Chapter 5
From Crisp Sets to Fuzzy Sets

5.1 Introduction

We are now starting the second part of this book. After the introduction to the Lisp programming language in part I, you are now ready to begin the phase of our travel that will require the most of you. Not because it is complex, but because it represents a shift of paradigm not only in the way we think about sets, but also in the form that we usually reason and analyze things. We shall start revisiting the essentials of classic sets theory, including the concepts of belonging to a set, union and intersection of sets, general properties of sets and the concepts of Cartesian Product and Relations between sets.

Hereafter, we shall leave the traditional way of thinking in sets theory and will start to introduce fuzzy sets. It will not be an abrupt quantum leap, but a smooth pathway towards the new paradigm. Here, the exposed material on crisp sets will be helpful for establishing a contrast between the two worlds, and the Lisp code from this chapter will help you not only as a pedagogical tool, but also to build your own fuzzy sets, that is, to experiment with Lisp expressions in your own area of expertise under the viewpoint of the new theory.

5.2 A Review of Crisp (Classical) Sets

A scientific theory does not appear suddenly. The development of a new scientific paradigm usually follows several phases. At a given time one or more scientists develop some ideas and soon they exchange them and derive new knowledge. It may happen that no clear, defined theory is immediately available, but the scientific community quickly recognizes that something new is in the air, creating an exciting scientific ambience. Later, more people is attracted by the fresh concepts and the theory gains momentum until arriving a point in time where the overall concepts are

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distilled into a new scientific frame. Sets theory is an example of this evolution in human thinking, appearing in the second half of century XIX in Germany, mainly from the works of Georg Cantor, Richard Dedekind and Ernst Zermelo, becoming a recognized branch of mathematical logic from 1915, approximately.

5.2.1 Definition of Sets and the Concept of Belonging

The definition of a set is in some way difficult because it is in itself an extremely simple concept. A set is a collection of objects, things, or put into a more formal mathematical language, elements. When put together, physically or conceptually, a collection of things becomes a set. As an example we can mention the set of odd numbers from a roulette, the set of Galilean Moons, the set of Messier objects, etc. In fact, we can describe a set by means of three ways:

1. By simple enumeration where you describe, one by one, the name of all the elements in a set. For example the set S of Galilean Moons can be mathematically represented as follows:

$$S = \{\text{Callisto, Ganymede, Europa, Io}\}$$

2. Conceptually, by means of an established condition:

$$S = \{x \mid x \text{ is one of the four biggest moons of Jupiter}\}$$

   This type of expressions can be read as "x such as x is … (condition)". In this case, (condition) equals to "one of the biggest moon of Jupiter".
3. By means of a Venn diagram:
   A Venn diagram, as shown in Fig. 5.1, consists in a circle or ellipse containing the elements of a set, becoming an excellent graphical method for representing them.

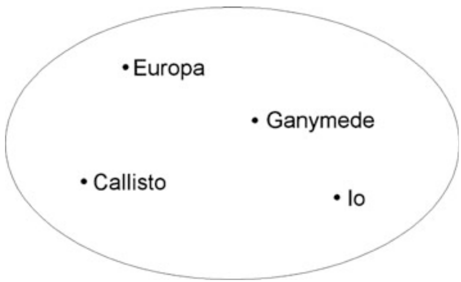


Fig. 5.1 A venn diagram representing the set S of Galilean Moons

4. Additionally, in this book we are going also to represent sets by means of Lisp expressions. If lists are composed by elements, it seems a natural step to use lists for representing sets:

(setq S '(Callisto Ganymede Europa Io))

Independently from the way of representing sets, we shall always use capital letters for naming them.

Intrinsically related to the very definition of a set is the concept of *belonging*. In crisp, classical sets, a given element x either belongs or does not belong to a given set A. Formally:

$$x \in A \text{ (x belongs to set A)} \tag{5-1}$$

$$x \notin A \text{(x does not belong to set A)} \tag{5-2}$$

In Lisp, we can use the user-defined predicate *(belongs?)*, shown in Code 5-1, for checking if a given element x belongs or not to a given set A:

```
;code 5-1
(define (belongs? x A)
    (if (or (intersect (list x) A) (= x '()))
          true
          nil
    )
)
```

Now, after typing *(setq S '(Callisto Ganymede Europa, Io))* at the Lisp prompt we can interrogate the system with, for example: *(belongs? 'Europe S)* → *true*, and, for example, *(belongs? 'Phobos S)* → *false*.

5.2.2 Subsets

If every element of a set A is also an element of a set B, we say that A is a subset of B. We can also say that A is included in B. Formally:

$$A \subset B \tag{5-3}$$

Conversely, if at least an element of set A is not an element of set B then we say that A is not a subset of B. We can also say that A is not included in B:

$$A \not\subset B \tag{5-4}$$

There is a peculiar set in Sets Theory, named “empty set” that is simply a set without any elements and is usually represented by the Greek letter ϕ or alternatively by { }. The empty set should not be strange for us because in Lisp we have extensively used the empty list () in many functions in the previous section of this book, proving its utility in real world applications. By definition, the empty set is included in every set. With these ideas on mind, we can write a Lisp predicate for checking if a set A is a subset of a set B, as shown in Code 5-2:

```
;code 5-2
(define (subset? A B)
    (if (or (= A (intersect A B)) (= A '()))
            true
            nil
    )
)
```

As an example, if we define the set U as the set of planets in the Solar System, U = {Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune}, H as the set of hard planets in the Solar System, H = {Mercury, Venus, Earth, Mars} and then G as the set of gaseous planets G = {Jupiter Saturn Uranus Neptune}, we can write:

$$H \subset U, G \subset U$$

Expressing these sets in Lisp notation: *(setq U ‘(Mercury Venus Earth Mars Jupiter Saturn Uranus Neptune))*, *(setq H ‘(Mercury, Venus, Earth, Mars))*, *(setq G ‘(Jupiter Saturn Uranus Neptune))*, then the following Lisp expressions hold: *(subset? H U) → true, (subset? G U) → true*. Please note that following the actual classification of the Astronomical Union, Pluto is not a planet of the Solar System.

Also by definition, every subset is included into itself, as we can see typing at the Lisp prompt: *(subset? G G) → true*.

Another peculiar set in sets theory is the Universal Set, usually represented by the capital letter U. The most trivial Universal Set would be the set of all things contained in the known universe, including atoms, quarks and any imaginable thing the reader can think in this moment. Needless to say, such a set can only be handled from a philosophical point of view, and from a mathematical and computer science perspective we define a Universal Set as the Set that contains all the objects under consideration, that is, the set of all the elements about a given subject matter. For example the Universal Set of numbers in a French roulette is the set U_1 of integer numbers from 0 to 36. The Universal Set of possible outcomes of throwing a dice is the set U_2 of integers from 1 to 6 and so on. Some example subsets of U_1 are, for example

$$S_1 = \{x \mid x \text{ is red}\}$$

$$S_2 = \{x \mid x \text{ belongs to the first column}\}$$

Remembering the Lisp code from the previous chapter, the set S_2 was already represented by the symbol **column1** as *(setq *column1* '(1 4 7 10 13 16 19 22 25 28 31 34))*. Needless to say, both S_1 and S_2 are included in U_1 and are subsets of it:

$$S_1 \subset U_1,\ S_2 \subset U_1$$

The imaginable set of "all things contained in the known universe" is, obviously uncountable, but uncountable sets can be represented mathematically, too. For example the set of real numbers between 1.0 and 10.0 can be expressed as:

$$S = \{x \mid x \geq 1.0 \text{ and } x \leq 10.0\},\ x \in R$$

Theoretically, Lisp can handle uncountable sets, too. The previous expression can be put into Lisp code using the expression *(if (and (>= x 1.0) (<= x 10.0)) (lisp-expression))*. Here *(lisp-expression)* would be the action that Lisp would follow if x belongs to S, so Lisp can also conceptually describe uncountable sets. However, it is common, both in mathematics and Lisp, to use countable sets. In this case, the number of elements contained in a set S is called *cardinality*, and it is represented by the Greek letter η, or simply by |S|. For example: $\eta(S_1) = |S_1| = 18$, and $\eta(S_2) = |S_2| = 12$.

Cardinality in Lisp is trivially expressed by Code 5-3:

```
;code 5-3
(define (cardinality S)
    (length S)
)
```

Although this function is simply a call to the function *(length)* it serves us well for establishing a continuum between Sets Theory and Lisp. Now, for example, *(cardinality *column1*) → 12*. Please note that in Lisp we can also re-name any function at the Lisp prompt:

> (setq cardinality length)
: length<1B3C6>

And then, as before, *(cardinality *column1*) → 12*.
We define sets S_1 and S_2 as equivalent if their cardinality is the same, that is:

$$\eta(S_1) = \eta(S_2) \tag{5-5}$$

(= (cardinality S1) (cardinality S2)) → true

S_1 and S_2 are equal if the elements in both elements are exactly the same. We then write:

$$S_1 = S_2 \tag{5-6}$$

Conversely, sets S_1 and S_2 are unequal if their cardinality is different or if their elements are not the same. Mathematically, we express it as follows:

$$S_1 \neq S_2 \tag{5-7}$$

From a Lisp point of view, it cannot be easier for checking if two sets are equal or unequal. Taking as example, *(setq A '(a b c)), (setq B '(a b c d)), (setq C '(x y z)), (setq D '(a b c))*, we have that *(= A B) → nil, (= A D) → true, (= C D) → nil*, but *(= (cardinality A) (cardinality D)) → true*. These expressions show us the road to write two simple Lisp predicates, shown in Code 5-3a and 5-3b:

```
;code 5-3a
(define (equivalent? A B)
    (if (= (cardinality A) (cardinality B)))
)

;code 5-3b
(define (equal? A B)
    (if (= A B))
)
```

The previous paragraph can be thus rewritten as: *(equal? A B) → nil, (equal? A D) → true, (equal? C D) → nil, (equivalent?A D) → true.*

5.2.3 Union, Intersection, Complement and Difference

The union $S_1 \cup S_2$ of two crisp sets S_1, S_2 is a set formed by all the elements of S_1 and all the elements of S_2 after eliminating any possible repeated element. Mathematically:

$$S_1 \cup S_2 = \{x \mid x \in S_1 \text{ or } x \in S_2\} \tag{5-8}$$

As an example, if we have two sets A and B composed by some lower-case letters:

$$A = \{a, b, c, d, e\},\ B = \{b, c, x, y, z\}$$

then,

$$A \cup B = \{a, b, c, d, e, x, y, z\}$$

NewLisp incorporates a function, not surprisingly named *(union)*, that returns the union of two sets as a list, that is, as another set. For example, if we create the

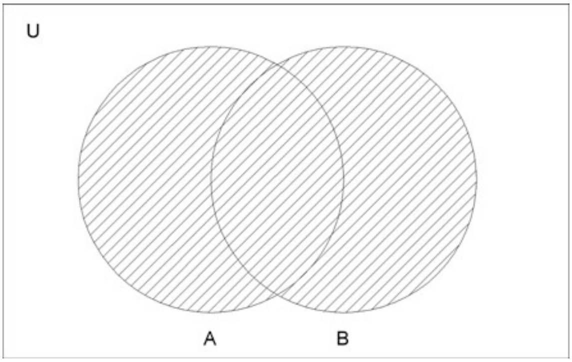


Fig. 5.2 A graphical representation of the union of two sets, A and B

sets A and B by typing *(setq A '(a b c d e))* and *(setq B '(b c x y z)),* then *(union A B)* → *(a b c d e x y z)*. Figure 5.2 shows graphically the union of two sets.

