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Lévy Matters – A subseries on Lévy Processes

Lars Nørvang Andersen · Søren Asmussen
Frank Aurzada · Peter W. Glynn
Makoto Maejima · Mats Pihlsgård
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Lévy Matters V

Functionals of Lévy Processes

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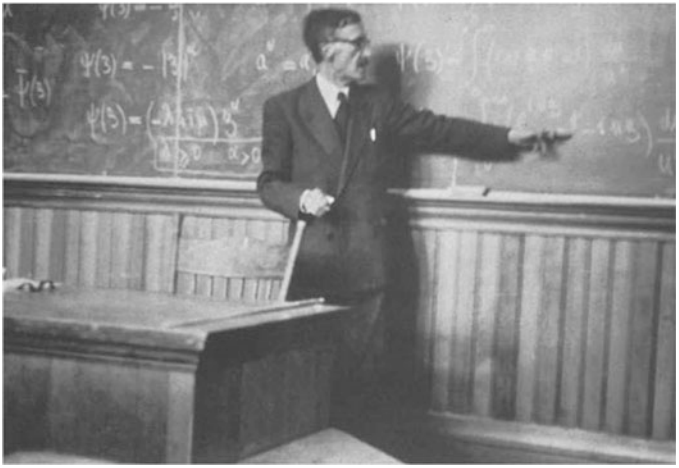

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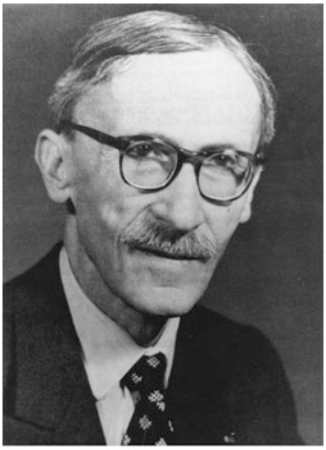
Preface to the Series Lévy Matters



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Over the past 10–15 years, we have seen a revival of general Lévy processes theory as well as a burst of new applications. In the past, Brownian motion or the Poisson process had been considered as appropriate models for most applications. Nowadays, the need for more realistic modelling of irregular behaviour of phenomena in nature and society such as jumps, bursts and extremes has led to a renaissance of the theory of general Lévy processes. Theoretical and applied researchers in fields as diverse as quantum theory, statistical physics, meteorology, seismology, statistics, insurance, finance and telecommunication have realized the enormous flexibility of Lévy models in modelling jumps, tails, dependence and sample path behaviour. Lévy processes or Lévy-driven processes feature slow or rapid structural breaks, extremal behaviour, clustering and clumping of points.

Tools and techniques from related but distinct mathematical fields, such as point processes, stochastic integration, probability theory in abstract spaces and differential geometry, have contributed to a better understanding of Lévy jump processes.

As in many other fields, the enormous power of modern computers has also changed the view of Lévy processes. Simulation methods for paths of Lévy processes and realizations of their functionals have been developed. Monte Carlo simulation makes it possible to determine the distribution of functionals of sample paths of Lévy processes to a high level of accuracy.

This development of Lévy processes was accompanied and triggered by a series of Conferences on Lévy Processes: Theory and Applications. The first and second conferences were held in Aarhus (1999, 2002, respectively), the third in Paris (2003), the fourth in Manchester (2005) and the fifth in Copenhagen (2007).

To show the broad spectrum of these conferences, the following topics are taken from the announcement of the Copenhagen conference:

- Structural results for Lévy processes: distribution and path properties
- Lévy trees, superprocesses and branching theory
- Fractal processes and fractal phenomena
- Stable and infinitely divisible processes and distributions
- Applications in finance, physics, biosciences and telecommunications
- Lévy processes on abstract structures
- Statistical, numerical and simulation aspects of Lévy processes
- Lévy and stable random fields

At the Conference on Lévy Processes: Theory and Applications in Copenhagen, the idea was born to start a series of Lecture Notes on Lévy processes to bear witness of the exciting recent advances in the area of Lévy processes and their applications. Its goal is the dissemination of important developments in theory and applications. Each volume will describe state-of-the-art results of this rapidly evolving subject with special emphasis on the non-Brownian world. Leading experts will present new exciting fields, or surveys of recent developments, or focus on some of the most promising applications. Despite its special character, each article is written in an expository style, normally with an extensive bibliography at the end. In this way, each article makes an invaluable comprehensive reference text. The intended audience are PhD and postdoctoral students, or researchers, who want to learn about recent advances in the theory of Lévy processes and to get an overview of new applications in different fields.

Now, with the field in full flourish and with future interest definitely increasing, it seemed reasonable to start a series of Lecture Notes in this area, whose individual volumes will appear over time under the common name "Lévy Matters", in tune with the developments in the field. "Lévy Matters" appears as a subseries of the Springer Lecture Notes in Mathematics, thus ensuring wide dissemination of the scientific material. The mainly expository articles should reflect the broadness of the area of Lévy processes.

We take the possibility to acknowledge the very positive collaboration with the relevant Springer staff and the editors of the LN series and the (anonymous) referees of the articles.

We hope that the readers of "Lévy Matters" enjoy learning about the high potential of Lévy processes in theory and applications. Researchers with ideas for contributions to further volumes in the Lévy Matters series are invited to contact any of the editors with proposals or suggestions.

Aarhus, Denmark — Ole E. Barndorff-Nielsen
Paris, France — Jean Bertoin
Paris, France — Jean Jacod
Munich, Germany — Claudia Küppelberg
June 2010

A Short Biography of Paul Lévy

A volume of the series "Lévy Matters" would not be complete without a short sketch about the life and mathematical achievements of the mathematician whose name has been borrowed and used here. This is more a form of tribute to Paul Lévy, who not only invented what we call now Lévy processes, but also is in a sense the founder of the way we are now looking at stochastic processes, with emphasis on the path properties.

Paul Lévy was born in 1886, and lived until 1971. He studied at the Ecole Polytechnique in Paris, and was soon appointed as professor of mathematics in the same institution, a position that he held from 1920 to 1959. He started his career as an analyst, with 20 published papers between 1905 (he was then 19 years old) and 1914, and he became interested in probability by chance, so to speak, when asked to give a series of lectures on this topic in 1919 in that same school: this was the starting point of an astounding series of contributions in this field, in parallel with a continuing activity in functional analysis.

Very briefly, one can mention that he is the mathematician who introduced characteristic functions in full generality, proving in particular the characterisation theorem and the first "Lévy's theorem" about convergence. This naturally led him to study more deeply the convergence in law with its metric, and also to consider sums of independent variables, a hot topic at the time: Paul Lévy proved a form of the 0-1 law, as well as many other results, for series of independent variables. He also introduced stable and quasi-stable distributions, and unravelled their weak and/or strong domains of attractions, simultaneously with Feller.

Then we arrive at the book "Théorie de l'addition des variables aléatoires", published in 1937, and in which he summaries his findings about what he called "additive processes" (the homogeneous additive processes are now called Lévy processes, but he did not restrict his attention to the homogeneous case). This book contains a host of new ideas and new concepts: the decomposition into the sum of jumps at fixed times and the rest of the process; the Poissonian structure of the jumps for an additive process without fixed times of discontinuities; the "compensation" of those jumps so that one is able to sum up all of them; the fact that the remaining continuous part is Gaussian. As a consequence, he implicitly gave the formula providing the form of all additive processes without fixed discontinuities, now called the Lévy-Itô Formula, and he proved the Lévy-Khintchine formula for the characteristic functions of all infinitely divisible distributions. But, as fundamental as all those results are, this book contains more: new methods, like martingales which, although not given a name, are used in a fundamental way; and also a new way of looking at processes, which is the "pathwise" way: he was certainly the first to understand the importance of looking at and describing the paths of a stochastic process, instead of considering that everything is encapsulated into the distribution of the processes.

This is of course not the end of the story. Paul Lévy undertook a very deep analysis of Brownian motion, culminating in his book "Processus stochastiques et mouvement brownien" in 1948, completed by a second edition in 1965. This is a remarkable achievement, in the spirit of path properties, and again it contains so many deep results: the Lévy modulus of continuity, the Hausdorff dimension of the path, the multiple points and the Lévy characterisation theorem. He introduced local time and proved the arc-sine law. He was also the first to consider genuine stochastic integrals, with the area formula. In this topic again, his ideas have been the origin of a huge amount of subsequent work, which is still going on. It also laid some of the basis for the fine study of Markov processes, like the local time again, or the new concept of instantaneous state. He also initiated the topic of multi-parameter stochastic processes, introducing in particular the multi-parameter Brownian motion.

As should be quite clear, the account given here does not describe the whole of Paul Lévy's mathematical achievements, and one can consult for many more details the first paper (by Michel Loève) published in the first issue of the Annals of Probability (1973). It also does not account for the humanity and gentleness of the person Paul Lévy. But I would like to end this short exposition of Paul Lévy's work by hoping that this series will contribute to fulfilling the program, which he initiated.

Paris, France Jean Jacod

Preface

This fifth volume of the series *Lévy Matters* consists of three chapters, each devoted to an important aspect of Lévy processes and their applications. They all concern distributions of certain functionals of Lévy processes, which appear naturally in different settings.

Historically, processes with independent increments have been considered by Paul Lévy to reveal the fine structure of infinitely divisible distributions; the paradigm being that a probability measure, say μ, is infinitely divisible if and only if there is a Lévy process $\{X_t\}$ such that X_1 has the law μ. In turn, the notion of infinite divisibility for probability measures arises naturally in the context of limit theorems for sums of triangular arrays. Well-known special cases of infinitely divisible laws include stable distributions, which describe the weak limits of certain properly rescaled random walks with heavy-tailed step distributions, and more generally self-decomposable distributions, which in turn arise similarly for sums of independent variables. The so-called Generalized Gamma Convolutions, or distributions in the Thorin class, lie somewhat in between the two former. The **first chapter** of this volume, by Makoto Maejima, surveys representations of the main sub-classes of infinitely divisible distributions in terms of mappings of certain Lévy processes via stochastic integration

$$\{X_t\} \mapsto \int_I f(t)dX_t,$$

where f is some specific deterministic function over some interval I. An important motivation for studying such mappings stems from free probability, and more specifically from the role of free cumulants in this area. The study of the compositions and the iterations of such mappings, and of their limits, then sheds light on the nested structure of those subclasses. A great variety of classical and not-yet classical examples of infinitely divisible distributions are then analyzed from this perspective. Overall, this chapter can be seen as a companion to the contribution by K. Sato "Fractional integrals and extensions of self-decomposability," which appeared in the first volume of this Series, and in which relations between many nested subclasses of infinitely divisible laws are discussed.

Reflecting a path at barriers is a fundamental concept for stochastic processes, both in theory and in applications to modeling physical phenomena. In the setting of one-dimensional Lévy processes, reflection at a single fixed barrier (usually 0) lies at the core of fluctuation theory and its connections with the Wiener-Hopf factorization. One-sided reflected Lévy processes can be used as basic models for stochastic storage processes; they arise in a variety of applications including queuing, dams, insurance, and data communication, to name just the main ones, and there is already a vast mathematical literature on this classical area. The **second chapter** of this volume, by Lars Nørvang Andersen, Søren Asmussen, Peter W. Glynn and Mats Pihlsgård, concerns real Lévy processes reflected at two barriers. Two-sided reflection can be used for modeling systems with a finite capacity, which is of course a crucial hypothesis to fit many real-life situations. Roughly speaking, as the two-sided reflected process V stays in a compact interval, it is positive recurrent. Thus, V possesses a stationary distribution, π, and the ergodic theorem applies. Theoretically, this should enable one to answer natural questions about the long-run behavior of the system; unfortunately in practice and except in some special situations that we shall discuss later on, the invariant measure π is hard to determine, even numerically. A most important quantity for the applications is the overflow, or the loss occurring at the upper barrier, which, for instance, in communication models, represents the number of bits, which are lost when the buffer is full. Explicit asymptotics of the average loss are obtained when the upper barrier goes to infinity, both in the discrete time (i.e., for random walks) and continuous time frameworks; different regimes occur depending on whether the tail distribution of a typical increment is light or heavy. Whereas in discrete time, the construction of a reflected chain raises no difficulty, the continuous time setting is somewhat less intuitive and requires a formulation *à la* Skorokhod. Therefore, stochastic calculus and martingale techniques, in particular using optional sampling for the Wald martingale and the Kella-Whitt martingale, provide fundamental tools for studying quantities related to two-sided reflected Lévy processes. As we mentioned previously, there are also important situations where handy expressions for the stationary distribution π can be obtained. Typically, this is the case whenever the two-sided exit problem can be solved, that is the probability that the Lévy process crosses the upper-barrier before the lower-barrier can be expressed explicitly as a function of its starting point. A first important situation when this occurs is when the jumps of X have a phase-type distribution, a situation that is amply discussed in this chapter. Another well-known case is when the Lévy process X is spectrally negative then a solution of the two-sided exit problem can be given in terms of the so-called scale function. Readers wishing to learn more about scale functions are invited to consult the contribution by Alexey Kuznetsov, Andreas Kyprianou, and Victor Rivero in the second volume of this series.

If now processes are killed rather than reflected when they cross the boundary, probably the most natural and important questions that one can ask concern the lifetime. Typically, for a one-dimensional process, say $\{Y_t\}_{t\geq 0}$, and when the domain is a semi-infinite interval (x, ∞), one is thus interested in the passage time $T_x := \inf\{t \geq 0 : Y_t > x\}$. The one-dimensional distributions of the process $\{T_x\}_{x>0}$ are characterized by those of the running supremum process $\{\sup_{0 \leq s \leq t} Y_s\}_{t \geq 0}$ through the basic identity

$$\mathbb{P}(T_x \leq t) = \mathbb{P}\left(\sup_{0 \leq s \leq t} Y_s > x\right)$$

that holds whenever x is not the location of a local maximum of Y. In general, these quantities are hard to compute explicitly, and merely determining the tail behavior of the survival probability $\mathbb{P}(T_x > t)$ is already highly challenging. In particular, one is interested in deciding whether the latter decays polynomially in t, i.e. if there is an exponent $\theta > 0$ with $\mathbb{P}(T_x > t) = t^{-\theta + o(1)}$ as $t \to \infty$. In that case, θ (which may depend on x) is called the persistence exponent. This is a fundamental issue in applications, notably for a number of models in physics, and the **third chapter** of this volume, by Frank Aurzada and Thomas Simon, is devoted to this question. Whereas for general Markov processes, this question is usually addressed by considering the eigenvalues and eigenfunctions of the infinitesimal generator of the killed process, for random walks or for Lévy processes, the deep formulas of fluctuation theory are the key to many classical results in this area. The problem is of course much harder for non-Markovian processes, and the main part of this chapter concerns the situation when the process Y is given by either the partial sum of a random walk or the integral of a Lévy process. Many very recent advances and developments are discussed in this setting.

We are confident that you will enjoy reading these new contributions to the theory of Lévy type processes and their applications, as much as those which already appeared in the preceding volumes of *Lévy Matters*.

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June 2015

Contents

Classes of Infinitely Divisible Distributions and Examples

Makoto Maejima

Abstract Bondesson (Generalized Gamma Convolutions and Related Classes of Distributions and Densities, Lecture Notes in Statistics, vol. 76, Springer, Berlin, 1992) said "Since a lot of the standard distributions now are known to be infinitely divisible, the class of infinitely divisible distributions has perhaps partly lost its interest. Smaller classes should be more in focus." This view was presented more than two decades ago, yet has not been fully addressed. Over the last decade, many classes of infinitely divisible distributions have been studied and characterized. In this article, we summarize such "smaller classes" and try to find classes which known infinitely divisible distributions belong to, as precisely as possible.

Keywords Generalized gamma convolution • Infinitely divisible distribution • Lévy measure • Mixture of exponential distributions • Nested subclasses • Selfdecomposable distribution • Stable distribution • Stochastic integral with respect to Lévy process

AMS Subject Classification 2000: Primary: 60E07, 60H05; Secondary: 60E10, 60G51

1 Introduction

The theory of infinitely divisible distributions has been a core topic of probability theory and the subject of extensive study over the years. One reason for that is the fact that many important distributions are infinitely divisible, such as Gaussian, stable, exponential, Poisson, compound Poisson and gamma distributions. Another reason is that the set of infinitely divisible distributions on $\mathbb{R}^d$ coincides with the set of distributions which are limits of distributions of sums $\sum_{j=1}^{k_n} \xi_{n,j}$ of $\mathbb{R}^d$-valued triangle arrays $\{\xi_{n,j}, 1 \leq j \leq k_n, n \geq 1\}, k_n \uparrow \infty$ as $n \to \infty$, where

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for each n, $\xi_{n,1}, \xi_{n,2}, \dots$ are independent, with the condition of infinite smallness that is $\lim_{n\to\infty} \max_{1\le j\le k_n} P(|\xi_{n,j}| \ge \varepsilon) = 0$ for any $\varepsilon > 0$. Suppose that $\xi_{n,k} = a_n^{-1}(\xi_j - b_j)$, for $a_n > 0$ with $\lim_{n\to\infty} a_n = \infty$, $\lim_{n\to\infty} a_{n+1}a_n^{-1} = 1$, $b_j \in \mathbb{R}^d$ and $k_n = n$. If $\{\xi_j\}$ are independent, then the resulting class is the class of selfdecomposable distributions, and if furthermore $\{\xi_j\}$ are identically distributed, then the resulting class is the class of stable distributions including Gaussians, These two classes are important classes of infinitely divisible distributions. Selfdecomposable distributions are known as marginal distributions of the stationary processes of Ornstein-Uhlenbeck type, which are stationary solutions of the Langevin equations with Lévy noise.

In 1977, Thorin [85, 86] introduced a class between the classes of stable and selfdecomposable distributions, called now the Thorin class, whose elements are called Generalized Gamma Convolutions (GGCs for short), when he wanted to prove the infinite divisibility of the Pareto and the log-normal distributions. Bondesson [16] published a monograph on this topic in 1992.

In 1983, Jurek and Vervaat [38] and Sato and Yamazato [79] showed that any selfdecomposable distribution $\tilde{\mu}$ can be characterized by stochastic integrals with respect to some Lévy process $\{X_t\}$ with $E[\log|X_1|] < \infty$ such as

$$\tilde{\mu} = \mathscr{L}\left(\int_0^\infty e^{-t} dX_t\right), \tag{1}$$

where $\mathscr{L}(X)$ is the law of a random variable of X. (The paper by Jurek [36] is a short historical survey on stochastic integral representations of classes of infinitely divisible distributions.) Since a Lévy process $\{X_t\}$ can be constructed one to one in law by some infinitely divisible distribution μ satisfying $\mathscr{L}(X_1) = \mu$, (1) can be regarded as a mapping Φ, say, from the class of infinitely divisible distributions with finite log-moments to the class of selfdecomposable distributions as

$$\tilde{\mu} = \Phi(\mu). \tag{2}$$

If we denote by $\{X_t^{(\mu)}\}$ a Lévy process such that $\mathscr{L}(X_1^{(\mu)}) = \mu$, (1) and (2) gives us

$$\tilde{\mu} = \Phi(\mu) = \mathscr{L}\left(\int_0^\infty e^{-t} dX_t^{(\mu)}\right).$$

Barndorff-Nielsen and Thorbjørnsen [7–10] introduced a mapping

$$\Upsilon(\mu) = \mathscr{L}\left(\int_0^1 \log(t^{-1}) dX_t^{(\mu)}\right)$$

related to the Bercovici-Pata bijection between free probability and classical probability. Then in Barndorff-Nielsen et al. [12], we investigated the range of the mapping Ψ and characterized several classes of infinitely divisible distributions in terms of the mappings Φ and Υ. Among others, we found that the composition of these two mappings produces the Thorin class. Since then, many mappings have been studied as mappings constructing classes of infinitely divisible distributions giving new probabilistic explanations of such classes and also as mappings themselves from a mathematical point of view.

Let us recall one sentence by Bondesson [16]. “Since a lot of the standard distributions now are known to be infinitely divisible, the class of infinitely divisible distributions has perhaps partly lost its interest. Smaller classes should be more in focus.” In this article, we survey such “smaller classes” and try to find classes which known infinitely divisible distributions belong to, as precisely as possible. All infinitely divisible distributions we treat here are finite dimensional and most of the examples are one-dimensional.

In Sect. 2, we give some preliminaries on infinitely divisible distributions on $\mathbb{R}^d$, Lévy processes and stochastic integrals with respect to Lévy processes.

In Sect. 3, we explain some known classes of infinitely divisible distributions and their relationships, and the characterization in terms of stochastic integral mappings is discussed in Sect. 4. Section 5 is devoted to some other mappings. These three sections form the first main subject of this article. Also, compositions of mappings are discussed in Sect. 6.

Since we have mappings to construct classes in hand, we can construct nested subclasses by the iteration of those mappings. This is the topic in Sect. 7. For the class of selfdecomposable distributions, these nested subclasses were already studied by Urbanik [90] and later by Sato [70].

Once we have a general theory for infinitely divisible distributions, it is necessary to provide specific examples. We know that many distributions are infinitely divisible. Then, the next question related to the above may be which classes such known infinitely distributions belong to. This is the second main subject of this article and is discussed in Sects. 8–10. Section 8 treats known distributions. After the monograph by Bondesson [16] and later a paper by James et al. [28], GGCs have been highlighted, and thus examples of GGCs recently appearing in quite different problems are explained separately in Sect. 9. Section 10 discusses new examples of α-selfdecomposable distributions.

We conclude the article with a short Sect. 11 on fixed points of the mapping for α-selfdecomposable distributions, offering a new perspective on the class of stable distributions.

Since this is a survey article, only a few statements have explicit proofs. However, even if the statements do not have proofs, readers may consult original proofs in the papers cited.

2 Preliminaries

2.1 Infinitely Divisible Distributions on $\mathbb{R}^d$

In the following, $\mathscr{P}(\mathbb{R}^d)$ is the set of all probability distributions on $\mathbb{R}^d$ and $\hat{\mu}(z) := \int_{\mathbb{R}^d} e^{i\langle z,x\rangle}\mu(dx), z \in \mathbb{R}^d$, is the characteristic function of $\mu \in \mathscr{P}(\mathbb{R}^d)$.

Definition 2.1 $\mu \in \mathscr{P}(\mathbb{R}^d)$ is *infinitely divisible* if, for any $n \in \mathbb{N}$, there exists $\mu_n \in \mathscr{P}(\mathbb{R}^d)$ such that $\hat{\mu}(z) = \hat{\mu}_n(z)^n$. $ID(\mathbb{R}^d)$ denotes the class of all infinitely divisible distributions on $\mathbb{R}^d$.

We also use

$$ID_{\mathrm{sym}}(\mathbb{R}^d) := \{\mu \in ID(\mathbb{R}^d) : \mu \text{ is symmetric on } \mathbb{R}^d\},$$
$$ID_{\log}(\mathbb{R}^d) := \{\mu \in ID(\mathbb{R}^d) : \int_{\mathbb{R}^d} \log^+ |x| \mu(dx) < \infty\}$$

and

$$ID_{\log^m}(\mathbb{R}^d) := \{\mu \in ID(\mathbb{R}^d) : \int_{\mathbb{R}^d} (\log^+ |x|)^m \mu(dx) < \infty\}, \quad m = 1, 2, \ldots,$$

where $\log^+ a = \max\{\log a, 0\}$.

The so-called Lévy-Khintchine representation of infinitely divisible distribution is provided in the following proposition.

Proposition 2.2 (The Lévy-Khintchine Representation; See e.g. Sato [73, Theorem 8.1])

(1) *If* $\mu \in ID(\mathbb{R}^d)$, *then*

$$\hat{\mu}(z) = \exp\left\{-2^{-1}\langle z, Az\rangle + i\langle \gamma, z\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle z,x\rangle} - 1 - \frac{i\langle z, x\rangle}{1+|x|^2}\right)\nu(dx)\right\}, \quad z \in \mathbb{R}^d, \tag{3}$$

where A *is a symmetric nonnegative-definite* $d \times d$ *matrix,* ν *is a measure on* $\mathbb{R}^d$ *satisfying*

$$\nu(\{0\}) = 0 \quad \textit{and} \quad \int_{\mathbb{R}^d} (|x|^2 \wedge 1)\nu(dx) < \infty, \tag{4}$$

and γ *is a vector in* $\mathbb{R}^d$.

(2) *The representation of* $\hat{\mu}$ *in* (1) *by* A, ν *and* γ *is unique.*

(3) *Conversely, if* A *is a symmetric nonnegative-definite* $d \times d$ *matrix,* ν *is a measure satisfying* (4) *and* $\gamma \in \mathbb{R}^d$, *then there exists a* $\mu \in ID(\mathbb{R}^d)$ *whose characteristic function is given by* (3).

A is called the Gaussian covariance matrix or the Gaussian part and ν is called the Lévy measure. The triplet (A, ν, γ) is called the Lévy-Khintchine triplet of μ. When we want to emphasize the Lévy-Khintchine triplet, we may write $\mu = \mu_{(A,\nu,\gamma)}$. If the Lévy measure ν of μ satisfies $\int_{|x|>1} |x|\nu(dx) < \infty$, then there exists the mean $\gamma^1 \in \mathbb{R}^d$ of μ such that

$$\hat{\mu}(z) = \exp\left\{-2^{-1}\langle z, Az\rangle + i\langle \gamma^1, z\rangle + \int_{\mathbb{R}^d}\left(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle\right)\nu(dx)\right\}.$$

In this case, we will write $\mu = \mu_{(A,\nu,\gamma^1)_1}$. If ν of μ satisfies $\int_{|x|\le 1} |x|\nu(dx) < \infty$, then there exists $\gamma^0 \in \mathbb{R}^d$ (called the drift of μ) such that

$$\hat{\mu}(z) = \exp\left\{-2^{-1}\langle z, Az\rangle + i\langle \gamma^0, z\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle z,x\rangle} - 1\right) \nu(dx)\right\}.$$

We write $\mu = \mu_{(A,\nu,\gamma^0)_0}$ in this case. We also write ν_μ for ν when ν is the Lévy measure of μ.

In the following, the notation 1_B denotes the indicator function of the set $B \in \mathscr{B}(\mathbb{R}^d)$. Here and in what follows, $\mathscr{B}(C)$ is the set of Borel sets in C.

Proposition 2.3 (Polar Decomposition of Lévy Measure; See e.g. Barndorff-Nielsen et al. [12, Lemma 2.1]) *Let ν_μ be the Lévy measure of some $\mu \in I(\mathbb{R}^d)$ with $0 < \nu_\mu(\mathbb{R}^d) \le \infty$. Then there exist a σ-finite measure λ on $S := \{\xi \in \mathbb{R}^d : |\xi| = 1\}$ with $0 \le \lambda(S) \le \infty$ and a family $\{\nu_\xi : \xi \in S\}$ of measures on $(0,\infty)$ such that*

$$\nu_\xi(B) \text{ is measurable in } \xi \text{ for each } B \in \mathscr{B}((0,\infty)), \tag{5}$$

$$0 < \nu_\xi((0,\infty)) \le \infty \quad \text{for each } \xi \in S, \tag{6}$$

$$\nu_\mu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi)\nu_\xi(dr) \quad \text{for } B \in \mathscr{B}(\mathbb{R}^d \setminus \{0\}). \tag{7}$$

Here λ and $\{\nu_\xi\}$ are uniquely determined by ν_μ in the following sense: if λ, $\{\nu_\xi\}$ and λ', $\{\nu'_\xi\}$ both have properties (5)–(7), then there is a measurable function $c(\xi)$ on S such that

$$0 < c(\xi) < \infty,\ \lambda'(d\xi) = c(\xi)\lambda(d\xi),\ c(\xi)\nu'_\xi(dr) = \nu_\xi(dr) \quad \text{for } \lambda\text{-a.e. } \xi \in S.$$

We call ν_ξ the radial component of ν_μ and when ν_ξ is absolute continuous, we call its density the Lévy density.

Definition 2.4 (The Cumulant of $\hat{\mu}$) For $\mu \in ID(\mathbb{R}^d)$, $C_\mu(z) = \log \hat{\mu}(z)$ is called the *cumulant* of μ, where log is the distinguished logarithm. (For the definition of the distinguished logarithm, see e.g. Sato [73], the sentence after Lemma 7.6.)

2.2 Stochastic Integrals with Respect to Lévy Processes

Definition 2.5 A stochastic process $\{X_t, t \ge 0\}$ on $\mathbb{R}^d$ is called a *Lévy process*, if the following conditions are satisfied.

(1) $X_0 = 0$ a.s.
(2) For any $0 \le t_0 < t_1 < \cdots < t_n, n \ge 1$, $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ are independent.

(3) For $h > 0$, the distribution of $X_{t+h} - X_t$ does not depend on t.
(4) For any $t \geq 0$ and $\varepsilon > 0$, $\lim_{h\to 0} P(|X_{t+h} - X_t| > \varepsilon) = 0$.
(5) For almost all ω, the sample paths $X_t(\omega)$ are right-continuous in $t \geq 0$ and have left limits in $t > 0$.

Dropping the condition (5) in Definition 2.5, we call any process satisfying (1)–(4) a *Lévy process in law*. In the following, "Lévy process" simply means "Lévy process in law". It is known (see e.g. Sato [73, Theorem 7.10(i)]) that if $\{X_t\}$ is a Lévy process on $\mathbb{R}^d$, then for any $t \geq 0, \mathscr{L}(X_t) \in ID(\mathbb{R}^d)$ and if we let $\mathscr{L}(X_1) = \mu$, then $\mathscr{L}(X_t) = \mu^{t*}$, where μ^{t*} is the distribution with characteristic function $\hat{\mu}(z)^t$. Thus the distribution of a Lévy process $\{X_t\}$ is determined by that of X_1. Further, a stochastic process $\{X_t, t \geq 0\}$ on $\mathbb{R}^d$ is called an *additive process* (in law), if (1), (2) and (4) are satisfied.

Proposition 2.6 (Stochastic Integral with Respect to Lévy Process; See Sato [77, Sect. 3.4]) *Let $\{X_t\}$ be a Lévy process on $\mathbb{R}^d$ with $\mathscr{L}(X_1) = \mu_{(A,\nu,\gamma)}$.*

(1) *Let $f(t)$ be a real-valued locally square integrable measurable function on $[0,\infty)$. Then the stochastic integral $X := \int_0^a f(t)dX_t$ exists and $\mathscr{L}(X) \in ID(\mathbb{R}^d)$. Its cumulant is represented as*

$$C_{\mathscr{L}(X)}(z) = \int_0^a C_\mu(f(t)z)dt.$$

The Lévy-Khintchine triplet (A_X, ν_X, γ_X) of $\mathscr{L}(X)$ is the following:

$$A_X = \int_0^a f(t)^2 A dt,$$

$$\nu_X(B) = \int_0^a dt \int_{\mathbb{R}^d} 1_B(f(t)x)\nu(dx), \quad B \in \mathscr{B}(\mathbb{R}^d \setminus \{0\}),$$

$$\gamma_X = \int_0^a f(t)dt \left(\gamma + \int_{\mathbb{R}^d} x \left(\frac{1}{1+|f(t)x|^2} - \frac{1}{1+|x|^2}\right)\nu(dx)\right).$$

(2) *The improper stochastic integral over $[0,\infty)$ is defined as follows, whenever the limit exists:*

$$X := \int_0^\infty f(t)dX_t = \lim_{a\to\infty}\int_0^a f(t)dX_t \quad \textit{in probability.}$$

Suppose $f(t)$ is locally square integrable on $[0,\infty)$. Then $\int_0^\infty f(t)dX_t$ exists if and only if $\lim_{a\to\infty}\int_0^a C_\mu(f(t)z)dt$ exists in $\mathbb{C}$ for all $z \in \mathbb{R}^d$. We have

$$C_{\mathscr{L}(X)}(z) = \lim_{a\to\infty}\int_0^a C_\mu(f(t)z)dt,$$

$$A_X = \int_0^\infty f(t)^2 A dt,$$

$$\nu_X(B) = \int_0^\infty dt \int_{\mathbb{R}^d} 1_B(f(t)x)\nu(dx), \quad B \in \mathscr{B}(\mathbb{R}^d \setminus \{0\}),$$

$$\gamma_X = \lim_{q\to\infty} \int_0^a f(t)dt \left(\gamma + \int_{\mathbb{R}^d} x \left(\frac{1}{1+|f(t)x|^2} - \frac{1}{1+|x|^2} \right) \nu(dx) \right).$$

Remark 2.7 We will treat many $f(t)$'s which have singularity at $t = 0$: (i) $f(t) = G^*_{\alpha,\beta}(t), t > 0$, the inverse function of $t = G_{\alpha,\beta}(s) = \int_s^\infty u^{-\alpha-1}e^{-u^\beta}du, s \geq 0$, in Sect. 5.2, which is specialized to the kernels of Υ-, Ψ-, $\mathscr{G}$- and $\mathscr{M}$-mappings in Sect. 4.1. (ii) $f(t) = t^{-1/\alpha}, t > 0$, which is the kernel of the stable mapping in Sect. 5.4.

3 Some Known Classes of Infinitely Divisible Distributions

As mentioned in Sect. 1, the main concern of this article is to discuss known and new classes of infinitely divisible distributions and characterize them in several ways. We start with some known classes in Sects. 3.2 and 3.3, and show the relationships among themselves in Sect. 3.4.

3.1 Completely Monotone Functions

In the following, the concept of completely monotone function plays an important role. So, we start with the definition of completely monotone function and some properties of it.

Definition 3.1 (Completely Monotone Function) A function $\varphi(x)$ on $(0, \infty)$ is completely monotone if it has derivatives $\varphi^{(n)}$ of all orders and $(-1)^n\varphi^{(n)}(x) \geq 0, n \in \mathbb{Z}_+, x > 0$.

Two typical examples of completely monotone functions are e^{-x} and $x^{-p}, p > 0$.

Proposition 3.2 (Bernstein's Theorem. See e.g. Feller [21, Chap. XIII, 4]) *A function φ on $(0, \infty)$ is completely monotone if and only if it is the Laplace transform of a measure μ on $(0, \infty)$.*

Proposition 3.3 (See e.g. Feller [21, Chap. XIII, 4])

(1) *The product of two completely monotone functions on $(0, \infty)$ is also completely monotone.*
(2) *If φ is completely monotone on $(0, \infty)$ and if ψ is a positive function with a completely monotone derivative on $(0, \infty)$, then the composed function $\varphi(\psi)$ is also completely monotone on $(0, \infty)$.*

3.2 The Classes of Stable and Semi-stable Distributions

Definition 3.4 Let $\mu \in ID(\mathbb{R}^d)$.

(1) It is called *stable* if, for any $a > 0$, there exist $b > 0$ and $c \in \mathbb{R}^d$ such that

$$\hat{\mu}(z)^a = \hat{\mu}(bz)e^{i\langle c,z\rangle}. \tag{8}$$

$S(\mathbb{R}^d)$ denotes the class of all stable distributions on $\mathbb{R}^d$.

(2) It is called *strictly stable* if, for any $a > 0$, there is $b > 0$ such that $\hat{\mu}(z)^a = \hat{\mu}(bz)$.

(3) It is called *semi-stable* if, for some $a > 0$ with $a \neq 1$, there exists $b > 0$ and $c \in \mathbb{R}^d$ satisfying (8). $SS(\mathbb{R}^d)$ denotes the class of all semi-stable distributions on $\mathbb{R}^d$.

(4) It is called *strictly semi-stable* if , for some $a > 0$ with $a \neq 1$, there exists $b > 0$ satisfying $\hat{\mu}(z)^a = \hat{\mu}(bz)$.

$\mu \in \mathscr{P}(\mathbb{R}^d)$ is called trivial if it is a distribution of a random variable concentrated at one point, otherwise it is called non-trivial. When this one point is $c \in \mathbb{R}^d$, we write $\mu = \delta_c$.

Theorem 3.5 (See e.g. Sato [73, Theorem 13.15] or Sato [72, Theorem 3.3]) *If μ is non-trivial stable, then there exists a unique $\alpha \in (0, 2]$ such that $b = a^{1/\alpha}$ in (8).*

In this case, we say that such a μ is α-stable. Gaussian distribution and Cauchy distribution are 2-stable and 1-stable, respectively. Note that any trivial distribution is stable in the sense that (8) is satisfied, and α is not uniquely determined. In the following, when we say α-stable distribution, we always include all trivial distributions. Also note that trivial distributions which are not δ_0 are not strictly stable except 1-stable distribution.

3.3 Some Known Classes of Infinitely Divisible Distributions

We start with the following six classes which are well-studied in the literature. We call Vx an elementary gamma random variable (resp. elementary mixed-exponential random variable, elementary compound Poisson random variable) on $\mathbb{R}^d$ if x is a nonrandom, nonzero element of $\mathbb{R}^d$ and V is a real random variable having gamma distribution (resp. a mixture of a finite number of exponential distributions, compound Poisson distribution whose jump size distribution is uniform on the interval $[0, a]$ for some $a > 0$).

(**1**) The class $U(\mathbb{R}^d)$ (the Jurek class): $\mu \in U(\mathbb{R}^d)$ if and only if $\mu \in ID(\mathbb{R}^d)$ and either $\nu_\mu = 0$ or $\nu_\mu \neq 0$ and, in case $\nu_\mu \neq 0$, the radial component ν_ξ of ν_μ is expressed as

$$\nu_\xi(dr) = \ell_\xi(r)dr, \tag{9}$$

where $\ell_\xi(r)$ is a nonnegative function measurable in $\xi \in S$ and nonincreasing on $(0,\infty)$ as a function of r.

The class $U(\mathbb{R}^d)$ was introduced by Jurek [31] and $\mu \in U(\mathbb{R}^d)$ is called s-selfdecomposable. Jurek [31] proved that $\mu \in U(\mathbb{R}^d)$ if and only if for any $b > 1$ there exists $\mu_b \in ID(\mathbb{R}^d)$ such that $\hat{\mu}(z) = \hat{\mu}(b^{-1}z)^{b^{-1}}\hat{\mu}_b(z)$. Sato [77] also formulated $U(\mathbb{R}^d)$ as the smallest class of distributions on $\mathbb{R}^d$ closed under convolution and weak convergence and containing all distributions of elementary compound Poisson random variables on $\mathbb{R}^d$.

(**2**) The class $B(\mathbb{R}^d)$ (the Goldie–Steutel–Bondesson class): $\mu \in B(\mathbb{R}^d)$ if and only if $\mu \in ID(\mathbb{R}^d)$ and either $\nu_\mu = 0$ or $\nu_\mu \neq 0$ and, in case $\nu_\mu \neq 0$, the radial component ν_ξ of ν_μ is expressed as

$$\nu_\xi(dr) = \ell_\xi(r)dr, \tag{10}$$

where $\ell_\xi(r)$ is a nonnegative function measurable in $\xi \in S$ and completely monotone on $(0,\infty)$ as a function of r.

Historically, Goldie [23] proved the infinite divisibility of mixtures of exponential distributions and Steutel [82] found the description of their Lévy measures. Then Bondesson [16] studied generalized convolutions of mixtures of exponential distributions on $\mathbb{R}_+$. It is the smallest class of distributions on $\mathbb{R}_+$ that contains all mixtures of exponential distributions and that is closed under convolution and weak convergence on $\mathbb{R}_+$. $B(\mathbb{R}^d)$ is its generalization by Barndorff-Nielsen et al. [12], where all mixtures of exponential distributions are replaced by all distributions of elementary mixed-exponential random variables on $\mathbb{R}^d$.

(**3**) The class $L(\mathbb{R}^d)$ (the class of selfdecomposable distributions) : $\mu \in L(\mathbb{R}^d)$ if and only if $\mu \in ID(\mathbb{R}^d)$ and either $\nu_\mu = 0$ or $\nu_\mu \neq 0$ and, in case $\nu_\mu \neq 0$, ν_μ is expressed as

$$\nu_\xi(dr) = r^{-1}k_\xi(r)dr, \tag{11}$$

where $k_\xi(r)$ is a nonnegative function measurable in $\xi \in S$ and nonincreasing on $(0,\infty)$ as a function of r.

It is known (see e.g. Sato [73, Theorem 15.10]) that $\mu \in L(\mathbb{R}^d)$ if and only if for any $b > 1$, there exists some $\rho_b \in \mathscr{P}(\mathbb{R}^d)$ such that

$$\hat{\mu}(z) = \hat{\mu}(b^{-1}z)\hat{\rho}_b(z). \tag{12}$$

This statement usually is used as the definition of the selfdecomposability. ρ_b in (12) can be shown to be infinitely divisible. Hence, we may replace $\rho_b \in \mathscr{P}(\mathbb{R}^d)$ by $\rho_b \in ID(\mathbb{R}^d)$ in the previous statement.

(**4**) The class $T(\mathbb{R}^d)$ (the Thorin class) : $\mu \in T(\mathbb{R}^d)$ if and only if $\mu \in ID(\mathbb{R}^d)$ and either $\nu_\mu = 0$ or $\nu_\mu \neq 0$ and, in case $\nu_\mu \neq 0$, the radial component ν_ξ of ν_μ is expressed as

$$\nu_\xi(dr) = r^{-1}k_\xi(r)dr, \tag{13}$$

where $k_\xi(r)$ is a nonnegative function measurable in $\xi \in S$ and completely monotone on $(0, \infty)$ as a function of r.

Originally this class was studied by Thorin [85, 86] when he wanted to prove the infinite divisibility of the Pareto and the log-normal distributions, as mentioned in Sect. 1. The class $T(\mathbb{R}_+)$ (resp. $T(\mathbb{R})$) is defined as the smallest class of distributions on $\mathbb{R}_+$ (resp. $\mathbb{R}$) that contains all positive (resp. positive and negative) gamma distributions and that is closed under convolution and weak convergence on $\mathbb{R}_+$ (resp. $\mathbb{R}$). The distributions in $T(\mathbb{R}_+)$ are called generalized gamma convolutions (GGCs) and those in $T(\mathbb{R})$ are called extended generalized gamma convolutions (EGGCs). Thorin showed that the Pareto and the log-normal distributions are GGCs, and thus are selfdecomposable and infinitely divisible. The infinite divisibility of the log-normal distribution was not known before the theory on *hyperbolic complete monotonicity* which was developed by Thorin.

$T(\mathbb{R}^d)$ is a generalization of $T(\mathbb{R})$ by Barndorff-Nielsen et al. [12], where all positive and negative gamma distributions are replaced by all distributions of elementary gamma random variable on $\mathbb{R}^d$.

(**5**) The class $G(\mathbb{R}^d)$ (the class of type G distributions): $\mu \in G(\mathbb{R}^d)$ if and only if $\mu \in ID(\mathbb{R}^d)$ and either $\nu_\mu = 0$ or $\nu_\mu \neq 0$ and, in case $\nu_\mu \neq 0$, the radial component ν_ξ of ν_μ is expressed as

$$\nu_\xi(dr) = g_\xi(r^2)dr, \tag{14}$$

where $g_\xi(r)$ is a nonnegative function measurable in $\xi \in S$ and completely monotone on $(0, \infty)$ as a function of r.

When $d = 1$, $\mu \in G(\mathbb{R}) \cap I_{\text{sym}}(\mathbb{R})$ if and only if $\mu = \mathscr{L}(V^{1/2}Z)$, where $V > 0$, $\mathscr{L}(V) \in I(\mathbb{R})$, Z is the standard normal random variable, and V and Z are independent. When $d \geq 1$, $\mu = \mu_{(A,\nu,\gamma)} \in G(\mathbb{R}^d) \cap ID_{\text{sym}}(\mathbb{R}^d)$ if and only if $\nu(B) = E[\nu_0(Z^{-1}B)]$ for some Lévy measure ν_0. (See Maejima and Rosiński [47].) Previously only symmetric distributions in $G(\mathbb{R}^d)$ were said to be of type G. In this article, however, we say that any distribution from $G(\mathbb{R}^d)$ is of type G.

(**6**) The class $M(\mathbb{R}^d)$ (Aoyama et al. [4]) : $\mu \in M(\mathbb{R}^d)$ if and only if $\mu \in ID(\mathbb{R}^d)$ and either $\nu_\mu = 0$ or $\nu_\mu \neq 0$ and, in case $\nu_\mu \neq 0$, the radial component ν_ξ of ν_μ is expressed as

$$\nu_\xi(dr) = r^{-1}g_\xi(r^2)dr, \tag{15}$$

where $g_\xi(r)$ is a nonnegative function measurable in $\xi \in S$ and completely monotone on $(0, \infty)$ as a function of r. (Originally, in Aoyama et al. [4], we defined the class $M(\mathbb{R}^d)$ restricted in $I_{\text{sym}}(\mathbb{R}^d)$. However, in this article, we do not assume the symmetry of $\mu \in M(\mathbb{R}^d)$. The requirement (15) is independent of the symmetry of the distribution.)

This class was introduced, being motivated by how the class will be if we replace $g_\xi(r^2)$ in (14) by $r^{-1}g_\xi(r^2)$ by multiplying an extra r^{-1} in the Lévy density which can been seen from (**1**) to (**3**) and from (**2**) to (**4**).

3.4 Relationships Among the Classes

With respect to relationships among the classes mentioned in Sect. 3.3, we have the following.

(1) $L(\mathbb{R}^d) \cup G(\mathbb{R}^d) \subsetneqq U(\mathbb{R}^d)$ and $T(\mathbb{R}^d) \subsetneqq L(\mathbb{R}^d)$. (By definition.)
(2) Each class in Sect. 3.3 includes $S(\mathbb{R}^d)$. This is because if $\mu \in S(\mathbb{R}^d)$, then either $A \neq 0$ and $\nu_\mu = 0$ or $A = 0$ and $\nu_\xi(dr) = r^{-1-\alpha}dr$ for some $\alpha \in (0,2)$, (see e.g. Sato [73, Theorem 14.3]).
(3) $T(\mathbb{R}^d) \subsetneqq B(\mathbb{R}^d) \subsetneqq G(\mathbb{R}^d)$. These inclusions follow from the properties of completely monotone functions. It follows from Proposition 3.3(1) that $T(\mathbb{R}^d) \subset B(\mathbb{R}^d)$. If we put $g_\xi(x) = l_\xi(x^{1/2})$, then it follows from Proposition 3.3(2) that $B(\mathbb{R}^d) \subset G(\mathbb{R}^d)$. The relation $\subsetneqq$ can be shown by choosing suitable Lévy densities.
(4) $T(\mathbb{R}^d) \subsetneqq M(\mathbb{R}^d) \subsetneqq L(\mathbb{R}^d) \cap G(\mathbb{R}^d)$. The proof is as follows (Aoyama et al. [4]):

We first show that $M(\mathbb{R}^d) \subsetneqq L(\mathbb{R}^d) \cap G(\mathbb{R}^d)$. Note that $r^{-1/2}$ is completely monotone and by Proposition 3.3(1) that the product of two completely monotone functions is also completely monotone. Thus by the definition of $M(\mathbb{R}^d)$, it is clear that $M(\mathbb{R}^d) \subset L(\mathbb{R}^d) \cap G(\mathbb{R}^d)$. To show that $M(\mathbb{R}^d) \neq L(\mathbb{R}^d) \cap G(\mathbb{R}^d)$, it is enough to construct $\mu \in ID(\mathbb{R}^d)$ such that $\mu \in L(\mathbb{R}^d) \cap G(\mathbb{R}^d)$ but $\mu \notin M(\mathbb{R}^d)$.

First consider the case $d = 1$. Let

$$\nu(dr) = r^{-1}g(r^2)dr, \quad r > 0.$$

For our purpose, it is enough to construct a function $g : (0,\infty) \mapsto (0,\infty)$ such that (a) $r^{-1/2}g(r)$ is completely monotone on $(0,\infty)$, (meaning that the corresponding μ belongs to $G(\mathbb{R})$), (b) $g(r^2)$ or, equivalently, $g(r)$ is nonincreasing on $(0,\infty)$, (meaning that the corresponding μ belongs to $L(\mathbb{R})$), and (c) $g(r)$ is not completely monotone on $(0,\infty)$, (meaning that the corresponding μ does not belong to $M(\mathbb{R})$). We show that

$$g(r) := r^{-1/2}h(r) := r^{-1/2}\left(e^{-0.9r} - e^{-r} + 0.1e^{-1.1r}\right), \quad r > 0,$$

satisfies the requirements (a)–(c) above.

(a) We have

$$r^{-1/2}g(r) = \int_{0.9}^{1} e^{-ru}du + 0.1\int_{1.1}^{\infty} e^{-ru}du,$$

which is a sum of two completely monotone functions, and thus $r^{-1/2}g(r)$ is completely monotone.

(b) If $h(r)$ is nonincreasing, then so is $g(r) = r^{-1/2}h(r)$. To show it, we have

$$h'(r) = -0.9e^{-0.9r} + e^{-r} - 0.11e^{-1.1r} = -0.9e^{-1.1r}\left[\left(e^{0.1r} - \frac{1}{1.8}\right)^2 - \frac{0.604}{3.24}\right]$$
$$\leq -0.9e^{-1.1r}\left[\left(1 - \frac{1}{1.8}\right)^2 - \frac{0.604}{3.24}\right] = -0.01e^{-1.1r} < 0, \quad r > 0.$$

(c) To show (c), we see that

$$h(r) = \int_0^\infty e^{-ru}Q(du),$$

where Q is a signed measure such that $Q = Q_1 + Q_2 + Q_3$ and

$$Q_1(\{0.9\}) = 1,\ Q_2(\{1\}) = -1,\ Q_3(\{1.1\}) = 0.1.$$

On the other hand,

$$r^{-1/2} = \pi^{-1/2}\int_0^\infty e^{-ru}u^{-1/2}du =: \int_0^\infty e^{-ru}R(du),$$

where

$$R(du) = (\pi u)^{-1/2}du.$$

Thus

$$g(r) = \int_0^\infty e^{-ru}R(du)\int_0^\infty e^{-rv}Q(dv) = \int_0^\infty e^{-rw}U(dw),$$

where

$$U(B) = \int_0^\infty Q(B-y)R(dy).$$

We are going to show that U is a signed measure, namely, for some interval (a,b), $U((a,b)) < 0$. If so, g is not completely monotone by Bernstein's theorem (Proposition 3.2). We have

$$U((a,b)) = \pi^{-1/2}\int_0^\infty Q((a-y,b-y))\,y^{-1/2}dy$$
$$= \pi^{-1/2}\sum_{j=1}^{3}\int_0^\infty Q_j((a-y,b-y))\,y^{-1/2}dy$$

$$= \pi^{-1/2}\left[\int_{a-0.9}^{b-0.9} y^{-1/2}dy - \int_{a-1}^{b-1} y^{-1/2}dy + 0.1\int_{a-1.1}^{b-1.1} y^{-1/2}dy\right]$$
$$= 2\pi^{-1/2}\Big[\left(\sqrt{b-0.9}-\sqrt{a-0.9}\right)-\left(\sqrt{b-1}-\sqrt{a-1}\right)$$
$$+0.1\left(\sqrt{b-1.1}-\sqrt{a-1.1}\right)\Big].$$

Take $(a,b) = (1.15, 1.35)$. Then

$$\begin{aligned}&U((1.15,1.35))\\&\quad = 2\pi^{-1/2}\left[(\sqrt{0.45}-\sqrt{0.25})-(\sqrt{0.35}-\sqrt{0.15})+0.1(\sqrt{0.25}-\sqrt{0.05})\right]\\&\quad < -0.01\pi^{-1/2} < 0.\end{aligned}$$

This concludes that g is not completely monotone.

A d-dimensional example of $\mu \in ID(\mathbb{R}^d)$ such that $\mu \in L(\mathbb{R}^d) \cap G(\mathbb{R}^d)$ but $\mu \notin M(\mathbb{R}^d)$ is given by taking the example of the Lévy measure on $\mathbb{R}$ constructed above as the radial component of a Lévy measure on $\mathbb{R}^d$. This completes the proof of $M(\mathbb{R}^d) \subsetneqq L(\mathbb{R}^d) \cap G(\mathbb{R}^d)$.

We next show that $T(\mathbb{R}^d) \subsetneqq M(\mathbb{R}^d)$. If $\mu \in T(\mathbb{R}^d)$, then the radial component of the Lévy measure of μ has the form $\nu_\xi(dr) = r^{-1}k_\xi(r)dr$, where k_ξ is completely monotone. By Proposition 3.3 and the fact that $\psi(r) = r^{1/2}$ has a completely monotone derivative, then $g_\xi(r) := k_\xi(r^{1/2})$ is completely monotone. Thus $\nu_\xi(dr)$ can be read as $r^{-1}g_\xi(r^2)dr$, where g_ξ is completely monotone, concluding that $\mu \in M(\mathbb{R}^d)$.

To show that $T(\mathbb{R}^d) \neq M(\mathbb{R}^d)$, it is enough to find a completely monotone function g_ξ such that $k_\xi(r) = g_\xi(r^2)$ is *not* completely monotone. However, the function $g_\xi(r) = e^{-r}$ has such a property. Although e^{-r} is completely monotone, $(-1)^2\frac{d^2}{dr^2}e^{-r^2} < 0$ for small $r > 0$. This completes the proof of the inclusion $T(\mathbb{R}^d) \subsetneqq M(\mathbb{R}^d)$.

Remark 3.6 It is important to remark that any distribution in $L(\mathbb{R})$ is unimodal, (a result by Yamazato [97]), which implies the unimodality of any distribution in $T(\mathbb{R})$, since $T(\mathbb{R}) \subset L(\mathbb{R})$.

The following are examples for non-inclusion among classes. (See Schilling et al. [80, Chap. 9].)

(5) $L(\mathbb{R}) \not\subset B(\mathbb{R})$. Let $\nu_\mu(dx) = x^{-1}1_{(0,1)}(x)dx,\ x > 0$. Then $k(x) = 1_{(0,1)}(x)$ is nonincreasing and thus $\mu \in L(\mathbb{R})$, but $\ell(x) = x^{-1}1_{(0,1)}(x)$ is not completely monotone and thus $\mu \notin B(\mathbb{R})$. Hence $L(\mathbb{R}) \not\subset B(\mathbb{R})$.
(6) $B(\mathbb{R}) \not\subset L(\mathbb{R})$. Let $\nu_\mu(dx) = e^{-x}dx,\ x > 0$. Then it is easy to see that $\mu \in B(\mathbb{R})$, but $\mu \notin L(\mathbb{R})$. Therefore $B(\mathbb{R}) \not\subset L(\mathbb{R})$.

4 Stochastic Integral Mappings and Characterizations of Classes (I)

This section is one of the main subjects of this article, as referred to in Sect. 1. In Sect. 4.1 we explain well-studied six stochastic integral mappings, and then in Sect. 4.2 we characterize them by stochastic integral mappings.

4.1 Six Stochastic Integral Mappings

For $\mu \in ID(\mathbb{R}^d)$, let $\{X_t^{(\mu)}, t \geq 0\}$ be the Lévy process with $\mathscr{L}(X_1^{(\mu)}) = \mu$. Let $f(t)$ be a real-valued square integrable measurable function on $[a, b]$, for any $0 < a < b < \infty$ and suppose that the stochastic integral $\int_0^\infty f(t) dX_t^{(\mu)}$ is definable in the sense of Proposition 2.6. Then we can define a mapping $\mu \mapsto \Phi_f(\mu)$. We denote the domain of Φ_f by $\mathfrak{D}(\Phi_f)$ that is the class of $\mu \in ID(\mathbb{R}^d)$ for which $\Phi_f(\mu)$ is definable. We also denote the range of Φ_f by $\mathfrak{R}(\Phi_f) = \Phi_f(\mathfrak{D}(\Phi_f))$.

Now the following are well-studied mappings.

(**1**) $\mathscr{U}$-mapping (Jurek [31]). For $\mu \in \mathfrak{D}(\mathscr{U}) = ID(\mathbb{R}^d)$, $\mathscr{U}(\mu) = \mathscr{L}\left(\int_0^1 t dX_t^{(\mu)}\right)$.

(**2**) Υ-mapping (Barndorff-Nielsen et al. [12]). For $\mu \in \mathfrak{D}(\Upsilon) = ID(\mathbb{R}^d)$, $\Upsilon(\mu) = \mathscr{L}\left(\int_0^1 \log(t^{-1}) dX_t^{(\mu)}\right)$.

(**3**) Φ-mapping (Jurek and Vervaat [38], Sato and Yamazato [79], Wolfe [96]). For $\mu \in \mathfrak{D}(\Phi) = ID_{\log}(\mathbb{R}^d)$, $\Phi(\mu) = \mathscr{L}\left(\int_0^\infty e^{-t} dX_t^{(\mu)}\right)$.

(**4**) Ψ-mapping (Barndorff-Nielsen et al. [12]). Let $p(s) = \int_s^\infty e^{-u} u^{-1} du, s > 0$, and denote its inverse function by $p^*(t)$. For $\mu \in \mathfrak{D}(\Psi) = ID_{\log}(\mathbb{R}^d)$, $\Psi(\mu) = \mathscr{L}\left(\int_0^\infty p^*(t) dX_t^{(\mu)}\right)$.

(**5**) $\mathscr{G}$-mapping (Maejima and Sato [48]). Let $g(s) = \int_s^\infty e^{-u^2} du, s > 0$, and denote its inverse function by $g^*(t)$. For $\mu \in \mathfrak{D}(\mathscr{G}) = ID(\mathbb{R}^d)$, $\mathscr{G}(\mu) = \mathscr{L}\left(\int_0^{\sqrt{\pi}/2} g^*(t) dX_t^{(\mu)}\right)$.

(**6**) $\mathscr{M}$-mapping (Maejima and Nakahara [44]). Let $m(s) = \int_s^\infty e^{-u^2} u^{-1} du, s > 0$, and denote its inverse function by $m^*(t)$. For $\mu \in \mathfrak{D}(\mathscr{M}) = ID_{\log}(\mathbb{R}^d)$, $\mathscr{M}(\mu) = \mathscr{L}\left(\int_0^\infty m^*(t) dX_t^{(\mu)}\right)$.

In the above, it is easy to see that the domains of the mappings are $ID(\mathbb{R}^d)$ when the intervals of the stochastic integrals are finite. However, in the cases where stochastic integrals are improper at infinity, we need the proofs. As an example, we show the case of the Φ-mapping below. (For (**4**), see Barndorff-Nielsen et al. [12, Theorem C], and for (**6**), see Maejima and Nakahara [44, Theorem 2.3], respectively.) Note that in the six examples above, the singularity of the kernel at $t = 0$ does not give any influence for determining the domains of the mappings.

For showing that $\mathfrak{D}(\Phi) = ID_{\log}(\mathbb{R}^d)$, we use Proposition 2.6(2) with $f(t) = e^{-t}$. Let (A, ν, γ) and $(\tilde{A}, \tilde{\nu}, \tilde{\gamma})$ be the Lévy-Khintchine triplets of μ and $\Phi(\mu)$, respectively. If we could show that $\tilde{A}$ and $\tilde{\gamma}$ are finite, and $\tilde{\nu}$ is a Lévy measure, then Proposition 2.6(2) ensures that the existence of the stochastic integral defining $\Phi(\mu)$.

(i) (Gaussian part) : $\tilde{A} = \int_0^\infty e^{-2t} A dt$ exists.
(ii) (Lévy measure) : We are going to show that

$$\tilde{\nu}(B) = \int_{\mathbb{R}^d} \nu(dx) \int_0^\infty 1_B(e^{-t}x) dt, \quad B \in \mathscr{B}(\mathbb{R}^d),$$

satisfies that $\int_{\mathbb{R}^d} (|x|^2 \wedge 1) \tilde{\nu}(dx) < \infty$. We have

$$\int_{\mathbb{R}^d} (|x|^2 \wedge 1) \tilde{\nu}(dx) = \int_{|x| \le 1} |x|^2 \tilde{\nu}(dx) + \int_{|x|>1} \tilde{\nu}(dx),$$

where

$$\begin{aligned} \int_{|x|\le 1} |x|^2 \tilde{\nu}(dx) &= \int_{\mathbb{R}^d} \nu(dx) \int_0^\infty |e^{-t}x|^2 1_{\{|e^{-t}x| \le 1\}} dt \\ &= \int_{\mathbb{R}^d} |x|^2 \nu(dx) \int_0^\infty e^{-2t} 1_{\{|x| \le e^t\}} dt = \int_{\mathbb{R}^d} |x|^2 \nu(dx) \int_{0 \vee \log|x|}^\infty e^{-2t} dt \\ &= \frac{1}{2} \int_{\mathbb{R}^d} |x|^2 (1 \wedge |x|^{-2}) \nu(dx) = \frac{1}{2} \int_{\mathbb{R}^d} (|x|^2 \wedge 1) \nu(dx) \end{aligned}$$

and

$$\begin{aligned} \int_{|x|>1} \tilde{\nu}(dx) &= \int_{\mathbb{R}^d} \nu(dx) \int_0^\infty 1_{\{|e^{-t}x|>1\}} dt = \int_{\mathbb{R}^d} \nu(dx) \int_0^\infty 1_{\{|x|>e^t\}} dt \\ &= \int_{|x|>1} \nu(dx) \int_0^\infty 1_{\{t < \log|x|\}} dt = \int_{|x|>1} \log|x| \nu(dx). \end{aligned}$$

Thus,

$$\int_{\mathbb{R}^d} (|x|^2 \wedge 1) \tilde{\nu}(dx) < \infty$$

if and only if

$$\int_{|x|>1} \log|x| \nu(dx) < \infty.$$

(iii) (γ-part) : To complete the proof, it is enough to show that

$$\tilde{\gamma} = \int_0^\infty e^{-t} dt \cdot \gamma + \int_{\mathbb{R}^d} \nu(dx) \int_0^\infty e^{-t} x \left(\frac{1}{1 + |e^{-t}x|^2} - \frac{1}{1 + |x|^2} \right) dt < \infty,$$

whenever $\int_{|x|>1} \log |x| \nu(dx) < \infty$. The first integral is trivial. As to the second integral, we have

$$\begin{aligned}
&\int_{\mathbb{R}^d} \nu(dx) \int_0^\infty \frac{e^{-t}|x|^3}{(1 + |e^{-t}x|^2)(1 + |x|^2)} dt \\
&= \left(\int_{|x|\le 1} + \int_{|x|>1} \right) \nu(dx) \int_0^\infty \frac{e^{-t}|x|^3}{(1 + |e^{-t}x|^2)(1 + |x|^2)} dt \\
&=: I_1 + I_2.
\end{aligned}$$

Here

$$I_1 \le \int_{|x|\le 1} |x|^3 \nu(dx) \int_0^\infty e^{-t} dt \le \int_{|x|\le 1} |x|^2 \nu(dx) < \infty$$

and

$$\begin{aligned}
I_2 &= \int_{|x|>1} \nu(dx) \int_0^\infty \frac{e^{-t}|x|^3}{(1 + |e^{-t}x|^2)(1 + |x|^2)} dt \\
&= \int_{|x|>1} \nu(dx) \left(\int_0^{\log |x|} + \int_{\log |x|}^\infty \right) \frac{e^{-t}|x|^3}{(1 + |e^{-t}x|^2)(1 + |x|^2)} dt =: I_3 + I_4,
\end{aligned}$$

where

$$\begin{aligned}
I_3 &= \int_{|x|>1} \frac{|x|^2}{1 + |x|^2} \nu(dx) \int_0^{\log |x|} \frac{e^{-t}|x|}{1 + |e^{-t}x|^2} dt, \\
&\le \int_{|x|>1} \frac{|x|^2}{1 + |x|^2} \nu(dx) \int_0^{\log |x|} \frac{1}{2} dt \le \int_{|x|>1} \log |x| \nu(dx) < \infty
\end{aligned}$$

and

$$\begin{aligned}
I_4 &\le \int_{|x|>1} \nu(dx) \int_{\log |x|}^\infty \frac{e^{-t}|x|^3}{1 + |x|^2} dt = \int_{|x|>1} \frac{|x|^3}{1 + |x|^2} e^{-\log |x|} \nu(dx) \\
&\le \int_{|x|>1} \frac{|x|^2}{1 + |x|^2} \nu(dx) < \infty.
\end{aligned}$$

The proof is completed.

4.2 Characterization of Classes as the Ranges of the Mappings

The six classes of infinitely divisible distributions in Sect. 3.3 can be characterized as the ranges of the mappings discussed in the previous section, as follows.

Proposition 4.1 *We have the following.*

(1) $U(\mathbb{R}^d) = \mathscr{U}(ID(\mathbb{R}^d))$. *(Jurek [31].)*
(2) $B(\mathbb{R}^d) = \Upsilon(ID(\mathbb{R}^d))$. *(Barndorff-Nielsen et al. [12].)*
(3) $L(\mathbb{R}^d) = \Phi(ID_{\log}(\mathbb{R}^d))$. *(Jurek and Vervaat [38], Sato and Yamazato [79], Wolfe [96].)*
(4) $T(\mathbb{R}^d) = \Psi(ID_{\log}(\mathbb{R}^d))$. *(Barndorff-Nielsen et al. [12].)*
(5) $G(\mathbb{R}^d) = \mathscr{G}(ID(\mathbb{R}^d))$. *(Aoyama and Maejima [3] for symmetric case and Maejima and Sato [48] for general case.)*
(6) $M(\mathbb{R}^d) = \mathscr{M}(ID_{\log}(\mathbb{R}^d))$. *(Aoyama et al. [4] for symmetric case and Maejima and Nakahara [44] for general case.)*

For the readers' convenience, we give here the proof of (3) $L(\mathbb{R}^d) = \Phi(ID_{\log}(\mathbb{R}^d))$ as an example. We show that $L(\mathbb{R}^d) \supset \Phi(ID_{\log}(\mathbb{R}^d))$ and that $L(\mathbb{R}^d) \subset \Phi(ID_{\log}(\mathbb{R}^d))$, separately.

(a) ($L(\mathbb{R}^d) \supset \Phi(ID_{\log}(\mathbb{R}^d))$): Suppose that $\mu = \mathscr{L}\left(\int_0^\infty e^{-t} dX_t\right)$ for some Lévy process $\{X_t\}$ satisfying that $\mathscr{L}(X_1) \in ID_{\log}(\mathbb{R}^d)$. Let $b > 1$ and let $\{\tilde{X}_t\}$ be an independent copy of $\{X_t\}$. In what follows, the notation $\stackrel{\mathrm{d}}{=}$ means the equality in law. We have

$$b^{-1}\int_0^\infty e^{-t} d\tilde{X}_t = \int_0^\infty e^{-(t+\log b)} d\tilde{X}_t \stackrel{\mathrm{d}}{=} \int_{\log b}^\infty e^{-t} dX_t,$$

and

$$\begin{aligned}\int_0^\infty e^{-t} dX_t &= \int_{\log b}^\infty e^{-t} dX_t + \int_0^{\log b} e^{-t} dX_t \\ &\stackrel{\mathrm{d}}{=} b^{-1}\int_0^\infty e^{-t} d\tilde{X}_t + \int_0^{\log b} e^{-t} dX_t,\end{aligned}$$

which shows the relation (12) and $\mu \in L(\mathbb{R}^d)$.

(b) ($L(\mathbb{R}^d) \subset \Phi(ID_{\log}(\mathbb{R}^d))$): We need a lemma on selfsimilar additive process.

Definition 4.2 Let $H > 0$. A stochastic process $\{X_t, t \ge 0\}$ on $\mathbb{R}^d$ is H-selfsimilar if for any $c > 0$, $\{X_{ct}\} \stackrel{\mathrm{d}}{=} \{c^H X_t\}$.

Lemma 4.3 (Sato [71]) $\mu \in L(\mathbb{R}^d)$ *if and only if there exists a* 1*-selfsimilar additive process* $\{Y_t\}$ *such that* $\mathscr{L}(Y_1) = \mu$.

The following proof is due to Jeanblanc et al. [29]. Let $\mu \in L(\mathbb{R}^d)$. By Lemma 4.3, there exists 1-selfsimilar additive process $\{Y_t\}$ such that $\mathscr{L}(Y_1) = \mu$. Define

$$X_t = \int_{e^{-t}}^{1} s^{-1} dY_s. \tag{16}$$

Since $\{Y_t\}$ is additive, $\{X_t\}$ is also additive. Further, for $h > 0$,

$$\begin{aligned} X_{t+h} - X_t &= \int_{e^{-(t+h)}}^{e^{-t}} s^{-1} dY_s \\ &= \int_{e^{-h}}^{1} (e^{-t}u)^{-1} dY_{e^{-t}u} \\ &\overset{\mathrm{d}}{=} \int_{e^{-h}}^{1} (e^{-t}u)^{-1} e^{-t} dY_u \quad (\text{by } \{Y_{cu}\} \overset{\mathrm{d}}{=} \{cY_u\}) \\ &= X_h. \end{aligned}$$

Thus $\{X_t\}$ is a Lévy process. By (16),

$$X_t = -\int_0^t (e^{-v})^{-1} d_v Y_{e^{-v}}$$

and thus

$$\int_0^\infty e^{-v} dX_v = -\int_0^\infty dY_{e^{-v}} = Y_1 - Y_0 = Y_1,$$

implying that $\mathscr{L}(X_1) \in \mathfrak{D}(\Phi)$ and $\mu = \mathscr{L}\left(\int_0^\infty e^{-v} dX_v\right)$ so that $\mu \in \Phi(ID_{\log}(\mathbb{R}^d))$.

5 Stochastic Integral Mappings and Characterizations of Classes (II)

Some other mappings in addition to the six mappings in Sect. 4.1 above will be explained in this section. Let

$$ID_\alpha(\mathbb{R}^d) = \left\{ \mu \in ID(\mathbb{R}^d) \colon \int_{\mathbb{R}^d} |x|^\alpha \mu(dx) < \infty \right\}, \quad \text{for } \alpha > 0,$$

$$ID_\alpha^0(\mathbb{R}^d) = \left\{ \mu \in ID_\alpha(\mathbb{R}^d) \colon \int_{\mathbb{R}^d} x\mu(dx) = 0 \right\}, \quad \text{for } \alpha \ge 1,$$

$$ID_1^*(\mathbb{R}^d) = \left\{ \mu = \mu_{(A,\nu,\gamma)} \in ID_1^0(\mathbb{R}^d) \colon \lim_{T\to\infty} \int_1^T t^{-1} dt \int_{|x|>t} x\nu(dx) \text{ exists in } \mathbb{R}^d \right\}.$$

5.1 Φ_α-Mapping

We define Φ_α-mapping as follows:

$$\Phi_\alpha(\mu) = \begin{cases} \mathscr{L}\left(\int_0^{-1/\alpha} (1+\alpha t)^{-1/\alpha} dX_t^{(\mu)}\right), & \text{when } \alpha < 0, \\ \mathscr{L}\left(\int_0^{\infty} e^{-t} dX_t^{(\mu)}\right), & \text{when } \alpha = 0, \\ \mathscr{L}\left(\int_0^{\infty} (1+\alpha t)^{-1/\alpha} dX_t^{(\mu)}\right), & \text{when } 0 < \alpha < 2. \end{cases}$$

The domain of Φ_α is given as

$$\mathfrak{D}(\Phi_\alpha) = \begin{cases} ID(\mathbb{R}^d), & \text{when } \alpha < 0, \\ ID_{\log}(\mathbb{R}^d), & \text{when } \alpha = 0, \\ ID_\alpha(\mathbb{R}^d), & \text{when } 0 < \alpha < 1, \\ ID_1^*(\mathbb{R}^d), & \text{when } \alpha = 1, \\ ID_\alpha^0(\mathbb{R}^d), & \text{when } 1 < \alpha < 2. \end{cases}$$

(For $\alpha \in (0,1) \cup (1,2)$, see Sato [75, Theorem 2.4], and for $\alpha = 1$, Sato [77, Theorem 4.4].)

Here we introduce a notion of "weak mean" for later use.

Definition 5.1 (The Weak Mean of $\mu \in ID(\mathbb{R}^d)$ [77, Definition 3.6]) Let $\mu = \mu_{(A,\nu,\gamma)} \in ID(\mathbb{R}^d)$. It is said that μ has weak mean m_μ if

$$\int_{1<|x|\le a} x\nu(dx) \text{ is convergent in } \mathbb{R}^d \text{ as } a \to \infty,$$

and

$$C_\mu(z) = -2^{-1}\langle z, Az\rangle + \lim_{a\to\infty} \int_{|x|\le a} (e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle)\nu(dx) + i\langle m_\mu, z\rangle.$$

The range of Φ_α is as follows.

Theorem 5.2 (Sato [77, Theorem 4.18]. $\mathfrak{R}(K_{1,\alpha})$ in the Notation There) *Let $0 < \alpha < 2$. Then $\mu \in \mathfrak{R}(\Phi_\alpha)$ if and only if $\mu \in ID(\mathbb{R}^d)$ and either $\nu_\mu = 0$ or $\nu_\mu \neq 0$, and in case $\nu_\mu \neq 0$, the radial component ν_ξ of ν_μ is expressed as, for some $k_\xi(r)$ which is a nonnegative function measurable in ξ and nonincreasing on $(0,\infty)$ as a function of r,*

(1) $(\alpha < 1)$ $\nu_\xi(dr) = r^{-\alpha-1}k_\xi(r)dr$,

(2) ($\alpha = 1$) $\nu_\xi(dr) = r^{-2}k_\xi(r)dr$, *and the weak mean of* μ *is* 0,
(3) ($1 < \alpha < 2$) $\nu_\xi(dr) = r^{-\alpha-1}k_\xi(r)dr$, *and the mean of* μ *is* 0.

We introduce the class $L^{\langle\alpha\rangle}(\mathbb{R}^d)$ (the class of α-selfdecomposable distributions). Let $\alpha \in \mathbb{R}$. We say that $\mu \in ID(\mathbb{R}^d)$ is α*-selfdecomposable*, if for any $b > 1$, there exists $\rho_b \in ID(\mathbb{R}^d)$ satisfying

$$\hat{\mu}(z) = \hat{\mu}(b^{-1}z)^{b^\alpha}\hat{\rho}_b(z), \quad z \in \mathbb{R}^d. \tag{17}$$

Theorem 5.3 (Maejima and Ueda [52])

(1) *For* $\beta < \alpha$, $\quad L^{\langle\beta\rangle}(\mathbb{R}^d) \supset L^{\langle\alpha\rangle}(\mathbb{R}^d)$.
(2) *For* $\alpha > 2 \quad L^{\langle\alpha\rangle}(\mathbb{R}^d) = \{\delta_\gamma : \gamma \in \mathbb{R}^d\}$.
(3) $L^{\langle 2\rangle}(\mathbb{R}^d) = \{$*all Gaussian distributions*$\}$.
(4) $L^{\langle\alpha\rangle}(\mathbb{R}^d)$ *is left-continuous in* $\alpha \in \mathbb{R}$, *namely,*

$$\bigcap_{\beta<\alpha} L^{\langle\beta\rangle}(\mathbb{R}^d) = L^{\langle\alpha\rangle}(\mathbb{R}^d) \quad \textit{for all } \alpha \in \mathbb{R}.$$

(5) *Let* $\alpha \in (-\infty, 2)$. *Then,* $\mu \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ *if and only if* $\mu \in ID(\mathbb{R}^d)$ *and either* $\nu_\mu = 0$ *or* $\nu_\mu \neq 0$ *and, in case* $\nu_\mu \neq 0$, *the radial component* ν_ξ *of* ν_μ *is expressed as*

$$\nu_\xi(dr) = r^{-\alpha-1}\ell_\xi(r)dr, \tag{18}$$

where $\ell_\xi(r)$ *is a nonnegative function which is measurable in* ξ, *and nonincreasing on* $(0, \infty)$ *as a function of* r.

