One Approach to Design the Fuzzy Fault Detection Filters for Takagi-Sugeno Models

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Abstract The paper relates a principle for designing the fuzzy fault detection filters devoted to a class of continuous-time nonlinear systems represented by Takagi-Sugeno models. The extension of the fuzzy reference model principle and the incremental quadratic constraints are proposed to obtain an approximation of H_∞/H_- criterion in the residual weight matrix parameter design for TS fuzzy fault detection filters. The design conditions are outlined in terms of linear matrix inequalities to posses a stable design framework.

Keywords Fault detection · Quadratic performance · Lyapunov function · Fuzzy models · Matrix formulation

1 Introduction

The fault detection filters (FDF) are mostly used to generate fault residual signals in active fault tolerant control systems. Because it is generally not possible to decouple fault effects from the perturbation influence in residuals [5, 7], the H_∞/H_- approach is used to tackle this conflict [10, 11]. Since faults are usually detected by setting a threshold on the residual signals, determination of the actual threshold is formulated as an adaptive threshold task [8]. Other approaches reduce FDF design to H_∞ problem to discriminate fault and disturbance effects in FDF signals by using the reference residual models (RRM) [3, 4].

By providing the possibility of weighting linear state-space representations of the class of nonlinear systems, the Takagi-Sugeno (TS) fuzzy approach [17], which

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avails mainly sector system dynamics approximation, have attracted noticeable penetration in fault detection [12, 15, 16] and estimation [9], usually utilizing the linear matrix inequality (LMI) design condition formulation. Building upon the theory of the systems whose some nonlinear time-varying terms satisfy the incremental quadratic constraints (IQC) [1, 2], in the paper the TS fuzzy models with local nonlinear terms are folded to design TS FDFs. In conjunction with RRM, it is demonstrated that IQC, parameterized by a multiplier matrix, can be reflected in LMI design conditions. The principle consists in generating the residual signals vector and estimating the subset of unmeasurable premise variables.

Throughout the paper $\boldsymbol{x}^T$, $\boldsymbol{X}^T$ denotes the transpose of the vector $\boldsymbol{x}$ and matrix $\boldsymbol{X}$, respectively, $\boldsymbol{X} = \boldsymbol{X}^T > 0$ means that $\boldsymbol{X}$ is a symmetric positive definite matrix, the symbol $\boldsymbol{I}_n$ indicates the nth order identity matrix, $I\!R$ denotes the set of real numbers, $I\!R^{n\times r}$ refers to the set of all $n \times r$ real matrices and $L_2\langle 0, +\infty)$ entails the space of square-integrable vector functions over $\langle 0, +\infty)$.

2 Takagi-Sugeno Fuzzy Fault Detection Filter

The systems under consideration belong to the class of MIMO nonlinear dynamic continuous-time systems, described by using TS approach as follows

$$\dot{\boldsymbol{q}}(t) = \sum_{i=1}^{s} \mathrm{h}_i(\boldsymbol{\theta}(t)) \left(\boldsymbol{A}_i \boldsymbol{q}(t) + \boldsymbol{B}_i \boldsymbol{u}(t) + \boldsymbol{E}_i \mathbf{p}(t) + \mathbf{B}_{fi} \boldsymbol{f}(t) + \mathbf{B}_{di} \boldsymbol{d}(t) \right) \tag{1}$$

$$\boldsymbol{y}(t) = \boldsymbol{C}\boldsymbol{q}(t) \tag{2}$$

where $\boldsymbol{q}(t) \in I\!R^n$, $\boldsymbol{u}(t) \in I\!R^r$, $\boldsymbol{y}(t) \in I\!R^m$ are vectors of the state, input, and output variables, respectively, $\boldsymbol{C} \in I\!R^{m\times n}$, $\boldsymbol{A}_i \in I\!R^{n\times n}$, $\boldsymbol{B}_i \in I\!R^{n\times r}$, $\boldsymbol{E}_i \in I\!R^{n\times r_p}$, $\boldsymbol{B}_{fi} \in I\!R^{n\times r_f}$, $\boldsymbol{B}_{di} \in I\!R^{n\times r_d}$, $i = 1, 2, \ldots s$, are constant matrices and $\boldsymbol{d}(t) \in I\!R^{r_d}$ is the disturbance input that belongs to $L_2\langle 0, +\infty)$. Here s, v are the numbers of sub-models and premise variables, the vector of premise variables is

$$\boldsymbol{\theta}(t) = \left[\theta_1(t)\ \theta_2(t) \cdots \theta_v(t) \right] \tag{3}$$

and $\mathrm{h}_i(\theta(t))$ is the i-th membership function satisfying the following properties

$$0 \le \mathrm{h}_i(\boldsymbol{\theta}(t)) \le 1, \quad \sum_{i=1}^{s} \mathrm{h}_i(\boldsymbol{\theta}(t)) = 1 \text{ for all } i \in \langle 1, \ldots, s \rangle \tag{4}$$

The nonlinear function $\mathbf{p}(t) \in I\!R^{r_p}$ is a bounded function of $\boldsymbol{q}(t)$, given as [2]

$$\mathbf{p}(t) = \varphi\left(\boldsymbol{U}\boldsymbol{q}(t) + \boldsymbol{W}\mathbf{p}(t)\right) \tag{5}$$

where $\boldsymbol{U} \in I\!R^{m_p \times n}$, $\boldsymbol{W} \in I\!R^{m_p \times r_p}$ are constant matrices. Note, if $\mathbf{p}(t)$ does not depends on the derivative of a system state variable then $\boldsymbol{W}$ is zero matrix.

It is considered that a fault $\boldsymbol{f}(t)$ may occur at an uncertain time, the size of the fault is unknown but bounded and all pairs $(\boldsymbol{A}_i, \boldsymbol{C})$, $i = 1, 2, \dots s$, are observable (more details can be found, e.g., in [12, 13]).

Considering (1) and (2) TS FDF is defined as

$$\dot{\boldsymbol{q}}_e(t) = \sum_{i=1}^{s} \mathrm{h}_i(\boldsymbol{\theta}(t)) \left(\boldsymbol{A}_i \boldsymbol{q}_e(t) + \boldsymbol{B}_i \boldsymbol{u}(t) + \boldsymbol{E}_i \mathbf{p}_e(t) + \boldsymbol{J}_i \left(\boldsymbol{y}(t) - \boldsymbol{y}_e(t) \right) \right) \tag{6}$$

$$\mathbf{p}_e(t) = \varphi \left(\boldsymbol{U} \boldsymbol{q}_e(t) + \boldsymbol{W} \mathbf{p}_e(t) + \boldsymbol{J}(\boldsymbol{y}(t) - \boldsymbol{y}_e(t)) \right) \tag{7}$$

$$\boldsymbol{r}(t) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t)) \boldsymbol{V}_i \left(\boldsymbol{y}(t) - \boldsymbol{y}_e(t) \right) , \quad \boldsymbol{y}_e(t) = \boldsymbol{C} \boldsymbol{q}_e(t) \tag{8}$$

where $\boldsymbol{q}_e(t) \in I\!R^n$ is estimate of $\boldsymbol{q}(t)$, $\boldsymbol{y}_e(t) \in I\!R^m$ is the observed output vector, $\mathbf{r}(t) \in I\!R^{m_r}$ is the residual signal, $\mathbf{p}_e(t) \in I\!R^{r_p}$ is estimate of $\mathbf{p}(t)$, and $\boldsymbol{J} \in I\!R^{m_p \times m}$, $\boldsymbol{J}_i \in I\!R^{n \times m}$, $\boldsymbol{V}_i \in I\!R^{m \times m}$, $i = 1, 2, \dots, s$, is the set of gains.

