represented by $\mu \leq \nu$. On the other hand, (1.3) implies the concavity of the function φ. Therefore the following correspondence between the concavity of the function φ and weighted quasi-arithmetic means μ and ν holds [11]:

$$\varphi'' \leq 0 \iff \mu \leq \nu. \tag{1.6}$$

In this paper, we investigate weighted quasi-arithmetic means on two-dimensional regions, and we discuss whether this kind of relation (1.6) still holds or does not on two-dimensional regions.

In Sect. 2 we discuss a *decision maker's attitude regarding risks*, for example *risk neutral*, *risk averse* and *risk loving* in micro-economics by the convexity and concavity of utility functions. In Sect. 3 we introduce weighted quasi-arithmetic means on two-dimensional regions, and we discuss conditions on utility functions to characterize the decision maker's attitude. In Sect. 4, we demonstrate risk premiums on two-dimensional regions, which is one of important concepts for risk management in economics. In Sect. 5 we give a few approaches to construct two-dimensional utilities from one-dimensional utilities, and we show a lot of examples of weighted quasi-arithmetic means.

2 Risk Neutral, Risk Averse and Risk Loving

In this section, we discuss the convexity and concavity of utility functions on two-dimensional regions to characterize the decision maker's attitude regarding risks.

Let a two-dimensional space $\mathbb{R}^2 = (-\infty, \infty)^2$ and let a domain D be a non-empty open convex subset of $\mathbb{R}^2$. Let a utility f be a twice continuously differentiable (C^2-class) function on D which is strictly increasing, i.e. $f_x(x, y) > 0$ and $f_y(x, y) > 0$ for $(x, y) \in D$. We introduce concepts about a decision maker's attitude regarding risks with his utility f. Let (Ω, P) be a probability space, where Ω is a non-empty sample space and P is a non-atomic probability measure on Ω. Let $\mathcal{X}$ be a family of all real valued random variables $X : \Omega \mapsto \mathbb{R}$. A pair of random vectors is called a random vector in this paper. Let $\mathcal{X}(D) = \{\text{random vectors } (X, Y) : \Omega \mapsto D,\ X, Y \in \mathcal{X}\}$.

Definition 2.1 (Risk and decision making, [2–5]).

(i) Decision making with a utility function $f : D \mapsto \mathbb{R}$ is called *risk neutral* if

$$f(E(X), E(Y)) = E(f(X, Y)) \tag{2.1}$$

for all random vectors $(X, Y) \in \mathcal{X}(D)$.

(ii) Decision making with a utility function $f : D \mapsto \mathbb{R}$ is called *risk averse* if

$$f(E(X), E(Y)) \geq E(f(X, Y)) \tag{2.2}$$

for all random vectors $(X, Y) \in \mathcal{X}(D)$.

(iii) Decision making with a utility function $f : D \mapsto \mathbb{R}$ is called *risk loving* if

$$f(E(X), E(Y)) \leq E(f(X, Y)) \tag{2.3}$$

for all random vectors $(X, Y) \in \mathcal{X}(D)$.

These decision maker's attitudes are related to the concavity and the convexity of his utility function f [9–11]. Hence we introduce the definitions of the concavity and the convexity of utility functions on two-dimensional regions [8].

Definition 2.2 (Concavity and convexity).

(i) A function $f : D \mapsto \mathbb{R}$ is called *concave* if

$$f((1-\theta)x_1 + \theta x_2, (1-\theta)y_1 + \theta y_2) \geq (1-\theta) f(x_1, y_1) + \theta f(x_2, y_2) \tag{2.4}$$

for all $(x_1, y_1), (x_2, y_2) \in D$ and all real numbers θ satisfying $0 \leq \theta \leq 1$.

(ii) A function $f : D \mapsto \mathbb{R}$ is called *strictly concave* if

$$f((1-\theta)x_1 + \theta x_2, (1-\theta)y_1 + \theta y_2) > (1-\theta) f(x_1, y_1) + \theta f(x_2, y_2) \tag{2.5}$$

for all $(x_1, y_1), (x_2, y_2) \in D$ satisfying $(x_1, y_1) \neq (x_2, y_2)$ and all real numbers θ satisfying $0 < \theta < 1$.