The intersection $S_1 \cap S_2$ of two crisp sets S_1, S_2 is a set formed by all the common elements of S_1 and S_2. Mathematically:

$$S_1 \cap S_2 = \{x \mid x \in S_1 \text{ and } x \in S_2\} \tag{5-9}$$

Following the previous example with sets A and B, we have:

$$A \cap B = \{b, c\}$$

Although the reader should be able to write a Lisp function for obtaining the intersection of two sets, NewLisp already incorporates it in its library of functions, and we only need to type the following at the Lisp prompt: *(intersect A B)* → *(b c)*. By the way, please note how we have seized the opportunity to use the function *(intersect)* for creating the functions *(belong?)* and *(subset?)* as shown in Code 5-1 and 5-2, respectively. Figure 5.3 shows graphically the intersection of two sets.

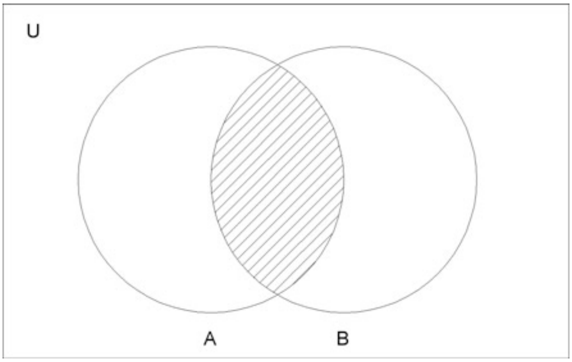


Fig. 5.3 A graphical representation of the intersection of two sets, A and B

If the intersection of two sets, A and B results into the empty set, ϕ, that is;

$$A \cap B = \phi = \{\} \tag{5-10}$$

we say that A and B are disjoint. For example, if A = {a, b, c}, B = {x, y, z} then A and B are disjoint because they do not share any element, that is, its intersection is null. Code 5-4 shows a simple Lisp predicate for testing if two sets are disjoint:

```
;code 5-4
(define (disjoint? A B)
     (if (= (intersect A B) '()))
)
```

Now, following the previous example, *(setq A '(a b c))*, *(setq B '(x y z)* then *(disjoint? A B)* → *true*. It is interesting to note that if two sets A and B are disjoint, then an obvious relationship does exist between the concepts of cardinality and union of sets:

$$\eta(A \cup B) = \eta(A) + \eta(B) \tag{5-11}$$

Using the last example for sets A and B in Lisp for trying expression (5.11), we have:

(cardinality (union A B)) → *6*
(+ (cardinality A) (cardinality B)) → *6*

In general, any two sets A and B, despite they are disjoint or not, satisfy the following property:

$$\eta(A \cup B) = \eta(A) + \eta(B) - \eta(A \cap B) \tag{5-12}$$

Again trying the two non-disjoint sets *(setq A '(a b c d e)), (setq B '(b c x y z)),* then we have: *(cardinality A)* → *5, (cardinality B)* → *5, (cardinality (union A B))* → *8, (cardinality (intersect A B))* → *2*. Expressed in only one line:

(= (cardinality (union A B)) (- (+ (cardinality A) (cardinality B)) (cardinality (intersect A B))) → *true*

After a simple manipulation of expression (5-12) we finally obtain:

$$\eta(A \cup B) + \eta(A \cap B) = \eta(A) + \eta(B) \tag{5-13}$$

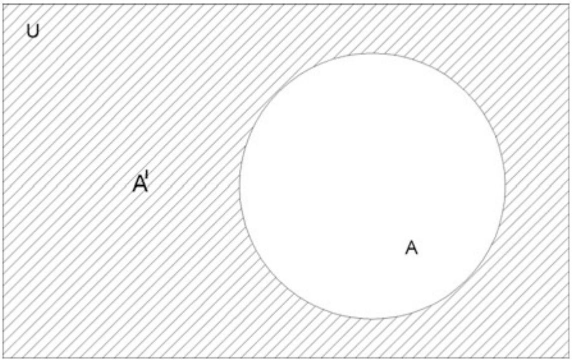


Fig. 5.4 A graphical representation of the complement of a set A, A′ with re-spect to an Universal set U

The complement of a set A with respect to a Universal set U is by definition the set composed by all the elements belonging to U that are not included in A. Formally:

$$A' = \{x \mid x \in U \, \text{and} \, x \notin A\} \tag{5-14}$$

A Venn diagram, shown in Fig. 5.4, will help to visualize expression (5-14).

As an example, let us take again the set U as the set of planets in the Solar System, U = {Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune} and then H as the set of hard planets in the Solar System, H = {Mercury, Venus, Earth, Mars}. Then the complementary of H, H′ is:

H′ = {Jupiter Saturn Uranus Neptune}

Code 5-5 shows a Lisp function for obtaining the complementary set of a set A with respect to a universal set U:

```
;code 5-5
(define (complement A U, lU i set-out)
     (setq set-out '())
     (setq lU (cardinality U))
     (setq i 0)

     (while (< i lU)
            (if (! = (belongs? (nth i U) A) true)
                   (setq set-out (cons (nth i U) set-out))
            )
            (++ i);this is equivalent to (setq i (+ 1 i))
     );end while
     (reverse set-out)
);end function
```

Expressing again sets U and H in Lisp we have: *(setq U '(Mercury Venus Earth Mars Jupiter Saturn Uranus Neptune))*, *(setq H '(Mercury, Venus, Earth, Mars))*, *(setq G '(Jupiter Saturn Uranus Neptune)).* And now:

(complement H U) → *(Jupiter Saturn Uranus Neptune*
(complement G U) → *(Mercury Venus Earth Mars)*

In this case, G and U are disjoint sets and its union covers the complete Universal Set:

(intersect H G) → *()*
(union H G) → *(Mercury Venus Earth Mars Jupiter Saturn Uranus Neptune)*

The difference between two sets A and B is another set whose elements belong to A but do not belong to B. Formally:

$$A - B = \{x \mid x \in A \text{ and } x \notin B\} \tag{5-15}$$

A graphic representation of the difference between sets can be seen in Fig. 5.5 with the help of Venn diagrams.

As can be inferred from Fig. 5.5, the difference between two sets A and B can be also described by the following expression:

$$A - B = A \cap B' \tag{5-16}$$

NewLisp incorporates a function named *(difference)* that automatically calculates the difference between two sets. As an example, let's take set A as the sets of satellites in the solar systems easily to observe with a small quality telescope: *(setq A '(Moon Callisto Ganymede Europa Io Titan))*, and B as the set of Galilean moons in Jupiter: *(setq B '(Callisto Ganymede Europa Io))*. Then, *(difference A B)* → *(Moon Titan)*. It is important to note that the difference between two sets is not commutative, that is: A − B ≠ B − A, for example: *(difference B A)* → *().*

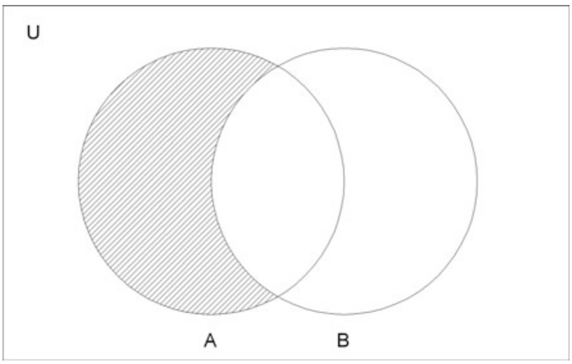


Fig. 5.5 A graphical representation of the difference between sets A and B

5.2.4 Set Properties

In this section we are going to expose the main properties of sets. Aside the formal description we shall give a simple Lisp example for each of the properties using the following expressions as sets: *(setq U '(0 1 2 3 4 5 6 7 8 9)), (setq A '(1 3 5 7)), (setq B '(5 6 7 8 9)), (setq C '(0 1 2 3 4))*. Each family of properties will be included into a single table as follows: Table 5.1 shows the Identity properties of sets, Table 5.2 the Idempotent ones, Table 5.3 the Complement ones, Table 5.4 the Associative ones, Table 5.5 the Commutative ones, Table 5.6 the Distributive ones and Table 5.7 shows the De Morgan's Laws.

Table 5.1 Identity properties of sets

$\mathbf{A \cup \phi = A}$
(union A '()) → (1 3 5 7)
$\mathbf{A \cup U = U}$
(union A U) → (1 3 5 7 0 2 4 6 8 9)
$\mathbf{A \cap U = A}$
(intersect A U) → (1 3 5 7)
$\mathbf{A \cap \phi = \phi}$
(intersect A '()) → ()

Table 5.2 Idempotent properties of sets

$\mathbf{A \cup A = A}$
(union A A) → (1 3 5 7)
$\mathbf{A \cap A = A}$
(intersect A A) → (1 3 5 7)
(intersect A '()) → ()

Table 5.3 Complement properties of sets

$\mathbf{A \cup A' = U}$
(union A (complement A U)) → (1 3 5 7 0 2 4 6 8 9)
$\mathbf{A \cap A' = \phi}$
(intersect A (complement A U)) → ()

Table 5.4 Associative properties of sets

$\mathbf{(A \cup B) \cup C = A \cup (B \cup C)}$
(union (union A B) C) → (1 3 5 7 6 8 9 0 2 4) *(union A (union B C)) → (1 3 5 7 6 8 9 0 2 4)*
$\mathbf{(A \cap B) \cap C = A \cap (B \cap C)}$
(intersect (intersect A B) C) → () *(intersect A (intersect B C)) → ()*

Table 5.5 Commutative properties of sets

A ∪ B = B ∪ A
(union A B) → (1 3 5 7 6 8 9) *(union B A) → (1 3 5 7 6 8 9)*
A ∩ B = B ∩ A
(intersect A B) → (5 7) *(intersect B A) → (5 7)*