Defining the notations

$$\boldsymbol{e}(t) = \boldsymbol{q}(t) - \boldsymbol{q}_e(t) , \quad \delta \mathbf{p}(t) = \mathbf{p}(t) - \mathbf{p}_e(t) , \quad \boldsymbol{A}_{ei} = \boldsymbol{A}_i - \boldsymbol{J}_i \boldsymbol{C} \tag{9}$$

the deviation form of TS FDF is

$$\dot{\boldsymbol{e}}(t) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t)) \left(\boldsymbol{A}_{ei} \boldsymbol{e}(t) + \boldsymbol{B}_{fi} \boldsymbol{f}(t) + \boldsymbol{B}_{di} \boldsymbol{d}(t) + \boldsymbol{E}_i \delta \mathbf{p}(t) \right) \tag{10}$$

$$\mathbf{p}_e(t) = \varphi \left(\boldsymbol{U} \boldsymbol{q}_e(t) + \boldsymbol{W} \mathbf{p}_e(t) + \boldsymbol{J} \boldsymbol{C} \boldsymbol{e}(t) \right) \tag{11}$$

$$\boldsymbol{r}(t) = \sum_{i=1}^{s} h_i(\theta(t)) \boldsymbol{V}_i \boldsymbol{C} \boldsymbol{e}(t) \tag{12}$$

In the following it is assumed that the premise variables associated with $\mathbf{p}(t)$ are unmeasurable while all others premise variables are measurable.

To explain basic relationships, the following lemma is presented.

Lemma 1 *If a matrix $\boldsymbol{M} \in \mathcal{M}$, where $\mathcal{M}$ is the set of real incremental multiplier matrices of dimension $(m_p + r_p) \times (m_p + r_p)$, then for the given matrices $\boldsymbol{U} \in I\!R^{m_p \times n}$, $\boldsymbol{W} \in I\!R^{m_p \times r_p}$, $\boldsymbol{J} \in I\!R^{m_p \times m}$ and $\boldsymbol{C} \in I\!R^{m \times n}$ IQC is*

$$\left[\boldsymbol{e}^T(t) \ \delta \mathbf{p}^T(t) \right] \boldsymbol{N} \begin{bmatrix} \boldsymbol{e}(t) \\ \delta \mathbf{p}(t) \end{bmatrix} \geq 0 \tag{13}$$

$$\boldsymbol{N} = \begin{bmatrix} (\boldsymbol{U}-\boldsymbol{J}\boldsymbol{C})^T & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{r_p} \end{bmatrix} \boldsymbol{Q}^T \boldsymbol{M} \boldsymbol{Q} \begin{bmatrix} \boldsymbol{U}-\boldsymbol{J}\boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{r_p} \end{bmatrix}, \quad \boldsymbol{Q} = \begin{bmatrix} \boldsymbol{I}_{m_p} & \boldsymbol{W} \\ \boldsymbol{0} & \boldsymbol{I}_{r_p} \end{bmatrix} \tag{14}$$

and $\boldsymbol{I}_{m_p} \in I\!R^{m_p \times m_p}$, $\boldsymbol{I}_{r_p} \in I\!R^{r_p \times r_p}$ *are identity matrices of given dimensions.*

Proof (compare [1, 14]) Writing (11) as follows

$$\begin{aligned} \mathbf{p}_e(t) &= \varphi\left(\boldsymbol{U}(\boldsymbol{q}(t)-\boldsymbol{e}(t)) + \boldsymbol{W}\mathbf{p}_e(t) + \boldsymbol{J}(\boldsymbol{C}\boldsymbol{q}(t) - \boldsymbol{C}(\boldsymbol{q}(t)-\boldsymbol{e}(t))\right) \\ &= \varphi\left(\boldsymbol{U}\boldsymbol{q}(t) + \boldsymbol{W}\mathbf{p}_e(t) - (\boldsymbol{U}-\boldsymbol{J}\boldsymbol{C})\boldsymbol{e}(t)\right) \end{aligned} \tag{15}$$

and introducing the variables

$$z_1(t) = \boldsymbol{U}\boldsymbol{q}(t) + \boldsymbol{W}\mathbf{p}(t), \quad z_2(t) = \boldsymbol{U}\boldsymbol{q}(t) + \boldsymbol{W}\mathbf{p}_e(t) - (\boldsymbol{U}-\boldsymbol{J}\boldsymbol{C})\boldsymbol{e}(t) \tag{16}$$

$$\delta z(t) = z_1(t) - z_2(t) = (\boldsymbol{U}-\boldsymbol{J}\boldsymbol{C})\boldsymbol{e}(t) + \boldsymbol{W}\delta\mathbf{p}(t) \tag{17}$$

then (15) and (16) implies

$$\delta\mathbf{p}(t) = \mathbf{p}(t) - \mathbf{p}_e(t) = \varphi(z_1(t)) - \varphi(z_2(t)) = \delta\varphi(t) \tag{18}$$

Writing (17) and (18) compactly, it yields for a symmetric $\boldsymbol{M} \in \mathcal{M}$

$$\begin{bmatrix} \delta z(t) \\ \delta\varphi(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{U}-\boldsymbol{J}\boldsymbol{C} & \boldsymbol{W} \\ \boldsymbol{0} & \boldsymbol{I}_{r_p} \end{bmatrix} \begin{bmatrix} \boldsymbol{e}(t) \\ \delta\mathbf{p}(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_{m_p} & \boldsymbol{W} \\ \boldsymbol{0} & \boldsymbol{I}_{r_p} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}-\boldsymbol{J}\boldsymbol{C} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{r_p} \end{bmatrix} \begin{bmatrix} \boldsymbol{e}(t) \\ \delta\mathbf{p}(t) \end{bmatrix} \tag{19}$$

$$\begin{bmatrix} \delta z^T(t) & \delta\varphi^T(t) \end{bmatrix} \boldsymbol{M} \begin{bmatrix} \delta z(t) \\ \delta\varphi(t) \end{bmatrix} = \begin{bmatrix} \boldsymbol{e}^T(t) & \delta\mathbf{p}^T(t) \end{bmatrix} \boldsymbol{N} \begin{bmatrix} \boldsymbol{e}(t) \\ \delta\mathbf{p}(t) \end{bmatrix} \geq 0 \tag{20}$$

where $\boldsymbol{N}$ takes the form (14). This concludes the proof. ■

3 Reference Residual Model

The reference residual model in the proposed structure provides a pattern that partly separates from the observer data model the interactions represented by cross-bonds in $\boldsymbol{d}(t)$ and $\boldsymbol{f}(t)$ and also those given by residual weight matrices $\boldsymbol{V}_i$ in $\boldsymbol{r}(t)$. Thus, by formalizing RRM in an appropriate mathematical framework, these interaction properties are modified for RRM design while the other related functions are keep together in defined layer of complexity and extensibility.

Taking into account the above, then (12) is reduced to

$$\boldsymbol{r}(t) = \boldsymbol{C}\boldsymbol{e}(t) \tag{21}$$

and (10) can be rewritten for $r_g = r_f = r_d$ as

$$\dot{\boldsymbol{e}}(t) = \sum_{i=1}^{s} h_i(\theta(t)) \left(\boldsymbol{A}_{ei}\boldsymbol{e}(t) + \boldsymbol{E}_i \mathbf{p}_e(t) + \begin{bmatrix} \boldsymbol{B}_{di} & -\boldsymbol{B}_{fi} \end{bmatrix} \begin{bmatrix} \boldsymbol{d}(t) \\ -\boldsymbol{f}(t) \end{bmatrix} \right) \tag{22}$$