Remark 5.4

(1) We have $L^{(-1)}(\mathbb{R}^d) = U(\mathbb{R}^d)$ and $L^{(0)}(\mathbb{R}^d) = L(\mathbb{R}^d)$. Thus by Theorem 5.3(1), if $\alpha < -1$, $L^{\langle\alpha\rangle}(\mathbb{R}^d) \supset U(\mathbb{R}^d) \supset L(\mathbb{R}^d)$.
(2) Another class bigger than $U(\mathbb{R}^d)$ is

$$A(\mathbb{R}^d) := \left\{\Phi_{\cos}(\mu) = \mathscr{L}\left(\int_0^1 \cos(2^{-1}\pi t)dX_t^{(\mu)}\right) \ : \ \mu \in I(\mathbb{R}^d)\right\}.$$

(See Maejima et al. [58, Theorem 2.6].)
(3) It is an open problem to find the relationship between $L^{\langle\alpha\rangle}(\mathbb{R}^d), \alpha < -1$, and $A(\mathbb{R}^d)$.

The relations between the mappings Φ_α and the classes $L^{\langle\alpha\rangle}(\mathbb{R}^d)$ are as follows. (The case $\alpha = 0$ is nothing but Proposition 4.1(3).)

Theorem 5.5 (Maejima et al. [57, Theorem 4.6]) *Let* $\alpha < 0$. $\tilde{\mu} \in L^{\langle\alpha\rangle}(\mathbb{R}^d)$ *if and only if* $\tilde{\mu} = \Phi_\alpha(\mu)$ *for some* $\mu \in ID(\mathbb{R}^d)$.

Theorem 5.6 (Maejima and Ueda [52, Theorem 5.1(ii) and (iv)]) *Let* $\alpha \in (0, 1) \cup (1, 2)$.

(1) *When* $0 < \alpha < 1$, $\tilde{\mu} \in L^{\langle \alpha \rangle}(\mathbb{R}^d)$ *if and only if*

$$\tilde{\mu} = \sigma_\alpha * \Phi_\alpha(\mu), \tag{19}$$

where $\mu \in ID_\alpha(\mathbb{R}^d)$ *and* σ_α *is a strictly* α*-stable distribution or a trivial distribution, where* $*$ *means convolution.*

(2) *When* $1 < \alpha < 2$, $\tilde{\mu} \in L^{\langle \alpha \rangle}(\mathbb{R}^d)$ *if and only if* (19) *holds for some* $\mu \in ID_\alpha^0(\mathbb{R}^d)$ *and some* α*-stable distribution* σ_α.

For the case $\alpha = 1$ we need slightly different mapping called the essential improper stochastic integrals introduced by Sato [74, 76] defined as

$$\Phi_{f,\mathrm{es}}(\mu) := \Bigg\{ \mathscr{L}\left(\operatorname*{p-lim}_{t\to\infty} \left(\int_0^t f(s)dX_s^{(\mu)} - q(t) \right) \right) : q \text{ is an } \mathbb{R}^d\text{-valued nonrandom function such that } \int_0^t f(s)dX_s^{(\mu)} - q(t) \text{ converges in probability as } t \to \infty \Bigg\}.$$

(The term "essentially improper stochastic integral" is changed to "essentially definable improper stochastic integral" in Sato [76].) When $f(t) = (1+t)^{-1}$, which is the integrand of $\Phi_1(\mu)$, we write $\Phi_{f,\mathrm{es}}(\mu)$ as $\Phi_{1,\mathrm{es}}(\mu)$.

Theorem 5.7 *When* $\alpha = 1$, $\tilde{\mu} \in L^{\langle 1 \rangle}(\mathbb{R}^d)$ *if and only if* $\tilde{\mu} = \sigma_1 * \tilde{\rho}$, *where* $\tilde{\rho} \in \Phi_{1,\mathrm{es}}(\rho)$ *for some* $\rho \in I_1(\mathbb{R}^d)$ *and* σ_1 *is a* 1*-stable distribution.*

Remark 5.8 The classes $L^{\langle \alpha \rangle}(\mathbb{R}^d), \alpha \in \mathbb{R}$, were already studied by many authors before Maejima and Ueda [52]. Alf and O'Connor [2] and O'Connor [62] studied the class of all infinitely divisible distributions on $\mathbb{R}$ with unimodal Lévy measures with mode 0, and showed that the class is equal to $L^{\langle -1 \rangle}(\mathbb{R})$, As to this class, Alf and O'Connor [2] studied stochastic integral characterizations with respect to Lévy processes. O'Conner [62] studied the decomposability (17) for $d = 1$ and $\alpha = -1$, and characterized this class by some limit theorem. O'Connor [61, 63] also studied the classes $L^{\langle \alpha \rangle}(\mathbb{R}), \alpha \in (-1, 2)$. He defined these classes by a condition of radial components of Lévy measures, and characterized these classes by stochastic integrals with respect to Lévy processes, by the decomposability (17) for $d = 1$, and by similar limit theorems to that in the case $L^{\langle -1 \rangle}(\mathbb{R})$. Jurek [30, 31, 35] and Iksanov et al. [26] defined and studied so-called s-selfdecomposable distributions on a real separable Hilbert space H. The totality of s-selfdecomposable distributions, denoted by $\mathscr{U}(H)$ in their papers, is equal to $L^{\langle -1 \rangle}(\mathbb{R}^d)$, when $H = \mathbb{R}^d$. Jurek [32–34] and Jurek and Schreiber [37] studied the classes $\mathscr{U}_\beta(Q), \beta \in \mathbb{R}$, of distributions on a real separable Banach space E, where Q is a linear operator on E with certain properties. These classes are equal to $L^{\langle -\beta \rangle}(\mathbb{R}^d)$ if $E = \mathbb{R}^d$ and Q is the identity operator. They defined the classes $\mathscr{U}_\beta(Q)$ by some limit theorems. As to these classes, they studied the decomposability similar to (17) and stochastic integral characterizations, although some results are only for the case that Q is the identity operator.

Remark 5.9 Maejima et al. [57] studied the classes $K_\alpha(\mathbb{R}^d), \alpha < 2$: $\mu \in K_\alpha(\mathbb{R}^d)$ if $\mu \in ID(\mathbb{R}^d)$ and either $\nu_\mu = 0$ or $\nu_\mu \neq 0$ and, in case $\nu_\mu \neq 0$, the radial component ν_ξ of ν_μ is expressed as

$$\nu_\xi(dr) = r^{-\alpha-1}\ell_\xi(r)dr, \tag{20}$$

where $\ell_\xi(r)$ is a nonnegative function which is measurable in ξ and nonincreasing on $(0,\infty)$ as a function of r, and $\ell_\xi(\infty) = 0$. The relation between $K_\alpha(\mathbb{R}^d)$ and $L^{\langle\alpha\rangle}(\mathbb{R}^d)$ for $\alpha < 2$ is

$$K_\alpha(\mathbb{R}^d) = L^{\langle\alpha\rangle}(\mathbb{R}^d) \cap \mathscr{C}_\alpha(\mathbb{R}^d),$$

where $\mathscr{C}_\alpha(\mathbb{R}^d)$ is the totality of $\mu \in ID(\mathbb{R}^d)$ whose Lévy measure ν_μ satisfies $\lim_{r\to\infty} r^\alpha \int_{|x|>r} \nu_\mu(dx) = 0$. (Maejima and Ueda [52], Maejima et al. [57].)

Remark 5.10 Recall that the difference between $U(\mathbb{R}^d)$ and $B(\mathbb{R}^d)$ in terms of Lévy measure is that $\ell_\xi(r)$ in (9) is nonincreasing and $\ell_\xi(r)$ in (10) is completely monotone. Also, the difference between $L(\mathbb{R}^d)$ and $T(\mathbb{R}^d)$ in terms of Lévy measure is that $k_\xi(r)$ in (11) is nonincreasing and $k_\xi(r)$ in (13) is completely monotone. From this point of view, the nonincreasing function $\ell_\xi(r)$ in (20) can be replaced by a completely monotone function $\ell_\xi(r)$ with $\ell_\xi(\infty) = 0$. Actually, if we do so, we can get (31) in Sect. 5.4 later, which leads to the tempered stable distribution by Rosiński [69].

5.2 $\Psi_{\alpha,\beta}$-Mapping

We define a more general notation of mapping, which we call $\Psi_{\alpha,\beta}$-mapping. Let

$$t = G_{\alpha,\beta}(s) = \int_s^\infty u^{-\alpha-1}e^{-u^\beta}du, \quad s \geq 0,$$

and let $s = G^*_{\alpha,\beta}(t)$ be its inverse function. Define $\Psi_{\alpha,\beta}$-mapping by

$$\Psi_{\alpha,\beta}(\mu) = \mathscr{L}\left(\int_0^{G_{\alpha,\beta}(0)} G^*_{\alpha,\beta}(t)dX_t^{(\mu)}\right),$$

with

$$G_{\alpha,\beta}(0) = \begin{cases} \beta^{-1}\Gamma(-\alpha\beta^{-1}), & \text{when } \alpha < 0, \\ \infty, & \text{when } \alpha \geq 0, \end{cases}$$

where $\Gamma(\cdot)$ is the gamma function. These mappings were introduced first by Sato [75] for $\beta = 1$ and later by Maejima and Nakahara [44] for general $\beta > 0$. Due to Sato [75] and Maejima and Nakahara [44], we see the domains $\mathfrak{D}(\Psi_{\alpha,\beta})$ as follows, which are independent of the value $\beta > 0$.

$$\mathfrak{D}(\Psi_{\alpha,\beta}) = \begin{cases} ID(\mathbb{R}^d), & \text{when } \alpha < 0, \\ ID_{\log}(\mathbb{R}^d), & \text{when } \alpha = 0, \\ ID_\alpha(\mathbb{R}^d), & \text{when } 0 < \alpha < 1, \\ ID_1^*(\mathbb{R}^d), & \text{when } \alpha = 1, \\ ID_\alpha^0(\mathbb{R}^d), & \text{when } 1 < \alpha < 2. \end{cases}$$

The six mappings in Sect. 4.1 are the special cases of the Φ_α- and $\Psi_{\alpha,\beta}$-mappings as follows.

Remark 5.11 $\mathscr{U} = \Phi_{-1}, \Upsilon = \Psi_{-1,1}, \Phi = \Phi_0, \Psi = \Psi_{0,1}, \mathscr{G} = \Psi_{-1,2}, \mathscr{M} = \Psi_{0,2}$.

5.3 $\Phi_{(b)}$-Mapping

Let $b > 1$. Define $\Phi_{(b)}$-mapping by

$$\Phi_{(b)}(\mu) = \mathscr{L}\left(\int_0^\infty b^{-[t]} dX_t^{(\mu)}\right), \quad \mathfrak{D}(\Phi_{(b)}) = ID_{\log}(\mathbb{R}^d),$$

where $[t]$ denotes the largest integer not greater than $t \in \mathbb{R}$.

$\mu \in ID(\mathbb{R}^d)$ is called *semi-selfdecomposable* if there exist $b > 1$ and $\rho \in ID(\mathbb{R}^d)$ such that $\hat{\mu}(z) = \hat{\mu}(b^{-1}z)\hat{\rho}(z)$. We call this b a span of μ, and we denote the class of all semi-selfdecomposable distributions with span b by $L(b, \mathbb{R}^d)$. From the definitions, $L(b, \mathbb{R}^d) \supsetneqq L(\mathbb{R}^d)$ and $L(\mathbb{R}^d) = \bigcap_{b>1} L(b, \mathbb{R}^d)$. $\mu \in L(b, \mathbb{R}^d)$ is also realized as a limiting distribution of normalized partial sums of independent random variables under the condition of infinite smallness when the limit is taken through a geometric subsequence. A typical example is a semi-stable distribution.

Theorem 5.12 *Fix any $b > 1$. Then, the range $\mathfrak{R}(\Phi_{(b)})$ is the class of all semi-selfdecomposable distributions with span b on $\mathbb{R}^d$, namely,*

$$\Phi_{(b)}\left(ID_{\log}(\mathbb{R}^d)\right) = L(b, \mathbb{R}^d).$$

(For the proof, see Maejima and Ueda [50].)

5.4 Stable Mapping

This section is from Maejima et al. [59]. Let $0 < \alpha < 2$. Define a mapping by

$$\Xi_\alpha(\mu) = \mathscr{L}\left(\int_0^\infty t^{-1/\alpha} dX_t^{(\mu)}\right). \tag{21}$$

Note that the kernel above has a singularity at $t = 0$ and is not square integrable around $t = 0$. This fact gives an influence when determining the domain of mappings. The following characterization of $\mathfrak{D}(\Xi_\alpha)$ follows from Proposition 5.3 and Example 4.5 of Sato [76].

Theorem 5.13

(1) *If* $0 < \alpha < 1$, *then*

$$\mathfrak{D}(\Xi_\alpha) = \left\{\mu = \mu_{(0,\nu,0)_0} \in ID(\mathbb{R}^d) \ : \ \int_{\mathbb{R}^d} |x|^\alpha \nu(dx) < \infty\right\}.$$

(2) *If* $\alpha = 1$, *then*

$$\mathfrak{D}(\Xi_1) = \Big\{\mu = \mu_{(0,\nu,0)_0} = \mu_{(0,\nu,0)_1} \in ID(\mathbb{R}^d) \ : \ \int_{\mathbb{R}^d} |x| \ \nu(dx) < \infty, \int_{\mathbb{R}^d} x \ \nu(dx) = 0,$$
$$\lim_{\varepsilon \downarrow 0} \int_{|x| \le 1} x \log(|x| \vee \varepsilon) \ \nu(dx) \ \textit{and} \ \lim_{T \to \infty} \int_{|x| > 1} x \log(|x| \wedge T) \ \nu(dx) \ \textit{exist}\Big\}.$$

(3) *If* $1 < \alpha < 2$, *then*

$$\mathfrak{D}(\Xi_\alpha) = \left\{\mu = \mu_{(0,\nu,0)_1} \in ID(\mathbb{R}^d) \ : \ \int_{\mathbb{R}^d} |x|^\alpha \nu(dx) < \infty\right\}.$$

Remark 5.14 There is a simple sufficient condition for μ in (2). Namely, $\mu = \mu_{(0,\nu,\gamma)} \in \mathfrak{D}(\Xi_1)$ if $\int_{\mathbb{R}^d} |x| \, |\log |x|| \ \nu(dx) < \infty$, $\int_{\mathbb{R}^d} x \ \nu(dx) = 0$, and $\gamma = \int_{\mathbb{R}^d} \frac{x}{1+|x|^2} \nu(dx)$.

The next theorem gives a full characterization of $\mathfrak{R}(\Xi_\alpha)$. $S_\alpha^0(\mathbb{R}^d)$ denotes the class of strictly α-stable distributions on $\mathbb{R}^d$. Note that in the case $\alpha = 1$ $\hat{\mu}$ can be written as follows:

$$\hat{\mu}(z) = \exp\left\{-\int_S \left(|\langle z, \xi\rangle| + i2\pi^{-1}\langle z, \xi\rangle \log |\langle z, \xi\rangle|\right) \lambda_1(d\xi) + i\langle z, \tau\rangle\right\}, \tag{22}$$

where λ_1 is a finite measure on S and $\tau \in \mathbb{R}^d$, and where $\int_S \xi \lambda_1(d\xi) = 0$. (See e.g. Sato [73, Theorem 14.10].)

Theorem 5.15 (Maejima et al. [59]) *Let* $0 < \alpha < 2$.

(1) *When* $\alpha \neq 1$, *we have*

$$\Xi_\alpha(\mathfrak{D}(\Xi_\alpha)) = S_\alpha^0(\mathbb{R}^d).$$

(2) *When* $\alpha = 1$, *we have*

$$\Xi_1(\mathfrak{D}(\Xi_1)) = \left\{\mu \in S_1^0(\mathbb{R}^d) \ : \ \tau \in \text{span supp}(\lambda_1)\right\},$$

where, respectively, λ_1 *and* τ *are those in* (22). *Here* $\text{supp}(\lambda_1)$ *denotes the support of* λ_1. *If* $\lambda_1 = 0$, *then we put* $\text{span supp}(\lambda_1) = \{0\}$ *by convention.*

6 Compositions of Stochastic Integral Mappings

The motivation for the paper by Barndorff-Nielsen et al. [12] was to see if the Thorin class can be realized as the composited mapping of Φ and Υ, where Φ produces the class of selfdecomposable distributions and Υ produces the Goldie-Steutel-Bondesson class. So, we believed that compositions of stochastic integral mappings would be important and useful in many aspects, which was verified by several observations. This is why we will discuss compositions of stochastic integral mappings.

Let Φ_f and Φ_g be two stochastic integral mappings. The composition of two mappings is defined as

$$(\Phi_f \circ \Phi_g)(\mu) = \Phi_f(\Phi_g(\mu)),$$

with

$$\mathfrak{D}(\Phi_f \circ \Phi_g) = \{\mu \in \mathfrak{D}(\Phi_g) \ : \ \Phi_g(\mu) \in \mathfrak{D}(\Phi_f)\}.$$

We have the following.

Theorem 6.1 *We have*

(1) $\Psi = \Phi \circ \Upsilon = \Upsilon \circ \Phi$,
(2) $\Upsilon \circ \mathscr{U} = \mathscr{U} \circ \Upsilon$,
(3) $\mathscr{G} \circ \mathscr{U} = \mathscr{U} \circ \mathscr{G}$,
(4) $\Phi \circ \mathscr{U} = \mathscr{U} \circ \Phi$.

Proof of (1) of Theorem 6.1 (Barndorff-Nielsen et al. [12, Theorem C(ii)]) Note that $\mathfrak{D}(\Psi)(z) = ID_{\log}(\mathbb{R}^d)$. If $\mu \in ID_{\log}(\mathbb{R}^d)$, then

$$C_{\Phi(\mu)}(z) = \int_0^\infty C_\mu(e^{-t}z)dt.$$

On the other hand, if $\mu \in ID(\mathbb{R}^d)$, then

$$C_{\Upsilon(\mu)}(z) = \int_0^1 C_\mu(\log(t^{-1})z)dt = \int_0^\infty e^{-s} C_\mu(sz)ds.$$

Also note that

$$\Upsilon(\mu) \in ID_{\log}(\mathbb{R}^d) \text{ if and only if } \mu \in ID_{\log}(\mathbb{R}^d), \tag{23}$$

(see Barndorff-Nielsen et al. [12, Theorem C(i)]). Thus, if $\mu \in ID_{\log}(\mathbb{R}^d)$, then $\Upsilon(\mu) \in ID_{\log}(\mathbb{R}^d)$ by (23), and hence

$$C_{(\Phi\circ\Upsilon)(\mu)}(z) = \int_0^\infty dt \int_0^\infty e^{-s} C_\mu(e^{-t}sz)ds$$

and

$$C_{(\Upsilon\circ\Phi)(\mu)}(z) = \int_0^\infty e^{-s}ds \int_0^\infty C_\mu(e^{-t}sz)dt.$$

If we could show

$$\int_0^\infty e^{-s}ds \int_0^\infty |C_\mu(e^{-t}sz)|dt < \infty, \quad \text{for each } z \in \mathbb{R}, \tag{24}$$

then we can apply Fubini's theorem to get

$$C_{(\Phi\circ\Upsilon)(\mu)}(z) = C_{(\Upsilon\circ\Phi)(\mu)}(z),$$

meaning $\Phi \circ \Upsilon = \Upsilon \circ \Phi$, and

$$\begin{aligned} C_{(\Phi\circ\Upsilon)(\mu)}(z) &= \int_0^\infty dt \int_0^\infty e^{-s} C_\mu(uz)e^{t-ue^t}du = \int_0^\infty C_\mu(uz)e^{-u}u^{-1}du \\ &= -\int_0^\infty C_\mu(uz)dp(u) = \int_0^\infty C_\mu(p^*(t)z)dt = C_{\Psi(\mu)}(z), \end{aligned}$$

concluding $\Phi \circ \Upsilon = \Psi$. It remains to prove (24). We need the following lemma.

Lemma 6.2 *Let $\mu = \mu_{(A,\nu,\gamma)} \in ID(\mathbb{R}^d)$ For each fixed $z \in \mathbb{R}$,*

$$|C_\mu(az)| \le c_z \left[a^2 + |a| + \int_{\mathbb{R}^d} \frac{|ax|^2}{1+|ax|^2}\nu(dx) + \int_{\mathbb{R}^d} \frac{(|a|+|a|^3)|x|^3}{(1+|x|^2)(1+|ax|^2)}\nu(dx) \right],$$

where $c_z > 0$ is a finite constant depending only on z.

Proof of Lemma 6.2 Let $g(z,x) = e^{i\langle z,x\rangle} - 1 - \frac{i\langle z,x\rangle}{1+|x|^2}$. Since

$$|C_\mu(z)| \le \frac{1}{2}(\mathrm{tr}A)|z|^2 + |\gamma||z| + \int_{\mathbb{R}^d} |g(z,x)|\nu(dx),$$

we have

$$|C_\mu(az)| \le c_z(a^2 + |a|) + \int_{\mathbb{R}^d} |g(z,ax)|\nu(dx) + \int_{\mathbb{R}^d} |g(az,x) - g(z,ax)|\nu(dx).$$

The inequalities

$$g(z,x) \le c_z \frac{|x|^2}{1+|x|^2}$$

and

$$|g(az,x) - g(z,ax)| \le c_z \frac{(|a| + |a|^3)|x|^3}{(1+|x|^2)(1+|ax|^2)}$$

conclude the proof of the lemma.

We then have, by Lemma 6.2 that

$$\begin{aligned}
&\int_0^\infty e^{-s}ds \int_0^\infty |C_\mu(e^{-t}sz)|dt \\
&\le \int_0^\infty e^{-s}ds \int_0^\infty c_z \Bigg[e^{-2t}s^2 + e^{-t}s + \int_{\mathbb{R}^d} \frac{|e^{-t}sx|^2}{1+|e^{-t}sx|^2}\nu(dx) \\
&\quad + \int_{\mathbb{R}^d} \frac{(e^{-t}s + e^{-3t}s^3)|x|^3}{(1+|x|^2)(1+|e^{-t}sx|^2)}\nu(dx) \Bigg] dt \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

$I_1 < \infty$ and $I_2 < \infty$ are trivial. As to I_3,

$$\begin{aligned}
I_3 &= c_z \int_0^\infty e^{-s}ds \int_0^\infty dt \int_{\mathbb{R}^d} \frac{|e^{-t}sx|^2}{1+|e^{-t}sx|^2}\nu(dx) \\
&= c_z \int_0^\infty e^{-s}ds \int_0^\infty dt \left(\int_{|x|\le 1} + \int_{|x|>1} \right) \frac{|e^{-t}sx|^2}{1+|e^{-t}sx|^2}\nu(dx),
\end{aligned}$$

where

$$\begin{aligned}
&\int_0^\infty e^{-s}ds \int_0^\infty dt \int_{|x|\le 1} \frac{|e^{-t}sx|^2}{1+|e^{-t}sx|^2}\nu(dx) \\
&\quad \le \int_0^\infty e^{-s}s^2 ds \int_0^\infty e^{-2t}dt \int_{|x|\le 1} |x|^2\nu(dx) < \infty
\end{aligned}$$

and

$$
\begin{aligned}
&\int_0^\infty e^{-s}ds \int_{|x|>1} \nu(dx) \int_0^\infty \frac{|e^{-t}sx|^2}{1+|e^{-t}sx|^2}dt \\
&\quad = \int_0^\infty e^{-s}ds \int_{|x|>1} \nu(dx) \left(\int_0^{\log|x|} + \int_{\log|x|}^\infty \right) \frac{|e^{-t}sx|^2}{1+|e^{-t}sx|^2}dt \\
&\quad \le \int_0^\infty e^{-s}ds \int_{|x|>1} \left(\log|x| + s^2|x|^2 \frac{1}{2} e^{-2\log|x|} \right) \nu(dx) < \infty.
\end{aligned}
$$

As to I_4, we omit the proof, since the basic ideas are the same as for I_3. Equation (24) is thus proved.

Proof of (2) of Theorem 6.1 We have

$$
\begin{aligned}
C_{(\Upsilon\circ\mathscr{U})(\mu)}(z) &= \int_0^1 C_{\mathscr{U}(\mu)}(\log(t^{-1})z)dt = \int_0^1 dt \int_0^1 C_\mu(\log(t^{-1})sz)ds \\
&\qquad \text{(by Fubini's theorem)} \\
&= \int_0^1 ds \int_0^1 C_\mu(\log(t^{-1})sz)dt = \int_0^1 C_{\Upsilon(\mu)}(sz)ds = C_{(\mathscr{U}\circ\Upsilon)(\mu)}(z).
\end{aligned}
$$

Proof of (3) of Theorem 6.1 We have

$$
\begin{aligned}
C_{(\mathscr{G}\circ\mathscr{U})(\mu)}(z) &= \int_0^{\sqrt{\pi}/2} C_{\mathscr{U}(\mu)}(g^*(t)z)dt = \int_0^{\sqrt{\pi}/2} dt \int_0^1 C_\mu(g^*(t)sz)ds \\
&\qquad \text{(by Fubini's theorem)} \\
&= \int_0^1 ds \int_0^{\sqrt{\pi}/2} C_\mu(g^*(t)sz)dt = \int_0^1 C_{\mathscr{G}(\mu)}(sz)ds = C_{(\mathscr{U}\circ\mathscr{G})(\mu)}(z).
\end{aligned}
$$

Proof of (4) of Theorem 6.1 We first show that $\mathscr{U}(\mu) \in ID_{\log}(\mathbb{R}^d)$ if and only if $\mu \in ID_{\log}(\mathbb{R}^d)$. Let $\mu \in ID(\mathbb{R}^d)$ and $\tilde{\mu} = \mathscr{U}(\mu)$. We have

$$
\begin{aligned}
\int_{|x|>2} \log|x| \nu_{\tilde{\mu}}(dx) &= \int_0^1 sds \int_{|x|>2/s} \log(s|x|)\nu_\mu(dx) \\
&= \int_{|x|>2} \nu_\mu(dx) \int_{2/|x|}^1 (s\log s + s\log|x|)\, ds \\
&=: \int_{|x|>2} h(x)\nu_\mu(dx),
\end{aligned}
$$

where $h(x) \sim \log|x|$ as $|x| \to \infty$. Thus, $\int_{|x|>2} \log|x|\nu_{\tilde{\mu}}(dx) < \infty$ if and only if $\int_{|x|>2} \log|x|\nu_\mu(dx) < \infty$. Then if $\mu \in ID_{\log}(\mathbb{R}^d)$, then using $\mathscr{U}(\mu) \in ID_{\log}(\mathbb{R}^d)$, we have

$$C_{(\Phi\circ\mathscr{U})(\mu)}(z) = \int_0^\infty C_{\mathscr{U}(\mu)}(e^{-t}z)dt = \int_0^\infty dt \int_0^1 C_\mu(se^{-t}z)ds$$

(by Fubini's theorem)

$$= \int_0^1 ds \int_0^\infty C_\mu(se^{-t}z)dt = \int_0^1 C_{\Phi(\mu)}(sz)ds = C_{(\mathscr{U}\circ\Phi)(\mu)}(z)$$

For the applicability of Fubini's theorem above, we have to check that

$$\int_0^1 ds \int_0^\infty |C_\mu(se^{-t}z)|dt < \infty.$$

By Lemma 6.2, we have

$$\int_0^1 ds \int_0^\infty |C_\mu(se^{-t}z)|dt \le \int_0^1 ds \int_0^\infty c_z\Bigg[e^{-2t}s^2 + e^{-t}s + \int_{\mathbb{R}^d} \frac{|e^{-t}sx|^2}{1+|e^{-t}sx|^2}\nu(dx)$$

$$+ \int_{\mathbb{R}^d} \frac{(e^{-t}s + e^{-3t}s^3)|x|^3}{(1+|x|^2)(1+|e^{-t}sx|^2)}\nu(dx)\Bigg]dt$$

$$=: I_1 + I_2 + I_3 + I_4.$$

$I_1 < \infty$ and $I_2 < \infty$ are trivial. We have

$$I_3 = \int_0^1 ds \int_0^\infty dt \left(\int_{|x|\le 1} + \int_{|x|>1}\right) \frac{|e^{-t}sx|^2}{1+|e^{-t}sx|^2}\nu(dx),$$

where

$$\int_0^1 ds \int_0^\infty dt \int_{|x|\le 1} \frac{|e^{-t}sx|^2}{1+|e^{-t}sx|^2}\nu(dx)$$

$$\le \int_0^1 s^2 \int_0^\infty e^{-2t}dt \int_{|x|\le 1} |x|^2\nu(dx) < \infty,$$

$$\int_0^1 ds \int_{|x|>1} \nu(dx) \int_0^\infty \frac{(e^{-t}s + e^{-3t}s^3)|x|^3}{(1+|x|^2)(1+|e^{-t}sx|^2)}dt$$

$$= \int_0^1 ds \int_{|x|>1} \nu(dx) \left(\int_0^{\log|x|} + \int_{\log|x|}^\infty\right) \frac{(e^{-t}s + e^{-3t}s^3)|x|^3}{(1+|x|^2)(1+|e^{-t}sx|^2)}dt$$

$$\le \int_0^1 ds \int_{|x|>1} \left(\log|x| + s^2|x|^2\frac{1}{2}e^{-2\log|x|}\right)\nu(dx) < \infty$$

I_4 can be handled similarly. This completes the proof of (4) *of Theorem 6.1*.

The following Proposition 6.3 is a special case of Theorem 3.1 of Sato [75] and Proposition 6.4 can be proved similarly, but we give a proof of Proposition 6.4 here.

Proposition 6.3 *Let*

$$k(s) = \int_s^\infty ue^{-u}du, \quad s \ge 0,$$

and let $k^*(t)$ *be its inverse function. Define a mapping* $\mathscr{K}$ *from* $\mathfrak{D}(\mathscr{K})$ *into* $ID(\mathbb{R}^d)$ *by*

$$\mathscr{K}(\mu) = \mathscr{L}\left(\int_0^1 k^*(t)dX_t^{(\mu)}\right).$$

Then $\mathfrak{D}(\mathscr{K}) = ID(\mathbb{R}^d)$ *and*

$$\Upsilon = \mathscr{K} \circ \mathscr{U} = \mathscr{U} \circ \mathscr{K}.$$

Proposition 6.4 *Let*

$$a(s) = 2\int_s^\infty u^2e^{-u^2}du, \quad s \ge 0,$$

and let $a^*(t)$ *be its inverse function. Define a mapping* $\mathscr{A}$ *from* $\mathfrak{D}(\mathscr{A})$ *into* $ID(\mathbb{R}^d)$ *by*

$$\mathscr{A}(\mu) = \mathscr{L}\left(\int_0^{\Gamma(3/2)} a^*(t)dX_t^{(\mu)}\right). \tag{25}$$

Then $\mathfrak{D}(\mathscr{A}) = ID(\mathbb{R}^d)$ *and*

$$\mathscr{G} = \mathscr{A} \circ \mathscr{U} = \mathscr{U} \circ \mathscr{A}.$$

Proof With respect to the domain of the $\mathscr{A}$-mapping, it is enough to show

$$\int_0^{\Gamma(3/2)} (a^*(t))^2dt < \infty, \tag{26}$$

but (26) follows from

$$\int_0^{\Gamma(3/2)} (a^*(t))^2dt = \int_0^\infty u^4e^{-u^2}du < \infty.$$

Next applying Proposition 2.6(1), we have

$$C_{\mathscr{U}(\mu)}(z) = \int_0^1 C_\mu(tz)dt,$$

$$C_{\mathscr{G}(\mu)}(z) = \int_0^{\sqrt{\pi}/2} C_\mu(h^*(t)z)dt$$

and

$$C_{\mathscr{A}(\mu)}(z) = \int_0^{\Gamma(3/2)} C_\mu(a^*(t)z)dt.$$

Thus,

$$C_{(\mathscr{A}\circ\mathscr{U})(\mu)}(z) = \int_0^{\Gamma(3/2)} dt \int_0^1 C_\mu(a^*(t)uz)du$$

and

$$C_{(\mathscr{U}\circ\mathscr{A})(\mu)}(z) = \int_0^1 dt \int_0^{\Gamma(3/2)} C_\mu(ta^*(u)z)du.$$

If we are allowed to exchange the order of the integrations by Fubini's theorem, then we have

$$C_{(\mathscr{A}\circ\mathscr{U})(\mu)}(z) = C_{(\mathscr{U}\circ\mathscr{A})(\mu)}(z),$$

implying $\mathscr{A}\circ\mathscr{U} = \mathscr{U}\circ\mathscr{A}$, and we have

$$\begin{aligned} C_{(\mathscr{A}\circ\mathscr{U})(\mu)}(z) &= \int_0^1 du \int_0^{\Gamma(3/2)} C_\mu(a^*(t)uz)dt = 2\int_0^1 du \int_0^\infty C_\mu(vuz)v^2 e^{-v^2}dv \\ &= 2\int_0^1 u^{-3}du \int_0^\infty C_\mu(yz)y^2 e^{-y^2/u^2}dy \\ &= 2\int_0^\infty C_\mu(yz)y^2 dy \int_0^1 u^{-3}e^{-y^2/u^2}du = 2\int_0^\infty C_\mu(yz)dy \int_y^\infty te^{-t^2}dt \\ &= \int_0^\infty C_\mu(yz)e^{-y^2}dy = \int_0^{\sqrt{\pi}/2} C_\mu(h^*(t)z)dt = C_{\mathscr{G}(\mu)}(z), \end{aligned}$$

concluding $\mathscr{A} \circ \mathscr{U} = \mathscr{G}$. In order to assure the exchange of the order of the integrations by Fubini's theorem, it is enough to show that

$$\int_0^1 du \int_0^\infty \left| C_\mu(uvz) \right| v^2 e^{-v^2} dv < \infty. \tag{27}$$

For $\mu = \mu_{(A,\nu,\gamma)} \in ID(\mathbb{R}^d)$, we have

$$|C_\mu(z)| \le 2^{-1}(\mathrm{tr}A)|z|^2 + |\gamma||z| + \int_{\mathbb{R}^d} |g(z,x)|\nu(dx),$$

where

$$g(z,x) = e^{i\langle z,x\rangle} - 1 - i\langle z,x\rangle(1+|x|^2)^{-1}.$$

Hence

$$\begin{aligned}|C_\mu(uvz)| &\le 2^{-1}(\mathrm{tr}A)u^2v^2|z|^2 + |\gamma||u||v||z| + \int_{\mathbb{R}^d} |g(z,uvx)|\nu(dx)\\ &\quad + \int_{\mathbb{R}^d} |g(uvz,x) - g(z,uvx)|\nu(dx) =: I_1 + I_2 + I_3 + I_4.\end{aligned}$$

The finiteness of $\int_0^1 du \int_0^\infty (I_1 + I_2)v^2 e^{-v^2} dv$ is trivial. Noting that $|g(z,x)| \le c_z|x|^2(1+|x|^2)^{-1}$ with a positive constant c_z depending on z, we have

$$\begin{aligned}&\int_0^1 du \int_0^\infty I_3 v^2 e^{-v^2} dv\\ &\le c_z \int_{\mathbb{R}^d} \nu(dx) \int_0^1 du \int_0^\infty \frac{(uv|x|)^2}{1+(uv|x|)^2} v^2 e^{-v^2} dv\\ &= c_z \left(\int_{|x|\le 1} \nu(dx) + \int_{|x|>1} \nu(dx) \right) \int_0^1 du \int_0^\infty \frac{(uv|x|)^2}{1+(uv|x|)^2} v^2 e^{-v^2} dv\\ &=: I_{31} + I_{32},\end{aligned}$$

where

$$I_{31} \le c_z \int_{|x|\le 1} |x|^2 \nu(dx) \int_0^1 u^2 du \int_0^\infty v^4 e^{-v^2} dv < \infty,$$

$$I_{32} \le c_z \int_{|x|>1} \nu(dx) \int_0^1 du \int_0^\infty v^2 e^{-v^2} dv < \infty.$$

As to I_4, note that for $a \in \mathbb{R}$,

$$|g(az,x) - g(z,ax)| = \frac{|\langle az,x\rangle||x|^2|1-a^2|}{(1+|x|^2)(1+|ax|^2)}$$
$$\le \frac{|z||x|^3(|a|+|a|^3)}{(1+|x|^2)(1+|ax|^2)} \le \frac{|z||x|^2(1+|a|^2)}{2(1+|x|^2)},$$

since $|b|(1+b^2)^{-1} \le 2^{-1}$. Then

$$\int_0^1 du \int_0^\infty I_4 v^2 e^{-v^2} dv \le 2^{-1}|z| \int_{\mathbb{R}^d} \frac{|x|^2}{1+|x|^2}\nu(dx) \int_0^1 du \int_0^\infty (1+u^2v^2) v e^{-v} dv < \infty.$$

This completes the proof of (27).

We can give a more general result than Propositions 6.3 and 6.4.

Theorem 6.5 (Maejima and Ueda [53]) *Let* $\mathscr{X}_\beta(\mu) = \mathscr{L}\left(X_\beta^{(\mu)}\right), \beta > 0$. *Then for* $\alpha \in (-\infty, 1) \cup (1, 2)$ *and* $\beta > 0$,

$$\Psi_{\alpha,\beta} = \mathscr{X}_\beta \circ \Phi_\alpha \circ \Psi_{\alpha-\beta,\beta} = \mathscr{X}_\beta \circ \Psi_{\alpha-\beta,\beta} \circ \Phi_\alpha.$$

(*Remark that* $\mathscr{X}_\beta(\mu) = \mu^{\beta *}$.)

Proposition 6.3 is the case of Theorem 6.5 with $\alpha = -1$ and $\beta = 1$, since $\mathscr{K} = \mathscr{X}_1 \circ \Psi_{-2,1} = \Psi_{-2,1}$, and Proposition 6.4 is the case of Theorem 6.5 with $\alpha = -1$ and $\beta = 2$, since $\mathscr{A} = \mathscr{X}_2 \circ \Psi_{-3,2}$.

7 Nested Subclasses of Classes of Infinitely Divisible Distributions

As mentioned in Sect. 1 already, once we have mappings in hand, it is natural to consider the iteration of mappings. In our case, this procedure gives us nested subclasses of the original class, which, without mapping, was already studied in the case of selfdecomposable distributions by Urbanik [90] and Sato [70].

7.1 Iteration of Mappings

Let Φ_f be a stochastic integral mapping. The iteration Φ_f^m is defined by $\Phi_f^1 = \Phi_f$ and

$$\Phi_f^{m+1}(\mu) = \Phi_f(\Phi_f^m(\mu))$$

with

$$\mathfrak{D}(\Phi_f^{m+1}) = \{\mu \in \mathfrak{D}(\Phi_f^m) \ : \ \Phi_f^m(\mu) \in \mathfrak{D}(\Phi_f)\}.$$

We have

$$\Phi_f^{m+1}(\mathfrak{D}(\Phi_f^{m+1})) = \Phi_f^m(\Phi_f(\mathfrak{D}(\Phi_f^{m+1}))),$$

implying

$$\Phi_f(\mathfrak{D}(\Phi_f^{m+1})) \subset \mathfrak{D}(\Phi_f^m),$$

and

$$\Phi_f^{m+1}(\mathfrak{D}(\Phi_f^{m+1})) \subset \Phi_f^m(\mathfrak{D}(\Phi_f^m)).$$

Therefore, if we write

$$K_m^f(\mathbb{R}^d) := \Phi_f^m(\mathfrak{D}(\Phi_f^m)),$$

$K_m^f(\mathbb{R}^d), m = 2, 3, \ldots,$ are nested subclasses of $K_1^f(\mathbb{R}^d) = \Phi_f(\mathfrak{D}(\Phi_f))$.

With respect to the domain of mappings, if Φ_f is a proper stochastic integral mapping, then $\mathfrak{D}(\Phi_f^m) = ID(\mathbb{R}^d)$ as mentioned before. For $\Phi_f = \Phi$ or Ψ, (which is an improper stochastic integral mapping), we have the following.

Lemma 7.1 *We have*

(1) $\mathfrak{D}(\Phi^m) = ID_{\log^m}(\mathbb{R}^d)$,
(2) $\mathfrak{D}(\Psi^m) = ID_{\log^m}(\mathbb{R}^d)$.

Proof

(1) See e.g. Rocha-Arteaga and Sato [67, Theorem 49].
(2) We first show that

$$\Upsilon(\mu) \in ID_{\log^m}(\mathbb{R}^d) \text{ if and only if } \mu \in ID_{\log^m}(\mathbb{R}^d).$$

Let $\mu \in ID(\mathbb{R}^d)$ and $\tilde{\mu} = \Upsilon(\mu)$. We have

$$\begin{aligned}\int_{|x|>2} \log^m |x| \nu_{\tilde{\mu}}(dx) &= \int_0^\infty e^{-s} ds \int_{|x|>2/s} \log^m(s|x|)\nu_\mu(dx) \\ &= \int_{\mathbb{R}^d} \nu_\mu(dx) \int_{2/|x|}^\infty e^{-s}(\log s + \log|x|)^m ds \\ &=: \int_{\mathbb{R}^d} h(x)\nu_\mu(dx).\end{aligned}$$

Here $h(x) = o(|x|^2)$ as $|x| \downarrow 0$ and $h(x) \sim \log^m |x|$ as $|x| \to \infty$. Thus, $\int_{|x|>2} \log^m |x|\nu_{\tilde{\mu}}(dx) < \infty$ if and only if $\int_{|x|>2} \log^m |x|\nu_\mu(dx) < \infty$. By

Theorem 6.1(1), we know that $\Psi^m = \Phi^m \circ \Upsilon^m$. Since $\mathfrak{D}(\Upsilon^m) = ID(\mathbb{R}^d)$ and $\mathfrak{D}(\Phi^m) = ID_{\log^m}(\mathbb{R}^d)$, we have $\mathfrak{D}(\Psi^m) = ID_{\log^m}(\mathbb{R}^d)$.

7.2 Definitions and Some Properties of Nested Subclasses

Put

$$U_0(\mathbb{R}^d) = U(\mathbb{R}^d), \quad B_0(\mathbb{R}^d) = B(\mathbb{R}^d), \quad L_0(\mathbb{R}^d) = L(\mathbb{R}^d),$$
$$T_0(\mathbb{R}^d) = T(\mathbb{R}^d), \quad G_0(\mathbb{R}^d) = G(\mathbb{R}^d), \quad M_0(\mathbb{R}^d) = M(\mathbb{R}^d).$$

Definition 7.2 For $m = 0, 1, 2, \ldots$, define
$U_m(\mathbb{R}^d) = \mathscr{U}^{m+1}(ID(\mathbb{R}^d))$,
$B_m(\mathbb{R}^d) = \Upsilon^{m+1}(ID(\mathbb{R}^d))$,
$L_m(\mathbb{R}^d) = \Phi^{m+1}(ID_{\log^{m+1}}(\mathbb{R}^d))$,
$T_m(\mathbb{R}^d) = \Psi^{m+1}(ID_{\log^{m+1}}(\mathbb{R}^d))$,
$G_m(\mathbb{R}^d) = \mathscr{G}^{m+1}(ID(\mathbb{R}^d))$,
$M_m(\mathbb{R}^d) = \mathscr{M}^{m+1}(ID(\mathbb{R}^d))$
and further $U_\infty(\mathbb{R}^d) = \bigcap_{m=0}^\infty U_m(\mathbb{R}^d), B_\infty(\mathbb{R}^d) = \bigcap_{m=0}^\infty B_m(\mathbb{R}^d), L_\infty(\mathbb{R}^d) = \bigcap_{m=0}^\infty L_m(\mathbb{R}^d)$, $T_\infty(\mathbb{R}^d) = \bigcap_{m=0}^\infty T_m(\mathbb{R}^d)$, $G_\infty(\mathbb{R}^d) = \bigcap_{m=0}^\infty G_m(\mathbb{R}^d)$, $M_\infty(\mathbb{R}^d) = \bigcap_{m=0}^\infty M_m(\mathbb{R}^d)$.

Distributions in $L_\infty(\mathbb{R}^d)$ are called completely selfdecomposable distributions.

We start with the following.

Definition 7.3 A class $H \subset ID(\mathbb{R}^d)$ is said to be *completely closed in the strong sense* (c.c.s.s.), if the following are satisfied.

(1) It is closed under convolution.
(2) It is closed under weak convergence.
(3) If X is an $\mathbb{R}^d$-valued random variable with $\mathscr{L}(X) \in H$, then $\mathscr{L}(aX + b) \in H$ for any $a > 0$ and $b \in \mathbb{R}^d$.
(4) $\mu \in H$ implies $\mu^{s*} \in H$ for any $s > 0$.

Proposition 7.4 (Maejima and Sato [48, Proposition 3.2]) *Fix* $0 < a < \infty$. *Suppose that* f *is square integrable on* $(0, a)$ *and* $\int_0^a f(t)dt \neq 0$. *Define a mapping* Φ_f *by*

$$\Phi_f(\mu) = \mathscr{L}\left(\int_0^a f(t)dX_t^{(\mu)}\right).$$

Then the following are true.

(1) $\mathfrak{D}(\Phi_f) = ID(\mathbb{R}^d)$.
(2) *If* H *is c.c.s.s., then* $\Phi_f(H) \subset H$.
(3) *If* H *is c.c.s.s., then* $\Phi_f(H)$ *is also c.c.s.s.*

Remark 7.5

(1) Note that Proposition 7.4 can be applied to Υ- and $\mathscr{G}$-mappings, because in those mappings the stochastic integral is proper, f is square integrable and $\int_0^a f(t)dt \neq 0$. Since $ID(\mathbb{R}^d)$ is c.c.s.s., $B(\mathbb{R}^d)$ and $G(\mathbb{R}^d)$ are c.c.s.s.
(2) Proposition 7.4(3) is not necessarily true when $a = \infty$. Namely, there is a mapping Φ_f defined by $\Phi_f(\mu) = \mathscr{L}\left(\int_0^\infty f(t)dX_t^{(\mu)}\right)$ such that $\Phi_f(H \cap \mathfrak{D}(\Phi_f))$ is not closed under weak convergence for some H which is c.c.s.s. Indeed, the mapping Ψ_α with $0 < \alpha < 1$ in Theorem 4.2 of Sato [75] serves as an example.
(3) However, it is known that when $\Phi_f = \Phi$, Proposition 7.4(2) and (3) are true with $\Phi_f(H)$ replaced by $\Phi(H \cap \mathfrak{D}(\Phi))$, even if $a = \infty$. See Lemma 4.1 of Barndorff-Nielsen et al. [12]. In particular, $L_m(\mathbb{R}^d)$ is c.c.s.s. for $m = 0, 1, \ldots$.
(4) We also have that $T_\infty(\mathbb{R}^d)$ is c.c.s.s. (Maejima and Sato [48, Lemma 3.8].)

Theorem 7.6 *We have the following.*

(1) $B_m(\mathbb{R}^d) \subset U_m(\mathbb{R}^d)$,
(2) $G_m(\mathbb{R}^d) \subset U_m(\mathbb{R}^d)$,
(3) $L_m(\mathbb{R}^d) \subset U_m(\mathbb{R}^d)$,
(4) $T_m(\mathbb{R}^d) \subset L_m(\mathbb{R}^d)$.

Proof

(1) We know that $B_0(\mathbb{R}^d) \subset U_0(\mathbb{R}^d)$. Suppose that $B_m(\mathbb{R}^d) \subset U_m(\mathbb{R}^d)$ for some $m \geq 0$, as the induction hypothesis. Then

$$\begin{aligned} B_{m+1}(\mathbb{R}^d) &= \Upsilon^{m+2}(ID(\mathbb{R}^d)) = \Upsilon(\Upsilon^{m+1}(ID(\mathbb{R}^d)) = \Upsilon(B_m(\mathbb{R}^d)) \\ &\subset \Upsilon(U_m(\mathbb{R}^d)) = \Upsilon(\mathscr{U}^{m+1}(ID(\mathbb{R}^d)) = \mathscr{U}^{m+1}(\Upsilon(ID(\mathbb{R}^d)) \\ &\qquad (\text{since } \Upsilon \circ \mathscr{U} = \mathscr{U} \circ \Upsilon \text{ (Theorem 6.1(2))}) \\ &= \mathscr{U}^{m+1}(\mathscr{U}(ID(\mathbb{R}^d)) = U_{m+1}(\mathbb{R}^d). \end{aligned}$$

(2) The same proof as above works if we apply the relation $\mathscr{G} \circ \mathscr{U} = \mathscr{U} \circ \mathscr{G}$ (Theorem 6.1(3)) instead of $\Upsilon \circ \mathscr{U} = \mathscr{U} \circ \Upsilon$.
(3) We know that $L_0(\mathbb{R}^d) \subset U_0(\mathbb{R}^d)$. Suppose that $L_m(\mathbb{R}^d) \subset U_m(\mathbb{R}^d)$ for some $m \geq 0$, as the induction hypothesis. Then

$$\begin{aligned} L_{m+1}(\mathbb{R}^d) &= \Phi^{m+2}(ID_{\log^{m+2}}(\mathbb{R}^d)) = \Phi(\Phi^{m+1}(ID_{\log^{m+2}}(\mathbb{R}^d))) \\ &\subset \Phi(\Phi^{m+1}(ID_{\log^{m+1}}(\mathbb{R}^d))) = \Phi(L_m(\mathbb{R}^d) \cap ID_{\log}(\mathbb{R}^d)) \\ &\subset \Phi(U_m(\mathbb{R}^d) \cap ID_{\log}(\mathbb{R}^d)) = \Phi(\mathscr{U}^{m+1}(ID(\mathbb{R}^d)) \cap ID_{\log}(\mathbb{R}^d)) \\ &= \Phi(\mathscr{U}^{m+1}(ID_{\log}(\mathbb{R}^d)) = \mathscr{U}^{m+1}(\Phi(ID_{\log}(\mathbb{R}^d)) \\ &\qquad (\text{by } \Phi \circ \mathscr{U} = \mathscr{U} \circ \Phi \text{ (Theorem 6.1 (4))}) \\ &= \mathscr{U}^{m+1}(L_0(\mathbb{R}^d)) \subset \mathscr{U}^{m+1}(U_0(\mathbb{R}^d)) = U_{m+1}(\mathbb{R}^d). \end{aligned}$$

(4) We show $T_m(\mathbb{R}^d) \subset L_m(\mathbb{R}^d)$. We can show that, for any $m \geq 0$,

$$T_m(\mathbb{R}^d) = (\Phi\Upsilon)^{m+1}(ID_{\log^{m+1}}(\mathbb{R}^d)) = (\Upsilon^{m+1}\Phi^{m+1})(ID_{\log^{m+1}}(\mathbb{R}^d))$$
$$= \Upsilon^{m+1}(L_m(\mathbb{R}^d)).$$

Then by Proposition 7.4(2) and Remark 7.5(3),

$$\Upsilon^{m+1}(L_m(\mathbb{R}^d)) \subset L_m(\mathbb{R}^d).$$

The proof is completed.

7.3 *Limits of Nested Subclasses*

The following is a main result on the limits of nested subclasses.

Theorem 7.7 (Maejima and Sato [48], Aoyama et al. [5]) *Let $\overline{S(\mathbb{R}^d)}$ be the closure of $S(\mathbb{R}^d)$, where the closure is taken under weak convergence and convolution. We have*

$$U_\infty(\mathbb{R}^d) = B_\infty(\mathbb{R}^d) = L_\infty(\mathbb{R}^d) = T_\infty(\mathbb{R}^d) = G_\infty(\mathbb{R}^d) = M_\infty(\mathbb{R}^d) = \overline{S(\mathbb{R}^d)}.$$

To prove this theorem, we start with the following two known results.

Theorem 7.8 (The Class of Completely Selfdecomposable Distributions. Urbanik [90] and Sato [70]) $L_\infty(\mathbb{R}^d) = \overline{S(\mathbb{R}^d)}$.

Theorem 7.9 (Jurek [35], See Also Maejima and Sato [48]) $U_\infty(\mathbb{R}^d) = L_\infty(\mathbb{R}^d)$.

We also have the following two propositions.

Proposition 7.10

$$T_\infty(\mathbb{R}^d) \subset U_\infty(\mathbb{R}^d), \quad B_\infty(\mathbb{R}^d) \subset U_\infty(\mathbb{R}^d) \quad \textit{and} \quad G_\infty(\mathbb{R}^d) \subset U_\infty(\mathbb{R}^d).$$

Proof Trivial from Theorem 7.6.

Proposition 7.11 *We have*

$$B_\infty(\mathbb{R}^d) \supset \overline{S(\mathbb{R}^d)}, \quad G_\infty(\mathbb{R}^d) \supset \overline{S(\mathbb{R}^d)} \quad \textit{and} \quad T_\infty(\mathbb{R}^d) \supset \overline{S(\mathbb{R}^d)}.$$

Proof It follows from Remark 7.5(1) that $B_\infty(\mathbb{R}^d)$ and $G_\infty(\mathbb{R}^d)$ are c.c.s.s., and from Remark 7.5 (4) that $T_\infty(\mathbb{R}^d)$ is also c.c.s.s. Thus, we have

$$B_\infty(\mathbb{R}^d) = \overline{B_\infty(\mathbb{R}^d)}, \quad G_\infty(\mathbb{R}^d) = \overline{G_\infty(\mathbb{R}^d)} \quad \text{and} \quad T_\infty(\mathbb{R}^d) = \overline{T_\infty(\mathbb{R}^d)}.$$

We know that each class includes $S(\mathbb{R}^d)$. Thus,

$$B_\infty(\mathbb{R}^d) \supset \overline{S(\mathbb{R}^d)}, \quad G_\infty(\mathbb{R}^d) \supset \overline{S(\mathbb{R}^d)} \quad \text{and} \quad T_\infty(\mathbb{R}^d) \supset \overline{S(\mathbb{R}^d)}.$$

The proof is completed.

Proof of Theorem 7.7 The statement follows from Theorems 7.8 and 7.9 and Propositions 7.10 and 7.11.

7.4 Limits of the Iterations of Stochastic Integral Mappings

A natural question is whether $L_\infty(\mathbb{R}^d)$ is the only class which can appear as the limit of iterations of stochastic integral mappings. In this section, we give an answer to this question. We start with the following.

Theorem 7.12 (A Characterization of $L_\infty(\mathbb{R}^d)$ (Sato [70])) *$\mu \in L_\infty(\mathbb{R}^d)$ if and only if $\mu \in ID(\mathbb{R}^d)$ and*

$$\nu_\mu(B) = \int_{(0,2)} \Gamma^\mu(d\alpha) \int_S \lambda_\alpha(d\xi) \int_0^\infty 1_B(r\xi) r^{-\alpha-1} dr, \quad B \in \mathscr{B}(\mathbb{R}^d),$$

where Γ^μ is a measure on $(0, 2)$ satisfying

$$\int_{(0,2)} \left(\frac{1}{\alpha} + \frac{1}{2-\alpha} \right) \Gamma^\mu(d\alpha) < \infty$$

and λ_α is a probability measure on S for each α and it is measurable in α. Here Γ^μ is unique and so it can be considered a characteristic of μ.

Definition 7.13 For $A \in \mathscr{B}((0, 2))$, define $L_\infty^A(\mathbb{R}^d) := \{\mu \in L_\infty(\mathbb{R}^d) \colon \Gamma^\mu((0, 2) \setminus A) = 0\}$.

Theorem 7.14 (Sato [78], Maejima and Ueda [54]) *We have*

$$\bigcap_{m=1}^\infty \mathfrak{R}(\Phi_\alpha^m) = \bigcap_{m=1}^\infty \mathfrak{R}(\Psi_{\alpha,1}^m)$$
$$= \begin{cases} L_\infty(\mathbb{R}^d), & \text{for } \alpha \in (-\infty, 0], \\ L_\infty^{(\alpha,2)}(\mathbb{R}^d), & \text{for } \alpha \in (0, 1), \\ \left\{\mu \in L_\infty^{(1,2)}(\mathbb{R}^d) : \text{the weak mean of } \mu \text{ is } 0\right\}, & \text{for } \alpha = 1, \\ \left\{\mu \in L_\infty^{(\alpha,2)}(\mathbb{R}^d) : \int_{\mathbb{R}^d} x\mu(dx) = 0\right\}, & \text{for } \alpha \in (1, 2). \end{cases}$$

7.5 Characterizations of Some Nested Subclasses

Here we treat three cases, $L_m(\mathbb{R}^d)$, $B_m(\mathbb{R}^d)$ and $G_m(\mathbb{R}^d)$.

Sato [70] characterized the classes $L_m(\mathbb{R}^d)$ in terms of ν_ξ as follows. Recall the functions k_ξ in (11). We call the function $h_\xi(u)$ defined by $h_\xi(u) = k_\xi(e^{-u})$ the h-function of μ.

Let f be a real-valued function on $\mathbb{R}$. For $\varepsilon > 0, n = 1, 2, \ldots,$ denote

$$\Delta_\varepsilon^n f(u) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n}{j} f(u + j\varepsilon).$$

Define $\Delta_\varepsilon^0 f = f$. We say that $f(u), u \in \mathbb{R}$, is monotone of order n if $\Delta_\varepsilon^j f \geq 0$ for $\varepsilon > 0, j = 0, 1, 2, \ldots, n$.

Theorem 7.15 (Sato [70, Theorem 3.2]) *Let $m = 1, 2, \ldots$. Then $\mu \in L_m(\mathbb{R}^d)$ if and only if $\mu \in L(\mathbb{R}^d)$ and the h-function of μ is monotone of order $m+1$ for λ-a.e. ξ, where λ is the measure appearing in (2.3).*

Another characterization of $L_m(\mathbb{R}^d)$ in terms of the decomposability is the following.

Theorem 7.16 (See Sato [70, Theorem 2.1] and Rocha-Arteaga and Sato [67, Theorem 49]) *For $m = 1, 2, \ldots, \infty$, $\mu \in L_m(\mathbb{R}^d)$ if and only if $\mu \in L_0(\mathbb{R}^d)$ and for any $b > 1$, there exists some $\rho_b \in L_{m-1}(\mathbb{R}^d)$ such that (12) is satisfied.*

For the characterization of $B_m(\mathbb{R}^d)$, we introduce a sequence of functions $\varepsilon_m(x), m = 0, 1, 2, \ldots$. For $x \geq 0$, let

$$\varepsilon_0(x) = e^{-x},$$

$$\varepsilon_1(x) = -\int_0^\infty e^{-x/u} d\varepsilon_0(u) > 0,$$

$$\ldots$$

$$\varepsilon_m(x) = -\int_0^\infty e^{-x/u} d\varepsilon_{m-1}(u) > 0.$$

We have

Theorem 7.17 (Maejima [43]) *Let $m = 1, 2, \ldots$. Then $\mu \in B_m(\mathbb{R}^d)$ if and only if $\mu \in B_0(\mathbb{R}^d)$ and ν_μ is either 0 or it is expressed as*

$$\nu_\mu(B) = -\int_0^\infty \nu_0(t^{-1}B) d\varepsilon_m(t)$$

for the Lévy measure ν_0 of some $\mu_0 \in ID(\mathbb{R}^d)$.

For the characterization of $G_m(\mathbb{R}^d)$, we restrict ourselves to the symmetric distributions, which is easier. For $m \in \mathbb{N}$, let $\phi_m(x)$ be the probability density function of the product of $(m+1)$ independent standard normal random variables.

Theorem 7.18 (Aoyama and Maejima [3]) *Let $\mu \in ID_{\mathrm{sym}}(\mathbb{R}^d)$. Then for each $m \in \mathbb{N}$, $\mu \in G_m(\mathbb{R}^d)$ if and only if $\mu \in G_0(\mathbb{R}^d)$ and ν_μ is either* 0 *or it is expressed as*

$$\nu_\mu(B) = \int_{-\infty}^{\infty} \nu_0(u^{-1}B)\phi_{m-1}(u)du,$$

where ν_0 is the Lévy measure of some $\mu_0 \in G_0(\mathbb{R}^d)$.

Another characterization is the following.

Theorem 7.19 (Aoyama and Maejima [3]) *Let $m \in \mathbb{N}$. A $\mu \in ID_{\mathrm{sym}}(\mathbb{R}^d)$ belongs to $G_m(\mathbb{R}^d)$ if and only if $\mu \in G_0(\mathbb{R}^d)$ and ν_μ is either* 0 *or it is expressed as*

$$\nu_\mu(B) = \int_S \lambda(d\xi) \int_0^{\infty} 1_B(r\xi) g_{m,\xi}(r^2)dr, \quad B \in \mathscr{B}_0(\mathbb{R}^d),$$

where λ is a symmetric measure on the unit sphere S on $\mathbb{R}^d$ and $g_{m,\xi}(r)$ is represented as

$$g_{m,\xi}(s) = \int_{-\infty}^{\infty} \phi_{m-1}(\sqrt{s}|r|^{-1})|r|^{-1} g_\xi(r^2)dr,$$

where $g_\xi(r)$ on $(0,\infty)$ is a jointly measurable function such that $g_\xi = g_{-\xi}, \lambda - a.e.$ for any fixed $\xi \in S$, $g_\xi(\cdot)$ is completely monotone on $(0,\infty)$ and satisfies

$$\int_0^{\infty} (1 \wedge r^2) g_\xi(r^2)dr = c \in (0,\infty)$$

with c independent of ξ.

7.6 Some Nested Subclasses Appearing in Finance

In Carr et al. [19], they discussed the problem of pricing options with Lévy processes and Sato processes (which are the selfsimilar additive processes) for asset returns. Then they showed the importance of the distributions in $L_1(\mathbb{R}_+)$ or $L_2(\mathbb{R}_+)$, and also $L_\infty(\mathbb{R}_+)$. Actually, some tempered stable distributions belong to $L_1(\mathbb{R}^d)$ and $L_2(\mathbb{R}^d)$, which will be seen in Sect. 5.4 later, and Rosiński [69] mentioned that tempered stable processes were introduced in mathematical finance to model stochastic volatility (see e.g. CGMY model in Carr et al. [18] discussed in Sect. 7.7 later), and that option pricing based on such processes were considered.

7.7 Nested Subclasses of $L(\mathbb{R}^d)$ with Continuous Parameter

We have discussed nested subclasses $L_m(\mathbb{R}^d), m = 1, 2, \ldots,$ of $L(\mathbb{R}^d)$. Nguyen Van Thu [91–93] extended $L_m(\mathbb{R}^d)$ to $L_p(\mathbb{R}^d)$ by replacing the integers m by positive real numbers $p > 0$. It turns out that his classes $L_p(\mathbb{R}^d)$ are special cases of $L_{p.\alpha}(\mathbb{R}^d)$, recently studied by Sato [77]. For $p > 0$ and $\alpha \in \mathbb{R}$, let

$$j_{p,\alpha}(s) = \frac{1}{\Gamma(p)} \int_s^1 (-\log u)^{p-1} u^{-\alpha-1} du, \quad 0 < s \le 1,$$

and denote its inverse function by $j^*_{p,\alpha}(t)$. Define

$$\Lambda_{p,\alpha} := \Phi_{j^*_{p,\alpha}} \quad \text{and} \quad L_{p,\alpha}(\mathbb{R}^d) := \mathfrak{R}(\Lambda_{p,\alpha}).$$

Then

$$L_m(\mathbb{R}^d) = L_{m+1,1}(\mathbb{R}^d), \quad m = 1, 2, \ldots,$$

and the classes $L_p(\mathbb{R}^d)$ by Nguyen Van Thu [91–93] are

$$L_p(\mathbb{R}^d) = L_{p,1}(\mathbb{R}^d), \quad p > 0.$$

For the details of $L_{p,\alpha}(\mathbb{R}^d)$, see Sato [77].

Also note that $\varepsilon^*_{\alpha,m}(t)$ in Maejima et al. [57] is the same as $j^*_{p,\alpha}(t)$ above with $p = m + 1$. Hence, $L_{m,\alpha}(\mathbb{R}^d), m = 1, 2, \ldots, \alpha < 2$, in Maejima et al. [57] is the special case of $L_{p,\alpha}(\mathbb{R}^d)$ in Sato [77].

8 Examples (I)

All examples in this section are one-dimensional distributions except in Sect. 8.5, and we show which classes such known distributions belong to.

8.1 Gamma, χ^2-, Student t- and F-Distributions

(a) Let $\Gamma_{c,\lambda}$ be a gamma random variable with parameters $c > 0$ and $\lambda > 0$. Namely, $P(\Gamma_{c.\lambda} \in B) = \lambda^c \Gamma(c)^{-1} \int_{B\cap(0,\infty)} x^{c-1} e^{-\lambda x} dx$. (When $c = 1$, it is exponential.) In its Lévy-Khintchine representation, the Gaussian part is 0 and the Lévy measure is $\nu(dr) = ce^{-\lambda r}r^{-1}1_{(0,\infty)}(r)dr$, (see e.g. Steutel and van Harn [83, Chap. III, Example 4.8]). Then $\mathscr{L}(\Gamma_{c,\lambda}) \in T(\mathbb{R}_+)$, (from the form of the Lévy measure of $\mathscr{L}(\Gamma_{c,\lambda})$), but $\mathscr{L}(\Gamma_{c,\lambda}) \notin L_1(\mathbb{R}_+)$, (Maejima et al. [56, Example 1(i)]).

(b) Let $n \in \mathbb{N}$ and let $Z_1, \dots, Z_n$ be independent standard normal random variables. The distribution of

$$\chi^2(n) := Z_1^2 + \cdots + Z_n^2$$

is called the χ^2-distribution with n degrees of freedom. It is known that

$$\mathscr{L}(\chi^2(n)) = \mathscr{L}(\Gamma_{n/2,1/2}),$$

and hence $\mathscr{L}(\chi^2(n)) \in T(\mathbb{R}_+)$.

(c) Let Z be the standard normal random variable and $\chi^2(n)$ a χ^2-random variable with n degrees of freedom and suppose that they are independent. Then the distribution of

$$t(n) := \frac{Z}{\sqrt{\chi^2(n)/n}} \tag{28}$$

is called Student t-distribution of n degrees of freedom. Its density is

$$\mu(dx) = \frac{1}{B(n/2, 1/2)\sqrt{n}} \left(1 + \frac{x^2}{n}\right)^{-(n+1)/2} dx,$$

where $B(\cdot, \cdot)$ is the Beta function. It is known that $\mathscr{L}(t(n)) \in L(\mathbb{R})$, (see Steutel and van Harn [83, Chap. VI, Theorem 11.15]).

(d) Let $\chi_1^2(n)$ and $\chi_2^2(m)$ be two independent χ^2-random variables with n and m degrees of freedom, respectively. Then the distribution of

$$F(n,m) := \frac{\chi_1^2(n)/n}{\chi_2^2(m)/m} \tag{29}$$

is called F-distribution, and its density is

$$\mu(dx) = \frac{1}{B(n/2, m/2)x} \left(\frac{nx}{nx+m}\right)^{n/2} \left(1 - \frac{nx}{nx+m}\right)^{m/2} dx, \quad x > 0.$$

It is known that $\mathscr{L}(F) \in I(\mathbb{R}_+)$, (Ismail and Kelker [27]), and more that $\mathscr{L}(F) \in T(\mathbb{R}_+)$, (Bondesson [16, Example 4.3.1]).

8.2 Logarithm of Gamma Random Variable

It is known that ν_ξ corresponding to $\log \Gamma_{c,\lambda}$ is $\nu_1 = 0$ and

$$\nu_{-1}(dr) = \frac{e^{-cr}}{r(1-e^{-r})}dr, \quad r > 0, \tag{30}$$

(see e.g. Linnik and Ostrovskii [41, Eq. (2.6.13)]). This does not depend on the parameter $\lambda > 0$.

(a) $\mathscr{L}(\log \Gamma_{c,\lambda}) \in L(\mathbb{R})$, (Shanbhag and Sreehari [81]). Shanbhag and Sreehari proved the selfdecomposability by showing (12) without using (30). However, once we know (30), we can show it by (11) and (30).
(b) $\mathscr{L}(\log \Gamma_{c,\lambda}) \in L_1(\mathbb{R})$ if $c \geq 1/2$, (Akita and Maejima [1]). It is enough to apply Theorem 7.15.
(c) $\mathscr{L}(\log \Gamma_{c,\lambda}) \in L_2(\mathbb{R})$ if $c \geq 1$, (Akita and Maejima [1]). It is enough to apply Theorem 7.15 again.
(d) $\mathscr{L}(\log \Gamma_{c,1}) \in T(\mathbb{R})$. (See Bondesson [16, p. 112].)

8.3 Symmetrized Gamma Distribution

The symmetrized gamma distribution with parameter $c > 0$ and $\lambda > 0$, is written as sym-gamma (c, λ). Its characteristic function is $\varphi(z) = \left(\lambda^2/(\lambda^2 + z^2)\right)^c$ and in its Lévy-Khintchine representation, the Gaussian part is 0 and the Lévy measure is $\nu(dr) = c|r|^{-1}e^{-\lambda|r|}dr,\ (r \neq 0)$. (See Steutel and van Harn [83, Chap. V, Example 6.17].) (When $c = 1$ it is the Laplace distribution.)

We have

(a) sym-gamma $(c, \lambda) \in T(\mathbb{R})$, (from the form of the Lévy measure above).

Thus

(b) sym-gamma $(c, \lambda) \in G(\mathbb{R})$, (see Rosiński [68]).

8.4 More Examples Related to Gamma Random Variables

(a) Product of independent gamma random variables. (Steutel and van Harn [83, Chap. VI, Theorem 5.20].) Let $\Gamma_1, \Gamma_2, \ldots \Gamma_n$ be independent gamma random variables, and let $q_1, q_2, \ldots, q_n \in \mathbb{R}$ with $|q_j| \geq 1$. Then

$$\mathscr{L}(\Gamma_1^{q_1}\Gamma_2^{q_2}\cdots\Gamma_n^{q_n}) \in L(\mathbb{R}_+).$$

(b) When $n = 1$ above, we can say more. Namely,

$$\mathscr{L}(\Gamma_1^{q_1}) \in T(\mathbb{R}_+).$$

(Thorin [87].)

(c) Power of gamma random variables. (Bosch and Simon [17].) Let Γ be a gamma random variable and $p \in (-1, 0)$. Then $\mathscr{L}(\Gamma^p) \in L(\mathbb{R}_+)$. The proof is as follows: Let

$$g(u) = \frac{u\Gamma(1 - p(u+1))}{\Gamma(1 - pu)},$$

and let $X = \{X_t\}$ be the Lévy process such that

$$E\left[e^{-uX_t}\right] = e^{-ug(u)}, \quad u, t \geq 0.$$

Then by an application of Proposition 2 of Bertoin and Yor [13] (see Bosch and Simon [17] for the details), we have

$$\Gamma^p \stackrel{\mathrm{d}}{=} \int_0^\infty e^{-X_t} dt (=: I).$$

Let $T_y = \inf\{t > 0 : X_t = y\}$ for every $y > 0$. The fact that $X_t \to \infty$ a.s. as $t \to \infty$ and the absence of positive jumps assure that $T_y < \infty$ a.s. We thus have

$$I = \int_0^{T_y} e^{-X_t} dt + \int_{T_y}^\infty e^{-X_t} dt \stackrel{\mathrm{d}}{=} \int_0^{T_y} e^{-X_t} dt + e^{-y} \int_0^\infty e^{-X'_t} dt,$$

where X' is an independent copy of X and the second equality follows from the Markov property at T_y. This shows that I satisfies (12), and hence $\Gamma^p (\stackrel{\mathrm{d}}{=} I)$ is selfdecomposable by (12). We remark here that $\mathscr{L}(\Gamma^p), p \in (0, 1)$ is not infinitely divisible. (See Bosch and Simon [17, p. 627].)

(d) Exponential function of gamma random variable. (Bondesson [16, p. 94].) Let X is denumerable convolution of gamma random variables Γ_{c_j,λ_j} with $c_j \geq 1$. Then

$$\mathscr{L}(e^X) \in T(\mathbb{R}_+).$$

(e) Let Γ be a gamma random variable and let $a, b \in \mathbb{R}$. Then

$$\mathscr{L}(a\Gamma + b\Gamma^2) \in T(\mathbb{R}).$$

(Privault and Yang [65].)

8.5 Tempered Stable Distribution

The tempered stable distributions were defined by Rosiński [69]. Let $0 < \alpha < 2$. T_α is called a tempered α-stable random variable on $\mathbb{R}^d$, if $\mathscr{L}(T_\alpha) = \mu_{(A,\nu,\gamma)}$ is such that $A = 0$ and ν_μ has polar decomposition

$$\nu_\mu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) r^{-\alpha-1} q_\xi(r) dr, \tag{31}$$

where $q_\xi(r)$ is completely monotone in r, measurable in ξ, and $\lambda(S) < \infty$, $q_\xi(\infty) = 0$. Because of the assumption that $q_\xi(\infty) = 0$, $q_\xi(r)$ cannot be constant, and thus an α-stable distribution is not tempered α-stable but tempered $\beta(< \alpha)$-stable.

We have the following. It is easy to see by checking (13) for (a) and Theorem 7.15 for (b)–(d). (See Barndorff-Nielsen et al. [12].)

(a) If $0 < \alpha < 2$, then $\mathscr{L}(T_\alpha) \in T(\mathbb{R}^d)$.
(b) If $1/4 \le \alpha < 2$, then $\mathscr{L}(T_\alpha) \in L_1(\mathbb{R}^d)$.
(c) If $0 < \alpha < 1/4$, and $q_\xi(r) = c(\xi)e^{-b(\xi)r}$ for all ξ in a set of positive λ-measure, where $c(\xi)$ and $b(\xi)$ are positive measurable functions of ξ, then $\mathscr{L}(T_\alpha) \notin L_1(\mathbb{R}^d)$.
(d) If $2/3 \le \alpha < 2$, then $\mathscr{L}(T_\alpha) \in L_2(\mathbb{R}^d)$.

8.6 Limits of Generalized Ornstein-Uhlenbeck Processes (Exponential Integrals of Lévy Processes)

(a) Let $\{(X_t, Y_t), t \ge 0\}$ be a two-dimensional Lévy process. Suppose that $\{X_t\}$ does not have positive jumps, $0 < E[X_1] < \infty$ and $\mathscr{L}(Y_1) \in ID_{\log}(\mathbb{R})$. Then

$$\mathscr{L}\left(\int_0^\infty e^{-X_{t-}} dY_t\right) \in L(\mathbb{R}).$$

(Bertoin et al. [15].)

(b) Let $\{N_t\}$ be a Poisson process, and let $\{Y_t\}$ be a strictly stable Lévy process or a Brownian motion with drift. Then

$$\mathscr{L}\left(\int_0^\infty e^{-N_{t-}} dY_t\right) \in L(\mathbb{R}).$$

(Kondo et al. [39].)