Table 5.6 Distributive properties of sets

A ∪ (B ∩ C) = (A ∪ B) ∩ (A ∪ C)
(union A (intersect B C)) → (1 3 5 7) *(intersect (union A B) (union A C)) → (1 3 5 7)*
A ∩ (B ∪ C) = (A ∩ B) ∪ (A ∩ C)
(intersect A (union B C)) → (1 3 5 7) *(union (intersect A B) (intersect A C)) → (5 7 1 3)*

Table 5.7 De Morgan's Laws

(A ∪ B)′ = A′ ∩ B′
(complement (union A B) U) → (0 2 4) *(intersect (complement A U) (complement B U)) → (0 2 4)*
(A ∩ B)′ = A′ ∪ B′
(complement (intersect A B) U) → (0 1 2 3 4 6 8 9) *(union (complement A U) (complement B U)) → (0 2 4 6 8 9 1 3)*

5.2.5 Cartesian Product and Relations

In sets theory, the Cartesian Product of two sets, A and B, denoted by A × B, is the set of all possible ordered pairs (x,y) whose first component x is a member of A and whose second component y is a member of B. Formally:

$$A \times B = \{(x,y) \mid x \in A \text{ and } x \in B\} \tag{5-17}$$

As an example, if A = {a, b, c}, and B = {1, 2, 3, 4}, then:

A × B = {(a,1), (a,2), (a,3), (a,4), (b,1), (b,2), (b,3), (b,4), (c,1), (c,2), (c,3), (c,4)}

Needless to say, the Cartesian Product is not commutative, that is A × B ≠ B × A. In fact, the commutative property for the Cartesian Product between two sets A and B only holds when A = B. Using the same previous example with sets A and B:

B × A = {(1,a), (1,b), (1,c), (2,a), (2,b), (2,c), (3,a), (3,b), (3,c), (4,a), (4,b), (4,c)}

Neither Lisp nor the NewLisp dialect incorporate a function for calculating the Cartesian Product of two sets, but it is not a hard undertaking to write one. Code 5-6 shows a simple Lisp implementation of such a function:

```
;code 5-6
(define (cartesian-product A B, lA lB i j set-out)
    (setq lA (cardinality A))
    (setq lB (cardinality B))
    (setq i 0 j 0);initializes i and j at the same time to
       zero
    (setq set-out '())

    (while (< i lA)
      (while (< j lB)
           (setq set-out (cons (list (nth i A) (nth j B))
                            set-out))
                  (++ j)
      );end while j
      (++ i)
      (setq j 0);reinitializes j
    );end while i
    (reverse set-out)
)
```

Then, making *(setq A ‘(a b c))* and *(setq B ‘(1 2 3 4))*, we only need to write at the Lisp prompt: *(setq U1 (cartesian-product A B))* and Lisp will answer:

((a 1) (a 2) (a 3) (a 4) (b 1) (b 2) (b 3) (b 4) (c 1) (c 2) (c 3) (c 4))

Conversely, the Lisp expression *(setq U2 (cartesian-product B A))* produces, as expected:

((1 a) (1 b) (1 c) (2 a) (2 b) (2 c) (3 a) (3 b) (3 c) (4 a) (4 b) (4 c))

A Cartesian Product can be represented in two dimensions using a simple two-axis graphic. Figure 5.6a, b show A × B and B × A, respectively. Please note from the simple observation of the figures that A × B ≠ B × A, as previously stated:

The definition of a Relation between a set A and a set B is simple: A Relation between sets A and B (or from A to B) is any subset R of the Cartesian Product A × B. After a Relation is established we can say that a ∈ A and b ∈ B are related by R. Using the same previous sets A = {a, b, c} and B = {1, 2, 3, 4} a Relation R_1 can be, for example:

$$R_1 = \{(a,3), (a,4), (b,1), (b,2), (b,3), (b,4)\}$$

R_1 can be represented graphically as shown in Fig. 5.7.

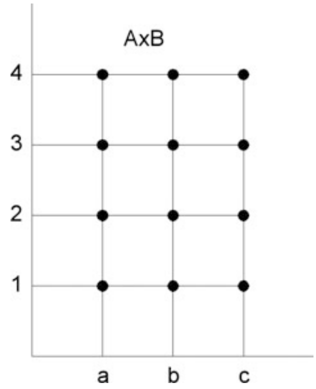


Fig. 5.6a A graphical representation of the Cartesian product A × B

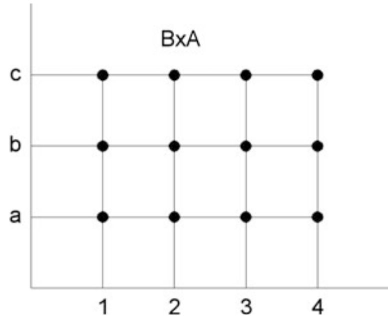


Fig. 5.6b A graphical representation of the Cartesian product B × A

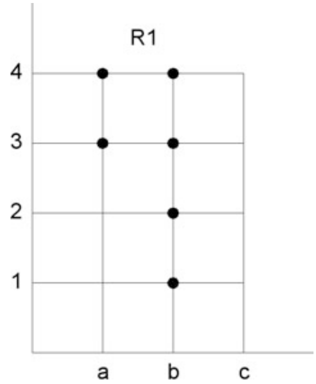


Fig. 5.7 A graphical representation of relation R1

Expressing this into Lisp can not be simpler*: (setq R1 '((a 3) (a 4) (b1) (b 2) (b 3) (b 4))*. For testing if this is a Relation from A to B, we only need to type at the Lisp Prompt: *(subset? R1 U1)* -> *true*, and, as expected, *(subset? R1 U2)* -> *nil,* since R1 is not a relation from B to A.

5.3 Moving Towards Fuzzy Sets

In the previous sections of this chapter we have seen that probably the most important concept in sets theory is the concept of belonging or membership of an element to a set. In fact, without this concept it would be impossible to describe sets. Aside enumerating the elements of a set or expressing it conceptually, or by means of a Venn diagram, we can also express the membership of an element x to a set A using what is known as a membership function, $\mu_A(x)$ or characteristic function. Such a function can only take two values in classic sets theory: 0 if an element x does not belong to a set A, or 1 if an element x certainly belongs to a set A. Formally:

$$\mu_A(x) = 1 \text{ for } x \in A \tag{5-18a}$$

$$\mu_A(x) = 0 \text{ for } x \notin A \tag{5-18b}$$

As an example, let us take the set A as the set of French people, and a very small subset E of European citizens such as E = {Klaus, Jean, John, Paolo, Maurice, Juan}. Only two elements from E belong to A. Using expressions (5-18a) and (5-18b) we have:

$$\mu_A(\text{Klaus}) = 0,\ \mu_A(\text{Jean}) = 1,\ \mu_A(\text{John}) = 0$$
$$\mu_A(\text{Paolo}) = 0,\ \mu_A(\text{Maurice}) = 1,\ \mu_A(\text{Juan}) = 0$$

This type of situations is perfectly covered by using classic, crisp sets. Other examples could be the set of cars that use V12 engines, the set of rockets used in manned spaceflight missions and so on, where the membership value $\mu(x)$ to every element of their respective universal sets is either one or zero. However, there are also examples, in fact the most of things we observe in nature, that do not adhere to this formal framework of yes/no, 1 or 0 belonging.

Let us take the age of a human being as an example, ranging from 0 to 80 years (here we use 80 years old as a top limit since that figure expresses well an average of life duration in developed countries from occidental societies). The key question here is simple at first sight: How do we define the set of old people? That is, where do we establish a sharp separation between young and old people, at which age? After a while, you, dear reader, maybe would answer: "well, maybe it would be better to divide the universal set representing age from 0 to 80 years old in several subsets such as 'child, young, mature and old' in order to have a better representation of the concept of age". It is not a bad answer. However, if you reflect a bit about it you will soon realize that your answer put the question to sleep for a while, but it does not solve the problem because, for example, it leads to questions such us "how do we establish the dividing line between mature and old?". Pressed by my questions you maybe would tell me, probably with a challenging tone of voice:

"Well, we can not do infinite partitions in the set that represents age, right?" This is not a bad reply, either.

For exposing clearly the nature of the problem at hand let us say that there is an agreement if I say that an 80 years old person is an old person. If we subtract one year we obtain 79, and a 79 years old person is again an old person. Following the procedure we have the sequence of numbers 80, 79, 78, 77, 76, … When do we stop enumerating the concept of old age? Since you are delighted with the classic set theory exposed until now in this chapter you decisively say: "ok, a decision must be made. Let us take 50 years old as the dividing line in such a way that we obtain a set A of young people and a set B of old people. Then for every man of woman on the entire Earth we can express formally:"

$$A = \{x \mid x \geq 0 \text{ and } x \leq 50\}$$
$$B = \{x \mid x > 50 \text{ and } x \leq 80\}$$

Sadly, this leads to another set of itching, irritating questions: Is it really a 51 years old person an old person?, or, do you really believe that a 49 years old person is a young person? Finally you exclaim "well Luis, at least you can not negate that a 75 years old person is older than a 55 years old person, and that a 23 years old person is younger than a 62 years old person!". No, I shall not negate that. In fact I think these are excellent observations.

5.3.1 The "Fuzzy Sets" Paper

The previous "age problem" is in fact an example of the well known "Sorites Paradox", a class of paradoxical arguments, based on little-by-little increments or decrements in quantity. A heap formed by sand grains is the primitive paradox (in fact "Sorites" means "heap" in ancient Greek) where decrementing the heap size grain by grain is impossible to establish when a heap turns into a no-heap. The same paradox arises when we try to describe the set of rich people, the set of ill patients in a hospital, the set of luminous galaxies, the set of beautiful women…, the number of instances of the Sorites paradox is quasi-infinite in the real world. These paradoxes can not be adequately solved by using classic sets theory.

Lofti Zadeh, the father of the fuzzy sets theory, was born in 1921 in Baku, Azerbaijan. Soon after graduating from the University of Tehran in electrical engineering in 1942 he emigrated to the United Sates, entering the Massachusetts Institute of Technology, MIT, in 1944 and getting an MS degree in electrical engineering in 1946. Not much later he moved again, this time to New York City, where he received his PhD degree in electrical engineering from the University of Columbia in 1949. After ten years of lecturing at Columbia he finally moved to Berkeley in 1959. While I am writing this book (2014) he still continues writing papers in the famous University on the eastern side of the San Francisco Bay.