(iii) A function $f : D \mapsto \mathbb{R}$ is called *convex* if

$$f((1-\theta)x_1 + \theta x_2, (1-\theta)y_1 + \theta y_2) \leq (1-\theta) f(x_1, y_1) + \theta f(x_2, y_2) \tag{2.6}$$

for all $(x_1, y_1), (x_2, y_2) \in D$ and all real numbers θ satisfying $0 \leq \theta \leq 1$.

(iv) A function $f : D \mapsto \mathbb{R}$ is called *strictly convex* if

$$f((1-\theta)x_1 + \theta x_2, (1-\theta)y_1 + \theta y_2) < (1-\theta) f(x_1, y_1) + \theta f(x_2, y_2) \tag{2.7}$$

for all $(x_1, y_1), (x_2, y_2) \in D$ satisfying $(x_1, y_1) \neq (x_2, y_2)$ and all real numbers θ satisfying $0 < \theta < 1$.

The concavity and the convexity of utility functions are characterized with differentials as follows (Rockafellar [8]).

Lemma 2.1 *Let a utility* $f : D \mapsto \mathbb{R}$ *be a* C^2*-class function on* D *such that* $f_x > 0$ *and* $f_y > 0$ *on* D.

(i) *The following (a)–(c) are equivalent:*
 (a) f *is concave (strictly concave respectively).*
 (b) *Its Hessian matrix*

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

 is negative semi-definite (negative definite).
 (c) f *satisfies*

$$f_{xx} \leq 0, \ \ f_{yy} \leq 0 \ \text{and} \ |H| = f_{xx} f_{yy} - f_{xy}^2 \geq 0 \tag{2.8}$$

$$(f_{xx} < 0, \ \ f_{yy} < 0 \ \text{and} \ |H| = f_{xx} f_{yy} - f_{xy}^2 > 0) \tag{2.9}$$

 on D.

(ii) *The following (a')–(c') are equivalent:*
 (a') f *is convex (strictly convex respectively).*
 (b') *Its Hessian matrix*

$$H = \begin{pmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{pmatrix}$$

 is positive semi-definite (positive definite).
 (c') f *satisfies*

$$f_{xx} \geq 0, \ \ f_{yy} \geq 0 \ \text{and} \ |H| = f_{xx} f_{yy} - f_{xy}^2 \geq 0 \tag{2.10}$$

$$(f_{xx} > 0, \ \ f_{yy} > 0 \ \text{and} \ |H| = f_{xx} f_{yy} - f_{xy}^2 > 0) \tag{2.11}$$

 on D.

By Jensen's inequality, we obtain the following lemma from Definitions 2.1 and 2.2.

Lemma 2.2

(i) *If a utility function* $f : D \mapsto \mathbb{R}$ *is linear, i.e.* $f(x, y) = \alpha x + \beta y + \gamma$ *for* $(x, y) \in D$ *where real constants* α, β, γ *satisfying* $\alpha > 0$ *and* $\beta > 0$, *then decision making with the utility* f *is risk neutral.*
(ii) *If a utility function* $f : D \mapsto \mathbb{R}$ *is concave, then decision making with the utility* f *is risk averse.*
(iii) *If a utility function* $f : D \mapsto \mathbb{R}$ *is convex, then decision making with the utility* f *is risk loving.*

3 Weighted Quasi-Arithmetic Means on Two-Dimensional Regions

In this section, we introduce weighted quasi-arithmetic means on two-dimensional regions, and we discuss conditions on the decision maker's utility functions to characterize his attitude for risks based on the weighted quasi-arithmetic means. Let a domain D be a non-empty open convex subset of $\mathbb{R}^2$, and let a utility f be a C^2-class strictly increasing function on D. Now we take a weighting w as a density function of a random vector (X, Y) in Definition 2.1, where we assume w is a once continuously differentiable (C^1-class) positive valued function on D. Denote a family of rectangle regions by $\mathcal{R}(D) = \{R = I \times J \mid I \text{ and } J \text{ are bounded closed intervals and } R \subset D\}$. For a rectangle region $R \in \mathcal{R}(D)$, *weighted quasi-arithmetic means* on region R with utility f and weighting w are given by a subset $M_w^f(R)$ of R as follows.