(c) Let $\{N_t\}_{t\ge 0}$ be a Poisson process such that $E[N_1] < 1$. Then

$$\mathscr{L}\left(\int_0^\infty e^{-(t-N_t)} dt\right) \in L(\mathbb{R}) \cap L_1(\mathbb{R})^c. \tag{32}$$

(Lindner and Maejima [40].) The proof of (32) is as follows: Let $X_t := t - N_t$ and $V := \int_0^\infty e^{-X_t} dt$. For $c > 0$, let $\tau_c := \inf\{t \geq 0 : X_t = c\}$. Since $X_t \to \infty$ a.s. as $t \to \infty$ and $\{X_t\}$ does not have positive jumps, $\tau_c < \infty$ almost surely. Then

$$V = \int_0^{\tau_c} e^{-X_t} dt + \int_{\tau_c}^{\infty} e^{-X_t} dt =: Y_c + V_c,$$

where V_c and Y_c are independent. We have

$$V_c = \int_{\tau_c}^{\infty} e^{-(X_t - X_{\tau_c})} e^{-X_{\tau_c}} dt = e^{-c} \int_{\tau_c}^{\infty} e^{-(X_t - X_{\tau_c})} dt.$$

Denote

$$V_c' := \int_{\tau_c}^{\infty} e^{-(X_t - X_{\tau_c})} dt.$$

By the strong Markov property, $\{X_t - X_{\tau_c}\}_{t > \tau_c}$ is independent of Y_c and has the same distribution as $\{X_t\}_{t>0}$. Thus we conclude that for all $c > 0$

$$V = Y_c + e^{-c} X_c',$$

where $\mathscr{L}(X_c') = \mathscr{L}(V)$. Thus $\mathscr{L}(V) \in L(\mathbb{R})$ by (12). But, in order that it is in $L_1(\mathbb{R})$, it is needed that $\mathscr{L}(Y_c) \in L(\mathbb{R})$ by Theorem 7.16. This, however, is not the case. For instance, we have

$$P\left(Y_1 = \int_0^1 e^{-t} dt\right) \geq P(N_t \text{ does not jump until time } 1) = e^{-E[N_1]} > 0.$$

This means that Y_1 has a point mass at $\int_0^1 e^{-t} dt = 1 - e^{-1}$, but is not a constant, namely, $\mathscr{L}(Y_1)$ is a non-trivial distribution with a point mass. Recall that any non-trivial selfdecomposable distribution on $\mathbb{R}$ must be absolutely continuous (see e.g. Sato [73, Theorem 27.13]), and thus $\mathscr{L}(Y_1) \notin L(\mathbb{R})$. We then conclude that $\mathscr{L}(V) \notin L_1(\mathbb{R})$.

8.7 Type S Random Variable

For $0 < \alpha < 2$, define $X := V^{1/\alpha} Z_\alpha$, where V is a positive infinitely divisible random variable and Z_α is a symmetric α-stable random variable on $\mathbb{R}$, and where V and Z_α are independent. We call X the type S random variable.

Here we explain subordination of Lévy processes.

Theorem 8.1 (Sato [73, Theorem 30.1]) *Let $\{V_t, t \geq 0\}$ be a subordinator (a nondecreasing Lévy process on $\mathbb{R}$) and let $\{Z_t, t \geq 0\}$ be a Lévy process on $\mathbb{R}^d$, independent of $\{V_t\}$. Then $X_t := Z_{V_t}$ is a Lévy process on $\mathbb{R}^d$, and $\mathscr{L}(X_t) \in ID(\mathbb{R}^d)$.*

The transformation of $\{Z_t\}$ to $\{X_t\}$ is called subordination by the subordinator $\{V_t\}$.

Theorem 8.2 *Let $\{V_t, t \geq 0\}$ be a subordinator and let $\{Z_\alpha(t)\}$ be a symmetric $\alpha(\in (0,2])$-stable Lévy process on $\mathbb{R}$, independent of $\{V_t\}$. Then if we write $V = V_1$,*

$$Z_\alpha(V) \stackrel{\mathrm{d}}{=} V^{1/\alpha} Z_\alpha. \tag{33}$$

Thus, $\mathscr{L}(V^{1/\alpha} Z_\alpha) \in ID(\mathbb{R})$, implying that type S random variables are infinitely divisible.

Proof We compare the characteristic functions on both sides of (33). Note that $E[e^{izZ_\alpha}] = \exp\{-c|z|^\alpha\}$ with some $c > 0$, and for the Lévy process $\{X_t\}$, $E[e^{izX_t}] = \left(E[e^{izX_1}]\right)^t$. We then have

$$E\left[\exp\{izZ_\alpha(V)\}\right] = E_V\left[\exp\{-cV|z|^\alpha\}\right]$$

and

$$E\left[\exp\left\{izV^{1/\alpha}Z_\alpha\right\}\right] = E_V\left[\exp\left\{-c|V^{1/\alpha}z|^\alpha\right\}\right],$$

implying that both sides of (33) are equal in law.

Notice that a symmetric stable random variable is of type G. For, we can check, by the characteristic functions,

$$Z_\alpha \stackrel{\mathrm{d}}{=} (Z^+_{\alpha/2})^{1/2} Z_2, \tag{34}$$

where $Z^+_{\alpha/2}$ is a positive $\alpha/2$-stable random variable.

Theorem 8.3 *Type S random variables are of type G.*

Proof By (34), we have

$$V^{1/\alpha} Z_\alpha \stackrel{\mathrm{d}}{=} V^{1/\alpha}(Z^+_{\alpha/2})^{1/2} Z_2 = (V^{2/\alpha} Z^+_{\alpha/2})^{1/2} Z_2.$$

It remains to show that $\mathscr{L}(V^{2/\alpha} Z^+_{\alpha/2}) \in I(\mathbb{R}_+)$, but this can be shown in the same way as in the proof of (33), completing the proof.

(a) If $V \stackrel{\mathrm{d}}{=} \Gamma_{1,\lambda}$, then $\mathscr{L}(Z_\alpha(V)) \in \mathscr{G}(T(\mathbb{R})) \subset T(\mathbb{R})$. (See Bondesson [16, p. 38].)
(b) Let $\lambda > 0$ and $\{B_t\}$ a standard Brownian motion, and let $\{Z_t\}$ be a symmetric stable Lévy process. Then $\int_0^\infty e^{-B_t - \lambda t} dZ_t$ is of type S. (See Maejima and Niiyama [45], Aoyama et al. [4] and Kondo et al. [39].)

8.8 Convolution of Symmetric Stable Distributions of Different Indexes

The characteristic function of the convolution of symmetric stable distributions of different indexes is $\varphi(z) = \exp\left\{\int_{(0,2)} -|z|^{\alpha} m(d\alpha)\right\}$, where m is a measure on the interval $(0, 2)$. It belongs to $L_{\infty}(\mathbb{R})$. (See e.g. Sato [67].)

8.9 Product of Independent Standard Normal Random Variables

Let $Z_1, Z_2, \ldots$ be independent standard normal random variables.

(a) $\mathscr{L}(Z_1 Z_2) \in T(\mathbb{R})$. This is because $\mathscr{L}(Z_1 Z_2) = \mathscr{L}(\text{sym-gamma}(1/2, 1))$, (see Steutel and van Harn [83, p. 504]),

(b) (Maejima and Rosiński [46, Example 5.1].) $\mathscr{L}(Z_1 \cdots Z_n) \in G(\mathbb{R}),\ n \geq 2$. The proof is as follows: Recall that if $V > 0, \mathscr{L}(V) \in I(\mathbb{R}), Z$ is the standard normal random variable and V and Z are independent, then $\mu = \mathscr{L}(V^{1/2}Z) \in G(\mathbb{R})$. Here we need a lemma.

Lemma 8.4 (Shanbhag and Sreehari [81, Corollary 4]) *Let Z be the standard normal random variable and Y a positive random variable independent of Z. Then $|Z|^p Y$ is infinitely divisible for any $p \geq 2$.*

We have $Z_1 \cdots Z_n \stackrel{\mathrm{d}}{=} Z_1 |Z_2 \cdots Z_n|$ and $|Z_2 \cdots Z_n|^2$ is infinitely divisible by Lemma 8.4, which implies that $\mathscr{L}(Z_1 \cdots Z_n) \in G(\mathbb{R}),\ n \geq 2$.

(c) When $n = 2$, we can say more, namely, $\mathscr{L}(Z_1 Z_2) \in G_1(\mathbb{R})$. (For the proof, see Maejima and Rosiński [46, Example 5.2].)

9 Examples (II)

In this section, we list examples of distributions in the classes $L(\mathbb{R}), B(\mathbb{R}), T(\mathbb{R})$ and $G(\mathbb{R})$, in addition to what we have explained in the previous section.

9.1 Examples in $L(\mathbb{R})$

There are many examples in $L(\mathbb{R})$. The following are some of them.

(a) Let Z be the standard normal random variable, $t(n)$ Student's t-random variable and let $F(n, m)$ be F-random variable. Then (i) $\mathscr{L}(\log|Z|) \in L(\mathbb{R})$, (ii)

$\mathscr{L}(\log|t|) \in L(\mathbb{R})$ and (iii) $\mathscr{L}(\log F) \in L(\mathbb{R})$. (Shanbhag and Sreehari [81].) These follow from the following facts:

(i) Since $|Z|^2 \stackrel{\mathrm{d}}{=} \chi^2(1)$, $\log|Z| \stackrel{\mathrm{d}}{=} \frac{1}{2}\log\chi^2(1)$.
(ii) By (28),

$$\log|t(n)| \stackrel{\mathrm{d}}{=} \log|Z| - \frac{1}{2}\log\Gamma_{n/2,1/2} + \frac{1}{2}\log n,$$

where Z and $\Gamma_{n/2,1/2}$ are independent.
(iii) By (29),

$$\log F(n,m) \stackrel{\mathrm{d}}{=} \log\Gamma_{n/2,1/2} - \log\Gamma_{m/2,1/2} - \log n + \log m,$$

where $\Gamma_{n/2,1/2}$ and $\Gamma_{m/2,1/2}$ are independent.

(b) Let E have a standard exponential random variable. Consider $X \stackrel{\mathrm{d}}{=} -\log E$. Then the distribution function G_1 of X is $G_1(x) = e^{-e^{-x}}$, $x \in \mathbb{R}$, called Gumbel distribution. (See Steutel and van Harn [83, Chap. IV, Example 11.1].) By Sect. 5.2(a), $\mathscr{L}(X) \in L(\mathbb{R})$. Also $G_2(x) = 1 - e^{-e^{x}}$, $x \in \mathbb{R}$, is selfdecomposable, because $G_2(x) = 1 - G_1(-x)$ and so $G_2 = \mathscr{L}(-X)$.
(c) Let Y be a beta random variable. Then $\mathscr{L}\left(\log Y(1-Y)^{-1}\right) \in L(\mathbb{R})$. (Barndorff-Nielsen et al. [11].)

9.2 Examples in $L_1(\mathbb{R}_+)$

The following is Maejima et al. [56, Example 1(ii)]. Let $\mu \in ID(\mathbb{R}_+)$ be such that $k_{+1}(r)$ in (11) is $cx^{-\alpha}e^{-ar}$, $r > 0$ with $a, c > 0$ and $0 < \alpha < 2$. Then $\mu \in L_1(\mathbb{R}_+)$. It is enough to apply Theorem 7.15.

9.3 Examples in $B(\mathbb{R})$

(a) (Bondesson [16, p. 143].) Let $\{Y_j\}$ be i.i.d. exponential random variables and N a Poisson random variable independent of $\{Y_j\}$. Put $X = \sum_{j=1}^{N} Y_j$. Then $\mathscr{L}(X) \in B(\mathbb{R}_+)$.
(b) (Bondesson [16, pp. 143–144].) Let $Y = Y(\alpha,\beta)$ be a beta random variable with parameters α and β and let $X = -\log Y$. Then

(b1) $\mathscr{L}(X) \in B(\mathbb{R}_+)$.
(b2) $\mathscr{L}(X) \in L(\mathbb{R}_+)$ if and only if $2\alpha + \beta \geq 1$.

9.4 Examples in $G(\mathbb{R})$

More examples of distributions in $G(\mathbb{R})$ are the following by Fukuyama and Takahashi [22]. Let $([0,1], \mathfrak{B}, \lambda)$ be the Lebesgue probability space with Lebesgue measure λ. For any $\mu \in G(\mathbb{R}) \cap ID_{\mathrm{sym}}(\mathbb{R})$, there exist $\{a_j\}$, $A_n(\to \infty)$ and $\{\beta_j\} \subset \mathbb{R}$ such that

$$\frac{1}{A_n} \sum_{j=1}^{n} a_j \cos\left(2\pi j(\omega + \beta_j)\right), \quad \omega \in [0,1],$$

converges weakly to μ on the Lebesgue probability space.

9.5 Examples in $T(\mathbb{R})$

There are many examples in $T(\mathbb{R})$. (See e.g. Bondesson [16].) The following are some of them.

(a) (Log-normal distribution.) Let Z be the standard normal random variable and put $X = e^Z$. The distribution of X is called the log-normal distribution, and its density is

$$\mu(dx) = \frac{1}{\sqrt{2\pi}} \frac{1}{x} \exp\left\{-\frac{1}{2}(\log x)^2\right\} 1_{(0,\infty)}(x)dx.$$

The log-normal distribution belongs to $T(\mathbb{R}_+)$. (See Steutel and van Harn [83, Chap. VI, Theorems 5.18 and 5.21].)

(b) (Pareto distribution.) Let $\Gamma_{1,1}$ and $\Gamma_{c,1}, c > 0$ be two independent gamma random variables and put $X = \Gamma_{1,1}/\Gamma_{c,1}$. Then its density is

$$\mu(dx) = \frac{1}{B(1,c)} \left(\frac{1}{1+x}\right)^{1+c} 1_{(0,\infty)}(x)dx,$$

and the corresponding distribution is called the Pareto distribution and belongs to $T(\mathbb{R})$. (See Steutel and van Harn [83, Chap. VI, Example 12.9 and Theorems 5.18 and 5.19(ii)].)

(c) Generalized inverse Gaussian distributions belong to $T(\mathbb{R})$. (See e.g. Bondesson [16, Example 4.3.2].)

(d) Let X_α be a positive α-stable random variable with $0 < \alpha < 1$. Then $\mathscr{L}(\log X_\alpha) \in T(\mathbb{R})$. (See Bondesson [16, Example 7.2.5].)

(e) (Lévy 's stochastic area X of the two-dimensional Brownian motion. See e.g. Sato [73, Example 15.15].) The density of X is $f(x) = (\pi \cosh x)^{-1}$ and $k_{\pm 1}(r)$ in (11) is $|2 \sinh r|^{-1}$. Since $|2 \sinh r|^{-1}$ is completely monotone in $r \in (0, \infty)$, we have $\mathscr{L}(X) \in T(\mathbb{R}_+)$. This distribution μ_1 with a bit different scaling (the density is $f_1(x) = (2\pi \cosh \frac{1}{2}x)^{-1}$) is called the hyperbolic cosine distribution, (see e.g. Steutel and van Harn [83, p. 505], for this and below). It is also known that μ_1 is $\mathscr{L}(\log(Y/Z))$ with independent Y and Z both of which are $\Gamma_{1/2,1}$. The distribution μ_2 with density $f_2(x) = (2\pi^2 \sinh \frac{1}{2}x)^{-1}x$ is called hyperbolic sine distribution. It is known that μ_2 is $\mathscr{L}(Y+Z)$ with independent Y and Z both of which are distributed as hyperbolic cosine distribution. $k_\xi(r)$ is $|\sinh r|^{-1}$ up to scaling, and thus also $\mu_2 \in T(\mathbb{R}_+)$.

9.6 Examples in $T(\mathbb{R}) \cap L_1(\mathbb{R})^c$ (Revisited)

(a) $\mathscr{L}(\Gamma_{c,\lambda})$. (Section 8.1.)
(b) $\mathscr{L}(T_\alpha)$ if $0 < \alpha < 1/4$ and $q_\xi(r) = c(\xi)e^{-b(\xi)r}$ for all ξ in a set of positive λ-measure, where $c(\xi)$ and $b(\xi)$ are positive measurable functions of ξ. (Section 8.5, (a) and (c).)

10 Examples (III)

The class of GGCs, which is the Thorin class, is generating renewed interest, since many examples have recently appeared in quite different problems. We explain some of them below.

10.1 The Rosenblatt Process and the Rosenblatt Distribution

Let $0 < D < 1/2$. The Rosenblatt process is defined, for $t \geq 0$, as

$$Z_D(t) = C(D) \int'_{\mathbb{R}^2} \left(\int_0^t (u-s_1)_+^{-(1+D)/2} (u-s_2)_+^{-(1+D)/2} du \right) dB_{s_1} dB_{s_2},$$

where $\{B_s, s \in \mathbb{R}\}$ is a standard Brownian motion, $\int'_{\mathbb{R}^2}$ is the Wiener-Itô multiple integral on $\mathbb{R}^2$ and $C(D)$ is a normalizing constant. The distribution of $Z_D(1)$ is called the Rosenblatt distribution.

The Rosenblatt process is H-selfsimilar with $H = 1 - D$ and has stationary increments. The Rosenblatt process lives in the so-called second Wiener chaos. Consequently, it is not a Gaussian process.

In the last few years, this stochastic process has been the object of several papers. (See Pipiras and Taqqu [64], Tudor [88], Tudor and Viens [89], Veillette and Taqqu [94] among others.)

Let

$$\mathscr{H}_D = \Big\{h : h \text{ is a complex-valued function on } \mathbb{R}, h(x) = \overline{h(-x)},$$
$$\int_{\mathbb{R}} |h(x)|^2 |x|^{D-1} dx < \infty\Big\}$$

and for every $t \geq 0$ define an integral operator A_t by

$$A_t h(x) = C(D) \int_{-\infty}^{\infty} \frac{e^{it(x-y)-1}}{i(x-y)} h(y)|y|^{D-1} dy, \quad h \in \mathscr{H}_D.$$

Since A_t is a self-adjoint Hilbert-Schmidt operator (see Dobrushin and Major [20]), all eigenvalues $\lambda_n(t), n = 1, 2, \ldots,$ are real and satisfy $\sum_{n=1}^{\infty} \lambda_n^2(t) < \infty$.

We start with the following.

Theorem 10.1 (Maejima and Tudor [49]) *For every* $t_1, \ldots, t_d \geq 0$,

$$(Z_D(t_1), \ldots, Z_D(t_d)) \stackrel{\mathrm{d}}{=} \left(\sum_{n=1}^{\infty} \lambda_n(t_1)(\varepsilon_n^2 - 1), \ldots, \sum_{n=1}^{\infty} \lambda_n(t_d)(\varepsilon_n^2 - 1) \right),$$

where $\{\varepsilon_n\}$ *are i.i.d. standard normal random variables.*

The case $d = 1$ was shown by Taqqu (see Proposition 2 of Dobrushin and Major [20]). The proof is enough to extend the idea of Taqqu from one dimension to multi-dimensions.

Theorem 10.2 (Maejima and Tudor [49]) *For every* $t_1, \ldots, t_d \geq 0$, *the law of* $(Z_D(t_1), \ldots, Z_D(t_d))$ *belongs to* $T(\mathbb{R}^d)$.

Proof By Theorem 10.1,

$$\begin{aligned}(Z_D(t_1), \ldots, Z_D(t_d)) &\stackrel{\mathrm{d}}{=} \left(\sum_{n=1}^{\infty} \lambda_n(t_1)(\varepsilon_n^2 - 1), \ldots, \sum_{n=1}^{\infty} \lambda_n(t_d)(\varepsilon_n^2 - 1) \right) \\ &= \sum_{n=1}^{\infty} \varepsilon_n^2 (\lambda_n(t_1), \ldots, \lambda_n(t_d)) - \left(\sum_{n=1}^{\infty} \lambda_n(t_1), \ldots, \sum_{n=1}^{\infty} \lambda_n(t_d) \right),\end{aligned}$$

where $\varepsilon_n^2(\lambda_n(t_1), \ldots, \lambda_n(t_d)), n = 1, 2, \ldots,$ are the elementary gamma random variables in $\mathbb{R}^d$. Since they are independent and since the class $T(\mathbb{R}^d)$ is closed under convolution and weak convergence, we see that the law of $\sum_{n=1}^{\infty} \varepsilon_n^2(\lambda_n(t_1), \ldots, \lambda_n(t_d))$ belongs to $T(\mathbb{R}^d)$, and so does the law of $(Z_D(t_1), \ldots, Z_D(t_d))$. This completes the proof.

In general, let $I_2^B(f)$ be a double Wiener-Itô integral with respect to standard Brownian motion B, where $f \in L^2_{\mathrm{sym}}(\mathbb{R}_+^2)$. Then we have a more general result as follows:

Proposition 10.3

$$I_2^B(f) \stackrel{\mathrm{d}}{=} \sum_{n=1}^{\infty} \lambda_n(f)(\varepsilon_n^2 - 1),$$

where the series converges in $L^2(\Omega)$ *and almost surely. Also*

$$\hat{\mu}_{I_2^B(f)}(z) = \exp\left\{\frac{1}{2}\int_{\mathbb{R}_+} (e^{izx} - 1 - izx)\frac{1}{x}\left(\sum_{n=1}^{\infty} e^{-x/\lambda_n}\right) dx\right\}.$$

Thus $\mathscr{L}\left(I_2^B(f))\right) \in T(\mathbb{R})$.

(For the proof, see e.g. Nourdin and Peccati [60].)

The Rosenblatt distribution is represented by double Wiener-Itô integrals. However, we have seen that it belongs to the Thorin class $T(\mathbb{R})$. The distributions in $T(\mathbb{R})$ have several stochastic integral representations with respect to Lévy processes. Here we take one example. We regard them as members of the class of selfdecomposable distributions, which is a larger class than the Thorin class. This allows us to obtain a new result related to the Rosenblatt distribution.

The following is known. (Aoyama et al. [6, Corollary 2.1].) If $\{\Gamma_{t,\lambda}, t \geq 0\}$ is a gamma process with parameter $\lambda > 0$, $\{N(t), t \geq 0\}$ is a Poisson process with unit rate and they are independent, then for any $c > 0, \lambda > 0$,

$$\Gamma_{c,\lambda} \stackrel{\mathrm{d}}{=} \int_0^{\infty} e^{-t} d\Gamma_{N(ct),\lambda}.$$

Let

$$Y_t = \Gamma_{N(t/2),1/2} - t.$$

Note that $\{Y_t, t \geq 0\}$ is a Lévy process. Then we have

$$\varepsilon_n^2 - 1 \stackrel{\mathrm{d}}{=} \Gamma_{1/2,1/2}^{(n)} - 1 \stackrel{\mathrm{d}}{=} \int_0^{\infty} e^{-t} dY_t^{(n)},$$

where $\Gamma_{1/2,1/2}^{(n)}$ and $\{Y_t^{(n)}\}$ are independent copies of $\Gamma_{1/2,1/2}$ and $\{Y_t\}$, respectively. Thus

$$Z_D \stackrel{\mathrm{d}}{=} \int_0^{\infty} e^{-t} d\left(\sum_{n=1}^{\infty} \lambda_n Y_t^{(n)}\right) =: \int_0^{\infty} e^{-t} dZ_t.$$

Remark 10.4 $\sum_{n=1}^{\infty} \lambda_n Y_t^{(n)}$ is convergent a.s. and in L^2 because

$$\sum_{n=1}^{\infty} E\left[\left(\lambda_n Y_t^{(n)}\right)^2\right] = E\left[Y_t^2\right] \sum_{n=1}^{\infty} \lambda_n^2 < \infty.$$

Remark 10.5 Since $\{Y_t^{(n)}\}, n = 1, 2, \ldots,$ are independent and identically distributed Lévy processes, their infinite weighted sum $\{Z_t\}$ is a Lévy process.

We thus finally have the following theorem.

Theorem 10.6 (Maejima and Tudor [49])

$$Z_D \stackrel{\mathrm{d}}{=} \int_0^{\infty} e^{-t} dZ_t,$$

where $\{Z_t\}$ *is a Lévy process in Remark 10.5.*

10.2 The Duration of Bessel Excursions Straddling Independent Exponential Times

This section is from Bertoin et al. [14].

Let $\{R_t, t \geq 0\}$ be a Bessel process with $R_0 = 0$, with dimension $d = 2(1-\alpha)$, $(0 < \alpha < 1$, equivalently $0 < d < 2)$. When $\alpha = 1/2$, $\{R_t\}$ is a Brownian motion. Let

$$g_t^{(\alpha)} := \sup\{s \leq t : R_s = 0\},$$
$$d_t^{(\alpha)} := \inf\{s \geq t : R_s = 0\}$$

and

$$\Delta_t^{(\alpha)} := d_t^{(\alpha)} - g_t^{(\alpha)},$$

which is the length of the excursion above 0, straddling t, for the process $\{R_u, u \geq 0\}$, and let ε be a standard exponential random variable independent of $\{R_u, u \geq 0\}$. Let $\Delta_\alpha := \Delta_\varepsilon^{(\alpha)}$, which is the duration of Bessel excursions straddling independent exponential times.

Theorem 10.7 $\mathscr{L}(\Delta_\alpha) \in T(\mathbb{R}_+)$.

The idea of the proof is the following. They showed that

$$E\left[e^{-s\Delta_\alpha}\right] = \exp\left\{-(1-\alpha)\int_0^{\infty} (1-e^{-sx}) \frac{E[e^{-xG_\alpha}]}{x} dx\right\}, \quad s > 0,$$

with a nonnegative random variable G_α on $[0, 1]$. (The density function of G_α is explicitly given.) Since $k(x) := E[e^{-xG_\alpha}]$ is completely monotone by Bernstein's theorem (Proposition 3.2), the statement of the theorem follows from (13).

10.3 Continuous State Branching Processes with Immigration

We start with some general theory on GGCs. Any GGC $\mu \in T(\mathbb{R}_+)$ has the Laplace transform:

$$\pi(s) := \int_0^\infty e^{-sx}\mu(dx) = \exp\left\{-\gamma s - \int_0^\infty (1 - e^{-sx})\frac{k(x)}{x}dx\right\}, \quad s > 0,$$

where $\gamma \geq 0$, $\int_0^\infty \frac{(1\wedge x)}{x}k(x)dx < \infty$ and $k(x)$ is completely monotone on $(0, \infty)$. By Bernstein's theorem (Proposition 3.2), there exists a positive measure σ such that

$$k(x) = \int_0^\infty e^{-xy}\sigma(dy).$$

We call this σ the Thorin measure, (see James et al. [28, Sect. 1.2.b]). Therefore, $\mu \in T(\mathbb{R}_+)$ can be parameterized by the pair (γ, σ). Recall

$$\pi(s) = \exp\left\{-\gamma s - \int_0^\infty (1 - e^{-sx})\frac{1}{x}\left(\int_0^\infty e^{-xy}\sigma(dy)\right)dx\right\}, \quad s > 0.$$

The integrability condition for the Lévy measure ν of GGC is, in terms of σ, turned out to be

$$\int_0^\infty \log\left(1 + \frac{s}{y}\right)\sigma(dy) < \infty \quad \text{for all } s > 0,$$

(see James et al. [28, Eq. (3)]) which is equivalent to

$$\int_{(0,1/2]} |\log y|\sigma(dy) + \int_{(1/2,\infty)} \frac{1}{y}\sigma(dy) < \infty.$$

The following is from Handa [24]. Consider continuous state branching processes with immigration (CBCI-process, in short) with quadruplet (a, b, ρ, δ) having the generator

$$L_\delta f(x) = axf''(x) - bxf'(x) + x\int_0^\infty [f(x+y) - f(x) - yf'(x)]\rho(dy) + \delta f'(x),$$

where ρ is a measure on $(0, \infty)$ satisfying $\int_0^\infty (y \wedge y^2)\rho(dy) < \infty$.

Theorem 10.8 *Let $\gamma \geq 0$ and suppose that σ is a non-zero Thorin measure.*

(1) *There exist (a, b, M) such that*

$$\gamma + \int \frac{1}{s+u}\sigma(du) = \frac{1}{a}s + b + \int \frac{s}{s+u}M(du), \quad s > 0.$$

(2) *Any GGC with pair (γ, σ) is a unique stationary solution of the CBCI-process with quadruplet $(a, b, \rho, 1)$, where ρ is a measure on $(0, \infty)$ defined by*

$$\rho(dy) = \left(\int_0^\infty u^2 e^{-yu} M(du) \right) dy.$$

10.4 Lévy Density of Inverse Local Time of Some Diffusion Processes

This section is from Takemura and Tomisaki [84].

Example 10.9 (Also, Shilling et al. [80, p. 201]) Let $I = (0, \infty)$ and $-1 < p < 0$. Let $\mathscr{G}^{(p)} = \frac{1}{2}\frac{d^2}{dx^2} + \frac{2p+1}{2x}\frac{d}{dx}$. Assume 0 is reflecting. Let $\mathbb{D}^{(p)}$ be the diffusion process on I with the generator $\mathscr{G}^{(p)}$ and $\ell^{(p)}$ the Lévy density of the inverse local time at 0 for $\mathbb{D}^{(p)}$. Then we have $\ell^{(p)}(x) = C\frac{1}{x}x^{-|p|}$, which is the Lévy density of a GGC.

Example 10.10 Let $I = (0, \infty)$ and $-1 < p < 0$. Let $\mathbb{D}^{(p)}$ be the diffusion process with the generator $\mathscr{G}^{(p)} = 2x\frac{d^2}{dx^2} + (2p + 2)\frac{d}{dx}$ and suppose that the end point 0 is reflecting. If $\ell^{(p)}$ is the Lévy density of the inverse local time at 0 for $\mathbb{D}^{(p)}$, then $\ell^{(p)}(x) = C\frac{1}{x}x^{-|p|}$, which is again the Lévy density of a GGC.

Example 10.11 Let $-1 < p < 1$ and $\beta > 0$. Let

$$\mathscr{G}^{(p,\beta)} = \frac{1}{2}\frac{d^2}{dx^2} + \left\{ \frac{1}{2x} + \sqrt{2\beta}\frac{K_p'(\sqrt{2\beta}x)}{K_p(\sqrt{2\beta}x)} \right\} \frac{d}{dx},$$

where $K_p(x)$ is the modified Bessel function and, let $\mathbb{D}^{(p,\beta)}$ be the diffusion process on I with the generator $\mathscr{G}^{(p,\beta)}$. Suppose that the end point 0 is reflecting. Then $\ell^{(p,\beta)}$, the Lévy density of the inverse local time at 0 for $\mathbb{D}^{(p,\beta)}$, satisfies

$$\ell^{(p,\beta)}(x) = C\frac{1}{x}x^{-|p|}e^{-\beta x},$$

which is the Lévy density of a GGC. (When $p = 0$, Shilling et al. [80, p. 202].)

Example 10.12 Let $-1 < p < 1$ and $\beta > 0$. Let

$$\mathscr{G}^{(p,\beta)} = 2x\frac{d^2}{dx^2} + 2\left\{1 + \sqrt{2\beta x}\frac{K_p'(\sqrt{2\beta x})}{K_p(\sqrt{2\beta x})}\right\}\frac{d}{dx}.$$

If $\mathbb{D}^{(p,\beta)}$ is the diffusion process with the generator $\mathscr{G}^{(p,\beta)}$ and the end point 0 is reflecting, then $\ell^{(p,\beta)}$, the Lévy density of the inverse local time at 0 for $\mathbb{D}^{(p,\beta)}$, is $\ell^{(p,\beta)}(x) = C\frac{1}{x}x^{-|p|}e^{-\beta x}$, which is a GGC.

10.5 GGCs in Finance

Lévy processes play an important role in asset modeling, and among others a typical pure jump Lévy process is a subordination of Brownian motion. One of them is the variance-gamma process $\{Y_t\}$ by Madan and Seneta [42], which is a time-changed Brownian motion $B = \{B_t\}$ on $\mathbb{R}$ subordinated by the gamma process $\Gamma = \{\Gamma(t)\}$; namely

$$Y_t = B_{\Gamma(t)}, \tag{35}$$

where the gamma process $\{\Gamma(t)\}$ is a Lévy process on $\mathbb{R}$ such that $\mathscr{L}(\Gamma(1))$ is the distribution of a gamma random variable $\Gamma_{1,\lambda}$. This is a special case of Example 30.8 of Sato [73], where B is a general Lévy process on $\mathbb{R}^d$, and when B is the standard Brownian motion on $\mathbb{R}$, for $z \in \mathbb{R}$,

$$E\left[e^{izY_t}\right] = \left(\frac{\lambda}{\lambda + z^2}\right)^t.$$

This is sym-gamma $(t, \sqrt{\lambda})$ in Sect. 5.3.

The variance-gamma processes, which are studied in finance, are generalized to the variance-GGC process. The variance-GGC process is $\{Y_t\}$ in (35) with the replacement of the gamma process Γ by the GGC process $\tilde{\Gamma} = \{\tilde{\Gamma}_t\}$, which is a Lévy process on $\mathbb{R}$ such that $\mathscr{L}(\tilde{\Gamma}_1)$ is a GGC. The following is known.

Proposition 10.13 (Privault and Yang [66]) *Let B is a Brownian motion with drift and $Y_t = B_{\tilde{\Gamma}_t}$. Then Y_t is decomposed as $Y_t = U_t - W_t$, where $\{U_t\}$ and $\{W_t\}$ are two independent GGC process, and thus $\mathscr{L}(Y_t) \in T(\mathbb{R})$.*

The next example is the so-called CGMY model (Carr et al. [18]). It is EGGC with the Lévy density $r^{-1}k_\xi(r)$ and

$$k_\xi(r) = \begin{cases} Ce^{-Gr}r^{-1-Y}, & \text{for } \xi = -1, \\ Ce^{-Mr}r^{-1-Y}, & \text{for } \xi = +1. \end{cases}$$

where $C > 0, G, M \geq 0, Y < 2$. The case $Y = 0$ is the special case of the variance gamma model. This model has been used as a new model for asset returns, which, in contrast to standard models like Black-Scholes model, allows for jump components displaying finite or infinite activity and variation.

11 Examples of α-Selfdecomposable Distributions

In this section, we give two examples of α-selfdecomposable distributions. The first one is two-dimensional.

11.1 The First Example

Many examples in $L^{\langle 0\rangle}(\mathbb{R}) = L(\mathbb{R})$ are known as selfdecomposable distributions, but we have less examples of distributions in $L^{\langle\alpha\rangle}(\mathbb{R}^d), \alpha \neq 0$. In this section, we give an example in $L^{\langle -2\rangle}(\mathbb{R}^2)$. This section is from Maejima and Ueda [51].

Let (Z_1, Z_2) be a bivariate Gaussian random variable, where Z_1 and Z_2 are standard Gaussian random variables with correlation coefficient $\sigma \in (-1, 1)$. Define a bivariate gamma random variable by $W = (Z_1^2, Z_2^2)$. Our concerns are whether W is selfdecomposable or not and if not, which class its distribution belongs to.

Theorem 11.1 *Suppose* $\sigma \neq 0$. *Then*

$$\mathscr{L}(W) \begin{cases} \in L^{\langle\alpha\rangle}(\mathbb{R}^2) & \text{for all } \alpha \leq -2, \\ \notin L^{\langle\alpha\rangle}(\mathbb{R}^2) & \text{for all } \alpha > -2. \end{cases}$$

Remark 11.2 This is an example showing that $L^{\langle\alpha\rangle}(\mathbb{R}^2)$ is not right-continuous in α at $\alpha = -2$, namely

$$L^{\langle -2\rangle}(\mathbb{R}^2) \supsetneqq \bigcup_{\beta > -2} L^{\langle\beta\rangle}(\mathbb{R}^2).$$

Proof of Theorem 11.1 Let $\overline{W} := \frac{1}{2}(W_1 + W_2)$, where W_1, W_2 are independent copies of W. Note that $\overline{W}$ is α-selfdecomposable if and only if W is α-selfdecomposable. Vere-Jones [95] gave the form of the moment generating function of $\overline{W}$. Then we can see that the Lévy measure ν of $\overline{W}$ is

$$\nu(B) = \int_S \lambda(d\xi) \int_0^\infty 1_B(r\xi) \frac{1}{r^{(-2)+1}} \ell_\xi(r) dr,$$

where

$$\ell_\xi(r) = \frac{|\sigma|}{r(1-\sigma^2)\sqrt{\cos\theta\sin\theta}} \exp\left\{-\frac{\cos\theta+\sin\theta}{1-\sigma^2} r\right\} I_1\left(\frac{2|\sigma|\sqrt{\cos\theta\sin\theta}}{1-\sigma^2} r\right),$$

$$(\xi = (\cos\theta, \sin\theta), \theta \in (0, \pi/2)),$$

where $I_1(\cdot)$ is the modified Bessel function of the first kind. To show $\mathscr{L}(\overline{W}) \in L^{\langle -2\rangle}(\mathbb{R}^2)$, it is enough to check that $\ell_\xi(r), r > 0$ is nonincreasing, which is proved in Maejima and Ueda [51].

To see that $\mathscr{L}(\overline{W}) \notin L^{\langle\alpha\rangle}(\mathbb{R}^2), \alpha > -2$, it is enough to check that for any $\beta > 0$, $r^\beta \ell_\xi(r), r > 0$ is "not" nonincreasing, which is easily shown.

11.2 The Second Example

This section is from Maejima and Ueda [55].

Remark 11.3

$$\mathscr{L}(\Gamma_{c,\lambda}) \begin{cases} \in L^{\langle 0\rangle}(\mathbb{R}), \\ \notin L^{\langle\alpha\rangle}(\mathbb{R}), \quad \alpha > 0. \end{cases}$$

Thus, $L^{\langle\alpha\rangle}(\mathbb{R})$ is not right-continuous at $\alpha = 0$.

Consider $\mathscr{L}(\log \Gamma_{c,\lambda})$. It is known that $\mathscr{L}(\log \Gamma_{c,\lambda}) \in L(\mathbb{R}) = L^{\langle 0\rangle}(\mathbb{R})$ (Sect. 5.2 (a)). Let

$$h(\alpha; r) := \frac{\alpha}{r} - \frac{e^{-r}}{1-e^r}, \quad r > 0,$$

$$k(r) := \frac{r^2 e^{-r}}{(1-e^{-r})^2}, \quad r > 0.$$

Write the solution of $k(r) = \alpha$ by $r = r_\alpha$. Let

$$A_1 = \{(c, \alpha) \in (0, \infty) \times \mathbb{R} \ : \ 0 < \alpha < 1, \ c \geq h(\alpha; r_\alpha)\}$$

and

$$A_2 = \{(c, \alpha) \in (0, \infty) \times \mathbb{R} \ : \ \alpha = 1, \ c \geq \frac{1}{2}\}.$$

Theorem 11.4

$$\mathscr{L}(\log \Gamma_{c,\lambda}) \begin{cases} \in L^{\langle\alpha\rangle}(\mathbb{R}), & \textit{if } (c,\alpha) \in ((0,\infty)\times(-\infty,0]) \cup A_1 \cup A_2, \\ \notin L^{\langle\alpha\rangle}(\mathbb{R}), & \textit{if } (c,\alpha) \notin ((0,\infty)\times(-\infty,0]) \cup A_1 \cup A_2. \end{cases}$$

Proof As we have seen in (30) in Sect. 5.2, ν_{-1} of $\log \Gamma_{c,\lambda}$ is $\nu_{-1}(dr) = \frac{e^{-cr}}{r(1-e^{-r})}dr, r > 0$. Thus

$$\nu_{-1}(dr) = \frac{1}{r^{\alpha+1}} \cdot \frac{r^{\alpha}e^{-cr}}{1-e^{-r}}dr =: \frac{1}{r^{\alpha+1}}\ell_{c,\alpha}(r)dr$$

and it is enough to check the monotonicity or non-monotonicity of $\ell_{c,\alpha}(r), r > 0$, depending on (c,α). (For the details of the proof, see Maejima and Ueda [55].)

Corollary 11.5 *$L^{\langle\alpha\rangle}(\mathbb{R})$ is not right-continuous at $\alpha \in (0,1]$.*

Remark 11.6

(i) For any $c > 0$, $\mathscr{L}(\log \Gamma_{c,\lambda}) \notin L^{\langle\alpha\rangle}(\mathbb{R}), \ \alpha > 1$.

(ii) Let E be an exponential random variable. Then

$$\mathscr{L}(\log E) \begin{cases} \in L^{\langle 1\rangle}(\mathbb{R}), \\ \notin L^{\langle\alpha\rangle}(\mathbb{R}), \ \alpha > 1. \end{cases}$$

(iii) Let Z be a standard normal random variable. Then

$$\mathscr{L}(\log |Z|) \begin{cases} \in L^{\langle 1\rangle}(\mathbb{R}), \\ \notin L^{\langle\alpha\rangle}(\mathbb{R}), \ \alpha > 1, \end{cases}$$

since $Z^2 \stackrel{\mathrm{d}}{=} \Gamma_{1/2,1/2}$.

12 Fixed Points of Stochastic Integral Mappings: A New Sight of $S(\mathbb{R}^d)$ and Related Topics

Following Jurek and Vervaat [38], Jurek [31] and Jurek [32], we define a fixed point μ under a mapping Φ_f as follows.

Definition 12.1 $\mu \in \mathfrak{D}(\Phi_f)$ is called a fixed point under the mapping Φ_f, if there exist $a > 0$ and $c \in \mathbb{R}^d$ such that

$$\Phi_f(\mu) = \mu^{a*} * \delta_c. \tag{36}$$

Remark 12.2 Given a mapping Φ_f, the natural definition of its fixed point may be μ satisfying $\Phi_f(\mu) = \mu$. However, if we restrict ourselves to the mapping Φ_α

for instance, only the Cauchy distribution satisfies $\Phi_\alpha(\mu) = \mu$. Then what is the meaning of (36)? We know that $\mu \in ID(\mathbb{R}^d)$ determines a Lévy process $\{X_t\}$ such that $\mu = \mathscr{L}(X_1)$, and $\mu^{a*} * \delta_c = \mathscr{L}(X_a + c)$. Therefore, (36) means that some Lévy process is a "fixed point" in some sense.

We consider here only Φ_α. The set of all fixed points under the mapping Φ_α is denoted by $\mathrm{FP}(\Phi_\alpha)$. For $0 < p \le 2$, let $S_p(\mathbb{R}^d)$ be the class of all p-stable distributions on $\mathbb{R}^d$ and thus $S(\mathbb{R}^d) = \bigcup_{0<p\le 2} S_p(\mathbb{R}^d)$. Furthermore, for $1 < p \le 2$, let $S_p^0(\mathbb{R}^d)$ be the class of p-stable distributions on $\mathbb{R}^d$ with mean 0.

Theorem 12.3 *We have*

$$\mathrm{FP}(\Phi_\alpha) = \begin{cases} S(\mathbb{R}^d), & \text{when } \alpha \le 0, \\ \bigcup_{p\in(\alpha,2]} S_p(\mathbb{R}^d), & \text{when } 0 < \alpha < 1, \\ \bigcup_{p\in(\alpha,2]} S_p^0(\mathbb{R}^d), & \text{when } 1 \le \alpha < 2. \end{cases}$$

Remark 12.4 Theorem 12.3 for $\alpha \le 0$ was already proved in Jurek and Vervaat [38], Jurek [31] and Jurek [32] even in a general setting of a real separable Banach space. The case for $0 < \alpha < 2$ is by Ichifuji et al. [25]. One meaning of this theorem is to give new characterizations of the classes $S(\mathbb{R}^d)$, $\bigcup_{p\in(\alpha,2]} S_p(\mathbb{R}^d)$ with $0 < \alpha < 1$ and $\bigcup_{p\in(\alpha,2]} S_p^0(\mathbb{R}^d)$ with $1 \le \alpha < 2$.

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Lévy Processes with Two-Sided Reflection

Lars Nørvang Andersen, Søren Asmussen, Peter W. Glynn, and Mats Pihlsgård

Abstract Let X be a Lévy process and V the reflection at boundaries 0 and $b > 0$. A number of properties of V are studied, with particular emphasis on the behaviour at the upper boundary b. The process V can be represented as solution of a Skorokhod problem $V(t) = V(0)+X(t)+L(t)-U(t)$ where L, U are the local times (regulators) at the lower and upper barrier. Explicit forms of V in terms of X are surveyed as well as more pragmatic approaches to the construction of V, and the stationary distribution π is characterised in terms of a two-barrier first passage problem. A key quantity in applications is the loss rate ℓ^b at b, defined as $\mathbb{E}_\pi U(1)$. Various forms of ℓ^b and various derivations are presented, and the asymptotics as $b \to \infty$ is exhibited in both the light-tailed and the heavy-tailed regime. The drift zero case $\mathbb{E}X(1) = 0$ plays a particular role, with Brownian or stable functional limits being a key tool. Further topics include studies of the first hitting time of b, central limit theorems and large deviations results for U, and a number of explicit calculations for Lévy processes where the jump part is compound Poisson with phase-type jumps.

Keywords Applied probability • Central limit theorem • Finite buffer problem • First passage problem • Functional limit theorem • Heavy tails • Integro-differential equation • Itô's formula • Linear equations • Local time • Loss rate • Martingale • Overflow • Phase-type distribution • Poisson's equation • Queueing theory • Siegmund duality • Skorokhod problem • Storage process

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1 Introduction

This article is concerned with a one-dimensional Lévy process $X = \{X(t)\}_{t\geq 0}$ reflected at two barriers $0, b$ and is a mixture of a literature survey and new results or proofs.

We denote the two-sided reflected process by $V = \{V(t)\}_{t\geq 0}$ (or V^b, when the dependence on b needs to be stressed). The discrete time counterpart of V is a two-sided reflected random walk $\big(V_n : n = 0, 1, 2, \ldots\big)$ defined by

$$V_n = \min\big(b, \max(0, V_{n-1} + Y_n)\big) \tag{1}$$

where $Y_1, Y_2, \ldots$ are i.i.d. (with common distribution say F) and initial condition $V_0 = v$ for some $v \in [0, b]$; the role of the Lévy process X is then taken by the random walk $X_n = Y_1 + \cdots + Y_n$.

The study of such processes in discrete or continuous time has a long history and numerous applications. For a simple example, consider the case $X(t) = \sum_1^{N(t)} Z_i - ct$ of a compound Poisson process with drift, where N is Poisson(λ) and the Z_i independent of N, i.i.d. and non-negative. Here one can think of V as the amount of work in a system with a server working at rate c, jobs arriving at Poisson rate λ and having sizes $Z_1, Z_2, \ldots$, and finite capacity b of storage. If a job of size $y > b - x$ arrives when the content is x, only $b - x$ of the job is processed and $x + y - b$ is lost. One among many examples is a data buffer, where the unit is number of bits (discrete in nature, but since both b and a typical job size z are huge, a continuous approximation is motivated).

Studies of systems with such finite capacity are numerous, and we mention here waiting time processes in queues with finite capacity [25, 48–50], and a finite dam or fluid model [11, 113, 132]. They are used in models of network traffic or telecommunications systems involving a finite buffer [79, 95, 144], and they also occur in finance, e.g. [60, 63]. In the queueing context, it should be noted that even if in the body of literature, there is no upper bound b on the state space, the reason is mainly mathematical convenience: the analysis of infinite-buffer systems is in many respects substantially simpler than that of finite-buffer systems. In real life, infinite waiting rooms or infinite buffers do not occur, so that the infinite-buffer assumption is really just an approximation.

In continuous time, there is no obvious analogue of the defining Eq. (1). We follow here the tradition of representation as solution of a Skorokhod problem

$$V(t) = V(0) + X(t) + L(t) - U(t) \tag{2}$$

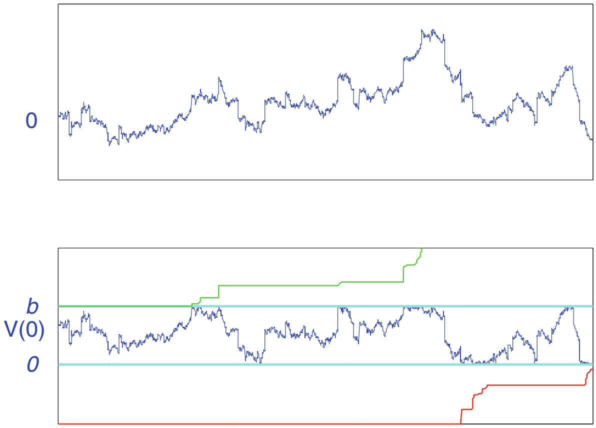


Fig. 1 The processes X (*blue*), V (*blue*), L (*green*), U (*red*)

where L, U are non-decreasing right-continuous processes such that

$$\int_0^\infty V(t)\,\mathrm{d}L(t) \;=\; 0\,, \quad \int_0^\infty \bigl(b - V(t)\bigr)\,\mathrm{d}U(t) \;=\; 0\,. \tag{3}$$

In other words, L can only increase when V is at the lower boundary 0, and U only when V is at the upper boundary b. Thus, L represents the 'pushing up from 0' that is needed to keep $V(t) \geq 0$ for all t, and U represents the 'pushing down from b' that is needed to keep $V(t) \leq b$ for all t. An illustration is given in Fig. 1, with the unreflected Lévy process in the upper panel, whereas the lower panel has the two-sided reflected process V (blue) in the middle subpanel, L (red) in the lower and U (green) in the upper. Questions of existence and uniqueness are discussed in Sects. 3 and 4.

As usual in applied probability, a first key question in the study of V is the long-run behavior. A trivial case is monotone sample paths. For example if in continuous time the underlying Lévy process X has non-decreasing and non-constant sample paths, then $V(t) = b$ for all large t. Excluding such degenerate cases, V regenerates at suitable visits to 0 (see Sect. 5.1 for more detail), and a geometric trial argument easily gives that the mean regeneration time is finite. Thus by general theory [11], a stationary distribution $\pi = \pi^b$ exists and

$$\frac{1}{N+1}\sum_{n=0}^{N} f(V_n) \;\to\; \pi(f)\,, \quad \frac{1}{T}\int_0^T f(V_t)\,\mathrm{d}t \;\to\; \pi(f) \tag{4}$$

a.s. in discrete, resp. continuous time whenever f is (say) bounded or non-negative. A further fundamental quantity is the overflow or loss at b which is highly relevant for applications; in the dam context, it represents the amount of lost water and in the data buffer context the number of lost bits. The loss at time n is $(Y_n+V_{n-1}-b)^+$ and so by a suitable LLN (e.g. [11, VI.3]) the long-run behavior of the loss in discrete time is given by

$$\frac{1}{N}\sum_{n=1}^{N}(Y_n+V_{n-1}-b)^+ \;\to\; \int_0^b \pi(\mathrm{d}x)\int_{b-x}^{\infty}(y+x-b)^+\,F(\mathrm{d}y)\,, \tag{5}$$

as follows by conditioning on $Y_n = y$ and using (4).

We denote by $\ell = \ell^b$ the limit on the r.h.s. of (5) and refer to it as the *loss rate*. For example, in data transmission models the loss rate can be interpreted as the bit loss rate in a finite data buffer.

The form of π in the general continuous-time Lévy case is discussed in Sect. 5. In general, π is not explicitly available (sometimes the Laplace transform is). The key result for us is a representation as a two-sided exit probability,

$$\pi[x,\infty) \;=\; \pi[x,b] \;=\; \mathbb{P}\big(V\big(\tau[x-b,x)\big)\geq x\big) \tag{6}$$

where $\tau[y,x) = \inf\{t\geq 0:\; X(t)\notin [y,x)\}$, $y\leq 0\leq x$.

In continuous time, the obvious definition of the loss rate is $\ell = \mathbb{E}_\pi U(1) = \mathbb{E}_\pi U(t)/t$. However, representations like (5) are not a priori obvious, except for special cases as X being compound Poisson where

$$\ell \;=\; \int_0^b \pi(\mathrm{d}x)\int_{b-x}^{\infty}(x+y-b)\lambda\, G(\mathrm{d}y)\,,$$

where λ is the Poisson rate and G the jump size distribution. To state the main result, we need to introduce the basic Lévy setup:

$$X(t) \;=\; ct+\sigma B(t)+J(t)\,,$$

where B is standard Brownian motion and J an independent jump process with Lévy measure ν and jumps of absolute size ≤ 1 compensated. That is, the Lévy exponent

$$\kappa(\alpha) \;=\; \log\mathbb{E}\mathrm{e}^{\alpha X(1)} \;=\; \frac{1}{t}\log\mathbb{E}\mathrm{e}^{\alpha X(t)}$$

(defined when $\mathbb{E}\mathrm{e}^{\Re(\alpha)X(1)}<\infty$) is given by

$$\kappa(\alpha) \;=\; c\alpha+\sigma^2\alpha^2/2+\int_{-\infty}^{\infty}\big(\mathrm{e}^{\alpha y}-1-y\mathbb{1}(|y|\leq 1)\big)\,\nu(\mathrm{d}y)\,, \tag{7}$$

and one often refers to (c, σ^2, ν) as the *characteristic triplet* of X (see the end of the section for further detail and references). We further write

$$m = \mathbb{E}X(1) = \mathbb{E}X(t)/t = \kappa'(0) = c + \int_{|y|>1} y\,\nu(\mathrm{d}y)$$

for the mean drift of X.

Theorem 1.1 *Assume that m is well-defined and finite. Then*

$$\ell^b = \frac{1}{2b}\left\{2m\,\mathbb{E}_\pi V + \sigma^2 + \int_0^b \pi(\mathrm{d}x)\int_{-\infty}^{\infty} \varphi(x,y)\nu(\mathrm{d}y)\right\},$$

where

$$\varphi(x,y) = \begin{cases} -(x^2+2xy) & \text{if } y \le -x, \\ y^2 & \text{if } -x < y < b-x, \\ 2y(b-x)-(b-x)^2 & \text{if } y \ge b-x. \end{cases}$$

Theorem 1.1 first appears in Asmussen and Pihlsgård [16], with a rather intricate and lengthy proof. Section 6 contains a more direct and shorter proof originating from Pihlsgård and Glynn [118]. In Sect. 8, we summarize the original approach of [16], and in Sect. 15, some new representations of ℓ^b are presented using yet another approach. Whereas the method in Sect. 8 uses asymptotic expansions of identities obtained by martingale optional stopping, the ones in Sects. 6 and 15 contain stochastic calculus as a main ingredient.

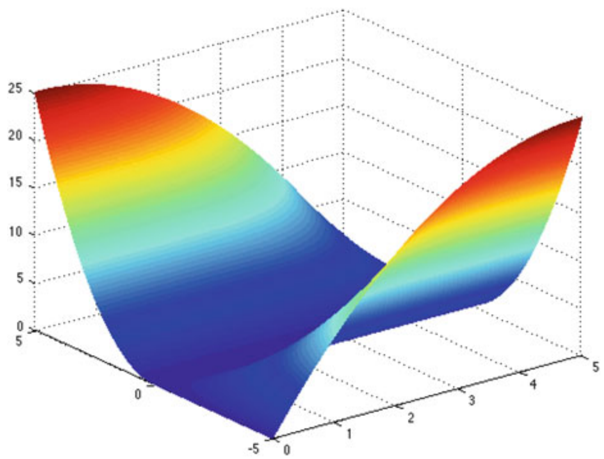


Fig. 2 The function $\varphi(x,y)$ plotted for $b = 5, 0 \le x \le b, -5 \le y \le 5$

The form of the function $\varphi(x, y)$ is illustrated in Fig. 2. Starting from Theorem 1.1, it is fairly straightforward to derive an alternative formula for ℓ^b, which can be convenient (note that the form (6) for the tail probability $\pi[x, b]$ has a nicer form than the one for $\pi(\mathrm{d}x)$ that follows by differentiation).

Corollary 1.2 *The loss rate ℓ^b can be written*

$$\ell^b = \frac{1}{2b}\Bigg\{2m\mathbb{E}V + \sigma^2 + \int_0^b y^2\nu(\mathrm{d}y) + \int_b^\infty (2yb - b^2)\nu(\mathrm{d}y)$$
$$- 2\int_0^b \Bigg\{\int_{-\infty}^{-x}(x+y)\nu(\mathrm{d}y) + \int_{b-x}^\infty (x+y-b)\nu(\mathrm{d}y)\Bigg\}\pi[x, b]\mathrm{d}x\Bigg\}.$$

Similar discussion applies to the underflow of 0, but for obvious symmetry reasons, it suffices to consider the situation at the upper barrier. One should note, however, that with $\ell^0 = \mathbb{E}_\pi L(1)$ one has

$$0 \;=\; m + \ell^0 - \ell^b\,. \tag{8}$$

Thus, ℓ^0 is explicit in terms of ℓ^b. Relation (8) follows by a rate conservation principle, since in order for $X + L - U$ to preserve stationarity, the drift must be zero. One may note that no moment conditions on X are needed for the existence of a stationary version of V. However, if $\mathbb{E}|X(1)| = \infty$, then one of ℓ^b or ℓ^0 is infinite (or both are).

In many applications, the upper buffer size b is large. This motivates that instead of going into the intricacies of exact computation of quantities like the loss rate ℓ^b, one may look for approximate expression for $b \to \infty$. Early references in this direction are Jelenković [79] who treated the random walk case with heavy tails, and Kim and Shroff [95], who considered the light-tailed case but only gave logarithmic asymptotics. Exact asymptotics for the light-tailed case is given in Asmussen and Pihlsgård [16] and surveyed in Sect. 10, whereas asymptotics for the heavy-tailed Lévy case first appears in Andersen [5] and is surveyed in Sect. 11. We assume negative drift, i.e. $m = \mathbb{E}X(1) < 0$, but by (8), the results can immediately be translated to positive drift. Note, however, that with negative drift one has $\ell^b \to 0$ as $b \to \infty$ (the results of Sects. 10 and 11 give the precise rates of decay), whereas with positive drift $\ell^b \to m$ (thus, (8) combined with Sects. 10 and 11 gives the convergence rate). The case of zero drift $m = 0$ has specific features as studied in Andersen and Asmussen [4], see Sect. 12; the key tool is here a functional limit theorem with either a Brownian or a stable process limit.

Going one step further in the study of the loss rate, one may ask for transient properties. One question is properties of the overflow time $\inf\{t > 0 : V^b(t) = b\}$ where one possible approach is regenerative process theory, Sect. 13 and another integro-differential equations, Sect. 14.2. Another question is properties of $U(t)$ for

$t < \infty$. From the above, U obeys the LLN $\mathbb{E}U(t)/t \to \ell$ as $t \to \infty$. Obvious questions are an associated CLT,

$$\sqrt{t}\big(U(t) - \ell\big) \ \to \ N(0, \sigma^2)$$

for some suitable σ^2, and large deviations properties like the asymptotics of

$$\mathbb{P}\big(U(t) > t(1+\varepsilon)\ell\big) \ \text{and} \ \mathbb{P}\big(U(t) < t(1-\varepsilon)\ell\big).$$

These topics are treated (for the first time) in Sects. 14.3 and 14.4.

Finally, the paper contains a number of explicit calculations for the special case where the jump part is compound Poisson with phase-type jumps. There is a considerable literature on this or closely related models. and we refer to Asmussen [12] for a survey and references. We have also included some material on one-sided reflection (Sect. 2) and two-sided reflection in discrete time (Sect. 3), which should serve both as a background and to give an understanding of the special problems that arise for the core topic of the paper, two-sided reflection in continuous time.

We conclude this introduction with some supplementary comments on the set-up on the Lévy model. Classical general references are Bertoin [28] and Sato [129], but see also Kyprianou [104] and Applebaum [7].

A simple case is jumps of bounded variation, which occurs if and only if $\int_{-\infty}^{\infty} |x|\, \nu(\mathrm{d}x) \ < \ \infty$. Then the expression (7) can be rewritten

$$\kappa(\alpha) \ = \ \tilde{c}\alpha + \sigma^2\alpha/2 + \int_{-\infty}^{\infty} (\mathrm{e}^{\alpha y} - 1)\, \nu(\mathrm{d}y)\,, \tag{9}$$

where

$$\tilde{c} \ = \ c - \int_{-1}^{1} y\, \nu(\mathrm{d}y)\,, \ \ m \ = \ \tilde{c} + \int_{-\infty}^{\infty} y\, \nu(\mathrm{d}y)\,. \tag{10}$$

With infinite variation, the integrals in (9) diverge, so that one needs the form (7).

To avoid trivialities, we assume throughout that the sample paths of X are non-monotone; in terms of the parameters of X, this means that either

(a) $\sigma^2 > 0$,
(b) $\sigma^2 = 0$ and X is of unbounded variation (i.e. $\int |y|\, \nu(\mathrm{d}y) = \infty$),
(c) $\sigma^2 = 0$, X is of bounded variation, and the Lévy measure ν has support both in $(-\infty, 0)$ and $(0, \infty)$,
(d) $\sigma^2 = 0$, X is of bounded variation, and either the Lévy measure ν has support in $(-\infty, 0)$ and $\tilde{c} > 0$ in (9), or ν has support in $(0, \infty)$ and $\tilde{c} < 0$ in (9).

2 One-Sided Reflection

Consider first the discrete time case and let $X_n = Y_1 + \cdots + Y_n$ where $Y_1, Y_2, \ldots$ are i.i.d. (with common distribution say F) so that X is a random walk. The random walk one-sided reflected at 0 (i.e., corresponding to $b = \infty$) is then defined by the recursion

$$V_n^\infty = (V_{n-1}^\infty + Y_n)^+ = \max\bigl(0, V_{n-1}^\infty + Y_n\bigr) \tag{11}$$

starting from $V_0^\infty \geq 0$. The process V^∞ also goes under the name a *Lindley process* (see [11, III.6] for a survey and many facts used in the following) and is a Markov chain with state space $[0, \infty)$.

For the following, it is important to note that the recursion (11) is explicitly solvable:

$$V_n^\infty = \max\bigl(V_0^\infty + X_n, X_n - X_1, \ldots, X_n - X_{n-1}, 0\bigr) \tag{12}$$

(for a proof, one may just note that the r.h.s. of (12) satisfies the recursion (11)). Reversing the order of $Y_1, \ldots, Y_n$ yields

$$V_n^\infty \stackrel{\mathscr{D}}{=} \max\bigl(V_0^\infty + X_n, X_{n-1}, \ldots, X_1\bigr)\,. \tag{13}$$

This shows in particular that V_n^∞ is increasing in stochastic order so that a limit in distribution V_∞^∞ exists. By a standard random walk trichotomy [11, VIII.2], one of the following possibilities arises:

(a) $X_n \to \infty$ so that $V_\infty^\infty = \infty$ a.s.;
(b) $\limsup_{n\to\infty} X_n = \infty$, $\liminf_{n\to\infty} X_n = -\infty$ so that $\max\bigl(V_0 + X_n, X_{n-1}, \ldots, X_1\bigr) \to \infty$ and $V_\infty^\infty = \infty$ a.s.;
(c) $X_n \to -\infty$ so that $V_\infty < \infty$ a.s.

For our purposes, it is sufficient to assume $\mathbb{E}|Y| < \infty$, and letting $m = \mathbb{E}Y$, the three cases then correspond to $m > 0$, $m = 0$, resp. $m < 0$, or, in Markov chain terms, roughly to the transient, null recurrent, resp. positive recurrent (ergodic) cases.

Consider from now on the ergodic case $m < 0$ (and, to avoid trivialities, assume that $\mathbb{P}(Y > 0) > 0$). Define $M = \max_{n\geq 0} X_n$. Since $X_n \to -\infty$, $V_0^\infty + X_n$ in (13) vanishes eventually, and letting $n \to \infty$ yields

$$V_\infty^\infty \stackrel{\mathscr{D}}{=} M\,. \tag{14}$$

It is often convenient to rewrite this in the form

$$\overline{\pi}^\infty(x) = \mathbb{P}(V_\infty^\infty > x) = \mathbb{P}\bigl(\tau(x) < \infty\bigr)\,, \tag{15}$$

where $\tau(x) = \inf\{n : X_n > x\}$ and π^∞ is the distribution of V_∞^∞.

Explicit or algorithmically tractable forms of π^∞ can only be found assuming some special structure, mainly skip-free properties or phase-type (or, more generally, matrix-exponential) forms, see [11, VIII.5]. Therefore asymptotics is a main part of the theory. The two main results are:

Theorem 2.1 (Light-Tailed Case) *Assume* $m < 0$*, that* F *is non-lattice and that there exists* $\gamma > 0$ *with* $\mathbb{E}e^{\gamma Y} = 1$*,* $\mathbb{E}\big[Ye^{\gamma Y}\big] < \infty$*. Then there exists* $0 < C < \infty$ *such that*

$$\overline{\pi}^\infty(x) \;=\; \mathbb{P}(V_\infty^\infty > x) \;\sim\; Ce^{-\gamma x}\,, \quad x \to \infty. \tag{16}$$

Theorem 2.2 (Heavy-Tailed Case) *Assume* $m < 0$*, that*

$$\overline{F}_I(x) \;=\; \int_x^\infty \overline{F}(y)\,\mathrm{d}y$$

is a subexponential tail[1] *and that* F *is long-tailed in the sense that* $\overline{F}(x+x_0)/\overline{F}(x) \to 1$ *for any* x_0*. Then*

$$\overline{\pi}^\infty(x) \;=\; \mathbb{P}(V_\infty^\infty > x) \;\sim\; \frac{1}{|m|}\overline{F}_I(x)\,, \quad x \to \infty. \tag{17}$$

Sketch of Proof of Theorem 2.1 We use a standard exponential change of measure technique [11, Ch. XIII]. Let $\tilde{\mathbb{P}}$, $\tilde{\mathbb{E}}$ refer to the case where X has c.d.f.

$$\tilde{F}(x) \;=\; \mathbb{E}\big[e^{\gamma X};\, X \le x\big]$$

rather than $F(x)$. Using (15) and standard likelihood ratio identities gives

$$\overline{\pi}^\infty(x) \;=\; \mathbb{P}\big(\tau(x) < \infty\big) \;=\; \tilde{\mathbb{E}}\big[e^{-\gamma X_{\tau(x)}};\, \tau(x) < \infty\big] \;=\; e^{-\gamma x}\tilde{\mathbb{E}}e^{-\gamma\xi(x)}\,, \tag{18}$$

where $\xi(x) = X_{\tau(x)} - x$ is the overshoot. Thus, the result follows with $C = \mathbb{E}e^{-\gamma\xi(\infty)}$ once it is shown that $\xi(x)$ has a proper limit $\xi(\infty)$ in distribution. This in turn follows by renewal theory by noting that $\xi(x)$ has the same distribution as the time until the first renewal after x in a renewal process with interarrivals distributed as $\xi(0)$ (the first ladder height). That $\xi(0)$ is non-lattice follows from F being so, and $\tilde{\mathbb{E}}\xi(0) < \infty$ follows from $\mathbb{E}[Xe^{\gamma X}] < \infty$. We omit the easy details. □

The computation of $C = \mathbb{E}e^{-\gamma\xi(\infty)}$ is basically of the same level of difficulty as the computation of π^∞ itself and feasible in essentially the same situations. Results

[1] By this we mean that there exists a subexponential distribution G such that $\overline{F}_I(x) = \overline{G}(x)$ for all large x. For background on heavy-tailed distributions, see e.g.[13, X.1], [62] and the start of Sect. 3.

of type Theorem 2.1 commonly go under the name *Cramér-Lundberg asymptotics*, and the equation $\mathbb{E}e^{\gamma Y} = 1$ is the *Lundberg equation*.

Discussion of Proofs of Theorem 2.2 The form of the result can be understood from the 'one big jump' heuristics, stating that large values of sums and random walks arise as consequence of one big Y_i, while the remaining Y_j are 'typical'; in particular, $X_{i-1} = \sum_{j=1}^{i-1} Y_j \approx im$ for large i. Splitting up after the value of i and noting that the contribution from a finite segment $1, \ldots, i_0$ is insignificant, we therefore get

$$\begin{aligned}\mathbb{P}(M > x) &= \sum_{i=1}^{\infty} \mathbb{P}\big(\tau(x) = i\big) \;\approx\; \sum_{i=1}^{\infty} \mathbb{E}\big[X_{i-1} \approx im, Y_i > x - im\big) \\ &\approx \sum_{i=1}^{\infty} \mathbb{P}(Y_i > x - im) \;=\; \sum_{i=1}^{\infty} \overline{F}(x - im) \\ &\approx \int_0^{\infty} \overline{F}(x - tm)\,\mathrm{d}t \;=\; \frac{1}{|m|}\int_x^{\infty} \overline{F}(u)\,\mathrm{d}u \;=\; \frac{1}{|m|}\overline{F}_I(x)\,.\end{aligned}$$

The rigorous verification of (17) traditionally follows a somewhat different line where the essential tool is ladder height representations. The first step is to show that the first ladder height $X_{\tau(0)}$ has a tail asymptotically proportional to $\overline{F}_I$, and next one uses the representation of M as a geometric sum of ladder heights to get the desired result. The details are not really difficult but too lengthy to be given here. See, e.g., [11, X.9] or [13, X.3]. A more recent proof by Zachary [140] (see also Foss et al. [62]) is, however, much more in line with the above heuristics. □

We next turn to continuous time where X is a Lévy process. There is no recursion of equal simplicity as (11) here, so questions of existence and uniqueness have to be treated by other means.

One approach simply adapts the representation (12) by rewriting the r.h.s. as

$$(V_0^{\infty} + X_n) \vee \max_{i=0,\ldots,n} (X_n - X_i) \;=\; X_n \;+\; \max\Big(V_0^{\infty}, -\min_{i=0,\ldots,n} X_i\Big)\,.$$

One then defines the continuous-time one-sided reflected process in complete analogue with the discrete case by

$$V^{\infty}(t) \;=\; X(t) + L(t) \tag{19}$$

where

$$L(t) \;=\; \max\Big(V^{\infty}(0), -\min_{0\le s\le t} X(s)\Big)\,. \tag{20}$$

(this can be motivated for example by a discrete skeleton approximation). Here L is often denoted the *local time at* 0, though this terminology is somewhat unfortunate because 'local time' in used in many different meanings in the probability literature. Often also the term *regulator* is used.

The second approach uses the Skorokhod problem: in (20), take L as a nondecreasing right-continuous process such that

$$\int_0^\infty V^\infty(t)\,\mathrm{d}L(t) \;=\; 0\,. \tag{21}$$

In other words, L can only increase when V^∞ is at the boundary 0. Thus, L represents the 'pushing up from 0' that is needed to keep $V^\infty(t) \geq 0$ for all t.

It is readily checked that the r.h.s. of (20) represents one possible choice of L. Thus, existence is clear. Uniqueness also holds:

Proposition 2.3 *Let $\{L^*(t)\}$ be any nondecreasing right-continuous process such that* (a) *the process $\{V^*(t)\}$ given by $V^*(0) = V(0)$, $V^*(t) = X_t + L^*(t)$ satisfies $V^*(t) \geq 0$ for all t,* (b) *L^* can increase only when $V^* = 0$, i.e. $\int_0^T V^*(t)\,\mathrm{d}L^*(t) = 0$ for all T. Then $L^*(t) = L(t)$, $V^*(t) = V^\infty(t)$.*

Proof Let $D(t) = L(t) - L^*(t)$, $\Delta D(s) = D(s) - D(s-)$. The integration-by-parts formula for a right-continuous process of bounded variation gives

$$\begin{aligned}
D^2(t) &= 2\int_0^t D(s)\mathrm{d}D(s) - \sum_{s\leq t}\big(\Delta D(s)\big)^2 \\
&= 2\int_0^t \big(L(s) - L^*(s)\big)\mathrm{d}L(s) - 2\int_0^t \big(L(s) - L^*(s)\big)\mathrm{d}L^*(s) \;-\; \sum_{s\leq t}\big(\Delta D(s)\big)^2 \\
&= 2\int_0^t \big(V^\infty(s) - V^*(s)\big)dL(s) - 2\int_0^t \big(V^\infty(s) - V^*(s)\big)\mathrm{d}L^*(s) \;-\; \sum_{s\leq t}\big(\Delta D(s)\big)^2 \\
&= -2\int_0^t V^*(s)\,\mathrm{d}L(s) \;-\; 2\int_0^t V^\infty(s)\mathrm{d}L^*(s) \;-\; \sum_{s\leq t}\big(\Delta D(s)\big)^2 .
\end{aligned}$$

Here the two first integrals are nonnegative since $V^*(s)$ and $V^\infty(s)$ are so, and also the sum is clearly so. Thus $D(t)^2 \leq 0$, which is only possible if $L(t) \equiv L^*(t)$. □

Define $M = \max_{0\leq t<\infty} X(t)$ and assume $m = \mathbb{E}X(1) < 0$. The argument for (14) then immediately goes through to get the existence of a proper limit $V^\infty(\infty)$ of $V^\infty(t)$ and the representation

$$V^\infty(\infty) \;\stackrel{\mathscr{D}}{=}\; M\,. \tag{22}$$

Equivalently,

$$\overline{\pi}^\infty(x) \;=\; \mathbb{P}(V^\infty(\infty) > x) \;=\; \mathbb{P}\big(\tau(x) < \infty\big)\,, \tag{23}$$

where $\tau(x) = \inf\{t > 0 : X(t) > x\}$ and π^∞ is the distribution of $V(\infty)$.

The loss rate $\ell = \ell^b$ is undefined in this setting since $b = \infty$. A closely related quantity is $\ell^0 = \mathbb{E}_{\pi^\infty} L(1)$ and one has

$$\ell^0 \;=\; -m\,. \tag{24}$$

This follows by a conservation law argument: in (19), take $t = 1$, consider the stationary situation and take expectations to get[2]

$$\mathbb{E}_{\pi^\infty} V(1) \;=\; \mathbb{E}_{\pi^\infty} V(0) + \mathbb{E}_{\pi^\infty} X(1) + \mathbb{E}_{\pi^\infty} L(1) \;=\; \mathbb{E}_{\pi^\infty} V(1) + m + \ell^0\,.$$

For an example of the relevance of ℓ^0, consider the M/G/1 workload process. Here ℓ^0 can be interpreted as the average unused capacity of the server or as the average idle time.

We next consider analogues of the asymptotic results in Theorems 2.1 and 2.2. The main results are the following two theorems (for a more complete treatment, see [13, XI.2]):

Theorem 2.4 (Light-Tailed Case) *Assume $m < 0$, that X is not a compound Poisson process with lattice support of the jumps, and that there exists $\gamma > 0$ with $\kappa(\gamma) = 0$, $\kappa'(\gamma) < \infty$. Then there exists $0 < C < \infty$ such that*

$$\overline{\pi}^\infty(x) \;=\; \mathbb{P}\big(V^\infty(\infty) > x\big) \;\sim\; C\mathrm{e}^{-\gamma x}\,, \quad x \to \infty. \tag{25}$$

Theorem 2.5 (Heavy-Tailed Case) *Assume $m < 0$, that*

$$\overline{\nu}(x) \;=\; \int_x^\infty \nu(\mathrm{d}y)$$

is a subexponential tail and that ν is long-tailed in the sense that $\overline{\nu}(x+x_0)/\overline{\nu}(x) \to 1$ for any x_0. Then

$$\overline{\pi}^\infty(x) \;=\; \mathbb{P}\big(V^\infty(\infty) > x\big) \;\sim\; \frac{1}{|m|}\overline{\nu}_I(x)\,, \quad x \to \infty, \tag{26}$$

where $\overline{\nu}_I(x) \;=\; \displaystyle\int_x^\infty \overline{\nu}(y)\,\mathrm{d}y$.

Sketch of Proof of Theorem 2.4 The most substantial (but small) difference from the proof of Theorem 2.1 is the treatment of the overshoot process ξ which has no longer the simple renewal process interpretation. However, the process ξ is regenerative with regeneration points $\omega(1), \omega(2), \ldots$ where one can take

$$\omega(k) \;=\; \inf\{t > \omega(k-1) + U_k : \xi(t) = 0\}\,,$$

[2] Strictly speaking, the argument requires $\mathbb{E}_{\pi^\infty} V(0) < \infty$ which amounts to a second moment assumption. For the general case, just use a truncation argument.

where $U_1, U_2, \ldots$ are independent uniform(0, 1) r.v.'s. One can then check that the non-compound Poisson property suffices for $\xi\bigl(\omega(1)\bigr)$ to be non-lattice and that $\kappa'(x) < \infty$ suffices for $\mathbb{E}\xi\bigl(\omega(1)\bigr) < \infty$. These two facts entail the convergence in distribution of $\xi(t)$ to a proper limit. □

As in discrete time, C can only be evaluated in special cases; general expressions are in Bertoin and Doney [29] but require the full (spatial) Wiener-Hopf factorization, a problem of equal difficulty. However, if X is upward skipfree (i.e., ν is concentrated on $(-\infty, 0)$), then $C = 1$ as is clear from $\xi(x) \equiv 0$. See also [13, XI.2] for the downward skipfree case as well as for related calculations, and [12] for the compound Poisson phase-type case.