Back in the summer of 1964, Zadeh was preparing a paper on pattern recognition for a conference to be held at the Wright-Patterson Air Force Base in Dayton. Ohio. The flight to Dayton made a stopover in New York, so Zadeh enjoyed an evening of free time, an evening free of academic and social encounters that conceded him the freedom of thinking at his best. In his own words:

> *I was by myself and so I started thinking about some of these issues (pattern recognition). And it was during that evening that the thought occurred to me that when you talk about patterns and things of this kind, ... that the thing to do is to use grades of membership. I remember that distinctly and once I got the idea, it became grow to be easy to develop it* (Seising 2007)

As it usually happens, inspiration comes when you are working hard into a problem and you have a strong knowledge not only in the discipline where the problem to solve is defined, but in other more or less parallel and related disciplines. Under these conditions the human brain tends to establish new connections from patterns to patterns of neurons. This neurophysiological process is in fact what we call inspiration. In the case of Zadeh, the nucleus of his inspiration can be summarized in only five words: "to use grades of membership" and that realization materialized into what is probably the most famous paper in the history of fuzzy-logic. This paper, unambiguously titled "Fuzzy Sets", symbolized a shift of paradigm in the theory of sets. I cannot renounce to remember the first words of the abstract of such an important tour de force in the history of computer science and mathematics:

> *A fuzzy set is a class of objects with a continuum of grades of membership. Such a set is characterized by a membership (characteristic) function which assigns to each object a grade of membership ranging between zero and one* (Zadeh 1965)

It is difficult to express the definition of a fuzzy set in a better way. Before presenting some mathematical and Lisp expressions for representing fuzzy sets, I think it is convenient to show a graphical representation of fuzzy sets. In Fig. 5.8a we can see a traditional Venn diagram showing a classic, crisp set A. It is a simple sketch drawn by hand with a pencil, but interesting enough for our discussion. For convenience we have shown it in black. If you wish so, you can imagine it

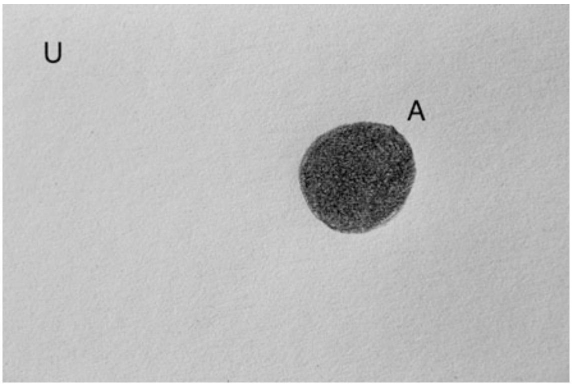


Fig. 5.8a A sketch of a crisp set, A

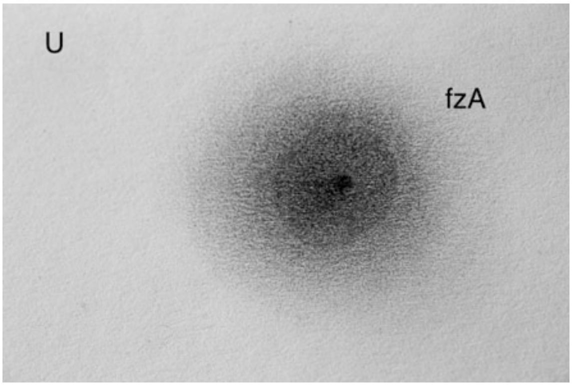


Fig. 5.8b A sketch of a fuzzy set, *fzA*. The *black* nucleus represents a whole membership degree to the fuzzy set

represents a set of old people from the traditional point of view of classic sets theory where every person x belonging to it satisfies the previous expression $\{x \mid x > 50$ and $x \leq 80\}$. In this way, every person older than 50 years old would be "located" inside the black Venn diagram. Needless to say, every young person, less than 50 years old, would be located outside it.

Zadeh's shift of paradigm can be appreciated from the simple observation of Fig. 5.8b. There, it is easy to observe how the blackness decreases from its nucleus towards the exterior in a continuous way, representing different grades of membership. An element x_1 representing an 80 years old person would be located just in the centre, showing a whole membership degree to the fuzzy set fzA, that is $\mu_{fzA} = 1$. As the age of a person decreases, the location of its corresponding element x_i would move away from the centre, getting diminishing values of μ_{fzA}, that is, $\mu_{fzA} < 1$, such as, for example $\mu_{fzA} = 0.7$, $\mu_{fzA} = 0.45$ or $\mu_{fzA} = 0.2$ for decreasing values of age. Just note that now there is no need to define a threshold value for separating in a sharp way the set of old people from the set of young people.

Definition A fuzzy set *A* is defined by a characteristic function $\mu_A(x)$ that maps every element x belonging to A to the closed interval of real numbers [0,1]. Formally we can write:

$$A = \{(x, \mu_A(x)) \mid x \in A, \mu_A(x) \in [0,1]\} \tag{5-19}$$

That is, we can create a fuzzy set by means of enumerating a collection of ordered pairs $(x_i, \mu_A(x_i))$ where $\mu_A(x_i)$ is the membership degree of an element x_i to the fuzzy set A. In general:

$$\mu_A : X \rightarrow [0,1] \tag{5-20}$$

In this expression, the function μA completely defines the fuzzy set A (Klir and Yuan 1995). Following the example of age in human beings we can enumerate a precise, however subjective, characterization of the fuzzy set *A* of old people as, for example: $\mu_A(35) = 0.1$; $\mu_A(45) = 0.2$; $\mu_A(55) = 0.4$; $\mu_A(65) = 0.7$; $\mu_A(75) = 0.9$; $\mu_A(80) = 1$. Using the formal representation given by (5-19) we would have:

$$A = \{(35,0.1), (45,0.2), (55,0.4), (65,0.7), (75,0.9), (80,1.0)\}$$

Since we are dealing with persons, and admitting Paul is 35 years old, John is 45, Mary is 55, Klaus is 65, Juan is 75 and Agatha is 80, we can also write:

$$A_{names} = \{(\text{Paul},0.1), (\text{John},0.2), (\text{Mary},0.4), (\text{Klaus},0.7), (\text{Juan},0.9), (\text{Agatha},1.0)\}$$

As the reader has quickly realized, representing these fuzzy sets using Lisp expressions is straightforward:

```
(setq A '((35 0.1) (45 0.2) (55 0.4) (65 0.7) (75 0.9) (80 1.0)))
(setq A-names '((Paul 0.1) (John 0.2) (Mary 0.4) (Klaus 0.7) (Juan 0.9)
                (Agatha 1.0)))
```

For graphically representing fuzzy sets, traditional Venn diagrams are not enough since they were designed for representing crisp sets. Inspired by the sketch shown in Fig. 5.8, we can use a radar type diagram as a sort of enhanced Venn diagram, as shown in Fig. 5.9. The inner the dot in the radar diagram, the higher is its membership degree to the set. Circles representing membership degrees are spaced every 0.2 units in the figure.

When elements from a fuzzy set are based on numbers, it is usually more convenient to use grid diagrams, as the one shown in Fig. 5.10. The vertical axis

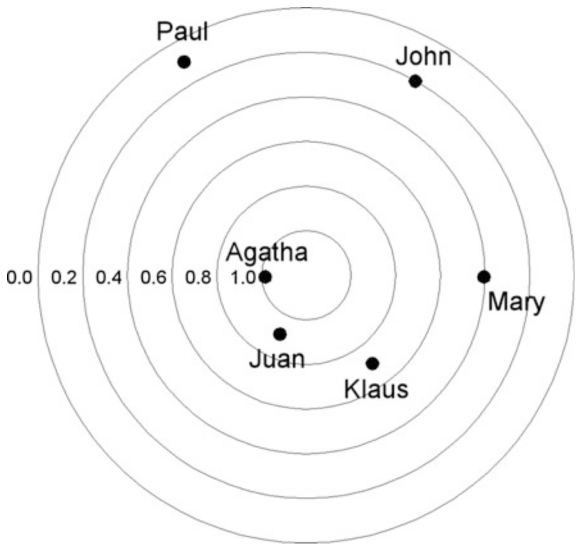


Fig. 5.9 A radar-type diagram for representing a fuzzy set. *Inner circles* represent higher membership degrees to the set

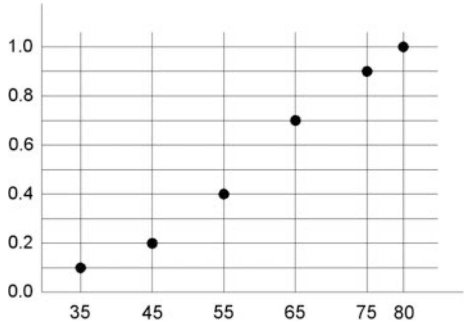


Fig. 5.10 A grid-type diagram for representing a fuzzy set. The vertical axis shows the membership degree to the set. The horizontal axis shows age in years

represents the membership degree to the set, while the horizontal axis shows the numerical elements. In this book we shall use the most suitable type of graphic for our needs. At the end of this chapter we shall make an intense use of grid-type diagrams.

It is especially important to remark that the subjective definition of a fuzzy set by means of its characteristic function is an advantage when modeling vague concepts because we can adapt, or better said, choose the most suitable one depending on context. Continuing with the fuzzy set of "old people" *A*, we have exposed a general example that suits well for a general population of human beings. However, if we are speaking about professional tennis players we can use, for example, the following fuzzy set *T*:

T = {(Dimitrov,0.2), (Djokovic,0.5), (Nadal,0.6), (Federer,0.9), (Borg,1.0)}

where Borg is definitely an old professional tennis player, despite he would have a membership degree about 0.5 to the set of old people *A* in 2014. In fact you should take into account that the definition of set *T* is valid from the point in time I am writing this book. If you are reading this book in 2035 all the elements from *T* will have a membership degree of 1.0, that is, all of them will be old for playing tennis professionally.

As it was the case with crisp sets, and as the reader has probably suspected, the concept of belonging to a fuzzy set deserves a dedicated Lisp function. It is shown in Code 5-7:

```
;code 5-7
(define (fz-belongs? x A)
     (if (assoc x A)
             (last (assoc x A))
             nil
     )
)
```

As can be observed after reading the code, the function *(fz-belongs?)* returns *nil* if a given element *x* is not a member to the fuzzy set *A*, else, that is, if *x* belongs to *A*, then it returns its membership degree. Taking again the Lisp definition of the fuzzy set *A-names*: *(setq A-names '((Paul 0.1) (John 0.2) (Mary 0.4) (Klaus 0.7) (Juan 0.9) (Agatha,1.0)))*, then we would have, for example: *(fz-belongs? 'Klaus A-names) → 0.7,* but *(fz-belongs? 'Paolo A-names) → nil.*

5.3.2 Union, Intersection and Complement of Fuzzy Sets

As we did in Sect. 5.2.3, we are going to explore now how the union, intersection and complement of fuzzy sets are defined. Basically the concepts remain the same, but the introduction of the idea of membership degree to a given fuzzy set adds important details that must be taken into account.