Inserting the same cross-bonds between $\boldsymbol{d}(t)$ and $\boldsymbol{f}(t)$ (22) can be redefined as

$$\dot{\boldsymbol{e}}^{\circ}(t) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t)) \left(\boldsymbol{A}_{ei}^{\circ}\boldsymbol{e}^{\circ}(t) + \boldsymbol{E}_i^{\circ} \mathbf{p}_e^{\circ}(t) + \begin{bmatrix} \boldsymbol{B}_{di} & -\boldsymbol{B}_{fi} \end{bmatrix} \boldsymbol{T}^{\circ} \begin{bmatrix} \boldsymbol{d}(t) \\ -\boldsymbol{f}(t) \end{bmatrix} \right) \tag{23}$$

$$\boldsymbol{A}_{ei}^{\circ} = \boldsymbol{A}_i - \boldsymbol{J}_i^{\circ}\boldsymbol{C}, \quad \boldsymbol{E}_i^{\circ} = \boldsymbol{E}_i \tag{24}$$

where the cross-bonds matrix $\boldsymbol{T}^{\circ}$ was selected as follows:

$$\boldsymbol{T}^{\circ} = \begin{bmatrix} \boldsymbol{I}_{r_g} & \boldsymbol{I}_{r_g} \\ \boldsymbol{I}_{r_g} & \boldsymbol{I}_{r_g} \end{bmatrix} = \begin{bmatrix} \boldsymbol{I}_{r_g} \\ \boldsymbol{I}_{r_g} \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_{r_g} & \boldsymbol{I}_{r_g} \end{bmatrix} = \boldsymbol{H}^{\circ}\boldsymbol{H}^{\circ T}, \quad \boldsymbol{H}^{\circ T} = \begin{bmatrix} \boldsymbol{I}_{r_g} & \boldsymbol{I}_{r_g} \end{bmatrix} \tag{25}$$

Denoting

$$\boldsymbol{G}_i^{\circ} = \begin{bmatrix} \boldsymbol{B}_{di} & -\boldsymbol{B}_{fi} \end{bmatrix} \boldsymbol{H}^{\circ}, \quad \boldsymbol{g}^{\circ}(t) = \boldsymbol{H}^{\circ T} \begin{bmatrix} \boldsymbol{d}(t) \\ -\boldsymbol{f}(t) \end{bmatrix} = \boldsymbol{d}(t) - \boldsymbol{f}(t) \tag{26}$$

where $\boldsymbol{G}_i^{\circ} \in I\!R^{n \times r_d}$, $\boldsymbol{g}^{\circ}(t) \in I\!R^{r_d}$, RRM can be written as

$$\dot{\boldsymbol{e}}^{\circ}(t) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t)) \left(\boldsymbol{A}_{ei}\boldsymbol{e}^{\circ}(t) + \boldsymbol{E}_i \delta\mathbf{p}^{\circ}(t) + \boldsymbol{G}_i^{\circ}\boldsymbol{g}^{\circ}(t) \right) \tag{27}$$

$$\boldsymbol{r}^{\circ}(t) = \boldsymbol{C}\boldsymbol{e}^{\circ}(t), \quad \delta\mathbf{p}^{\circ}(t) = \mathbf{p}(t) - \mathbf{p}_e^{\circ}(t) \tag{28}$$

$$\mathbf{p}_e^{\circ}(t) = \varphi \left(\boldsymbol{U}\boldsymbol{q}_e^{\circ}(t) + \boldsymbol{W}\mathbf{p}_e^{\circ}(t) + \boldsymbol{J}^{\circ}\boldsymbol{C}\boldsymbol{e}^{\circ}(t) \right) \tag{29}$$

The observer parameters $\boldsymbol{J}^{\circ} \in I\!R^{m_p \times m}$, $\boldsymbol{J}_i^{\circ} \in I\!R^{n \times m}$ for $i = 1, 2, \dots, s$ have to be designed in such a way that

$$\|\boldsymbol{r}^{\circ}(t)\|_{\infty}^{2} \le \gamma^{\circ} \|\boldsymbol{g}^{\circ}(t)\|_{\infty}^{2} \tag{30}$$

for the square of H_{∞} norm γ° is as small as possible. Since, with respect to (26),

$$\|\boldsymbol{\Gamma}^{\circ}\|_{\infty} = \|\boldsymbol{\Gamma}_d^{\circ} - \boldsymbol{\Gamma}_f^{\circ}\|_{\infty} \ge \|\boldsymbol{\Gamma}_d^{\circ}\|_{\infty} - \|\boldsymbol{\Gamma}_f^{\circ}\|_{\infty} \tag{31}$$

where $\|\boldsymbol{\Gamma}^{\circ}\|_{\infty}$ is the H_{∞} norm of the residual transfer function matrix with respect to $\boldsymbol{g}^{\circ}$, then minimizing γ° means maximizing $\|\boldsymbol{\Gamma}_{f}^{\circ}\|_{\infty}$ while minimizing $\|\boldsymbol{\Gamma}_{d}^{\circ}\|_{\infty}$. This minimizes impact of disturbance on the residual signal amplitude.

Remark 1 In order to use the above forms when $r_f \neq r_d$, the corresponding degenerative input matrix is extended by $|r_f - r_d|$ zero columns and the corresponding vector by $|r_f - r_d|$ zero elements to the common dimension $r_g = \max(r_f, r_d)$.

Theorem 1 *The reference residual model (28) and (29) is stable with the quadratic performance γ° if there exist symmetric positive definite matrices $\boldsymbol{P}^{\circ} \in I\!R^{n \times n}$, $\boldsymbol{X}^{\circ} \in I\!R^{m_p \times m_p}$, $\boldsymbol{Y}^{\circ} \in I\!R^{r_p \times r_p}$, matrices $\boldsymbol{Z}^{\circ} \in I\!R^{m_p \times m}$, $\boldsymbol{Z}_i^{\circ} \in I\!R^{n \times m}$, $i = 1, 2, \ldots, s$, and a positive scalar $\gamma^{\circ} \in I\!R$ such that for all i*

$$\boldsymbol{P}^{\circ} = \boldsymbol{P}^{\circ T} > 0,\ \boldsymbol{X}^{\circ} = \boldsymbol{X}^{\circ T} > 0,\ \boldsymbol{Y}^{\circ} = \boldsymbol{Y}^{\circ T} > 0,\ \gamma^{\circ} > 0 \tag{32}$$

$$\begin{bmatrix} \boldsymbol{P}^{\circ}\boldsymbol{A}_i + \boldsymbol{A}_i^T\boldsymbol{P}^{\circ} - \boldsymbol{Z}_i^{\circ}\boldsymbol{C} - \boldsymbol{C}^T\boldsymbol{Z}_i^{\circ T} & * & * & * & * \\ \boldsymbol{E}_i^{\circ T}\boldsymbol{P}^{\circ} & -\boldsymbol{Y}^{\circ} & * & * & * \\ \boldsymbol{G}_i^{\circ T}\boldsymbol{P}^{\circ} & \boldsymbol{0} & -\gamma^{\circ}\boldsymbol{I}_{r_g} & * & * \\ \boldsymbol{C} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I}_m & * \\ \boldsymbol{X}^{\circ}\boldsymbol{U} - \boldsymbol{Z}^{\circ}\boldsymbol{C} & \boldsymbol{X}^{\circ}\boldsymbol{W} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{X}^{\circ} \end{bmatrix} < 0 \tag{33}$$

When the above conditions hold, the reference model gain matrices are given as

$$\boldsymbol{J}_i^{\circ} = (\boldsymbol{P}^{\circ})^{-1}\boldsymbol{Z}_i^{\circ}, \quad \boldsymbol{J}^{\circ} = (\boldsymbol{X}^{\circ})^{-1}\boldsymbol{Z}^{\circ} \tag{34}$$

Hereafter, $*$ denotes the symmetric item in a symmetric matrix.