$$M_w^f(R) = \left\{ (\tilde{x}, \tilde{y}) \in R \mid f(\tilde{x}, \tilde{y}) \iint_R w(x,y)\, dx\, dy = \iint_R f(x,y) w(x,y)\, dx\, dy \right\}. \quad (3.1)$$

Then we have $M_w^f(R) \neq \emptyset$ since f is continuous on R and

$$\min_{(\tilde{x},\tilde{y}) \in R} f(\tilde{x}, \tilde{y}) \leq \iint_R f(x,y) w(x,y)\, dx\, dy \Big/ \iint_R w(x,y)\, dx\, dy \leq \max_{(\tilde{x},\tilde{y}) \in R} f(\tilde{x}, \tilde{y}).$$

Since $f_x > 0$ and $f_y > 0$ on R, there exists an implicit function ϕ which satisfies an equation

$$f(\tilde{x}, \phi(\tilde{x})) \iint_R w(x,y)\, dx\, dy = \iint_R f(x,y) w(x,y)\, dx\, dy, \quad (3.2)$$

and then ϕ is strictly decreasing since $\phi' = -\frac{f_x}{f_y} < 0$ on $M_w^f(R)$. Thus the set of weighted quasi-arithmetic means $M_w^f(R) = \{(x,y) \in R \mid y = \phi(x)\}$ becomes a continuous strictly decreasing curve segment on R. This curve is called *indifference curve* for utility function f in economics.

Lemma 3.1 *Let a rectangle region $R \in \mathcal{R}(D)$, and let a utility f be a C^2-class strictly increasing function on D. Let ϕ be an implicit function for (3.2). Then the following (i) and (ii) hold:*

(i) If f is concave (strictly concave), then its implicit function ϕ is convex i.e. $\phi'' \geq 0$ (strictly convex i.e. $\phi'' > 0$ respectively).
(ii) If f is convex (strictly convex), then its implicit function ϕ is concave i.e. $\phi'' \leq 0$ (strictly concave i.e. $\phi'' < 0$ respectively).

From Definition 2.1 we introduce the following concept depending on a rectangle region and a weighting function.

Definition 3.1 Let a rectangle region $R \in \mathcal{R}(D)$. Let a utility function $f : D \mapsto \mathbb{R}$ and let a weighting function $w : D \mapsto (0, \infty)$.

(i) Decision making with a utility f is called *risk neutral on R with weighting w* if

$$f(\overline{x}_R, \overline{y}_R) \iint_R w(x,y)\,dx\,dy = \iint_R f(x,y)w(x,y)\,dx\,dy, \tag{3.3}$$

where we define a point $(\overline{x}_R, \overline{y}_R)$ on R by weighted means

$$\overline{x}_R = \iint_R x\,w(x,y)\,dx\,dy \Big/ \iint_R w(x,y)\,dx\,dy, \tag{3.4}$$

$$\overline{y}_R = \iint_R y\,w(x,y)\,dx\,dy \Big/ \iint_R w(x,y)\,dx\,dy. \tag{3.5}$$

(ii) Decision making with a utility f is called *risk averse on R with weighting w* if

$$f(\overline{x}_R, \overline{y}_R) \iint_R w(x,y)\,dx\,dy \geq \iint_R f(x,y)w(x,y)\,dx\,dy. \tag{3.6}$$

(iii) Decision making with a utility f is called *risk loving on R with weighting w* if

$$f(\overline{x}_R, \overline{y}_R) \iint_R w(x,y)\,dx\,dy \leq \iint_R f(x,y)w(x,y)\,dx\,dy. \tag{3.7}$$

Hence we investigate the weighted means (3.4) and (3.5) in Definition 3.1. Let a rectangle region $R \in \mathcal{R}(D)$. Let a weighting function $w : D \mapsto (0, \infty)$. From Lemma 2.2(i), we can give a risk neutral utility function $g : D \mapsto \mathbb{R}$ by a linear function: $g(x,y) = \alpha x + \beta y + \gamma$ for $(x,y) \in D$ with real constants α, β, γ satisfying $\alpha > 0$ and $\beta > 0$. Then its weighted quasi-arithmetic means are reduced to

$$M_w^g(R) = \{(x,y) \in R \mid \alpha(x - \overline{x}_R) + \beta(y - \overline{y}_R) = 0\}, \tag{3.8}$$

where $(\overline{x}_R, \overline{y}_R)$ is defined by (3.4) and (3.5). In (3.8), it holds clearly that $(\overline{x}_R, \overline{y}_R) \in M_w^g(R)$ for linear risk neutral utility functions g with any real parameters α, β, γ satisfying $\alpha > 0$ and $\beta > 0$. Therefore $(\overline{x}_R, \overline{y}_R)$ is called an *invariant risk neutral point on R with weighting w*.