Definition The union $C = A \cup B$ of two fuzzy sets *A* and *B*, determined respectively by their characteristic functions $\mu_A(x)$, $\mu_B(x)$, is defined formally by the following expression:

$$C = A \cup B = \mu_C(x) = \max[\mu_A(x), \mu_B(x)] \tag{5-21}$$

As an example, let us take the fuzzy sets *A* and *B*:

A = {(1,0.7), (2,0.1), (3,0.3), (4,0.9), (5,0.2)}
B = {(1,0.1), (2,0.8), (3,0.9), (4,0.2), (5,1)}

Figure 5.11a, b show a grid representation of these fuzzy sets:
Then, the union $C = A \cup B$ is:

C = {(1,0.7), (2,0.8), (3,0.9), (4,0.9), (5,1)}

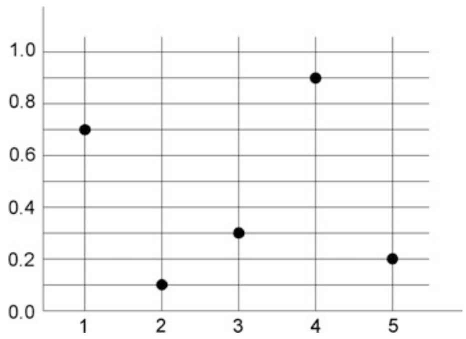


Fig. 5.11a Grid representation of fuzzy set *A*

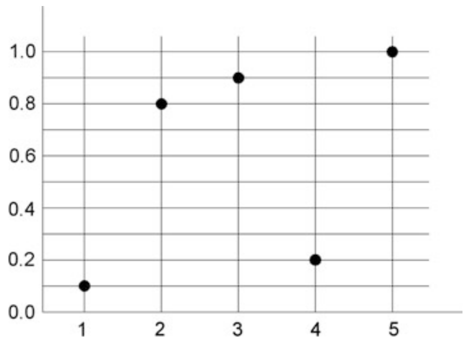


Fig. 5.11b Grid representation of fuzzy set *B*

In order to have a Lisp function for obtaining the union of two fuzzy sets (and also for obtaining its intersection), it is convenient to have first an auxiliary function, named *(clean-md)* that given a fuzzy set *A* returns a related crisp set, that is, a set that conserves all the elements *x* in *A* but cleans all its membership values. Such a function is shown in Code 5-8:

```
;code 5-8.
(define (clean-md A, lA i set-out)
     (setq lA (cardinality A))
     (setq i 0)
     (setq set-out '())

     (while (< i lA)
             (setq set-out (cons (first (nth i A)) set-out))
             (++ i)
     ); end while i
     (reverse set-out)
)
```

Then, for example, is we take *(setq A '((1 0.7) (2 0.1) (3 0.3) (4 0.9) (5 0.2)))* and *(setq B '((1 0.1) (2 0.8) (3 0.9) (4 0.2) (5 1.0)))*, then *(clean-md A)* → *(1 2 3 4 5)* and also *(clean-md B)* → *(1 2 3 4 5)*. Now, we can easily write the function *(fz-union)*, as shown in Code 5-9:

```
;code 5-9
(define (fz-union A B, temp lA lB lt i element md-a md-b
set-out)
     (setq temp (union (clean-md A) (clean-md B)))
     (setq lA (cardinality A))
     (setq lB (cardinality B))
     (setq lt (cardinality temp))
     (setq i 0)
     (setq set-out '())
```

```
        (while (< i lt)
                (setq element (nth i temp))
                (setq md-a (assoc element A))
                (setq md-b (assoc element B))
                (if (> = md-a md-b)
                        (setq  set-out  (cons  md-a  set-out));
                        else:
                        (setq set-out (cons md-b set-out))
                )
                (++ i)
        ); end while i
        (reverse set-out)
)
```

Now, for testing the function, we only need to type at the Lisp prompt: *(fz-union A B)*, obtaining: *((1 0.7) (2 0.8) (3 0.9) (4 0.9) (5 1))*. Figure 5.12a, b show a graphic representation of the union of *A* and *B*.

Please observe how for both figures the membership degree of each element in the resulting union of sets get "higher" in its respective graphic. In the grid representation, this raising is clear, while in the radar-type graphic, the elements 1, 2, 3, 4 and 5 are closer to the centre. Imagining the curves representing membership degrees as contour lines in a terrain, we can appreciate that the resulting elements are closer to the top.

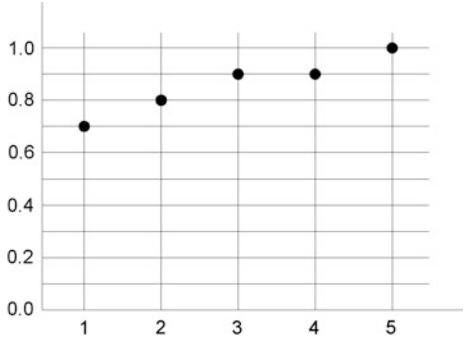
Definition The intersection $C = A \cap B$ of two fuzzy sets *A* and *B*, determined respectively by their characteristic functions $\mu_A(x)$, $\mu_B(x)$, is defined formally by the following expression:

$$C = A \cap B = \mu_C(x) = \min[\mu_A(x), \mu_B(x)] \tag{5-22}$$

Taking again the same example sets A and B, we have the intersection $C = A \cap B$:

$$C = \{(1,0.1), (2,0.1), (3,0.3), (4,0.2), (5,0.2)\}$$

Fig. 5.12a A grid representation of $A \cup B$



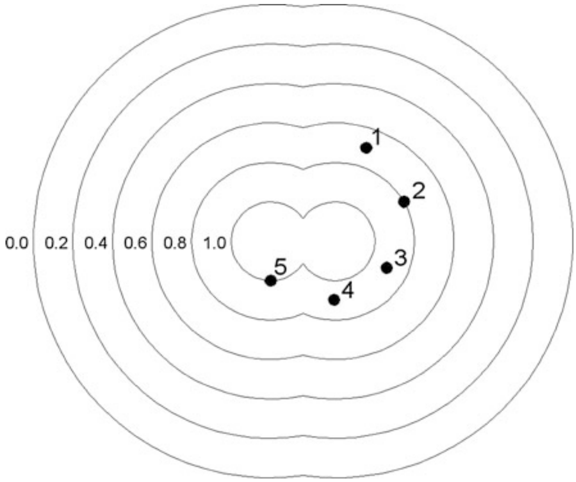


Fig. 5.12b A radar-type representation of $A \cup B$

Translating this to Lisp results into the function *(fz-intersect)*, shown in Code 5-10:

```
;code5-10
(define (fz-intersect A B,
          temp lA lB lt i element md-a md-b set-out)
    (setq temp (intersect (clean-md A) (clean-md B)))
    (setq lA (cardinality A))
    (setq lB (cardinality B))
    (setq lt (cardinality temp))
    (setq i 0)
    (setq set-out '())
    (while (< i lt)
          (setq element (nth i temp))
          (setq md-a (assoc element A))
          (setq md-b (assoc element B))

          (if (<= md-a md-b)
                (setq set-out (cons md-a set-out));
                  else:
                (setq set-out (cons md-b set-out))
          )
          (++ i)
    ); end while i
    (reverse set-out)
)
```

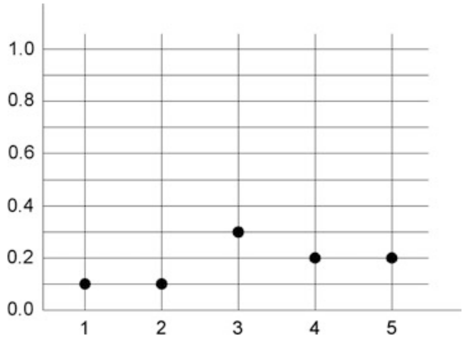


Fig. 5.13a A grid representation of $A \cap B$

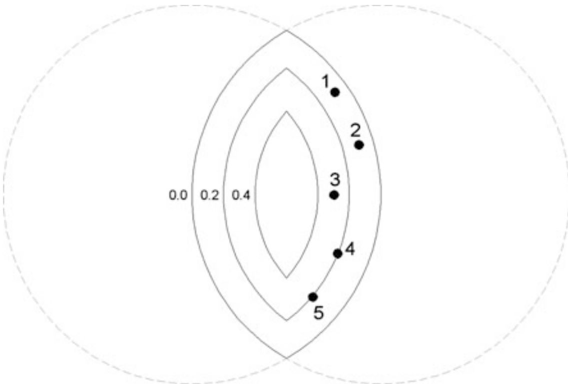


Fig. 5.13b A radar-type representation of $A \cap B$

And testing the function at the Lisp prompt we get: *(fz-intersect A B)* → *((1 0.1) (2 0.1) (3 0.3) (4 0.2) (5 0.2))*. Figure 5.13a, b show a grid-type and a radar-type representation of the intersection of fuzzy sets A and B, respectively.

The dashed curves in Fig. 5.13b show the original external shapes (that is, 0.0 membership degrees) of the Venn diagrams corresponding to A and B. Since $A \cap B$ has resulted into a set with low membership degrees, you can visualize it as two radar-type Venn diagrams relatively separated with a common part (continuous lines). If the resulting fuzzy intersection had elements with high membership degrees, this fact would imply that the radar-type Venn diagrams would be closer. In fact, $A \cap A \rightarrow A$, and then the separation would be inexistent. Expressing it into Lisp: *(fz-intersect A A)* → *((1 0.7) (2 0.1) (3 0.3) (4 0.9) (5 0.2))*.

Definition The complement A' of a fuzzy set A determined by its characteristic function $\mu_A(x)$ is defined formally by the following expression:

$$A' = 1 - \mu_A(x) \tag{5-23}$$

The translation of expression (5-23) is not especially complex, and it is shown in Code 5-11:

```
;code 5-11
(define (fz-complement A, lA i set-out element)
     (setq lA (cardinality A))
     (setq i 0)
   (setq set-out '())

     (while (< i lA)
             (setq element (nth i A))
             (setq set-out (cons (list (first element)
                     (sub 1.0 (last element))) set-out))
             (++ i)
     ); end while i
     (reverse set-out)
)
```

Remembering the fuzzy sets *A* and *A-names* in Sect. 5.3.1 of this chapter representing membership degrees to the concept of "old people": *(setq A '((35 0.1) (45 0.2) (55 0.4) (65 0.7) (75 0.9) (80 1.0))), (setq A-names '((Paul 0.1) (John 0.2) (Mary 0.4) (Klaus 0.7) (Juan 0.9) (Agatha 1.0)))*, then we have the following calls to the function *(fz-complement)*:

(fz-complement A) → ((35 0.9) (45 0.8) (55 0.6) (65 0.3) (75 0.1) (80 0))
(fz-complement A-names) → ((Paul 0.9) (John 0.8) (Mary 0.6) (Klaus 0.3) (Juan 0.1) (Agatha 0))

Now it is easy to realize that the fuzzy complements *A′* and *A-names'* represent the concept of "young people". For example, a 65 years old person had a 0.7 membership degree to the fuzzy set of old people and now has a 0.3 membership degree to its complement. Juan (75 years old) had a 0.9 membership degree to *A-names*, but only a 0.1 membership degree to its fuzzy complement and so on. Figure 5.14a, b show a grid and a radar-type representation of *A′* and *A-names*', respectively.