Proof Defining the Lyapunov function candidate

$$v(\boldsymbol{e}^{\circ}(t)) = \boldsymbol{e}^{\circ T}(t)\boldsymbol{P}^{\circ}\boldsymbol{e}^{\circ}(t) + \int_0^t \left(\boldsymbol{r}^{\circ T}(x)\boldsymbol{r}^{\circ}(x) - \gamma^{\circ}\,\boldsymbol{g}^{\circ T}(x)\boldsymbol{g}^{\circ}(x)\right) \mathrm{d}x \tag{35}$$

then, after evaluation the derivative of (35), it is obtained

$$\dot{v}(\boldsymbol{e}^{\circ}(t)) = \dot{\boldsymbol{e}}^{\circ T}(t)\boldsymbol{P}^{\circ}\boldsymbol{e}^{\circ}(t) + \boldsymbol{e}^{\circ T}(t)\boldsymbol{P}^{\circ}\dot{\boldsymbol{e}}^{\circ}(t) + \boldsymbol{r}^{\circ T}(t)\boldsymbol{r}^{\circ}(t) - \gamma^{\circ}\,\boldsymbol{g}^{\circ T}(t)\boldsymbol{g}^{\circ}(t) < 0 \tag{36}$$

Substitution of (27) and (28) into (36) gives

$$\begin{aligned} \dot{v}(\boldsymbol{e}^{\circ}(t)) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))(\boldsymbol{e}^{\circ T}(t)\boldsymbol{P}^{\circ}\boldsymbol{E}_i^{\circ}\delta\mathbf{p}^{\circ}(t) + \delta\mathbf{p}^{\circ T}(t)\boldsymbol{E}_i^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{e}^{\circ}(t)) - \\ -\gamma^{\circ}\boldsymbol{g}^{\circ T}(t)\boldsymbol{g}^{\circ}(t)) + \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\boldsymbol{e}^{\circ T}(t)(\boldsymbol{A}_{ei}^{\circ T}\boldsymbol{P}^{\circ} + \boldsymbol{P}^{\circ}\boldsymbol{A}_{ei}^{\circ} + \boldsymbol{C}^T\boldsymbol{C})\boldsymbol{e}^{\circ}(t) + \\ + \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))(\boldsymbol{e}^{\circ T}(t)\boldsymbol{P}^{\circ}\boldsymbol{G}_i^{\circ}\boldsymbol{g}^{\circ}(t) + \boldsymbol{g}^{\circ T}(t)\boldsymbol{G}_i^{\circ T}\boldsymbol{P}^{\circ}\boldsymbol{e}^{\circ}(t)) < 0 \end{aligned} \tag{37}$$

Thus, defining the composed vector

$$\boldsymbol{e}_c^{\circ T}(t) = \left[\boldsymbol{e}^{\circ T}(t)\ \delta\mathbf{p}^{\circ T}(t)\ \boldsymbol{g}^{\circ T}(t)\right] \tag{38}$$

(37) can be defined as follows

$$\dot{v}(\boldsymbol{e}^{\circ}(t)) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\boldsymbol{e}_c^{\circ T}(t)\boldsymbol{P}_{ci}^{\circ}\boldsymbol{e}_c^{\circ}(t) \le -\boldsymbol{e}_c^{\circ T}(t)\boldsymbol{N}^{\diamond}\boldsymbol{e}_c^{\circ}(t) < 0 \tag{39}$$

$$\boldsymbol{P}_{ci}^{\circ} = \begin{bmatrix} \boldsymbol{P}^{\circ}\boldsymbol{A}_{ei}^{\circ} + \boldsymbol{A}_{ei}^{\circ T}\boldsymbol{P}^{\circ} + \boldsymbol{C}^T\boldsymbol{C} & * & * \\ \boldsymbol{E}_i^{\circ T}\boldsymbol{P}^{\circ} & \boldsymbol{0} & * \\ \boldsymbol{G}_i^{\circ T}\boldsymbol{P}^{\circ} & \boldsymbol{0} & -\gamma^{\circ}\boldsymbol{I}_{r_g} \end{bmatrix}, \quad \boldsymbol{N}^{\diamond} = \mathrm{diag}\left[\boldsymbol{N}^{\circ}\ \boldsymbol{0}\right] \tag{40}$$

which gives

$$\dot{v}(\boldsymbol{e}^{\circ}(t)) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t))\boldsymbol{e}_c^{\circ T}(t)(\boldsymbol{P}_{ci}^{\circ} + \boldsymbol{N}^{\diamond})\boldsymbol{e}_c^{\circ}(t) < 0 \tag{41}$$

Introducing $\boldsymbol{U}_e^{\circ}$ and defining the incremental multiplier matrix as follows

$$\boldsymbol{U}_e^{\circ} = \boldsymbol{U} - \boldsymbol{J}^{\circ}\boldsymbol{C}, \quad \boldsymbol{M}^{\circ} = \mathrm{diag}\left[\boldsymbol{X}^{\circ}\ -\boldsymbol{Y}^{\circ}\right] \tag{42}$$

where $\boldsymbol{X}^{\circ} \in I\!R^{m_p \times m_p}$, $\boldsymbol{Y}^{\circ} \in I\!R^{r_p \times r_p}$ are symmetric positive definite matrices, then (14) and (19) implies

$$\begin{aligned} \boldsymbol{N}^{\circ} &= \begin{bmatrix} \boldsymbol{U}_e^{\circ T} & \boldsymbol{0} \\ \boldsymbol{W}^T & \boldsymbol{I}_{r_p} \end{bmatrix} \begin{bmatrix} \boldsymbol{X}^{\circ} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{Y}^{\circ} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_e^{\circ} & \boldsymbol{W} \\ \boldsymbol{0} & \boldsymbol{I}_{r_p} \end{bmatrix} = \\ &= \begin{bmatrix} \boldsymbol{U}_e^{\circ T}\boldsymbol{X}^{\circ} \\ \boldsymbol{W}^T\boldsymbol{X}^{\circ} \end{bmatrix} (\boldsymbol{X}^{\circ})^{-1} \left[\boldsymbol{X}^{\circ}\boldsymbol{U}_e^{\circ}\ \boldsymbol{X}^{\circ}\boldsymbol{W}\right] - \begin{bmatrix} \boldsymbol{0} \\ \boldsymbol{I}_{r_p} \end{bmatrix} \boldsymbol{Y}^{\circ} \left[\boldsymbol{0}\ \boldsymbol{I}_{r_p}\right] \end{aligned} \tag{43}$$

Thus, (41) is negative if $(\boldsymbol{P}_{ci}^{\circ} + \boldsymbol{N}^{\diamond})$ is negative definite, that is the matrix sum

$$\begin{bmatrix} \boldsymbol{U}_e^T\boldsymbol{X}^{\circ} \\ \boldsymbol{W}^T\boldsymbol{X}^{\circ} \\ \boldsymbol{0} \end{bmatrix} (\boldsymbol{X}^{\circ})^{-1} \left[\boldsymbol{X}^{\circ}\boldsymbol{U}_e\ \boldsymbol{X}^{\circ}\boldsymbol{W}\ \boldsymbol{0}\right] + \begin{bmatrix} \boldsymbol{P}^{\circ}\boldsymbol{A}_{ei}^{\circ} + \boldsymbol{A}_{ei}^{\circ T}\boldsymbol{P}^{\circ} + \boldsymbol{C}^T\boldsymbol{C} & * & * \\ \boldsymbol{E}_i^{\circ T}\boldsymbol{P}^{\circ} & -\boldsymbol{Y}^{\circ} & * \\ \boldsymbol{G}_i^{\circ T}\boldsymbol{P}^{\circ} & \boldsymbol{0} & -\gamma^{\circ}\boldsymbol{I}_{r_g} \end{bmatrix} \tag{44}$$

has to be negative definite. Applying twice the Schur complement property and using, with respect to (24) and (42), the notations

$$\boldsymbol{Z}_i^{\circ} = \boldsymbol{P}^{\circ}\boldsymbol{J}_i^{\circ}, \quad \boldsymbol{Z}^{\circ} = \boldsymbol{X}^{\circ}\boldsymbol{J}^{\circ} \tag{45}$$

then (44) implies (33). This concludes the proof. ■

4 FDF Design Condition

To rebind $\boldsymbol{r}(t)$ with RRM, the overall FDF model, incorporating (27) and (28) with $\boldsymbol{T}^{\circ} = \boldsymbol{I}_{2r_g}$, can be expressed as

$$\dot{\boldsymbol{e}}^{\bullet}(t) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t)) \left(\boldsymbol{A}^{\bullet}_{ci} \boldsymbol{e}^{\bullet}(t) + \boldsymbol{G}^{\bullet}_i \boldsymbol{g}^{\bullet}(t) + \boldsymbol{E}^{\bullet}_i \delta \mathbf{p}^{\bullet}(t) \right) \tag{46}$$

$$\boldsymbol{r}^{\bullet}(t) = \boldsymbol{r}(t) - \boldsymbol{r}^{\circ}(t) = \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t)) \boldsymbol{V}^{\star}_i \boldsymbol{C}^{\star} \boldsymbol{e}^{\bullet}(t) \tag{47}$$