Example 3.1 Fix a domain $D = \mathbb{R}^2$ and a rectangle region $R = [0,1]^2$ and fix a weighting function $w = 1$. Then the invariant risk neutral point is $(\overline{x}_R, \overline{y}_R) = (\frac{1}{2}, \frac{1}{2})$. Hence we investigate the following three cases.

(i) (Strictly concave utility f). Take a utility function f as

$$f(x,y) = -x^2 - y^2 + 3x + 3y \tag{3.9}$$

for $(x,y) \in \mathbb{R}^2$. Then $f_x > 0$ and $f_x > 0$ on R. We can easily check

$$f(\overline{x}_R, \overline{y}_R) \iint_R dx\,dy = \iint_R f(x,y)\,dx\,dy + \frac{1}{6}.$$

Thus the utility f is risk averse on R with weighting w. Its Hessian matrix is

$$H = \begin{pmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{xy}(x,y) & f_{yy}(x,y) \end{pmatrix} = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}$$

and the determinant is $|H| = 4 > 0$. Therefore the utility f is strictly concave.

(ii) (Non-concave utility f). Take a utility function f as

$$f(x,y) = x^2 - 2y^2 + x + 5y \tag{3.10}$$

for $(x,y) \in \mathbb{R}^2$. Then $f_x > 0$ and $f_x > 0$ on R. We can easily check

$$f(\overline{x}_R, \overline{y}_R) \iint_R dx\,dy = \iint_R f(x,y)\,dx\,dy + \frac{1}{12}.$$

Thus the utility f is risk averse on R with weighting w. Its Hessian matrix is

$$H = \begin{pmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{xy}(x,y) & f_{yy}(x,y) \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -4 \end{pmatrix}$$

and the determinant is $|H| = -8 < 0$. Therefore the utility f is not concave.

(iii) (Non-convex utility f). Take a utility function f as

$$f(x,y) = 2x^2 - y^2 + x + 3y \tag{3.11}$$

for $(x,y) \in \mathbb{R}^2$. Then $f_x > 0$ and $f_x > 0$ on R. We can easily check

$$f(\overline{x}_R, \overline{y}_R) \iint_R dx\,dy = \iint_R f(x,y)\,dx\,dy - \frac{1}{12}.$$

Thus the utility f is risk loving on R with weighting w. Its Hessian matrix is

$$H = \begin{pmatrix} f_{xx}(x,y) & f_{xy}(x,y) \\ f_{xy}(x,y) & f_{yy}(x,y) \end{pmatrix} = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$$

and the determinant is $|H| = -8 < 0$. Therefore the utility f is not convex.

Example 3.1(ii) shows the concavity of utility f may not be a necessary condition for the risk averse on R, and Example 3.1(iii) also shows the convexity of utility f may not be a necessary condition for the risk loving on R. However it is possible to give the following necessary conditions for risk averse and risk loving (Fig. 1).

Theorem 3.1 *Let a utility f be a C^2-class strictly increasing function on D.*

(i) If decision making with utility f is risk averse on any rectangle region $R \in \mathcal{R}(D)$ with any C^1-class positive valued weighting function w on D, then it holds that

$$f_{xx} \leq 0 \text{ and } f_{yy} \leq 0 \quad \text{on } D. \tag{3.12}$$

(ii) If decision making with utility f is risk loving on any rectangle region $R \in \mathcal{R}(D)$ with any C^1-class positive valued weighting function w on D, then it holds that