It is interesting to compare these figure to Figs. 5.9 and 5.10, especially the radar-type one. Surprisingly at first sight, the geometrical positions of Agatha, Juan, Klaus, Mary, John and Paul are exactly the same in Figs. 5.9 and 5.14b. However, if you observe meticulously both figures you will soon realize that the values of the membership degrees associated to every contour line are reversed. In other words: while the radar-type diagram of Fig. 5.9 increases the values of its contour lines from the outside to the center of the diagram, in Fig. 5.14b the external line represents the maximum membership degree (1.0) and the inner the position in the diagram of an element the lesser its membership degree to the set. Any hypothetical

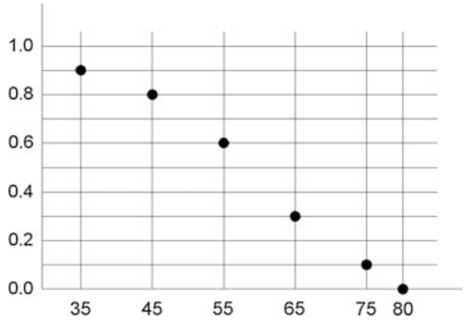


Fig. 5.14a Grid diagram for representing the complement fuzzy set A'

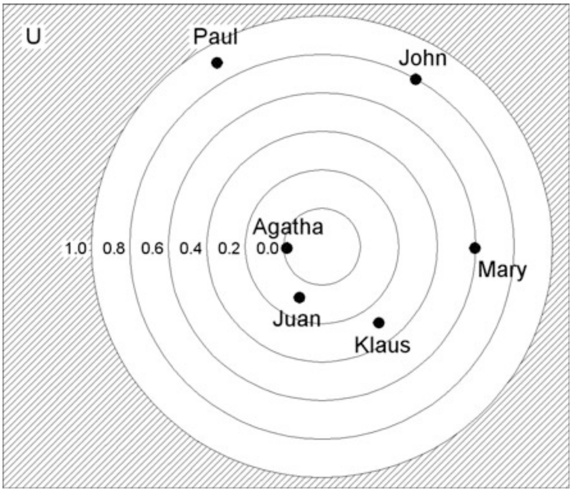


Fig. 5.14b Radar-type diagram for representing the complement fuzzy set *A-names'*

point located outside the external circle, that is, located on the shaded area, would have a 1.0 membership degree to *A-names'*.

5.3.3 *Fuzzy Sets Properties*

Fuzzy sets do not satisfy every property of classic sets as shown in Sect. 5.2.4. While the identity, idempotent, associative, commutative, De Morgan's Laws and distributive properties are perfectly satisfied by fuzzy sets, the complement properties are not. Especially interesting is the second expression from Table 5.3, named 'law of non contradiction' in logic (Trillas 2009):

$$A \cap A' = \phi$$

Let us take again the fuzzy set *A* as the set representing the concept of "old people", then: *A* → *((35 0.1) (45 0.2) (55 0.4) (65 0.7) (75 0.9) (80 1))* and *(fz-complement A)* → *((35 0.9) (45 0.8) (55 0.6) (65 0.3) (75 0.1) (80 0))*. When we intersect these fuzzy sets, that is, after typing *(fz-intersect A (fz-complement A))* at the Lisp prompt, we obtain the following fuzzy set:

((35 0.1) (45 0.2) (55 0.4) (65 0.3) (75 0.1) (80 0))

And this set is far from being equal to the empty set, ϕ. As we have already seen, since *A* represents the concept of "old people", its complement, *A′*, represents the concept of "young people". If we were dealing with crisp sets, the intersection of A and A′ would be the empty set, that is, every person is either young or old. However, when dealing with fuzzy sets we usually have:

$$A \cap A' \neq \phi$$

So, what is the meaning of this expression translated to normal language? If we take the example obtained from evaluating *(fz-intersect A (fz-complement A))* at the Lisp prompt, it means that the elements belonging both to the fuzzy sets A and A′ are at the same time young and old people! How can this be possible? Let us examine some elements of $A \cap A'$: For an 80 years old person her membership degree to the intersection is zero, so this person is not young and old at the same time, because from the definition of *A* she is "entirely" old. In a similar way, a 35 years old person has only a 0.1 membership degree to the intersection. We could say that he is mainly young, but some oldness has already started to appear in his physiology, in his organic development. Let us analyze now the element representing a 55 years old person. We can see that he has a 0.4 membership degree to $A \cap A'$, so he is clearly young and old at the same time. Is this statemen false? Under classical sets theory it certainly is. However, it is perfectly true when we use fuzzy sets, and what is especially interesting: it describes the real world in a very sensible way because a 55 years old person, while not young at all is still far from being old. He is transiting in time from young to old, and this transition is what the use of fuzzy sets represents perfectly well. At the beginning of Sect. 5.3 we defined the crisp set A as the set of young people and B as the crisp set of old people in the following way:

$$A = \{x \mid x \geq 0 \text{ and } x \leq 50\}$$
$$B = \{x \mid x > 50 \text{ and } x \leq 80\}$$

And as you can remember, these crisp definitions were generating serious irritating questions. Now, by means of using fuzzy sets the itching has disappeared. This key concept, the concept of transition between infinites shades of gray in the [0,1] interval of real numbers is what makes fuzzy sets theory so attractive for modelling systems from the real world. We use mathematics, physics, biology, astronomy and other sciences to model Nature, written here with capital letter at the beginning of the word to remark the broad meaning of the term, and we usually get a good representation of the systems we observe. However, when we introduce fuzzy sets in these sciences we usually get an even better representation of natural systems and in some cases we obtain a representation that is impossible to obtain by means of the simple use of crisp sets.

We are now immersed in the conceptual nucleus of this book, so it is a good time to remark another interesting matter that usually arises from using the [0,1] interval of real numbers for describing the membership degree on an element to a fuzzy set *A*, as already shown in expressions (5-19), (5-20): Membership degrees are not probabilities. As the reader already knows, probability is a mathematical concept for measuring the likeliness that an event E will happen. Formally:

$$p(\mathrm{E}) \rightarrow [0,1]$$

When $p(\mathrm{E}) = 0$ we say that an event E will not happen, for example, the probability that I will finish this book just tomorrow is 0 (I am now writing Chapter five), while when we write $p(\mathrm{E}) = 1$ we are expressing that an event E will occur with absolute certainty. For example, the probability that the Sun will be a bright celestial object tomorrow is 1 (and it will continue being one well after the extinction of the human race). Any other value of p between 0 and 1 is an attempt to measure the likeliness that an event E will happen. Just observe the grammar I have used in this last sentence: As a general feature of probability, this branch of mathematics deals with things that can or cannot happen in the future. Membership degrees in fuzzy sets, on the other hand, do not need to deal with future events. They exclusively deal with actual facts, that is, with intrinsic features of an existing natural system. When we say, for example that Mary (55 years old) has a 0.4 membership degree to the fuzzy set of old persons we are expressing something that actually exists in reality without any mention to the future. However, if we say that the likelihood of Mary to live until 75 years old is $p = 0.85$ we are speaking about probabilities, since it is a measure of something that will or will not happen in the future and that actually we do not know for sure. Another example, taken from the famous potable drink problem by Bezdek (2013), will help to perfectly distinguish between membership degrees to a fuzzy set and probabilities.

Just imagine you have decided to spend your holidays in the Sahara desert. Don't ask me why, but after a few days and some discomforting adventures you are suddenly alone and lost in the hot African sands, and what is even worst: without water. After a while your luck seems to change a bit and you arrive to a small, very special oasis where you find two bottles with exactly the same shape placed over a table. Both are big and full of liquid, and the only difference between the two bottles is their labels: One of them, let us say, bottle A, says: "The liquid in this bottle has a 0.75 membership degree to the set of potable drinks". On the other hand, bottle B has a label that says: "The liquid in this bottle has a 0.9 probability of being potable". Now the important question is: Which bottle would you choose for drinking?

Bottle B seems attractive, especially if you do not like mathematics (we must recognize that since you are reading this book this is unlikely) because after all, 0.9 is a bigger number than 0.75 and everybody knows that the bigger the better. However, if p(B) = 0.9, it implies that p(B′) = 0.1, that is, it means that there is also a 0.1 probability value that the content of bottle B is not potable, being, for example, poisoned water. If you decide to drink from bottle B you are making a bet with two possible outcomes: To replenish your body with potable, pure water, with an associated probability p = 0.9 or to drink some poison and die, with an associated probability p = 0.1. Now, let us think about bottle A. The label tells us that it has a μ = 0.75 membership degree to the set of potable drinks. Well, as any person versed in fuzzy sets theory, you know that the label of bottle A is informing you that its content is not pure water (that would have a 1.0 membership degree to the set of potable drinks). Maybe it is a commercial drink with lots of sugar, colorants and the like, but it is, intrinsically, a potable liquid so if you drink from bottle A you will have more time to find someone in the Sahara and eventually escape the desert. Even more: before drinking from bottle A you already know that you have find a suitable solution to your problems of thirst. If you choose to drink from bottle B you do not know beforehand what will happen. Especially interesting is what happens after, let us say, one hour after you drink: Bottle A will continue to have a μ = 0.75 membership degree to the set of potable drinks because membership degrees are intrinsic to the features of a given element in a set. However, the probability value associated with bottle B has disappeared: Time has passed by and you now have a complete knowledge about its content if it was potable liquid. If it was not a potable liquid, say, poisoned water, then now you have not knowledge at all.