$$\boldsymbol{A}^{\bullet}_{ei} = \boldsymbol{A}^{\bullet}_i - \boldsymbol{J}^{\bullet}_i \boldsymbol{C}^{\bullet}_i,\ \boldsymbol{E}^{\bullet}_i = \text{diag}\left[\boldsymbol{E}_i \ \ \boldsymbol{E}_i \right], \quad \boldsymbol{V}^{\star}_i = \left[\boldsymbol{V}_i \ -\boldsymbol{I}_m \right] \tag{48}$$

$$\boldsymbol{A}^{\bullet}_i = \text{diag}\left[\boldsymbol{A}_i \ \ \boldsymbol{A}_i \right], \quad \boldsymbol{C}^{\bullet}_i = \text{diag}\left[\boldsymbol{C}\ \boldsymbol{J}^{\circ}_i \boldsymbol{C} \right], \quad \boldsymbol{J}^{\bullet}_i = \text{diag}\left[\boldsymbol{J}_i\ \boldsymbol{I}_n \right] \tag{49}$$

$$\boldsymbol{C}^{\star} = \text{diag}\left[\boldsymbol{C}\ \boldsymbol{C} \right], \quad \boldsymbol{U}^{\star}_e = \boldsymbol{U} - \boldsymbol{J}\boldsymbol{C}, \quad \boldsymbol{U}^{\circ}_e = \boldsymbol{U} - \boldsymbol{J}^{\circ}\boldsymbol{C} \tag{50}$$

$$\boldsymbol{e}^{\bullet}(t) = \begin{bmatrix} \boldsymbol{e}(t) \\ \boldsymbol{e}^{\circ}(t) \end{bmatrix},\ \delta\mathbf{p}^{\bullet}(t) = \begin{bmatrix} \delta\mathbf{p}(t) \\ \delta\mathbf{p}^{\circ}(t) \end{bmatrix},\ \boldsymbol{g}^{\bullet}(t) = \begin{bmatrix} \boldsymbol{f}(t) \\ \boldsymbol{d}(t) \end{bmatrix},\ \boldsymbol{G}^{\bullet}_i = \begin{bmatrix} \boldsymbol{B}_{fi}\ \boldsymbol{B}_{di} \\ \boldsymbol{B}_{fi}\ \boldsymbol{B}_{di} \end{bmatrix} \tag{51}$$

and $\boldsymbol{e}^{\bullet}(t) \in I\!R^{2n}$, $\boldsymbol{g}^{\bullet}(t) \in I\!R^{2r_d}$, $\boldsymbol{G}^{\bullet}_i \in I\!R^{2n\times 2r_d}$, $\boldsymbol{A}^{\bullet}_i \in I\!R^{2n\times 2n}$, $\boldsymbol{V}^{\star}_i \in I\!R^{m\times 2m}$, $\boldsymbol{J}^{\bullet}_i \in I\!R^{2n\times(m+n)}$, $\boldsymbol{E}^{\bullet}_i \in I\!R^{2n\times 2r_p}$, $\boldsymbol{C}^{\bullet}_i \in I\!R^{(m+n)\times 2n}$, $\boldsymbol{U}^{\star}_e, \boldsymbol{U}^{\circ}_e \in I\!R^{m_p\times n}$, $\boldsymbol{C}^{\star} \in I\!R^{m\times 2n}$. It should be noted that these matrix structures must be defined for the existence of structured LMI matrix variables.

Remark 2 According to (47), formulation of the optimization criterion means that the double summation through membership functions occurs in the product $\boldsymbol{r}^{\bullet T}(t)\boldsymbol{r}^{\bullet}(t)$. Since $\sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t)) = 1$, for solving this optimization problem the following approximation is applied (proof see, e.g., in [13])

$$\begin{aligned} \boldsymbol{r}^{\bullet T}(t)\boldsymbol{r}^{\bullet}(t) = \boldsymbol{e}^{\bullet T}(t) \sum_{i=1}^{s}\sum_{j=1}^{s} h_i(\boldsymbol{\theta}(t)) h_j(\boldsymbol{\theta}(t)) \boldsymbol{C}^{\star T} \boldsymbol{V}^{\star T}_j \boldsymbol{V}^{\star}_i \boldsymbol{C}^{\circ} \boldsymbol{e}^{\bullet}(t) \leq \\ \leq \boldsymbol{e}^{\bullet T}(t) \sum_{i=1}^{s} h_i(\boldsymbol{\theta}(t)) \boldsymbol{C}^{\star T} \boldsymbol{V}^{\star T}_i \boldsymbol{V}^{\star}_i \boldsymbol{C}^{\star} \boldsymbol{e}^{\bullet}(t) \end{aligned} \tag{52}$$

Theorem 2 *The fault detection filter (46) and (47) associated with RRM (28) and (29) is stable with the quadratic performance* $\gamma^{\bullet}$ *if there exist symmetric positive definite matrices* $\boldsymbol{P}^{\star}_1, \boldsymbol{P}^{\star}_2 \in I\!R^{n\times n}$, $\boldsymbol{X}^{\star} \in I\!R^{m_p\times m_p}$, $\boldsymbol{Y}^{\star} \in I\!R^{r_p\times r_p}$, *matrices* $\boldsymbol{Z}^{\star} \in I\!R^{m_p\times m}$, $\boldsymbol{Z}^{\star}_i \in I\!R^{n\times m}$, $\boldsymbol{V}_i \in I\!R^{m\times n}$, $i = 1, 2, \ldots, s$ *and a positive scalars* $\gamma^{\bullet} \in I\!R$ *such that for all i*

$$\boldsymbol{P}^{\star}_1 = \boldsymbol{P}^{\star T}_1 > 0,\ \boldsymbol{P}^{\star}_2 = \boldsymbol{P}^{\star T}_2 > 0,\ \boldsymbol{X}^{\star} = \boldsymbol{X}^{\star T} > 0,\ \boldsymbol{Y}^{\star} = \boldsymbol{Y}^{\star T} > 0,\ \gamma^{\bullet} > 0 \tag{53}$$

$$\begin{bmatrix} \boldsymbol{P}^{\bullet}\boldsymbol{A}_i^{\bullet} + \boldsymbol{A}_i^{\bullet T}\boldsymbol{P}^{\bullet} - \boldsymbol{Z}_i^{\bullet}\boldsymbol{C}_i^{\bullet} - \boldsymbol{C}_i^{\bullet T}\boldsymbol{Z}_i^{\bullet T} & * & * & * & * \\ \boldsymbol{E}_i^{\bullet T}\boldsymbol{P}^{\bullet} & -\boldsymbol{I}^{\bullet}\boldsymbol{Y}^{\bullet}\boldsymbol{I}^{\bullet} & * & * & * \\ \boldsymbol{G}_i^{\bullet T}\boldsymbol{P}^{\bullet} & \boldsymbol{0} & -\gamma^{\bullet}\boldsymbol{I}_{2r_g} & * & * \\ \boldsymbol{V}_i^{\star}\boldsymbol{C}^{\star} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{I}_m & * \\ \boldsymbol{X}^{\bullet}\boldsymbol{U}^{\bullet} - \boldsymbol{Z}^{\bullet}\boldsymbol{C}^{\bullet} & \boldsymbol{X}^{\bullet}\boldsymbol{W}^{\bullet} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{X}^{\bullet} \end{bmatrix} < 0 \tag{54}$$

$$\boldsymbol{P}^{\bullet} = diag\left[\boldsymbol{P}_1^{\star}\ \boldsymbol{P}_2^{\star}\right], \quad \boldsymbol{Z}_i^{\bullet} = diag\left[\boldsymbol{Z}_i^{\star}\ \boldsymbol{P}_2^{\star}\right], \quad \boldsymbol{Z}^{\bullet} = diag\left[\boldsymbol{Z}^{\star}\ \boldsymbol{I}_{m_p}\right] \tag{55}$$