$$f_{xx} \geq 0 \text{ and } f_{yy} \geq 0 \quad \text{on } D. \tag{3.13}$$

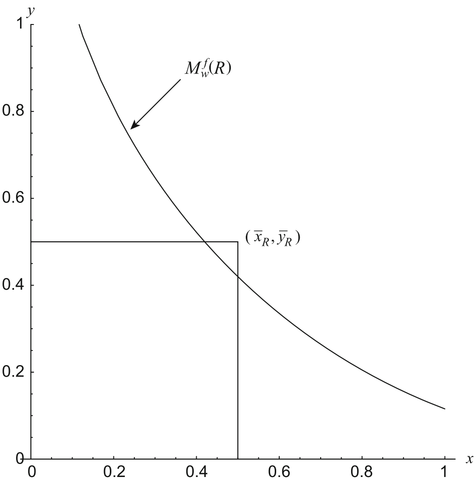

Fig. 1. Weighted quasi-arithmetic means $M_w^f(R)$ ($f(x,y) = -x^2 - y^2 + 3x + 3y$ and $w(x,y) = 1$ on $R = [0,1]^2$)

4 Risk Premiums on Two-Dimensional Regions

Risk premiums are one of important concepts in financial theory. In this section we discuss risk premiums on two-dimensional regions. For this purpose, we introduce the following natural ordering on $\mathbb{R}^2$.

Definition 4.1 (A partial order $\preceq$ on $\mathbb{R}^2$). For two points $(\underline{x}, \underline{y}), (\overline{x}, \overline{y}) (\in \mathbb{R}^2)$, an order $(\underline{x}, \underline{y}) \preceq (\overline{x}, \overline{y})$ implies $\underline{x} \leq \overline{x}$ and $\underline{y} \leq \overline{y}$.

Let a domain D be a non-empty open convex subset of $\mathbb{R}^2$. Let a utility f be a C^2-class strictly increasing function on D, and let a weighting w be a C^1-class positive valued function on D. We introduce the following concept from [5].

Definition 4.2 A vector $\pi_w^f(R) \in [0,\infty)^2$ satisfying the following equation is called a *risk premium for utility* f:

$$f((\overline{x}_R, \overline{y}_R) - \pi_w^f(R)) \iint_R w(x,y)\,dx\,dy = \iint_R f(x,y)w(x,y)\,dx\,dy. \tag{4.1}$$

We denote the set of risk premiums satisfying (4.1) by the following $\Pi_w^f(R)$:

$$\Pi_w^f(R) = \{\pi_w^f(R) \mid (\overline{x}_R, \overline{y}_R) - \pi_w^f(R) \in M_w^f(R),\ \mathbf{0} \preceq \pi_w^f(R)\}, \tag{4.2}$$

where $\mathbf{0}$ is the zero vector on $\mathbb{R}^2$.

Hence $\Pi_w^f(R)$ is also written as

$$\Pi_w^f(R) = \{(\overline{x}_R, \overline{y}_R) - (x,y) \mid (x,y) \in M_w^f(R) \cap R_-^{(\overline{x}_R, \overline{y}_R)}\}, \tag{4.3}$$

where $R_-^{(\overline{x}_R,\overline{y}_R)}$ is *a subregion dominated by the invariant risk neutral point* $(\overline{x}_R,\overline{y}_R)$ which is defined by

$$R_-^{(\overline{x}_R,\overline{y}_R)} = \{(x,y) \in R \mid (x,y) \preceq (\overline{x}_R,\overline{y}_R)\}. \tag{4.4}$$

Since f is strictly increasing, from (3.6) and (4.1) we obtain the following theorem.

Theorem 4.1 *If decision making with utility f is risk averse on R with weighting w, then there exists a risk premium for utility f, i.e. $\Pi_w^f(R) \neq \emptyset$.*

Let $\|\cdot\|$ be a norm on $\mathbb{R}^2$. Now we estimate risk premiums by norm $\|\cdot\|$. Since $M_w^f(R) \cap R_-^{(\overline{x}_R,\overline{y}_R)}$ is a continuous curve, the estimated risk premiums become a closed interval:

$$\{\|\pi_w^f(R)\| \mid \pi_w^f(R) \in \Pi_w^f(R)\} = [\underline{\Pi}_w^f(R), \overline{\Pi}_w^f(R)], \tag{4.5}$$

where the maximum risk premium $\overline{\Pi}_w^f(R)$ and the minimum risk premium $\underline{\Pi}_w^f(R)$ are given from (4.3) as follows:

$$\overline{\Pi}_w^f(R) = \max_{(x,y)\in M_w^f(R)\cap R_-^{(\overline{x}_R,\overline{y}_R)}} \|(\overline{x}_R,\overline{y}_R) - (x,y)\|, \tag{4.6}$$

$$\underline{\Pi}_w^f(R) = \min_{(x,y)\in M_w^f(R)\cap R_-^{(\overline{x}_R,\overline{y}_R)}} \|(\overline{x}_R,\overline{y}_R) - (x,y)\|. \tag{4.7}$$