For the proof of Theorem 2.5, we need a lemma:

Lemma 2.6 $\mathbb{P}\bigl(X(1) > x\bigr) \sim \overline{\nu}(x)$.

Proof Write $X = X' + X'' + X'''$ where the characteristic triplets of X', X'', X''' are (c, σ^2, ν'), $(0, 0, \nu'')$ and $(0, 0, \nu''')$, resp., with ν', ν'', ν''' being the restrictions of ν to $[-1, 1]$, $(-\infty, -1)$ and $(1, \infty)$, respectively.

With $\beta''' = \overline{\nu}(1)$, the r.v. $X(1)'''$ is a compound Poisson sum of r.v.'s, with Poisson parameter β''' and distribution ν'''/β'''. Standard heavy-tailed estimates (e.g. [13, X.2]) then give

$$\mathbb{P}(X'''(1) > x) \sim \beta''' \frac{\overline{\nu}'''(x)}{\beta'''} = \overline{\nu}(x)\,, \quad x > 1\,.$$

The independence of $X''(1)$ and $X'''(1) > 0$ therefore implies

$$\mathbb{P}\bigl(X''(1) + X'''(1) > x\bigr) \sim \overline{\nu}(x)\,,$$

cf. the proof of [13, X.3.2]. It is further immediate that $\kappa'(r) < \infty$ for all r. In particular, $X'(1)$ is light-tailed, and the desired estimate for $X(1) = X'(1) + X''(1) + X'''(1)$ then follows by [13, X.1.11]. □

Proof of Theorem 2.5 Define

$$M^d = \sup_{n=0,1,2,\ldots} X(n)\,.$$

Then

$$\mathbb{P}(M^d > u) \sim \frac{1}{|\mathbb{E}X(1)|} \int_u^\infty \overline{\nu}(y)\,\mathrm{d}y \tag{27}$$

by Theorem 2.4 and Lemma 2.6. Also clearly $\mathbb{P}(M^d > u) \leq \mathbb{P}(M > u) = \psi(u)$. Given $\varepsilon > 0$, choose $a > 0$ with $\mathbb{P}\bigl(\inf_{0 \leq t \leq 1} X(t) > -a\bigr) \geq 1 - \varepsilon$. Then

$$\mathbb{P}(M^d > u - a) \geq (1 - \varepsilon)\mathbb{P}(M > u)\,.$$

But by subexponentiality, $\mathbb{P}(M^d > u - a) \sim \mathbb{P}(M^d > u)$. Putting these estimates together completes the proof. □

The proof of Theorem 2.5 is basically a special case of what is called *reduced load equivalence*. This principle states that if X has negative drift and $X = X_1 + X_2$, where X_1 has heavy-tailed increments and X_2 has increments with lighter tails, then $M = \sup_t X(t)$ has the same tail behavior as $\sup_t \big(X_1(t) + \mathbb{E}X_2(t)\big)$. For precise versions of the principle, see e.g. Jelenković et al. [80].

3 Loss Rate Asymptotics for Two-Sided Reflected Random Walks

We recall from Sect. 1 that a two-sided reflected random walk $\{V_n\}_{n=0,1,2,\ldots}$ is defined by the recursion

$$V_n \;=\; \min\big(b, \max\big(0, V_{n-1} + Y_n\big)\big) \tag{28}$$

where $Y_1, Y_2, \ldots$ are i.i.d. (with common distribution say F) and initial condition $V_0 = v$ for some $v \in [0, b]$. Let $X_n \;=\; Y_1 + \cdots + Y_n$ so that X is a random walk.

Existence of V is not an issue in discrete time because of the recursive nature of (28). Recall from (6) that the stationary distribution π^b can be represented in terms of two-sided exit probabilities as

$$\mathbb{P}(V \geq x) \;=\; \pi^b[x, \infty) \;=\; \pi^b[x, b] \;=\; \mathbb{P}\big(X_{\tau[x-b,x)} \geq x\big) \tag{29}$$

where V is a r.v. having the stationary distribution and $\tau[y, x) \;=\; \inf\big\{k \geq 0 : X_k \notin [y, x)\big\}$, $y \leq 0 \leq x$ (we defer the proof of this to Sect. 5).

The loss rate in discrete time as defined as the limit in (5) may be written as

$$\ell^b = \mathbb{E}(V + Y - b)^+ = \mathbb{E}\max(V + Y - b, 0) \tag{30}$$

where V is the stationary r.v. For later use we note the alternative form

$$\ell^b = \mathbb{E}(Y - b)^+ + \int_0^b \mathbb{P}(Y > b - y)\,\overline{\pi}(y)\,\mathrm{d}y, \tag{31}$$

which follows by partial integration in (30).

From now on we assume that $-\infty < m = \mathbb{E}Y < 0$. The following two results on the asymptotics of ℓ^b are close analogues of Theorems 2.1 and 2.2:

Theorem 3.1 *Under the assumptions on Y and γ in Theorem 2.1,*

$$\ell^b \sim D\mathrm{e}^{-\gamma b},\; b \to \infty,$$

where D is a constant given in (34) below.

Theorem 3.2 *Let $Y_1, Y_2, \cdots$ be an i.i.d. sequence with mean $m < 0$ and let ℓ^b be the loss rate at b of the associated random walk $X_n = Y_1 + \cdots + Y_n$, reflected in* 0 *and b. Assume $\overline{F}(x) \sim \overline{B}(x)$ for some distribution $B \in \mathscr{S}^*$. Then*

$$\ell^b \sim \overline{F}_I(b),\ b \to \infty,\ \textit{where}\ \overline{F}_I(b\ ) \ = \ \int_b^\infty \overline{F}(y)\,\mathrm{d}y \ = \ \mathbb{E}(Y-b)^+.$$

We used here the standard notation for the classes $\mathscr{L}, \mathscr{S}$ and $\mathscr{S}^*$ of heavy-tailed distributions (see e.g. [97] or [13]): If B is a distribution on $[0, \infty)$ we have $B \in \mathscr{L}$ (B is *long-tailed*) iff

$$\lim_{x\to\infty} \frac{\overline{B}(x+y)}{\overline{B}(x)} = 1, \qquad \text{for all } y\,,$$

where $\overline{B}(x) = 1 - B(x)$. The class $\mathscr{S}$ of subexponential distributions is defined by the requirement

$$\lim_{x\to\infty} \frac{\overline{B^{*n}}(x)}{\overline{B}(x)} = n \qquad n = 2, 3, \cdots$$

where B^{*n} denotes the nth convolution power of B. A subclass of $\mathscr{S}$ is $\mathscr{S}^*$, where we require that the mean μ_B of B is finite and

$$\lim_{x\to\infty} \int_0^x \frac{\overline{B}(x-y)}{\overline{B}(x)} \overline{B}(y)\,\mathrm{d}y \ = \ 2\mu_B.$$

The classes are related by $\mathscr{S}^* \subseteq \mathscr{S} \subseteq \mathscr{L}$. More generally, we will say a measure ν belongs to, say, $\mathscr{S}$ if it is tail equivalent to a distribution in $\mathscr{S}$, that is $\nu([x, \infty)) \sim \overline{B}(x)$ for some B in $\mathscr{S}$.

Theorem 3.1 is from Pihlsgård [117]. Theorem 3.2 was originally proved in Jelenković [79], but we provide a shorter proof by taking advantage of the representation of the stationary distribution provided by (6).

3.1 *Proof of Theorem 3.1 (Light Tails)*

We introduce the following notation (standard in random walk theory):

$M = \sup_{k\geq 0} X_k$.
$\tau_+^s(u) = \inf\{k \geq 1 : X_k > u\},\ \tau_+^w(u) = \inf\{k \geq 1 : X_k \geq u\},\ u \geq 0$.
$G_+(x) = \mathbb{P}(X_{\tau_+^s(u)} \leq x),\ G_+^w(x) = \mathbb{P}(X_{\tau_+^w(u)}(u)) \leq x)$.
$\tau_-^s(-u) = \inf\{k \geq 1 : X_k < -u\},\ u \geq 0$.
The overshoot of level u, $B(u) = X_{\tau_+^s(u))} - u,\ u \geq 0$.

The weak overshoot of level u, $B^w(u) = X_{\tau_+^w(u)} - u,\ u \geq 0$.
$B(\infty)$, a r.v. having the limiting distribution (if it exists) of $B(u)$ as $u \to \infty$.
$B^w(\infty)$, a r.v. having the limiting distribution (if it exists) of $B^w(u)$ as $u \to \infty$.

Recall that $\kappa(\alpha) = \log \mathbb{E}e^{\alpha Y_1}$ and that $\gamma > 0$ is the root of the Lundberg equation $\kappa(\alpha) = 0$ with $\kappa'(\gamma) < \infty$. We let $\mathbb{P}_L$ and $\mathbb{E}_L$ correspond to a measure which is exponentially tilted by γ, i.e.,

$$\mathbb{P}(G) = \mathbb{E}_L[e^{-\gamma X_\tau}; G] \tag{32}$$

when τ is a stopping time w.r.t. $\{\mathscr{F}(n) = \sigma(Y_1, Y_2, \ldots, Y_n)\}$ and $G \in \mathscr{F}(\tau)$, $G \subseteq \{\tau < \infty\}$ where $\mathscr{F}(\tau)$ is the stopping time σ-field. Note that $\mathbb{E}_L Y = \kappa'(\gamma) > 0$ by convexity.

Lemma 3.3 *Assume that Y is non-lattice. Then, for each $v \geq 0$,*

$$\mathbb{P}\big(\tau_-^s(-v) > \tau_+^w(u)\big) \sim e^{-\gamma u}\mathbb{E}_L e^{-\gamma B(\infty)}\mathbb{P}_L\big(\tau_-^s(-v) = \infty)\big),\ u \to \infty.$$

Proof We first note that $\tau_+^w(u)$ is a stopping time w.r.t. $\mathscr{F}(n)$ and that $\{\tau_-^s(-v) > \tau_+^w(u)\} \in \mathscr{F}(\tau_+^w(u))$. Then (32) gives

$$\begin{aligned}\mathbb{P}\big(\tau_-^s(-v) > \tau_+^w(u)\big) &= \mathbb{E}_L\big[e^{-\gamma Z(\tau_+^w(u))}; \tau_-^s(-v) > \tau_+^w(u)\big] \\ &= e^{-\gamma u}\mathbb{E}_L\big[e^{-\gamma B(u)}; \tau_-^s(-v) > \tau_+^s(u)\big]\mathbb{P}_L\big(\tau_+^w(u) = \tau_+^s(u)\big) \\ &\quad + e^{-\gamma u}\mathbb{P}_L\big[\tau_-^s(-v) > \tau_+^w(u) \,\big|\, \tau_+^w(u) \neq \tau_+^s(u)\big]\mathbb{P}_L\big(\tau_+^w(u) \neq \tau_+^s(u)\big).\end{aligned}$$

Since Y is non-lattice, it follows that G_+^w is so (see [11, Lemma 1.3, p. 222]) and then the renewal theorem (see [11, Theorem 4.6, p. 155]) applied to the renewal process governed by G_+^w, in which the forward recurrence time process coincides with the overshoot process $B^w = B^w(u)$, yields $B^w(u) \xrightarrow{\mathscr{D}} B^w(\infty)$ w.r.t. $\mathbb{P}_L$ where $B^w(\infty)$ has a density. Thus 0 is a point of continuity of $B^w(\infty)$ and we then get that $\mathbb{P}_L(\tau_+^w(u) \neq \tau_+^s(u)) = \mathbb{P}_L(B^w(u) = 0) \to 0$ and $\mathbb{P}_L(\tau_+^w(u) = \tau_+^s(u)) \to 1,\ u \to \infty$. We now use that $B(u) \to B(\infty)$, $\{\tau_-^s(-v) > \tau_+^w(u)\} \uparrow \{\tau_-^s(-v) = \infty\}$ in $\mathbb{P}_L$-distribution and apply the argument used in the proof of Corollary 5.9, p. 368 in [11] saying that $B(u)$ and $\{\tau_-^s(-v) > \tau_+^w(u)\}$ are asymptotically independent. □

In the representation of ℓ^b in (31), It follows from the assumption $\kappa'(\gamma) < \infty$ that $\mathbb{E}(Y - b)^+ = o(e^{-\gamma b})$. In the second term we make the change of variables $v = b - y$ and get

$$\begin{aligned}\int_0^b \mathbb{P}(Y > b - y)\overline{\pi}(y)dy &= \int_0^\infty \mathbb{1}(v \leq b)\mathbb{P}(Y > v)\mathbb{P}\big(\tau_-^s(-v) > \tau_+^w(b - v)\big)dv \\ = e^{-\gamma b}\int_0^\infty & e^{\gamma v}\mathbb{1}(v \leq b)\mathbb{P}(Y > v)e^{\gamma(b-v)}\mathbb{P}\big(\tau_-^s(-v) > \tau_+^w(b - v)\big)\, dv.\end{aligned} \tag{33}$$

Further, we have that $\mathbb{P}(\tau_-^s(-v) > \tau_+^w(b-v)) \le \mathbb{P}(M \ge b-v) \le e^{-\gamma(b-v)}$ (the last inequality is just a variant of Lundberg's inequality), so

$$e^{\gamma v}\mathbb{1}(v \le b)\mathbb{P}(Y > v)e^{\gamma(b-v)}\mathbb{P}\big(\tau_-^s(-v) > \tau_+^w(b-v)\big) \le e^{\gamma v}\mathbb{P}(Y > v)$$

and since $\int_0^\infty e^{\gamma v}\mathbb{P}(Y > v)\mathrm{d}v < \infty$, the assertion follows with

$$D = \mathbb{E}_L e^{-\gamma B(\infty)} \int_0^\infty e^{\gamma v}\mathbb{P}(Y > v)\mathbb{P}_L(\tau_-^s(-v) = \infty)\,\mathrm{d}v \tag{34}$$

by (33), Lemma 3.3 and dominated convergence. □

Remark 3.4 The constants occurring in D and above are standard in Wiener-Hopf theory for random walks. Note that alternative expressions for D are in [117].

3.2 Proof of Theorem 3.2 (Heavy Tails)

By (31), we need to prove that

$$\limsup_{b\to\infty} I(b) = 0 \text{ where } I(b) = \int_0^b \frac{\mathbb{P}(Y > b-y)\overline{\pi}^b(y)}{\overline{F}_I(b)}\,\mathrm{d}y\,. \tag{35}$$

For any $A > 0$

$$\limsup_{b\to\infty}\int_0^A \frac{\mathbb{P}(Y > b-y)\overline{\pi}^b(y)}{\overline{F}_I(b)}\,\mathrm{d}y \le \limsup_{b\to\infty}\frac{\mathbb{P}(Y > b-A)}{\overline{F}_I(b)}\int_0^A \overline{\pi}^b(y)\,\mathrm{d}y = 0$$

so therefore

$$\limsup_{b\to\infty}\int_0^b I(b) = \limsup_{b\to\infty}\int_A^b \frac{\mathbb{P}(Y > b-y)\overline{\pi}^b(y)}{\overline{F}_I(b)}\,\mathrm{d}y\,. \tag{36}$$

Define $m^+ = \int_0^\infty \mathbb{P}(Y > t)\,\mathrm{d}t$ and $F_e(y) = (1/m^+)\int_0^x \mathbb{P}(Y > t)\,\mathrm{d}t$. According to (17) we have $\overline{\pi}^\infty(y)|m| \sim \overline{F}_I(y)$ so that for large A and $y > A$

$$\overline{\pi}^\infty(y) \le 2\overline{F}_I(y)/|m| = 2m^+\overline{F}_e(y)/|m|\,.$$

From Proposition 11.6 in Sect. 11 (proved there for Lévy processes but valid also for random walks as it only relies on the representation (29) of π as a two-barrier passage time probability),

$$0 \le \overline{\pi}^\infty(x) - \overline{\pi}^b(x) \le \overline{\pi}^\infty(b)\,. \tag{37}$$

Using this, we have:

$$\begin{aligned}\limsup_{b\to\infty}\int_A^{b-A}\frac{\mathbb{P}(Y>b-y)\overline{\pi}^b(y)}{\overline{F}_I(b)}\,\mathrm{d}y &\le 2\limsup_{b\to\infty}\int_A^{b-A}\frac{m^+\mathbb{P}(Y>b-y)\overline{F}_e(y)}{|m|\overline{F}_I(b)}\,\mathrm{d}y\\ &= 2\limsup_{b\to\infty}\int_A^{b-A}\frac{\mathbb{P}(Y>b-y)\overline{F}_e(y)}{|m|\overline{F}_e(b)}\,\mathrm{d}y\\ &= 2\limsup_{b\to\infty}\frac{\overline{F}_e^{*2}(b)}{\overline{F}_e(b)}\int_A^{b-A}\frac{\mathbb{P}(Y>b-y)\overline{F}_e(y)}{|m|\overline{F}_e^{*2}(b)}\,\mathrm{d}y\\ &= 4\limsup_{b\to\infty}\int_A^{b-A}\frac{\mathbb{P}(Y>y)\overline{F}_e(b-y)}{|m|\overline{F}_e^{*2}(b)}\,\mathrm{d}y\\ &= 4\limsup_{b\to\infty}\frac{m^+}{|m|}\mathbb{P}(A<U\le b-A\mid U+V>b)\\ &= \frac{2m^+}{|m|}\overline{F}_e(A)\,.\end{aligned}$$

where U and V are independent with $U\overset{\mathscr{D}}{=}V\overset{\mathscr{D}}{=}F_e$ and we have used that for i.i.d. random variables in $\mathscr{S}$

$$\mathbb{P}(A<Y_1<b-A\mid Y_1+Y_2>b)\to\frac{1}{2}\overline{F}(A),\quad b\to\infty$$

(cf. [13, pp. 294, 296], slightly adapted). By combining the result above with (36) we have

$$\limsup_{b\to\infty} I(b)\le\frac{2m^+}{|m|}\overline{F}_e(A)+\limsup_{b\to\infty}\int_{b-A}^{b}\frac{\mathbb{P}(Y>b-y)\overline{\pi}^b(y)}{\overline{F}_I(b)}\,\mathrm{d}y\,. \tag{38}$$

Here the integral equals

$$\limsup_{b\to\infty}\int_0^A\frac{\mathbb{P}(Y>y)\overline{\pi}^b(b-y)}{\overline{F}_I(b)}\,\mathrm{d}y\le\limsup_{b\to\infty}\frac{\overline{\pi}^b(b-A)}{\overline{F}_I(b)}\int_0^A\mathbb{P}(X>y)\,\mathrm{d}y\,.$$

If we define $\sigma_A=\inf\{n\ge 0\mid X_n<-A\}$, $M_n=\max_{k\le n}X_k$ and use the representation (29) of the stationary distribution we have:

$$\overline{\pi}^b(b-A)=\mathbb{P}(M_{\sigma_A}>b-A)\,.$$

By Theorem 1 of [61] we have $\mathbb{P}(M_{\sigma_A}>b-A)\sim\mathbb{E}\sigma_A\overline{F}(b)$ and therefore $\overline{\pi}^b(b-A)/\overline{F}_I(b)\to 0$ since the tail of F is lighter than that of the integrated tail. Using (38) it thus follows that we can bound $\limsup I(b)$ by $2m^+\overline{F}_e(A)/|m|$. Letting $A\to\infty$ completes the proof.

4 Construction and the Skorokhod Problem

In discrete time, the definition of the two-sided reflected process $V = V^b$ is straightforward via the recursion (1). In this section, we consider how to rigorously proceed in continuous time.

First, we note that there is a simple pragmatic solution: Let $y \in [0, b]$ be the initial value. For $y < b$, take the segment up to the first hitting time $\tau(b)$ of b as the initial segment of V^∞ (the one-sided reflected process started from y) until (b, ∞) is hit; we then let $V\big(\tau(b)\big) = b$. For $y = b$, we similarly take the segment up to the first hitting time $\tau^*(0)$ of 0 by using the one-sided reflection operator (with the sign reversed and change of origin) as constructed in Sect. 2; at time $\tau^*(0)$ where this one-sided reflected (at b) process hits $(-\infty, 0]$, we let $V\big(\tau^*(0)\big) = 0$. The whole process V is then constructed by glueing segments together in an obvious way. Glueing also local times together, we obtain the desired solution of the Skorokhod problem. Uniqueness of this solution may be established using a proof nearly identical to that of Proposition 2.3.

Before we proceed to a more formal definition of V we restate the Skorokhod problem: Given a cadlag process $\{X(t)\}$ we say a triplet $(\{V(t)\}, \{L(t)\}, \{U(t)\})$ of processes is the *solution to the Skorokhod problem on* $[0, b]$ if $V(t) = X(t) + L(t) - U(t) \in [0, b]$ for all t and

$$\int_0^\infty V(t)\,\mathrm{d}L(t) = 0 \quad \text{and} \quad \int_0^\infty (b - V(t))\,\mathrm{d}U(t) = 0\,.$$

Note that the Skorokhod problem as introduced above is a purely deterministic problem. We refer to the mapping which associates a triplet $(\{V(t)\}, \{L(t)\}, \{U(t)\})$ to a cadlag process $X(t)$ as the *Skorokhod map*.

Remark 4.1 The Skorokhod problem on $[0, b]$ is a particular case of reflection of processes in convex regions of $\mathbb{R}^n$, which is treated in Tanaka [136] where a proof of existence and uniqueness is provided given that the involved processes are continuous or step functions. This is extended in [6] to include cadlag processes, which covers what is needed in this article. Apart from the generalizations to larger classes of functions, other papers have focused on more general domains than convex subsets of $\mathbb{R}^n$, e.g. Lions and Snitzman [110] and Saisho [126]. The case of Brownian motion in suitable regions has received much attention in recent decades, see e.g. Harrison and Reiman [74] and Chen and Yao [43]. In [73, Chap. 2, Sect. 4], the Skorokhod problem on $[0, b]$ is introduced as the *two-sided regulator* and is used to treat Brownian motion with two-sided reflection; another early reference on two-sided reflection problems is Chen and Mandelbaum [42]. A comprehensive treatment of the Skorokhod map and its continuity properties, as well as other reflection mappings and their properties, is given in Whitt [138].

Various formulas for the Skorokhod map have appeared in the literature, among them Cooper et al. [49]. See [101] for a survey of these formulas and the relation between them. An alternative approach to estimation of stationary quantities is to take advantage of the integral representation of the one-dimensional Skorokhod reflection, see Konstantopoulos and Last [99], Anantharam and Konstantopoulos [2], and Buckingham et al. [40]. This is applicable when considering processes of finite variation, so that we can write $S(t) = A(t) + B(t)$ for non-decreasing cadlag processes A and B. It is then possible to write $V(t)$ as an integral with respect to $A(\mathrm{d}t)$. This representation can for example be used to derive the Laplace transform of V in terms of the Palm measure. □

As described in Sect. 2, specifically (19) and (20), an explicit expression for $V(t)$ is available when one is concerned with one-sided reflection. This is also the case when dealing with Skorokhod problem on $[0, b]$. Indeed, from Kruk et al. [101] we have:

$$V^b(t) = X(t) - \Big((V(0) - b)^+ \wedge \inf_{u \in [0,t]} X(u) \Big) \vee \sup_{s \in [0,t]} \Big((V(0) - b) \wedge \inf_{u \in [s,t]} X(u) \Big) . \tag{39}$$

We shall assume $V(0) = 0$ a.s and in this case we have following simplification, which was originally proved in [5].

Theorem 4.2 *If* $V(0) = 0$*, then*

$$V^b(t) = \sup_{s \in [0,t]} \Big(\big(X(t) - X(s) \big) \wedge \inf_{u \in [s,t]} \big(b + X(t) - X(u) \big) \Big) . \tag{40}$$

Remark 4.3 Before we provide a rigorous proof, we note the following intuitive explanation for the expression (40): For $v > 0$ consider the process $\{V_v(t)\}_{t > v}$ obtained by reflecting $X_v(t) = X(t) - X(v)$ at b from below (in terms of recursions like (11)) this is $V_n^v = b \vee (V_{n-1}^v + Y_n)$ applied to the increments with $n > v$ and with $V_v^v = 0$). Similarly to (19) and (20) we obtain $V_v(t) = X_v(t) \wedge \inf_{v<u<t}(b + X_v(t) - X_v(u))$. Then obviously $V_v(t) \le V(t)$ but since $V_v(t^*) = V(t^*)$ for $t^* = \sup_{0<u<t} V(u) = 0$, we have $V(t) = \sup_{0<v<t} V_v(t)$. □

The proof of (40) proceeds as follows: First we prove Proposition 4.4 and 4.5 which are the discrete time equivalents of (39) and (40). Then we prove Lemma 4.6, which states that the implied mapping of $X(t)$ in (40) is Lipschitz-continuous in the J_1 topology which is combined with an piecewise constant approximation to obtain the equivalence of (40) and (39). To emphasize the deterministic nature of the Skorokhod problem and for explicit treatment of the involved mappings, we switch notation and let $\mathbf{y} = \{y_n\}_{n=1}^{\infty}$ be a sequence in $\mathbb{R}^{\infty}$ and consider the sequences $\mathbf{x}$ and $\mathbf{v}$ obtained by respectively taking cumulative sums of $\mathbf{y}$ and applying two-sided reflection, that is $x_n = y_1 + \cdots + y_n$ and $v_n = \min(b, \max(0, v_{n-1} + y_n)$ with $x_0 = v_0 = 0$. We let $\Gamma_{0,b}$ denote the two-sided reflection mapping, that is $\Gamma_{0,b}(\mathbf{x}) = \mathbf{v}$.

Proposition 4.4 *The solution of the two-sided reflection is given by*

$$\Gamma_{0,b}(\mathbf{x})(n) = \max_{k\in\{0,\cdots,n\}} \left(\min_{j\in\{k,\cdots,n\}} (x_n - x_k, b + x_n - x_j) \right). \tag{41}$$

Proof We prove the claim by induction. The case $n = 1$ is trivial, so we assume (41) holds for some n, and consider the cases $y_{n+1} \leq 0$ and $y_{n+1} > 0$ separately. For the former case we have

$$\begin{aligned} \Gamma_{0,b}(\mathbf{x})(n+1) &= v_{n+1} = 0 \vee (v_n + y_{n+1}) \wedge b = 0 \vee (v_n + y_{n+1}) \\ &= 0 \vee \left(\max_{k\in\{0,\cdots,n\}} \left(\min_{j\in\{k,\cdots,n\}} (x_n - x_k, b + x_n - x_j) \right) + y_{n+1} \right) \\ &= 0 \vee \left(\max_{k\in\{0,\cdots,n\}} \left(\min_{j\in\{k,\cdots,n\}} (x_{n+1} - x_k, b + x_{n+1} - x_j) \right) \right). \end{aligned} \tag{42}$$

Since $y_{n+1} \leq 0$, we have

$$\min_{j\in\{k,\cdots,n+1\}} x_{n+1} - x_j = \min_{j\in\{k,\cdots,n\}} x_{n+1} - x_j,$$

so that (42) equals

$$\begin{aligned} &0 \vee \left(\max_{k\in\{0,\cdots,n\}} \left(\min_{j\in\{k,\cdots,n+1\}} (x_{n+1} - x_k, b + x_{n+1} - x_j) \right) \right) \\ &= \max_{k\in\{0,\cdots,n+1\}} \left(\min_{j\in\{k,\cdots,n+1\}} (x_{n+1} - x_k, b + x_{n+1} - x_j) \right), \end{aligned} \tag{43}$$

as desired. The case $y_{n+1} > 0$ is similar:

$$\begin{aligned} v_{n+1} &= 0 \vee (v_n + y_{n+1}) \wedge b = (v_n + y_{n+1}) \wedge b \\ &= \left(\max_{k\in\{0,\cdots,n\}} \left(\min_{j\in\{k,\cdots,n\}} (x_n - x_k, b + x_n - x_j) \right) + y_{n+1} \right) \wedge b \\ &= \max_{k\in\{0,\cdots,n\}} \left(\min_{j\in\{k,\cdots,n\}} (x_{n+1} - x_k, b + X_{n+1} - x_j) \wedge b \right), \end{aligned}$$

which equals (43) as well. This completes the proof. □

Proposition 4.4 provides the discrete-time analogue of (40). Next, we provide the discrete-time analogue for (39), in the case $v_0 = 0$.

Proposition 4.5 *The solution of the two-sided reflection is given by*

$$\Gamma_{0,b}(\mathbf{x})(n) = \min_{k\in\{0,\ldots,n\}} \left[\left((x_n - x_k + b) \wedge \max_{i\in\{0,\ldots,n\}} (x_n - x_i) \right) \vee \max_{i\in\{k,\ldots,n\}} (x_n - x_i) \right]. \tag{44}$$

Proof The proof is again by induction and again the case $n = 1$ is straightforward, so we assume the stated holds for some n. Then we have

$$\begin{aligned}
\Gamma_{0,b}(\mathbf{x})(n+1) &= 0 \vee (v_n + y_{n+1}) \wedge b \\
&= 0 \vee \left(\min_{k\in\{0,\dots,n\}} \left[\left((x_n - x_k + b) \wedge \max_{i\in\{0,\dots,n\}} (x_n - x_i) \right) \vee \max_{i\in\{k,\dots,n\}} (x_n - x_i) \right] + y_{n+1} \right) \wedge b \\
&= 0 \vee \min_{k\in\{0,\dots,n\}} \left[\left((x_{n+1} - x_k + b) \wedge \max_{i\in\{0,\dots,n\}} (x_{n+1} - x_i) \right) \vee \max_{i\in\{k,\dots,n\}} (x_{n+1} - x_i) \right] \wedge b \\
&= \min_{k\in\{0,\dots,n\}} \left[\left((x_{n+1} - x_k + b) \wedge \max_{i\in\{0,\dots,n\}} ((x_{n+1} - x_i) \vee 0) \right) \vee \max_{i\in\{k,\dots,n\}} ((x_{n+1} - x_i) \vee 0) \right] \wedge b \\
&= \min_{k\in\{0,\dots,n\}} \left[\left((x_{n+1} - x_k + b) \wedge \max_{i\in\{0,\dots,n+1\}} (x_{n+1} - x_i) \right) \vee \max_{i\in\{k,\dots,n+1\}} (x_{n+1} - x_i) \right] \wedge b.
\end{aligned} \tag{45}$$

We notice that

$$\begin{aligned}
&\left((x_{n+1} - x_k + b) \wedge \max_{i\in\{0,\dots,n+1\}} (x_{n+1} - x_i) \right) \vee \max_{i\in\{k,\dots,n+1\}} (x_{n+1} - x_i) \\
&= \begin{cases} \max_{i\in\{0,\dots,n+1\}} (x_{n+1} - x_i) & \text{if } k = 0 \\ \max_{i\in\{0,\dots,n+1\}} (x_{n+1} - x_i) \wedge b & \text{if } k = n+1, \end{cases}
\end{aligned}$$

so that (45) equals

$$\min_{k\in\{0,\dots,n+1\}} \left[\left((x_{n+1} - x_k + b) \wedge \max_{i\in\{0,\dots,n+1\}} (x_{n+1} - x_i) \right) \vee \max_{i\in\{k,\dots,n+1\}} (x_{n+1} - x_i) \right].$$

This proves the claim. □

We now proceed to the proof of (40). Let $\psi \in \mathscr{D}[0,\infty)$. From [101] we have:

$$\Gamma_{0,b}(\psi)(t) = \psi(t) - \sup_{s\in[0,t]} \left[\left((\psi(s) - b)) \vee \inf_{u\in[0,t]} \psi(u) \right) \wedge \inf_{u\in[s,t]} \psi(u) \right], \tag{46}$$

when the process is started at 0. In view of the two previous propositions it seems reasonable to conjecture that $\Gamma_{0,b} = \Xi$, where

$$\Xi[\psi](t) = \sup_{s\in[0,t]} \left[(\psi(t) - \psi(s)) \wedge \inf_{u\in[s,t]} (b + \psi(t) - \psi(u)) \right]. \tag{47}$$

We prove this by first showing that Ξ is Lipschitz-continuous in the J_1 topology.

Lemma 4.6 *The mapping Ξ is Lipschitz-continuous in the uniform and J_1 metrics as a mapping from $D[0,T]$ for $T \in [0,\infty]$, with constant 2.*

Proof We follow the proof of Corollary 1.5 in [100] closely. Fix $T < \infty$. We start by proving Lipschitz-continuity in the uniform metric. Define

$$R_t[\psi](s) \;=\; \left[(-\psi(s)) \wedge \inf_{u\in[s,t]}(b-\psi(u))\right]; \quad S[\psi](t) \;=\; \sup_{s\in[0,t]} R_t[\psi](s). \tag{48}$$

For $\psi_1, \psi_2 \in D[0,T]$ we have

$$\begin{aligned} S[\psi_1](t) - S[\psi_2](t) &\leq \sup_{s\in[0,t]} (R_t[\psi_1](s) - R_t[\psi_2](s)) \\ &\leq \sup_{s\in[0,t]} \left[|-\psi_1(s) - (-\psi_2(s))| \vee \left| \inf_{u\in[s,t]}(b-\psi_1(u)) - \inf_{u\in[s,t]}(b-\psi_2(u)) \right| \right] \\ &\leq \|\psi_1 - \psi_2\|_T \,. \end{aligned}$$

The same inequality applies to $S[\psi_2](t) - S[\psi_2](t)$, so that taking the supremum leads to

$$\| S[\psi_1] - S[\psi_2] \|_T \leq \| \psi_1 - \psi_2 \|_T \,,$$

and this proves Lipschitz-continuity, with constant 2:

$$\| \Xi[\psi_1] - \Xi[\psi_2] \|_T \;\leq \| \psi_1 - \psi_2 \| + \| S[\psi_1] - S[\psi_t] \|_T \leq 2 \, \| \psi_1 - \psi_2 \|_T \,.$$

We now turn to the J_1-metric, and we let $\mathscr{M}$ denote the class of strictly increasing continuous functions from $[0,T]$ onto itself with continuous inverse. An elementary verification yields that for $\psi \in D[0,T]$ and $\boldsymbol{\lambda} \in \mathscr{M}$ we have $\Xi[\psi \circ \boldsymbol{\lambda}] = \Xi[\psi] \circ \boldsymbol{\lambda}$. With $\boldsymbol{e}$ being the identity, this leads to

$$\begin{aligned} d_{J_1}(\Xi[\psi_1], \Xi[\psi_2]) &= \inf_{\boldsymbol{\lambda}\in\mathscr{M}} \{\| \Xi[\psi_1] \circ \boldsymbol{\lambda} - \Xi[\psi_2] \|_T \vee \| \boldsymbol{\lambda} - \boldsymbol{e} \|_T\} \\ &= \inf_{\boldsymbol{\lambda}\in\mathscr{M}} \{\| \Xi[\psi_1 \circ \boldsymbol{\lambda}] - \Xi[\psi_2] \|_T \vee \| \boldsymbol{\lambda} - \boldsymbol{e} \|_T\} \\ &\leq \inf_{\boldsymbol{\lambda}\in\mathscr{M}} \{2 \, \| \psi_1 \circ \boldsymbol{\lambda} - \psi_2 \|_T \vee \| \boldsymbol{\lambda} - \boldsymbol{e} \|_T\} \leq 2 d_{J_1}(\psi_1, \psi_2), \end{aligned}$$

where we used the Lipschitz-continuity in the uniform metric. This proves Lipschitz-continuity in the J_1 metric, again with constant 2; it is valid for every $T < \infty$ and hence also for $T = \infty$. □

We are now ready to prove that $\Gamma_{0,b} = \Xi$.

Theorem 4.7 *For $\psi \in D[0,\infty)$ we have $\Gamma[\psi](t) = \Xi[\psi](t)$.*

Proof Let $\psi \in D[0,\infty)$ be given, and define $\boldsymbol{\gamma}_n$ and ψ_n by $\gamma_n(t) = \lfloor nt \rfloor / n$, $\psi_n(t) = \psi(\gamma_n(t))$. Since $\boldsymbol{\gamma}_n \to \boldsymbol{e}$ in the uniform topology, we have $\boldsymbol{\gamma}_n \to_{d_{J_1}} \boldsymbol{e}$ and hence $(\psi, \boldsymbol{\gamma}_n) \to (\psi, \boldsymbol{e})$ in the strong version of the J_1 topology (see p. 83 in [138]). Since $\boldsymbol{e}$ is strictly increasing we may apply Theorem 13.2.2 in [138] to obtain $\psi_n \to_{d_{J_1}} \psi$. Fix $t < T$, and consider ψ as element of $D[0,T]$. Since the image $\psi_n([0,T])$ is finite, we may apply Propositions 4.4 and 4.5, in conjunction with (46), to obtain $\Gamma_{0,b}[\psi_n] = \Xi[\psi_n]$. Finally, we let $n \to \infty$ and use the J_1-continuity of the $\Gamma_{0,b}$ mapping proved in [100], and the J_1-continuity of Ξ proved in Lemma 4.6 to finish the proof. □

Remark 4.8 Letting $b \to \infty$ yields $\sup_{s\in[0,t]} [(\psi(t) - \psi(s))]$, which is indeed the standard one-sided reflection from (19) and (20). □

5 The Stationary Distribution

5.1 Ergodic Properties

The following observation is easy but basic:

Proposition 5.1 *The two-sided reflected Lévy process $V = V^b$ admits a unique stationary distribution $\pi = \pi^b$. Furthermore, for any initial distribution V converges in distribution and total variation* (t.v.) *to π.*

Proof We appeal to the theory of regenerative processes [11, Ch. VI]. The classical definition of a stochastic process to be regenerative means in intuitive terms that the process can be split into i.i.d. *cycles* (with the first cycle having a possibly different distribution). There is usually a multitude of ways to define a cycle. The naive approach in the case of V^b is to take the instants of visits to state 0 (say) as regeneration points, but these will typically have accumulation points (cf. the theory of Brownian zeros!) and so a bit more sophistication is needed. Instead we may, e.g., define the generic cycle length T as starting at level 0 at time 0, waiting until level b is hit and taking the cycle termination time T as the next hitting time of 0 ('up to b from 0 and down again'). That is,

$$T \;=\; \inf\bigl\{t > \inf\{s > 0 : \; V^b(s) = b\} : \; V^b(t) = 0 \,\big|\, V^b(0) = 0\bigr\}\,.$$

The regenerative structure together with the easily verified fact $\mathbb{E}T < \infty$ then immediately gives the existence of π^b.

T.v. convergence just follows from coupling V^b with the stationary version $\hat{V}^b$ (cf. [11, VII.1]). Indeed, we may assume that V^b and $\hat{V}^b$ both have the same driving process X. Then $\hat{V}^b(t) \geq V(t)$ for all t, and so $\tau = \inf\{t > 0 : \; V^b(t) = \hat{V}^b(t)\}$ is bounded by T_1, hence a.s. finite. □

Remark 5.2 T.v. convergence in distribution is often alternatively established by verifying that the distribution of T is spread-out [11, VI.1]. In the present context, this is slightly more tedious but goes like this. T decomposes as the independent sum $T_1 + T_2$ where T_1 is passage time from 0 to b and T_2 the passage from b to 0 so that it suffices to verify that one of T_1, T_2 is spread-out. This is obvious for Brownian motion since there T_1, T_2 are both absolutely continuous. In the case $\nu \neq 0$ of a non-vanishing jump component, suppose, e.g., that ν does not vanish on $(0, \infty)$. Then b may be hit by a jump, i.e. $\mathbb{P}_0\big(\Delta X(T_1) > 0\big) > 0$. Then also $\mathbb{P}_0\big(\Delta X(T_1) > \varepsilon\big) > 0$ for some $\varepsilon > 0$ and so $\mathbb{P}_0(T_1 \in \cdot, \Delta X(T_1) > \varepsilon\big) > 0$ serves as candidate for the absolutely continuous part of T_1.

Another approach is to take advantage of the fact that V^b is a Markov process on a compact state space with a semi-group with easily verified smoothness properties, cf. [94] for some general theory, and yet another to invoke Harris recurrence in continuous time, cf. [23, 24]. We omit the details. □

Remark 5.3 The process V^b is in fact geometrically ergodic, i.e.

$$\sup_A \Big|\mathbb{P}_x(V^b(t) \in A) - \pi^b(A)\Big| = \mathrm{O}(\mathrm{e}^{-\varepsilon t}) \tag{49}$$

for some $\varepsilon > 0$ where the O term is uniform in x. This follows again from the coupling argument by bounding the l.h.s. of (49) by $\mathbb{P}(\tau > t)$ and checking that τ has exponential moments (geometric trials argument!).

It is easy to derive rough bounds on the tail of τ and thereby lower bounds on ε. To get the exact rate of decay in (49) seems more difficult, as is typically the case in Markov process theory (but see Linetsky [109] for the Brownian case). □

5.2 First Passage Probability Representation

The main result on the stationary distribution π^b is as follows and states that π^b can be computed via two-sided exit probabilities for the Lévy process.

Theorem 5.4 *The stationary distribution of the two-sided reflected Lévy process $V = V^b$ is given by*

$$\pi^b[x, b] = \mathbb{P}\big(V(\infty) \geq x\big) = \mathbb{P}\big(X\big(\tau[x-b, x)\big) \geq x\big) \tag{50}$$

where $\tau[u, v) = \inf\big\{t \geq 0 : X(t) \notin [u, v)\big\}$, $u \leq 0 \leq v$.

Note that in the definition of $\tau[u, v)$ we write $t \geq 0$, not $t > 0$.

We remark that in the case of spectrally negative Lévy processes the evaluation of $\mathbb{P}\big(X\big(\tau[x-K, x)\big) \geq x\big)$ is a special case of scale function calculations. For such a process, the scale function W^q is usually defined as the function with Laplace transform

$$\int_0^{\infty} \mathrm{e}^{-sx} W^q(x)\,\mathrm{d}x \;=\; \frac{1}{\kappa(-s)-q}\,.$$

However, it has a probabilistic interpretation related to (50) by means of

$$\mathbb{E}\big[\mathrm{e}^{-q\tau[a,b)}\mathbb{1}\big(X(\tau[a,b)) \geq b\big)\big] \;=\; \frac{W^q(|a|)}{W^q(|a|+b)} \tag{51}$$

The present state of the area of scale functions is surveyed in Kuznetsov et al. [102]. Classically, there have been very few explicit examples, but a handful more, most for quite special structures, have recently emerged (see, e.g., Hubalek and Kyprianou [76] and Kyprianou and Rivero [106]).

We shall present two approaches to the proof of Theorem 5.4. One is direct and specific for the model, the other uses general machinery for certain classes of stochastic processes with certain monotonicity properties.

5.3 Direct Verification

Write $V_0(t)$ for V started from $V_0(0) = 0$, let T be fixed and for $0 \leq t \leq T$, let $\{R_x(t)\}$ be defined as $R_x(t) = x - X(T) + X(T-t)$ until $(-\infty, 0]$ or (b,∞) is hit; the value is then frozen at 0, resp. ∞. We shall show that

$$V_0(T) \geq x \iff R_x(T) = 0; \tag{52}$$

this yields

$$\mathbb{P}\big(V_0(T) \geq x\big) \;=\; \mathbb{P}\big(\tau[x-b,x) \leq T,\, X\big(\tau[x-b,x)\big) \geq x\big)$$

and the proposition then follows by letting $T \to \infty$.

Let $\sigma = \sup\{t \in [0,T] : V_0(t) = 0\}$ (well-defined since $V_0(0) = 0$). Then $V_0(T) = X(T) - X(\sigma) + U(\sigma) - U(T)$, so if $V_0(T) \geq x$ then $X(T) - X(\sigma) \geq x$, and similarly, for $t \geq \sigma$

$$x \;\leq\; V_0(T) \;=\; V_0(t) + X(T) - X(t) + U(t) - L(T) \;\leq\; b + X(T) - X(t),$$

implying $R_x(T-t) \leq b$. Thus absorbtion of $\{R_x(t)\}$ at ∞ is not possible before $T-\sigma$, and $X(T) - X(\sigma) \geq x$ then yields $R_x(T-\sigma) = 0$ and $R_x(T) = 0$.

Assume conversely $R_x(T) = 0$ and write the time of absorbtion in 0 as $T - \sigma$. Then $x - X(T) + X(\sigma) \leq 0$, and $R_x(t) \leq b$ for all $t \leq T - \sigma$ implies $x - X(T) + X(t) \leq b$

for all $t \geq \sigma$. If $V_0(t) < b$ for all $t \in [\sigma, T]$, then $U(T) - U(t) = 0$ for all such t and hence

$$V_0(T) = V_0(\sigma) + X(T) - X(\sigma) + L(T) - L(\sigma) \geq V_0(\sigma) + X(T) - X(\sigma) \geq 0 + x.$$

If $V_0(t) = b$ for some $t \in [\sigma, T]$, denote by ω the last such t. Then $U(T) = U(\omega)$ and hence

$$V_0(T) = V_0(\omega) + X(T) - X(\omega) + L(T) - L(\omega) \geq b + X(T) - X(\omega) + 0 \geq x.$$

□

5.4 Siegmund Duality

Now consider the general approach. Let $\mathbb{T} = \mathbb{N}$ or $\mathbb{T} = [0, \infty)$, let $\{V(t)\}_{t \in \mathbb{T}}$ be a general Markov process with state space $E = [0, \infty)$ or $E = \mathbb{N}$, and let $V_x(t)$ be the version starting from $V_x(0) = x$. We write interchangeably $\mathbb{P}(V_x(t) \in A)$ and $\mathbb{P}_x(V(t) \in A)$. Then $\{V_x(t)\}$ is *stochastically monotone* if $x \leq y$ implies $V_x(t) \leq_{\text{so}} V_x(t)$ (stochastical ordering) for all $t \in \mathbb{T}$, i.e. if $\mathbb{P}_x(V(t) \geq z) \leq \mathbb{P}_y(V(t) \geq z)$ for all t and z.

Proposition 5.5 *The existence of a Markov process $\{R(t)\}_{t \in \mathbb{T}}$ on $E \cup \{\infty\}$ such that*

$$\mathbb{P}_x\big(V(t) \geq y\big) = \mathbb{P}_y\big(R(t) \leq x\big) \tag{53}$$

is equivalent to (i) *$\{V(t)\}$ is stochastically monotone and* (ii) *$\mathbb{P}_x\big(V(t) \geq y\big)$ is a right-continuous function of x for all t and y.*

Proof If $\{R(t)\}$ exists, the l.h.s. of (53) is nondecreasing and right-continuous in x and so necessity of (i), (ii) is clear. If conversely (i), (ii) hold, then the r.h.s. of (53) defines a probability measure for each y that we can think of as the element $P^t(y, \cdot)$ of a transition kernel P^t (thus $P^t(y, \{\infty\}) = 1 - \lim_{x \to \infty} \mathbb{P}_x\big(V(t) \geq y\big)$), and we shall show that the Chapman-Kolmogorov equations $P^{t+s} = P^t P^s$ hold. This follows since

$$\begin{aligned} P^{t+s}\big(y, [0, x]\big) &= \mathbb{P}_x\big(V(t+s) \geq y\big) = \int_E \mathbb{P}_x\big(V(t) \in \mathrm{d}z\big)\mathbb{P}_z\big(V(s) \geq y\big) \\ &= \int_E \mathbb{P}_x\big(V(t) \in \mathrm{d}z\big) \int_0^z P^s(y, \mathrm{d}u) = \int_0^z P^s(y, \mathrm{d}u)\mathbb{P}_x\big(V(t) \geq u\big) \\ &= \int_0^z P^s(y, \mathrm{d}u) P^t\big(u, [0, x]\big) = (P^t P^s)\big(y, [0, x]\big). \end{aligned}$$

□

Theorem 5.6 *The state* 0 *is absorbing for* $\{R(t)\}$*. Furthermore, letting*

$$\tau \;=\; \inf\{t > 0 : R_x(t) \le 0\} \;=\; \inf\{t > 0 : R_x(t) = 0\} \;,$$

one has

$$\mathbb{P}_0\big(V(T) \ge x\big) \;=\; \mathbb{P}_x(\tau \le T), \tag{54}$$

and if $V(t)$ *converges in total variation, say to* $V = V(\infty)$*, then*

$$\mathbb{P}_0(V \ge x) \;=\; \mathbb{P}_x(\tau < \infty), \tag{55}$$

Proof Taking $x = y = 0$ in (53) yields $\mathbb{P}_0\big(R(t) \le 0\big) = \mathbb{P}_0\big(V(t) \ge 0\big) = 1$ so that indeed 0 is absorbing for $\{R(t)\}$. We then get

$$\mathbb{P}_x(\tau \le T) \;=\; \mathbb{P}_x\big(R(T) \le 0\big) \;=\; \mathbb{P}_0\big(V(T) \ge x\big).$$

Letting $T \to \infty$ concludes the proof. □

5.5 *Dual Recursions*

We turn to a second extension of (53), (55) which does not require the Markov property but, however, works more easily when $\mathbb{T} = \mathbb{N}$ than when $\mathbb{T} = [0, \infty)$. We there assume that $\{V_n\}_{n\in\mathbb{N}}$ is generated by a recursion of the form

$$V_{n+1} \;=\; f(V_n, U_n), \tag{56}$$

where $\{U_n\}$ (the *driving sequence*) is a stationary sequence of random elements taking values in some arbitrary space F and $f : E \times F \to E$ is a function. The (time-homogeneous) Markov case arises when the U_n are i.i.d. (w.l.o.g., uniform on $F = (0, 1)$), but also much more general examples are incorporated. We shall need the following lemma, which summarizes the standard properties of generalized inverses as occurring in, e.g., quantile functions.

Lemma 5.7 *Assume that* $f(x, u)$ *is continuous and nondecreasing in* x *for each fixed* $u \in F$ *and define* $g(x, u) = \inf\{y : f(y, u) \ge x\}$*. Then for fixed* u*,* $g(x, u)$ *is left-continuous in* x*, nondecreasing in* x *and strictly increasing on the interval* $\{x : 0 < g(x, u) < \infty\}$*. Further,* $f(y, u) = \sup\{x : g(x, u) \le y\}$ *and*

$$g(x, u) \le y \iff f(y, u) \ge x. \tag{57}$$

W.l.o.g., we can take $\{U_n\}$ with doubly infinite time, $n \in \mathbb{Z}$, and define the dual process $\{R_n\}_{n\in\mathbb{N}}$ by

$$R_{n+1} \;=\; g(R_n, U_{-n}), \quad n \in \mathbb{N}; \tag{58}$$

when the initial value $x = R_0$ is important, we write $R_n(x)$.

Theorem 5.8 *Equations* (53) *and* (55) *also hold in the set-up of* (56) *and* (58).

Proof For $T \in \mathbb{N}$, define $V_0^{(T)}(y) = y$,

$$V_1^{(T)}(y) = f\big(V_0^{(T)}(y), U_{-(T-1)}\big), \;\ldots\;, V_T^{(T)}(y) = f\big(V_{T-1}^{(T)}(y), U_0\big).$$

We shall show by induction that

$$V_T^{(T)}(y) \ge x \iff R_T(x) \le y \tag{59}$$

(from this (53) follows by taking expectations and using the stationarity; since $g(0,u) = 0$, (54) then follows as above). The case $T = 0$ of (59) is the tautology $y \ge x \iff x \le y$. Assume (59) shown for T. Replacing y by $f(y, U_{-T})$ then yields

$$V_T^{(T)}\big(f(y, U_{-T})\big) \ge x \iff R_T(x) \le f(y, U_{-T}).$$

But $V_T^{(T)}\big(f(y, U_{-T})\big) = V_{T+1}^{(T+1)}(y)$ and by (57),

$$R_T(x) \le f(y, U_{-T}) \iff R_{T+1}(x) \;=\; g(R_T(x), U_{-T}) \;\le\; y.$$

Hence (59) holds for $T + 1$. □

Example 5.9 Consider the a discrete time random walk reflected at 0, $V_{n+1} = (V_n + \xi_n)^+$ with increments $\xi_0, \xi_1, \ldots$ which are i.i.d. or, more generally, stationary.

In the set-up of Proposition 5.5 and Theorem 5.6, we need (for the Markov property) to assume that $\xi_0, \xi_1, \ldots$ are i.i.d. We take $E = [0, \infty)$ and for $y > 0$, we then get

$$\mathbb{P}_y(R_1 \le x) \;=\; \mathbb{P}_x(V_1 \ge y) \;=\; \mathbb{P}(x + \xi_0 \ge y) \;=\; \mathbb{P}(y - \xi_0 \le x).$$

For $y = 0$, we have $\mathbb{P}_0(R_1 = 0) = 1$. These two formulas show that $\{R_n\}$ evolves as a random walk $\check{X}_n = -\xi_0 - \xi_{-1} - \cdots - \xi_{-n+1}$ with increments $-\xi_0, -\xi_1, \ldots$ as long as $R_n > 0$, i.e. $R_n(x) = x - \check{X}_n$, $n < \tau$, $R_n(x) = 0$, $n \ge \tau$; when $(-\infty, 0]$ is hit, the value is instantaneously reset to 0 and $\{R_n\}$ then stays in 0 forever. We see further that we can identify τ and $\tau(x)$, and thus (55) is the same as the maximum representation (15) of the stationary distribution of V.

Consider instead the approach via Theorem 5.8 (which allows for increments that are just stationary). We let again $E = [0, \infty)$, take $U_k = \xi_k$ and $f(x, u) = (x + u)^+$.

It is easily seen that $g(y,u) = (y-u)^+$ and so $\{R_n\}$ evolves as $\breve{X}$ as long as $R_n > 0$, while 0 is absorbing. With, $X_n^* = -\breve{X}_n$ it follows that $\tau = \inf\{n : x + \breve{X}_n \le 0\} = \inf\{n : X_n^* \ge x\}$. This last expression shows that (54) is the same as a classical result in queueing theory known as *Loynes' lemma*, [11, IX.2c]. □

Example 5.10 Consider two-sided reflection in discrete time,

$$V_{n+1} = \min\big[b, (V_n + \xi_n)^+\big]. \tag{60}$$

For Theorem 5.6, we take $\xi_0, \xi_1, \ldots$ i.i.d. and $E = [0,\infty)$ (not $[0,b]$!). For $y > B$, we then get

$$\mathbb{P}_y(R_1 \le x) = \mathbb{P}_x(V_1 \ge y) \le \mathbb{P}_x(V_1 > b) = 0$$

for all x, i.e. $\mathbb{P}_y(R_1 = \infty) = 1$. For $0 \le y \le b$, $\mathbb{P}_y(R_1 \le x) = \mathbb{P}_x(V_1 \ge y)$ becomes

$$\mathbb{P}\big((x+\xi_0)^+ \ge y\big) = \begin{cases} 1 & y = 0, \\ \mathbb{P}(y - X_0 \le x) & 0 < y \le b. \end{cases}$$

Combining these facts show that $\{R_n\}$ evolves as $\breve{X}$ as long as $R_n \in (0,b]$. States 0 and ∞ are absorbing, and from $y > b\,\{R_n\}$ is in the next step absorbed at ∞. Thus for $R_0 = x \in (0,b]$, absorbtion at 0 before N, i.e. $\tau \le N$, cannot occur if (b,∞) is entered and with $X_n = \xi_0 + \cdots + \xi_{n-1}$, $\tau[u,v) = \inf\{n \ge 0 : X_n \notin [u,v)\}$, $u \le 0 < v$, we get

$$\begin{aligned} \mathbb{P}_0(V_N \ge x) &= \mathbb{P}_x(\tau \le N) \\ &= \mathbb{P}\big(\tau[x-b,x) \le N,\, X_{\tau[x-b,x)} \ge x\big), \end{aligned} \tag{61}$$

$$\mathbb{P}(V \ge x) = \mathbb{P}(X_{\tau[x-b,x)} \ge x) \tag{62}$$

(note that $\tau[x-b,x)$ is always finite). □

The Markov process approach of Theorem 5.6 is from Siegmund [130], and the theory is often referred to as *Siegmund duality*, whereas the recursive approach of Theorem 5.8 is from Asmussen and Sigman [18]. None of the approaches generalizes readily to higher dimension, as illustrated by Blaszczyszyn and Sigman [33] in their study of many–server queues. For stochastic recursions in general, see Brandt et al. [38] and Borovkov and Foss [37].

The two-barrier formula (62) is implicit in Lindley [108] and explicit in Siegmund [130], but has often been overlooked so that there are a number of alternative treatments of stationarity in two-barrier models around.

When applying Siegmund duality when $\mathbb{T} = [0,\infty)$, it is often more difficult to rigorously identify $\{R_t\}$ than when $\mathbb{T} = \mathbb{N}$. Asmussen [9] gives a Markov-modulated generalization for $\mathbb{T} = [0,\infty)$, and there is some general theory for the recursive setting in Ryan and Sigman [131].

Example 5.11 An early closely related and historically important example is Moran's model for the dam [113], which is discrete-time with the analogue of Y_k having the form $Y_k = A_k - c$. The inflow sequence $\{A_n\}$ is assumed i.i.d. and the release is constant, say c per time unit (if the content just before the release is $x < c$, only the amount x is released), and we let b denote the capacity of the dam. We will consider a slightly more general model where also the release at time n is random, say B_n rather than c (the sequence $\{B_n\}$ is assumed i.i.d. and independent of $\{A_n\}$).

We let Q_n^A denote the content just before the nth input (just after the $(n-1)$th release) and Q_n^B the content just after that (just before the $(n+1)$th release). Then

$$Q_n^A = \big[Q_{n-1}^B - B_{n-1}\big]^+, \tag{63}$$

$$Q_n^B = \big(Q_n^A + A_n\big) \wedge K, \tag{64}$$

$$Q_n^A = \big[(Q_{n-1}^A + A_{n-1}) \wedge K - B_{n-1}\big]^+, \tag{65}$$

$$Q_n^B = \big([Q_{n-1}^B - B_{n-1}]^+ + A_n\big) \wedge K. \tag{66}$$

The recursions (65), (66) are obviously closely related to (28), but not a special case.

The stationary distributions of the recursions (65), (66) can be studied by much the same methods as used for (28). Consider e.g. (66) which can be written as $Q_n^B = f(Q_{n-1}^B, \boldsymbol{U}_{n-1})$ where $\boldsymbol{u} = (a, b), f(x, \boldsymbol{u}) = ([x-b]^+ + a) \wedge K$ and $\boldsymbol{U}_{n-1} = (A_n, B_{n-1})$. The inverse function g of f in the sense of Proposition 5.7 is then given by

$$g(x, a, b) = \begin{cases} 0 & x = 0 \text{ or } x \in (0, b],\ a \geq b \\ x - (a - b) & x \in (0, b],\ a < b \\ \infty & x > b \end{cases}.$$

It follows that the dual process $\{R_n\}$ started from x evolves as the unrestricted random walk $\big\{(x - A_0)^+ - S_n\big\}$, starting from $(x - A_0)^+$ and having random walk increments $Z_n = A_n - B_{n-1}$, and that $\mathbb{P}_e(Q_n^B \geq x)$ is the probability that this process will exit $(0, K]$ to the right. □

5.6 Further Properties of π^b

We first ask when π^b has an atom at b, i.e., when $\pi^b\{b\} > 0$ so that there is positive probability of finding the buffer full. The dual question is whether $\pi^b\{0\} > 0$. For the answers, we need the fact that in the finite variation case, the underlying Lévy process X has the form

$$X(t) = \theta t + S_1(t) - S_2(t) \tag{67}$$

where S_1, S_2 are independent subordinators.

Theorem 5.12

(i) *In the infinite variation case,* $\pi^b\{b\} = \pi^b\{0\} = 0$.
In the finite variation case (67)*:*
(ii) $\pi^b\{b\} > 0$ *and* $\pi^b\{0\} = 0$ *when* $\theta > 0$*;*
(iii) $\pi^b\{b\} = 0$ *and* $\pi^b\{0\} > 0$ *when* $\theta < 0$*;*

Proof We have $\pi^b\{b\} = \mathbb{P}\big(X\big(\tau[0,b)\big) \geq b\big)$. In the unbounded variation case, $(-\infty, 0)$ is regular for X, meaning that $(-\infty, 0)$ is immediately entered when starting from $X(0) = 0$ [104, p. x], so that in this case $X\big(\tau[0,b)\big) = 0$ and $\pi^b\{b\} = 0$. Thus $\pi^b\{b\} > 0$ can only occur in the bounded variation case which is precisely (67). Similarly for $\pi^b\{0\}$.

One has

$$S_1(t)/t \overset{\text{a.s.}}{\to} 0\,, \quad S_2(t)/t \overset{\text{a.s.}}{\to} 0, \quad t \to 0 \tag{68}$$

(cf. [11, p. 254]). Thus if $\theta < 0$, X takes negative values arbitrarily close to $t = 0$ so that $\tau[0,b) = 0$, $X\big(\tau[0,b)\big) = 0$ and $\pi^b\{b\} = 0$.

If $\theta > 0$, we get $X(t) > 0$ for $0 < t < \varepsilon$ for some ε. This implies that X has a chance to escape to $[b, \infty)$ before hitting $(-\infty, 0)$ which entails $\mathbb{P}\big(X\big(\tau[0,b)\big) \geq b\big) > 0$ and $\pi^b\{b\} > 0$.

Combining these facts with a sign reversion argument yields (ii), (iii). □

Corollary 5.13 *In the spectrally positive (downward skipfree) case,* $\pi^b\{b\} = 0$.

Proof The conclusion follows immediately from Theorem 5.12(i) in the infinite variation case. In the finite variation case where $S_2 \equiv 0$, our basic assumption that the paths of X are non-monotonic implies $\theta < 0$, and we can appeal to Theorem 5.12(iii). □

Remark 5.14 Corollary 5.13 can alternatively be proved by identifying $\pi^b\{b\}$ as the limiting average of the time spent in b before t and noting that the Lebesgue measure of this time is 0 because X leaves b instantaneously, cf. the remark after (68). The same argument also yields $\pi^\infty\{b\} = 0$. □

The next result relates one- and two-sided reflection (see also [11, XIV.3] for some related discussion).

Theorem 5.15 *Assume* $m = \mathbb{E}X(1) < 0$ *so that* π^∞ *exists, and that* X *is spectrally positive with* $\nu\{b\} = 0$*. Then* π^b *is* π^∞ *conditioned to* $[0,b]$*, i.e.* $\pi^b(A) = \pi^\infty(A)/\pi^\infty[0,b]$ *for* $A \subseteq [0,b]$*. Equivalently,* π^b *is the distribution of* $M = \sup_{t\geq 0} X(t)$ *conditioned on* $M \leq b$*.*

Proof For $x \in (0,b)$, define

$$p_1(x) = \mathbb{P}\big(X\big(\tau[x-b,x)\big) \geq x\big)\,, \quad p_2(x) = \mathbb{P}\big(X\big(\tau(-\infty,x)\big) \geq x\big)\,.$$

Then spectral positivity implies that X is downward skipfree so that

$$p_2(x) = p_1(x) + \big(1 - p_1(x)\big)p_2(b), \quad p_1(x) = \frac{p_2(x) - p_2(b)}{1 - p_2(b)}.$$

In terms of stationary distributions, this means

$$\pi^b[x,b] = \frac{\pi^\infty[x,b)}{\pi^\infty[0,b)} = \frac{\pi^\infty[x,b]}{\pi^\infty[0,b]},$$

where the last equality follows from Remark 5.14. □

Corollary 5.16 *Assume $m = \mathbb{E}X(1) > 0$ and that X is spectrally negative with $\nu\{-b\} = 0$. Then π^b is the distribution of $b - \underline{M}$ conditioned on $\underline{M} \geq -b$ where $\underline{M} = \inf_{t \geq 0} X(t)$.*

6 The Loss Rate via Itô's Formula

The identification of the loss rate ℓ^b of a Lévy process X first appeared in Asmussen and Pihlsgård [16]. The derivation is based on optional stopping of the Kella-Whitt martingale followed by lots of tedious algebra, see Sect. 8. In this section we will follow an alternative more natural approach presented in Pihlsgård and Glynn [118]. One important point of that paper is that the dynamics of the two-sided reflection are governed by stochastic integrals involving the feeding process. Thus, all that is required is that stochastic integration makes sense. Hence, the natural framework is to take the input X to be a semimartingale. What we will do in the current section is to solve the more general problem of explicitly identifying the local times L and U (in terms of X and V) when the feeding process X is a semimartingale. The main result in [16] follows easily from what will be presented below.

We start with a brief discussion about semimartingales. A stochastic process X is a semimartingale if it is adapted, cadlag and admits a decomposition

$$X(t) = X(0) + N(t) + B(t)$$

where N is a local martingale, B a process of a.s. finite variation on compacts with $N(0) = B(0) = 0$. Alternatively, a semimartingale is a stochastic process for which the stochastic integral

$$\int H(s)\mathrm{d}X(s) \tag{69}$$

is well defined for H belonging to a satisfactory rich class of processes (more precisely, the predictable processes). In (69), we will in this exposition take H to be an adapted process with left continuous paths with right limits. The class of semimartingales forms a vector space and contains e.g. all adapted processes with cadlag paths of finite variation on compacts and Lévy processes. For a thorough introduction to semimartingales we refer to Protter [123].

Let X and Y be semimartingales. $[X,X]$ denotes the quadratic variation process of X and $[X,X]^c$ is the continuous part of $[X,X]$. $[X,Y]$ is the quadratic covariation process (by some authors referred to as the bracket process) of X and Y.

Section 4 contains a discussion concerning the existence and uniqueness of the solution (V,L,U) to the underlying Skorokhod problem in which no assumptions about the structure of X are made, so it applies to the case where X is a general semimartingale. We will start by presenting two preliminary results.

Lemma 6.1 *V, L and U are semimartingales.*

Proof Since L and U are cadlag, increasing and finite (thus of bounded variation) it follows that they are semimartingales. Since $V = X + L - U$ and X is a semimartingale, the proof is concluded by noting that semimartingales form a vector space. □

Lemma 6.2 *It holds that* $[V,V]^c = [X,X]^c$.

Proof $L-U$ is cadlag of bounded variation and it follows by Theorem 26, p. 71, in [123] that $[L-U,L-U]^c = 0$ which is well known to imply $[X,L-U]^c = 0$, see, e.g., Theorem 28, p. 75 in [123]. The claim now follows from

$$[V,V]^c = [X+L-U,X+L-U]^c = [X,X]^c+[L-U,L-U]^c+2[X,L-U]^c = [X,X]^c.$$

□

We now establish the link between (L,U) and X. We choose to mainly focus on the local time U, by partly eliminating L, but it should be obvious how to obtain the corresponding results for L.

Theorem 6.3 *Let X be a semimartingale which is reflected at* 0 *and b. Then the following relationship holds.*

$$2bU(t) = V(0)^2 - V(t)^2 + 2\int_{0+}^{t} V(s-)\,\mathrm{d}X(s) + [X,X]^c(t) + J_R(t) \tag{70}$$

where J_R is pure jump, increasing and finite with

$$J_R(t) = \sum_{0<s\le t} \varphi\big(V(s-), \varDelta X(s)\big), \tag{71}$$

where

$$\varphi(x,y) = \begin{cases} -(x^2+2xy) & \textit{if } y \le -x, \\ y^2 & \textit{if } -x < y < b-x, \\ 2y(b-x)-(b-x)^2 & \textit{if } y \ge b-x. \end{cases}$$

Proof By the definition of the quadratic variation process $[V, V]$ and Lemma 6.2,

$$V(t)^2 - V(0)^2 - 2\int_{0+}^{t} V(s-)\mathrm{d}V(s) = [V, V](t) = [X, X]^c(t) + \sum_{0<s\le t} (\Delta V(s))^2. \tag{72}$$

Furthermore,

$$\mathrm{d}V(t) = \mathrm{d}X(t) + \mathrm{d}L(t) - \mathrm{d}U(t) \text{ and } V(s-) = V(s) - \Delta V(s),$$

so it follows by the formulation of the Skorokhod problem that

$$\begin{aligned}
&\int_{0+}^{t} V(s-)\mathrm{d}V(s) \\
&= \int_{0+}^{t} V(s-)\mathrm{d}X(s) + \int_{0+}^{t} (V(s) - \Delta V(s))\mathrm{d}L(s) - \int_{0+}^{t} (V(s) - \Delta V(s))\mathrm{d}U(s) \\
&= \int_{0+}^{t} V(s-)\mathrm{d}X(s) - \int_{0+}^{t} \Delta V(s)\mathrm{d}L(s) - bU(t) + \int_{0+}^{t} \Delta V(s)\mathrm{d}U(s) \\
&= \int_{0+}^{t} V(s-)\mathrm{d}X(s) - \sum_{0<s\le t} \Delta V(s)\Delta L(s) - bU(t) + \sum_{0<s\le t} \Delta V(s)\Delta U(s).
\end{aligned} \tag{73}$$

Then (70) and (71) follow by combining (72) and (73) with the fact that

$$\begin{aligned}
\Delta V(s) &= \max(\min(\Delta X(s), b - V(s-)), 0) + \min(\max(\Delta X(s), -V(s-)), 0), \\
\Delta V(s)\Delta L(s) &= -V(s-)(-\min(\Delta X(s) + V(s-), 0)), \\
\Delta V(s)\Delta U(s) &= (b - V(s-)) \max(\Delta X(s) + V(s-) - b, 0).
\end{aligned}$$

Since $0 \le \varphi(x, y) \le y^2$ it follows that $J_R(t)$ is increasing and that

$$J_R(t) \le \sum_{0<s\le t} (\Delta X(s))^2 \le [X, X](t) < \infty.$$

□

We will need the next result in order to go from the path-by-path representation in Theorem 6.3 to the loss rate ℓ^b.

Lemma 6.4 *Suppose that X is a Lévy process with characteristic triplet (μ, σ, ν) and $\mathbb{E}|X(1)| < \infty$. Let*

$$I(t) = \int_{0+}^{t} V(s-)\,\mathrm{d}X(s).$$

Then in the stationary case it holds that $\mathbb{E}_\pi I(t) = tm\mathbb{E}_\pi V(0)$.

Proof Let $\tilde{X}(t) = X(t) - \sum_{0<s\le t} \Delta X(s)\mathbb{1}(|\Delta X(s)| \ge 1)$, so that

$$I(t) = \int_{0+}^{t} V(s-)\mathrm{d}\tilde{X}(s) + \sum_{0<s\le t} V(s-)\Delta X(s)\mathbb{1}(|\Delta X(s)| \ge 1).$$

We let $\tilde{Y}(t) = \tilde{X}(t) - t\mathbb{E}_\pi \tilde{X}(1)$. Then $\tilde{Y}$ is a martingale (and thus a local martingale) and it follows by Theorem 29, p. 128, in [123] that

$$J(t) = \int_{0+}^{t} V(s-)\,\mathrm{d}\tilde{Y}(s)$$

is also a local martingale. Theorem 29, p. 75, in [123] tells us that

$$\begin{aligned}[J,J](t) = \int_{0+}^{t} V(s-)^2\mathrm{d}[\tilde{Y},\tilde{Y}](s) \;&=\; \int_{0+}^{t} V(s-)^2\mathrm{d}[\tilde{X},\tilde{X}](s) \;\le\; b^2[\tilde{X},\tilde{X}](t)\\ &=\; b^2\Big(\sigma^2 t + \sum_{0<s\le t} (\Delta X(s))^2\mathbb{1}(|\Delta X(s)| < 1)\Big)\end{aligned}$$

and it follows that $\mathbb{E}_\pi[J,J](t) < \infty$ for all $t \ge 0$, which implies that J is a martingale, see Corollary 3, p. 73 in [123]. Then $\mathbb{E}_\pi J(t) = \mathbb{E}_\pi J(0) = 0$, and thus

$$\mathbb{E}_\pi \int_{0+}^{t} V(s-)\,\mathrm{d}\tilde{X}(s) \;=\; \mathbb{E}_\pi \int_{0+}^{t} V(s-)\mathbb{E}_\pi\tilde{X}(1)\,\mathrm{d}s \;=\; t\mathbb{E}_\pi V(0)\mathbb{E}\tilde{X}(1).$$

Furthermore, since $\sum_{0<s\le t} \Delta X(s)\mathbb{1}(|\Delta X(s)| \ge 1)$ is a compound Poisson process and $V(s-)$ is independent of $\Delta X(s)$, we get that

$$\mathbb{E}_\pi \sum_{0<s\le t} V(s-)\Delta X(s)\mathbb{1}\big(|\Delta X(s)| \ge 1\big) \;=\; t\mathbb{E}_\pi V(0)\bigg(\int_1^{\infty} x\,\nu(\mathrm{d}x) + \int_{-\infty}^{-1} x\,\nu(\mathrm{d}x)\bigg)$$

and it follows that

$$\mathbb{E}_\pi I(t) \;=\; t\mathbb{E}_\pi V(0)\mathbb{E}\tilde{X}(1) + t\mathbb{E}_\pi V(0)\left(\int_1^{\infty} x\,\nu(\mathrm{d}x) + \int_{-\infty}^{-1} x\,\nu(\mathrm{d}x)\right) \;=\; tm\mathbb{E}_\pi V(0).$$

□

Remark 6.5 In the proof of Lemma 6.4, we used the intuitively obvious fact that $V(s-)$ and $\Delta X(s)$ are independent. For a formal proof, one may appeal to Campbell's formula (e.g. [28, p. 7]). More precisely, write the sum

$$\sum_{0\le s\le t} V(s-)\Delta X(s)\mathbb{1}\big(|\Delta X(s)| \ge 1\big)$$

as

$$\sum_{0\le s<\infty} H_s(\Delta X(s)\mathbb{1}(\Delta X(s)\ge 1)) - \sum_{0\le s<\infty} H_s(\Delta X(s)\mathbb{1}(\Delta X(s)\le -1)), \tag{74}$$

where $H_s(x) = |x|V(s-)\mathbb{1}(s \le t)$. We see that H_s (viewed as a process indexed by s taking values in the space of nonnegative measurable functions on $\mathbb{R}$) is predictable. This is the formal equivalent to the independence argument used above. By applying Campbell's formula separately to each part in (74), we get

$$\begin{aligned}
&\mathbb{E}_\pi \sum_{0\le s\le t} V(s-)\Delta X(s)\mathbb{1}(|\Delta X(s)|\ge 1)\\
&= \mathbb{E}_\pi \sum_{0\le s<\infty} H_s(\Delta X(s)\mathbb{1}(\Delta X(s)\ge 1)) - \mathbb{E}_\pi \sum_{0\le s<\infty} H_s(\Delta X(s)\mathbb{1}(\Delta X(s)\le -1))\\
&= \mathbb{E}_\pi \int_0^\infty \mathrm{d}s \int_1^\infty \nu(\mathrm{d}x)H_s(x) - \mathbb{E}_\pi \int_0^\infty \mathrm{d}s \int_{-\infty}^{-1} \nu(\mathrm{d}x)H_s(x)\\
&= t\mathbb{E}_\pi V(0)\int_1^\infty x\nu(\mathrm{d}x) + t\mathbb{E}_\pi V(0)\int_{-\infty}^{-1} x\nu(\mathrm{d}x).
\end{aligned}$$

Similar arguments are tacitly used in other parts of the paper, in particular in the proofs of Corollary 6.6, Lemma 8.4, Eqs. (106)–(108), (142), Theorem 14.3, and the calculations leading to (201). □

The next corollary is an easy consequence of Theorem 6.3 and Lemma 6.4 and is precisely the main result in the paper [16]. Note that as we keep b fixed it is no restriction to assume that the support of ν is $[-a,\infty)\setminus\{0\}$ for some $a \ge b$. Otherwise we just truncate ν at $-a$ (we then get a point mass of size $\nu((-\infty,-a])$ at $-a$). The truncation does not affect V and hence not ℓ^b.