$$\boldsymbol{X}^{\bullet} = diag\left[\boldsymbol{X}^{\star}\ \boldsymbol{I}_{m_p}\right], \quad \boldsymbol{Y}^{\bullet} = diag\left[\boldsymbol{Y}^{\star}\ \boldsymbol{I}_{r_p}\right], \quad \boldsymbol{V}_i^{\star} = \left[\boldsymbol{V}_i\ \ -\boldsymbol{I}_m\right] \tag{56}$$

$$\boldsymbol{U}^{\bullet} = diag\left[\boldsymbol{U}\ \sqrt{\boldsymbol{X}^{\circ}}\boldsymbol{U}_e^{\circ}\right], \quad \boldsymbol{C}^{\bullet} = diag\left[\boldsymbol{C}\ \sqrt{\boldsymbol{X}^{\circ}}\boldsymbol{U}_e^{\circ}\right] \tag{57}$$

$$\boldsymbol{W}^{\bullet} = diag\left[\boldsymbol{W}\ \sqrt{\boldsymbol{X}^{\circ}}\boldsymbol{W}\right], \quad \boldsymbol{I}^{\bullet} = diag\left[\boldsymbol{I}_{r_p}\ \sqrt{\boldsymbol{Y}^{\circ}}\right] \tag{58}$$

where $\boldsymbol{P}^{\bullet} \in I\!R^{2n\times 2n}$, $\boldsymbol{Z}_i^{\bullet} \in I\!R^{2n\times(m+n)}$, $\boldsymbol{Y}^{\bullet} \in I\!R^{2r_p\times 2r_p}$, $\boldsymbol{V}_i^{\star} \in I\!R^{m_r\times(m+m_r)}$, $\boldsymbol{X}^{\bullet} \in I\!R^{m_p\times 2m_p}$, $\boldsymbol{Z}^{\bullet} \in I\!R^{m_p\times 2m_p}$ *are structured matrix variables and all remaining matrix parameters are defined in (48)–(51).*

When the above conditions are satisfied, then

$$\boldsymbol{J}_i = (\boldsymbol{P}_1^{\star})^{-1}\boldsymbol{Z}_i^{\star}, \quad \boldsymbol{J} = (\boldsymbol{X}^{\star})^{-1}\boldsymbol{Z}^{\star}, \quad \boldsymbol{V}_i = \boldsymbol{V}_i^{\star}\left[\boldsymbol{I}_m\ \boldsymbol{0}\right]^T \tag{59}$$

Proof Now the Lyapunov function candidate is defined as

$$v(\boldsymbol{e}^{\bullet}(t)) = \boldsymbol{e}^{\bullet T}(t)\boldsymbol{P}^{\bullet}\boldsymbol{e}^{\bullet}(t) + \int_0^t \left(\boldsymbol{r}^{\bullet T}(x)\boldsymbol{r}^{\bullet}(x) - \gamma^{\bullet}\boldsymbol{g}^{\bullet T}(x)\boldsymbol{g}^{\bullet}(x)\right) \mathrm{d}x \tag{60}$$

and its time derivative is

$$\begin{aligned} &\dot{v}(\boldsymbol{e}^{\bullet}(t)) \\ &= \dot{\boldsymbol{e}}^{\bullet T}(t)\boldsymbol{P}^{\bullet}\boldsymbol{e}^{\bullet}(t) + \boldsymbol{e}^{\bullet T}(t)\boldsymbol{P}^{\bullet}\dot{\boldsymbol{e}}^{\bullet}(t) + \boldsymbol{r}^{\bullet T}(t)\boldsymbol{r}^{\bullet}(t) - \gamma^{\bullet}\boldsymbol{g}^{\bullet T}(t)\boldsymbol{g}^{\bullet}(t) < 0 \end{aligned} \tag{61}$$

where the structure of $\boldsymbol{A}_i^{\bullet}$, $\boldsymbol{C}_i^{\bullet}$ implies the structure of $\boldsymbol{P}^{\bullet}$ in (58).

Considering the property (52) and substituting (46) and (47) into (61) results in

$$\begin{aligned} \dot{v}(\boldsymbol{e}^{\bullet}(t)) \le &-\gamma^{\bullet}\boldsymbol{g}^{\bullet T}(t)\boldsymbol{g}^{\bullet}(t) + \sum_{i=1}^{s} \mathrm{h}_i(\boldsymbol{\theta}(t))\boldsymbol{e}^{\bullet T}(t)\boldsymbol{C}^{\star T}\boldsymbol{V}_i^{\star T}\boldsymbol{V}_i^{\star}\boldsymbol{C}^{\star}\boldsymbol{e}^{\bullet}(t) \\ &+ \sum_{i=1}^{s} \mathrm{h}_i(\boldsymbol{\theta}(t))(\boldsymbol{A}_{ei}^{\bullet}\boldsymbol{e}^{\bullet}(t) + \boldsymbol{G}_i^{\bullet}\boldsymbol{g}^{\bullet}(t) + \boldsymbol{E}_i^{\bullet}\delta\mathbf{p}^{\bullet}(t))^T\boldsymbol{P}^{\bullet}\boldsymbol{e}^{\bullet}(t) \\ &+ \sum_{i=1}^{s} \mathrm{h}_i(\boldsymbol{\theta}(t))\boldsymbol{e}^{\bullet T}(t)\boldsymbol{P}^{\bullet}(\boldsymbol{A}_{ei}^{\bullet}\boldsymbol{e}^{\bullet}(t) + \boldsymbol{G}_i^{\bullet}\boldsymbol{g}^{\bullet}(t) + \boldsymbol{E}_i^{\bullet}\delta\mathbf{p}^{\bullet}(t)) < 0 \end{aligned} \tag{62}$$

and with the notation

$$\boldsymbol{e}_c^{\bullet T}(t) = \left[\boldsymbol{e}^{\bullet T}(t)\ \delta\mathbf{p}^{\bullet T}(t)\ \ \boldsymbol{g}^{\bullet T}(t) \right] \tag{63}$$

the time derivative of $v(\boldsymbol{e}^{\bullet}(t))$ can be prescribe (in analogy with (41)) as

$$\dot{v}(\boldsymbol{e}^{\bullet}(t)) \leq \sum_{i=1}^{s} \mathrm{h}_i(\boldsymbol{\theta}(t))\boldsymbol{e}_c^{\bullet T}(t)(\boldsymbol{P}_{ci}^{\bullet} + \boldsymbol{N}^{\rhd})\boldsymbol{e}_c^{\bullet}(t) < 0 \tag{64}$$

where $\boldsymbol{N}^{\rhd}$ is a positive definite matrix, $(\boldsymbol{P}_{ci}^{\bullet} + \boldsymbol{N}^{\rhd})$ is negative definite and

$$\boldsymbol{P}_{ci}^{\bullet} = \begin{bmatrix} \boldsymbol{P}^{\bullet}\boldsymbol{A}_{ei}^{\bullet} + \boldsymbol{A}_{ei}^{\bullet T}\boldsymbol{P}^{\bullet} + \boldsymbol{C}^{\star T}\boldsymbol{V}_i^{\star T}\boldsymbol{V}_i^{\star}\boldsymbol{C}^{\circ} & * & * \\ \boldsymbol{E}_i^{\bullet T}\boldsymbol{P}^{\bullet} & \boldsymbol{0} & * \\ \boldsymbol{G}_i^{\bullet T}\boldsymbol{P}^{\bullet} & \boldsymbol{0} & -\gamma^{\bullet}\boldsymbol{I}_{2r_g} \end{bmatrix} \tag{65}$$