Hence we define sets of points, which are pairs of two-dimensional weighted quasi-arithmetic means, to attain the maximum risk premium and minimum risk premium as follows:

$$\overline{M}_w^f(R) = \arg \max_{(x,y)\in M_w^f(R)\cap R_-^{(\overline{x}_R,\overline{y}_R)}} \|(\overline{x}_R,\overline{y}_R) - (x,y)\|, \tag{4.8}$$

$$\underline{M}_w^f(R) = \arg \min_{(x,y)\in M_w^f(R)\cap R_-^{(\overline{x}_R,\overline{y}_R)}} \|(\overline{x}_R,\overline{y}_R) - (x,y)\|. \tag{4.9}$$

Using these tools, we can estimate risk premiums for risk averse utility. In Definition 2.1, risk neutral decision making is included in risk averse decision making and risk loving decision making. As a special case of Theorem 4.1, therefore, if decision making is risk neutral then the corresponding risk premium is $\Pi_w^f(R) = \{\mathbf{0}\}$. Then $\overline{\Pi}_w^f(R) = \underline{\Pi}_w^f(R) = 0$ and $\overline{M}_w^f(R) = \underline{M}_w^f(R) = (\overline{x}_R,\overline{y}_R)$. The following Example 4.1 illustrates this concepts.

Example 4.1 We calculate risk premiums for Example 3.1(i). Take a domain $D = \mathbb{R}^2$ and a rectangle region $R = [0,1]^2$, and take a weighting function $w = 1$. Then the invariant risk neutral point is $(\overline{x}_R,\overline{y}_R) = (\frac{1}{2},\frac{1}{2})$. Take a strictly concave increasing utility function f as (3.9). Then the utility f is risk averse on R with weighting w. From (4.3), we obtain risk premiums

$$\Pi_w^f(R) = \left\{ \left(\frac{1}{2} - x, \frac{1}{2} - y \right) \mid -x^2 - y^2 + 3x + 3y = \frac{7}{3}, 0 \le x \le \frac{1}{2}, 0 \le y \le \frac{1}{2} \right\}. \quad (4.10)$$

Take a norm $\|(x,y)\| = \sqrt{x^2+y^2}$ for $(x,y) \in \mathbb{R}^2$. Then the estimated risk premiums becomes a closed interval:

$$\{\|\pi_w^f(R)\| \mid \pi_w^f(R) \in \Pi_w^f(R)\} = [\underline{\Pi}_w^f(R), \overline{\Pi}_w^f(R)] = \left[\frac{\sqrt{78}}{6} - \sqrt{2}, \frac{\sqrt{42}}{6} - 1 \right]. \quad (4.11)$$

Hence the maximum risk premium $\overline{\Pi}_w^f(R) = \frac{\sqrt{42}}{6} - 1 = 0.0801234\cdots$ is attained by two-dimensional weighted quasi-arithmetic means $\left(\frac{\sqrt{42}}{6} - 1, 0\right), \left(0, \frac{\sqrt{42}}{6} - 1\right) \in \overline{M}_w^f(R)$, and the minimum risk premium $\underline{\Pi}_w^f(R) = \frac{\sqrt{78}}{6} - \sqrt{2} = 0.0577466\cdots$ is attained by a two-dimensional weighted quasi-arithmetic mean $\left(\frac{\sqrt{39}}{6} - 1, \frac{\sqrt{39}}{6} - 1\right) \in \underline{M}_w^f(R)$.

5 Construction of Two-Dimensional Utilities from One-Dimensional Utilities

A lot of examples of utility functions on one-dimensional domains are known ([9–11]). Hence we give a few methods, which are easily checked from the definitions, to construct utility functions g on two-dimensional regions from utility functions on one-dimensional domains.