5.3.4 Fuzzy Relations

As we have seen in Sect. 5.2.4, the Cartesian Product A × B between two sets A and B composes a new set of ordered pairs (x,y) where the first component of each element belongs to A and the second component belongs to B, as shown by expression (5-17). Now, we can extend this idea to the realm of fuzzy sets by using expression (5-24):

$$R = \{((x,y),\ \mu_R(x,y))|(x,y) \in A \times B, \mu_R(x,y) \in [0,1]\} \qquad (5\text{-}24)$$

A fuzzy relation is a mapping from the Cartesian Product A × B to the closed interval [0,1]. The membership degree of the Relation is given by the function $\mu_R(x, y)$, that is, the value of $\mu_R(x, y)$ expresses the strength of the relation between the elements x and y of the pairs (x, y). Let us take, as an example, the following sets A and B:

A = {a, b, c}
B = {x, y, z}

Table 5.8 Basic operations on fuzzy relations

Operation	Expression
Union	$\mu_{R\cup S}(x,y) = \max(\mu_R(x,y),\ \mu_S(x,y))$
Intersection	$\mu_{R\cap S}(x,y) = \min(\mu_R(x,y),\ \mu_S(x,y))$
Complement	$(\mu_R(x,y))' = 1 - \mu_R(x,y)$

Then, a possible fuzzy relation, *R1*, between A and B could be:

$$R1 = \{(a,x,0.5), (a,z,0.8), (b,y,0.3), (b,z,1.0), (c,x,0.6)\}$$

Expressing fuzzy relations in Lisp is, needless to say, straightforward. *R1* can be represented as *(setq R1 '((a x 0.5) (a z 0.8) (b y 0.3) (b z 1.0) (c x 0.6)))*.

The basic operations on fuzzy relations defined in the Cartesian space A × B are given in Table 5.8.

5.3.4.1 Fuzzy Cartesian Product

Especially interesting is the situation where two sets *A* and *B* related by a fuzzy relation *R* are also fuzzy. In this case, every element from the pairing (x,y), $x \in A$, $y \in B$, already carries a membership degree. For obtaining the resulting ordered triples (x, y, μ_R(x,y)) we use the Fuzzy Cartesian Product *A* × *B*, given by:

$$\mu_R(x,y) = \min(\mu_A(x),\ \mu_B(y)) \tag{5-25}$$

As an example, let us take the following two fuzzy sets *A* and *B*:

$$A = \{(x1,0.4), (x2,0.7)\}$$
$$B = \{(y1,0.8), (y2,0.6), (y3,0.4)\}$$

Then, the Fuzzy Cartesian Product *A* × *B* is:

$$A \times B = \{(x1,y1,0.4), (x1,y2,0.4), (x1,y3,0.4), (x2,y1,0.7), (x2,y2,0.6), (x2,y3,0.4)\}$$

In order to automatically calculating the Fuzzy Cartesian Product of two fuzzy sets, we only need to translate expression (5-25) into Lisp, as shown in Code 5-12:

```
;code 5-12
(define (fz-cartesian-product A B, lA lB i j set-out)
   (setq lA (cardinality A))
   (setq lB (cardinality B))
   (setq i 0 j 0)
   (setq set-out '())
```

```
    (while (< i lA)
      (while (< j lB)
          (setq set-out (cons (list (first (nth i A))
           (first (nth j B)) (min (last (nth i A))
            (last (nth j B)))) set-out))

           (++ j)
      );end while j
      (++ i)
      (setq j 0);reinitializes j
    );end while i
    (reverse set-out)
  )
```

Using the previous example as a test, we only need to type at the Lisp prompt: *(setq A '((x1 0.4) (x2 0.7)))*, *(setq B '((y1 0.8) (y2 0.6) (y3 0.4)))*, and then *(fz-cartesian-product A B)* → *((x1 y1 0.4) (x1 y2 0.4) (x1 y3 0.4) (x2 y1 0.7) (x2 y2 0.6) (x2 y3 0.4))*.

Since the concept of fuzzy relations and fuzzy Cartesian product is of pivotal importance for the future material in this book, we shall see another example. Let us say two women, Ana and Mary, belong to the fuzzy set *A* of highly communicating woman with membership degrees 0.9 and 0.1, respectively. On the other hand, John and Paul belong to the fuzzy set *B* of highly communicating men with membership degrees 0.8 and 0.4, respectively. The fuzzy Cartesian product between *A* and *B* will give us a fuzzy relation that expresses the strength of the possible communication links between all the members from A and B. Let us type at the Lisp prompt the following expressions: *(setq A '((Ana 0.9) (Mary 0.1)))*, *(setq B '((John 0.8) (Paul 0.4)))*. Now we obtain: *(fz-cartesian-product A B)* → *((Ana John 0.8) (Ana Paul 0.4) (Mary John 0.1) (Mary Paul 0.1))*. That is, Ana and John will be able to exchange a lot of ideas because their inherent communicating abilities, while Mary will not be able to communicate well nor with John neither with Paul. Figure 5.15 shows a graphical representation of $A \times B$:

The three dimensional appearance of the graphical representation of the fuzzy relation between *A* and *B* is important. Since the grade of complexity has grown from the relations in classic sets theory, a new (third) dimension is needed to correctly represent fuzzy relations. The vertical axis represents the membership degree, $\mu_R(x,y)$, of the relation. The other two axes represent the membership degrees $\mu_A(x)$ and $\mu_B(y)$ to the fuzzy sets *A* and *B*, respectively. In future chapters in this book we shall seize the opportunity of using 3D representations as a tool for a better understanding of new concepts.

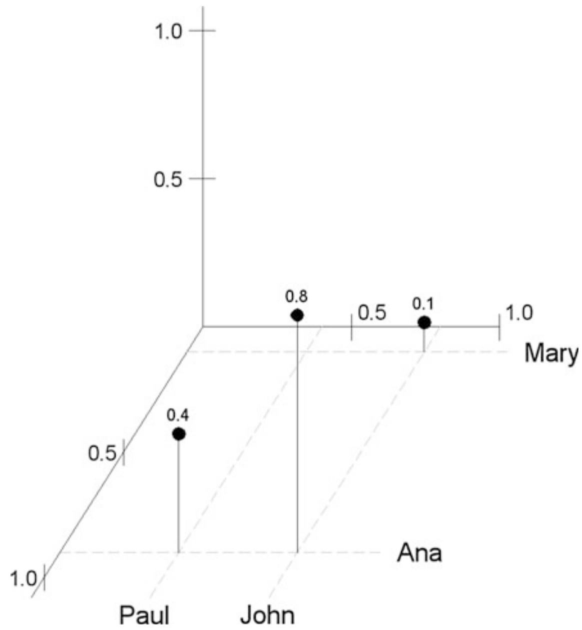


Fig. 5.15 3D representation of the fuzzy relation between fuzzy sets *A* and *B*. Please note that the membership degree of the pair "Mary-Paul" has not be in-cluded in order to improve the clarity of the image

5.4 Membership Degrees: An Example Application in Medicine

Since this book has a strong practical vocation we are going to expose in this section a practical and complete use of the concept of membership degrees, in this case applied to the medical practice. We have already discussed the fuzzy set of old people. Now we can improve it in such a way that we are ready to speak about illness along the life of a person. For this we shall introduce the concept of Life Illness Curves, learning at the same time how a lot of phenomena in nature can be modelled by means of membership degrees and time.

We define a Life Illness Curve, LIC, as a graphical representation of the membership degree that a person has to the fuzzy set *I* of Illness over time, that is:

$$y = \mu_I(x), t \tag{5-26}$$

In these parametric graphics, the vertical axis represents the membership degree $\mu_I(x)$, defined as usually in the interval [0,1], while the horizontal axis represents time in years, from 0 to 80 years old. The value $\mu_I(x) = 0$ means an absolutely absence of illness that is experienced by the human being only at birth, when no congenital disorder is present. We emphasize the condition "only at birth" because cellular deterioration begins just with life, albeit usually extremely slowly. The value $\mu_I(x) = 1$ means an integral and definitive presence of illness that happens only at the individual's *exitus*. In the next paragraphs we present some examples of Life Illness Curves.

Figure 5.16 shows a LIC of a normal, healthy individual. The value $\mu_I(x)$ remains low for almost the entire life of the person and only in the last months of his/her life the organism looses its healthy state.

Figure 5.17 represents the evolution in time of a patient affected by amyotrophic lateral sclerosis (ALS), where the value $\mu_I(x)$ remains low and normal until the disease's debut that leads into a relatively quick outcome (Brown 2010). This figure represents exactly the case of the famous baseball player Lou Gehrig, which suffered this condition from 1938 to 1941. Interestingly, Fig. 5.18 shows the LIC of the same illness, this time representing the case of the known cosmologist Stephen Hawking. The curve shows the debut of the condition in his twenties, a tracheotomy

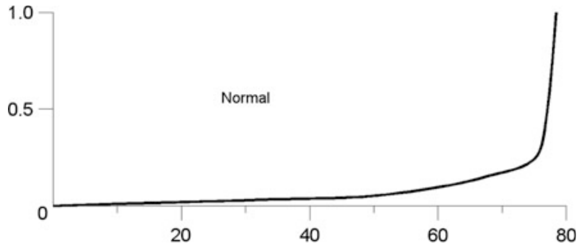


Fig. 5.16 LIC of a healthy person

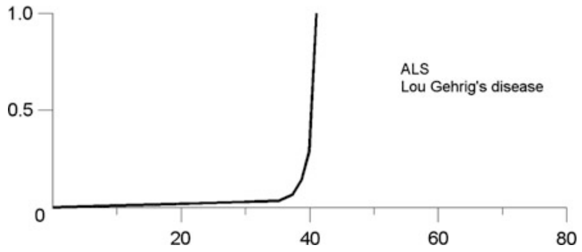


Fig. 5.17 LIC of a patient suffering Amyotrophic Lateral Sclerosis. Lou Gehrig's case

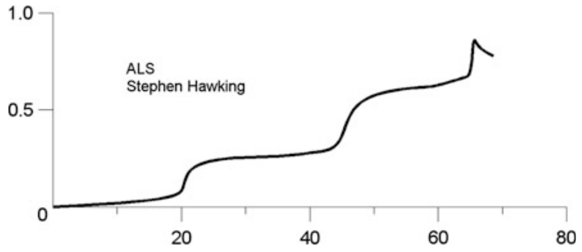


Fig. 5.18 LIC of a patient suffering Amyotrophic Lateral Sclerosis. Stephen Hawking's case

that resulted into a permanent aphonia in his forties and some severe infectious disorders at his sixties.

Figure 5.19 shows the LIC of a patient affected by type-1 diabetes mellitus that debuts at 15 years old, is correctly diagnosed and is treated adequately by means of insulin therapy, diet and exercise. As can be seen, under such circumstances the values $\mu_I(x), t$ remain relatively low through his life and only from his sixties he could start to suffer diabetes-related complications that spark the apparition of other conditions such as blindness, kidney failure and so on (Guyton 1996).

Figure 5.20 shows two possible evolutions of patients suffering acquired immune deficiency syndrome (AIDS): one of them living in a third world country (the one with the less favourable LIC) and another one living in a modern country. The difference in LICs is due to both a correct diagnostic and an appropriate treatment.

Figure 5.21 shows the LIC of a patient that has suffered a car accident in his thirties, resulting in permanent spine damage. Two main regions can be seen for the LIC: the one before the accident and the one after it. Despite the sharp increase in the $\mu_I(x)$ value resulting from the accident, the shape of each region resembles that of a normal life, like in Fig. 5.16.