Defining the incremental multiplier matrix with respect to the structure of (63) as follows

$$\boldsymbol{M}^{\bullet} = \mathrm{diag}\left[\boldsymbol{X}^{\star}\ \boldsymbol{X}^{\circ}\ -\boldsymbol{Y}^{\star}\ -\boldsymbol{Y}^{\circ} \right] \tag{66}$$

where $\boldsymbol{X}^{\star} \in I\!R^{m_p \times m_p}$, $\boldsymbol{Y}^{\star} \in I\!R^{r_p \times r_p}$ are symmetric positive definite matrices and $\boldsymbol{X}^{\circ}$, $\boldsymbol{Y}^{\circ}$ are the constant matrices satisfying (32) and (33), then it is

$$\boldsymbol{N}^{\bullet} = \begin{bmatrix} \boldsymbol{U}_e^{\star T} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{U}_e^{\circ T} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{W}^T & \boldsymbol{0} & \boldsymbol{I}_{r_p} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{W}^T & \boldsymbol{0} & \boldsymbol{I}_{r_p} \end{bmatrix} \begin{bmatrix} \boldsymbol{X}^{\star} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{X}^{\circ} & \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{Y}^{\star} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & -\boldsymbol{Y}^{\circ} \end{bmatrix} \begin{bmatrix} \boldsymbol{U}_e^{\star} & \boldsymbol{0} & \boldsymbol{W} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{U}_e^{\circ} & \boldsymbol{0} & \boldsymbol{W} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_{r_p} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_{r_p} \end{bmatrix} \tag{67}$$

and (67) can be separated in the two components

$$\boldsymbol{N}_1^{\bullet} = \begin{bmatrix} \boldsymbol{U}_e^{\star T}\boldsymbol{X}^{\star} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{U}_e^{\circ T}\sqrt{\boldsymbol{X}^{\circ}} \\ \boldsymbol{W}^T\boldsymbol{X}^{\star} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{W}^T\sqrt{\boldsymbol{X}^{\circ}} \end{bmatrix} \begin{bmatrix} \boldsymbol{X}^{\circ} & \\ \boldsymbol{0} & \boldsymbol{I}_{m_p} \end{bmatrix}^{-1} \begin{bmatrix} \boldsymbol{X}^{\star}\boldsymbol{U}_e^{\star} & \boldsymbol{0} & \boldsymbol{X}^{\star}\boldsymbol{W} & \boldsymbol{0} \\ \boldsymbol{0} & \sqrt{\boldsymbol{X}^{\circ}}\boldsymbol{U}_e^{\circ} & \boldsymbol{0} & \sqrt{\boldsymbol{X}^{\circ}}\boldsymbol{W} \end{bmatrix} \tag{68}$$

$$\boldsymbol{N}_2^{\bullet} = \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \\ \boldsymbol{I}_{r_p} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{I}_{r_p} \end{bmatrix} \begin{bmatrix} -\boldsymbol{Y}^{\star} & \boldsymbol{0} \\ \boldsymbol{0} & -\boldsymbol{Y}^{\circ} \end{bmatrix} \begin{bmatrix} \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_{r_p} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{0} & \boldsymbol{I}_{r_p} \end{bmatrix} \tag{69}$$

To obtain relationships that allow to use structured matrix variables, it can be written with $\boldsymbol{U}_e^{\star}$, $\boldsymbol{U}_e^{\circ}$ given in (50)

$$\begin{bmatrix} \boldsymbol{X}^{\star}\boldsymbol{U}_e^{\star} & \mathbf{0} \\ \mathbf{0} & \sqrt{\boldsymbol{X}^{\circ}}\boldsymbol{U}_e^{\circ} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}^{\star} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{U} & \mathbf{0} \\ \mathbf{0} & \sqrt{\boldsymbol{X}^{\circ}}\boldsymbol{U}_e^{\circ} \end{bmatrix} - \begin{bmatrix} \boldsymbol{X}^{\star}\boldsymbol{J} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{C} & \mathbf{0} \\ \mathbf{0} & \sqrt{\boldsymbol{X}^{\circ}}\boldsymbol{U}_e^{\circ} \end{bmatrix} \tag{70}$$

$$\begin{bmatrix} \boldsymbol{X}^{\star}\boldsymbol{W} & \mathbf{0} \\ \mathbf{0} & \sqrt{\boldsymbol{X}^{\circ}}\boldsymbol{W} \end{bmatrix} = \begin{bmatrix} \boldsymbol{X}^{\star} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_{m_p} \end{bmatrix} \begin{bmatrix} \boldsymbol{W} & \mathbf{0} \\ \mathbf{0} & \sqrt{\boldsymbol{X}^{\circ}}\boldsymbol{W} \end{bmatrix} \tag{71}$$

and with the notations (56)–(58), where $\boldsymbol{Z}^{\star} = \boldsymbol{X}^{\star}\boldsymbol{J}$, (68) can be written as

$$\boldsymbol{N}_1^{\bullet} = \begin{bmatrix} (\boldsymbol{X}^{\bullet}\boldsymbol{U}^{\bullet} - \boldsymbol{Z}^{\bullet}\boldsymbol{C}^{\bullet})^T \\ \boldsymbol{W}^{\bullet T}\boldsymbol{X}^{\bullet} \end{bmatrix} (\boldsymbol{X}^{\bullet})^{-1} \begin{bmatrix} \boldsymbol{X}^{\bullet}\boldsymbol{U}^{\bullet} - \boldsymbol{Z}^{\bullet}\boldsymbol{C}^{\bullet} & \boldsymbol{X}^{\bullet}\boldsymbol{W}^{\bullet} \end{bmatrix} \tag{72}$$

Since the connecting matrix element in (69) can be factorized as

$$-\begin{bmatrix} \boldsymbol{Y}^{\star} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{Y}^{\circ} \end{bmatrix} = -\begin{bmatrix} \boldsymbol{I}_{r_p} & \mathbf{0} \\ \mathbf{0} & \sqrt{\boldsymbol{Y}^{\circ}} \end{bmatrix} \begin{bmatrix} \boldsymbol{Y}^{\star} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{I}_{r_p} \end{bmatrix} \begin{bmatrix} \boldsymbol{I}_{r_p} & \mathbf{0} \\ \mathbf{0} & \sqrt{\boldsymbol{Y}^{\circ}} \end{bmatrix} = -\boldsymbol{I}^{\bullet}\boldsymbol{Y}^{\bullet}\boldsymbol{I}^{\bullet} \tag{73}$$

then, using the same procedure as in redaction of the matrix (44), from (65), (72) and (69), (73), the following can be obtained

$$\begin{bmatrix} \boldsymbol{P}^{\bullet}\boldsymbol{A}_{ei}^{\bullet} + \boldsymbol{A}_{ei}^{\bullet T}\boldsymbol{P}^{\bullet} & * & * & * & * \\ \boldsymbol{E}_i^{\bullet T}\boldsymbol{P}^{\bullet} & -\boldsymbol{I}^{\bullet}\boldsymbol{Y}^{\bullet}\boldsymbol{I}^{\bullet} & * & * & * \\ \boldsymbol{G}_i^{\bullet T}\boldsymbol{P}^{\bullet} & \mathbf{0} & -\gamma\boldsymbol{I}_{2r_g} & * & * \\ \boldsymbol{V}_i^{\star}\boldsymbol{C}^{\star} & \mathbf{0} & \mathbf{0} & -\boldsymbol{I}_m & * \\ \boldsymbol{X}^{\bullet}\boldsymbol{U}^{\bullet} - \boldsymbol{Z}^{\bullet}\boldsymbol{C}^{\bullet} & \boldsymbol{X}^{\bullet}\boldsymbol{W}^{\bullet} & \mathbf{0} & \mathbf{0} & -\boldsymbol{X}^{\bullet} \end{bmatrix} < 0 \tag{74}$$

Using the block diagonal matrix $\boldsymbol{P}^{\bullet}$ and $\boldsymbol{A}_{ei}^{\bullet}$ given in (48), (49) and (55), where $\boldsymbol{Z}_i^{\star} = \boldsymbol{P}_1^{\star}\boldsymbol{J}_i$, then it can be written

$$\boldsymbol{P}^{\bullet}\boldsymbol{A}_{ei}^{\bullet} = \begin{bmatrix} \boldsymbol{P}_1^{\star} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{P}_2^{\star} \end{bmatrix} \begin{bmatrix} \boldsymbol{A}_i & \mathbf{0} \\ \mathbf{0} & \boldsymbol{A}_i \end{bmatrix} - \begin{bmatrix} \boldsymbol{P}_1^{\star}\boldsymbol{J}_i & \mathbf{0} \\ \mathbf{0} & \boldsymbol{P}_2^{\star} \end{bmatrix} \begin{bmatrix} \boldsymbol{C} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{J}_i^{\circ}\boldsymbol{C} \end{bmatrix} = \boldsymbol{P}^{\bullet}\boldsymbol{A}_i^{\bullet} - \boldsymbol{Z}_i^{\bullet}\boldsymbol{C}^{\bullet} \tag{75}$$

Thus (74) with (75) implies (54). This concludes the proof.