Lemma 5.1 *Let D be a non-empty open domain in $\mathbb{R}^2$, and let a rectangle region $R \in \mathcal{R}(D)$. Let I and J be closed sub-intervals of $\mathbb{R}$. Let g be a C^2-class concave (strictly concave) function on D. Let a pair of utilities $(\xi, \eta) : I \times J \mapsto D$ be C^2-class such that $\xi' > 0$ and $\xi'' \le 0$ on I and $\eta' > 0$ and $\eta'' \le 0$ on J. Then*

$$f(x,y) = g(\xi(x), \eta(y)) \quad (5.1)$$

is a C^2-class concave (strictly concave resp.) utility function on $I \times J$.

Corollary 5.1 *Let I and J be closed sub-intervals of $\mathbb{R}$. Let α and β be positive constants. Let two utilities $\xi : I \mapsto \mathbb{R}$ and $\eta : J \mapsto \mathbb{R}$ be C^2-class such that $\xi' > 0$ and $\xi'' \le 0$ on I and $\eta' > 0$ and $\eta'' \le 0$ on J. Then*

$$f(x,y) = \alpha\xi(x) + \beta\eta(y) \quad (5.2)$$

is a C^2-class concave utility function on $I \times J$.

Lemma 5.2 *Let D be a non-empty open domain in $\mathbb{R}^2$, and let I be a closed sub-interval of $\mathbb{R}$. Let $g : D \mapsto I$ be a C^2-class concave (strictly concave) utility function on D. Let a utility $\varphi : I \mapsto \mathbb{R}$ be C^2-class such that $\varphi' > 0$ and $\varphi'' \le 0$ on I. Then*

$$f(x,y) = \varphi(g(x,y)) \quad (5.3)$$

is a C^2-class concave (strictly concave) utility function on D.

Corollary 5.2 *Let a rectangle region $I \times J \in \mathcal{R}(D)$ and let α and β be positive constants. Let K be a closed sub-interval of $\mathbb{R}$ such that $K = \{\alpha x + \beta y \mid x \in I, y \in J\}$. Let a utility $\varphi : K \mapsto \mathbb{R}$ be C^2-class such that $\varphi' > 0$ and $\varphi'' < 0$ on K. Then*

$$f(x, y) = \varphi(\alpha x + \beta y) \tag{5.4}$$

is a C^2-class concave utility function on $I \times J$.

Example 5.1 In Table 1 we list up some economic utility functions φ on one-dimensional domains [10,11], and then from (5.2) and (5.4) we can construct utility functions on two-dimensional regions by combining these functions. For example, from (5.2) and Table 1 we can obtain a utility function on two-dimensional domain $(0, \infty)^2$ by

$$f(x, y) = \alpha \log x + \beta \log y \tag{5.5}$$

for $(x, y) \in (0, \infty)^2$ with positive constants α and β. On the other hand from 5.4 and Table 1 we can give a utility function on two-dimensional domain $\mathbb{R}^2$ by

$$f(x, y) = 1 - e^{-(\alpha x + \beta y)} \tag{5.6}$$

for $(x, y) \in \mathbb{R}^2$ with positive constants α and β.

Table 1. Strictly concave utility functions φ on one-dimensional domains

Utility function, domain and parameters	$\varphi(x)$
Power utility $(0, \infty); 0 < \lambda < 1$	$\dfrac{x^\lambda}{\lambda}$
Logarithmic utility $(0, \infty); \lambda > 0$	$\lambda \log x$
Exponential utility $(-\infty, \infty); \lambda > 0$	$\dfrac{1 - e^{-\lambda x}}{\lambda}$
Quadratic utility $(0, \lambda); \lambda > 0$	$\lambda x - \dfrac{1}{2} x^2$
Sigmoid utility $(0, \infty); \lambda > 0$	$\dfrac{1}{1 + e^{-\lambda x}}$

Concluding Remark. Lemma 2.2 shows that the concavity of utility functions is a sufficient condition for the risk averse. However, in Example 3.1(ii) we found that the concavity of utility functions is not a necessary and sufficient condition for the risk averse. We need to find other conditions instead of the determinant condition for the Hessian in Lemma 2.1(i):

$$|H| = f_{xx} f_{yy} - f_{xy}^2 \geq 0 \tag{5.7}$$

on D.

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