Corollary 6.6 *Let X be a Lévy process with characteristic triplet (μ,σ,ν). If $\int_1^\infty y\nu(\mathrm{d}y) = \infty$, then $\ell^b = \infty$ and otherwise*

$$\ell^b = \frac{1}{2b}\left\{2m\mathbb{E}V + \sigma^2 + \int_0^b \pi(\mathrm{d}x)\int_{-\infty}^{\infty} \varphi(x,y)\nu(\mathrm{d}y)\right\}. \tag{75}$$

Proof The first part is obvious. The second part follows immediately from (70) and (71) and Lemma 6.4 if we note that for a Lévy process $[X,X]^c(t) = \sigma^2 t$. □

7 Two Martingales

We will need nothing more sophisticated here than taking the property of $\{M(t)\}_{t\geq 0}$ to be a martingale as

$$\mathbb{E}\big[M(t+s)\,\big|\,\mathscr{F}(t)\big] \;=\; M(t),\quad t\geq 0,\; s>0\,, \tag{76}$$

where $\{\mathscr{F}(t)\}_{t\geq 0}$ is the natural filtration generated by the Lévy process, i.e. $\mathscr{F}(t) = \sigma\big(X(v) : 0 \leq v \leq t\big)$.

The applications of martingales in the present context are typically optional stopping, i.e. the identity $\mathbb{E}M(\tau) = M(0)$ for a stopping time τ when $M(0)$ is deterministic or $\mathbb{E}M(\tau) = \mathbb{E}M(0)$ in the general case. This is not universally true, but conditions need to be verified, for example

$$\mathbb{E}\sup_{t\leq\tau}\big|M(t)\big| \;<\; \infty\,. \tag{77}$$

7.1 The Wald Martingale

A classical example in the area of Lévy processes is the Wald martingale given by

$$M(t) = \mathrm{e}^{\alpha X(t) - t\kappa(\alpha)}\,. \tag{78}$$

The proof that this is a martingale is elementary using the property of independent stationary increments and the definition of the Lévy exponent κ.

Remark 7.1 For the Wald martingale $\mathrm{e}^{\theta X(t)-t\kappa(\theta)}$, there is an usually easier approach to justify stopping than (77): consider the exponentially tilted Lévy process with $\kappa_\theta(\alpha) = \kappa(\alpha+\theta) - \kappa(\theta)$. Then optional stopping is permissible if and only if $\mathbb{P}_\theta(\tau<\infty) = 1$. See [11, p. 362]. □

Example 7.2 Consider Brownian motion with drift μ and variance constant σ^2, and the problem of computing the two-sided exit probability

$$\mathbb{P}\big(X\big(\tau[x-b,x)\big)\geq x\big) \;=\; \pi^b[x,b]$$

occurring in the calculation of the stationary distribution π^b.

We have $\kappa(\alpha) = \alpha\mu + \alpha^2\sigma^2/2$, and take $\alpha = \gamma = -2\mu/\sigma^2$ as the root of the Lundberg equation $\kappa(\alpha) = 0$. Then the martingale is $\mathrm{e}^{\gamma X(t)}$. Condition (77) holds for $\tau = \tau[x-b,x)$ since $x-b\leq X(t)\leq x$ for $t\leq\tau[x-b,x)$. Letting

$$p^+(x) \;=\; \mathbb{P}\big(X\big(\tau[x-b,x)\big)\geq x\big) \;=\; \mathbb{P}\big(X\big(\tau[x-b,x)\big)=x\big)\,,$$
$$p^-(x) \;=\; \mathbb{P}\big(X\big(\tau[x-b,x)\big)< x-b\big) \;=\; \mathbb{P}\big(X\big(\tau[x-b,x)\big)=x-b\big)$$

(note the path properties of Brownian motion for the second expression!), optional stopping thus gives

$$1 \;=\; M(0) \;=\; \mathbb{E}M\big(\tau[x-b,x)\big) \;=\; p^+(x)\mathrm{e}^{\gamma x} + p^-(x)\mathrm{e}^{\gamma(x-b)}\,.$$

Together with $1 = p^+(x) + p^-(x)$ this gives

$$p^+(x) \;=\; \frac{1-\mathrm{e}^{\gamma(x-b)}}{\mathrm{e}^{\gamma x}-\mathrm{e}^{\gamma(x-b)}} \;=\; \frac{\mathrm{e}^{-\gamma x}-\mathrm{e}^{-\gamma b}}{1-\mathrm{e}^{-\gamma b}}\,. \tag{79}$$

The last expression identifies π^b as the distribution of an exponential r.v. W conditioned to $[0,b]$ when $\gamma > 0$, (i.e. $\mu < 0$) and of $b - W$ when $\gamma < 0$ (i.e. $\mu > 0$). □

Example 7.3 Consider again the Brownian setting, but now with the problem of computing quantities like

$$r^+ = \mathbb{E}[\mathrm{e}^{-q\tau}; X(\tau) = v]\,, \quad r^- = \mathbb{E}[\mathrm{e}^{-q\tau}; X(\tau) = u]\,, \quad r = r^+ + r^- = \mathbb{E}\mathrm{e}^{-q\tau}$$

where $\tau = \inf\{t : X(t) \notin [u, v]\}$ with $u < 0 < v$ and $q > 0$, as occurring in the calculation of the scale function.

We take α as root of

$$q \;=\; \kappa(\alpha) \;=\; \alpha\mu + \alpha^2\sigma^2/2$$

(rather than the Lundberg equation $\kappa(\alpha) = 0$). Since $q > 0$, there are two roots, one positive and one negative,

$$\theta^+ \;=\; \theta^+(q) \;=\; \frac{-\mu + \sqrt{\mu^2 + 2\sigma^2 q}}{2}\,, \quad \theta^- \;=\; \theta^-(q) \;=\; \frac{-\mu - \sqrt{\mu^2 + 2\sigma^2 q}}{2}\,.$$

We therefore have two Wald martingales at disposal, $\mathrm{e}^{\theta^+ X(t)-qt}$ and $\mathrm{e}^{\theta^- X(t)-qt}$.

Instead of verifying condition (77) (trivial for θ^+ and by a symmetry argument also for θ^-!), it is easier to note that in the present context, we have $\tau < \infty$ for all μ, and this implies the conditions of Remark 7.1. Optional stopping thus gives

$$1 \;=\; r^+\mathrm{e}^{\theta^+ v} + r^+ r^- \mathrm{e}^{\theta^+ v} \quad 1 \;=\; r^+\mathrm{e}^{\theta^- v} + r^+ r^- \mathrm{e}^{\theta^- v}\,.$$

These two linear equations can immediately be solved for r^+, r^-, and then also $r = r^+ + r^-$ is available. □

Example 7.4 Consider again the two-sided exit problem, but now with exponential(δ) jumps at rate λ in the positive directions added to the Brownian motion.

Inspired by Examples 7.2 and 7.3 we look for solutions of the Lundberg equation

$$0 \;=\; \kappa(\alpha) \;=\; \alpha\mu + \alpha^2\sigma^2/2 + \lambda\frac{\delta}{\delta-\alpha}\,.$$

This is a cubic, which looks promising since we have three unknowns, the probability of exit below at u, the probability of continuous exit above at v, and the probability of exit above by a jump. However, only two of the three roots θ satisfy $\mathbb{E}\mathrm{e}^{\Re(\alpha)X(1)} < \infty$ and so the third does not lead to a permissible Wald martingale. Thus, additional ideas are needed to deal with this example. This is done next. □

7.2 *The Kella-Whitt Martingale*

Consider a modification $Z(t) = X(t) + B(t)$ of the Lévy process, where $\{B(t)\}_{t\ge 0}$ is adapted with D-paths, locally bounded variation, continuous part $\{B^c(t)\}$, and jumps $\Delta B(s) = B(s) - B(s-)$. The Kella-Whitt martingale is given by

$$\begin{aligned}\kappa(\alpha)\int_0^t \mathrm{e}^{\alpha Z(s)}\,\mathrm{d}s \;+\; \mathrm{e}^{\alpha Z(0)} \;-\; \mathrm{e}^{\alpha Z(t)}& \\ +\;\alpha\int_0^t \mathrm{e}^{\alpha Z(s)}\,\mathrm{d}B^c(s) \;+\; \sum_{0\le s\le t}\mathrm{e}^{\alpha Z(s)}\big(1-\mathrm{e}^{-\alpha\Delta B(s)}\big)\,.& \qquad (80)\end{aligned}$$

Since the Kella-Whitt martingale (80) is less standard than the Wald martingale (78), we add some discussion and references. The first occurrence is in Kella and Whitt [91] where it was identified as a rewriting of the stochastic integral

$$\int_0^t \exp\big\{\alpha\big(X(s-)+B(s-)\big) + s\kappa(\alpha)\big\}\,\mathrm{d}W(s)$$

where W is the Wald martingale. The stochastic integral representation immediately gives the *local* martingale property. To proceed from this, much subsequent work next shows the *global* martingale property by direct calculations specific for the particular application. However, recently Kella and Boxma [88] showed that this is automatic under minor conditions.

A simple but still useful case is the Kella-Whitt martingale with $B(t) \equiv 0$,

$$\kappa(\alpha)\int_0^t \mathrm{e}^{\alpha X(s)}\,\mathrm{d}s \;+\; \mathrm{e}^{\alpha x} \;-\; \mathrm{e}^{\alpha X(t)} \qquad (81)$$

A survey of applications of the Kella-Whitt martingale is in Asmussen [11, IX.3]; see also Kyprianou [104] and [105].

8 The Loss Rate via the Kella-Whitt Martingale

In this section we summarize the original derivation of the loss rate $\ell = \ell^b$ which is presented in Asmussen and Pihlsgård [16]. It is essentially based on optional stopping of the Kella-Whitt martingale for V. As stated in Sects. 1 and 6, this is less straightforward than the direct Itô integration method used in Sect. 6. It is not difficult to see why the latter approach leads more directly to the result: the Kella-Whitt martingale, see Kella and Whitt [91], is itself obtained as a stochastic integral with respect to the Wald martingale (indexed by, say, α) for V, so this method implicitly relies on Itô's formula and, more importantly, there is introduced an arbitrariness via α which is removed by letting $\alpha \to 0$. This requires a delicate analysis, which is to a large extent based on Taylor expansions and tedious algebra, and hence of limited probabilistic interest. This is perhaps the most serious drawback of the original approach. However, the martingale technique also has advantages. E.g., if the process X is such that the equation $\kappa(\alpha) = 0$ has a non-zero root γ, we obtain an alternative formula for ℓ, see Theorem 8.6 below, which turns out to be very useful when we derive asymptotics for ℓ^b as $b \to \infty$ when X is light tailed. We see no immediate way of deriving this result directly via Itô's formula.

To follow the exposition in [16], we need to introduce some further notation. First, we split L and U into their continuous and jump parts, i.e.,

$$L(t) = L_c(t) + L_j(t) \text{ and } U(t) = U_c(t) + U_j(t) \tag{82}$$

where $L_c(t)$ is the continuous part of L, $L_j(t)$ the jump part etc., i.e., $L_j(t) = \sum_{0\le s\le t} \Delta L(s)$ and $L_c(t) = L(t) - L_j(t)$. Further, we treat the contributions to L and U coming from small and large jumps of X separately: let

$$\begin{aligned}\underline{\Delta} L(s) &= \Delta L(s)\mathbb{1}\big(-k \le \Delta X(s) \le 0\big), \quad \overline{\Delta} L(s) = \Delta L(s)\mathbb{1}\big(\Delta X(s) < -k\big), \\ \underline{\Delta} U(s) &= \Delta U(s)\mathbb{1}\big(0 \le \Delta X(s) \le k\big), \quad \overline{\Delta} U(s) = \Delta U(s)\mathbb{1}\big(\Delta X(s) > k\big)\end{aligned}$$

where k is a constant such that $k > \max(1, b)$. Further, we let

$$\ell_j^b = \mathbb{E} U_j(1), \quad \ell_c^b = \mathbb{E} U_c(1), \quad \underline{\ell}_j^b = \mathbb{E} \sum_{0\le s\le 1} \underline{\Delta} U(s), \quad \overline{\ell}_j^b = \mathbb{E} \sum_{0\le s\le 1} \overline{\Delta} U(s),$$

and similarly at 0. Clearly $\ell_j^b = \underline{\ell}_j^b + \overline{\ell}_j^b$ and $\ell_j^0 = \underline{\ell}_j^0 + \overline{\ell}_j^0$. The Lévy exponent $\kappa(\alpha)$ can be rewritten as

$$c_k\alpha + \sigma^2\alpha^2/2 + \int_{-\infty}^{\infty} \big[e^{\alpha x} - 1 - \alpha x \mathbb{1}\big(|x| \le k\big)\big]\, \nu(dx), \tag{83}$$

where $c_k = c + \int_1^k y\, \nu(dy) + \int_{-k}^{-1} y\, \nu(dy)$.

The paper [16] relies on the original reference on the Kella-Whitt local martingale associated with Lévy processes, [91], and on Asmussen and Kella [15] for the generalisation to a multidimensional local martingale associated with Markov additive processes with finite state space Markov modulation. However, it was recently discovered in Kella and Boxma [88] that without any further assumptions, these local martingales are in fact martingales. This very useful result makes it possible to keep the treatment below slightly shorter than what was presented in [16].

The first step in the analysis is to show that ℓ^0 and ℓ^b are well defined if the process X is sufficiently well behaved.

Lemma 8.1 *If* $\mathbb{E}|X(1)| < \infty$ *then* $\mathbb{E}L(t) < \infty$ *and* $\mathbb{E}U(t) < \infty$ *(for all t).*

Proof Assume (without loss of generality) that $V(0) = 0$. Let $\tau(0) = 0$ and, for $j \geq 1$, $\upsilon(j) = \inf\{t > \tau(j-1) : V(t) = b\}$, $\tau(j) = \inf\{t > \upsilon(j) : V(t) = 0\}$. We view V as regenerative with ith cycle equal to $[\tau(i-1), \tau(i))$. Let $n(t)$ denote the number of cycles completed in $[0, t]$. Clearly, $\mathbb{E}(\tau(i) - \tau(i-1)) > 0$ and this implies that $\mathbb{E}n(t) < \infty$, see Proposition 1.4, p. 140 in [11]. We have

$$L(t) = \sum_{i=1}^{n(t)} C_i + R(t) \tag{84}$$

(we use the convention that $\sum_{i=1}^{0} C_i = 0$) where C_i is the contribution to $L(t)$ from the ith cycle and $R(t)$ is what comes from $[\tau(n(t)), t]$. Let $m(t)$ be the local time corresponding to X one-sided reflected at 0 ($m(t) = -\inf_{0 \leq s \leq t} X(s)$). By the strong Markov property of X and the fact that V and the process starting from 0 at time $\tau(i-1)$ resulting from one-sided reflection of X coincide on $[\tau(i-1), \upsilon(i))$, we have $C_i \stackrel{\mathscr{D}}{=} m(\upsilon(1)) + J_1$, where J_1 comes from a jump of X ending the cycle. For fixed t, the initial parts of the cycles (from 0 up to b) can influence $L(t)$ only through what occurs during $[\tau(i-1), (\tau(i-1) + t) \wedge \upsilon(i)]$, so in (84) we may replace C_i by $C_i(t)$ where $C_i(t) \stackrel{\mathscr{D}}{=} m(\upsilon(1) \wedge t) + J_1$. Now,

$$J_1 \leq 1 \vee \max_{0 \leq s \leq t} |\Delta X(s)| \mathbb{1}\big(|\Delta X(s)| \geq 1\big) \leq 1 + \sum_{0 \leq s \leq t} |\Delta X(s)| \mathbb{1}\big(|\Delta X(s)| \geq 1\big),$$

so $\mathbb{E}J_1 \leq 1 + \mathbb{E}\sum_{0 \leq s \leq t} |\Delta X(s)| \mathbb{1}\big(|\Delta X(s)| \geq 1\big) = 1 + t\int_{|x| \geq 1} |x| \nu(\mathrm{d}x) < \infty$ (recall that we assume that $\mathbb{E}|X(1)| < \infty$). It is known, see Lemma 3.3, p. 256 in [11], or Theorem 25.18, p. 166 in [129], that $\mathbb{E}m(t) < \infty$, so $\mathbb{E}m(\upsilon(1) \wedge t) \leq \mathbb{E}m(t) < \infty$, which together with $\mathbb{E}J_1 < \infty$ yields $\mathbb{E}C_i(t) < \infty$. In a similar way, we see that $\mathbb{E}R(t) \leq \mathbb{E}m(t) < \infty$. It now follows from Wald's identity that

$$\begin{aligned} \mathbb{E}L(t) &= \mathbb{E}\sum_{i=1}^{n(t)} C_i(t) + \mathbb{E}R(t) \leq \mathbb{E}\sum_{i=1}^{n(t)+1} C_i(t) + \mathbb{E}R(t) \\ &= (\mathbb{E}n(t) + 1)\mathbb{E}C_1(t) + \mathbb{E}R(t) < \infty. \end{aligned}$$

$\mathbb{E}U(t) < \infty$ is now immediate from the formulation of the Skorokhod problem. □

The next step is the construction of the Kella-Whitt martingale for the reflected process V.

Proposition 8.2 *Assume that $\mathbb{E}|X(1)| < \infty$. For each t, let $M(t)$ be the random variable*

$$\kappa(\alpha)\int_0^t e^{\alpha V(s)}ds + e^{\alpha V(0)} - e^{\alpha V(t)} + \alpha\int_0^t e^{\alpha V(s)}dL_c(s) + \sum_{0\le s\le t} e^{\alpha V(s)}(1-e^{-\alpha\Delta L(s)})$$
$$-\alpha\int_0^t e^{\alpha V(s)}dU_c(s) + \sum_{0\le s\le t} e^{\alpha V(s)}(1-e^{\alpha\Delta U(s)}).$$

Then

$$M(t) = \kappa(\alpha)\int_0^t e^{\alpha V(s)}ds + e^{\alpha V(0)} - e^{\alpha V(t)} + \alpha L_c(t) + \sum_{0\le s\le t}(1-e^{-\alpha\Delta L(s)})$$
$$-\alpha e^{\alpha b}U_c(t) + e^{\alpha b}\sum_{0\le s\le t}(1-e^{\alpha\Delta U(s)}) \tag{85}$$

and M is a zero mean martingale.

Proof L and U solve the Skorokhod problem stated in Sect. 1, so the first claim is clearly true. $L-U$ is of bounded variation and it follows by what was proved in [88] that M is a martingale. □

We proceed by stating two lemmas.

Lemma 8.3 *ℓ^b satisfies the following equation:*

$$\alpha(1-e^{\alpha b})\ell^b = -\kappa(\alpha)\mathbb{E}_\pi e^{\alpha V(0)} + \alpha\mathbb{E}_\pi X(1) - \alpha e^{\alpha b}\overline{\ell}_j^b + \alpha\overline{\ell}_j^0 + \frac{\alpha^2}{2}\mathbb{E}_\pi\sum_{0\le s\le 1}(\underline{\Delta}U(s))^2$$
$$+\frac{\alpha^2}{2}\mathbb{E}_\pi\sum_{0\le s\le 1}(\underline{\Delta}L(s))^2 - e^{\alpha b}\mathbb{E}_\pi\sum_{0\le s\le 1}(1-e^{\alpha\overline{\Delta}U(s)})$$
$$-\mathbb{E}_\pi\sum_{0\le s\le 1}(1-e^{-\alpha\overline{\Delta}L(s)}) + o(\alpha^2), \tag{86}$$

where $o(\alpha^2)/\alpha^2 \to 0$ if $\alpha \to 0$.

Proof If we take $t = 1$ in Proposition 8.2 and use the stationarity of V, we get

$$0 = \kappa(\alpha)\mathbb{E}_\pi e^{\alpha V(0)} + \alpha\ell_c^0 + \mathbb{E}_\pi\sum_{0\le s\le 1}(1-e^{-\alpha\Delta L(s)}) - \alpha e^{\alpha b}\ell_c^b + e^{\alpha b}\mathbb{E}_\pi\sum_{0\le s\le 1}(1-e^{\alpha\Delta U(s)}). \tag{87}$$

We write

$$\sum_{0\le s\le 1}(1-\mathrm{e}^{\alpha\Delta U(s)}) = \sum_{0\le s\le 1}(1-\mathrm{e}^{\alpha\underline{\Delta}U(s)}) + \sum_{0\le s\le 1}(1-\mathrm{e}^{\alpha\overline{\Delta}U(s)}), \quad (88)$$

$$\sum_{0\le s\le 1}(1-\mathrm{e}^{-\alpha\Delta L(s)}) = \sum_{0\le s\le 1}(1-\mathrm{e}^{-\alpha\underline{\Delta}L(s)}) + \sum_{0\le s\le 1}(1-\mathrm{e}^{-\alpha\overline{\Delta}L(s)}) \quad (89)$$

and apply the expansion

$$\mathrm{e}^{\alpha x} = 1+\alpha x+\frac{(\alpha x)^2}{2}+\frac{(\alpha x)^3}{6}\mathrm{e}^{\theta\alpha x},\ \theta\in(0,1) \quad (90)$$

to the first parts of the r.h.s. of (88) and (89) and get for the part in (88):

$$\begin{aligned}
\mathrm{e}^{\alpha b}\mathbb{E}_\pi\sum_{0\le s\le 1}(1-\mathrm{e}^{\alpha\underline{\Delta}U(s)}) &= \mathrm{e}^{\alpha b}\mathbb{E}_\pi\Big(-\alpha\sum_{0\le s\le 1}\underline{\Delta}U(s)-\frac{\alpha^2}{2}\sum_{0\le s\le 1}(\underline{\Delta}U(s))^2\Big)+\mathrm{o}(\alpha^2)\\
&= -\alpha\mathrm{e}^{\alpha b}\underline{\ell}_j^b-\mathrm{e}^{\alpha b}\frac{\alpha^2}{2}\mathbb{E}_\pi\sum_{0\le s\le 1}(\underline{\Delta}U(s))^2+\mathrm{o}(\alpha^2)\\
&= -\alpha\mathrm{e}^{\alpha b}(\ell_j^b-\overline{\ell}_j^b)-\frac{\alpha^2}{2}\mathbb{E}_\pi\sum_{0\le s\le 1}(\underline{\Delta}U(s))^2+\mathrm{o}(\alpha^2),
\end{aligned} \quad (91)$$

because $\mathbb{E}_\pi\sum_{0\le s\le 1}\alpha^3(\underline{\Delta}U(s))^3\mathrm{e}^{\theta\alpha\underline{\Delta}U(s)}/6=\mathrm{o}(\alpha^2)$, $\ell_j^b=\underline{\ell}_j^b+\overline{\ell}_j^b$ and $\mathrm{e}^{\alpha b}\alpha^2/2=\alpha^2/2+\mathrm{o}(\alpha^2)$. We proceed similarly for the part in (89) and get

$$\mathbb{E}_\pi\sum_{0\le s\le 1}(1-\mathrm{e}^{-\alpha\underline{\Delta}L(s)})=\alpha(\ell_j^0-\overline{\ell}_j^0)-\frac{\alpha^2}{2}\mathbb{E}_\pi\sum_{0\le s\le 1}(\underline{\Delta}L(s))^2+\mathrm{o}(\alpha^2). \quad (92)$$

If we combine (87)–(89), (91) and (92) we get

$$\begin{aligned}
0 &= \kappa(\alpha)\mathbb{E}_\pi\mathrm{e}^{\alpha V(0)}+\alpha\ell^0-\alpha\mathrm{e}^{\alpha b}\ell^b-\alpha\overline{\ell}_j^0+\alpha\mathrm{e}^{\alpha b}\overline{\ell}_j^b-\frac{\alpha^2}{2}\mathbb{E}_\pi\sum_{0\le s\le 1}(\underline{\Delta}U(s))^2\\
&\quad -\frac{\alpha^2}{2}\mathbb{E}_\pi\sum_{0\le s\le 1}(\underline{\Delta}L(s))^2+\mathrm{e}^{\alpha b}\mathbb{E}_\pi\sum_{0\le s\le 1}(1-\mathrm{e}^{\alpha\overline{\Delta}U(s)})\\
&\quad +\mathbb{E}_\pi\sum_{0\le s\le 1}(1-\mathrm{e}^{-\alpha\overline{\Delta}L(s)})+\mathrm{o}(\alpha^2).
\end{aligned}$$

The claim now follows if we make the substitution $\ell^0=\ell^b-\mathbb{E}_\pi X(1)$ and rearrange terms. □

Lemma 8.4 *Let, for $x > 0$, $\overline{\nu}(x) = \nu(x,\infty)$ and, for $x < 0$, $\underline{\nu}(x) = \nu(-\infty,x)$. In stationarity it then holds that as $\alpha \to 0$,*

$$\kappa(\alpha)\mathbb{E}_\pi e^{\alpha V(t)} = o(\alpha^2) + \int_0^b e^{\alpha x}\pi(dx)\int_{-\infty}^{\infty} e^{\alpha y}\mathbb{1}(|y|\geq k)\nu(dy)$$
$$- \int_{-\infty}^{\infty} I(|y|\geq k)\nu(dy) + \alpha\left(c_k - \int_0^b x\pi(dx)\int_{-\infty}^{\infty}\mathbb{1}(|y|\geq k)\nu(dy)\right)$$
$$+ \alpha^2\left(c_k\int_0^b x\pi(dx) + \sigma^2/2 + \int_{-k}^{k} y^2/2\nu(dy)\right.$$
$$\left. - \int_0^b x^2/2\pi(dx)\int_{-\infty}^{\infty}\mathbb{1}(|y|\geq k)\nu(dy)\right), \tag{93}$$

$$e^{\alpha b}\mathbb{E}_\pi \sum_{0\leq s\leq 1}(1 - e^{\alpha\overline{\Delta}U(s)}) = e^{\alpha b}\int_0^b \pi(dx)\int_k^{\infty}(1 - e^{\alpha(y-b+x)})\nu(dy)$$
$$= (1 + \alpha b + \alpha^2 b^2/2)\overline{\nu}(k)$$
$$- \int_0^b e^{\alpha x}\pi(dx)\int_k^{\infty} e^{\alpha y}\nu(dy) + o(\alpha^2), \tag{94}$$

$$\mathbb{E}_\pi \sum_{0\leq s\leq 1}(1 - e^{-\alpha\overline{\Delta}L(s)}) = \int_0^b \pi(dx)\int_{-\infty}^{-k}(1 - e^{\alpha(x+y)})\nu(dy)$$
$$= \underline{\nu}(-k) - \int_0^b e^{\alpha x}\pi(dx)\int_{-\infty}^{-k} e^{\alpha y}\nu(dy) + o(\alpha^2), \tag{95}$$

$$\alpha e^{\alpha b}\overline{\ell}_j^b = \alpha e^{\alpha b}\int_0^b \pi(dx)\int_k^{\infty}(y - b + x)\nu(dy)$$
$$= (\alpha + \alpha^2 b)\int_0^b \pi(dx)\int_k^{\infty}(y - b + x)\nu(dy) + o(\alpha^2), \tag{96}$$

$$\alpha\overline{\ell}_j^0 = -\alpha\int_0^b \pi(dx)\int_{-\infty}^{-k}(x + y)\nu(dy), \tag{97}$$

$$\alpha m = \alpha c_k + \alpha\int_{-\infty}^{\infty} y\mathbb{1}(|y|\geq k)\nu(dy), \tag{98}$$

$$\frac{\alpha^2}{2}\mathbb{E}_\pi \sum_{0\le s\le 1} (\underline{\Delta} U(s))^2 = \frac{\alpha^2}{2}\int_0^b \pi(\mathrm{d}x)\int_{b-x}^{k} (y-b+x)^2\nu(\mathrm{d}y), \tag{99}$$

$$\frac{\alpha^2}{2}\mathbb{E}_\pi \sum_{0\le s\le 1} (\underline{\Delta} L(s))^2 = \frac{\alpha^2}{2}\int_0^b \pi(\mathrm{d}x)\int_{-k}^{-x} (x+y)^2\nu(\mathrm{d}y). \tag{100}$$

Proof Clearly $\mathbb{E}_\pi \mathrm{e}^{\alpha V(s)} = \int_0^b \mathrm{e}^{\alpha x}\pi(\mathrm{d}x)$. Equation (93) follows if we use the representation for $\kappa(\alpha)$ in (83) and expand the integrands corresponding to the compact sets $[-k,k]$ and $[0,b]$ according to (90). The remaining statements all follow by conditioning on $V(s-)$ and applying (90) where appropriate. □

We are now ready to identify ℓ^b in terms of π and (c,σ^2,ν). We recall the remark just before Corollary 6.6.

Theorem 8.5 *If* $\int_1^\infty y\nu(\mathrm{d}y) = \infty$, *then* $\ell^b = \infty$ *and otherwise*

$$\ell^b = \frac{1}{2b}\left\{2m\mathbb{E}_\pi V + \sigma^2 + \int_0^b \pi(\mathrm{d}x)\int_{-\infty}^{\infty} \varphi(x,y)\nu(\mathrm{d}y)\right\}, \tag{101}$$

where

$$\varphi(x,y) = \begin{cases} -(x^2+2xy) & \text{if } y\le -x,\\ y^2 & \text{if } -x<y<b-x,\\ 2y(b-x)-(b-x)^2 & \text{if } y\ge b-x. \end{cases}$$

Proof The first claim is obvious. We use (86) and identify the terms in the right hand side via Lemma 8.4 and get,

$$\begin{aligned}\alpha(1-\mathrm{e}^{\alpha b})\ell^b = &-c_k\alpha^2\int_0^b x\pi(\mathrm{d}x) - \frac{\sigma^2\alpha^2}{2} - \frac{\alpha^2}{2}\int_0^b \pi(\mathrm{d}x)\int_0^{b-x} y^2\nu(\mathrm{d}y)\\ &- \frac{\alpha^2}{2}\int_0^b \pi(\mathrm{d}x)\int_{-x}^{0} y^2\nu(\mathrm{d}y) + (\overline{\nu}(k)+\underline{\nu}(-k))\frac{\alpha^2}{2}\int_0^b x^2\pi(\mathrm{d}x)\\ &+ \frac{\alpha^2}{2}\int_0^b \pi(\mathrm{d}x)\int_{b-x}^{k} ((x-b)^2+2y(x-b))\nu(\mathrm{d}y)\\ &- \alpha^2 b\int_k^\infty y\nu(\mathrm{d}y) + \frac{\alpha^2}{2}\int_0^b \pi(\mathrm{d}x)\int_{-k}^{-x} (x^2+2xy)\nu(\mathrm{d}y)\\ &+ \alpha^2 b\int_0^b (b-x)\pi(\mathrm{d}x)\overline{\nu}(k) + \mathrm{o}(\alpha^2).\end{aligned} \tag{102}$$

We divide both sides of (102) by $\alpha(1-\mathrm{e}^{\alpha b})$ and let first $\alpha\to 0$ and then $k\to\infty$ and get the limit (101) (note that $c_k \to \mathbb{E}X(1)$ as $k\to\infty$). □

The next result, which follows almost directly from the proof of Lemma 8.3, gives an alternative expression for ℓ^b whenever we can find a non-zero root γ of $\kappa(\alpha) = 0$ (a genuine root in the sense that $\mathrm{e}^{\gamma X(1)}$ has finite expectation, cf. Lemma 10.7 where the meaning of $\kappa(\alpha) = 0$ is different). Note that in the original version (Theorem 3.2 in [16]) it is required that γ is real but this is not necessary.

Theorem 8.6 *Assume that there exists a non-zero root γ of the equation $\kappa(\alpha) = 0$. Then*

$$\ell^b = \frac{1}{\mathrm{e}^{\gamma b} - 1}\left\{\mathrm{e}^{\gamma b} I_1 + I_2 - \mathbb{E}X(1)\right\} \tag{103}$$

where

$$I_1 = \int_0^b \pi(\mathrm{d}x) \int_{b-x}^{\infty} ((y - b + x) + \gamma^{-1}(1 - \mathrm{e}^{\gamma(y-b+x)}))\nu(\mathrm{d}y)$$

$$I_2 = \int_0^b \pi(\mathrm{d}x) \int_{-\infty}^{-x} ((x + y) + \gamma^{-1}(1 - \mathrm{e}^{\gamma(x+y)}))\nu(\mathrm{d}y)$$

Proof Let $\varepsilon > 0$. We truncate the Lévy measure at ε and $-\varepsilon$. By arguing precisely as when we derived (86) and taking $\alpha = \gamma$, we get

$$\gamma(\mathrm{e}^{\gamma b} - 1)\ell^b = -\gamma\mathbb{E}X(1) + \mathrm{e}^{\gamma b} I_1^{\varepsilon} + I_2^{\varepsilon} + \mathrm{O}(\varepsilon), \tag{104}$$

where

$$I_1^{\varepsilon} = \int_0^b \pi(\mathrm{d}x) \int_{(b-x)\vee\varepsilon}^{\infty} (\gamma(y - b + x) + (1 - \mathrm{e}^{\gamma(y-b+x)}))\nu(\mathrm{d}y),$$

$$I_2^{\varepsilon} = \int_0^b \pi(\mathrm{d}x) \int_{-\infty}^{-(x\vee\varepsilon)} (\gamma(x + y) + (1 - \mathrm{e}^{\gamma(x+y)}))\nu(\mathrm{d}y),$$

and the claim follows if we divide both sides of (104) by $\gamma(\mathrm{e}^{\gamma b} - 1)$, let $\varepsilon \downarrow 0$ and apply monotone convergence. □

As we have seen in Sect. 3.1, the identification of ℓ^b is almost trivial in the discrete time case. However, the continuous time case is much more involved and less intuitive, no matter the choice of method for deriving the expression(s) for ℓ^b (the direct Itô approach presented in Sect. 6 or the methods used in the current section). In order to provide the presentation with some intuition, we present an alternative heuristic derivation of the formula for ℓ^b as given in (103). Recall the definitions of ℓ_c^b, ℓ_j^b etc. given above. We will derive four equations involving ℓ_c^b, ℓ_j^b, ℓ_c^0 and ℓ_j^0 and solve for the unknowns. The first equation follows directly from the Skorokhod problem formulation and the stationarity of V:

$$\ell_c^0 + \ell_j^0 - \ell_c^b - \ell_j^b = -m. \tag{105}$$

The second equation is

$$\ell_j^b = \int_0^b \pi(\mathrm{d}x) \int_{b-x}^{\infty} (y - b + x)\nu(\mathrm{d}y) \tag{106}$$

and the third is

$$\ell_j^0 = -\int_0^b \pi(\mathrm{d}x) \int_{-\infty}^{-x} (x + y)\nu(\mathrm{d}y). \tag{107}$$

In order to obtain the fourth equation, we take $\alpha = \gamma$ in (85), which yields

$$\begin{aligned} \gamma\ell_c^0 - \gamma \mathrm{e}^{\gamma b}\ell_c^b = -\mathrm{e}^{\gamma b} \int_0^b \pi(\mathrm{d}x) \int_{b-x}^{\infty} (1 - \mathrm{e}^{\gamma(y-b+x)})\nu(\mathrm{d}y) \\ - \int_0^b \pi(\mathrm{d}x) \int_{-\infty}^{-x} (1 - \mathrm{e}^{\gamma(x+y)})\nu(\mathrm{d}y). \end{aligned} \tag{108}$$

Equations (106)–(108) apply at least if the jump part of X is of bounded variation, i.e., if $\int_{-1}^{1} |x|\nu(\mathrm{d}x) < \infty$. In this case they follow by straightforward conditioning on the value of V immediately prior to a jump of X. By combining (105)–(108), we may identify the unknowns and the expression for ℓ^b given in (103) follows from $\ell^b = \ell_c^b + \ell_j^b$. Note that if we take X to be a compound Poisson process with intensity β and jump distribution F, we get $\ell^b = \ell_j^b = \int_0^b \pi(\mathrm{d}x) \int_{b-x}^{\infty} (y - b + x)\beta F(\mathrm{d}y)$. Thus in this case the expression for ℓ^b is the same as in discrete time. If we compare this expression to (101), we see that for a compound Poisson process it must hold that

$$\int_0^b \pi(\mathrm{d}x) \int_{(b-x)}^{\infty} (y - b + x)F(\mathrm{d}y) = \int_0^b \pi(\mathrm{d}x) \int_{-\infty}^{\infty} (2b)^{-1}(2xy + \varphi(x, y))F(\mathrm{d}y).$$

Example 8.7 Assume that X is Brownian motion with drift μ and variance σ^2, i.e., $\kappa(\alpha) = \mu\alpha + \sigma^2\alpha^2/2$. Then $\gamma = -2\mu/\sigma^2$ and Theorem 8.6 gives us $\ell^b = -\mu/(\mathrm{e}^{-2b\mu/\sigma^2} - 1)$.

Example 8.8 Suppose that X is a strictly stable Lévy process with index $\alpha \in (0, 2) \setminus \{1\}$ (note that if $\alpha = 1$, then $\ell^b = \infty$ and if $\alpha = 2$, then $\ell^b = \sigma^2/2b$), i.e.,

$$\nu(\mathrm{d}x) = \begin{cases} c_+ x^{-(\alpha+1)}\mathrm{d}x & \text{if } x > 0, \\ c_- |x|^{-(\alpha+1)}\mathrm{d}x & \text{if } x < 0, \end{cases}$$

where $c_+, c_- \geq 0$ are such that $c_+ + c_- > 0$; see, e.g., Bertoin [28, pp. 216–218]. Let $\beta = (c_+ - c_-)/(c_+ + c_-)$ and $\rho = 1/2 + (\pi\alpha)^{-1} \arctan(\beta \tan(\pi\alpha/2))$.

If $\alpha \in (0, 1)$ then ℓ^b is 0 if $\beta = -1$ (then X is the negative of a subordinator) and ∞ otherwise. We now consider the case $\alpha \in (1, 2)$, which implies that

$\mathbb{E}X(1) = \sigma = 0$. It follows from Theorem 1 in Kyprianou [103] and some rescaling manipulations that if X is not spectrally one-sided, i.e., if $\rho \in (1 - 1/\alpha, 1/\alpha)$, then

$$\pi(\mathrm{d}x) = (bB(\alpha\rho, \alpha(1-\rho)))^{-1}(1 - x/b)^{\alpha\rho-1}(x/b)^{\alpha(1-\rho)-1}\mathrm{d}x,$$

where $B(\cdot,\cdot)$ is the beta function. Further, it turns out that in this example,

$$\int_{-\infty}^{\infty} \varphi(x,y)\nu(\mathrm{d}y) = 2(\alpha(\alpha-1)(2-\alpha))^{-1}(c_- x^{2-\alpha} + c_+(b-x)^{2-\alpha}),$$

and it follows from Theorem 8.5 (Theorem 3.1 in [16]) that

$$\ell^b = \frac{c_- B(2-\alpha\rho, \alpha\rho) + c_+ B(2-\alpha(1-\rho), \alpha(1-\rho))}{B(\alpha\rho, \alpha(1-\rho))\alpha(\alpha-1)(2-\alpha)b^{\alpha-1}}.$$

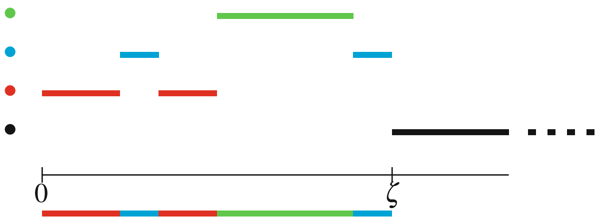
9 Phase-Type Jumps

A key step in the analysis of two-sided reflection is the computation of the stationary distribution or equivalently two-sided exit probabilities. This is not possible in general (at least there are no known methods), but requires additional structure. One example is the spectrally negative case with the scale function available. Another one, that we concentrate on here, is phase-type jumps in both directions and an added Brownian component. This class of Lévy models has the major advantage of being dense (in the sense of D-convergence) in the class of all Lévy processes. Further, not only are explicit computations available for two-sided exit probabilities but also in a number of other problems standard in fluctuation theory for Lévy processes, see the survey in Asmussen [12] and the extensive list of references there.

9.1 Phase-Type Distributions

Phase-type distributions are absorption time distributions in finite continuous-time Markov processes (equivalently, lifelength distributions in terminating finite Markov processes). Let $\{J(t)\}_{t\geq 0}$ be Markov with a finite state space $E \cup \{\Delta\}$ such that Δ is absorbing and the rest transient. That is, the process ends eventually up in Δ so that the absorption time (lifetime) $\zeta = \inf\{t : J(t) = \Delta\}$ is finite a.s. For $i, j \in E$, $i \neq j$, write t_{ij} for the transition rate $i \to j$ and t_i for the transition rate $i \to \Delta$. Define $t_{ii} = -t_i - \sum_{j\neq i} t_{ij}$ and let $\mathbf{T}$ be the $E \times E$ matrix with ijth element t_{ij}. If $\boldsymbol{\alpha}$ is an E-row vector with elements α_i summing to 1, we then define a phase-type (PH) distribution F with representation $(E, \boldsymbol{\alpha}, \mathbf{T})$ (or just $(\boldsymbol{\alpha}, \mathbf{T})$) as the distribution of ζ corresponding to the initial distribution $\mathbb{P}_{\boldsymbol{\alpha}}$ of $\{J(t)\}$ given by $\mathbb{P}_{\boldsymbol{\alpha}}\big(J(0) = i\big) = \alpha_i$.

The situation is illustrated in the following figure, where we have represented the states by colored bullets, such that Δ corresponds to black. The process can be illustrated by a traditional graph, as above the horizontal line, or, as below, as a line of length ζ with segments colored according to the sample path. This last representation is the one to be used in subsequent figures.



Analytic expressions for PH distributions, say for the p.d.f., c.d.f., etc., typically have matrix form. All that will matter to us is the form of the m.g.f. of PH$(\boldsymbol{\alpha}, \mathbf{T})$,

$$\mathbb{E}e^{s\zeta} = \boldsymbol{\alpha}(-s\mathbf{I} - \mathbf{T})^{-1}\mathbf{t}\,, \tag{109}$$

where $\mathbf{t}$ is the column vector with elements t_i (the exit rate vector).

The exponential distribution corresponds to E having only one state, a mixture of exponentials to $t_{ij} = 0$ for $i \neq j$, and an Erlang(p, δ) distribution (a gamma(p, δ) distribution) to $E = \{1, \dots, p\}$, $t_{i(i+1)} = \delta$ for $i < p$, all other off-diagonal elements 0, $\boldsymbol{\alpha} = (1\ 0\ \dots\ 0)$.

9.2 The PH Lévy Model

Any one-point distribution, say at $z > 0$, is the limit as $p \to \infty$ of the Erlang$(p, p/z)$ distribution. The PH class is closed under mixtures, and so its closure contains all distributions on $(0, \infty)$ with finite support. Hence the PH class is dense.

The class of compound Poisson processes is dense in D in the class of Lévy processes. Hence the denseness properties of PH imply that the class of differences of two compound Poisson processes with PH jumps are dense. In our key examples, we will work in this class with an added Brownian component,

$$X(t) = \mu t + \sigma B(t) + \sum_{i=1}^{N^+(t)} Y_i^+ - \sum_{j=1}^{N^-(t)} Y_j^- \tag{110}$$

where $N^{\pm}$ is Poisson($\lambda^{\pm}$) and the $Y^{\pm}$ PH($E^{\pm}, \boldsymbol{\alpha}^{\pm}, \mathbf{T}^{\pm}$), with $n^{\pm}$ states. Then by (109), we have (in obvious notation) that

$$\kappa(s) = \mu s + \sigma^2 s^2/2 \\ + \lambda^+\big[\boldsymbol{\alpha}^+(-s\mathbf{I}^+ - \mathbf{T}^+)^{-1}\mathbf{t}^+ - 1\big] + \lambda^-\big[\boldsymbol{\alpha}^-(s\mathbf{I}^- - \mathbf{T}^-)^{-1}\mathbf{t}^- - 1\big]$$

where $\mathbf{t}^+, \mathbf{t}^-$ are the vectors of rates of transitions to the absorbing state.

Expanding the inverses as ratios between minors and the determinant, it follows that $\kappa(s) = r_1(s)/r_2(s)$ where r_1, r_2 are polynomials, with degree $n^+ + n^-$ of

$$r_2(s) = \det\big(-s\mathbf{I}^+ - \mathbf{T}^+\big)\det\big(s\mathbf{I}^- - \mathbf{T}^-\big)$$

and degree $n^+ + n^- + 2$ of r_1 if $\sigma^2 > 0$, $n^+ + n^- + 1$ if $\sigma^2 = 0$, $\mu \neq 0$, and $n^+ + n^-$ if $\sigma^2 = 0$, $\mu = 0$. Obviously, $\kappa(s)$ therefore has an analytic continuation to the whole of the complex plane with the zeros of r_2 removed. This representation is fundamental for the paper. We further let

$$\Theta = \big\{s \in \mathbb{C} : \mathbb{E}e^{\Re(s)X(1)} < \infty\big\};$$

then Θ is a strip of the form $\Theta = \big\{s \in \mathbb{C} : \underline{\theta} < \Re(s) < \overline{\theta}\big\}$ for suitable $\underline{\theta} < 0 < \overline{\theta}$ ($-\overline{\theta}$ is the eigenvalue of largest real part of $\mathbf{T}^+$ and $\underline{\theta}$ the eigenvalue of largest real part of $\mathbf{T}^-$).

The situation is illustrated in Fig. 3. The green-shaded area is the strip $\Theta \subset \mathbb{C}$ where the m.g.f. converges. The red squares are the singularities, i.e. the roots of r_2 or, equivalently, the union of the sets of roots of $\det\big(-s\mathbf{I}^+ - \mathbf{T}^+\big)$ and $\det\big(s\mathbf{I}^- - \mathbf{T}^-\big)$.

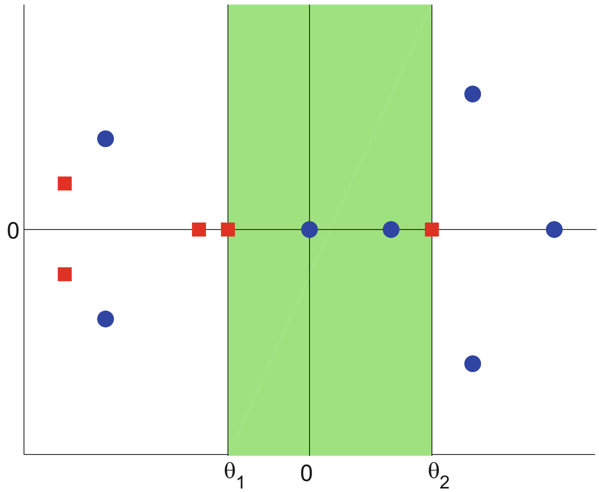


Fig. 3 Features of κ

The blue circles are the roots of r_1 or, equivalently, of κ which will show up in numerous computational schemes of the paper.

To avoid tedious distinctions between the various cases arising according to whether σ^2, μ are non-zero or not, we will assume that $\sigma^2 > 0$. This assumption has a further motivation from a common procedure (e.g. Asmussen and Rosinski [17]) of replacing small jumps by a Brownian motion with the same mean and variance.

9.3 Two-Sided Exit

Recall that $\tau[a,b) = \inf\{t \geq 0 : X(t) \notin [a,b)\}$ with $a \leq 0 < b$; we want to compute $\mathbb{P}\big(X\big(\tau[a,b)\big) \geq b\big)$.

Write

$$
\begin{aligned}
p_c^+ &= \mathbb{P}\big(X\big(\tau[a,b)\big) = b\big)\,,\quad p_c^- = \mathbb{P}\big(X\big(\tau[a,b)\big) = a\big)\,,\\
p_i^+ &= \mathbb{P}\big(X\big(\tau[a,b)\big) > b,\ \text{upcrossing occurs in phase } i\big)\,,\ i = 1,2,\ldots,n^+,\\
p_j^- &= \mathbb{P}\big(X\big(\tau[a,b)\big) < a,\ \text{downcrossing occurs in phase } j\big)\,,\ j = 1,2,\ldots,n^-\,.
\end{aligned}
$$

In more detail, we can imagine each upward jump of the process to be governed by a terminating Markov process J with generator $\mathbf{T}^+$, and if the first exit time from $[a,b)$ is t, 'upcrossing in phase i' then means $J\big(b - X(\tau[a,b)-)\big) = i$ (similarly for the downward jumps). See Fig. 4 where F^+ has two phases, `red` and `green`, and F^- just one, `blue` (we denote by $F^\pm$ the distributions of $Y^\pm$); thus on the figure, there is upcrossing in the `green` phase.

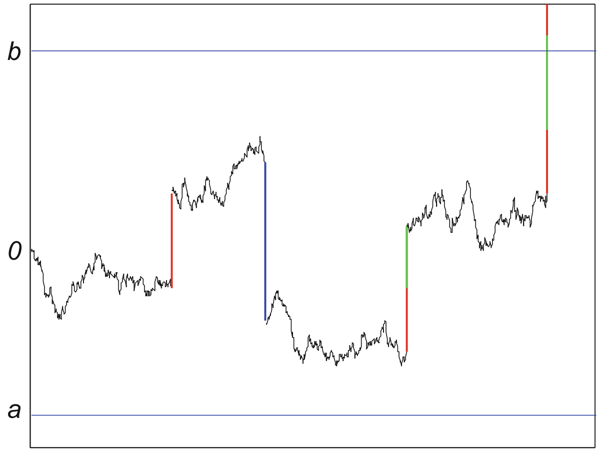


Fig. 4 The two-sided exit problem

We have $\mathbb{P}\big(X\big(\tau[a,b)\big) \geq b\big) = p_c^+ + p_1^+ + \cdots + p_{n^+}^+$ and need $n^+ + n^- + 2$ equations to be able to solve for the unknowns. The first equation is the obvious

$$p_c^+ + \sum_{i=1}^{n^+} p_i^+ + p_c^- + \sum_{j=1}^{n^-} p_j^- = 1.$$

The following notation will be used. Let $\mathbf{e}_i^+, \mathbf{e}_i^-$ denote the ith unit row vectors and let $\hat{F}_i^{\pm}[s] = \mathbf{e}_i^{\pm}(-s\mathbf{I}^{\pm} - \mathbf{T}^{\pm})^{-1}\mathbf{t}^{\pm}$ denote the m.g.f. of the phase-type distributions $F_i^{\pm}$ with initial vector $\mathbf{e}_i^{\pm}$ and phase generator $\boldsymbol{T}^{\pm}$. Let further $0 = \rho_1, \rho_2, \ldots, \rho_{n^+ + n^+ + 2}$ denote the roots of $\kappa(\rho) = 0$, i.e., of the polynomial equation $r_1(\rho) = 0$.

9.3.1 Heuristics via the Wald Martingale

If the drift $\kappa'(0)$ is non-zero, a $\gamma \neq 0$ with $\mathbb{E}\mathrm{e}^{\gamma X(1)} < \infty, \kappa(\gamma) = 0$ exists and we can take $\rho_1 = 0$, $\rho_2 = \gamma$. Thus $\mathrm{e}^{\rho_2 X(t)}$ is an (integrable) martingale. Optional stopping at $\tau[a,b)$ then yields $1 = \mathbb{E}\mathrm{e}^{\rho_2 X(\tau[a,b))}$, which, taking over- and undershoots into account, means

$$1 = \mathrm{e}^{\rho_k b}\Big(p_c^+ + \sum_{i=1}^{n^+} p_i^+ \hat{F}_i^+[\rho_k]\Big) + \mathrm{e}^{\rho_k a}\Big(p_c^- + \sum_{j=1}^{n^-} p_j^- \hat{F}_j^-[-\rho_k]\Big) \tag{111}$$

for $k = 2$. This is one equation more, but only one. If ρ_k, $k > 2$, is one of the remaining $n^+ + n^-$ roots and $\mathbb{E}|\mathrm{e}^{\rho_k X(1)}| < \infty$, we can then proceed as for ρ_k to conclude that (111) holds also for this k, and get in this way potentially the needed $n^+ + n^-$ remaining equations. But the problem is that typically $\mathbb{E}\big|\mathrm{e}^{\rho_k X(1)}\big| < \infty$ fails. Now both sides of (111) are analytic functions. But the validity for two k is not enough to apply analytic continuation.

9.3.2 Computation via the Kella-Whitt Martingale

We will use the simple form (81) of the Kella-Whitt martingale. This gives that K defined according to

$$K(t) = \kappa(\alpha)\int_0^{t\wedge\tau[a,b)} \mathrm{e}^{\alpha X(s)}\mathrm{d}s + 1 - \mathrm{e}^{\alpha X(t\wedge\tau[a,b))},\ \alpha \in \Theta,$$

is a local martingale. In fact, K is a martingale as follows from Kella and Boxma [88]. Further, we have the bound

$$|K(t)| \leq |\kappa(\alpha)|t\mathrm{e}^{|\alpha|\max(|a|,b)} + 1 + \mathrm{e}^{|\alpha|(x+V^+)} + \mathrm{e}^{|\alpha|(b-x+V^-)}$$

where V^+ and V^- (the overshoot and undershoot of b and a, respectively, at $\tau[a,b)$) are phase-type distributed and so have finite exponential means. From $\mathbb{E}\tau[a,b) < \infty$ we then get $\mathbb{E}\sup_{t\le\tau[a,b)}|K(t)| < \infty$, so optional stopping at $\tau[a,b)$ is permissible. Letting $\phi(\alpha) = \mathbb{E}\int_0^{\tau[a,b)} \mathrm{e}^{\alpha X(s)}\mathrm{d}s$, this gives

$$\begin{aligned} 0 &= \kappa(\alpha)\phi(\alpha) + 1 \\ &\quad - \mathrm{e}^{\alpha b}\Big(p_c^+ + \sum_{i=1}^{n^+} p_i^+ \hat{F}_i^+[\alpha]\Big) - \mathrm{e}^{\alpha a}\Big(p_c^- + \sum_{j=1}^{n^-} p_j^- \hat{F}_j^-[-\alpha]\Big), \end{aligned} \tag{112}$$

It is easily seen that the function $\phi(\alpha)$ is well defined for all $\alpha \in \mathbb{C}$, not just for $\alpha \in \Theta$, and analytic when the common singularities of κ and the $\hat{F}_i^+, \hat{F}_i^-$ are removed. Therefore by analytic continuation (112) is valid for all α in this domain. In particular we may take α as any of the ρ_k to obtain (111) for $k = 1,\ldots,n^+ + n^- + 2$.

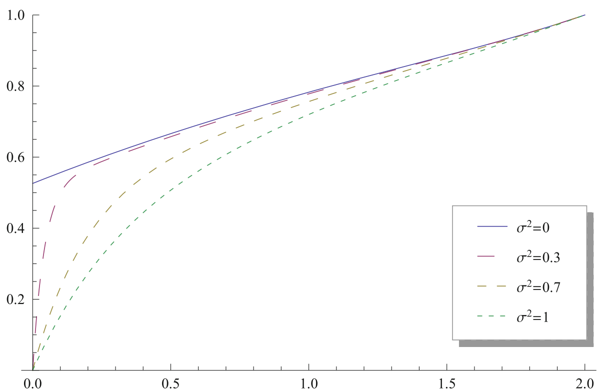
Example 9.1 Take as a simple example all jumps to be exponentially distributed (with parameters μ^+, μ^-) and $\mu = -1$. Then

$$\kappa(\alpha) = \frac{\lambda^+\alpha}{\mu^+ - \alpha} - \frac{\lambda^-\alpha}{\mu^- + \alpha} + \frac{\sigma^2\alpha^2}{2} - \alpha.$$

The method described above allows us to explicitly compute the c.d.f. $\pi^b[0,x]$ (in terms of the parameters of the model and b). Even for this simple case, the resulting expressions are quite complicated and rather than presenting them, we display numerical results in Fig. 5 in the form of plots of the c.d.f. of π^b, taking $\lambda^+ = \lambda^- = \mu^+ = \mu^- = 1$, $b = 2$, and letting σ^2 vary.

□

Fig. 5 C.d.f. of π^b



9.4 The Scale Function

Though the scale function does not appear in the rest of the paper, we give for the sake of completeness a sketch of its computation in the PH model. In view of (51), we need to evaluate

$$\mathbb{E}\Big[\mathrm{e}^{-q\tau[a,b)}\mathbb{1}(X(\tau[a,b)) \geq b)\Big]. \tag{113}$$

To this end, we use the Kella-Whitt martingale with $B(t) = -qt/\alpha$ which takes the form

$$\kappa(\alpha)\int_0^t \mathrm{e}^{\alpha X(s)-qs}\,\mathrm{d}s \;+\; 1 \;-\; \mathrm{e}^{\alpha X(t)-qt} \;-\; q\int_0^t \mathrm{e}^{\alpha X(s)-qs}\,\mathrm{d}s$$

Optional stopping at $\tau[a,b)$ gives

$$\begin{aligned} 1 = \; & -\big(\kappa(\alpha)-q\big)\mathbb{E}\int_0^{\tau[a,b)} \mathrm{e}^{\alpha X(s)-qs}\,\mathrm{d}s \\ & + \mathrm{e}^{\alpha b}\Big(p_c^+y_c^+ + \sum_{i=1}^{n^+} p_i^+\hat{F}_i^+[\alpha]y_i^+\Big) \; + \mathrm{e}^{\alpha a}\Big(p_c^-y_c^- + \sum_{j=1}^{n^-} p_j^-\hat{F}_j^-[-\alpha]y_j^-\Big) \end{aligned}$$

where y_c^+ is the expectation of $\mathrm{e}^{-q\tau[a,b)}$ given continuous exit above, y_i^+ the expectation of $\mathrm{e}^{-q\tau[a,b)}$ given exit above in phase i, and similarly for the y_c^-, y_j^-. As in Sect. 9.3.2, we may now choose $\rho_1^q, \ldots, \rho_{n^++n^-+2}^q$ as the roots of $\kappa(s) = q$ to conclude that

$$1 \;=\; \mathrm{e}^{\rho_k^q b}\Big(p_c^+y_c^+ + \sum_{i=1}^{n^+} p_i^+\hat{F}_i^+[\rho_k^q]y_i^+\Big) \; + \mathrm{e}^{\rho_k^q a}\Big(p_c^-y_c^- + \sum_{j=1}^{n^-} p_j^-\hat{F}_j^-[-\rho_k^q]y_j^-\Big)$$

for $k = 1, \ldots, n^+ + n^- + 2$. These linear equations may be solved for the $p_c^+y_c^+, p_i^+y_i^+, p_c^-y_c^-, p_j^-y_j^-$, and (113) can then be computed as $p_c^+y_c^+ + \sum p_i^+y_i^+$.

9.5 The Loss Rate

As before, we take $N^\pm$ to be Poisson($\lambda^\pm$) and $Y^\pm$ to be PH($E^\pm, \boldsymbol{\alpha}^\pm, \mathbf{T}^\pm$) with $n^\pm$ phases, respectively. If we let $x \to b$ and $x - b \to a$ in (111), we obtain, for $k = 1, \ldots, n^+ + n^- + 2$,

$$\mathrm{e}^{-\rho_k x} = p_c^+ + \sum_{i=1}^{n^+} p_i^+\hat{F}_i^+[\rho_k] + \mathrm{e}^{-\rho_k b}\Big(p_c^- + \sum_{j=1}^{n^-} p_i^-\hat{F}_j^-[-\rho_k]\Big) \tag{114}$$

(where, as above, we let $\rho_1 = 0$). We let $\mathbf{a}_k$ be the vector

$$\mathbf{a}_k = \big(1\ \hat{F}_1^+(\rho_k)\ \dots\ \hat{F}_{n^+}^+(\rho_k)\ \mathrm{e}^{-\rho_k b}\ \mathrm{e}^{-\rho_k b}\hat{F}_1^-(-\rho_k)\ \dots\ \mathrm{e}^{-\rho_k b}\hat{F}_{n^-}^-(-\rho_k)\big)$$

and construct the matrix $\mathbf{A}$ according to

$$\mathbf{A} = \begin{pmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_{n^+ + n^- + 2} \end{pmatrix}.$$

If we let

$$\mathbf{p} = \big(p_c^+\ p_1^+\ \dots\ p_{n^+}^+\ p_c^-\ p_1^-\ \dots\ p_{n^-}^-\big)^{\mathsf{T}}$$

and take $\mathbf{e}_+$ to be a row vector with the first $n^+ + 1$ elements equal to one and zero otherwise, we may compute $\overline{\pi}(x) = p_c^+ + \sum_{i=1}^{n^+} p_i^+$ as $\mathbf{e}_+\mathbf{p}$ where $\mathbf{p}$ solves the set of linear equations $\mathbf{A}\mathbf{p} = \mathbf{h}(x)$, where

$$\mathbf{h}(x) = (\mathrm{e}^{-\rho_1 x}\ \dots\ \mathrm{e}^{-\rho_{n^+ + n^- + 2} x})^{\mathsf{T}},$$

i.e., formally

$$\overline{\pi}(x) = \mathbf{g}\mathbf{h}(x) = \mathbf{g}\exp\{\mathbf{H}x\}\mathbf{e}, \tag{115}$$

where $\mathbf{g} = \mathbf{e}_+\mathbf{A}^{-1}$ and $\mathbf{H} = \mathrm{diag}(-\rho_1, -\rho_2, \dots, -\rho_{n^+ + n^- + 2})$ (the rightmost part in (115) will prove itself useful below).

With this formula for $\overline{\pi}(x)$ at hand we may proceed to the computation of ℓ^b. We will take as a starting point the alternative formula for ℓ^b which is presented in Sect. 1, i.e.,

$$\ell^b = \frac{1}{2b}\big(2m\,\mathbb{E}V + \sigma^2 + J_1 + J_2 - 2J_3 - 2J_4\big) \tag{116}$$

where

$$\begin{aligned} J_1 &= J_1(b) = \int_0^b y^2 \nu(\mathrm{d}y)\,, \\ J_2 &= J_2(b) = \int_b^\infty (2yb - b^2)\nu(\mathrm{d}y)\,, \\ J_3 &= J_3(b) = \int_0^b \int_{-\infty}^{-x} (x+y)\nu(\mathrm{d}y)\,\overline{\pi}(x)\,\mathrm{d}x\,, \\ J_4 &= J_4(b) = \int_0^b \int_{b-x}^{\infty} (x+y-b)\nu(\mathrm{d}y)\,\overline{\pi}(x)\,\mathrm{d}x\,. \end{aligned}$$

It thus remains to identify $m, \mathbb{E}V$ and J_1, J_2, J_3, J_4. If we note that

$$\nu(\mathrm{d}x) = \begin{cases} \lambda^+ \boldsymbol{\alpha}^+ \exp\{\mathbf{T}^+ x\}\mathbf{t}^+ \mathrm{d}x & \text{if } x > 0, \\ \lambda^- \boldsymbol{\alpha}^- \exp\{-\mathbf{T}^- x\}\mathbf{t}^- \mathrm{d}x & \text{if } x < 0, \end{cases}$$

we see that the computation of ℓ^b is more or less a matter of routine (though tedious!). However, for the sake of completeness and clarity we will perform the calculations in some detail anyway. Clearly,

$$\begin{aligned} m &= \mu - \lambda^+ \boldsymbol{\alpha}^+ (\mathbf{T}^+)^{-1}\mathbf{e} + \lambda^- \boldsymbol{\alpha}^- (\mathbf{T}^-)^{-1}\mathbf{e}, \\ \mathbb{E}V &= \int_0^b \overline{\pi}(x)\mathrm{d}x = \int_0^b \mathbf{g}\mathbf{h}(x)\mathrm{d}x = \mathbf{g}\mathbf{k}, \end{aligned}$$

where

$$\mathbf{k} = \begin{pmatrix} b \\ \rho_2^{-1}(1 - \mathrm{e}^{-\rho_2 b}) \\ \vdots \\ \rho_{n^+ + n^- + 2}^{-1}(1 - \mathrm{e}^{-\rho_{n^+ + n^- + 2} b}) \end{pmatrix}.$$

Let $\otimes$ and $\oplus$ denote Kronecker matrix multiplication and addition, respectively, where $\oplus$ is defined for square matrices by $\mathbf{A}_1 \oplus \mathbf{A}_2 = \mathbf{A}_1 \otimes \mathbf{I} + \mathbf{I} \otimes \mathbf{A}_2$. It is not difficult to show that

$$\int \nu(\mathrm{d}y) = \begin{cases} \lambda^+ \mathbf{a}^+ (\mathbf{T}^+)^{-1} \exp\{\mathbf{T}^+ y\}\mathbf{t}^+ & \text{if } y > 0, \\ -\lambda^- \mathbf{a}^- (\mathbf{T}^-)^{-1} \exp\{-\mathbf{T}^- y\}\mathbf{t}^- & \text{if } y < 0, \end{cases} \tag{117}$$

$$\int y\nu(\mathrm{d}y) = \begin{cases} \lambda^+ \mathbf{a}^+ (\mathbf{T}^+)^{-1} (y\mathbf{I} - (\mathbf{T}^+)^{-1}) \exp\{\mathbf{T}^+ y\}\mathbf{t}^+ & \text{if } y > 0, \\ -\lambda^- \mathbf{a}^- (\mathbf{T}^-)^{-1} (y\mathbf{I} - (\mathbf{T}^-)^{-1}) \exp\{-\mathbf{T}^- y\}\mathbf{t}^- & \text{if } y < 0, \end{cases} \tag{118}$$

$$\int y^2\nu(\mathrm{d}y) = \lambda^+ \mathbf{a}^+ (\mathbf{T}^+)^{-1} (y^2\mathbf{I} - 2y(\mathbf{T}^+)^{-1} + 2(\mathbf{T}^+)^{-2}) \exp\{\mathbf{T}^+ y\}\mathbf{t}^+, \text{ if } y > 0. \tag{119}$$

[note that when we write $\int f(y)\mathrm{d}y$ (without integration limits) for some function f we mean the primitive (indefinite integral), i.e. $\int f(y)\mathrm{d}y$ is a function such that its derivative with respect to y equals $f(y)$]. It follows from (115), (117), (118) and the fact that all eigenvalues of $\mathbf{T}^+$ and $\mathbf{T}^-$ have negative real part, see e.g. [11, p. 83], that

$$\begin{aligned} J_3 &= \int_0^b \lambda^- \mathbf{a}^- (\mathbf{T}^-)^{-2} \exp\{\mathbf{T}^- x\}\mathbf{t}^- \, \overline{\pi}(x)\, \mathrm{d}x \\ &= \int_0^b \lambda^- \mathbf{a}^- (\mathbf{T}^-)^{-2} \exp\{\mathbf{T}^- x\}\mathbf{t}^- \, \mathbf{g} \exp\{\mathbf{H}x\}\mathbf{e}\, \mathrm{d}x \end{aligned}$$

$$
\begin{aligned}
&= \lambda^{-}\big[(\mathbf{a}^{-}(\mathbf{T}^{-})^{-2}) \otimes \mathbf{g}\big]\Big[\int_0^b \exp\{(\mathbf{T}^{-} \oplus \mathbf{H})x\}\mathrm{d}x\Big][\mathbf{t}^{-} \otimes \mathbf{e}] \\
&= \lambda^{-}\big[(\mathbf{a}^{-}(\mathbf{T}^{-})^{-2}) \otimes \mathbf{g}\big]\big[(\mathbf{T}^{-} \oplus \mathbf{H})^{-1}\big(\exp\{(\mathbf{T}^{-} \oplus \mathbf{H})b\} - \mathbf{I}\big)\big][\mathbf{t}^{-} \otimes \mathbf{e}]
\end{aligned}
$$

where we used the standard identities

$$
\begin{aligned}
&(\mathbf{A}_1\mathbf{B}_1\mathbf{C}_1)(\mathbf{A}_2\mathbf{B}_2\mathbf{C}_2) = (\mathbf{A}_1 \otimes \mathbf{A}_2)(\mathbf{B}_1 \otimes \mathbf{B}_2)(\mathbf{C}_1 \otimes \mathbf{C}_2) \\
&\exp\{\mathbf{R}x\} \otimes \exp\{\mathbf{S}x\} = \exp\{(\mathbf{R} \oplus \mathbf{S})x\}.
\end{aligned}
$$

Similarly, J_4 becomes

$$
-\lambda^{+}\big[(\mathbf{a}^{+}(\mathbf{T}^{+})^{-2}\exp\{\mathbf{T}^{+}b\}) \otimes \mathbf{g}\big]\big[(-\mathbf{T}^{+} \oplus \mathbf{H})^{-1}(\exp\{(-\mathbf{T}^{+} \oplus \mathbf{H})b\} - \mathbf{I})\big][\mathbf{t}^{+} \otimes \mathbf{e}].
$$

Finally, it follows easily from (117)–(119), that

$$
\begin{aligned}
J_1 &= \lambda^{+}\mathbf{a}^{+}(\mathbf{T}^{+})^{-1}\big[(b^2\mathbf{I} - 2b(\mathbf{T}^{+})^{-1} + 2(\mathbf{T}^{+})^{-2})\exp\{\mathbf{T}^{+}b\} - 2(\mathbf{T}^{+})^{-2}\big]\mathbf{t}^{+} \\
J_2 &= -\lambda^{+}\mathbf{a}^{+}(\mathbf{T}^{+})^{-1}\big[b^2\mathbf{I} - 2b(\mathbf{T}^{+})^{-1}\big]\exp\{\mathbf{T}^{+}b\}\mathbf{t}^{+},
\end{aligned}
$$

and thereby all terms in (116) have been evaluated.

10 Loss Rate Asymptotics: Light Tails

In this section we derive asymptotics of ℓ^b as $b \to \infty$ when X is assumed to be light-tailed with $-\infty < \mathbb{E}X(1) < 0$. By light-tailed, we simply mean that the set $\Theta = \{\alpha \in \mathbb{R} : \mathbb{E}e^{\alpha X(1)} < \infty\}$ has a non-empty intersection with $(0, \infty)$.

We start by introducing the following notation.

$M(t) = \sup_{0 \le s \le t} X(s)$, $M(\infty) = \sup_{0 \le t < \infty} X(t)$.
$\tau_{+}(u) = \inf\{t > 0 : X(t) > u\}$, $\tau_{+}^{w}(u) = \inf\{t > 0 : X(t) \ge u\}$, $u \ge 0$.
$\tau_{-}(-v) = \inf\{t > 0 : X(t) < -v\}$, $v \ge 0$.
The overshoot of level u, $B(u) = X(\tau_{+}(u)) - u$, $u \ge 0$.
The weak overshoot of level u, $B^{w}(u) = X(\tau_{+}^{w}(u)) - u$, $u \ge 0$.
$B(\infty)$, a r.v. having the limiting distribution (if it exists) of $B(u)$ as $u \to \infty$.

Furthermore, we will assume that the Lundberg equation $\kappa(\alpha) = 0$ has a solution $\gamma > 0$ with $\kappa'(\gamma) < \infty$. We let $\mathbb{P}_L$ and $\mathbb{E}_L$ ($\mathbb{P}_\gamma$ and $\mathbb{E}_\gamma$ in earlier notation) correspond to a measure which is exponentially tilted by γ, i.e.,

$$
\mathbb{P}(G) = \mathbb{E}_L(e^{-\gamma X(\tau)}; G) \tag{120}
$$

when τ is a stopping time and $G \in \mathscr{F}(\tau)$, $G \subseteq \{\tau < \infty\}$. Note that $\mathbb{E}_L X(1) = \kappa'(\gamma) > 0$ by convexity of κ.

We need the following two lemmas. The first is just a reformulation of Theorem 8.6 and the second describes the asymptotic probability, as $u \to \infty$, of the event that X's first exit of the set $[-v, u)$ occurs at the upper barrier.

Lemma 10.1 *For the integrals I_1 and I_2 in Theorem 8.6 we have the following alternative formulas.*

$$I_1 = \int_b^\infty ((y-b)+\gamma^{-1}(1-\mathrm{e}^{\gamma(y-b)}))\nu(\mathrm{d}y) + \int_0^b \overline{\pi}(x)\mathrm{d}x \int_{b-x}^\infty (1-\mathrm{e}^{\gamma(y-b+x)})\nu(\mathrm{d}y),$$

$$I_2 = \int_{-\infty}^0 (y+\gamma^{-1}(1-\mathrm{e}^{\gamma y}))\nu(\mathrm{d}y) + \int_0^b \overline{\pi}(x)\mathrm{d}x \int_{-\infty}^{-x} (1-\mathrm{e}^{\gamma(x+y)})\nu(\mathrm{d}y).$$

Proof Just change order of integration and perform partial integration. Then switch back to the original order of integration. □

Lemma 10.2 *Assume that X is not compound Poisson with lattice jump distribution. Then, for each $v \geq 0$,*

$$\mathbb{P}\big(\tau_-(-v) > \tau_+^w(u)\big) \sim \mathrm{e}^{-\gamma u}\mathbb{E}_L \mathrm{e}^{-\gamma B(\infty)}\mathbb{P}_L\big(\tau_-(-v) = \infty\big),\ u \to \infty.$$

Proof It is easily seen that $\tau_+^w(u)$ is a stopping time and that $\{\tau_-(-v) > \tau_+^w(u)\} \in \mathscr{F}\big(\tau_+^w(u)\big)$. By the Blumenthal zero-one law (e.g. [28, p. 19]), it follows that $\mathbb{P}\big(\tau_+(0) = 0\big)$ is either 0 or 1. In the first case the sample paths of M are step functions a.s. and it follows in the same way as in the proof of Lemma 3.3 (Lemma 2.3 in [117]) that $\mathbb{P}(\tau_+^w(u) \neq \tau_+(u)) \to 0,\ u \to \infty$. In the second case it follows by the strong Markov property applied at $\tau_+^w(u)$ that $\mathbb{P}(\tau_+^w(u) \neq \tau_+(u)) = 0$. From (120) we then get,

$$\begin{aligned}\mathbb{P}\big(\tau_-(-v) > \tau_+^w(u)\big) &= \mathbb{E}_L\big[\mathrm{e}^{-\gamma X(\tau_+^w(u))}; \tau_-(-v) > \tau_+^w(u)\big]\\ &= \mathrm{e}^{-\gamma u}\mathbb{E}_L\big[\mathrm{e}^{-\gamma B(u)}; \tau_-(-v) > \tau_+(u)\big]\mathbb{P}\big(\tau_+^w(u) = \tau_+(u)\big)\\ &\quad + \mathrm{e}^{-\gamma u}\mathbb{P}_L\big(\tau_-(-v) > \tau_+^w(u) \,\big|\, \tau_+^w(u) \neq \tau_+(u)\big)\mathbb{P}_L\big(\tau_+^w(u) \neq \tau_+(u)\big)\\ &\sim \mathrm{e}^{-\gamma u}\mathbb{E}_L \mathrm{e}^{-\gamma B(\infty)}\mathbb{P}_L\big(\tau_-(-v) = \infty\big).\end{aligned}$$

In the last step we used $B(u) \to B(\infty)$, see [29] and [117], $\{\tau_-(-v) > \tau_+(u)\} \uparrow \{\tau_-(-v) = \infty\}$ (both in $\mathbb{P}_L$-distribution) and asymptotic independence between $B(u)$ and $\{\tau_-(-v) > \tau_+(u)\}$, see the proof of Corollary 5.9, p. 368, in [11]. □

Remark 10.3 In the proof of Lemma 10.2 above we had to treat the cases $\mathbb{P}(\tau_+(0) = 0) = 1$ and $\mathbb{P}(\tau_+(0) = 0) = 0$ (corresponding to completely different short time behaviors of X) in slightly different ways. In traditional terminology, these cases correspond to whether the point 0 is *regular*, or *irregular*, for the set $(0, \infty)$, see [28, p. 104] or [129, pp. 313, 353]. As a small digression, we shall briefly discuss this issue. It turns out that 0 is regular for $(0,\infty)$ if and only if

$$\int_0^1 t^{-1}\mathbb{P}(X(t) > 0)\,\mathrm{d}t \;=\; \infty,$$

see Theorem 47.2 and the remark at the bottom of p. 353 in [129]. Perhaps more interestingly, we can characterize the short time behavior of X via its Lévy triplet. We will not give a complete account for all types of Lévy processes (this is done in Theorem 47.5 on p. 355 in [129]), but note that whenever the paths of X are of infinite variation then 0 is regular for $(0,\infty)$ and if X is the sum of a compound Poisson process and a non-positive drift then 0 is irregular for $(0,\infty)$. □

Next, we state the main result about the asymptotics for ℓ^b.