We define a Life Quality Curve (LQC) as a graphical representation of the membership degree that a person has to the fuzzy set L of good quality of life over time. This curve is defined by the expression:

$$y = \mu_L(x) = (1 - \mu_I(x)),\ t \tag{5-27}$$

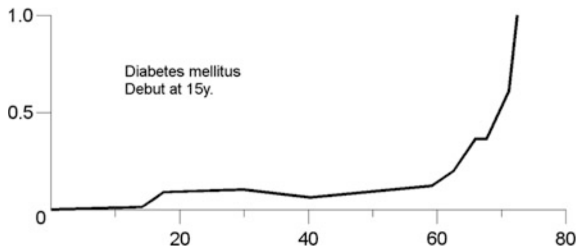


Fig. 5.19 LIC of a patient affected by diabetes with good diagnostic and treatment

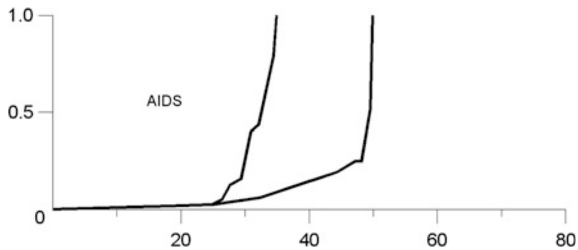


Fig. 5.20 LIC of a patient affected by AIDS. Two posible outcomes

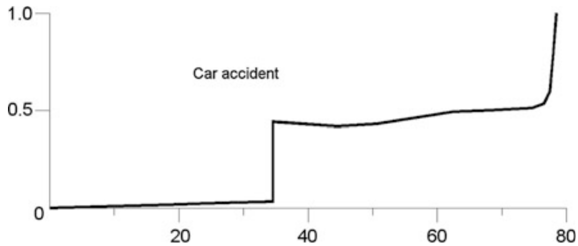
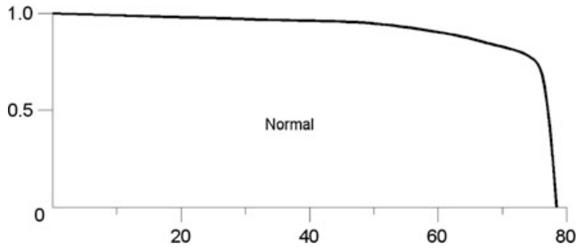


Fig. 5.21 LIC of a patient with permanent spine damage caused by an accident

Fig. 5.22 LQC of a normal individual



As can be easily seen, such an expression generates a symmetrical curve from LICs whose axis of symmetry is $y = 0.5$. In other words, LQCs represent the fuzzy complement of LICs. Since the meaning of such type of curves is immediate after having exposed LICs, we shall offer only three examples of LQCs, shown in Figs. 5.22, 5.23 and 5.24:

At this point we must note an important remark: both LIC and LQC curves are fuzzy and are not carved in stone, as we can immediately realize from Figs. 5.17 and 5.18, where the same illness, ALS, show two dramatically different behaviours in two different patients, although we must concede that Hawking's case is certainly rare. In any case, it's really interesting to note that the shape of LIC and LQC curves are affected by the perception of the person who observes the condition. Let us take again as an example the LQC of Stephen Hawking as shown in Fig. 5.25: While we can interpret the bold curve as the perception of a neurologist, it is more than likely

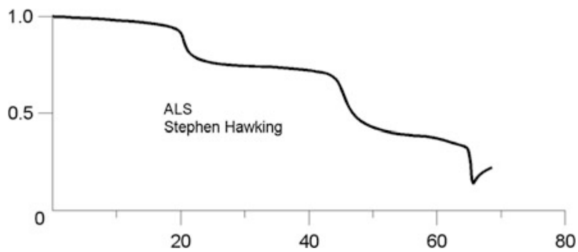


Fig. 5.23 LQC in Amyotrophic Lateral Sclerosis, Stephen Hawking case

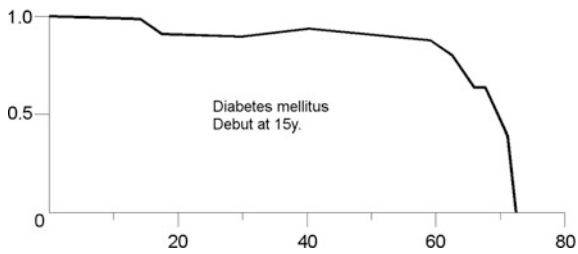


Fig. 5.24 LQC of a patient affected by diabetes with good diagnostic and treatment

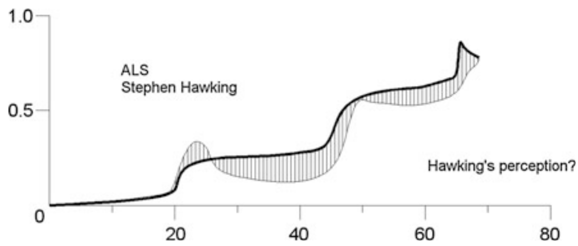


Fig. 5.25 Two possible LQC perceptions for ALS: Physician and patient

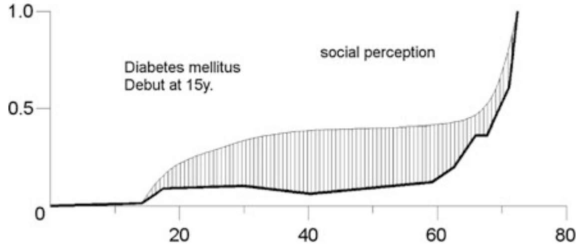


Fig. 5.26 Two possible LIC perceptions for diabetes: Patient and society

that the own patient's perception is different, expressed as an example by the thin curve in the graphic.

The difference in perception between patient and physician is not the only one at play. Society also usually perceives a given condition from a different point of view that the one from the affected person, as we can observe in Fig. 5.26 for diabetes, where the bold line shows a patient's possible own perception of the condition, used to daily subcutaneous insulin injections, while the fine line expresses a possible generalized external perception as a result of social lack of information about diabetes

Needless to say, the shape of Life Illness Curves and Life Quality Curves are not only dependent on the perception of the patient, physician or society, but also from the family environment, social class, economic scenario, politics, etc.

5.5 As a Summary

Lofti Zadeh was the man that back in 1965 realized that the membership of an element x to a set *A* can be expressed as a real number between 0 and 1. Since in mathematics, any closed interval of real numbers [a,b] contains infinite numbers, then this holds too for the closed interval [0,1] and hence, there are infinite membership degrees between 0 (meaning no membership to a set) and 1 (meaning a whole membership degree to a set). These types of sets are named fuzzy sets.

The seminal paper, written in late 1964 was published in 1965 and titled "Fuzzy Sets". The first words of its abstract say: "A fuzzy set is a class of objects with a continuum of grades of membership. Such a set is characterized by a membership (characteristic) function which assigns to each object a grade of membership ranging between zero and one".

Nowadays, a usual definition of a fuzzy set is the following one: A fuzzy set *A* is defined by a characteristic function $\mu_A(x)$ that maps every element x belonging to A to the closed interval of real numbers [0,1]. Formally:

$$A = \{(x, \mu_A(x)) \mid x \in A, \mu_A(x) \in [0,1]\}$$

That is, we can create a fuzzy set by means of enumerating a collection of ordered pairs $(x_i, \mu_A(x_i))$ where $\mu_A(x_i)$ is the membership degree of an element x_i to the fuzzy set A.

Geometry is an excellent tool for understanding fuzzy sets. In fact, grid and radar-type graphics are convenient and expressive tools for visualizing the meaning of fuzzy sets. When using grid-type graphics, the elements of the set are represented on the horizontal axis, while their respective membership degrees are shown on the vertical axis. Radar-type graphics are an enhancement of classic Venn diagrams where several contour lines show the different membership degrees. The external contour line usually shows the 0.0 membership degree, and then, the inner the circle, the bigger the membership degree until reaching a 1.0 value, located at the centre of the diagram.

In general, fuzzy sets satisfies all the properties of crisp sets, that is, the identity, idempotent, associative, commutative, De Morgan's Laws and distributive properties are perfectly satisfied by fuzzy sets. However, the complement properties are not satisfied in fuzzy sets theory. Especially significant is the so named 'law of non contradiction'. In classic sets theory it holds that: $A \cap A' = \phi$, but when operating with fuzzy sets, generally it holds that $\boldsymbol{A \cap A' \neq \phi}$. That is, some elements of a fuzzy set *A* belong both to *A* and to its complement *A′*. As examples, some men can be young and old at the same time, some cars can be fast and slow, some houses can be expensive and cheap at the same time, etc. Since we use membership degrees in fuzzy sets there is no contradiction in these statements. In fact these types of statements reflect many times just the things we observe in nature. We can remember in this moment the famous words by A. Einstein: "as far as the laws of mathematics refer to reality, they are not certain, and as far they are certain, they do not refer to reality".

Another important point in this chapter is the fact that membership degrees are not probabilities. If I say that an element x has a membership degree of 0.7 to a fuzzy set *A*, that is, $\mu_A(x) = 0.7$, I'm affirming something that actually exists or is intrinsic to an existing system. I'm in fact describing in a meaningful way a property of x with respect to the fuzzy set *A*. On the other hand, if I say that an event x has an associated probability 0.7, that is, $p(x) = 0.7$, I'm giving a measure of the likelihood that this event will happen in the future. As an example: let us imagine a black bag containing seven red balls and three black balls. The probability of extracting a red ball is $p(x) = 0.7$, but this value only exists before extracting the ball. While in probability there is always a random substratum, fuzzy sets theory deals with descriptions of existing, observable features of reality.

Fuzzy relations are an extension of classic, crisp relations between sets. While the classic theory of sets tells us if an element x of a set X is or is not related to an element y in a set Y, fuzzy relations inform us of the strength of the relation between elements, expressed in the closed interval [0,1]. Fuzzy relations can be established between elements from crisp sets or from elements belonging to fuzzy sets. This latter type of relation is the most interesting one for us in this book.

An engaging application of membership degrees in fuzzy sets is the construction of parametric curves of the type $y = \mu(x)$, t where t represents time along the horizontal axis and $\mu(x)$ represents membership degrees along the vertical axis. That is, this type of curves show the membership degree of an element x to a fuzzy set *A* along time. In this chapter we have seen as an example how the Life Illness Curves, ILC, and Life Quality Curves, LQC, can be implemented in such a way.

In the following chapter we shall continue exposing material from the fuzzy sets theory. I am sure many questions have flourished in the reader's mind after reading this chapter. I hope at least some of them will find its answer in the next one.

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