5 Illustrative Example

As a simple illustrative model, the nonlinear dynamics of the ball-and-beam system, represented by the nonlinear state-space model, was taken from [6], where

$$\begin{aligned} \dot{q}_1(t) &= q_2(t)\,, & z(t) &= q_1(t) \\ \dot{q}_2(t) &= a\left(q_1(t)q_4^2(t) - g\sin(q_3(t))\right), & y_1(t) &= q_1(t) \\ \dot{q}_3(t) &= q_4(t)\,, & y_2(t) &= q_3(t) \\ \dot{q}_4(t) &= b\left(-q_1(t) + gu(t)\right) \end{aligned}$$

while the input variable $u(t)$ is the angular acceleration of the beam [rad/s^2], the output variable $z(t)$ is equal $q_1(t)$ and the measured variables are $q_1(t)$ and $q_3(t)$, while $q_1(t)$ is the position of the ball [m], $q_2(t)$ is the velocity of the ball [m/s], $q_3(t)$ is the angle of the beam [rad] and $q_4(t)$ is the angular velocity of the beam [rad/s]. The model parameters are

$$a = \frac{m}{m + \frac{J}{r^2}} = 0.7143, \quad b = \frac{m}{J + J_b} = 1.1$$

J— the ball inertia moment $1.76\,10^{-5}$ kg m^2, m—the mass of the ball 0.11 kg
J_b—the beam inertia moment 0.1 kg m^2, r—the radius of the ball 0.02 m
g—the gravitational constant 9.81 m/s^2.

Introducing the premise variable $\theta(t) = q_1(t)q_4(t)$ which is bounded in the sector $q_1(t)q_4(t) \in \langle -d, d\rangle = \langle -5, 5\rangle$, the associated membership functions are

$$h_2(\theta(t)) = \begin{cases} 1, & \theta(t) \ge d \\ \frac{1}{d}\theta(t), & 0 < \theta(t) < d, \\ 0, & \theta(t) \le 0 \end{cases} \quad h_3(\theta(t)) = \begin{cases} 0, & \theta(t) \ge d \\ -\frac{1}{d}\theta(t), & 0 > \theta(t) > -d \\ 1, & \theta(t) \le -d \end{cases}$$

$$h_1(\theta(t)) = 1 - h_2(\theta(t)) - h_3(\theta(t))$$

It is supposed that FDF is designed to support the fault detection in the structure with unmeasurable $q_4(t)$ and the nonlinear function $\mathbf{p}(t)$ is therefore given as

$$\mathbf{p}(t) = \sin(q_4(t)) = \sin\left(\begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix} \boldsymbol{q}(t)\right) = \sin(\boldsymbol{U}\boldsymbol{q}(t) + \boldsymbol{W}\mathbf{p}(t))\,,$$

$$\boldsymbol{q}^T(t) = \begin{bmatrix} q_1(t) & q_2(t) & q_3(t) & q_4(t) \end{bmatrix}, \quad \boldsymbol{U} = \begin{bmatrix} 0 & 0 & 0 & 1 \end{bmatrix}, \quad \boldsymbol{W} = 0$$

Consequently, the representation in the TS fuzzy system model gives for all i

$$\boldsymbol{A}_1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -b & 0 & 0 & 0 \end{bmatrix}, \boldsymbol{A}_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & ad \\ 0 & 0 & 0 & 1 \\ -b & 0 & 0 & 0 \end{bmatrix}, \boldsymbol{A}_3 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -ad \\ 0 & 0 & 0 & 1 \\ -b & 0 & 0 & 0 \end{bmatrix}, \boldsymbol{C}^T = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

$$\boldsymbol{B}_i = \boldsymbol{B}_{fi} = \boldsymbol{B} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ bg \end{bmatrix}, \quad \boldsymbol{B}_{di} = \boldsymbol{B}_d = \begin{bmatrix} 0.1 \\ 0.2 \\ 0.5 \\ 0.4 \end{bmatrix}, \quad \boldsymbol{E}_i = \boldsymbol{E} = \begin{bmatrix} 0 \\ -ag \\ 0 \\ 0 \end{bmatrix}$$

Within the above parameters and Theorem 1 and by solving LMIs (32) and (33) with SeDuMi packet, the RRM gains were found as

$$\gamma^\circ = 4.2758\,, \quad \boldsymbol{J}^\circ = \begin{bmatrix} -0.0071 \; 0.0916 \end{bmatrix}$$

$$\boldsymbol{J}_1^\circ = \begin{bmatrix} 12.0282 & 0.3613 \\ 120.9501 & 2.9155 \\ 1.0973 & 17.4208 \\ 0.6048 & 108.0026 \end{bmatrix}, \; \boldsymbol{J}_2^\circ = \begin{bmatrix} 11.7658 & 0.4128 \\ 117.9370 & 3.4767 \\ -5.8114 & 17.2653 \\ -46.7773 & 106.9042 \end{bmatrix}$$

$$\boldsymbol{J}_3^\circ = \begin{bmatrix} 12.2317 & 0.3081 \\ 123.3139 & 2.3357 \\ 7.9978 & 17.5818 \\ 47.9626 & 109.1395 \end{bmatrix}$$

Subsequently, by Theorem 2, the parameters of the stable FDF were computed as follows

$$\gamma^\bullet = 1.7725, \quad \boldsymbol{J} = \begin{bmatrix} -0.0064 \; 0.1231 \end{bmatrix}$$

$$\boldsymbol{J}_1 = \begin{bmatrix} 15.3748 & 0.3219 \\ 107.4360 & 1.7553 \\ 0.0468 & 17.0285 \\ -2.3693 & 138.1231 \end{bmatrix}, \; \boldsymbol{J}_2 = \begin{bmatrix} 15.2233 & 0.5052 \\ 106.2246 & 3.1328 \\ -4.9619 & 16.9679 \\ -45.6834 & 137.4746 \end{bmatrix}$$

$$\boldsymbol{J}_3 = \begin{bmatrix} 15.4152 & 0.1386 \\ 107.8083 & 0.3748 \\ 5.0541 & 17.0252 \\ 40.9506 & 138.0520 \end{bmatrix}$$

$$\boldsymbol{V}_1 = \begin{bmatrix} 0.0005 & -0.0006 \\ -0.0003 & 0.1467 \end{bmatrix}, \; \boldsymbol{V}_2 = \begin{bmatrix} 0.0006 & 0.0241 \\ 0.0008 & 0.1473 \end{bmatrix}, \; \boldsymbol{V}_3 = \begin{bmatrix} 0.0008 & -0.0263 \\ -0.0014 & 0.1459 \end{bmatrix}$$

It should be noted that this example at first explains the inclusion of RRM and incremental quadratic constraints in FDF design.

6 Concluding Remarks

The introduced nonlinear fuzzy FDF design method based on RRM is presented in the paper. This is achieved by application of Lyapunov function and incremental quadratic constraints parameterized by a symmetric multiplier matrix. In the presented version, the sensitivity of the reference residual model and FDF stability problem is solved, considering premise variables determined from the subsets of measurable and unmeasurable state variables. It is obvious that the adaptation methodology proposed to TS fuzzy state observer is imminent.

It should be pointed out that the proposed technique using TS fuzzy models with nonlinear terms might give more conservative results than the existing approaches in some cases, but the advantage of them lies in designing a problem oriented fuzzy FDF with fewer rules and less computational burden. It is clear that in specific cases it is necessary to have a compromise between the complexity of the implemented method and the number of LMIs to be solved.

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