Theorem 10.4 *Suppose that X fulfills the conditions in Lemma 10.2. Then, as $b\to\infty$, $\ell^b \sim C\mathrm{e}^{-\gamma b}$ where γ is the solution to the Lundberg equation and*

$$\begin{aligned}C = {} & -m + \mathbb{E}_L \mathrm{e}^{-\gamma B(\infty)}\int_0^\infty \mathrm{e}^{\gamma x}\mathbb{P}_L\big(\tau_-(-x)=\infty\big)\int_x^\infty (1-\mathrm{e}^{\gamma(y-x)})\,\nu(\mathrm{d}y)\,\mathrm{d}x \\ & + \int_{-\infty}^0 \big[y+\gamma^{-1}(1-\mathrm{e}^{\gamma y})\big]\,\nu(\mathrm{d}y) \\ & + \int_0^\infty \mathbb{P}\big(\tau_+^w(x)<\infty\big)\int_{-\infty}^{-x}(1-\mathrm{e}^{\gamma(x+y)})\,\nu(\mathrm{d}y)\,\mathrm{d}x. \end{aligned} \tag{121}$$

Proof It follows from Lemma 10.1 and $\kappa'(\gamma)<\infty$ that

$$\begin{aligned}\mathrm{e}^{\gamma b} I_1 \;&=\; \mathrm{o}(1) + \mathrm{e}^{\gamma b}\int_0^b \mathbb{P}\big(\tau_-(x-b) > \tau_+^w(x)\big)\,\mathrm{d}x\int_{b-x}^\infty (1-\mathrm{e}^{\gamma(y-b+x)})\,\nu(\mathrm{d}y) \\ &=\; \mathrm{o}(1) + \int_0^b \mathrm{e}^{\gamma z}\mathrm{e}^{\gamma(b-z)}\mathbb{P}\big(\tau_-(-z) > \tau_+^w(b-z)\big)\,\mathrm{d}z\int_z^\infty (1-\mathrm{e}^{\gamma(y-z)})\,\nu(\mathrm{d}y) \\ &\to\; \mathbb{E}_L\mathrm{e}^{-\gamma B(\infty)}\int_0^\infty \mathrm{e}^{\gamma x}\mathbb{P}_L\big(\tau_-(-x)=\infty\big)\int_x^\infty (1-\mathrm{e}^{\gamma(y-x)})\,\nu(\mathrm{d}y)\,\mathrm{d}x,\quad b\to\infty.\end{aligned}$$

The convergence follows from the pointwise convergence in Lemma 10.2 and dominated convergence, which is applicable because

$$\mathrm{e}^{\gamma b}\overline{\pi}(b-x)\mathbb{1}(x\le b) \;\le\; \mathrm{e}^{\gamma b}\mathbb{P}\big(M(\infty) > b-x\big)\mathbb{1}(x\le b) \;\le\; \mathrm{e}^{\gamma x}$$

and

$$\int_0^\infty \mathrm{e}^{\gamma x}\,\mathrm{d}x\int_x^\infty (1-\mathrm{e}^{\gamma(y-x)})\,\nu(\mathrm{d}y) \;=\; \int_0^\infty \big(\gamma^{-1}\mathrm{e}^{\gamma y} - y\mathrm{e}^{\gamma y} - \gamma^{-1}\big)\,\nu(\mathrm{d}y) \;>\; -\infty.$$

In I_2 we bound $\overline{\pi}(x)\mathbb{1}(x \le b)$ by 1, note that

$$\int_0^\infty \mathrm{d}x \int_{-\infty}^{-x} (1 - \mathrm{e}^{\gamma(x+y)})\nu(\mathrm{d}y) = \int_{-\infty}^0 (-y - \gamma^{-1} + \gamma^{-1}\mathrm{e}^{\gamma y})\nu(\mathrm{d}y) < \infty$$

and apply dominated convergence which together with $\overline{\pi}(x) \to \mathbb{P}\big(\tau_+^w(x) < \infty\big)$ gives

$$I_2 \to \int_{-\infty}^0 \big[y + \gamma^{-1}(1 - \mathrm{e}^{\gamma y})\big]\, \nu(\mathrm{d}y) + \int_0^\infty \mathbb{P}\big(\tau_+^w(x) < \infty\big) \int_{-\infty}^{-x} [1 - \mathrm{e}^{\gamma(x+y)}]\, \nu(\mathrm{d}y)\, \mathrm{d}x.$$

The assertion now follows from Theorem 8.6. □

If X is spectrally one-sided the constant in Theorem 10.4 simplifies significantly.

Corollary 10.5 *Let X satisfy the conditions in Lemma 10.2. If $\nu(-\infty, 0) = 0$, then*

$$C = -m\left\{1 + \frac{1}{\mathbb{E}_L X(1)} \int_0^\infty (\mathrm{e}^{\gamma x} - 1) \int_x^\infty (1 - \mathrm{e}^{\gamma(y-x)})\nu(\mathrm{d}y)\mathrm{d}x\right\}.$$

If $\nu(0, \infty) = 0$, then

$$C = -m + \int_{-\infty}^0 \big[y + \gamma^{-1}(1 - \mathrm{e}^{\gamma y})\big]\, \nu(\mathrm{d}y) + \int_0^\infty \mathrm{e}^{-\gamma x} \int_{-\infty}^{-x} [1 - \mathrm{e}^{\gamma(x+y)}]\, \nu(\mathrm{d}y)\, \mathrm{d}x.$$

Proof In the spectrally positive case we have that $\mathbb{E}_L \mathrm{e}^{-\gamma B(\infty)} = -m/\mathbb{E}_L X(1)$, see, e.g., Bertoin and Doney [29], and that

$$\mathbb{P}_L\big(\tau_-(-x) = \infty\big) = 1 - \mathbb{P}_L\big(\tau_-(-x) < \infty\big) = 1 - \mathbb{E}\mathrm{e}^{\gamma X(\tau_-(-x))} = 1 - \mathrm{e}^{-\gamma x}.$$

In the spectrally negative case,

$$\mathbb{P}\big(\tau_+^w(x) < \infty\big) = \mathbb{E}_L \mathrm{e}^{-\gamma X(\tau_+^w(x))} = \mathrm{e}^{-\gamma x}.$$

The claim now follows from Theorem 10.4. □

We next turn our attention towards asymptotics for ℓ^b as $b \to \infty$ in the PH example. In principle, we should be able to describe the asymptotics by carefully analyzing what comes out of (116), but we prefer to apply Theorem 10.4. Recall that we assume negative drift of the feeding process X, i.e. $\mathbb{E}X(1) < 0$. This means that $X(t) \overset{\text{a.s.}}{\to} -\infty,\ t \to \infty$, and that there exists a real positive root γ of the equation $\kappa(\alpha) = 0$ such that $\mathbb{E}\mathrm{e}^{\gamma X(1)} = 1$, i.e. γ is a genuine root of the Lundberg equation corresponding to X.

Theorem 10.6 *In the PH Lévy model,*

$$
\begin{aligned}
C &= \mathbf{e}^{\mathsf{T}}\mathbf{B}^{-1}\mathbf{e}_1^{\mathsf{T}}\gamma\lambda^{+} \\
&\quad\times\left\{-\boldsymbol{\alpha}^{+}(\gamma\mathbf{I}+\mathbf{T}^{+})^{-2}\mathbf{e}+((\mathbf{e}^{\mathsf{T}}\tilde{\mathbf{B}}^{-1})\otimes(\boldsymbol{\alpha}^{+}(\gamma\mathbf{I}+\mathbf{T}^{+})^{-1}))\left\{\tilde{\mathbf{J}}\oplus(\gamma\mathbf{I}+\mathbf{T}^{+})\right\}^{-1}\mathbf{e}\right\} \\
&\quad+\lambda^{-}\boldsymbol{\alpha}^{-}\left\{(\mathbf{T}^{-})^{-1}-(-\gamma\mathbf{I}+\mathbf{T}^{-})^{-1}\right\}\mathbf{e}+(-\mu-\lambda^{+}\boldsymbol{\alpha}^{+}(\mathbf{T}^{+})^{-1}\mathbf{e}+\lambda^{-}\boldsymbol{\alpha}^{-}(\mathbf{T}^{-})^{-1}\mathbf{e}) \\
&\quad+\gamma\lambda^{-}((\mathbf{e}^{\mathsf{T}}\mathbf{B}^{-1})\otimes(\boldsymbol{\alpha}^{-}(-\gamma\mathbf{I}+\mathbf{T}^{-})^{-1}))(\mathbf{J}\oplus\mathbf{T}^{-})^{-1}\mathbf{e}\,. \qquad (122)
\end{aligned}
$$

For the proof, we need two lemmas. The first is classical and relates to the locations in the complex plane of the roots of $\kappa(\alpha) = q$, $q \geq 0$ (see [12, 51] and references there).

Lemma 10.7 *Let X be defined according to* (110)*.*

(i) *Consider the equation $\kappa(\alpha) = 0$. If $m \leq 0$ then 0 is the only root with zero real part. There are n^- roots with negative real part and $n^+ + 1$ roots with positive real part.*
(ii) *Consider the equation $\kappa(\alpha) = q$ with $q > 0$. Then (regardless of the value of m) there are no roots with zero real part, $n^- + 1$ with negative real part and $n^+ + 1$ with positive real part.*

Lemma 10.8 *Assume $m < 0$. Then $\gamma > 0$ is a simple root, i.e. of algebraic multiplicity 1, and if ρ is any other root with $\Re(\rho) > 0$, then $Re(\rho) > \gamma$.*

Proof Part (i) in Lemma 10.7 tells us that there are $n^+ + 1$ roots of $\kappa(\alpha) = 0$ with positive real part. Clearly, γ is one of these. Let $\rho = \Re(\rho) + \mathrm{i}\Im(\rho)$ be one of the remaining roots (with positive real part) and suppose that $0 < \Re(\rho) \leq \gamma$. Now,

$$
\begin{aligned}
1 = \mathbb{E}\mathrm{e}^{\rho X(1)} &= \mathbb{E}\mathrm{e}^{\Re(\rho)X(1)}\left(\cos(\Im(\rho)X(1)) + \mathrm{i}\sin(\Im(\rho)X(1))\right) \\
&= \mathbb{E}\mathrm{e}^{\Re(\rho)X(1)}\cos(\Im(\rho)X(1)) + \mathrm{i}\mathbb{E}\mathrm{e}^{\Re(\rho)X(1)}\sin(\Im(\rho)X(1)). \qquad (123)
\end{aligned}
$$

From (123), the elementary inequality $|\cos(\Im(\rho)X(1))| \leq 1$ and the convexity of $\kappa(\cdot)$ in $(0, \gamma]$, it follows that $\Re(\rho) < \gamma$ is impossible (no matter the distribution of $X(1)$). Note that in the case under consideration, $\kappa(\alpha)$ is a rational function (i.e. $\kappa(\alpha) = p(\alpha)/q(\alpha)$ where p and q are polynomials) and from the fact that $0 < \kappa'(\gamma) < \infty$, see Sect. 10, we may conclude that the algebraic multiplicity of the root γ equals one, i.e. $p(\alpha) = (\alpha-\gamma)r(\alpha)$ where $r(\alpha)$ does not contain the factor $(\alpha-\gamma)$. If $\Re(\rho) = \gamma$ and $\Im(\rho) \neq 0$ then it is easily seen that $1 = \mathbb{E}\mathrm{e}^{\Re(\rho)X(1)}\cos(\Im(\rho)X(1))$ is possible provided that $X(1)$ is lattice with span $2\pi/|\Im(\rho)|$, a case which is clearly ruled out by the structure of X. □

Proof of Theorem 10.6 We have to compute $\mathbb{E}_L\mathrm{e}^{-\gamma B(\infty)}$, $\mathbb{P}_L\big(\tau_-(-x) = \infty\big)$ and $\mathbb{P}\big(\tau_+^w(x) < \infty\big)$ for $x > 0$, see Theorem 10.4. Because of (thanks to!) the Brownian component in X we need not to distinguish between $\tau_+^w(x)$ and $\tau_+(x)$, cf. Remark 10.3. Define

$$p_c^+(t) \;=\; \mathbb{P}(X(\tau_+(x)) = x, \tau_+(x) \le t),$$
$$p_i^+(t) \;=\; \mathbb{P}(X(\tau_+(x)) > x, \tau_+(x) \le t, \text{ upcrossing occurs in state } i)\,,$$

$i = 1, 2, \ldots, n^+$. If we let $\phi(\alpha, t) = \mathbb{E}\int_0^{\tau_+(x)\wedge t} \mathrm{e}^{\alpha X(s)} \mathrm{d}s$ it follows by optional stopping of the Kella-Whitt martingale at $\tau_+(x)$ that

$$0 = \kappa(\alpha)\phi(\alpha, t) + 1 - \mathrm{e}^{\alpha x}\left(p_c^+(t) + \sum_{i=1}^{n^+} p_i^+(t)\hat{F}_i^+(\alpha)\right)$$
$$- \mathbb{E}\left[\mathrm{e}^{\alpha X(t)}; t < \tau_+(x)\right],\ \alpha \in \Theta. \tag{124}$$

Let $\rho_2, \rho_3, \ldots, \rho_{n^+ +2}$ denote the roots with positive real part (we tacitly assume that these are distinct and ordered so that $\rho_2 = \gamma$). If we mimic the derivation of (111), and take $\alpha = \rho_k$, we get

$$0 = 1 - \mathrm{e}^{\rho_k x}\left(p_c^+(t) + \sum_{i=1}^{n^+} p_i^+(t)\hat{F}_i^+(\rho_k)\right) - \mathbb{E}\left[\mathrm{e}^{\rho_k X(t)}; t < \tau_+(x)\right]. \tag{125}$$

If we let $t \to \infty$ in (125), it follows by $X(t) \overset{\text{a.s.}}{\to} -\infty$ and dominated convergence that

$$\mathrm{e}^{-\rho_k x} = p_c^+ + \sum_{i=1}^{n^+} p_i^+ \hat{F}_i^+(\rho_k),\ k = 2, 3, \ldots, n^+ + 2. \tag{126}$$

Let $\mathbf{B}$ be the matrix with kth row equal to $(1\ \hat{F}_1^+(\rho_k)\ \ldots\ \hat{F}_{n^+}^+(\rho_k))$. Then it is easily seen that

$$\mathbb{P}(\tau_+(x) < \infty) \;=\; \mathbf{e}^{\mathsf{T}}\mathbf{B}^{-1}\exp\{\mathbf{J}x\}\mathbf{e}, \tag{127}$$

where $\mathbf{J} = \mathrm{diag}(-\rho_2, \ldots, -\rho_{n^+ +2})$. Since $\mathbb{P}(\tau_+(x) < \infty) = \mathbb{P}(M(\infty) > x)$ and we know that $\mathbb{P}(M(\infty) > x) \sim \mathbb{E}_L \mathrm{e}^{-\gamma B(\infty)} \mathrm{e}^{-\gamma x}$, $x \to \infty$, we can use (127) and what we know about the elements of $\mathbf{J}$ to identify $\mathbb{E}_L \mathrm{e}^{-\gamma B(\infty)}$ as the sum of the elements in the first column of $\mathbf{B}^{-1}$. Now, it is well known, see e.g. [21], that w.r.t. $\mathbb{P}_L$, X is still the sum of a Brownian motion with drift and a compound Poisson process with phase-type distributed jumps, with Lévy exponent $\kappa_L(\alpha) = \kappa(\alpha + \gamma)$. Furthermore, if we define $\mathbf{d} = (\gamma\mathbf{I} - \mathbf{T}^-)^{-1}\mathbf{t}^-$ and let $\mathbf{D}$ be the diagonal matrix with the d_i on the diagonal then (w.r.t. $\mathbb{P}_L$) the intensity matrix corresponding to negative jumps is $\mathbf{T}_\gamma^- = \mathbf{D}^{-1}\mathbf{T}^-\mathbf{D} - \gamma\mathbf{I}$. It is clear that the equation $\kappa_L(\alpha) = 0$ has $n^- + 1$ roots with negative real part $\tilde{\rho}_k$, $k = 1, 2, \ldots, n^- + 1$ (all of the form $\tilde{\rho}_k = \rho - \gamma$ where

$\kappa(\rho) = 0$ and $\Re(\rho) \le 0$). Define $\hat{\tilde{F}}_i^-$ in the same way as $\hat{F}_i^-$ with $\mathbf{T}^-$ replaced by $\mathbf{T}_\gamma^-$. In a fashion similar to the derivation of (111), we obtain (in the obvious notation)

$$0 = 1 - e^{-\tilde{\rho}_k x}\left(\tilde{p}_c^-(t) + \sum_{i=1}^{n^-}\tilde{p}_i^-(t)\hat{\tilde{F}}_i^-(-\tilde{\rho}_k)\right) - \mathbb{E}_L\left[e^{\tilde{\rho}_k X(t)}; t < \tau_-(-x)\right]. \quad (128)$$

From (128) it follows that

$$e^{\tilde{\rho}_k x} = \tilde{p}_c^- + \sum_{i=1}^{n^-}\tilde{p}_i^-\hat{\tilde{F}}_i^-(-\tilde{\rho}_k),\ k = 1, 2, \ldots, n^- + 1, \quad (129)$$

and if we define $\tilde{\mathbf{B}}$ as the matrix with kth row equal to $\big(1\ \hat{\tilde{F}}_1^-(-\tilde{\rho}_k)\ \ldots\ \hat{\tilde{F}}_{n^-}^-(-\tilde{\rho}_k)\big)$, it is clear that

$$\mathbb{P}_L\big(\tau_-(-x) = \infty\big) = 1 - \mathbf{e}^{\mathsf{T}}\tilde{\mathbf{B}}^{-1}\exp\{\tilde{\mathbf{J}}x\}\mathbf{e}, \quad (130)$$

where $\tilde{\mathbf{J}} = \mathrm{diag}(\tilde{\rho}_1, \ldots, \tilde{\rho}_{n^-+1})$. All that now remains in order to describe the asymptotics of ℓ^b is to evaluate the integrals in (121); we omit the details. □

An important lesson to learn from this example is that the case where X is spectrally one-sided is much easier than the general case. In fact, if we e.g. take X to be spectrally positive then according to Corollary 10.5, $\ell^b \sim Ce^{-\gamma b}$, $b \to \infty$, where

$$C = -m\big\{1 - \gamma\lambda^+\boldsymbol{\alpha}^+(\gamma\mathbf{I} + \mathbf{T}^+)^{-1}\big\{(\gamma\mathbf{I} + \mathbf{T}^+)^{-1} - (\mathbf{T}^+)^{-1}\big\}\,\mathbf{e}/\kappa'(\gamma)\big\},$$

i.e. we need only to know γ to compute C (the same thing holds when X is spectrally negative, but C comes out in a slightly different way, again see Corollary 10.5), whereas in the general case all roots of $\kappa(\alpha) = 0$ are required in order to completely describe the asymptotic behavior of ℓ^b.

11 Loss Rate Asymptotics: Heavy Tails

The main result of this section states that under some heavy-tailed conditions, $\ell^b \sim \int_b^\infty \overline{\nu}(y)\mathrm{d}y$ which in view of Lemma 2.6, can be interpreted as stating that Theorem 3.2 still holds when the random walk is replaced by a Lévy process. More precisely:

Theorem 11.1 *Let X be a Lévy process with Lévy measure $\nu \in \mathscr{S}$ and finite negative mean $m = \mathbb{E}X(1) < 0$. Consider the conditions*

A $\mathbb{E}X(1)^2 < \infty$ *and* $\int_b^\infty \overline{\nu}_I(y)\,\mathrm{d}y/\overline{\nu}_I(b) = \mathrm{O}(b)$.

B $\overline{\nu}(b) \sim L(b)b^{-\alpha}$ *where L is a locally bounded slowly varying function and* $1 < \alpha < 2$.

If either **A** *or* **B** *holds, then*

$$\ell^b \sim \int_b^\infty \overline{\nu}(y)\mathrm{d}y. \tag{131}$$

It is worth noting, that the requirement on the tail of ν in **A** is very weak. Indeed, suppose $\overline{\nu}_I(x) \sim \overline{B}(x)$ where B is either lognormal, Benktander or heavy-tailed Weibull. Then we recognize $a(x) = \int_x^\infty \overline{B}(y)\,\mathrm{d}y/\overline{B}(x)$ as the mean-excess function and it is known (see [69]), that $a(x) = \mathrm{o}(x)$. Furthermore, it is easily checked that the condition is satisfied when B is a Pareto or Burr distribution, provided that the second moment is finite. Another remark is that we may use the results of Embrechts et al. [56] to express sufficient conditions for Theorem 11.1 in terms of the distribution of $X(1)$.

We will also derive Theorem 11.3 below, which gives an expression for the m.g.f. of the stationary distribution in the case of one-sided reflection. This result is of some independent interest and is useful in the proof of Theorem 11.1; see further Remark 11.4 below. Recall the decomposition of the one-sided reflected process, $V^\infty(t) = V^\infty(0) + X(t) + L(t)$, let $L^c(t)$ and $L^j(t)$ denote the continuous and jump parts of the local time, respectively, and recall that $\Theta = \{\alpha \in \mathbb{C} : \mathbb{E}\mathrm{e}^{\Re(\alpha)X(1)} < \infty\}$.

Lemma 11.2 *Consider a Lévy process X, let V^∞ be the process one-sided reflected at 0 and let L^c and L^j be the continuous and jump part of the corresponding local time L, respectively. Then, for $\alpha \in \Theta$ and $V^\infty(0) = x \geq 0$,*

$$M(t) = \kappa(\alpha)\int_0^t \mathrm{e}^{\alpha V^\infty(s)}\,\mathrm{d}s + \mathrm{e}^{\alpha x} - \mathrm{e}^{\alpha V^\infty(t)} + \alpha L^c(t) + \sum_{0\leq s\leq t}(1 - \mathrm{e}^{-\alpha \Delta L(s)}) \tag{132}$$

is a martingale.

Proof The proof is similar to (but slightly easier than) the proof of Proposition 8.2, once we note that L can increase only when V is zero. □

Theorem 11.3 *Suppose $-\infty < m = \mathbb{E}X(1) < 0$, so that $V^\infty(\infty) = \lim_{t\to\infty} V^\infty(t)$ exists in distribution. For $\alpha \in \Theta$ we have*

$$\mathbb{E}\mathrm{e}^{\alpha V^\infty(\infty)} = -\frac{1}{\kappa(\alpha)}\Big(\alpha \mathbb{E}_{\pi^\infty} L^c(1) + \mathbb{E}_{\pi^\infty}\sum_{0\leq s\leq 1}(1 - \mathrm{e}^{-\alpha\Delta L(s)})\Big). \tag{133}$$

Proof Replacing x by a r.v. distributed as $V^{\infty}(\infty)$ in (132) and taking expectations at $t = 1$ gives

$$0 \;=\; \kappa(\alpha)\,\mathbb{E}_{\pi^\infty}\int_0^1 \mathrm{e}^{\alpha V^{\infty}(s)}\,\mathrm{d}s + \alpha\mathbb{E}_{\pi^\infty}L^c(1) + \mathbb{E}_{\pi^\infty}\sum_{0\le s\le 1}\left(1-\mathrm{e}^{-\alpha\Delta L(s)}\right).$$

Now just note that the expectation of the integral equals $\mathbb{E}\mathrm{e}^{\alpha V^{\infty}(\infty)}$. □

Remark 11.4 If X has no negative jumps, the term $\mathbb{E}_{\pi^\infty}\sum_{0\le s\le 1}(1-\mathrm{e}^{-\alpha\Delta L(s)})$ disappears, and $\mathbb{E}_{\pi^\infty}L^c(1) = \mathbb{E}_{\pi^\infty}L(1) = -m$, and we see that Theorem 11.3 indeed is a generalization of Corollary 3.4 in Chap. IX [11] which is itself a generalization of the Pollaczeck-Khinchine formula.

The expression provided by Theorem 11.3 can be compared to related identities from fluctuation theory (see Chap. VI in [28] or Chap. 6 in [104]). Indeed, in view of (22), we may let $q \downarrow 0$ in equation (1), VI in [28] to conclude the that l.h.s. of (133) is equal to $\phi(0)/\phi(-\alpha)$ where ϕ is the Laplace exponent of the upward ladder height processes. Furthermore, letting $\hat{\phi}(\alpha)$ denote the Laplace exponent of the downward ladder height processes, we obtain from the Wiener-Hopf factorization (Eq. (4) IV in [28]) that $-\kappa(\alpha) = \phi(-\alpha)\hat{\phi}(\alpha)$, for $\alpha \in \Theta$ in the case where the process is not compound Poisson. Thus we arrive at

$$\mathbb{E}\mathrm{e}^{\alpha V^{\infty}(\infty)} \;=\; \frac{\phi(0)}{\phi(-\alpha)} \;=\; -\frac{\phi(0)\hat{\phi}(\alpha)}{\kappa(\alpha)}.$$

□

Next, we use the results above to obtain an expression for the mean of the stationary distribution in the case of one-sided reflection.

Corollary 11.5 *If X is square integrable then V^{∞} is integrable and we have*

$$\mathbb{E}V^{\infty} \;=\; \frac{1}{2m}\Big(\mathbb{E}_{\pi^\infty}\sum_{0\le s\le 1}\Delta L(s)^2 - \mathbb{V}ar(X(1))\Big) \tag{134}$$

$$\;=\; \frac{1}{2m}\Big(\int_{-\infty}^{\infty} y^2\nu(\mathrm{d}y) + \sigma^2 - \int_0^{\infty}\int_{-\infty}^{-x}(x+y)^2\nu(\mathrm{d}y)\pi^{\infty}(\mathrm{d}x)\Big). \tag{135}$$

Proof Since $X(1)$ is non-degenerate, we have by Lemma 4 Chap. XV.1 in [59] that there exists $\varepsilon > 0$ such that $\kappa(\mathrm{i}t) \neq 0$ for $t \in (-\varepsilon, \varepsilon)\setminus\{0\}$, and we may use (133) to obtain the characteristic function ψ of V. We wish to show that ψ is differentiable at 0. Define

$$g(t) \;=\; \mathbb{E}_{\pi^\infty}\sum_{0\le s\le 1}(1-\mathrm{e}^{-\mathrm{i}t\Delta L(s)}),\;\; \ell_1 = \mathbb{E}_{\pi^\infty}L^c(1).$$

By Doob's inequality, we have that $\mathbb{E}X(1)^2 < \infty$ implies $\mathbb{E}L^2(1) < \infty$ and therefore $\mathbb{E}_{\pi^\infty} L^2(1) < \infty$, which in turn implies that g is twice differentiable at 0. We see that

$$g'(0) = \mathrm{i}\,\mathbb{E}_{\pi^\infty} \sum_{0 \le s \le 1} \Delta L(s) = \mathrm{i}\,\mathbb{E}_{\pi^\infty} L^j(1)\,, \quad g''(0) = \mathbb{E}_{\pi^\infty} \sum_{0 \le s \le 1} \Delta L(s)^2\,,$$

$\mathrm{i}\ell_1 + g'(0) = \mathrm{i}\mathbb{E}_{\pi^\infty} L(1) = -\mathrm{i}m$. Since X is square integrable, we may use formula (2.4.1) p. 27 in [111] to get $\kappa(\mathrm{i}t) = \kappa'(0)\mathrm{i}t + \mathrm{o}(t)$. By combining this with equation (133), we conclude that

$$\lim_{t\to 0} \frac{\mathbb{E}e^{\mathrm{i}tV} - 1}{t} = \lim_{t\to 0} \frac{-t\mathrm{i}\ell_1 - g(t) - \kappa(\mathrm{i}t)}{t\kappa(\mathrm{i}t)} = \lim_{t\to 0} \frac{-t\mathrm{i}\ell_1 - g(t) - \kappa(\mathrm{i}t)}{\kappa'(0)\mathrm{i}t^2}\,,$$

provided that the limit exists. We may confirm that this is true, by applying l'Hospital's rule twice to the real and imaginary part separately

$$\begin{aligned}\lim_{t\to 0} \frac{-t\mathrm{i}\ell_1 - g(t) - \kappa(\mathrm{i}t)}{\kappa'(0)\mathrm{i}t^2} &= \lim_{t\to 0} \frac{-\mathrm{i}\ell_1 - g'(t) - \mathrm{i}\kappa'(\mathrm{i}t)}{2\mathrm{i}\kappa'(0)t} \\ &= \lim_{t\to 0} \frac{-g''(t) + \kappa''(\mathrm{i}t)}{2\mathrm{i}\kappa'(0)} = \frac{-g''(0) + \kappa''(0)}{2\mathrm{i}\kappa'(0)}.\end{aligned}$$

We see that ψ is differentiable. In itself, this does not entail integrability of V, but a short argument using the Law of Large Numbers and the fact that V is non-negative, yields that V is integrable. The first moment is

$$\mathbb{E}V = \frac{-g''(0) + \kappa''(0)}{2(-1)\kappa'(0)}$$

which is (134). We obtain (135) by conditioning on the value of the process prior to a jump. □

We proceed to the proof of Theorem 11.1. In order to establish (131), we need to prove that 1 is a lower bound for $\liminf_b \ell^b/\overline{\upsilon}_I(b)$ and an upper bound for $\limsup_b \ell^b/\overline{\upsilon}_I(b)$. The former is established in Proposition 11.7 and is seen to hold without the conditions assumed in Theorem 11.1. In the proof of the latter, we use Proposition 11.6 to establish the inequality

$$\frac{m}{b}\int_0^b \overline{\pi}^b(x)\,\mathrm{d}x \le \frac{m}{b}\int_0^b \overline{\pi}^\infty(x)\,\mathrm{d}x - m\overline{\pi}^\infty(b) \tag{136}$$

and the proof then follows two distinct routes depending on which of the conditions **A** or **B** is assumed. Under assumption **A**, we are allowed to rewrite the integral on the right-hand side of (136) as $\int_0^\infty \overline{\pi}^\infty(x)\,\mathrm{d}x - \int_b^\infty \overline{\pi}^\infty(x)\,\mathrm{d}x$. The first of these integrals is the mean of the stationary distribution in the case of one-sided reflection. This observation and Corollary 11.5 are important keys to the proof in this case. Under assumption **B** the proof essentially consists of combining the inequality (136) with repeated applications of Karamata's theorem.

Proposition 11.6 *Let X be a Lévy process, and let $\overline{\pi}^{\infty}(y), \overline{\pi}^{b}(y)$ be the tails of the reflected (one/two-sided) distributions. Then we have the following inequalities for $x > 0, b > 0$*

$$0 \leq \overline{\pi}^{\infty}(x) - \overline{\pi}^{b}(x) \leq \overline{\pi}^{\infty}(b)\,. \tag{137}$$

Proof The inequalities in (137) are trivial for $x > b$. Let $0 \leq x \leq b$. The inequality $\overline{\pi}^{b}(x) \leq \overline{\pi}^{\infty}(x)$ follows from the representations (6) and (23). The inequality $\overline{\pi}^{\infty}(x) - \overline{\pi}^{b}(x) \leq \overline{\pi}^{\infty}(b)$, follows by dividing the sample paths of X which cross above x into those which do so by first passing below $x - b$, and those which stay above $x - b$. To be precise, define $\tau(y) = \inf\{t > 0 : X(t) \geq y\}$ and $\sigma(y) = \inf\{t > 0 : X(t) < y\}$ to be the first passage times above and below y respectively. Then we can consider the event that a path crosses below $x - b$ before eventually passing above x, and since such a path must pass an interval of length at least b, we find that

$$\begin{aligned}\mathbb{P}\big(\sigma(x-b) < \tau(x) < \infty\big) \;&\leq\; \mathbb{P}\Big(\sup_{t>0} X\big(\sigma(x-b)+t\big) - X\big(\sigma(x-b)\big) > b\Big)\\ &=\; \mathbb{P}\big(\tau(b) < \infty\big)\,.\end{aligned}$$

where we used the strong Markov property in the last equality. Next, we apply (23) to find

$$\begin{aligned}\overline{\pi}^{\infty}(x) \;=\; \mathbb{P}\big(\tau(x) < \infty\big) \;&=\; \mathbb{P}\big(\tau(x) < \sigma(x-b)\big) + \mathbb{P}\big(\sigma(x-b) < \tau(x) < \infty\big)\\ &\leq\; \overline{\pi}^{b}(x) + \mathbb{P}\big(\tau(b) < \infty\big) = \overline{\pi}^{b}(x) + \overline{\pi}^{\infty}(b)\,,\end{aligned}$$

where we have used the equality $\mathbb{P}\big(\tau(x) < \sigma(x-b) \leq \infty\big) = \overline{\pi}^{b}(x)$, which is a restatement of (6). □

Proposition 11.7 *For any Lévy process we have* $1 \leq \liminf_{b\to\infty} \dfrac{\ell^b}{\overline{\nu}_I(b)}$.

Proof We have

$$\int_0^b \pi^b(\mathrm{d}x) \int_b^{\infty} (y - b + x)\, \nu(\mathrm{d}y) \;\leq\; \ell^b$$

since the left-hand side is the contribution to ℓ^b by the jumps larger than b. Now just note that

$$\overline{\nu}_I(b) \le \int_b^\infty (y-b)\nu(\mathrm{d}y) + \int_0^b x\pi^b(\mathrm{d}x)\overline{\nu}(b) = \int_0^b \pi^b(\mathrm{d}x)\int_b^\infty (y-b+x)\nu(\mathrm{d}y)\,.$$

□

We are now ready for the proof of Theorem 11.1.

Proof Thanks to Proposition 11.7, we only need to prove

$$\limsup_b \ell^b/\overline{\nu}_I(b) \le 1\,. \tag{138}$$

Define

$$\mathscr{I}_1 = \frac{m}{b}\int_0^b x\pi^b(\mathrm{d}x)\,,\ \mathscr{I}_2 = \frac{\sigma^2}{2b}\,,\ \mathscr{I}_3 = \frac{1}{2b}\int_0^b \pi^b(\mathrm{d}x)\int_{-\infty}^\infty \varphi_b(x,y)\nu(\mathrm{d}y)\,.$$

where the function $\varphi_b(\cdot,\cdot)$ is that of Theorem 1.1, with the dependence on b made explicit. From Proposition 11.6 we have $\overline{\pi}^\infty(x) - \overline{\pi}^\infty(b) \le \overline{\pi}^b(x)$ and since m is assumed to be negative, we have $m\overline{\pi}^b(x) \le m(\overline{\pi}^\infty(x) - \overline{\pi}^\infty(b))$. Applying this inequality to expression for the loss rate in Theorem 1.1 we obtain the following inequality:

$$\ell^b \le \frac{m}{b}\int_0^b \overline{\pi}^\infty(x)\,\mathrm{d}x - m\overline{\pi}^\infty(b) + \mathscr{I}_2 + \mathscr{I}_3\,. \tag{139}$$

First, we assume **A** holds. By (26) we have

$$\lim_b \frac{-m\overline{\pi}^\infty(b)}{\overline{\nu}_I(b)} = 1\,, \tag{140}$$

so we will be done if we can show

$$\limsup_b \frac{1}{\overline{\nu}_I(b)}\left[\frac{m}{b}\int_0^b \overline{\pi}^\infty(y)\,\mathrm{d}y + \mathscr{I}_2 + \mathscr{I}_3\right] = 0\,. \tag{141}$$

We start by rewriting the term in the brackets above. Using Corollary 11.5 and the assumption that $\mathbb{E}X(1)^2 < \infty$ we have that $\int_0^\infty \overline{\pi}^\infty(y)\mathrm{d}y < \infty$ and using (134)

$$\begin{aligned}\frac{m}{b}\int_0^b \overline{\pi}^\infty(y)\,\mathrm{d}y &= \frac{m}{b}\int_0^\infty \overline{\pi}^\infty(y)\,\mathrm{d}y - \frac{m}{b}\int_b^\infty \overline{\pi}^\infty(y)\,\mathrm{d}y\\ &= \frac{\mathbb{E}_{\pi^\infty}[\sum_{0\le s\le 1}\Delta L(s)^2] - \mathbb{V}ar(X(1))}{2b} + \frac{|m|}{b}\int_b^\infty \overline{\pi}^\infty(y)\,\mathrm{d}y\,.\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mathscr{I}_2 + \mathscr{I}_3 &= \frac{\sigma^2}{2b} + \frac{1}{2b}\int_0^b \pi^b(\mathrm{d}x)\Big(\int_{-\infty}^{-x} -(x^2+2xy)\nu(\mathrm{d}y) + \int_{-x}^{b-x} y^2\nu(\mathrm{d}y) \\
&\quad + \int_{b-x}^{\infty} \big[2y(b-x)-(b-x)^2\big]\nu(\mathrm{d}y)\Big) \\
&= \frac{\sigma^2}{2b} + \frac{1}{2b}\int_{-\infty}^{\infty} y^2\nu(\mathrm{d}y) + \frac{1}{2b}\int_0^b \pi^b(\mathrm{d}x)\int_{-\infty}^{-x}\big[-(x^2+2xy)-y^2\big]\nu(\mathrm{d}y) \\
&\quad + \frac{1}{2b}\int_0^b \pi^b(\mathrm{d}x)\int_{b-x}^{\infty}\big[2y(b-x)-(b-x)^2-y^2\big]\nu(\mathrm{d}y) \\
&= \frac{\sigma^2}{2b} + \frac{1}{2b}\int_{-\infty}^{\infty} y^2\nu(\mathrm{d}y) - \frac{1}{2b}\int_0^b \pi^b(\mathrm{d}x)\int_{-\infty}^{-x}(x+y)^2\nu(\mathrm{d}y) \\
&\quad - \frac{1}{2b}\int_0^b \pi^b(\mathrm{d}x)\int_{b-x}^{\infty}(y-(b-x))^2\nu(\mathrm{d}y) \\
&= \frac{\mathbb{V}ar(X(1)) - \mathbb{E}_{\pi^b}\sum_{0\le s\le 1}\Delta L(s)^2}{2b} - \frac{1}{2b}\int_0^b \pi^b(\mathrm{d}x)\int_{b-x}^{\infty}(y-(b-x))^2\nu(\mathrm{d}y).
\end{aligned}$$

The last equation follows from Example 25.12, p. 163 in [129], as does the following:

$$\begin{aligned}
\mathbb{E}_{\pi^b}\sum_{0\le s\le 1}\Delta L(s)^2 &= \mathbb{E}_{\pi^b}\sum_{0\le s\le 1}(V(s-)+\Delta X(s))^2\mathbb{1}(V(s-)+\Delta X(s)<0) \\
&= \int_0^b\left(\mathbb{E}\sum_{0\le s\le 1}(x+\Delta X(s))^2\mathbb{1}(x+\Delta X(s)<0)\right)\pi^b(\mathrm{d}x) \\
&= \int_0^b \pi^b(\mathrm{d}x)\int_{-\infty}^{-x}(x+y)^2\nu(\mathrm{d}y)\,, \qquad (142)
\end{aligned}$$

where we use Theorem 2.7, p. 41 in [104] in the last equation. Next, we note the fact that

$$\mathbb{E}_{\pi^\infty}\sum_{0\le s\le 1}\Delta L(s)^2 \le \mathbb{E}_{\pi^b}\sum_{0\le s\le 1}\Delta L(s)^2,$$

which can be verified using partial integration and (37). Using this in the last equation above, we may continue our calculation and obtain

$$\begin{aligned}
\mathscr{I}_2 + \mathscr{I}_3 \;\le\; & \frac{\mathbb{V}ar(X(1)) - \mathbb{E}_{\pi^\infty}\sum_{0\le s\le 1}\Delta L(s)^2}{2b} \\
& - \frac{1}{2b}\int_0^b \pi^b(\mathrm{d}x)\int_{b-x}^{\infty}(y-(b-x))^2\nu(\mathrm{d}y)\,.
\end{aligned}$$

Comparing the expressions above we see that fractions cancel, and the expression in the brackets in (141) is less than

$$\frac{|m|}{b}\int_b^\infty \overline{\pi}^\infty(y)\,\mathrm{d}y - \frac{1}{2b}\int_0^b\int_{b-x}^\infty (y-(b-x))^2\nu(\mathrm{d}y)\pi^b(\mathrm{d}x)\,.$$

Applying partial integration

$$\begin{aligned}
&\frac{|m|}{b}\int_b^\infty \overline{\pi}^\infty(y)\,\mathrm{d}y - \frac{1}{2b}\int_0^b\int_{b-x}^\infty (y-(b-x))^2\nu(\mathrm{d}y)\pi^b(\mathrm{d}x)\\
&= \frac{|\mathbb{E}X(1)|}{b}\int_b^\infty \overline{\pi}^\infty(y)\,\mathrm{d}y - \frac{1}{2b}\int_b^\infty (y-b)^2\nu(\mathrm{d}y) - \frac{1}{b}\int_0^b \overline{\pi}^b(x)\overline{\nu}_I(b-x)\mathrm{d}x\\
&\le \frac{|m|}{b}\int_b^\infty \overline{\pi}^\infty(y)\,\mathrm{d}y - \frac{1}{2b}\int_b^\infty (y-b)^2\nu(\mathrm{d}y)\\
&= \frac{|m|}{b}\int_b^\infty \overline{\pi}^\infty(y)\,\mathrm{d}y - \frac{1}{b}\int_b^\infty \overline{\nu}_I(y)\,\mathrm{d}y.
\end{aligned}$$

Returning to (141) and applying the results above we get

$$\begin{aligned}
&\limsup_b \frac{1}{\overline{\nu}_I(b)}\left[\frac{m}{b}\int_0^b \overline{\pi}^\infty(y)\,\mathrm{d}y + \mathscr{I}_2 + \mathscr{I}_3\right]\\
&\quad\le \limsup_b \frac{1}{\overline{\nu}_I(b)}\left[\frac{|m|}{b}\int_b^\infty \overline{\pi}^\infty(y)\,\mathrm{d}y - \frac{1}{b}\int_b^\infty \overline{\nu}_I(y)\,\mathrm{d}y\right]\\
&\quad= \limsup_b \frac{\int_b^\infty \overline{\nu}_I(y)\,\mathrm{d}y}{b\overline{\nu}_I(b)}\left[\frac{\int_b^\infty |m|\overline{\pi}^\infty(y)\,\mathrm{d}y}{\int_b^\infty \overline{\nu}_I(y)\,\mathrm{d}y} - 1\right] = 0,
\end{aligned}$$

where the last equality follows since the term in the brackets tends to 0, and the fraction outside it is bounded by assumption. This proves that (131) holds under condition **A**.

We now assume condition **B** and start by noticing the following consequences of the assumptions

$$\int_b^\infty \overline{\nu}(y)\,\mathrm{d}y \sim \int_b^\infty \frac{L(y)}{y^\alpha}\,\mathrm{d}y \sim \frac{b^{-\alpha+1}L(b)}{\alpha-1}\,, \qquad b\to\infty\,, \tag{143}$$

where the last equivalence follows by Proposition 1.5.10 of [32] and the fact that $\alpha > 1$. Since by Proposition 1.3.6 of [32], we have $b^{-\alpha+2}L(b)\to\infty$, (143) implies $b\overline{\nu}_I(b)\to\infty$.

The inequality (139) still holds, as does the limit in (140), so we proceed to analyze $m\int_0^b \overline{\pi}^\infty(y)\mathrm{d}y/(\overline{\nu}_I(b)b)$. Since $b\overline{\nu}_I(b)\to\infty$ as $b\to\infty$ we see that for any A

$$\lim_{b\to\infty} \frac{m}{b\overline{\nu}_I(b)} \int_0^A \overline{\pi}^{\infty}(y)\,\mathrm{d}y = 0\,. \tag{144}$$

Because of the result above we have for any A

$$\lim_{b\to\infty} \frac{m}{b\overline{\nu}_I(b)} \int_0^b \overline{\pi}^{\infty}(y)\,\mathrm{d}y = \lim_{b\to\infty} \frac{m}{b\overline{\nu}_I(b)} \int_A^b \overline{\pi}^{\infty}(y)\,\mathrm{d}y$$

and using $|m|\overline{\pi}^{\infty}(b) \sim \overline{\nu}_I(b) \sim b^{-\alpha+1}L(b)/(\alpha-1)$ we have

$$\begin{aligned}\lim_{b\to\infty} \frac{m}{b\overline{\nu}_I(b)} \int_A^b \overline{\pi}^{\infty}(y)\,\mathrm{d}y &= \lim_{b\to\infty} -\frac{1}{b\overline{\nu}_I(b)} \int_A^b \overline{\nu}_I(y)\,\mathrm{d}y \\ &= -\lim_{b\to\infty} \frac{1}{b\overline{\nu}_I(b)} \int_A^b \frac{y^{-\alpha+1}L(y)}{(\alpha-1)}\,\mathrm{d}y\end{aligned}$$

in the sense that if either limit exits so does the other and they are equal. Furthermore, since $-\alpha+1 > -1$ and L is locally bounded, we may apply Proposition 1.5.8 in [32] to obtain

$$-\lim_{b\to\infty} \frac{1}{b\overline{\nu}_I(b)} \int_A^b \frac{y^{-\alpha+1}L(y)}{(\alpha-1)}\,\mathrm{d}y = -\lim_{b\to\infty} \frac{1}{b\overline{\nu}_I(b)} \frac{b^{-\alpha+2}L(b)}{(-\alpha+2)(\alpha-1)} = -\frac{1}{-\alpha+2}.$$

That is, we obtain

$$\lim_{b\to\infty} \frac{m}{b\overline{\nu}_I(b)} \int_0^b \overline{\pi}^{\infty}(y)\,\mathrm{d}y = -\frac{1}{-\alpha+2}. \tag{145}$$

Returning to (139) we have

$$\begin{aligned}\limsup_b \frac{\ell^b}{\overline{\nu}_I(b)} &= \limsup_b \left[\frac{m}{b\overline{\nu}_I(b)} \int_0^b \overline{\pi}^{\infty}(y)\,\mathrm{d}y - \frac{m\overline{\pi}^{\infty}(b)}{\overline{\nu}_I(b)} + \frac{\mathscr{I}_2}{\overline{\nu}_I(b)} + \frac{\mathscr{I}_3}{\overline{\nu}_I(b)}\right] \\ &= -\frac{1}{-\alpha+2} + 1 + \limsup_b \left[\frac{\mathscr{I}_2}{\overline{\nu}_I(b)} + \frac{\mathscr{I}_3}{\overline{\nu}_I(b)}\right].\end{aligned} \tag{146}$$

Since $b\overline{\nu}_I(b) \to \infty$ we have

$$\limsup_b \mathscr{I}_2/\overline{\nu}_I(b) = \limsup_b \frac{\sigma^2}{2b\overline{\nu}_I(b)} = 0\,,$$

and we may continue our calculation from (146)

$$\frac{1}{-\alpha+2}+1+\limsup_b\Big[\frac{\mathscr{I}_2}{\overline{\nu}_I(b)}+\frac{\mathscr{I}_3}{\overline{\nu}_I(b)}\Big]=\frac{1}{-\alpha+2}+1+\limsup_b\Big[\frac{\mathscr{I}_3}{\overline{\nu}_I(b)}\Big] \tag{147}$$

So we turn our attention to $\mathscr{I}_3$. First we divide the integral into two:

$$2b\mathscr{I}_3=\underbrace{\int_0^b \pi^b(\mathrm{d}x)\Big(\int_{-\infty}^{-x}-(x^2+2xy)\nu(\mathrm{d}y)+\int_{-x}^{0}y^2\nu(\mathrm{d}y)\Big)}_{A(b)} \tag{148}$$

$$+\underbrace{\int_0^b \pi^b(\mathrm{d}x)\Big(\int_0^{b-x}y^2\nu(\mathrm{d}y)+\int_{b-x}^{\infty}2(b-x)y-(b-x)^2\nu(\mathrm{d}y)\Big)}_{B(b)}. \tag{149}$$

We may assume ν is bounded from below; otherwise truncate ν at $-L$ for some $L>0$ chosen large enough to ensure that the mean of $X(1)$ remains negative. This truncation may increase the loss rate, which is not a problem, since we are proving an upper bound. Thus, we may assume that $A(b)$ is bounded:

$$A(b)\le\int_0^b \pi^b(\mathrm{d}x)\int_{-\infty}^{0}y^2\nu(\mathrm{d}y)\le\int_{-\infty}^{0}y^2\nu(\mathrm{d}y)<\infty\,.$$

And therefore, since $b\overline{\nu}_I(b)\to\infty$, we have

$$\frac{A(b)}{2b\overline{\nu}_I(b)}\to 0\,. \tag{150}$$

Turning to $B(b)$, we first perform partial integration

$$\begin{aligned}
B(b)&=\int_0^b y^2\nu(\mathrm{d}y)+\int_b^{\infty}2by-b^2\nu(\mathrm{d}y)-\int_0^b\overline{\nu}_I(b-x)\overline{\pi}^b(x)\,\mathrm{d}x\\
&\le\int_0^b y^2\nu(\mathrm{d}y)+\int_b^{\infty}2by-b^2\nu(\mathrm{d}y)\\
&=\int_0^b 2y\overline{\nu}(y)\,\mathrm{d}y-b^2\overline{\nu}(b)+\int_b^{\infty}2by-b^2\nu(\mathrm{d}y)\\
&=\int_0^b 2y\overline{\nu}(y)\,\mathrm{d}y+2b\int_b^{\infty}\overline{\nu}(y)\,\mathrm{d}y\,.
\end{aligned}$$

Since $y\overline{\nu}(y) \sim y^{-\alpha+1}L(y)$ way may apply Proposition 1.5.8 from [32] to get

$$\int_0^b 2y\overline{\nu}(y)\,\mathrm{d}y \sim 2\frac{L(b)b^{-\alpha+2}}{2-\alpha}\,,$$

and therefore

$$\lim_b \frac{1}{2b\overline{\nu}_I(b)} \int_0^b 2y\overline{\nu}(y)\,\mathrm{d}y = \frac{\alpha-1}{2-\alpha}\,.$$

Combining this with our inequality for $B(b)$ above, we have

$$\limsup_{b\to\infty} \frac{B(b)}{2b\overline{\nu}_I(b)} \;\le\; \frac{\alpha-1}{2-\alpha} + 1 \;=\; \frac{1}{2-\alpha}\,.$$

Finally, by combining this with (146), (150) and (147), we obtain (138). □

12 Loss Rate Symptotics: No Drift

In both Sects. 10 and 11 it was assumed that the underlying stochastic process had negative mean, and as discussed in Sect. 1 this also gives the asymptotic behavior in the case of positive drift. Thus, it remains to give an asymptotic expression as $b \to \infty$ for the loss rate in the zero-mean case. The result is as follows:

Theorem 12.1

a) *Let $\{X(t)\}$ be a Lévy process with $m = \mathbb{E}X(1) = 0$ and*

$$\psi^2 \;=\; \mathbb{V}ar\big(X(1)\big) \;=\; \kappa''(0) \;=\; \sigma^2 + \int_{-\infty}^{\infty} y^2\,\nu(\mathrm{d}y) \;<\infty\,.$$

Then

$$\ell^b \sim \frac{1}{2b}\mathbb{V}ar(X(1))\,, \qquad b\to\infty\,. \tag{151}$$

b) *Let $\{X(t)\}$ be a Lévy process with Lévy measure ν. Assume $\mathbb{E}X(1) = 0$ and that for some $1 < \alpha < 2$, there exist slowly varying functions $L_1(x)$ and $L_2(x)$ such that for $L(x) = L_1(x) + L_2(x)$, we have*

$$\overline{\nu}(x) = x^{-\alpha}L_1(x) \qquad \nu(-x) = x^{-\alpha}L_2(x) \qquad \lim_{x\to\infty}\frac{L_1(x)}{L(x)} = \frac{\beta+1}{2} \tag{152}$$

where $\nu(x) = \nu(-\infty, x])$ and $\overline{\nu}(x) = \nu[x,\infty)$. Then, setting

$$\rho = 1/2 + (\pi\alpha)^{-1}\arctan(\beta\tan(\pi\alpha/2))\,,\;\; c_+ = (\beta+1)/2\,,\;\; c_- = (1-\beta)/2\,,$$

we have $\ell^b \sim \gamma L(b)/b^{\alpha-1}$ *where*

$$\gamma = \frac{c_- B(2-\alpha\rho, \alpha\rho) + c_+ B(2-\alpha(1-\rho), \alpha(1-\rho))}{B(\alpha\rho, \alpha(1-\rho))(\alpha-1)(2-\alpha)}$$

and $B(x,y) = \Gamma(x)\Gamma(y)/\Gamma(x+y)$ *is the Beta function.*

By comparing to Example 8.8, we see that the loss rate behaves asymptotically like that of a stable Lévy process.

To prove Theorem 12.1, we will use the fact that by properly scaling our Lévy process we may construct a sequence of Lévy processes which converges weakly to either a Brownian Motion or a stable process. Since ℓ^b has been calculated for both Brownian Motion and stable processes in Examples 8.7 and 8.8, we may use this convergence to obtain loss rate asymptotics in the case of zero drift, provided that the loss rate is continuous in the sense, that weak convergence (in the sense of Proposition 12.4 below) of the involved processes implies convergence of the associated loss rates. The required continuity results are established in Theorems 12.2 and 12.3.

Theorem 12.2 *Let* $\{X^n\}_{n=0,1,\dots}$ *be a sequence of Lévy processes with associated loss rates* $\ell^{b,n}$. *Suppose* $X^n \overset{\mathscr{D}}{\to} X^0$ *in* $D[0,\infty)$ *and that the family* $\left(X(1)^n\right)_{n=1}^{\infty}$ *is uniformly integrable. Then* $\ell^{b,n} \to \ell^{b,0}$ *as* $n \to \infty$.

We shall also need:

Theorem 12.3 *Let* $\{X_n\}_{n=1,2,\dots}$ *be a sequence of weakly convergent infinitely divisible random variables, with characteristic triplets* (c_n, σ_n, ν_n). *Then for* $\alpha > 0$*:*

$$\lim_{a\to\infty} \sup_n \int_{[-a,a]^c} |y|^\alpha \nu_n(\mathrm{d}y) = 0 \iff \left(\left(|X_n|^\alpha\right)_{|n\geq 1} \text{ is uniformly integrable.}\right.$$

The result is certainly not unexpected and appears in Andersen and Asmussen [4] in this form. It is, however, a special case of more general results on uniform integrability and infinitely divisible measures on Banach Spaces given by Theorem 2 in Jurek and Rosinski [81]. Cf. also Theorem 25.3 in [129].

12.1 Weak Convergence of Lévy Processes

We prove here Theorems 12.2 and 12.3. We will need the following weak convergence properties, where $D[0,\infty)$ is the metric space of cadlag functions on $[0,\infty)$ endowed with the Skorokhod topology (see Chap. 3, Sect. 16 in [31] or Chap. 3 in [138]).

Proposition 12.4 *Let $X^0, X^1, X^2, \ldots$ be Lévy processes with characteristic triplet (c_n, σ_n, ν_n) for X^n. Then the following properties are equivalent:*

(i) $X(t)^n \xrightarrow{\mathscr{D}} X(t)^0$ *for some* $t > 0$*;*
(ii) $X(t)^n \xrightarrow{\mathscr{D}} X(t)^0$ *for all t;*
(iii) $\{X(t)^n\} \xrightarrow{\mathscr{D}} \{X(t)^0\}$ *in* $D[0, \infty)$*;*
(iv) $\tilde{\nu}_n \to \tilde{\nu}_0$ *weakly, where* $\tilde{\nu}_n$ *is the bounded measure*

$$\tilde{\nu}_n(\mathrm{d}y) \;=\; \sigma_n^2 \delta_0(\mathrm{d}y) + \frac{y^2}{1+y^2}\nu_n(\mathrm{d}y) \tag{153}$$

and $\tilde{c}_n \to \tilde{c}_0$ *where*

$$\tilde{c}_n \;=\; c_n + \int \left(\frac{y}{1+y^2} - y\mathbb{1}|y| \le 1 \right) \nu_n(\mathrm{d}y)$$

See e.g. [82, pp. 244–248], in particular Lemmas 13.15 and 13.17. If one of (i)–(iv) hold, we write simply $X^n \xrightarrow{\mathscr{D}} X^0$.

The following proposition is standard:

Proposition 12.5 *Let $p > 0$ and let $X_n \in L^p$, $n = 0, 1, \ldots,$ such that $X_n \xrightarrow{\mathscr{D}} X_0$. Then $\mathbb{E}|X_n|^p \to \mathbb{E}|X_0|^p$ if and only if the family $\left(|X_n|^p\right)_{n \ge 1}$ is uniformly integrable.*

First, we prove Theorem 12.3. This is achieved through several preliminary results, of which the first is Lemma 12.6 which essentially states we may disregard the behavior of the Lévy measures on the interval $[-1, 1]$ in questions regarding uniform integrability. It is therefore sufficient to prove Theorem 12.3 for compound Poisson distributions, which is done in Proposition 12.8.

We start by examining the case where the Lévy measures have uniformly bounded support, i.e., there exists $A > 0$ such that $\nu_n([-A, A]^c) = 0$ for all n. We know from Lemmas 25.6 and 25.7 in [129] that this implies the existence of finite exponential moments of X_n and therefore the mth moment of X_n exists and is finite for all $n, m \in \mathbb{N}$.

Lemma 12.6 *Suppose $X_n \xrightarrow{\mathscr{D}} X_0$ and the Lévy measures have uniformly bounded support. Then $\mathbb{E}[(X_n)^m] \to \mathbb{E}[(X_0)^m]$ for $m = 1, 2, \cdots$. In particular* (cf. Proposition 12.5) *the family $\left(|X_n|^\alpha\right)_{n \ge 1}$ is uniformly integrable for all $\alpha > 0$.*

Proof Since the Lévy measures are uniformly bounded, the characteristic exponent from (7) is

$$\kappa_n(t) = c_n t + \sigma_n^2 t^2/2 + \int_{-A}^{A} \left(\mathrm{e}^{ty} - 1 - ty\mathbb{1}|y| \le 1\right)\nu_n(\mathrm{d}y)\,. \tag{154}$$

With the aim of applying Proposition 12.4 we rewrite (154) as

$$\kappa_n(t) = \tilde{c}_n t + \int_{-A}^{A} \left(e^{ty} - 1 - \frac{ty}{1+y^2} \right) \frac{1+y^2}{y^2} \tilde{\nu}_n(dy) . \tag{155}$$

(the integrand is defined to be 0 at $y = 0$) where $\tilde{\nu}_n$ is given by (153) and

$$\tilde{c}_n = c_n + \int_{-A}^{A} \left(\frac{y}{1+y^2} - y \mathbb{1}|y| \le 1 \right) \nu_n(dy) .$$

According to Proposition 12.4 the weak convergence of $\{X_n\}_{n \ge 1}$ implies $\tilde{c}_n \to \tilde{c}_0$ and $\tilde{\nu}_n \stackrel{\mathscr{D}}{\to} \tilde{\nu}_0$. Since the integrand in (155) is bounded and continuous, this implies that $\kappa_n(t) \to \kappa_0(t)$, which in turn implies that all exponential moments converge. In particular, the family $\left(e^{X_n} + e^{-X_n}\right)_{n \ge 1}$ is uniformly integrable, which implies that $\left(|X_n|^\alpha\right)_{n \ge 1}$ is so.

Next, we express the condition of uniform integrability using the tail of the involved distributions. We will need the following lemma on weakly convergent compound Poisson distributions.

Lemma 12.7 *Let $U_0, U_1, \ldots$ be a sequence of positive independent random variables such that $U_n > 1$, and let $N_0, N_1, \ldots$ be independent Poisson random variables with rates $\lambda_0, \lambda_1 \ldots$ Set $X_n = \sum_1^{N_n} U_{i,n}$ (empty sum = 0) with the $U_{i,n}$ being i.i.d for fixed n with $U_{i,n} \stackrel{\mathscr{D}}{=} U_n$. Then $X_n \stackrel{\mathscr{D}}{\to} X_0$ if and only if $U_n \stackrel{\mathscr{D}}{\to} U_0$ and $\lambda_n \to \lambda_0$.*

Proof The 'if' part follows from the continuity theorem for characteristic functions. For the converse, we observe that $e^{-\lambda_n} \to e^{-\lambda_0} = \mathbb{P}(X_0 \le 1/2)$ since $1/2$ is a continuity point of X_0 (note that $\mathbb{P}(X_0 \le x) = \mathbb{P}(X_0 = 0)$ for all $x < 1$). Taking logs yields $\lambda_n \to \lambda_0$ and the necessity of $U_n \stackrel{\mathscr{D}}{\to} U_0$ then is obvious from the continuity theorem for characteristic functions. □

Using the previous result, we are ready to prove part of Theorem 12.3 for a class of compound Poisson distributions:

Proposition 12.8 *Let $U_0, U_1, \ldots, N_0, N_1, \ldots,$ and $X_0, X_1, \ldots$ be as in Lemma 12.7. Assume $X_n \stackrel{\mathscr{D}}{\to} X_0$. Then for $\alpha > 0$.*

$$\lim_{a \to \infty} \sup_n \mathbb{E}\left[X_n^\alpha \mathbb{1} X_n > a\right] = 0 \iff \lim_{a \to \infty} \sup_n \mathbb{E}\left[U_n^\alpha \mathbb{1} U_n > a\right] = 0 .$$

Proof To prove that the l.h.s. implies the r.h.s., we let $G_n(x) = \mathbb{P}(X_n \le x)$, $F_n(x) = \mathbb{P}(U_n \le x)$, $\overline{F}_n(x) = 1 - F_n(x)$, $\overline{G}_n(x) = 1 - G_n(x)$, and let $F_n^{*m}(x), G_n^{*m}(x)$ denote the m-fold convolutions. Then

$$\overline{G}_n(x) = \sum_{m=1}^{\infty} \frac{\lambda_n^m}{m!} e^{-\lambda_n} \overline{F_n^{*m}}(x) , \qquad x > 0$$

which implies $\overline{G}_n(x) \geq \lambda_n e^{-\lambda_n}\overline{F}_n(x)$. Letting $\beta = \sup_n e^{\lambda_n}/\lambda_n$, which is finite by Lemma 12.7, we get $\overline{F}_n(x) \leq \beta\overline{G}_n(x)$. Therefore:

$$\mathbb{E}[U_n^\alpha \mathbb{1} U_n > a] = \int_0^\infty \alpha t^{\alpha-1}\mathbb{P}(U_n > a \vee t)\mathrm{d}t \;=\; a^\alpha \overline{F}_n(a) + \alpha\int_a^\infty t^{\alpha-1}\overline{F}_n(t)\mathrm{d}t$$

$$\leq \beta a^\alpha \overline{G}_n(a) + \beta\alpha\int_a^\infty t^{\alpha-1}\overline{G}_n(t)\mathrm{d}t \;=\; \beta\mathbb{E}[X_n^\alpha \mathbb{1} X_n > a]\,.$$

Taking supremum and limits completes the first part of the proof.

For the converse we note that by Lemma 12.7 we have $F_n^{*1} \xrightarrow{\mathscr{D}} F_0^{*1}$ and it follows from the continuity theorem for characteristic functions that $F_n^{*m} \xrightarrow{\mathscr{D}} F_0^{*m}$. Fix $m \in \mathbb{N}$. Since $\left(\sum_{i=1}^m U_{i,n}\right)^\alpha \leq m^\alpha \sum_{i=1}^m U_{i,n}^\alpha$ and the family $\left(m^\alpha \sum_{i=1}^m U_{i,n}^\alpha\right)_{n\geq 1}$ is uniformly integrable, we have that also the family $\left(\sum_{i=1}^m U_{i,n}\right)_{n\geq 1}^\alpha$ is uniformly integrable. As noted above we have $\sum_{i=1}^m U_{i,n} \xrightarrow{\mathscr{D}} \sum_{i=1}^m U_{i,0}$, so Proposition 12.5 implies $\mathbb{E}\left(\sum_{i=1}^m U_{i,n}\right)^\alpha \to \mathbb{E}\left(\sum_{i=1}^m U_{i,0}\right)^\alpha$.

We next show $\mathbb{E}X_n^\alpha \to \mathbb{E}X_0^\alpha$ and thereby the assertion of the proposition. We have:

$$\lim_n \mathbb{E}X_n^\alpha \;=\; \lim_n \sum_{m=0}^\infty \mathbb{E}\Big(\sum_{i=1}^m U_{i,n}\Big)^\alpha \frac{\lambda_n^m}{m!}e^{-\lambda_n} \;=\; \sum_{m=0}^\infty \lim_n \mathbb{E}\Big(\sum_{i=1}^m U_{i,n}\Big)^\alpha \frac{\lambda_n^m}{m!}e^{-\lambda_n}$$

$$= \sum_{m=0}^\infty \mathbb{E}\Big(\sum_{i=1}^m U_{i,0}\Big)^\alpha \frac{\lambda_0^m}{m!}e^{-\lambda_0} \;=\; \mathbb{E}X_0^\alpha\,,$$

where we used dominated convergence with the bound

$$\mathbb{E}\Big(\sum_{i=1}^m U_{i,n}\Big)^\alpha \frac{\lambda_n^m}{m!}e^{-\lambda_n} \leq \gamma m^{\alpha+1}\beta^m/m!\,,$$

where $\gamma = \sup_n \mathbb{E}U_n^\alpha$ and $\beta = \sup_n \lambda_n$, and we used $\left[\sum_1^m u_i\right]^\alpha \leq m^\alpha u_i^\alpha$. □

Proof of Theorem 12.3 Using the Lévy-Khinchine representation, we may write

$$X_n = X_n^{(1)} + X_n^{(2)} + X_n^{(3)}\,, \tag{156}$$

where the $\left(X_n^{(i)}\right)_{n\geq 1}$ are sequences of infinitely divisible distributions having characteristic triplets $\left(0, 0, [\nu_n]_{\{y<-1\}}\right)$, $(c_n, \sigma_n, [\nu_n]_{\{|y|\leq 1\}})$ and $(0, 0, [\nu_n]_{\{y>1\}})$, respectively, which are independent for each n. Assume the family $\left(|X_n|^\alpha\right)_{n\geq 1}$ is uniformly integrable. We wish to apply Proposition 12.8 to the family $\left((X_n^{(3)})^\alpha\right)_{n\geq 1}$, and therefore we need to show that this family is uniformly integrable. First, we rewrite (156) as $X_n - X_n^{(2)} = X_n^{(1)} + X_n^{(3)}$ and use Lemma 12.6 together with the inequality $|x-y|^\alpha \le 2^\alpha(|x|^\alpha + |y|^\alpha)$ to conclude that the family $\big(|X_n - X_n^{(2)}|^\alpha\big)_{n\ge 1}$ is uniformly integrable, which in turn implies that the family $\big(|X_n^{(1)} + X_n^{(3)}|^\alpha\big)_{n\ge 1}$ is uniformly integrable.

Assuming w.l.o.g. that 1 is a continuity point of ν_0, we have that $X_n^{(1)}$ is weakly convergent and therefore tight. This implies that there exists $r > 0$ such that $\mathbb{P}\big(|X_n^{(1)}| \le r\big) \ge 1/2$ for all n, which implies that for all n and for all t so large that $(t^{1/\alpha} - r)^\alpha > t/2$, we have:

$$\begin{aligned}(1/2)\mathbb{P}\big((X_n^{(3)})^\alpha > t\big) &\le \mathbb{P}\big(|X_n^{(1)}| \le r\big)\mathbb{P}\big(X_n^{(3)} > t^{1/\alpha}\big)\\ &= \mathbb{P}\big(|X_n^{(1)}| \le r, X_n^{(3)} > t^{1/\alpha}\big) \le \mathbb{P}\big(X_n^{(1)} + X_n^{(3)} > t^{1/\alpha} - r\big)\\ &\le \mathbb{P}\big(|X_n^{(1)} + X_n^{(3)}|^\alpha > (t^{1/\alpha} - r)^\alpha\big) \le \mathbb{P}\big(|X_n^{(1)} + X_n^{(3)}|^\alpha > t/2\big)\,.\end{aligned}$$

This implies that $\big((X_n^{(3)})^\alpha\big)$ is uniformly integrable, since $\big(|X_n^{(1)} + X_n^{(3)}|^\alpha\big)$ is so. Applying Proposition 12.8 yields

$$\lim_a \sup_n \int_a^\infty y^\alpha \nu_n(\mathrm{d}y) = 0\,. \tag{157}$$

Together with a similar relation for $\int_{-\infty}^{-a}$ this gives

$$\lim_{a\to\infty} \sup_n \int_{[-a,a]^c} |y|^\alpha \nu_n(\mathrm{d}y) = 0\,.$$

For the converse, we assume $\lim_a \sup_n \int_{[-a,a]^c} |y|^\alpha \nu_n(\mathrm{d}y) = 0$, and return to our decomposition (156). As before, we apply Lemma 12.6 to obtain that the family $\big(X_n^{(2)}\big)$ is uniformly integrable. Furthermore, applying Proposition 12.8, we obtain that the families $\big(|X_n^{(1)}|^\alpha\big)$ and $\big(|X_n^{(3)}|^\alpha\big)$ are uniformly integrable, and since $|X_n|^\alpha \le 3^\alpha\big(|X_n^{(1)}|^\alpha + |X_n^{(2)}|^\alpha + |X_n^{(3)}|^\alpha\big)$, the proof is complete. □

Next, we prove Theorem 12.2.

We consider a sequence of Lévy processes $\{X^n\}$ such that $X^n \xrightarrow{\mathscr{D}} X^0$ and use obvious notation like $\ell^{b,n}, \pi^{b,n}$ etc. Furthermore, we let $\tau^n(A)$ denote the first exit time of X^n from A. Here A will always be an interval.

We first show that weak convergence of X^n implies weak convergence of the stationary distributions.

Proposition 12.9 $X^n \xrightarrow{\mathscr{D}} X^0 \Rightarrow \pi^{b,n} \xrightarrow{\mathscr{D}} \pi^{b,0}$.

Proof According to Theorem 13.17 in [82] we may assume $\Delta_{n,t} = \sup_{v \le t} |X^n(v) - X^0(v)| \xrightarrow{\mathbb{P}} 0$. Then

$$\begin{aligned}
&\mathbb{P}\big(X^0_{\tau^0[y+\varepsilon-b,y+\varepsilon)} \ge y+\varepsilon,\ \tau^0[y+\varepsilon-b,y+\varepsilon) \le t\big) \\
&\quad \le\ \mathbb{P}\big(X^n_{\tau^n[y-b,y)} \ge y,\ \tau^n[y-b,y) \le t\big)\ +\ \mathbb{P}(\Delta_{n,t} > \varepsilon) \\
&\quad \le\ \mathbb{P}\big(X^n_{\tau^n[y-b,y)} \ge y\big)\ +\ \mathbb{P}(\Delta_{n,t} > \varepsilon)\,.
\end{aligned}$$

Letting first $n \to \infty$ gives

$$\liminf_{n\to\infty} \overline{\pi}^{b,n}(y)\ \ge\ \mathbb{P}\big(X^0_{\tau^0[y+\varepsilon-b,y+\varepsilon)} \ge y+\varepsilon,\ \tau^0[y+\varepsilon-b,y+\varepsilon) \le t\big)\,,$$

and letting next $t \to \infty$, we obtain

$$\liminf_{n\to\infty} \overline{\pi}^{b,n}(y)\ \ge\ \overline{\pi}^{b,0}(y+\varepsilon)\,. \tag{158}$$

Similarly,

$$\begin{aligned}
&\mathbb{P}\big(X^n_{\tau^n[y-b,y)} \ge y,\ \tau^n[y-b,y) \le t\big)\ \le\ \mathbb{P}\big(X^0_{\tau^0[y-\varepsilon-b,y-\varepsilon)} \ge y-\varepsilon\big)\ +\ \mathbb{P}(\Delta_{n,t} > \varepsilon)\,, \\
&\limsup_{n\to\infty} \mathbb{P}\big(X^n_{\tau^n[y-b,y)} \ge y,\ \tau^n[y-b,y) \le t\big)\ \le\ \overline{\pi}^{b,0}(y-\varepsilon)\,. \qquad (159)
\end{aligned}$$

However,

$$\mathbb{P}\big(\tau^n[y-b,y) > t\big)\ \le\ \mathbb{P}\big(\tau^0[y-\varepsilon-b,y+\varepsilon) > t\big)\ +\ \mathbb{P}(\Delta_{n,t} > \varepsilon)\,,$$

so that

$$\limsup_{n\to\infty} \mathbb{P}\big(\tau^n[y-b,y) > t\big)\ \le\ \mathbb{P}\big(\tau^0[y-\varepsilon-b,y+\varepsilon) > t\big)\,.$$

Since the r.h.s. can be chosen arbitrarily small, it follows by combining with (159) that

$$\limsup_{n\to\infty} \overline{\pi}^{b,n}(y)\ =\ \limsup_{n\to\infty} \mathbb{P}\big(X^n_{\tau^n[y-b,y)} \ge y\big)\ \le\ \overline{\pi}^{b,0}(y-\varepsilon)\,.$$

Combining with (158) shows that $\overline{\pi}^{b,n}(y) \to \overline{\pi}^{b,0}(y)$ at each continuity point y of $\overline{\pi}^{b,0}$, which implies convergence in distribution. □

The following elementary lemma gives two properties of the function $\varphi = \varphi_b$ from Theorem 1.1. The proof is omitted.

Lemma 12.10 *The function* $\varphi_b(x, y)$ *is continuous in the region* $(x, y) \in [0, b] \times \mathbb{R}$ *and satisfies* $0 \le \varphi_b(x, y) \le 2y^2 \wedge 2b|y|$.

We are now ready to prove Theorem 12.2.

Proof Recall the definition (153) of the bounded measure $\tilde{\nu}$ and let $\tilde{\varphi}_b(x, y) = \varphi(x, y)(1 + y^2)/y^2$ for $y \neq 0$, $\tilde{\varphi}_b(x, 0) = 1$. Note that $\tilde{\varphi}_b(x, y)$ is continuous on $(0, b) \times \mathbb{R}$, but discontinuous at $y = 0$ if $x = 0$ or $x = b$. We also get

$$\int_{-\infty}^{\infty} \tilde{\varphi}_b(x, y)\tilde{\nu}_n(\mathrm{d}y) = \sigma_n^2 + \int_{-\infty}^{\infty} \varphi_b(x, y)\nu_n(\mathrm{d}y)\,,$$

so that

$$a_n = \sigma_n^2 + \int_0^b \pi^{b,n}(\mathrm{d}x) \int_{-\infty}^{\infty} \varphi_b(x, y)\nu_n(\mathrm{d}y) \;=\; \int_0^b \pi^{b,n}(\mathrm{d}x) \int_{-\infty}^{\infty} \tilde{\varphi}_b(x, y)\tilde{\nu}_n(\mathrm{d}y)\,.$$

Let $\tilde{\nu}_n^1, \tilde{\nu}_n^2$ denote the restrictions of $\tilde{\nu}_n$ to the sets $|y| \le a$, resp. $|y| > a$. Using $0 \le \varphi_b(x, y) \le 2b|y|$, and uniform integrability (Theorem 12.3) we can choose a such that

$$0 \le \int_{[-a,a]^c} \tilde{\varphi}_b(x, y)\tilde{\nu}_n^2(\mathrm{d}y) < \varepsilon$$

for all x and n (note that $\tilde{\nu}_n \le \nu_n$ on $\mathbb{R}\setminus\{0\}$). We may also further assume that a and $-a$ are continuity points of ν_0 which implies $\tilde{\nu}_n^1 \to \tilde{\nu}_0^1$ weakly. In particular,

$$\sup_n \tilde{\nu}_n^1([-a, a]) < \infty. \tag{160}$$

Define

$$f_n(x) = \int_{-a}^{a} \varphi_b(x, y)\nu_n(\mathrm{d}y) + \sigma_n^2 = \int_{-a}^{a} \tilde{\varphi}_b(x, y)\tilde{\nu}_n^1\mathrm{d}y$$

we wish to prove that $\int f_n\,\mathrm{d}\pi^{b,n} \to \int f_0\,\mathrm{d}\pi^{b,0}$ which, by using the generalized continuous-mapping theorem (e.g. [138]), will follow if

$$\pi^{b,0}(F) = 0 \tag{161}$$

where

$$F \;=\; \left\{x \mid \exists (x_n)_{n\ge1} : x_n \to x, f_n(x_n) \not\to f_0(x)\right\}. \tag{162}$$

The proof of this follows different routes depending on whether or not σ_0^2 is zero. First, we assume $\sigma_0^2 = 0$ and consider the functions

$$f_n^-(x) = \sigma_n^2 + \int_{(-\infty,0]} \varphi_b(x,y)\nu_n^1(\mathrm{d}y) = \int_{(-\infty,0]} \tilde{\varphi}_b(x,y)\tilde{\nu}_n^1(\mathrm{d}y)\,,$$

$$f_n^+(x) = \int_{(0,\infty)} \varphi_b(x,y)\nu_n^1(\mathrm{d}y) = \int_{(0,\infty)} \tilde{\varphi}_b(x,y)\tilde{\nu}_n^1(\mathrm{d}y)\,.$$

It follows from the definition of $\tilde{\nu}_n^1$, that the assumption $\sigma_0^2 = 0$ implies that $\tilde{\nu}_n^1$ has no mass at 0, and since this is the only possible discontinuity point of the integrands, we have $f_n^-(x) \to f_0^-(x)$ and $f_n^+(x) \to f_0^+(x)$ for $x \in [0,b]$. Furthermore, it can be checked that $x \mapsto f_n^-(x)$ is increasing, $x \mapsto f_n^+(x)$ is decreasing, and, using the bound $\varphi_b(x,y) \le 2y^2$, that both functions are uniformly bounded. That is, the functions f_n^- and f_n^+ form two uniformly bounded sequences of continuous, monotone functions which converge to a continuous limit and as such, they converge uniformly. From this we get

$$\begin{aligned}\sup_{0\le y\le b} |f_n(y) - f_0(y)| &= \sup_{0\le y\le b} |f_n^-(y) - f_0^-(y) + f_n^+(y) - f_0^+(y)| \\ &\le \sup_{0\le y\le b} |f_n^-(y) - f_0^-(y)| + \sup_{0\le y\le b} |f_n^+(y) - f_0^+(y)| \to 0\,.\end{aligned}$$

Using the calculation above, we see that if we consider any $x \in [0,b]$ and any sequence $(x_n)_{n\ge1}$ converging to x, we have

$$\begin{aligned}|f_n(x_n) - f_0(x)| &\le |f_n(x_n) - f_0(x_n)| + |f_0(x_n) - f_0(x)| \\ &\le \sup_{0\le y\le b} |f_n(y) - f_0(y)| + |f_0(x_n) - f_0(x)| \to 0\,,\end{aligned} \tag{163}$$

where we use continuity of f_0 in the last part of the statement. This gives us that F in (162) is the empty set, and hence we obtain (161) in the case $\sigma_0^2 = 0$.

Next, we consider the case where $\sigma_0^2 > 0$. We note that $\sigma_0^2 > 0$ implies that $\{X^0\}$ is a process of unbounded variation and using Theorem 6.5 in [104], this implies that 0 is regular for $(0,\infty)$. By comparing this to the representation (6) of the stationary distribution, we see that this implies $\pi^{b,0}(\{0,b\}) = 0$. Consider $x \in (0,b)$ and a sequence $(x_n)_{n\ge1}$ converging to x. Assume w.l.o.g. that $x_n \in [\varepsilon, b-\varepsilon]$ for some $\varepsilon > 0$. Since $\tilde{\varphi}_b(x,y)$ is continuous on the compact set $[\varepsilon, b-\varepsilon] \times [-a,a]$, we can use (161) to see that given ε_1, there exists ε_2 such that $|f_n(x') - f_n(x'')| < \varepsilon_1$ for all n whenever $|x' - x''| < \varepsilon_2$ and $x', x'' \in [\varepsilon, b-\varepsilon]$. Since $x_n \to x$ this means, that given any $\varepsilon_1 > 0$, we may use an inequality similar to (163) to conclude that for n large enough

$$|f_n(x_n) - f_0(x)| \;\le\; \varepsilon_1 + |f_n(x) - f_0(x)|$$

and by taking $\limsup_n$ we see that the convergence $f_n(x_n) \to f_0(x)$ holds when $x \in (0, b)$, and can only fail $x = 0$ or $x = b$. Using that $\{0, b\}$ has $\pi^{b,0}$-measure 0, we have $\int f_n \, \mathrm{d}\pi^{b,n} \to \int f_0 \, \mathrm{d}\pi^{b,0}$ in this case as well. By combining this with the uniform integrability estimate above, we get that for any $\varepsilon > 0$: $|a_n - a_0| \leq |f_n - f_0| + 2\varepsilon$, (note that f_i depends on ε) and hence $\limsup_n |a_n - a_0| \leq 2\varepsilon$, which implies $a_n \to a_0$.

By uniform integrability $\mathbb{E}X^n(1) \to \mathbb{E}X^0(1)$, and further $\pi^{b,n} \overset{\mathscr{D}}{\to} \pi^{b,0}$ implies $\int_0^b \overline{\pi}^{b,n}(y)\mathrm{d}y \to \int_0^b \overline{\pi}^{b,0}(y)\mathrm{d}y$. Remembering $a_n \to a_0$ and inspecting the expression (75) for the loss rate shows that indeed $\ell^{b,n} \to \ell^{b,0}$. □

12.2 *Proof of Theorem 12.1*

First we note the effect that scaling and time-changing a Lévy process has on the loss rate:

Proposition 12.11 *Let $\beta, \delta > 0$ and define $X^{\beta,\delta}(t) = X(\delta t)/\beta$. Then the loss rate $\ell^{b/\beta}(X^{\beta,\delta})$ for $X^{\beta,\delta}$ equals δ/β times the loss rate $\ell^b(X) = \ell^b$ for X.*

Proof It is clear that scaling by β results in the same scaling of the loss rate. For the effect of δ, note that the loss rate is the expected local time in stationarity per unit time and that one unit of time for $X^{\beta,\delta}$ corresponds to δ units of time for X. □

Proof of Theorem 12.1 a) Define $X^b(t) = X(tb^2)/b$. Then by Proposition 12.11 we have

$$b\ell^b(X) = \ell^1(X^b)$$

By the central limit theorem we have $X^b(1) \overset{\mathscr{D}}{\to} N(0, \psi^2)$ as $b \to \infty$. By Proposition 12.4, this is equivalent to $X^b \overset{\mathscr{D}}{\to} \psi B$ where B is standard Brownian motion. We may apply Theorem 12.2, since

$$\mathbb{E}\big[(X^b(1))^2\big] = \mathbb{V}ar(X^1(1))\,,$$

that is, $\{X^b(1)\}_{b=1}^{\infty}$ is bounded in L^2 and therefore uniformly integrable. Thus

$$\lim_b b\ell^b(X) = \lim_b \ell^1\big(X^b\big) = \ell^1(\psi B) = \psi^2/2\,,$$

where the last equality follows directly from the expression for the loss rate in Theorem 1.1. □

Proof of Theorem 12.1 b) First we note that the stated conditions implies that the tails of ν are regularly varying, and therefore they are subexponential. Then by Embrechts et al. [56] we have that the tails of $\mathbb{P}(X(1) < x)$ are equivalent to those of ν and hence we may write

$$\mathbb{P}(X(1) > x) = x^{-\alpha} L_1(x) g_1(x)\,, \quad \mathbb{P}(X(1) < -x) = x^{-\alpha} L_2(x) g_2(x)$$

where $\lim_{x\to\infty} g_i(x) = 1$, $i = 1, 2$. The next step is to show that the fact that the tails of the distribution function are regularly varying allows us to apply the stable central limit theorem. Specifically, we show that the assumptions of Theorem 1.8.1 in [128] are fulfilled.

We notice that if we define $M(x) = L_1(x)g_1(x) + L_2(x)g_2(x)$ then $M(x)$ is slowly varying and

$$x^{\alpha}\big[\mathbb{P}(X(1) < -x) + \mathbb{P}(X(1) > x)\big] = M(x)\,. \tag{164}$$

Furthermore:

$$\frac{P(X(1) > x)}{\mathbb{P}(X(1) < -x) + \mathbb{P}(X(1) > x)} = L_1(x)g_1(x)/M(x) \sim L_1(x)/L(x) \to \frac{\beta + 1}{2}\,, \tag{165}$$

as $x \to \infty$ since $L(x) \sim M(x)$. Define $L_0(x) = L(x)^{(-1/\alpha)}$ and let $L_0^{\#}(x)$ denote the de Bruin conjugate of L_0 (cf. [32, p. 29]) and set $f(n) = n^{(1/\alpha)} L_0^{\#}(n^{(1/\alpha)})$. Let $f^{\leftarrow}$ be the generalized inverse of f. By asymptotic inversion of regularly varying functions [32, pp. 28–29] we have $f^{\leftarrow}(n) \sim (nL_0(n))^{\alpha}$ which implies

$$\frac{f^{\leftarrow}(n)L(n)}{n^{\alpha}} \sim \frac{(nL_0(n))^{\alpha}L(n)}{n^{\alpha}} = 1$$

and since $f^{\leftarrow}(f(n)) \sim n$ we have

$$\frac{nM(f(n))}{f(n)^{\alpha}} \sim \frac{nL(f(n))}{f(n)^{\alpha}} \sim \frac{f^{\leftarrow}(f(n))L(f(n))}{f(n)^{\alpha}} \to 1 \tag{166}$$

and therefore, if we define $\sigma = (\Gamma(1-\alpha)\cos(\alpha\pi/2))^{1/\alpha}$ we have

$$\frac{nM(\sigma^{-1}f(n))}{(\sigma^{-1}f(n))^{\alpha}} \sim \frac{nM(f(n))}{(\sigma^{-1}f(n))^{\alpha}} \to \sigma^{\alpha} \tag{167}$$

using slow variation of M. By combining (164), (165) and (167) we may apply the stable CLT Theorem 1.8.1 [128][3] to obtain $X^b/f(b) \xrightarrow{\mathscr{D}} Z$ where Z is a r.v. with characteristic function ψ, where

$$\psi(u) = \exp(-|\sigma u|^{\alpha}(1 - \mathrm{i}\beta\,\mathrm{sgn}(u)\tan(\alpha\pi/2))\,. \qquad u \in \mathbb{R}$$

[3]Note that the constants there should be replaced by their inverses.

Recalling that κ is the characteristic exponent of X, this is equivalent to

$$\mathrm{e}^{b\kappa(\mathrm{i}u/f(b))} \to \psi(u)$$

and therefore

$$\mathrm{e}^{(bL_0(b))^{\alpha}\kappa(\mathrm{i}u/f(f^{\leftarrow}(b)))} \sim \mathrm{e}^{f^{\leftarrow}(b)\kappa(\mathrm{i}u/f(f^{\leftarrow}(b)))} \to \psi(u)$$

that is, for $\tilde{X}^b(t) = X(t(bL_0(b))^{\alpha})/f(f^{\leftarrow}(b))$ we have $\tilde{X}^b(1) \xrightarrow{\mathscr{D}} Z$, and using $f(f^{\leftarrow}(b)) \sim b$ as well as the definition of $L_0(b)$, we see that the same applies to $X^b(t) = X(t(b^{\alpha}/L(b)))/b$. Setting $d = (\beta+1)/2$ and $c = (1-\beta)/2$ we calculate (cf. [128])

$$-|\sigma t|^{\alpha}(1 - \mathrm{i}\beta\,\mathrm{sgn}(t)\tan(\alpha\pi/2)) = -|\sigma t|^{\alpha}(1 + \mathrm{i}(d-c)\,\mathrm{sgn}(t)\tan(\alpha\pi/2)$$
$$d\alpha\int_{-\infty}^{0}(\mathrm{e}^{\mathrm{i}vt} - 1 - ivt)(-t)^{-\alpha-1}\,\mathrm{d}t + c\alpha\int_{0}^{\infty}(\mathrm{e}^{\mathrm{i}vt} - 1 - ivt)t^{-\alpha-1}\,\mathrm{d}t\,.$$

That is, the characteristic triplet of Z is $(\tau, 0, \nu)$, where

$$\nu(\mathrm{d}u) = \begin{cases} \dfrac{\alpha c}{(-u)^{\alpha+1}}\,\mathrm{d}u & u < 0 \\ \dfrac{\alpha d}{u^{\alpha+1}}\,\mathrm{d}u & u > 0 \end{cases} \tag{168}$$

and τ is a centering constant. We wish to use Theorem 12.2 and have to prove uniform integrability. Note that by combining Proposition 11.10 and Corollary 8.3 in [129], we have that the Lévy measure of X^b is ν_b, where

$$\nu_b(B) = b^{\alpha}L(b)^{-1}\nu(\{x : b^{-1}x \in B\})\,.$$

Using the assumptions in (152), this implies

$$\overline{\nu}_b(a) = b^{\alpha}L(b)^{-1}\overline{\nu}(ab) = L(b)^{-1}a^{-\alpha}L_1(ab)\,,\;\; \nu_b(-a) = L(b)^{-1}a^{-\alpha}L_2(ab)\,.$$

Using partial integration and the remarks above, we find:

$$\int_{[-a,a]^c}|y|\nu_b(\mathrm{d}y) = a\overline{\nu}_b(a) + \int_a^{\infty}\overline{\nu}_b(t)\mathrm{d}t + a\nu_b(-a) + \int_{-\infty}^{-a}\nu_b(t)\mathrm{d}t$$
$$= a^{-\alpha+1}L(b)^{-1}\alpha L(ab) + \int_a^{\infty}t^{-\alpha}L(b)^{-1}L(tb)\mathrm{d}t\,.$$

Furthermore, using Potter's Theorem (Theorem 1.5.6 in [32]) we have that for $\delta > 0$ such that $1+\delta < \alpha$ there exists $\xi > 0$ such that

$$\frac{L(ab)}{L(b)} \le 2\max(a^{\delta}, a^{-\delta}) \qquad ab > \xi, b > \xi\,.$$

Using this, we get that

$$\lim_a \sup_{b>\xi} a^{-\alpha+1}\frac{L(ab)}{L(b)} \le 2\lim_a a^{-\alpha+1}\max(a^{\delta}, a^{-\delta}) = 0 \tag{169}$$

and similarly for the integral:

$$\sup_{b>\xi}\lim_a \int_a^\infty t^{-\alpha}\frac{L(tb)}{L(b)}\mathrm{d}t \le 2\lim_a \int_a^\infty t^{-\alpha}\max(t^{\delta}, t^{-\delta})\mathrm{d}t = 0\,. \tag{170}$$

By combining (169) and (170) we get

$$\lim_{a\to\infty}\sup_{b>\xi}\int_{[-a,a]^c} |y|\nu_b(\mathrm{d}y) = 0\,.$$

By Proposition 12.11 we have $b^{\alpha-1}L(b)^{-1}\ell^b(X) = \ell^1\big(X^b\big)$, and since we have proved uniform integrability, we may apply Theorem 12.2. Letting $b \to \infty$ and using Example 8.8 which states that the loss rate for our stable distribution is γ (see also [103]), yields the desired result. □

13 The Overflow Time

We define the overflow time as

$$\omega(b,x) \;=\; \inf\big\{t > 0 :\; V^b(t) = b \,\big|\, V^b(0) = x\big\}\,,\;\; 0 \le x < b.$$

It can also be interpreted in terms of the one-sided reflected process as

$$\omega(b,x) \;=\; \inf\big\{t > 0 :\; V^\infty(t) \ge b \,\big|\, V^\infty(0) = x\big\}\,,\;\; 0 \le x \le b\,.$$

It has received considerable attention in the applied literature; among many references, see e.g. [47, 75, 93, 98]. We consider here evaluation of characteristics of $\omega(b,x)$, in particular expected values and distributions, both exact and asymptotically as $b \to \infty$. When no ambiguity exists, we write ω instead of $\omega(b,x)$.

As may be guessed, the Brownian case is by far the easiest:

Example 13.1 Let X be $\mathrm{BM}(\mu,\sigma^2)$ with $\mu \ne 0$ [the case $\mu = 0$ requires a separate treatment which we omit]. Consider the Kella-Whitt martingale with $B(t) = x +$

$L(t) - qt/\alpha$ where L is the local time at 0 for the one-sided reflected process,

$$\kappa(\alpha)\int_0^t e^{\alpha V^\infty(s)-qs}\,ds + e^{\alpha x} - e^{\alpha V^\infty(t)-qt} + \alpha\int_0^t e^{-qs}\,dL(s) - q\int_0^t e^{\alpha V^\infty(s)-qs}\,ds$$

where we used that L can only increase when V^∞ is at 0 and so

$$\int_0^t e^{\alpha V^\infty(s)-qs}\,dL(s) = \int_0^t e^{-qs}\,dL(s)\,.$$

Take first $\gamma = -2\mu/\sigma$ as the root of $0 = \kappa(s) = s\mu + s^2\sigma^2/2$ and $q = 0$. Optional stopping at ω then gives $0 = e^{\gamma x} - e^{\gamma b} + \gamma\mathbb{E}L(\omega)$. Using $V^\infty = x + B + L$ and $\mathbb{E}B(\omega) = \mu\mathbb{E}\omega$ then gives

$$\mathbb{E}\omega(b,x) = \frac{b - x - (e^{\gamma b} - e^{\gamma x})/\gamma}{\mu} \tag{171}$$

Take next $q > 0$ and $\theta^\pm$ as the two roots of $\kappa(\alpha) = q$, cf. Example 7.3. We then get

$$0 = e^{\theta^+ x} - e^{\theta^+ x}\mathbb{E}e^{-q\omega} - q\,\mathbb{E}\int_0^\omega e^{\alpha V^\infty(s)-qs}\,ds\,.$$

Together with the similar equation with θ^- this can then be solved to obtain $\mathbb{E}e^{-q\omega}$ (the other unknown is $\mathbb{E}\int_0^\omega e^{\alpha V^\infty(s)-qs}\,ds$).

Early calculations of these and some related quantities are in Glynn and Iglehart [66] who also discuss the probabilistically obvious fact that $\omega(b,0)$ is exponentially distributed in the Brownian case (as in the spectrally positive Lévy case), cf. Athreya and Werasinghee [22]. □

13.1 Exact Results in the PH Model

Recall from Sect. 3 that the process V^∞ with one-sided reflection at 0 can be constructed as $V(t) = V(0) + X(t) + L(t)$, where

$$L(t) = -\min_{0\le s\le t}(V(0) + X(s))$$

is the local time. For our phase-type model with a Brownian component, $L(t)$ decomposes as $L^c(t) + L^d(t)$, where L_c is the continuous part (the contribution to L from the segments between jumps where V behaves as a reflected Brownian motion) and $L^d(t)$ the compensation of jumps of X that would have taken V below 0.

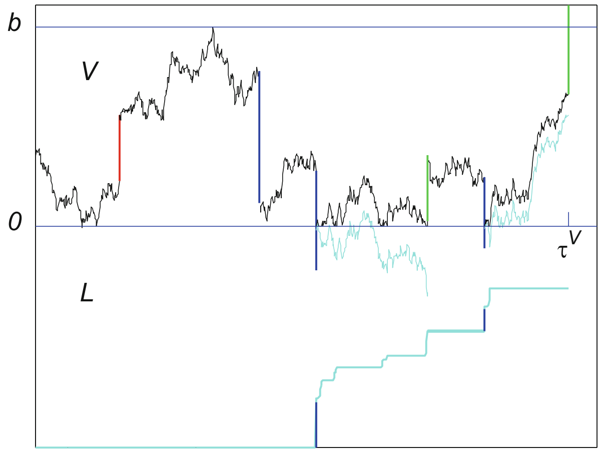


Fig. 6 One-sided reflected process $V = V^\infty$ and local time L

The situation is illustrated on Fig. 6. We have again phases `red, green` for F^+ and `blue` for F^-. The `cyan` Brownian segments are how Brownian motion would have evolved without reflection. In the lower panel, the `cyan` segments of L correspond to compensation when the Brownian motion would otherwise have taken V^∞ below 0, and the `blue` jumps are the compensation from jumps of X that would otherwise have taken V^∞ below 0.

To compute the Laplace transform of ω, we use the Kella-Whitt martingale with $B(t) = V(0) + L(t) - qt/\alpha$. Thus $\alpha Z(t) = \alpha V(t) - qt$, and the martingale takes the form

$$\kappa(\alpha)\int_0^t e^{\alpha V(s)-qs}\,ds \;+\; e^{\alpha V(0)} \;-\; e^{\alpha V(t)-qt}$$

$$+\,\alpha\int_0^t e^{\alpha V(s)-qs}\left(dL^c(s) - q\,ds/\alpha\right) + \sum_{0\le s\le t} e^{\alpha V(s)-qs}(1 - e^{-\alpha\Delta L^d(s)})$$

$$= \left(\kappa(\alpha) - q\right)\int_0^t e^{\alpha V(s)-qs}\,ds \;+\; e^{\alpha V(0)} \;-\; e^{\alpha V(t)-qt}$$

$$+\,\alpha\int_0^t e^{-qs}\,dL^c(s) \;+\; \sum_{0\le s\le t} e^{-qs}(1 - e^{-\alpha\Delta L^d(s)})\,,$$

where in the last step we used that L can only increase when V is at 0. Now introduce the following unknowns: z_c^+, the expectation of $e^{-q\omega}$ evaluated on the event of continuous upcrossing of level b only; z_i^+, the expectation of $e^{-q\omega}$ evaluated on the event of upcrossing in phase $i = 1, \ldots, n^+$ only; $\ell_c = \mathbb{E}\int_0^\omega e^{-qs}\,dL^c(s)$; and m_j, the expected value of the sum of the e^{-qs} with $s \le \omega$ such that at time s there is a downcrossing of level 0 in phase $j = 1, \ldots, n^-$. Optional stopping then gives

$$0 = \big(\kappa(\alpha) - q\big)\mathbb{E}\int_0^\omega \mathrm{e}^{\alpha V(s) - qs}\,\mathrm{d}s + \mathrm{e}^{\alpha V(0)} \\ - \mathrm{e}^{\alpha b}\Big(z_c^+ + \sum_{i=1}^{n^+} \hat{F}_i^+[\alpha] z_i^+\Big) + \alpha \ell_c + \sum_{j=1}^{n^-} m_j (1 - \hat{F}_j^-[-\alpha]) .$$

Taking α as one of the same roots as in Sect. 9.4, we get

$$0 = \mathrm{e}^{\rho_k^q V(0)} - \mathrm{e}^{\rho_k^q b}\Big(z_c^+ + \sum_{i=1}^{n^+} \hat{F}_i^+[\rho_k^q] z_i^+\Big) + \rho_k^q \ell_c + \sum_{j=1}^{n^-} m_j (1 - \hat{F}_j^-[-\rho_k^q]) ,$$

a set of linear equations from which the unknowns and hence $\mathbb{E}\mathrm{e}^{-q\omega} = z_c^+ + z_1^+ + \cdots + z_{n^+}^+$ can be computed.

13.2 Asymptotics via Regeneration

The asymptotic study of $\omega(b,x)$ is basically a problem in extreme value theory since

$$\mathbb{P}\big(\omega(b,x) \le t\big) = \mathbb{P}_x\Big(\max_{0 \le s \le t} V^\infty(s) \ge b\Big) . \tag{172}$$

This is fairly easy if $m = \mathbb{E}X(1) > 0$ since then the max in (172) is of the same order as $X(t)$ which is in turn of order mt. Hence we assume $m < 0$ in the following.

For processes with dependent increments such as V^∞, the asymptotic study of the quantities in (172) is most often (with Gaussian processes as one of the exceptions) done via regeneration, cf. [11, VI.4]

For V^∞, we define (inspired by the discussion in Sect. 5.1) a cycle by starting at level 0, waiting until level 1 (say) has been passed and taking the cycle termination time T as the next hitting time of 0 ('up to 1 from 0 and down again'). That is,

$$T = \inf\big\{t > \inf\{s > 0 : V^\infty(s) \ge 1\} : V^\infty(t) = 0 \,\big|\, V^\infty(0) = 0\big\} .$$

The key feature of the regenerative setting is that the asymptotic discussion can be reduced to the study of the behavior within a regenerative cycle. The quantities needed are

$$m_T = \mathbb{E}_0 T , \quad a(z) = \mathbb{P}_0\Big(\max_{0 \le s \le T} V^\infty(s) \ge z\Big) .$$

Indeed one has by [11, VI.4] that:

Theorem 13.2 *As $b \to \infty$, it holds for any fixed x that $a(b)\mathbb{E}\omega(b,x) \to m_T$ and that $a(b)\omega(b,x)/m_T$ has a limiting standard exponential distribution.*

For the more detailed implementation, we note:

Proposition 13.3

(a) *Assume that the Lévy measure ν is heavy-tailed, more precisely that $\overline{\nu}(z) = \int_z^\infty \nu(\mathrm{d}y)$ is a subexponential tail. Then $a(z) \sim m_T\overline{\nu}(x)$ as $z \to \infty$;*

(b) *Assume that the Lévy measure ν is light-tailed, more precisely that the Lundberg equation $\kappa(\gamma) = 0$ has a solution $\gamma > 0$ with $\kappa'(\gamma) < \infty$. Then $a(z) \sim C_T \mathrm{e}^{-\gamma z}$ for some constant C_T as $z \to \infty$*

Sketch of Proof For (a), involve 'the principle of one big jump' saying that exceedance of z occurs as a single jump of order z (which occurs at rate $\overline{\nu}(z)$). The rigorous proof, using the regenerative representation $\overline{\pi}^\infty(x) = a(x)/m_T$ of the stationary distribution of V^∞ and known results on π^∞, can be found in [10, 20].

For (b), let $\mathbb{P}_\gamma, \mathbb{E}_\gamma$ refer to the exponentially tilted case $\kappa_\gamma(\alpha) = \kappa(\alpha+\gamma) - \kappa(\alpha)$ with $V^\infty(0) = 0$. By standard likelihood ratio identities,

$$a(z) \;=\; \mathbb{P}_0\big(\omega(z,0) < T\big) \;=\; \mathbb{E}_\gamma\Big[\exp\big\{-\gamma X\big(\omega(z,0)\big)\big\};\, \omega(z,0) < T\Big] \qquad (173)$$

Now $m_\gamma = \kappa'(\gamma) > 0$ so that $\mathbb{P}_\gamma\big(V^\infty(t) \to \infty\big) = 1$. Hence $\big\{\omega(z,0) < T\big\} \uparrow \{T = \infty\}$ where $\mathbb{P}_\gamma(T = \infty) > 0$, and

$$X\big(\omega(z,0)\big) + L\big(\omega(z,0)\big) = V^\infty\big(\omega(z,0)\big) = z + \xi(z)$$

where $\xi(z)$, the overshoot, converges in $\mathbb{P}_\gamma$-distribution to a limit $\xi(\infty)$ (in fact, the same as when overshoot distribution are taken w.r.t. X, not V^∞), and $L\big(\omega(z,0)\big)$ converges in $\mathbb{P}_\gamma$-distribution to the finite r.v. $L(\infty)$. Combining with (173) and suitable independence estimates along the lines of Stam's lemma [11, pp. 368–369], the result follows with

$$C_T \;=\; \mathbb{E}_\gamma \mathrm{e}^{-\gamma\xi(\infty)} \cdot \mathbb{E}_\gamma\big[\mathrm{e}^{\gamma L(\infty)};\, T = \infty\big].$$

That $C_T < \infty$ is seen by a comparison with Theorem 2.1 since clearly

$$a(z) \;\le\; \mathbb{P}_0\Big(\max_{0\le s\le T} V^\infty(s) \ge z\Big) \;=\; \overline{\pi}^\infty(z).$$

□

In the heavy-tailed case, Theorem 13.2 and Proposition 13.3 determine the order of $\omega(b,x)$ as $\overline{\nu}(b)$. In the light-tailed case, we are left with the computation of the constant C_T. In general, one can hardly hope for an explicit expression beyond special cases. Note, however, that for the spectrally negative case one can find the Laplace transform $\mathbb{E}_x \mathrm{e}^{-q\omega(z,x)}$ as $Z^{(q)}(x)/Z^{(q)}(b)$, where

$$Z^{(q)}(x) \;=\; 1 + q\int_0^x W^{(q)}(y)\,\mathrm{d}y$$

is the 'second scale function'. See Pistorius [119] and Kyprianou [104, p. 228], with extensions in Ivanovs and Palmowski [78]. We return to $\mathbb{E}\omega$ in Sect. 14.2.

14 Studying V as a Markov Process

14.1 Preliminaries

An alternative approach to computing probabilities and expectations associated with V and its loss process U is to take advantage of the fact that V is a Markov jump-diffusion process. As a consequence, a great number of probabilities/expectations can be computed by solving linear integro-differential equations, subject to suitable boundary conditions related to the boundary behaviour of V and the specific functional under consideration.

Our exposition is somewhat simpler if we require that the jump component of X be of bounded variation (BV). So, we will henceforth assume that

$$\int_{|y|\le 1} |y|\,\nu(\mathrm{d}y) \;<\; \infty\,. \tag{174}$$

In this setting,

$$X(t) - X(0) \;=\; \mu t + \sigma B(t) + \sum_{0<s\le t} \Delta X(s)\,,$$

where $\Delta X(s) = X(s) - X(s-)$ and the sum converges absolutely for each $t < \infty$ because of (174). Cf. the discussion at the end of Sect. 1. Without (174), the jump part would a.s. have unbounded variation. In order to deal with Lévy processes having non-BV jumps, one needs to modify the equations and arguments of this section slightly. We discuss this non-BV extension briefly at the end in Sect. 14.6.

The key to establishing suitable integro-differential equations in this context is the systematic use of Itô's formula in the form

$$\begin{aligned} f\big(V(s)\big) &- f\big(V(s)\big) - \sum_{0<s\le t}\big[f\big(V(s)\big) - f\big(V(s-)\big)\big] \\ &= \int_0^t \Big[\mu f'\big(V(s)\big) + \frac{\sigma^2}{2} f''\big(V(s)\big)\Big]\,\mathrm{d}s + \int_0^t f'\big(V(s)\big)\,\mathrm{d}B(s) \\ &\quad + \int_0^t f'\big(V(s)\big)\,\mathrm{d}L_c(s) - \int_0^t f'\big(V(s)\big)\,\mathrm{d}U_c(s) \end{aligned} \tag{175}$$

[note that in our context, $V(s)$ could be replaced by b in the last integral and by 0 i the next-to-last]. This follows for compound Poisson jumps by using Itô's formula in a form involving boundary modifications (see [46]) on intervals between jumps of X (where V is continuous), and in the general case by approximation by such a process. Equation (174) is the basic form of Itô's formula that we will systematically apply in what follows.

With the aid of Itô's formula, we will illustrate the use of Markov process arguments in deriving various integro-differential equations associated with V and its loss process U. In fact, we will generalize from consideration of U to additive functionals of the form

$$\Lambda(t) \;=\; \int_0^t f\bigl(V(s)\bigr)\,\mathrm{d}s \;+\; \sum_{0<s\le t} \tilde{f}\bigl(V(s-),\Delta X(s)\bigr) \;+\; r_1 L_c(t) \;+\; r_2 U_c(t)\,. \tag{176}$$

We recall that $\Lambda = \bigl(\Lambda(t)\,:\, t \ge 0\bigr)$ is an additive functional of X if it can be represented as $\Lambda(t) = g_t\bigl(X(u)\,:\, 0 \le u \le t\bigr)$ where

$$g_{t+s}\bigl(X(u)\,:\,0\le u\le t+s\bigr) \;=\; g_s\bigl(X(u)\,:\,0\le u\le s\bigr) \;+\; g_t\bigl(X(s+u)\,:\,0\le u\le t\bigr)\,;$$

see also [34]. Note that we recover U in (176) if we set $f \equiv 0, \tilde{f}(x,y) = [x+y-b]^+$, $r_1 = 0$ and $r_2 = 1$. We assume throughout that f is bounded, that $\tilde{f}(x,0) = 0$, and that

$$\sup_{0\le x\le b} \int \bigl|\tilde{f}(x,y)\bigr|\,\nu(\mathrm{d}y) \;<\; \infty\,.$$

An additional notational simplification will be useful: we set

$$r(x,y) \;=\; \begin{cases} 0 & x+y\le 0 \\ x+y & 0\le x+y\le b \\ b & x+y\ge b \end{cases}$$

and observe that $V(s) = r\bigl(V(s-),\Delta X(s)\bigr)$ whenever $\Delta X(s) \neq 0$.

The integro-differential equations to follow are typically expressed in terms of the operator $\mathscr{L}$ defined on twice differentiable functions $\varphi\,:\,[0,b]\to\mathbb{R}$ and given by

$$(\mathscr{L}\varphi)(x) \;=\; \mu\varphi'(x) + \frac{\sigma^2}{2}\varphi''(x) + \int_{\mathbb{R}} \bigl[\varphi\bigl(r(x,y)\bigr) - \varphi(x)\bigr]\,\nu(\mathrm{d}y)\,,$$

The expression on the r.h.s. is familiar from the theory of generators of Markov processes, but given the multitude of formulations of this theory, we will not pursue this aspect. The interested reader can find a good discussion of generators in the diffusion setting in Karlin and Taylor [85, pp. 246–309], and in the jump-diffusion setting in Øksendahl and Sulem [115, pp. 10–11]. The generator view may sometimes be helpful to heuristically understand the form of the results. For example, we will study the expectation $s(x) = \mathbb{E}_x T_z$ where $T_z = \inf\{t > 0 : V(t) \geq z\}$ is a level crossing time of V (and as usual $\mathbb{P}_x, \mathbb{E}_x$ refer to the case $V(0) = x$). Intuitively, one should have

$$s(x) \approx h + \mathbb{E}_x s\bigl(x + V(h)\bigr) \approx h + s(x) + h\,(\mathscr{L}s)(x)$$

for small h, so that the equation to solve for computing s should be $(\mathscr{L}s)(x) = -1$ (subject to suitable boundary conditions). Our detailed analysis aims at making this rigorous.

We will encounter functions $h(\theta, x)$ depending on two arguments and write then as usual $h_\theta(\theta, x)$, $h_x(\theta, x)$ for the partial derivatives. When working with h as a function of x for a fixed θ, we write $h(\theta)$ rather than $h(\theta, \cdot)$.

14.2 Level Crossing Times

Consider the level crossing time

$$T_z = \inf\bigl\{t > 0 : V(t) \geq z\bigr\}$$

defined for V (with $0 < z \leq b$) and

$$\tau(w) = \inf\bigl\{t > 0 : \Lambda(t) \geq w\bigr\}$$

defined (with $w > 0$) for the additive functional Λ in (176). In this section, we formulate the integro-differential equations appropriate for computing characteristics of these quantities. Our approach closely follows that used by Karatzas and Shreve [84] and Harrison [73] as a means of calculating various expectations associated with Markov processes.

Theorem 14.1 *Fix $\theta \leq 0$. Suppose that there exists a function $h(\theta) = h(\theta, \cdot) : [0, b] \to \mathbb{R}$ that is twice continuously differentiable in $[0, z]$ satisfying the integro-differential equation*

$$\bigl(\mathscr{L}h(\theta)\bigr)(x) + \theta h(\theta, x) = 0 \tag{177}$$

for $0 \leq x \leq z$, subject to the boundary conditions

$$h(\theta, x) = 1 \ \textit{for}\ x \geq z, \quad h_x(\theta, 0) = 0\,.$$

Then $h(\theta, x) = \mathbb{E}_x e^{\theta T_z}$.

Proof Observe that $U_c(T_z) = 0$. Hence, Itô's formula yields

$$\mathbb{E}\mathrm{e}^{\theta(T_z\wedge t)}h\big(\theta, V(T_z\wedge t)\big) - h\big(\theta, V(0)\big) \;=\; \sum_{j=1}^{4} T_j,$$

where

$$\begin{aligned}
T_1 \;&=\; \int_0^{T_z\wedge t} \mathrm{e}^{bs}\big[\big(\mathscr{L}h(\theta)\big)\big(V(s)\big) + \theta h\big(\theta, V(s)\big)\big]\,\mathrm{d}s\;,\\
T_2 \;&=\; \int_0^{T_z\wedge t} \mathrm{e}^{bs}h_x\big(\theta, V(s)\big)\,\mathrm{d}s\,,\\
T_3 \;&=\; \int_0^{T_z\wedge t} \mathrm{e}^{bs}h_x\big(\theta, V(s)\big)\,\sigma\,\mathrm{d}B(s)\,\mathrm{d}s\,,\\
T_4 \;&=\; \sum_{0<s\le T_z\wedge t} \mathrm{e}^{bs}\big[h\big(\theta, V(s)\big) - h\big(\theta, V(s-)\big)\big]\,,\\
&\quad - \int_0^{T_z\wedge t} \mathrm{e}^{bs}\int_{\mathbb{R}}\big[h\big(\theta, r(V(s), y)\big) - h\big(\theta, V(s)\big)\big]\,\nu(\mathrm{d}y)\,\mathrm{d}s
\end{aligned}$$

Here $T_1 + T_2 = 0$ because of the integro-differential equation (177) and the boundary condition at $x = 0$, whereas $T_3 + T_4$ form a martingale. Hence

$$\mathbb{E}\mathrm{e}^{\theta(T_z\wedge t)}h\big(\theta, V(T_z\wedge t)\big) \;=\; h(\theta, x)$$

for $t \ge 0$. By monotone convergence and the boundary condition for $x \ge z$,

$$\mathbb{E}\big[\mathrm{e}^{\theta T_z}h\big(\theta, V(T_z)\big);\, T_z \le t\big] \;\uparrow\; \mathbb{E}\big[\mathrm{e}^{\theta T_z};\, T_z < \infty\big]$$

as $t \to \infty$. On the other hand, the boundedness of $h(\theta)$ and the fact that $\theta \le 0$ ensure that

$$\mathbb{E}\big[\mathrm{e}^{\theta t}h\big(\theta, V(t)\big);\, T_z > t\big] \;\to\; 0\,.$$

The conclusion follows by noting that $T_z < \infty$ a.s. because of the reflection at 0. □

We can now formally obtain our integro-differential equation for $\mathbb{E}_x T_z$ by differentiating (177) and the boundary conditions. In particular note that $\mathbb{E}_x T_z = h_\theta(0, x)$. Formally differentiating (177) w.r.t. θ yields

$$(\mathscr{L}h_\theta)(x) + h(\theta, x) + \theta h_\theta(\theta, x) \;=\; 0$$

for $0 \le x \le z$, subject to

$$h_\theta(\theta, x) = 0 \;\text{ for } x \ge z, \quad h_{x\theta}(\theta, 0) \;=\; 0\,.$$

Letting $\tilde{h}(x) = \mathbb{E}T_z$, we conclude from $h(0,x) = 1$ for $0 \le x \le z$ that $\tilde{h}$ should satisfy

$$(\mathscr{L}\tilde{h})(x) = -1$$

for $0 \le x \le z$, subject to

$$\tilde{h}(x) = 0 \text{ for } x \ge z, \quad \tilde{h}'(0) = 0\,.$$

By working with the martingale

$$\tilde{h}\big(V(T_z \wedge t)\big) + T_z \wedge t\,,$$

this can be rigorously verified by sending $t \to \infty$, using the boundedness of $\tilde{h}$, and exploiting the fact that $\tilde{h}\big(V(T_z)\big) = 0$.

Bounds on $\mathbb{E}_x T_z$ can be obtained similarly, in the presence of a non-negative twice continuously differentiable function γ for which

$$(\mathscr{L}\gamma)(x) \le -1$$

for $0 \le x \le z$. In this case,

$$\gamma\big(V(T_z \wedge t)\big) + T_z \wedge t$$

is a non-negative supermartingale. Because γ is non-negative, $\mathbb{E}_x T_z \wedge t \le \gamma(x)$ for $t \ge 0$, yielding the bound

$$\mathbb{E}_x T_z \le \gamma(z)$$

for $0 \le x \le z$ upon application of the monotone convergence theorem.

We next turn to the computation of $\mathbb{E}_x \mathrm{e}^{\theta\tau(w)}$ with $\Lambda(t)$ as in (176) (note for the following result the quantities $f, \tilde{f}, r_1, r_2$ occurring in the definition).

Theorem 14.2 *Fix $\theta \le 0$. Suppose that there exists a function $k(\theta) : [0,b] \times \mathbb{R} \to \mathbb{R}$ of x, λ that is twice continuously differentiable in x and continuously differentiable in λ on $[0,b] \times (-\infty, w]$, and satisfies*

$$\begin{aligned} 0 \;=\; & \mu k_x(\theta,x,\lambda) + \frac{\sigma^2}{2} k_{xx}(\theta,x,\lambda) + k_\lambda(\theta,x,\lambda) f(x) + \theta k(\theta,x,\lambda) \\ & + \int_{\mathbb{R}} \big[k\big(\theta, r(x,y), \lambda + \tilde{f}(x,y)\big) - k(\theta,x,\lambda)\big]\, \nu(\mathrm{d}y) \qquad (178) \end{aligned}$$

for $0 \le x \le b$, $\lambda \le w$, *subject to the boundary conditions*

$$r_1 k_\lambda(\theta,x,\lambda) + k_x(\theta,x,\lambda) = 0 \ \ \textit{and} \ \ r_2 k_\lambda(\theta,x,\lambda) - k_x(\theta,x,\lambda) = 0$$

for $\lambda \le w$. *If* $\mathbb{E}_\pi \Lambda(1) \ge 0$, *then* $k(\theta,x,\lambda) \ = \ \mathbb{E}_x \mathrm{e}^{\theta\tau(w)}$.

Proof An application of Itô's formula guarantees that

$$\mathrm{e}^{\theta(\tau(w)\wedge t)} k\big(\theta, V(\tau(w)\wedge t)), \Lambda(\tau(w)\wedge t)\big) - k\big(\theta, V(0), 0\big) \ = \ \sum_{j=1}^{6} T_j$$

where

$$\begin{aligned}
T_1 &= \int_0^{\tau(w)\wedge t} \mathrm{e}^{bs} \int_{\mathbb{R}} \Big[k\big(\theta, r(V(s),y), \Lambda(s) + \tilde{f}(V(s),y)\big) - k\big(\theta, V(s), \Lambda(s)\big)\Big]\, \nu(\mathrm{d}y)\,\mathrm{d}s\,,\\
T_2 &= \int_0^{\tau(w)\wedge t} \mathrm{e}^{bs}\Big[\mu k_x\big(\theta, V(s), \Lambda(s)\big) \frac{\sigma^2}{2} k_{xx}\big(\theta, V(s), \Lambda(s)\big)\\
&\quad + k_\lambda\big(\theta, V(s), \Lambda(s)\big) f\big(V(s)\big) + \theta k_\lambda\big(\theta, V(s), \Lambda(s)\big)\Big]\,\mathrm{d}s\,,\\
T_3 &= \int_0^{\tau(w)\wedge t} \mathrm{e}^{bs}\big[k_x\big(\theta, V(s), \Lambda(s)\big) + r_1 k_\lambda\big(\theta, V(s), \Lambda(s)\big)\big]\,\mathrm{d}L_c(s)\,,\\
T_4 &= \int_0^{\tau(w)\wedge t} \mathrm{e}^{bs}\big[-k_x\big(\theta, V(s), \Lambda(s)\big) + r_2 k_\lambda\big(\theta, V(s), \Lambda(s)\big)\big]\,\mathrm{d}U_c(s)\,,\\
T_5 &= \int_0^{\tau(w)\wedge t} \mathrm{e}^{bs} k_x\big(\theta, V(s), \Lambda(s)\big)\,\sigma\,\mathrm{d}B(s)\,,\\
T_6 &= \sum_{0<s\le t} \mathrm{e}^{bs}\big[k\big(\theta, r(V(s-), \Delta X(s)), \Lambda(s-) + \tilde{f}(V(s-), \Delta X(s))\big)\\
&\quad - k\big(\theta, V(s), \Lambda(s-)\big)\big]\,,\\
&\quad - \int_0^{\tau(w)\wedge t} \mathrm{e}^{bs} \int_{\mathbb{R}} \big[k\big(\theta, r(V(s-),y), \Lambda(s-) + \tilde{f}(V(s-),y)\big)\\
&\quad - k\big(\theta, V(s-), \Lambda(s-)\big)\big]\,\mathrm{d}s\,.
\end{aligned}$$

Here $T_1 + T_2 = 0$ because of the integro-differential equation (178), $T_3 = T_4 = 0$ because of the boundary conditions, and T_5, T_6 are martingales. Consequently,

$$k\big(\theta, V(0), 0\big) \ = \ \mathbb{E}_x\big[\mathrm{e}^{\theta\tau(w)} k\big(\theta, V(\tau(w)\wedge t)), \Lambda(\tau(w)\wedge t)\big)\big]$$

Now $\tau(w) < \infty$ a.s. because $\mathbb{E}_\pi \Lambda(1) < \infty$. Because $\theta \le 0$ and k is bounded, the r.h.s. converges to

$$\mathbb{E}_x\big[\mathrm{e}^{\theta\tau(w)}k\big(\theta, V(\tau(w))), \Lambda(\tau(w))\big).$$

The proof is completed upon recognizing that $k(\theta, x, \lambda) = 1$ for $\lambda \ge w$. □

14.3 Poisson's Equation and the CLT

A natural complement to the computation of the loss rate ℓ^b is the development of a central limit theorem (CLT) for the cumulative loss. In particular, we wish to obtain a CLT of the form

$$\frac{U(t) - \ell^b t}{\sqrt{t}} \overset{\mathscr{D}}{\to} \eta N(0,1) \tag{179}$$

as $t \to \infty$. This CLT lends itself to the approximation

$$U(t) \overset{\mathscr{D}}{\approx} \ell^b t + \eta\sqrt{t}N(0,1) \tag{180}$$

when t is large, where $\overset{\mathscr{D}}{\approx}$ means 'has approximately the same distribution as' [and carries no rigorous meaning, other than through (179)]. The key new parameter to be computed in the approximation (180) is the *time-average variance constant* η^2. Computing η^2, in turn, involves representing $U(t)$ in terms of the solution to *Poisson's equation* which is well-known to play a fundamental role for Markov process CLTs (cf. e.g. Bhattacharyya [30], Glynn [65], Glynn and Meyn [67], [11, I.7, II.4d]). See also Williams [139] for the CLT for U in the Brownian case.

We develop the theory in terms of a general additive functional V of the form (176) and its associated boundary processes L and U. Given a function $g: [0,b] \to \mathbb{R}$ and a scalar c, we say that the pair (g,c) is a solution to Poisson's equation for the additive functional Λ if

$$g\big(V(t)\big) + \Lambda(t) - ct$$

is a martingale. The martingale is a generalization of the Dynkin martingale that arises if $r_1 = r_2 = 0 \equiv \tilde{f}$ below; see [85, p. 299]. When $\Lambda = U$, c must clearly equal ℓ^b.

Theorem 14.3 *Assume that f is bounded and that*

$$\sup_{0\le x\le b} \int_{\mathbb{R}} \big(|\tilde{f}(x,y)| + \tilde{f}(x,y)^2\big)\, \nu(\mathrm{d}y) < \infty. \tag{181}$$

If there exists a twice continuously differentiable function $g : [0, b] \to \mathbb{R}$ *satisfying*

$$\sup_{0\le x\le b} \int_{\mathbb{R}} \big|g\big(r(x,y)\big)\big|\, \nu(\mathrm{d}y) \;<\; \infty\,. \tag{182}$$

and a scalar c such that the pair (g, c) *satisfies the integro-differential equation*

$$(\mathscr{L}g)(x) \;=\; -\Big(f(x) + \int_{\mathbb{R}} \tilde{f}(x,y)\, \nu(\mathrm{d}y) - c\Big) \tag{183}$$

for $0 \le x \le b$*, subject to the boundary conditions*

$$g'(0) = -r_1\,, \qquad g'(b) = r_2\,, \tag{184}$$

then

$$g\big(V(t)\big) + \Lambda(t) - ct$$

is a martingale. Furthermore,

$$\frac{\Lambda(t) - \ell^b t}{\sqrt{t}} \;\overset{\mathscr{D}}{\to}\; \eta N(0,1)$$

as $t \to \infty$*, where*

$$\eta^2 \;=\; \int_0^b \Big[\sigma^2 g'(x)^2 + \int_{\mathbb{R}} \big(\tilde{f}(x,y) + g\big(r(x,y)\big) - g(x)\big)^2\big]\, \nu(\mathrm{d}y)\Big]\, \pi(\mathrm{d}x)\,.$$

Proof We note that Itô's formula guarantees that

$$\begin{aligned}
& g\big(V(t)\big) - g\big(V(0)\big) + \Lambda(t) - ct \\
&\quad = \int_0^t (\mathscr{L}g)\big(V(s)\big)\, \mathrm{d}s \;+ \int_0^t g'\big(V(s)\big)\sigma\, \mathrm{d}B(s) \\
&\qquad + \sum_{0<s\le t} \big[g\big(V(s)\big) - g\big(V(s-)\big)\big] \\
&\qquad\quad - \int_0^t \int_{\mathbb{R}} \big[g\big(r(V(s-),y)\big) - g\big(V(s-)\big)\big]\, \nu(\mathrm{d}y)\, \mathrm{d}s \\
&\qquad + \int_0^t f\big(X(s)\big)\, \mathrm{d}s \;+ \int_0^t \int_{\mathbb{R}} \tilde{f}\big(V(s-),y\big)\, \nu(\mathrm{d}y)\, \mathrm{d}s \\
&\qquad + \sum_{0<s\le t} f\big(V(s-), \Delta X(s)\big) \;-\; \int_0^t \int_{\mathbb{R}} \tilde{f}\big(V(s-),y\big)\, \nu(\mathrm{d}y)\, \mathrm{d}s
\end{aligned}$$

$$
\begin{aligned}
&+\ r_1 L_c(t)\ + r_2 U_c(t) - ct\ +\ g'(0)L_c(t) - g'(b)U_c(t)\\
&= \int_0^t g'\big(V(s)\big)\sigma\,\mathrm{d}B(s)\\
&\quad + \sum_{0<s\le t}\big[g\big(V(s)\big) - g\big(V(s-)\big)\ +\ \tilde f(V(s-),\Delta X(s))\big]\\
&\qquad - \int_0^t\int_{\mathbb{R}}\big[g\big(r(V(s-),y)\big) - g\big(V(s)\big) + \tilde f\big(V(s-),y\big)\big]\,\nu(\mathrm{d}y)\,\mathrm{d}s\\
&= M(t)\ \ (\text{say})\,,
\end{aligned}
$$

where (181) and (182) were used to obtain the second equality. In the presence of (181), (182), and the boundedness of g and g', it follows that $M(t)$ is a martingale. Furthermore, the quadratic variation has the form

$$
\begin{aligned}
[M,M](t)\ &=\ \int_0^t g'\big(V(s)\big)^2\sigma^2\,\mathrm{d}s\\
&\quad + \sum_{0<s\le t}\big[g\big(V(s)\big) - g\big(V(s-)\big)\big\} + \tilde f(V(s-),\Delta X(s))\big]^2\\
&= \int_0^t\Big[\sigma^2 g\big(r(V(s-),y)\big)^2\\
&\quad + \int_{\mathbb{R}}\big[g\big(V(s)\big) - g\big(V(s-)\big)\ +\ \tilde f(V(s-),\Delta X(s))\big]^2\big]\,\nu(\mathrm{d}y)\,\mathrm{d}s\\
&\quad + M_1(t)
\end{aligned}
$$

where $M_1(t)$ is a martingale. It is easily seen that $[M,M](t)/t \to \eta^2$ a.s. as $t\to\infty$. Finally, to verify condition a) of the martingale CLT in [58, p. 340], we need to show that

$$
\frac{1}{\sqrt{t}}\mathbb{E}_\pi \sup_{0\le s\le t}\big|M(s)-M(s-)\big|\ \to\ 0 \tag{185}
$$

as $t\to\infty$ (this needs only to be verified for $V(0)$ distributed as π because we can couple V to the stationary version from any initial distribution). Of course, a sufficient condition for (185) is to establish that

$$
\frac{1}{t}\mathbb{E}_\pi \sup_{0\le s\le t}\big|M(s)-M(s-)\big|^2\ \to\ 0\,. \tag{186}
$$

It is well known that (186) is immediate if

$$
\mathbb{E}_\pi \sup_{0\le s\le 1}\big(M(s)-M(s-)\big)^2\ <\ \infty\,. \tag{187}
$$

But (187) is bounded by

$$\begin{aligned}
&\mathbb{E}_\pi \sum_{0\le s\le 1} \big[M(s) - M(s-)\big]^2 \\
&= \mathbb{E}_\pi \sum_{0\le s\le 1} \big[g\big(V(s)\big) - g\big(V(s-)\big) + \tilde{f}(V(s-), \Delta X(s))\big]^2 \\
&= \mathbb{E}_\pi \int_{\mathbb{R}} \big[g\big(r(V(s-), y)\big) - g\big(V(s-)\big) + \tilde{f}(V(s-), y)\big]^2 \nu(\mathrm{d}y)\,\mathrm{d}s \\
&= \int_0^b \int_{\mathbb{R}} \big[g\big(r(x, y)\big) - g\big(x,\big) + \tilde{f}(x, y)\big]^2 \nu(\mathrm{d}y)\,\pi(\mathrm{d}x)
\end{aligned}$$

due to the boundedness of g and condition (181). The martingale CLT then yields the desired conclusion. □

Theorem 14.3 therefore provides the CLT for general additive functionals associated with V, provided that one can solve the integro-differential equation (183) subject to the boundary condition (184). Finally, we note that the fact that $g\big(V(t)\big) + \Lambda(t) - ct$ is, in great generality, a martingale, implies that

$$\mathbb{E}_x \Lambda(t) \;=\; ct + g(x) - \mathbb{E}_x g\big(V(t)\big)\,,$$

where, as usual, $\mathbb{E}_x$ refers to the case $V(0) = x$. Since

$$\mathbb{E}_x g\big(V(t)\big) \;\to\; \mathbb{E}_\pi g\big(V(t)\big)$$

as $t \to \infty$ (since V is regenerative with absolutely continuous cycles), we conclude that

$$\mathbb{E}_x \Lambda(t) \;=\; ct + g(x) - \mathbb{E}_\pi g\big(V(t)\big) \;+\; \mathrm{o}(1)\,,$$

as $t \to \infty$. Hence, the solution g to Poisson's equation also provides a 'correction' to the value of $\mathbb{E}_x \Lambda(t)$ that reflects the influence of the initial condition on the expected value of an additive functional.

14.4 Large Deviations for the Loss Process

We turn next to obtaining a family of integro-differential equations from which the large deviations behaviour of the additive functional $\Lambda(\cdot)$ can be derived (for earlier work in this direction in the Brownian case, see Zhang and Glynn [141] and Forde et al. [60]). The key to the analysis is the following result:

Theorem 14.4 *Fix $\theta \in \mathbb{R}$. Suppose that*

$$\sup_{0 \le x \le b} \int_{\mathbb{R}} \mathrm{e}^{\theta \tilde{f}(x,y)} \, \nu(\mathrm{d}y) \ < \ \infty \, . \tag{188}$$

If there exists a positive twice differentiable function $u(\theta) : [0, b] \to \mathbb{R}$ and a scalar $\psi(\theta)$ such that the pair $\big(u(\theta), \psi(\theta)\big)$ satisfies the integro-differential equation

$$\begin{aligned} 0 \ &= \ \mu u_x(\theta, x) + \frac{\sigma^2}{2} u_{xx}(\theta, x) + \big(\theta f(x) - \psi(\theta)\big) u(\theta, x) \\ &\quad + \int_{\mathbb{R}} \big[\mathrm{e}^{\theta \tilde{f}(x,y)} u\big(\theta, r(x,y)\big) - u(\theta, x)\big] \, \nu(\mathrm{d}y) \end{aligned} \tag{189}$$

for $0 \le x \le b$, subject to the boundary conditions

$$u_x(\theta, 0) = -r_1 \theta \, , \quad u_x(\theta, b) = r_2 \theta \, , \tag{190}$$

then $M(\theta, t) = \mathrm{e}^{\theta \Lambda(t)} u\big(\theta, V(t)\big)$ is a martingale.

Proof Define $A(s) = \exp\big\{\theta \Lambda(s) - \psi(\theta) s\big\}$ and

$$S(t) \ = \sum_{0 < s \le t} A(s) \big[\exp\big\{\theta \tilde{f}\big(V(s-), \Delta X(s)\big)\big\} u\big(\theta, V(s)\big) - u\big(\theta, V(s-)\big)\big] \, .$$

Itô's formula shows that $M(\theta, t) - M(\theta, 0)$ equals

$$\begin{aligned} S(t) \ &+ \ \int_0^t \big[\theta f\big(V(s)\big) - \psi(\theta)\big] A(s) u\big(\theta, V(s-)\big) \, \mathrm{d}s \\ &+ \int_0^t r_1 \theta A(s) \, \mathrm{d}L_c(s) \ + \ \int_0^t r_2 \theta A(s) \, \mathrm{d}U_c(s) \\ &+ \int_0^t \Big[\mu u_x\big(\theta, V(s)\big) + \frac{\sigma^2}{2} u_{xx}\big(\theta, V(s)\big)\Big] A(s) \mathrm{d}s \\ &= \int_0^t A(s) u_x\big(\theta, V(s)\big) \sigma \, \mathrm{d}B(s) \ + \ S(t) \\ &\quad - \int_0^t A(s) \int_{\mathbb{R}} \big[\exp\big\{\theta \tilde{f}\big(V(s), y\big)\big\} u\big(\theta, r\big(V(s), y\big)\big) - u\big(\theta, V(s)\big)\big] \, \nu(\mathrm{d}y) \, \mathrm{d}s \, , \end{aligned}$$

where the second equality uses the fact that $\big(u(\theta), \psi(\theta)\big)$ satisfy (189) and (190). Given the boundedness of $u(\theta)$ and (188), the fact that $M(\theta, t)$ is integrable and is a martingale is clear. □

As a consequence of the martingale property and the fact that $u(\theta)$ is bounded above and below by finite positive constants, it is straightforward to establish that

$$\frac{1}{t}\log \mathbb{E}_x \mathrm{e}^{\theta \Lambda(t)} \;\to\; \psi(\theta)$$

as $t \to \infty$. Suppose that there exists $\theta^* > 0$ for which $\psi(\cdot)$ exists in a neighbourhood of θ^* and is continuously differentiable there. If we let $a = \psi'(\theta^*)$, then

$$\frac{1}{t}\log \mathbb{P}_x\big(\Lambda(t) \geq at\big) \;\to\; \psi(\theta^*) - \theta^* a\,;$$

see, for example, the proof of the Gärtner-Ellis theorem in [53, 45–51]. Hence, the integro-differential equation (189) is intimately connected to the study of large deviations for Λ.

14.5 Discounted Expectations for Additive Functionals

As our final illustration of how integro-differential equations naturally arise when computing expectations of additive functionals of reflected Lévy processes, we consider the calculation of an infinite horizon discounted expectation. Specifically, we let the discounting factor at t be

$$\Gamma(t) \;=\; \int_0^t g\big(V(s)\big)\,\mathrm{d}s \;+\; \sum_{0<s\leq t} \tilde{g}\big(V(s-), \Delta X(s)\big) \;+\; u_1 L_c(s) + u_2 U_c(t)$$

for given functions $g, \tilde{g}$ (where $\tilde{g}$ is such that $\tilde{g}(x,0) = 0$ for $0 \leq x \leq b$), and set

$$D \;=\; \int_0^\infty \mathrm{e}^{-\Gamma(s)}\,\mathrm{d}\Lambda(s)\,.$$

As for $f, \tilde{f}$, we assume that g is bounded and that

$$\sup_{0\leq x\leq b} \int_{\mathbb{R}} \big|\tilde{g}(x,y)\big|\, \nu(\mathrm{d}y) \;<\; \infty.$$

Theorem 14.5 *Assume that $f, \tilde{f}$, $g, \tilde{g}$, u_1, u_2 are non-negative with g strictly positive. If there exists a twice continuously differentiable function $k : [0,b] \to [0,\infty)$*

satisfying the integro-differential equation

$$\begin{aligned}0 &= \mu k'(x) + \frac{\sigma^2}{2} k''(x) - g(x)k(x) \\ &\quad + \int_{\mathbb{R}} \big[\mathrm{e}^{-\tilde{g}(x,y)} k\big(r(x,y)\big) - k(x)\big]\, \nu(\mathrm{d}y) \\ &\quad + f(x) + \int_{\mathbb{R}} \tilde{f}(x,y)\, \nu(\mathrm{d}y)\end{aligned}$$

for $0 \le x \le b$*, subject to the boundary conditions*

$$k'(0) - u_1 k(0) = -r_1\,, \quad k'(b) + u_2 k(b) = -r_2\,,$$

then $\mathbb{E}_x D = k(x)$ *for* $0 \le x \le b$*.*

Proof Itô's formula ensures that

$$D(t) + \mathrm{e}^{-\Gamma(t)} k\big(V(t)\big) - k\big(V(0)\big) = \sum_{j=1}^{8} T_j$$

where

$$\begin{aligned}
T_1 &= \int_0^t \mathrm{e}^{-\Gamma(s)} \big[\mu k'\big(V(s)\big) + \frac{\sigma^2}{2} k'\big(V(s)\big) - g(x)k\big(V(s)\big)\big]\,, \\
T_2 &= \int_0^t \mathrm{e}^{-\Gamma(s)} \int_{\mathbb{R}} \big[\mathrm{e}^{-\tilde{g}\left(V(s),y\right)} k\big(r\big(V(s),y\big)\big) - k\big(V(s)\big)\big]\, \nu(\mathrm{d}y)\,, \\
T_3 &= \int_0^t \mathrm{e}^{-\Gamma(s)} f\big(V(s)\big)\, \mathrm{d}s + \int_0^t \mathrm{e}^{-\Gamma(s)} \int_{\mathbb{R}} \tilde{f}\big(V(s),y\big)\, \nu(\mathrm{d}y)\, \mathrm{d}s\,, \\
T_4 &= \int_0^t \mathrm{e}^{-\Gamma(s)} \big[r_1 - u_1 k\big(V(s)\big) + k'\big(V(s)\big)\big]\, \mathrm{d}L_c(s)\,, \\
T_5 &= \int_0^t \mathrm{e}^{-\Gamma(s)} \big[r_2 - u_2 k\big(V(s)\big) - k'\big(V(s)\big)\big]\, \mathrm{d}U_c(s)\,, \\
T_6 &= \int_0^t k'\big(V(s)\big)\, \sigma\, \mathrm{d}B(s)\,, \\
T_7 &= \sum_{0<s\le t} \mathrm{e}^{-\Gamma(s)} \tilde{f}\big(V(s), \Delta X(s)\big) - \int_0^t \mathrm{e}^{-\Gamma(s)} \tilde{f}\big(V(s),y\big)\, \nu(\mathrm{d}y)\, \mathrm{d}s\,, \\
T_8 &= \sum_{0<s\le t} \mathrm{e}^{-\Gamma(s)} \big[\mathrm{e}^{-\tilde{g}(V(s-),\Delta X(s))} k\big(r\big(V(s), \Delta X(s)\big)\big) - k\big(V(s-)\big)\big]\,, \\
&\quad - \int_0^t \mathrm{e}^{-\Gamma(s)} \int_{\mathbb{R}} \big[\mathrm{e}^{-\tilde{g}(V(s-),y)} k\big(r\big(V(s),y\big)\big) - k\big(V(s-)\big)\big]\, \nu(\mathrm{d}y)\, \mathrm{d}s\,.
\end{aligned}$$

Here $T_1 + T_2 + T_3 = 0$ because of the integro-differential equation, $T_4 = T_5 = 0$ because of the boundary conditions satisfied by k, and T_6, T_7, T_8 are all martingales. Consequently,

$$k(x) \;=\; \mathbb{E}_x \int_0^t \mathrm{e}^{-\Gamma(s)}\,\mathrm{d}\Lambda(s) \;+\; \mathbb{E}_x \mathrm{e}^{-\Gamma(t)} k\big(V(t)\big)\,.$$

Sending $t \to \infty$, the non-negativity assumption ensures that

$$\mathbb{E}_x \int_0^t \mathrm{e}^{-\Gamma(s)}\,\mathrm{d}\Lambda(s) \;\uparrow\; \mathbb{E}_x D\,,$$

while the non-negativity of g, $\tilde{g}$, u_1, u_2, positivity of g and boundedness of k ensure that

$$\mathbb{E}_x \mathrm{e}^{-\Gamma(t)} k\big(V(t)\big) \;\to\; 0\,,$$

proving the theorem. □

14.6 Jumps of Infinite Variation

Lévy processes are permitted to have a jump part of infinite variation as long as the FV condition (174) is weakened to

$$\int_{|y|<1} y^2\,\nu(\mathrm{d}y) \;<\; \infty\,. \tag{191}$$

In this setting, one must compensate the small jumps, by considering the random measure

$$\int_{|y|<1} y\big[\mu(\mathrm{d}y, \mathrm{d}s) - \nu(\mathrm{d}y)\,\mathrm{d}s\big] < \;\infty \tag{192}$$

where μ is the Poisson random measure having intensity measure $\nu \otimes m$ (where m is Lebesgue measure). The centered random measure is well-defined, and forms a square-integrable martingale when integrated over s [due to (191)]. Thus, in the non-BV jump setting we can write the Lévy process X as

$$X(t) - X(0) = at + \sigma B(t) + \sum_{0<s\le t} \Delta X(s) \mathbb{1}\big(\big|\Delta X(s)\big| \ge 1\big) + \int_0^t \int_{|y|<1} y\big[\mu(\mathrm{d}y, \mathrm{d}s) - \nu(\mathrm{d}y)\,\mathrm{d}s\big]$$

for some suitably defined constant a; observe that when the stronger FV condition (174) holds,

$$a \;=\; \mu \;+\; \int_{|y|<1} y\,\nu(\mathrm{d}y)\,.$$

In order to develop an Itô-type formula in this setting, we note than when $\nu[-\varepsilon,\varepsilon]=0$ for some $\varepsilon>0$, we can write (for f twice differentiable)

$$\begin{aligned}
&f\big(V(t)\big)-f\big(V(0)\big)\\
&\quad= \int_0^t\int_{\mathbb{R}}\big[f\big(V(s-)+y\big)-f\big(V(s-)\big)\big]\mu(\mathrm{d}y,\mathrm{d}s) \;+\; \frac{\sigma^2}{2}\int_0^t f''\big(V(s)\big)\,\mathrm{d}s\\
&\qquad+\int_0^t f'\big(V(s)\big)\Big[a\,\mathrm{d}s \;+\; \sigma\,\mathrm{d}B(s)-\int_{|y|<1} y\,\nu(\mathrm{d}y)\,\mathrm{d}s+\mathrm{d}L_c(s)-\,\mathrm{d}U_c(s)\Big]\\
&\quad= \int_0^t\int_{\mathbb{R}}\big[f\big(V(s-)+y\big)-f\big(V(s-)\big)\big]\big(\mu(\mathrm{d}y,\mathrm{d}s)-\nu(\mathrm{d}y)\,\mathrm{d}s\big)\\
&\qquad+\int_0^t\int_{|y|\ge 1}\big[f\big(V(s-)+y\big)-f\big(V(s-)\big)\big]\nu(\mathrm{d}y)\,\mathrm{d}s\\
&\qquad+\int_0^t\int_{|y|<1}\big[f\big(V(s-)+y\big)-f\big(V(s-)\big)-yf'\big(V(s-)\big)\big]\nu(\mathrm{d}y)\,\mathrm{d}s \qquad (193)\\
&\qquad+\int_0^t f'\big(V(s)\big)\,\sigma\,\mathrm{d}B(s)+f'(0)\big(L_c(t)-L_c(0)\big)\,-\,f'(b)\big(U_c(t)-U_c(0)\big)\,.
\end{aligned}$$

By sending $\varepsilon\downarrow 0$ and utilising (191), we find that this formula extends to the general case in the general Lévy setting. We note that the smoothness of f guarantees that

$$f\big(V(s-)+y\big)-f\big(V(s-)\big)-yf'\big(V(s-)\big)$$

is of order y^2 when y is small, thereby guaranteeing that the term (193) on the r.h.s. is well-defined. As a consequence of the martingale property of the centered stochastic integral,

$$\mathbb{E}_x f\big(V(t)\big)-\mathbb{E}_x f\big(V(0)\big) \;=\; \int_0^t (\tilde{\mathscr{L}}f)\big(V(s)\big)\,\mathrm{d}s\,,$$

where for some suitable $\tilde{\mu}$

$$\begin{aligned}
(\tilde{\mathscr{L}}\varphi)(x) &= \tilde{\mu}\varphi'(x)+\frac{\sigma^2}{2}\varphi''(x)+\int_{|y|>1}\big[\varphi\big(r(x,y)\big)-\varphi(x)\big]\,\nu(\mathrm{d}y)\\
&\qquad+\int_{|y|\le 1}\big[\varphi\big(r(x,y)\big)-\varphi(x)-y\varphi'(x)\big]\,\nu(\mathrm{d}y)\,,
\end{aligned}$$

provided that $\varphi'(0 = \varphi'(b) = 0$. The integro-differential operator $\tilde{\mathscr{L}}$ replaces the operator $\mathscr{L}$ that appeared earlier in the BV case (it can be easily verified that $\tilde{\mathscr{L}} = \mathscr{L}$ in the BV case). For example, to compute $\mathbb{E}_x T_z$, the Itô argument above establishes that if h satisfies $(\tilde{\mathscr{L}}h)(x) = -1$ subject to $h'(0) = 0$ and $h(x) = 0$ for $x \geq z$, then $h(x) = \mathbb{E}_x T_z$. In a similar fashion all the other integro-differential equations derived earlier in this section can be generalised to Lévy processes having non-BV jumps.

15 Additional Representations for the Loss Rate

In Sects. 6 and 8, two representations for ℓ^b were provided, in which ℓ^b was represented in terms of an integral against the stationary distribution π for the 'interior process' V. In this section, we return to the computation of ℓ^b and provide a simple argument establishing that there are infinitely many such representations of ℓ^b in terms of π.

The notation is the same as in Sect. 14; recall in particular the function $r(x, y)$ associated with two-sided reflection and the integro-differential operator $\mathscr{L}$.

We first write the local time $U(t)$ at b in terms of the jump component and its continuous component, so that

$$U(t) - U(0) = \sum_{0<s\leq t} \Delta U(s) + U_c(t),$$

where, as usual, $\Delta U(s) = U(s) - U(s-)$ for $s > 0$. Clearly,

$$\ell^b = \ell_j^b + \ell_c^b,$$

where

$$\ell_j^b = \lim_{t\to\infty} \frac{1}{t} \sum_{0<s\leq t} \Delta U(s), \quad \ell_c^b = \lim_{t\to\infty} \frac{1}{t} U_c(t) \text{ a.s.}$$

We now show how ℓ_j^b and ℓ_c^b can be individually calculated in terms of π. Dealing with ℓ_j^b is easy. Note that

$$\tilde{M}(t) = \sum_{0<s\leq t} \Delta U(s) - \int_0^t \int_{\mathbb{R}} [V(s) + y - b]^+ \, \nu(\mathrm{d}y)\, \mathrm{d}s$$

is a martingale (see p. 6 of [115]), and hence

$$\mathbb{E}\frac{1}{t} \sum_{0\leq s\leq t} \Delta U(s) = \mathbb{E}\frac{1}{t} \int_0^t \int_{\mathbb{R}} [V(s) + y - b]^+ \, \nu(\mathrm{d}y)\, \mathrm{d}s.$$

Consequently,

$$\ell_j^b \;=\; \int_0^b \int_{\mathbb{R}} \big[x+y-b\big]^+ \,\nu(\mathrm{d}y)\,\pi(\mathrm{d}x)\,.$$

It remains only to compute ℓ_c^b. For a given twice differentiable function $h : [0,b] \to \mathbb{R}$, Itô's formula (see p. 7 of [115]) ensures that

$$\begin{aligned} h\big(V(t)\big) - h\big(V(0)\big) \;=\; & \sum_{0<s\le t} \big[h\big(V(s)\big) - h\big(V(s-)\big)\big] \\ & + \int_0^t \Big[\mu h'\big(V(s)\big) + \frac{\sigma^2}{2} h''\big(V(s)\big)\Big]\,\mathrm{d}s \\ & + \sigma \int_0^t h'\big(V(s)\big)\,\mathrm{d}B(s) + h'(0)L_c(t) - h'(b)U_c(t)\,, \end{aligned} \tag{194}$$

where $L_c(\cdot)$ is the continuous component of $L(\cdot)$. Letting

$$\begin{aligned} M(t) \;=\; & \sum_{0<s\le t} \big[h\big(V(s)\big) - h\big(V(s-)\big)\big] + \sigma \int_0^t h'\big(V(s)\big)\,\mathrm{d}B(s) \\ & - \int_0^t \int_{\mathbb{R}} \big[h\big(r\big(V(s-),y\big)\big) - h\big(V(s-)\big)\big]\,\nu(\mathrm{d}y)\,\mathrm{d}s\,, \end{aligned}$$

and rewriting (194) in terms of $\mathscr{L}$, we get

$$h\big(V(t)\big) - h\big(V(0)\big) \;=\; M(t) + \int_0^t (\mathscr{L}h)\big(V(s)\big)\,\mathrm{d}s + h'(0)L_c(t) - h'(b)U_c(t)$$

Further, $M(\cdot)$ is a square integrable martingale, and since h and its derivatives are bounded, it follows by taking stationary expectations at $t=1$ that

$$0 \;=\; \int_0^b (\mathscr{L}h)(x)\,\pi(\mathrm{d}x) + h'(0)\ell_c^0 - h'(b)\ell_c^b\,, \tag{195}$$

where $\ell_c^0 \;=\; \lim_{t\to\infty} \frac{1}{t} L_c(t)$ a.s.

As a consequence, we can now compute ℓ_c^0 and ℓ_c^b by choosing two (twice differentiable) functions h_1 and h_2. According to (195),

$$\begin{pmatrix} h_1'(b) & -h_1'(0) \\ h_2'(b) & -h_2'(0) \end{pmatrix} \begin{pmatrix} \ell_c^b \\ \ell_c^0 \end{pmatrix} \;=\; \begin{pmatrix} \displaystyle\int_0^b (\mathscr{L}h_1)(x)\,\pi(\mathrm{d}x) \\ \displaystyle\int_0^b (\mathscr{L}h_2)(x)\,\pi(\mathrm{d}x) \end{pmatrix} \tag{196}$$

Thus, if h_1 and h_2 are chosen so that the coefficient matrix on the l.h.s. of (196) is non-singular, this yields formulae for ℓ_c^0 and ℓ_c^b in terms of π. Consequently, there are infinitely many representations of ℓ^b in terms of π (two of which have been introduced in Sects. 6 and 8).

Even in situations where π is not easily computable, the above approach provides a mechanism for easily computing bounds on ℓ^b. For example, by choosing h_1 so that $h_1'(0) = 0$ and $h_1'(b) = 1$ (and $h_2(\cdot)$ arbitrarily), we can compute bounds on ℓ_c^b in terms of the supremum of $(\mathscr{L}h_1)(x)$.

16 Markov-Modulation

Models with the parameters varying according to the state of a finite Markov chain or -process have a long history and are popular in many areas: in statistics, they go under the name of *hidden Markov models* (e.g. Cappé et al. [41]), in finance the term *Markov regime switching* is used (e.g. Elliott et al. [55]), and in queueing the first occurrence was with the *Markov-modulated Poisson process*. We consider here Lévy processes with the characteristic triplet (c_i, σ_i^2, ν_i) depending on the state $J(t) = i$ of an underlying finite ergodic Markov process J, with the extension that additional jumps may occur at state changes of J. This is important since then the model class becomes dense in the whole of $D[0,\infty)$, cf. [11, Chap. XI] where also the connection to Markov additive processes is explained.

In this section we generalize the results from Sects. 6 and 8 to hold for a Markov-modulated Lévy process X. We will use the same technique as in Sect. 6 (a direct application of Ito's formula for general semimartingales) to derive a formula for ℓ^b. In [16] an approach based on optional stopping of a multi-dimensional version of the Kella-Whitt martingale is used to obtain ℓ^b, but this will not be presented here, since it is very complicated and does not really shed any probabilistic light upon the underlying Skorokhod problem. Further, the direct Ito approach leads directly to an easier expression for ℓ^b.

We start by constructing X. We assume that we are given an underlying probability space with filtration $\mathscr{F}$, which satisfies the usual conditions, i.e., it is augmented and right-continuous. Let J (the modulating process) be a right-continuous irreducible Markov process with state space $\{1,\dots,p\}$, intensity matrix $\mathbf{Q} = (q_{ij})$ and stationary row vector $\alpha = (\alpha_i)$. Let $X^1,\dots,X^p$ be Lévy processes (with respect to $\mathscr{F}$) with characteristic triplets (c_i, σ_i^2, ν_i), $i = 1,\dots,p$, which are independent of J and each other and satisfy $\mathbb{E}|X^i(1)| < \infty$, $i = 1,\dots,p$. Further, let $\{U^{ij} : 1 \le i,j \le p\}$ and $\{U_n^{ij} : n \ge 1,\ 1 \le i,j \le p\}$ be independent random variables which are also independent of $X^1,\dots,X^p$ and J, such that for each i,j,n, U^{ij} and U_n^{ij} are identically distributed with distribution H^{ij} and $\mathbb{E}|U^{ij}| < \infty$. Let $T_0, T_1, \dots$ be the jump epochs of J (with $T_0 = 0$). It is assumed that for every i,j,n, U_n^{ij} is measurable with respect to $\mathscr{F}(T_n)$ and that $U^{ij} \in \mathscr{F}(0)$. We then define the process X according to

$$X(t) = \sum_{n\ge 1}\sum_{\substack{1\le i,j\le p\\ i\ne j}} (X^i(T_n) - X^i(T_{n-1}) + U_n^{ij})\mathbb{1}(J(T_{n-1}) = i, J(T_n) = j, T_n \le t)$$
$$+ \sum_{n\ge 1}\sum_{i=1}^{p}(X^i(t) - X^i(T_{n-1}))\mathbb{1}(J(T_{n-1}) = i, T_{n-1} \le t < T_n), \qquad (197)$$

or, equivalently, $X(0) = 0$ and

$$\mathrm{d}X(s) = \sum_{i=1}^{p} \mathbb{1}(J(s) = i)\mathrm{d}X^i(s) + \sum_{n\ge 1}\sum_{\substack{1\le i,j\le p\\ i\ne j}} U_n^{ij}\mathbb{1}(s = T_n, J(T_n-) = i, J(T_n) = j). \qquad (198)$$

We denote the stationary measure of (V,J) by $\pi(\cdot,\cdot)$ ((V,J) is assumed to be stationary throughout this section). Let $\tilde{H}^{ij} = H^{ji}$ and $\tilde{J}$ be time-reversed version of J (note that $\tilde{J}$ has intensity matrix $\tilde{\mathbf{Q}} = \mathbf{A}^{-1}\mathbf{Q}^{\mathsf{T}}\mathbf{A}$ where $\mathbf{A}$ is the diagonal matrix with α on the diagonal, and that α is also stationary for $\tilde{J}$). $\tilde{X}$ is constructed by using (197) with H^{ij} replaced by $\tilde{H}^{ij}$ and J replaced by $\tilde{J}$. In the same way as in Proposition 2.11 in [11, p. 314], we obtain the following representation of π in the Markov-modulated case.

$$\pi([y,b],i) = \alpha_i\mathbb{P}_i(\tilde{X}(\tau[y-b,y)) \ge y), \qquad (199)$$

where $\tau[u,v) = \inf\{t \ge 0 : \tilde{X}(t) \notin [u,v)\},\ u \le 0 \le v$, and $\mathbb{P}_i(\cdot) = \mathbb{P}(\cdot\,|\,\tilde{J}(0) = i)$.

Now, we turn our attention towards the identification of ℓ^b. The only differences between the Markov-modulated case and the standard Lévy process case are that we now have to treat time segments corresponding to different states of J separately and that state changes in J generate jumps in X. In particular, we get the following equivalent to (72) [where $\mathrm{d}X(s)$ is given by (198)]

$$V(t)^2 - V(0)^2 - \int_{0+}^{t} 2V(s-)\mathrm{d}X(s)$$
$$= -2bU(t) + \int_{0+}^{t} \mathrm{d}[X,X]^c(s) + \sum_{0<s\le t}\{-2\Delta V(s)\Delta L(s) + 2\Delta V(s)\Delta U(s)$$
$$+ (\Delta V(s))^2\}.$$

where (cf. Corollaries 2.5 and 2.9 on p. 313 in [11])

$$m = \mathbb{E}_\pi X(1) = \sum_{i=1}^{p}\alpha_i\Big(m_i + \sum_{j\ne i} q_{ij}\mathbb{E}U^{ij}\Big) \qquad (200)$$

with $m_i = \mathbb{E}X^i(1)$. Thus, we have

$$-2m\mathbb{E}V = -2b\ell^b + \mathbb{E}_\pi \int_{0+}^1 \mathrm{d}[X,X]^c(s)$$
$$+ \mathbb{E}_\pi \sum_{0<s\le 1} \{-2\Delta V(s)\Delta L(s) + 2\Delta V(s)\Delta U(s) + (\Delta V(s))^2\}.$$

What remains is to identify terms which is fairly straightforward. It is easily seen that

$$\mathbb{E}\int_{0+}^1 \mathrm{d}[X,X]^c(s) = \sum_{i=1}^p \alpha_i\sigma_i^2\,, \quad \mathbb{E}V = \sum_{i=1}^p \int_0^b x\pi(\mathrm{d}x,i).$$

For the sum of jumps we get (condition on $\big(V(s-),J(s-)\big)$),

$$\mathbb{E}\sum_{0<s\le 1}(\Delta V(s))^2$$
$$= \sum_{i=1}^p \int_0^b \pi(\mathrm{d}x,i)\int_{-\infty}^{\infty} y^2 \mathbb{1}(-x<y<b-x)\Big(\nu_i(\mathrm{d}y) + \sum_{j\ne i} q_{ij}H^{ij}(\mathrm{d}y)\Big)$$
$$+ \sum_{i=1}^p \int_0^b \pi(\mathrm{d}x,i)\int_{-\infty}^{\infty} (b-x)^2 \mathbb{1}(y\ge b-x)\Big(\nu_i(\mathrm{d}y) + \sum_{j\ne i} q_{ij}H^{ij}(\mathrm{d}y)\Big)$$
$$+ \sum_{i=1}^p \int_0^b \pi(\mathrm{d}x,i)\int_{-\infty}^{\infty} x^2 \mathbb{1}(y\le -x)\Big(\nu_i(\mathrm{d}y) + \sum_{j\ne i} q_{ij}H^{ij}(\mathrm{d}y)\Big).$$

$$\mathbb{E}\sum_{0<s\le 1}\Delta V(s)\Delta L(s)$$
$$= \sum_{i=1}^p \int_0^b \pi(\mathrm{d}x,i)\int_{-\infty}^{\infty} x(x+y) \mathbb{1}(y\le -x)\Big(\nu_i(\mathrm{d}y) + \sum_{j\ne i} q_{ij}H^{ij}(\mathrm{d}y)\Big),$$

$$\mathbb{E}\sum_{0<s\le 1}\Delta V(s)\Delta U(s)$$
$$= \sum_{i=1}^p \int_0^b \pi(\mathrm{d}x,i)\int_{-\infty}^{\infty} (b-x)(y-b+x) \mathbb{1}(y\ge b-x)\Big(\nu_i(\mathrm{d}y) + \sum_{j\ne i} q_{ij}H^{ij}(\mathrm{d}y)\Big),$$

Putting the pieces together, we get the final expression for ℓ^b in the Markov-modulated case,

$$\ell^b = \frac{1}{2b}\Biggl\{2\Biggl(\sum_{i=1}^{p}\int_0^b x\pi(\mathrm{d}x,i)\Biggr)\Biggl(\sum_{i=1}^{p}\alpha_i\Bigl(m_i+\sum_{j\neq i}q_{ij}\mathbb{E}U^{ij}\Bigr)\Biggr) + \sum_{i=1}^{p}\alpha_i\sigma_i^2 + \sum_{i=1}^{p}\int_0^b \pi(\mathrm{d}x,i)\int_{-\infty}^{\infty}\varphi(x,y)\Bigl(\nu_i(\mathrm{d}y)+\sum_{j\neq i}q_{ij}H^{ij}(\mathrm{d}y)\Bigr)\Biggr\}. \tag{201}$$

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Persistence Probabilities and Exponents

Frank Aurzada and Thomas Simon

Abstract This article deals with the asymptotic behavior as $t \to +\infty$ of the survival function $\mathbb{P}[T > t]$, where T is the first passage time above a non negative level of a random process starting from zero. In many cases of physical significance, the behavior is of the type $\mathbb{P}[T > t] = t^{-\theta+o(1)}$ for a known or unknown positive parameter θ which is called the persistence exponent. The problem is well understood for random walks or Lévy processes but becomes more difficult for integrals of such processes, which are more related to physics. We survey recent results and open problems in this field.

Keywords First passage time • Gaussian process • Integrated process • Lévy process • Lower tail probability • Persistence • Random walk • Stable process

AMS Subject Classification 2000: 60F99, 60G10, 60G15, 60G18, 60G50, 60G52, 60K35, 60K40

1 Introduction

Let $\{X_t,\ t \geq 0\}$ be a real stochastic process in discrete or continuous time, starting from zero. The analysis of the first passage time $T_x = \inf\{t > 0,\ X_t > x\}$ above a non-negative level x is a classical issue in probability. In this paper we will

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be concerned with the asymptotic behavior as $t \to +\infty$ of the survival function $\mathbb{P}[T_x > t]$ for a class of processes related to random walks and Lévy processes. This problem has attracted some interest in the recent literature under the denomination persistence probability. In a self-similar framework, that is if $\{X_{ct}, t \geq 0\}$ and $\{c^H X_t, t \geq 0\}$ have the same distribution for some $H > 0$, the question is equivalent to the lower tail probability problem, which is the study of the quantity

$$\mathbb{P}[\sup_{0 \leq s \leq 1} X_s < x], \qquad \text{as } x = t^{-H} \to 0.$$

In many situations of interest it turns out that the behavior is polynomial: one has

$$\mathbb{P}[T_x > t] \;=\; t^{-\theta + o(1)} \tag{1}$$

for a non-negative parameter θ called the persistence exponent, which usually does not depend on $x \geq 0$ and often belongs to [0, 1]. The study of asymptotic behaviors like (1) has gained some attraction over the last years in the physics literature as well, where the parameter θ is often called survival exponent. Estimates of the type (1) also appear in reliability theory, a subject that we shall not discuss here—see [43] and the references therein for an account. In the following we will mostly be concerned with positive persistence exponents, that is the process ends up in crossing the level x. This contrasts sharply with classical risk theory, where the survival analysis deals with the probability for a given process *never* to cross a fixed level—see [1].

If X is a random walk or a Lévy process, studying the law of T_x is a special part of fluctuation theory and the persistence probabilities are then well understood. For example if X_1 is attracted to a stable law, classical fluctuation identities entail that the persistence exponent is the positivity parameter of the latter. There are many accounts on fluctuation theory and we refer in particular to [18] for random walks and to [38] for Lévy processes. In the first part of this article we focus on the results of this theory dealing with the asymptotic behavior of $\mathbb{P}[T_x > t]$, and we try to be as exhaustive as possible. In general, the behavior is the same in discrete and continuous time, except of course for $x = 0$ where the problem becomes different for Lévy processes. We notice that even though a posteriori the resulting exponents turn out to be the same for a Lévy process and the respective random walk, there is no simple approximation argument that would yield this a priori, and for this reason we have to consider the discrete and continuous time situations separately—a feature that we encounter also for more complicated processes. The results for random walks and Lévy processes, all classical, are presented here in order to give some insight into more complex situations where X, though constructed upon a Lévy process or a random walk, is not a Markov process anymore.

Some of these more complex situations, which are called non-trivial in the physical literature, are the matter of the second part of this article. We first consider integrated random walks and integrated Lévy processes. It turns out that for such simple constructions, the computation of the persistence exponent is not quite easy in general. We display recent and less recent results where the persistence probabilities are estimated with various degrees of precision. When this paper was accepted for publication, we believed in a reasonable universal rule that the persistence exponent of the underlying process should be twice the persistence exponent of its integral, and stated several conjectures in this direction. However, during the publication process, it turned out that this intuition was false, at least for integrated stable Lévy processes where the persistence exponent also depends on the self-similarity parameter, in a non-obvious way. An interesting problem is now to understand whether this exponent is the same for integrated random walks in the corresponding stable domain of attraction. This is true in the situation with no negative jumps, but the other cases are still open. In particular, the situation with no positive jumps, where classical fluctuation theory usually becomes much simpler, is still open for integrated random walks. As in the non-integrated case, it does not seem to us that persistence exponents in discrete and continuous time could be simply deduced from one another in the same stable framework.

We then consider fractionally integrated processes, where the situation is yet more difficult. The case when the underlying Lévy process is Brownian motion yields self-similar Gaussian processes which can be changed into the respective stationary ones by the Lamperti transformation. The sought after persistence exponent is then directly related to an estimate on the probability of non-zero crossings for these Lamperti transforms, a problem known to be hard in spite of the Itô-Rice formula which gives some information at the expectation level in the smooth case. We present some universality and monotonicity results, and also some partially heuristic comparisons with the fractional Brownian motion case, whose persistence exponent can be computed explicitly. We also display other explicit computations of persistence exponents for related processes such as weighted random walks, iterated processes, and autoregressive processes.

The third and last part of this article deals with some applications of the persistence probabilities in mathematical physics. We first deal with Lagrangian regular points of the inviscid Burgers equation with random self-similar initial data. The link between the Hausdorff dimension of such points and the persistence exponent of integrated processes dates back to the original paper [83]. We recall here that the problem is still open in the fractional Brownian motion case and state a plausible conjecture when the initial data is a two-sided stable Lévy process. Second, we consider the zero-crossings of a peculiar Gaussian stationary process which is related to the positivity of Kac polynomials with large even degree. This connection, which was discovered in [30], has further ramifications with the persistence exponents of integrals of Brownian motion with higher order and of a certain diffusion equation with white noise initial conditions in the plane, and we make a brief account on the subject. We last consider three different interacting statistical systems whose analysis hinges significantly upon the persistence probabilities of integrated processes: wetting models with Laplacian interaction, fluctuating interfaces with Langevin dynamics, and sticky particles on the line with Poissonian initial conditions.

Some open problems stated in the present paper are believed to be challenging and we think that they could catch the attention of some colleagues. We finally point out that we have not exhausted here all implications of persistence in physics and that the persistence exponent of many other models remains unknown—see [21, 27, 63, 66] and the references therein.

2 Classical Results

2.1 Random Walks

Let $\{X_n,\ n \ge 1\}$ be a sequence of i.i.d. real random variables with common distribution μ, and $S_n = X_1 + \ldots + X_n$ be the associated random walk. Consider

$$T_x \;=\; \inf\{n \ge 1,\ S_n > x\}$$

the first-passage time above $x \ge 0$. Recall the following basic rule, to be found in e.g. Chap. XII.1–2 of [42]:

$$T_x \text{ is a.s. finite for every } x \ge 0 \;\Leftrightarrow\; T_0 \text{ is a.s. finite,} \tag{2}$$

and that the latter is also equivalent to the fact that S_n does not drift to $-\infty$. In this case one has $\mathbb{P}[T_x > n] \to 0$ as $n \to +\infty$ for every $x \ge 0$, and the difficulty to estimate the speed of convergence comes from the fact that the event $\{T_x > n\} = \{S_1 \le x, \ldots, S_n \le x\}$ depends on n correlated random variables.

If μ is concentrated on $\mathbb{R}^+$, then $\mathbb{P}[T_x > n] = \mathbb{P}[S_n < x]$ and the problem becomes one-dimensional. The straightforward inequality

$$\mathbb{P}[S_n < x] \;\le\; e^{x+n\log(\mathbb{E}[e^{-X_1}])}$$

shows that $\mathbb{P}[T_x > n]$ tends to zero at least exponentially fast (unless μ is degenerate at zero). Notice that the rate might also be superexponential and depend on x at the logarithmic scale: if X_1 has a positive strictly α-stable law ($0 < \alpha < 1$) for instance, de Bruijn's Tauberian theorem—see Theorem 4.12.9 in [19]—leads to

$$\begin{aligned} -\log\mathbb{P}[T_x > n] &= -\log\mathbb{P}[S_n < x] \\ &= -\log\mathbb{P}[S_1 < xn^{-1/\alpha}] \sim \kappa_\alpha x^{-\alpha/(1-\alpha)} n^{1/(1-\alpha)} \end{aligned} \tag{3}$$

for every $x > 0$, with some explicit $\kappa_\alpha > 0$—see also [89]. Here and below $a_n \sim b_n$ is defined by $\lim_{n\to\infty} a_n/b_n = 1$ and analogously for other limits. Of course, one has $T_0 = 1$ a.s. whenever μ does not charge 0. If $\mu\{0\} > 0$, Jain and Pruitt's general uniform results on renewal sequences express the asymptotic behavior of $\mathbb{P}[S_n < x]$ in terms of quantities related to μ, at the logarithmic scale—see Theorem 2.1 in

[52]—and also at the exact scale, under some conservativeness assumption—see Theorem 4.1 in [52].

If μ is not concentrated on $\mathbb{R}^+$, it is easy to see that the asymptotics of $\mathbb{P}[T_x > n]$ will not drastically depend on x: choosing $\varepsilon > 0$ such that $\mu_\varepsilon = \mu(-\infty, -\varepsilon) > 0$, the Markov property entails $\mathbb{P}[T_x > n] \geq \mathbb{P}[T_0 > n] \geq \mathbb{P}[T_\varepsilon > n-1]\mathbb{P}[X_1 < -\varepsilon] \geq \mu_\varepsilon \mathbb{P}[T_\varepsilon > n] \geq \mu_\varepsilon^k \mathbb{P}[T_x > n]$ as soon as $k\varepsilon \geq x$, so that

$$\mathbb{P}[T_x > n] \asymp \mathbb{P}[T_0 > n]$$

for every $x \geq 0$. Here and below $a_n \asymp b_n$ is defined by $0 < \liminf_{n\to\infty} a_n/b_n$ and $\limsup_{n\to\infty} a_n/b_n < \infty$ and analogously for other limits.

The following formula computes the generating function of $\{\mathbb{P}[T_0 > n],\ n \geq 0\}$ in terms of that of the sequence $\{n^{-1}\mathbb{P}[S_n \leq 0],\ n \geq 1\}$, and is true for any random walk. It is a cornerstone for further investigations.

Sparre Andersen's Formula For every $z \in]-1, 1[$ one has

$$\sum_{n\geq 0} z^n \mathbb{P}[T_0 > n] \;=\; \exp\left[\sum_{n\geq 1} \frac{z^n}{n}\mathbb{P}[S_n \leq 0]\right]. \tag{4}$$

This result is obtained after simple rearrangements from Theorem XII.7.1 in [42], whose proof has a combinatorial character and depends heavily on the independence and stationarity of the increments of the random walk. A simpler method relying on elementary Fourier analysis—see Chap. 3.7 in [39]—yields the more general

Spitzer's Formula For every $z \in]-1, 1[$ and $\lambda \geq 0$, one has

$$\sum_{n\geq 0} z^n \mathbb{E}[e^{-\lambda M_n}] \;=\; \exp\left[\sum_{n\geq 1} \frac{z^n}{n}\mathbb{E}[e^{-\lambda S_n^+}]\right] \tag{5}$$

with the notation $M_n = \max(0, S_1, \ldots, S_n)$ and $S_n^+ = \max(0, S_n)$.

Indeed, one obtains (4) as a consequence of (5) in letting $\lambda \to +\infty$. It is beyond the peculiar scope of the present paper to discuss the full strength and the various generalisations of Spitzer's formula such as Baxter-Spitzer's formula or the Wiener-Hopf factorization, and we refer e.g. to [18, 38, 42] for more on this subject. Let us simply remark that Sparre Andersen's formula entails easily—see e.g. Theorem XII.7.2 in [42]—the following characterization of (2):

$$S_n \text{ drifts to } -\infty \;\Leftrightarrow\; \sum_{n\geq 1} \frac{1}{n}\mathbb{P}[S_n > 0] \;<\; +\infty.$$

From now on, we will suppose that S_n does not drift to $-\infty$ and that μ is not concentrated on $\mathbb{R}^+$. This entails that $\mathbb{P}[S_n > 0] > 0$ and $\mathbb{P}[S_n < 0] > 0$ for every $n \geq 1$.

A remarkable consequence of (4) is that when $\mathbb{P}[S_n \leq 0] = \rho \in (0,1)$ for every $n \geq 1$, one obtains an explicit formula for $\mathbb{P}[T_0 > n]$ depending only on ρ and n : one has

$$\sum_{n\geq 0} z^n \mathbb{P}[T_0 > n] \;=\; \frac{1}{(1-z)^\rho} \;=\; \sum_{n\geq 0} \frac{\Gamma(n+\rho)}{n!\Gamma(\rho)} z^n,$$

so that

$$\mathbb{P}[T_0 > n] \;=\; \frac{\Gamma(n+\rho)}{n!\Gamma(\rho)} \;\sim\; \frac{n^{\rho-1}}{\Gamma(\rho)}. \tag{6}$$

For example, symmetric random walks such that $\mathbb{P}[S_n = 0] = 0$ for every $n \geq 1$ (this latter property is true when μ is non atomic, for instance) all enjoy the property that

$$\mathbb{P}[T_0 > n] \;=\; \frac{\Gamma(n+1/2)}{\sqrt{\pi}n!} \;\sim\; \frac{1}{\sqrt{\pi n}}.$$

Recall in passing that the estimate is slightly different for the simple random walk where $\mu\{1\} = \mu\{-1\} = 1/2$, since then $\mathbb{P}[S_{2n} = 0] \neq 0$. The classical computation, to be found e.g. at the beginning of the monograph [75]

$$\begin{aligned}\sum_{n\geq 0} z^n \mathbb{P}[T_0 > n] &= \exp\left[\sum_{n\geq 1} \frac{z^n}{n}\frac{1}{2} - \sum_{n\geq 1} \frac{z^{2n}}{2n2^{2n}}\binom{2n}{n}\right] \\ &= \left(\frac{1}{1-z} - \frac{1-\sqrt{1-z^2}}{z(1-z)}\right) \;\sim\; \sqrt{\frac{2}{1-z}} \;\text{ as } z \uparrow 1,\end{aligned}$$

and the Tauberian theorem for monotonic sequences entail $\mathbb{P}[T_0 > n] \;\sim\; \sqrt{2/(\pi n)}$.

Another remarkable consequence of (5) is the exact computation of the persistence exponent whenever $\{S_n, n \geq 1\}$ fulfils the so-called Spitzer's condition

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} \mathbb{P}[S_k < 0] \;=\; \rho \in [0,1].$$

The latter turns out to be equivalent—see [15, 37], or Chap. 7 in [38]—to

$$\lim_{n\to\infty} \mathbb{P}[S_n < 0] \;=\; \rho, \tag{7}$$

and a theorem of Rogozin shows that (7) entails

$$\mathbb{P}[T_x > n] \sim \frac{c_x n^{\rho-1} l(n)}{\Gamma(\rho)} = n^{\rho-1+o(1)} \tag{8}$$

for every $x \geq 0$ with $l(n)$ some slowly varying sequence and c_x some explicit positive constant. Besides, as explained in Theorem 8.9.12 of [19], this asymptotic behavior is actually equivalent to (7) for $\rho \in (0,1)$. When $\mathbb{P}[S_n < 0] \to 0$ various behaviors are possible, contrary to the above. For example if $n^\rho \mathbb{P}[S_n < 0]$ is slowly varying for some $\rho \in (0,1)$, Theorem 8.9.14 in [19] and Theorem XVII.5.1 in [42] yield

$$\mathbb{P}[T_0 > n] \sim l(n) n^{-(1+\rho)} \tag{9}$$

with $l(n)$ some slowly varying sequence. Various other behaviors also appear when μ has positive expectation and we refer to [36] for precise results. When $\mathbb{P}[S_n < 0] \to 1$ and S_n does not drift to $-\infty$ (that is, when $\sum_n (1 - \mathbb{P}[S_n \leq 0])/n = \infty$), the behavior of $\mathbb{P}[T_0 > n]$ does not seem to be known, although one might expect that the persistence exponent is always zero.

We conclude this paragraph with the so-called upward skip-free or right-continuous random walks on $\mathbb{Z}$ viz. such that Supp $\mu \subseteq \{1, 0, -1, -2, -3, \ldots\}$, for which our survival analysis does not require the use of (4). Indeed, the distribution of T_k is then given by Kemperman's formula [54] which reads:

$$\mathbb{P}[T_k = n] = \frac{k+1}{n} \mathbb{P}[S_n = k+1] \quad \text{for every } n > k. \tag{10}$$

In this particular case, we are reduced to the asymptotical behavior of the one-dimensional probability $\mathbb{P}[S_n = 1]$. Notice—see [55]—that there is a universal upper bound $\mathbb{P}[S_n = 1] \leq c n^{-1/2}$ leading to $\mathbb{P}[T_k > n] \leq 2c(k+1)n^{-1/2}$, but the exact behavior of $\mathbb{P}[S_n = 1]$ depends on μ. For example, if μ is in the respective domain of normal attraction of some strictly α-stable law—with $\alpha \in (1,2]$ since otherwise S_n would drift to $-\infty$ by the law of large numbers, Gnedenko's local limit theorem—see e.g. Theorem 8.4.1 in [19]—yields $\mathbb{P}[S_n = 1] \sim c_\alpha n^{-1/\alpha} l(n)$ for some explicit c_α, so that by (10)

$$\mathbb{P}[T_k > n] \sim \alpha c_\alpha (k+1) l(n) n^{-1/\alpha}. \tag{11}$$

Recall—see Proposition 8.9.16 in [19] and recall that we deal with right-continuous walks—that μ is in the domain of attraction of some strictly α-stable law with $\alpha \in (1,2]$ if and only if (7) holds with $\rho = 1 - 1/\alpha$, so that (11) is also actually a consequence of (8). On the other hand, the slowly varying term $l(n)$ can be removed if S_n is in the respective domain of normal attraction (viz. when $S_n/n^{1/\alpha}$ converges in law to some non degenerate limit—see e.g. the concluding remark of Chap. XVII.5 in [42]), and this degree of precision is not given by Rogozin's theorem.

2.2 Lévy Processes

Let $\{Z_t,\ t \geq 0\}$ be a non-degenerate real Lévy process starting from 0 and

$$T_x \;=\; \inf\{t > 0,\ Z_t > x\}$$

be its first passage time above $x \geq 0$. Consider $\{Z_n,\ n \geq 1\}$ the associated random walk. The inequality

$$\mathbb{P}[T_x > t] \;\leq\; \mathbb{P}[\tilde{T}_x > [t]] \tag{12}$$

with the notation $\tilde{T}_x \;=\; \inf\{n \geq 1,\ Z_n > x\}$, yields a rough upper bound for $\mathbb{P}[T_x > t]$ which can be made more precise as a function of t and x in applying the results of the previous paragraph. This upper bound however does not yield enough information in general if $x = 0$. For example Rogozin's criterion—see Proposition VI.11 in [13] and the remark thereafter—shows that $T_0 = 0$ a.s. when Z has unbounded variation, whereas the function $t \mapsto \mathbb{P}[Z_1 \leq 0, \ldots, Z_{[t]} \leq 0]$ has a positive limit at $+\infty$ if Z drifts to $-\infty$. Recall also that in the bounded variation case, the regularity of the half-line for Z is characterized in terms of the Lévy measure and the drift—see Theorem 22 in [38], and again $\mathbb{P}[Z_1 \leq 0, \ldots, Z_{[t]} \leq 0]$ might have a positive limit when $t \to +\infty$ even though $T_0 = 0$ a.s.

If $x > 0$ however, it turns out that the two quantities in (12) are often comparable. First of all, for every $x > 0$ one has

$$\mathbb{P}[T_x = +\infty] \;>\; 0 \;\Leftrightarrow\; Z_t \to -\infty \text{ a.s. } \;\Leftrightarrow\; \int_1^\infty \frac{1}{t}\,\mathbb{P}[Z_t > 0]\,\mathrm{d}t \;<\; +\infty \tag{13}$$

(see e.g. Corollary 4.4(iii) in [38]), and a simple analysis shows that $Z_t \to -\infty$ a.s. is equivalent to $Z_n \to -\infty$ a.s. which is itself equivalent to

$$\sum_{n\geq 1} \frac{1}{n}\,\mathbb{P}[Z_n > 0] \;<\; +\infty,$$

so that with the above notation $\mathbb{P}[T_x > t] \to 0$ if and only if $\mathbb{P}[\tilde{T}_x > t] \to 0$ as $t \to +\infty$. In the following, we will see that in many examples one has

$$\mathbb{P}[T_x > t] \;\asymp\; \mathbb{P}[\tilde{T}_x > [t]].$$

However, it does not seem easy to prove this estimate a priori, which probably does not hold in full generality.

We will suppose henceforth that (13) does not hold. Set μ for the law of Z_1, $M_t = \sup\{Z_s, s \leq t\}$ for the running maximum of Z, and recall that

$$\mathbb{P}[T_x > t] \;=\; \mathbb{P}[M_t \leq x].$$

If μ is concentrated on $\mathbb{R}^+$ then Z is a.s. increasing and one is reduced to the random walk case because

$$\mathbb{P}[Z_{[t]+1} \leq x] \;\leq\; \mathbb{P}[T_x > t] \;=\; \mathbb{P}[Z_t \leq x] \;=\; \mathbb{P}[M_t \leq x] \;\leq\; \mathbb{P}[Z_{[t]} \leq x].$$

If μ is not concentrated on $\mathbb{R}^+$ then an argument analogous to that of the previous paragraph shows that

$$\mathbb{P}[T_x > t] \;\asymp\; \mathbb{P}[T_y > t]$$

as $t \to +\infty$ for every $x, y > 0$, as in the discrete time framework. The classical approach to obtain more information on $\mathbb{P}[T_x > t]$ relies on a particular case of the first fluctuation identity—see e.g. Theorem VI.5 in [13], which is an analogue of Spitzer's formula in continuous time:

Baxter-Donsker's Formula For every $\lambda \geq 0$ and $q > 0$, one has

$$\mathbb{E}[e^{-\lambda M_{e_q}}] \;=\; \exp - \left[\int_0^\infty \frac{e^{-qt}}{t} \, \mathbb{E}[(1 - e^{-\lambda Z_t^+})] \, \mathrm{d}t \right] \tag{14}$$

with the notation $Z_t^+ = \max(0, Z_t)$ and $e_q \sim \text{Exp}(q)$ an independent random time.

An important consequence of this formula is a theorem of Rogozin—see e.g. Theorem VI.18 in [13]—which shows that if the following Spitzer's condition

$$\frac{1}{t} \int_1^t \mathbb{P}[Z_s \geq 0] ds \;\to\; \rho \in (0,1), \quad t \to +\infty \tag{15}$$

holds, then for any $x > 0$

$$\mathbb{P}[T_x > t] \;\sim\; c_x l(t) t^{-\rho} \tag{16}$$

with $c_x > 0$ and $l(t)$ some slowly varying function not depending on x. Besides, one can show that (15) and (16) are actually equivalent—see again Theorem VI.18 in [13]. Recall—see Chap. 7 in [38]—that (15) is also equivalent to (7) for the random walk $\{-Z_n, n \geq 1\}$. The estimate (16) can be refined for strictly α-stable processes, which all enjoy the property that

$$\mathbb{P}[Z_t \geq 0] \;=\; \rho \in (0,1) \quad \text{for every } t > 0. \tag{17}$$

An asymptotic analysis of the so-called Darling integral—see Theorem 3b in [16]—entails then

$$\mathbb{P}[T_x > t] \;\sim\; c x^{\alpha\rho} t^{-\rho} \tag{18}$$

for such processes, where c is an explicit constant. Notice that there are other Lévy processes enjoying the property (17), like subordinate stable processes, for which no refinement of (16) seems available in the literature. In [12], the precise estimate

$$\mathbb{P}[T_x > t] \sim c_x t^{-1/2}$$

was obtained for every centered Lévy processes with finite variance and every $x > 0$. We finally remark that Sect. 3 in [57] provides uniform estimates for $\mathbb{P}[T_x > t]$ in terms of the renewal function of Z, the latter being non-explicit in general.

We conclude this paragraph with spectrally negative Lévy processes, where our survival analysis amounts to the study of a one-dimensional probability as in the discrete framework, and one does not really require (14). Indeed, the passage-time process $\{T_x, x \geq 0\}$ is then a subordinator with infinite lifetime if Z does not drift to $-\infty$, whose Laplace exponent $\Phi(\lambda) = -\log \mathbb{E}[e^{-\lambda T_1}]$ is characterized by the law of Z_1—see Chap. VII in [13] or Chap. 9 in [38] for these basic facts. In particular, if $\Phi(\lambda) \sim \lambda^\rho l(\lambda)$ as $\lambda \to 0$ for some $\rho \in (0,1)$ and $l(\lambda)$ some slowly varying function, then Theorem XIII.5.4. in [42] yields

$$\mathbb{P}[T_x > t] \sim \frac{x}{\Gamma(\rho)} l(t^{-1}) t^{-\rho}.$$

Actually, the above condition on Φ is equivalent to (15)—see Proposition VII.6 in [13] and notice that then necessarily $\rho \geq 1/2$. Hence, the above estimate is just a consequence of Rogozin's theorem with an explicit constant c_x. Spectrally negative Lévy processes also enjoy the following peculiar property, which follows easily from the strong Markov property and the absence of positive jumps:

$$M_{e_q} \stackrel{d}{=} Z_{e_q} \mid Z_{e_q} > 0. \tag{19}$$

In particular, one has $M_t \stackrel{d}{=} Z_t \,|\, Z_t > 0$ for every $t > 0$ if $\mathbb{P}[X_t > 0] = \rho \geq 1/2$ does not depend on t (which is true only for strictly $(1/\rho)$-stable process). This latter identity which can be shown in many different ways—see Sect. 8 in [17] and the references therein—recovers the estimate (18) in this particular case: one has

$$\mathbb{P}[T_x > t] \sim \frac{cx}{\Gamma(\rho)} t^{-\rho}$$

for some explicit constant $c > 0$ which is $\sqrt{2}$ for the standard Brownian motion. We finally stress that (19) can be useful for spectrally negative processes such that $\mathbb{P}[X_t > 0] \to 1$. For example, if X is an α-stable process with positive drift then (19) shows after a simple analysis that

$$\mathbb{P}[T_x > t] \sim cxt^{-\alpha}$$

for some explicit $c > 0$, an estimate which cannot be obtained directly neither from Rogozin's theorem nor from the behavior of Φ at zero, and which is also coherent with (9) since $\mathbb{P}[X_t < 0] \sim ct^{1-\alpha}$.

3 Recent Advances

3.1 Integrated Random Walks

An integrated random walk is the sequence of partial sums $A_n = S_1 + \cdots + S_n$, where $\{S_n, \ n \geq 1\}$ is a random walk. As above we write $S_n = X_1 + \cdots + X_n$ and we denote by μ the law of the increment $X_1 = S_1$. Let

$$T_x \;=\; \inf\{n \geq 1, \; A_n > x\}$$

be the first-passage time above $x \geq 0$. Since we have

$$\{S_1 \leq 0, \ldots, S_n \leq 0\} \subseteq \{A_1 \leq 0, \ldots, A_n \leq 0\},$$

the discussion made in Sect. 2.1 shows that $\mathbb{P}[T_x = +\infty] \geq \mathbb{P}[T_0 = +\infty] > 0$ as soon as S_n drifts to $-\infty$. If μ is concentrated on $\mathbb{R}^+$, then A_n has non-negative increments and since $A_n = nX_1 + (n-1)X_2 \cdots + X_n \geq (n/2)S_{[n/2]}$, one has

$$\mathbb{P}[T_x > n] \;=\; \mathbb{P}[A_n < x] \;\leq\; \mathbb{P}[nS_{[n/2]} < 2x] \;\leq\; e^{2x + n\log(\mathbb{E}[e^{-[n/2]X_1}])},$$

which shows that $\mathbb{P}[T_x > n]$ tends to zero superexponentially fast, unless μ is degenerate. If μ is not concentrated on $\mathbb{R}^+$, then choosing ε such that $\mu(-\infty, -\varepsilon) > 0$ and using the same argument as in Sect. 2.1 entail $\mathbb{P}[T_x > n] \geq \mathbb{P}[T_0 > n] \geq (\mu(-\infty, -\varepsilon))^k \mathbb{P}[T_x > n]$ for every $x \geq 0$ as soon as $k\varepsilon \geq x$, so that

$$\mathbb{P}[T_x > n] \;\asymp\; \mathbb{P}[T_0 > n]$$

for every $x \geq 0$.

Henceforth we will suppose that S_n does not drift to $-\infty$ and that μ is not concentrated on $\mathbb{R}^+$. This entails that $\mathbb{P}[S_n > 0] > 0$ and $\mathbb{P}[S_n < 0] > 0$ for every $n \geq 1$. We are interested in the rate of decay of $\mathbb{P}[T_0 > n]$ to zero, and we will see that far much less is known than for random walks.

The case of integrated simple random walks was first considered by Sinai [82], who showed the following

Theorem 3.1 (Sinai) *Suppose that* $\mu\{+1\} = \mu\{-1\} = 1/2$. *Then*

$$\mathbb{P}[T_0 > n] \;\asymp\; n^{-1/4}. \tag{20}$$

The main idea lying behind this theorem is very simple. Let $\{\tau_k, k \geq 0\}$ be the a.s. infinite sequence of return times of S_n to zero viz. $\tau_0 = 0$ and $\tau_k = \inf\{n > \tau_{k-1},\ S_n = 0\}$. On the one hand, the simplicity assumption entails that $\{A_{\tau_n}, n \geq 1\}$ is an integer-valued symmetric random walk, and that the a.s. identification

$$\{A_{\tau_1} \leq 0, \ldots, A_{\tau_n} \leq 0\} \;=\; \{A_i \leq 0,\ \forall\, i = 1 \ldots \tau_n\}$$

holds. By Gnedenko's local limit theorem, one has $\mathbb{P}[A_{\tau_n} = 0] \sim k n^{-1/2}$ for some $k > 0$, so that the series $\sum_{n \geq 1} n^{-1} \mathbb{P}[A_{\tau_n} = 0]$ is convergent. Sparre-Andersen's formula combined with the Tauberian theorem for monotonic sequences shows then that

$$\mathbb{P}[T_0 > \tau_n] \;\sim\; \frac{c}{\sqrt{n}}\cdot$$

On the other hand, the sequence τ_n grows like n^2 at infinity (more precisely, $n^{-2}\tau_n$ converges in law to some positive $(1/2)$-stable law), so that after some residual analysis on the bivariate random walk $\{A_{\tau_n}, \tau_n\}$, one obtains the desired result. Let us stress that the difficulty of the analysis stems from the fact that (τ_n) and (A_{τ_n}) are not independent.

Since then several authors [5, 24, 31, 34, 85, 87] have tried to extend the validity of the estimate (20) to more general random walks. The following general result was proved very recently in [31], solving conjectures made in [24, 84]:

Theorem 3.2 (Dembo-Ding-Gao) *Suppose that μ has finite second moment and zero mean. Then for any $x \in \mathbb{R}$*

$$\mathbb{P}[T_x > n] \;\asymp\; n^{-1/4}.$$

The method used by [31] is completely different from Sinai's and relies on a decomposition of the integrated walk at its supremum, which is somehow reminiscent of Sparre Andersen's argument, and too involved to be discussed here in detail. This method also allows for an elementary proof of the formula (6) in the symmetric and absolutely continuous case—see Proposition 1.4 therein.

The result in Theorem 3.2 was further refined in [34].

Theorem 3.3 (Denisov-Wachtel) *Suppose that μ has finite $(2+\delta)$ moment for some $\delta > 0$ and zero mean. Then there is a constant $c = c(\mu) > 0$ such that for $x \in \mathbb{R}$*

$$\mathbb{P}[T_x > n] \;\sim\; c n^{-1/4}.$$

Denisov and Wachtel use a new approach, based on their earlier results for other problems in [32]. They show that a typical scenario for $T_0 > n$ is that A becomes quite far negative over a short time period of order $n^{1-\varepsilon}$. From there, the pair (S_n, A_n) essentially looks like a rescaled pair of Brownian motion and integrated Brownian motion $(B_t, \int_0^t B_s ds)$, which is formally obtained by coupling techniques similar to the idea applied in [5]. For the remaining time, $n - n^{1-\varepsilon} \sim n$, one requires the second component of the rescaled pair $(B_t, \int_0^t B_s ds)$ to be negative, which can be precisely analyzed using [47]. The paper [34] is also the first to apply potential-theoretic tools for persistence probabilities outside the classical random walk and Lévy process setting.

Sinai's method of decomposing the integrated path along the excursions away from zero of the underlying random walk can be extended under some extra assumptions on the law of $-X_1$ conditioned to be positive, which we denote by μ^-. This idea had been used in [85] to show the same result as [31] in several situations like double-sided exponential, double-sided geometric, left-continuous, or lazy simple random walks. Recently in [87], this technique is combined with local limit theorems to obtain the following more precise result.

Theorem 3.4 (Vysotsky) *Suppose that either μ_- is exponential or that μ is left-continuous. If μ has finite variance and zero mean then there exists $c = c(\mu) > 0$ such that*

$$\mathbb{P}[T_0 > n] \ \sim \ c\, n^{-1/4}. \tag{21}$$

Notice that there is still a small gap in that the existence of the constant in (21) is only shown under the conditions in Theorems 3.3 and 3.4, while in view of these results one would certainly expect it to hold under the sole assumption of finite variance. Even more, as can be seen from the central limit theorem, random walks with zero mean and finite variance are such that $\mathbb{P}[S_n > 0] \to 1/2$. In view of the discussion made in Sect. 2.1, it is hence natural to raise the more general

Conjecture 1 Suppose that $\mathbb{P}[S_n > 0] \to 1/2$. Then

$$\mathbb{P}[T_0 > n] \ = \ n^{-1/4+o(1)}.$$

In this formulation, no assumption is made on the moments of μ and this enhances sharply the difficulty of the problem, which probably requires more combinatorial tools than the one used in the above references.

We now turn to some situations where the persistence exponent of integrated random walks is not 1/4. We denote by $\mathscr{D}(\alpha)$ the set of probability measures attracted to some strictly α-stable law with $\alpha \neq 2$ and we refer e.g. to Chap. XVII.5 in [42] for more on the subject. Recall that if $\mu \in \mathscr{D}(\alpha)$, then $\mathbb{E}[|X_1|^s] < \infty$ for every $s \in [0, \alpha)$ and that $\mathbb{E}[X_1] = 0$ if $\alpha > 1$. The following is a consequence of the main result in [31]:

Theorem 3.5 (Dembo-Ding-Gao) *Suppose that* $\mu \in \mathscr{D}(\alpha)$ *for some* $\alpha \in (1,2)$. *Then there exists an explicit constant* K *such that*

$$\mathbb{P}[T_0 > n] \;\le\; Kn^{-(1-1/\alpha)/2}, \quad n \ge 1.$$

If in addition μ *is attracted to a spectrally positive* α*-stable law, then*

$$\mathbb{P}[T_0 > n] \;\asymp\; n^{-(1-1/\alpha)/2}.$$

The result in [31] is formulated in a different manner and is actually more general. In the case when $\mu \in \mathscr{D}(\alpha)$ however, the additional assumption made therein for the lower bound is equivalent to the spectral positivity of the attracting law—see e.g. Theorem XVII.5.1 in [42]. With the help of an extension of Sinai's method, the estimate was recently refined in [87]:

Theorem 3.6 (Vysotsky) *Suppose that* μ_- *is exponential or that* μ *is left-continuous. If* μ *is centered, attracted to some spectrally positive* α*-stable law with* $\alpha \in (1,2)$, *and if*

$$\sum_{n\ge1} \frac{1}{n}\left(\mathbb{P}[S_n < 0] - \frac{1}{\alpha}\right) \;<\; +\infty,$$

then there exists $c = c(\mu) > 0$ *such that*

$$\mathbb{P}[T_0 > n] \;\sim\; c\, n^{-(1-1/\alpha)/2}.$$

In view of our final discussion made in Sect. 2.1, it is very surprising that the case where μ is right-continuous and normally attracted to some spectrally negative α-stable law seems to be more difficult to handle than the dual situation where μ is left-continuous and normally attracted to some spectrally positive α-stable law.

It is natural to ask what the persistence exponent should be when the limit stable law does have negative jumps. Let us hence denote by $\mathscr{D}(\alpha, \rho)$ the set of probability measures attracted to a strictly α-stable law with positivity parameter ρ. Notice—see e.g. [89] for details and recall that we excluded the one-sided case—that $\rho \in (0,1)$ for $\alpha \in (0,1]$, that $\rho \in [1 - 1/\alpha, 1/\alpha]$ for $\alpha \in (1,2)$, and that spectrally positive α-stable laws with $\alpha \in (1,2)$ are such that $\rho = 1 - 1/\alpha$. Besides, one has $\mathbb{P}[S_n > 0] \to \rho$ whenever $\mu \in \mathscr{D}(\alpha, \rho)$—see e.g. Theorem XVII.5.1 in [42]. In view of the discussion made in Sect. 2.1 and Theorem 3.9 stated below, it is natural to raise the general.

Conjecture 2 Suppose that $\mu \in \mathscr{D}(\alpha, \rho)$. Then

$$\mathbb{P}[T_0 > n] \; = \; n^{-\rho/(1+\alpha(1-\rho))+o(1)}.$$

Under the stronger assumption that μ is normally attracted, one may also wonder if a more precise behavior could not be obtained, as in [87].

3.2 Integrated Lévy Processes

In this section we consider the process

$$A_t \; = \; \int_0^t Z_s \, \mathrm{d}s, \quad t \geq 0,$$

where $\{Z_t, \; t \geq 0\}$ is a real Lévy process starting from zero. We set

$$T_x \; = \; \inf\{t > 0, \; A_t > x\} \; = \; \inf\{t > 0, \; A_t = x\}$$

for its first passage time above $x \geq 0$. Contrary to (12), there is no straightforward bound between $\mathbb{P}[T_x \geq t]$ and an analogous quantity involving some iterated random walk, so that the results of the previous paragraph cannot be used, except in the Brownian case, see [31, Remark 1.5].

The process (Z, A) is Fellerian and we set $\mathbb{P}_{(z,a)}$ for its law starting from (z, a), with the simplified notation $\mathbb{P} = \mathbb{P}_{(0,0)}$. It is clear by the right-continuity of Z that

$$\mathbb{P}_{(z,0)}[T_0 = 0] = 1 \text{ or } 0 \quad \text{according as} \quad z > 0 \text{ or } z < 0.$$

Since T_0 is the first passage time into the positive half-space for the Fellerian process (Z, A), one has also $\mathbb{P}[T_0 = 0] = 0$ or 1 by the 0–1 law, but to obtain a criterion for the regularity of the upper half-plane for (Z, A) is an open problem which does not seem obvious. Since integrated Lévy process all have finite variation, one might wonder whether this criterion would not be different from the aforementioned Theorem 22 in [38].

If Z drifts to $+\infty$, then it is clear that $T_x < +\infty$ a.s. for every $x > 0$. On the other hand, if Z drifts to $-\infty$ then its last passage time above zero can be made arbitrarily small so that one will have $\mathbb{P}[T_x = +\infty] > 0$ for every $x > 0$. When Z oscillates then probably one has $\mathbb{P}[T_x = +\infty] = 0$ for every $x > 0$, but there is no direct answer to this question. In general there is no result of basic fluctuation theory available for integrated Lévy processes.

In this paragraph we will consider two examples where the persistence exponent can be computed. The first one is the integrated Brownian motion and originates from Kolmogoroff [56], in relation with the two-dimensional generator

$$\frac{1}{2}\,\frac{\partial^2}{\partial x^2} \;+\; x\,\frac{\partial}{\partial y}$$

and the associated Fokker-Planck equation. Notice that [56] actually deals with the more general n-times integrated Brownian motion. The process (B, A) is a Gaussian Markov process whose transition density can be computed explicitly via the covariance matrix. By the (3/2)-self-similarity of A, one has

$$T_x \;\stackrel{d}{=}\; x^{2/3} T_1$$

under $\mathbb{P}$, so that our persistence problem amounts to finding the asymptotic of $\mathbb{P}[T_1 > t]$, a question which dates back to Uhlenbeck and Wang in 1945. Notice that the above identity also yields $\mathbb{P}[T_0 = 0] = 1$. Among other formulæ, the following was obtained by McKean in an analytical way—see (3.1) in [65]:

$$\begin{aligned}&\mathbb{P}_{(0,-1)}\left[T_0 \in \,\mathrm{d}t, B_{T_0} \in \,\mathrm{d}x\right]\\ &\quad= \frac{3x}{\pi\sqrt{2\pi}\,t^2} e^{-(2/t)(1-x+x^2)} \left(\int_0^{4x/t} e^{-3y/2} y^{-1/2}\,\mathrm{d}y\right) \mathbf{1}_{\{x\ge 0\}}\,\mathrm{d}t\,\mathrm{d}x.\end{aligned}$$

This formula is the key argument to the following result which is proved separately in [45, 51]:

Theorem 3.7 (Goldman, Isozaki-Watanabe) *With the above notations there exists a* $c > 0$ *such that*

$$\mathbb{P}[T_1 > t] \;\sim\; c\,t^{-1/4}. \tag{22}$$

Notice that by self-similarity this result has the more general formulation

$$\mathbb{P}[A_s \le x,\ \forall\, s \in [0,t]] \;=\; \mathbb{P}[T_x > t] \;\sim\; c\,x^{1/6} t^{-1/4} \quad \text{as } t/x^{2/3} \to +\infty.$$

In particular, one has

$$\mathbb{P}[A_s \le x,\ \forall\, s \in [0,1]] \;\sim\; c\,x^{1/6} \quad \text{as } x \to 0,$$

which is a lower tail probability statement as mentioned in the introduction. By approximation and his result on integrated simple random walks, Sinai had obtained in [82] the rougher estimate

$$\mathbb{P}[T_1 > t] \;\asymp\; t^{-1/4}.$$

The argument in [45] is analytical and relies on integral equations. It also yields a complicated explicit expression for the law of T_1:

$$\mathbb{P}[T_1 \in \mathrm{d}t] = \mathbb{P}_{(0,-1)}[T_0 \in \mathrm{d}t]$$
$$= \left(\frac{3\sqrt{3}}{2\sqrt{2\pi}t^{5/2}} e^{-3/(2t^3)} + \frac{2\sqrt{3}}{\pi} \int_0^\infty \int_0^\infty \int_0^t \mathbb{P}_{(-z,0)}[t - T_0 \in ds, B_{T_0} \in \mathrm{d}x] \right.$$
$$\left. \cdot e^{-6/s^3 - 2z^2/s^2} \sinh(6z/s^2) z dz ds \right) \mathrm{d}t,$$

where the quantity $\mathbb{P}_{(-z,0)}[t - T_0 \in ds, B_{T_0} \in \mathrm{d}x]$ can be expressed via McKean's above formula and self-similarity. Notice that both McKean and Goldman's formulæ have been generalized by Lachal [59], who obtained an explicit formula for $\mathbb{P}_{(b,a)}[T_0 \in \mathrm{d}t, B_{T_0} \in \mathrm{d}x]$ for any (b, a). The asymptotic analysis which is carried out in Proposition 2 of [45] gives the right speed of convergence for $\mathbb{P}[T \in \mathrm{d}t]$, but the resulting value of the constant c in (22) is erroneous because of some inaccuracies in the change of variable. The right value is

$$c = \frac{3^{4/3}\Gamma(2/3)}{\pi 2^{13/12}\Gamma(3/4)} \sim 0.718$$

and follows also from the simpler probabilistic method of [51], which relies on the Markov property and a Tauberian argument. This method yields the more general estimate

$$\mathbb{P}_{(b,0)}[T_x > t] \sim c_{b,x} t^{-1/4}$$

with some explicit $c_{b,x} > 0$—see (1.12) in [51]. It also allows to handle first-passage time asymptotics for fluctuating homogeneous additive functionals of Brownian motion, with the persistence exponent depending smoothly on the skewness of the functional—see Corollary 1 in [50]. We finally notice that the estimate (22) was also obtained (with an erroneous constant c) in [23] after solving some Krein-Kramers differential equation in the context of semiflexible polymers in the half-plane. This latter method was generalised in [20] to give another computation of the persistence exponent for fluctuating homogeneous additive functionals of Brownian motion, in the context of survival of a diffusing particle in a transverse shear flow. Let us also mention the paper [47], which obtains precise information on the law of integrated Brownian motion conditioned to stay positive.

We now turn to integrated strictly α-stable Lévy processes, which form the natural generalisation of integrated Brownian motion. The bivariate process (Z, A) is then a stable Markov process, where the stability property has the general meaning which is given in the monograph [76]. In particular, one can check from the Lévy-Khintchine formula—see e.g. Proposition 3.4.1 in [76]—that

$$A_1 \stackrel{d}{=} (1+\alpha)^{-1/\alpha} Z_1$$

for every $t \geq 0$, so that A_1 is a α-stable variable with the same positivity parameter as Z_1. On the other hand there does not exist an explicit formula for the density of the bivariate random variable (Z_1, A_1) except in the Gaussian case $\alpha = 2$. The univariate process A is $(1 + 1/\alpha)$-self-similar, so that with the above notations one has

$$T_x \stackrel{d}{=} x^{\alpha/(\alpha+1)} T_1$$

under $\mathbb{P}$ and we need only to study the asymptotic behavior of $\mathbb{P}[T_1 > t]$. Notice that again this identity yields $\mathbb{P}[T_0 = 0] = 1$. As a consequence of the main result of [80], the persistence exponent of A can be computed within a specific sub-class:

Theorem 3.8 (Simon) *Let* $\{Z_t,\ t \geq 0\}$ *be a strictly* α*-stable Lévy process with* $\alpha \in (1, 2)$. *With the above notations, there exists a positive constant* K *such that*

$$\mathbb{P}[T_1 > t] \ \leq\ K\, t^{-(1-1/\alpha)/2}, \quad t > 0.$$

If in addition Z *is spectrally positive, then*

$$\mathbb{P}[T_1 > t] \ =\ t^{-(1-1/\alpha)/2+o(1)}.$$

The main result in [80] deals with more general homogeneous functionals of stable Lévy processes, extending the results of Isozaki [49]. It also provides some explicit lower bound with a logarithmic term, which entails the following criterion for the finiteness of fractional moments of T_1 in the spectrally positive case:

$$\mathbb{E}[T_1^s] < \infty \ \Leftrightarrow\ -(\alpha + 1) < s < (\alpha - 1)/2\alpha.$$

The method of Simon [80] is an adaptation of Sinai's in continuous time, relying on the bivariate Lévy process $\{(\tau_t, A_{\tau_t}),\ t \geq 0\}$ with $\{\tau_t,\ t \geq 0\}$ the inverse local time at zero (which exists because $\alpha > 1$). The upper bound relies on the Wiener-Hopf factorization method as in [49], and the fact that A_{τ_1} is a symmetric $(\alpha - 1)/(\alpha + 1)$-stable variable whatever A_1's positivity parameter is. For the lower bound, the crucial fact is that a.s.

$$\{A_{\tau_s} \leq 1,\ \forall s \in [0, t]\} \ =\ \{A_s \leq 1,\ \forall s \in [0, \tau_t]\},$$

which allows to study the probability of the right event with the help of (18). This latter identity is true only in the spectrally positive case.

It is a natural question to find the persistence exponent for all integrated stable Lévy processes. If Z is an α-stable subordinator, then A is an increasing process and one gets the same superexponential behavior as in (3). This superexponential speed is coherent with the previous discussion for integrated one-sided random walks. If $-Z$ is an α-stable subordinator, then $T_x = +\infty$ a.s. for every $x \geq 0$. The general problem was only solved very recently, and after this paper had been accepted. We will only give a brief account.

Theorem 3.9 (Profeta-Simon) *Let Z be a strictly α-stable Lévy process such that* $\mathbb{P}[Z_1 > 0] = \rho \in (0, 1)$. *Then*

$$\mathbb{P}[T_1 > t] \;=\; t^{-\rho/(1+\alpha(1-\rho))+o(1)}.$$

The proof of this theorem relies on the exact computation of the harmonic measure

$$\mathbb{P}_{z,a}[Z_{T_0} \in dx]$$

where (z, a) belongs to one of the two coordinate axes, which leads to the unexpected behaviour

$$\mathbb{P}_{z,a}[|Z_{T_0}| > x] \;\sim\; c\, x^{-\alpha\rho/(1+\alpha(1-\rho))}$$

when (z, a) belongs to the lower open half-plane, with an explicit constant c. This computation is a generalization of previous results by McKean [65] and Gor'kov [46] for integrated Brownian motion, made possible by a combined use of Fresnel integrals, Mellin transform and the Markov property, which is too involved to be reproduced here in detail. The last part of the proof consists in showing that the intuitive identity in law

$$|Z_{T_0}| \;\stackrel{d}{=}\; |Z_1| \times T_0^{1/\alpha}$$

with an independent product on the right-hand side, though obviously false rigorously, becomes true in approximation for large values of the random variables and hence leads to the non-trivial persistence exponent

$$\frac{\rho}{1+\alpha(1-\rho)}.$$

See [73] for all details and also [74] for related results on the windings of the stable Kolmogorov process. In view of Theorem 3.9, Conjecture 2 and (16), we are naturally led to the following general question on integrated Lévy processes.

Conjecture 3 Let Z be a Lévy process such that $Z_1 \in \mathscr{D}(\alpha, \rho)$ with $\rho \in (0, 1)$. Then for every $x > 0$

$$\mathbb{P}[T_x > t] \;=\; t^{-\rho/(1+\alpha(1-\rho))+o(1)}.$$

3.3 Fractionally Integrated Lévy Processes

In this section we consider processes of the type

$$A_t^\beta \;=\; \frac{1}{\Gamma(\beta+1)} \int_0^t (t-s)^\beta \;\mathrm{d}Z_s, \quad t \geq 0, \tag{23}$$

where $\{Z_t,\ t \geq 0\}$ is a real Lévy process starting from zero and $\beta > -1$. The above convolution product makes sense for every $\beta \geq 0$, and can also be defined for some negative β depending on the law of Z. If Z is strictly α-stable for instance, then A^β is well-defined for every $\beta > -(1 \wedge 1/\alpha)$ and is then a stable process in the sense of [76], with continuous paths iff $\alpha = 2$ or $\beta > 0$ and with a.s. locally unbounded paths iff $\alpha < 2$ and $\beta < 0$—see Chap. 10 in [76]. When Z is strictly α-stable, it is customary to write $\beta = H - 1/\alpha$ with $H > 0$ the so-called Hurst parameter, and A^β is then an H-self-similar process. One can view β-fractionally integrated Lévy processes as the natural generalization of n times integrated Lévy processes, since an integration by parts shows that the latter form the subclass $\beta = n$. In this particular case the $(n+1)$-dimensional process $(Z, A^1, \ldots, A^n)$ is a strong Markov process, but there is no multidimensional Markov property when β is not an integer because then the fractional integration takes the whole memory of the driving process into account. In the literature, fractionally integrated Lévy processes are often called Riemann-Liouville processes, a denomination which is originally due to Lévy.

In the Brownian case $Z = B$, the process A^β is closely connected to the fractional Brownian motion $\{B_t^H,\ t \geq 0\}$, which we recall to be the centered Gaussian process with covariance function

$$\mathbb{E}\left[B_t^H B_s^H\right] = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \qquad t, s \geq 0.$$

Fractional Brownian motion can be written as the independent sum

$$B_t^H = c_H \left(A_t^{H-1/2} + \int_0^\infty ((t+s)^{H-1/2} - s^{H-1/2})\, \mathrm{d}\tilde{B}_s \right), \quad t \geq 0, \tag{24}$$

with $c_H = (H 2^{2H} \pi / (\Gamma(H+1/2)\Gamma(1-H)))^{1/2}$ the normalization constant and $\tilde{B}$ a Brownian motion independent of B, which shows that its paths are continuous a.s. Notice that B^H gives insight on the process of the long-range increments of $A^{H-1/2}$ in view of the immediate representation of the latter as an independent sum

$$A_{t+u}^{H-1/2} - A_u^{H-1/2} \stackrel{d}{=} A_t^{H-1/2} + \int_0^u ((t+s)^{H-1/2} - s^{H-1/2})\, \mathrm{d}\tilde{B}_s, \quad t \geq 0,$$

which entails

$$\{A_{t+u}^{H-1/2} - A_u^{H-1/2},\ t \geq 0\} \stackrel{d}{\to} \left\{c_H^{-1} B_t^H,\ t \geq 0\right\}, \quad u \to +\infty \tag{25}$$

(with, of course, an equality in law for every u when $H = 1/2$). Recall that B^H is defined for $H \in (0, 1]$ only and that B^1 is simply the linear function $t \mapsto tN$ with N a standard normal variable. Fractional Brownian motion can be shown [76] to be the unique H-self-similar Gaussian process with stationary increments, whence its greater importance in modeling than fractionally integrated Lévy processes. Setting

$$T_x^H \;=\; \inf\left\{t > 0,\; B_t^H > x\right\} \;=\; \inf\left\{t > 0,\; B_t^H = x\right\}$$

for every $x \geq 0$, one has $T_x^H \stackrel{d}{=} x^{1/H} T_1^H$ by self-similarity, which shows that $T_0^H = 0$ a.s. and that the survival analysis of B^H is reduced to the behavior of $\mathbb{P}[T_1^H > t]$ only. Among other results, the following was obtained in [67]:

Theorem 3.10 (Molchan) *For every* $H \in (0, 1]$ *one has*

$$\mathbb{P}[T_1^H > t] \;=\; t^{H-1+o(1)}.$$

The main argument of Molchan [67], partly inspired by Brownian fluctuation theory, is to quantify the correlation between T_1^H, the last zero of B^H on $[0, 1]$, the positive sojourn time of B^H on $[0, 1]$, and the inverse exponential functional

$$t \;\mapsto\; J_t^H \;=\; \left(\int_0^t e^{B_s^H}\,\mathrm{d}s\right)^{-1},$$

whose asymptotic behavior in expectation can be precisely analysed. The general link between J_t^H and T_1^H is explained by the heuristical fact that if $T_1^H > t$ then B^H has drifted towards $-\infty$ rather rapidly, so that J_t^H is big. Conversely if $T_1^H < t$ then B^H has been close zero for a positive fraction of time, so that J_t^H is small. However, the analysis of $\{\mathbb{E}[J_t^H],\; t \geq 0\}$ which is performed in [67] is very specific to the stationary increments of fractional Brownian motion, and does not seem to be suitable for fractionally integrated Lévy processes. However, it can indeed be used for other self-similar, stationarity increment processes with sufficiently high integrability properties, as for example done in [25], where the following process is considered:

$$\varDelta_t = \int_{\mathbb{R}} L_t(x)dW(x), \qquad t \geq 0,$$

with W being a two-sided Brownian motion and L_t the local time of an α-stable Lévy process ($\alpha \in (1, 2]$) independent of W. This process is $H = (1-1/(2\alpha))$-self-similar, has stationary increments, and can be analyzed with the techniques from [2, 67] to show that the persistence exponent is $1/(2\alpha) = 1 - H$. Similarly, Castell et al. [26] consider more general types of processes yielding the local time process L. We further mention [4, 72], where the persistence problem with moving boundary is studied, as well as moments of J_T and probabilities of scenarios that are very similar to persistence.

Continuing our discussion on FBM, note that it is clear that $\mathbb{P}[T_1^1 > t] \to 1/2$ by the above remark on the case $H = 1$, and it is natural to raise the

Conjecture 4 For every $H \in (0, 1)$ one has

$$\mathbb{P}[T_1^H > t] \asymp t^{H-1}.$$

In view of the previously stated precise results, one may also ask if $\mathbb{P}[T_1^H > t] \sim c\, t^{H-1}$.

The above conjecture was recently addressed in [2], where the following partial result is obtained by a refinement of Molchan's method:

Theorem 3.11 (Aurzada) *There exists* $c > 0$ *such that*

$$(\log t)^{-c} t^{H-1} \le \mathbb{P}[T_1^H > t] \le (\log t)^c t^{H-1}, \quad t \to +\infty.$$

We now turn to first passage asymptotics for a class of Gaussian stationary processes (GSP) which are related to the persistence of fractionally integrated Brownian motion. In [62] the Lamperti transformation

$$L_t^H = e^{-tH} B_{e^t}^H, \quad t \in \mathbb{R}, \tag{26}$$

a centered GSP, is studied in connection with the persistence exponent of B^H and it is shown that

$$\log \mathbb{P}\left[L_s^H \le 0, \ \forall\, s \in [0, t]\right] = \log \mathbb{P}\left[B_s^H \le 0, \ \forall\, s \in [1, e^t]\right] \sim t(H-1).$$

Since Molchan's theorem means that $\log \mathbb{P}\left[B_s^H \le 1, \ \forall\, s \in [0, e^t]\right] \sim t(H-1)$, one simply needs to switch the 0 and the 1 in the latter probability to obtain this result, which is justified in [62] through a refined use of Slepian's lemma. The Lamperti transformation is also a fruitful method in the reverse direction, and this was observed in [5] to investigate the persistence exponents of fractionally integrated Lévy processes. Assume that Z in (23) is a Brownian motion and introduce the notation

$$T_x^\beta = \inf\left\{t > 0, \ A_t^\beta > x\right\} = \inf\left\{t > 0, \ A_t^\beta = x\right\}$$

for every $\beta, x \ge 0$. Notice that $T_x^\beta \stackrel{d}{=} x^{2/(1+2\beta)} T_1^\beta$ by self-similarity, which shows that $T_0^\beta = 0$ a.s. and that the survival analysis of A^β is reduced to the behavior of $\mathbb{P}[T_1^\beta > t]$ only. The process

$$Y_t^H = e^{-tH} A_{e^t}^{H-1/2}, \quad t \in \mathbb{R}, \tag{27}$$

a centered GSP for every $H > 0$ which reduces to the stationary Ornstein-Uhlenbeck process when $H = 1/2$, plays a key rôle in the following theorem [5].

Theorem 3.12 (Aurzada-Dereich) *There exists a non-increasing function* $\beta \mapsto \theta(\beta)$ *such that*

$$\mathbb{P}[T_1^\beta > t] \;=\; t^{-\theta(\beta)+o(1)}$$

for every $\beta \geq 0$. *Besides one has* $\theta(\infty) = -\lim_{t\to\infty} t^{-1} \log \mathbb{P}\,[Y_s < 0,\ \forall\, s \in [0,t]]$, *where* Y *is the centered* GSP *with correlation* $\mathbb{E}[Y_0 Y_t] = 1/\cosh(t/2)$.

The important process Y is mentioned in [63] in the context of the diffusion equation with white noise initial condition, and in [30] in relation with the positivity of random polynomials with large even degree. More details will be given in the next section. In [62], the process Y is also viewed as the Lamperti transformation of the curious smooth (1/2)-self-similar Gaussian process

$$X_t \;=\; \sqrt{2}t^2 \int_0^\infty B_u e^{-ut} du, \quad t \geq 0,$$

which shares the same time inversion property $\{X_t,\, t > 0\} \stackrel{d}{=} \{t^{-1}X_{t^{-1}},\, t > 0\}$ as Brownian motion. In [5] only the upper bound

$$-\lim_{t\to\infty} t^{-1} \log \mathbb{P}\,[Y_s < 0,\ \forall\, s \in [0,t]] \;\leq\; \theta(\infty) \tag{28}$$

is proved via Slepian's lemma, but one can easily show that $Y^H \stackrel{d}{\to} Y$ as $H \to \infty$ in analysing the covariance function, so that the inequality in (28) is actually an equality, the limit on the left-hand side being a supremum. This also follows from the more general Theorem 1.6 in [29].

Theorem 3.12 has several interesting consequences. First, it shows that the persistence exponent is a non-increasing function of the order of integration for Brownian fractionally integrated processes. The fact that smoother processes have more probability to survive is believed to be a kind of universal feature. Second, it entails that $\theta(\beta) \geq \theta(1) = 1/4$ for every $\beta \leq 1$, so that by Molchan's result the persistence exponents of B^H and $A^{H-1/2}$ do not coincide whenever $H > 3/4$. We actually believe that $\theta(H - 1/2) > 1 - H$ for every $H \in (1/2, 1)$ and some reasons for that will be given soon afterwards. Last, it entails that $\theta(\infty) \leq \theta(1) = 0.25$, which improves the bound $\theta(\infty) < 0.325$ obtained in [62] via another Slepian's inequality. In [63] the numerical value $\theta(\infty) \sim 0.1875$ is suggested, whereas in [30] the value $\theta(\infty) = 0.19 \pm 0.01$ is obtained by simulations. It is a tantalizing question to compute the function $\theta(\beta)$ for every positive $\beta \notin \{0, 1\}$, as well as its limit $\theta(\infty)$. The lower bound $\theta(\infty) > 0.125$ is obtained in [61] with the help of a certain Gaussian comparison inequality, and in [69] this lower bound is improved into $\theta(\infty) > 1/(4\sqrt{3}) > 0.144$, in comparing Y with a linear time-change of the so-called Wong process [88], which is the GSP associated with integrated Brownian motion.

Let us give some more details on the correlation function $C_H(t) = \mathbb{E}[Y_0^H Y_t^H]$ of the Lamperti transform introduced in (27). It is given by

$$C_H(t) \;=\; 2e^{-Ht} \int_0^1 (u(u + e^t - 1))^{H-1/2}\, \mathrm{d}u$$

for every $t \geq 0$, and a simple analysis shows that for every $H \in (0, 1]$ and $t \geq 0$ one has

$$C_H(t) \;\leq\; \cosh(Ht) \;-\; 2^{2H-1}(\sinh(t/2))^{2H} \;=\; \mathbb{E}[L_0^H L_t^H],$$

where L^H is the Lamperti transform defined in (26). By Slepian's lemma and Molchan's theorem, this entails

$$\theta(H - 1/2) \;\geq\; 1 - H \tag{29}$$

for every $H \in [1/2, 1]$, with an equality if $H = 1/2$. From the above we know that (29) is strict if $H > 3/4$. It is very likely that $\theta(H - 1/2)$ exists for every $H \in (0, 1/2)$ and that (29) is strict for every $H \neq 1/2$. In view of the above discussion, it is natural to raise the

Conjecture 5 The function $\beta \mapsto \theta(\beta)$ is convex decreasing.

We now go back to general fractionally integrated Lévy processes and state the following result of Aurzada and Dereich [5], which is obtained by strong approximation and drift transformations of Gaussian processes:

Theorem 3.13 (Aurzada-Dereich) *For every $\beta \geq 0$, the persistence exponent $\theta(\beta)$ is the same among all β-fractionally integrated centered Lévy processes such that $t \mapsto \mathbb{E}[e^{tZ_1}]$ is finite in an open neighborhood of zero.*

The main result of Aurzada and Dereich [5], to which we refer for details, is more general and handles fractionally integrated random walks as well as more general Volterra kernels. The finiteness of exponential moments plays an important rôle therein due to the use of strong approximation, but in view of the aforementioned results and conjectures on integrated random walks and Lévy processes, it is natural to raise the

Conjecture 6 For every $\beta \geq 0$, the exponent $\theta(\beta)$ is the same among all β-fractionally integrated centered Lévy processes with finite variance.

We remark that there are many other types of fractional Lévy processes than fractionally integrated ones—see e.g. [40] for a recent account, but up to the knowledge of the authors no results for their persistence probabilities are known.

Let us conclude this paragraph with the integrated fractional Brownian motion

$$I_t^H = \int_0^t B_s^H \, ds, \quad t \geq 0.$$

This is a centered Gaussian process with positive correlation function, and with the help of its Lamperti transformation it is shown easily using Slepian's inequality, subadditivity, and e.g. Proposition 1.6 in [5] that for every $H \in (0,1)$ there exists $\rho(H) > 0$ such that

$$\mathbb{P}\left[I_s^H \leq 1, \; \forall\, s \in [0,t]\right] = t^{-\rho(H)+o(1)}.$$

Numerical simulations [70] suggest the following:

Conjecture 7 (Khokhlov-Molchan) One has $\rho(H) = H(1-H)$.

This expected value, which is symmetric with respect to $H = 1/2$, is very surprising because it is known that fractional Brownian motions with Hurst index smaller or greater than 1/2 are very different processes from several viewpoints. This does not match either the above heuristic discussion on smooth GSP's since the correlation function F^H of the Lamperti transform of I^H has a first-order expansion at zero

$$F^H(t) = 1 - \frac{(1-H^2)t^2}{2} + o(t^2).$$

In particular, the number N_t^H of zero-crossings of this Lamperti process has an expectation $\mathbb{E}[N_t^H] = t\sqrt{1-H^2}/\pi$ which decreases with H, and one might think that $\rho(H)$ also decreases. However integrated fractional Brownian motion may well be more the exception than the rule for this kind of questions, because of its complicated correlation structure. In [70] it is argued that the difference between $H < 1/2$ and $H > 1/2$ should be observed at the logarithmic level and it is shown in [68], with a detailed analysis, that $cH(1-H) \leq \rho(H) \leq 1-H$ for some $c \in (0,1)$. Other bounds such as $\rho(H) \geq \rho(1-H)$ for every $H \leq 1/2$ were also recently presented in [69] (with numerical explanations), neither proving nor disproving the above conjecture.

3.4 *Other Processes*

Integrated random walks which were considered previously can be written as the weighted sum

$$A_n = \sum_{i=1}^{n} (n+1-i)X_i,$$

where $\{X_i,\ i \geq 1\}$ is an i.i.d. sequence. The above weights depend on i and n so that the increments of A_n are neither stationary nor independent. It is natural to consider persistence problem for other weighted sums like weighted random walks:

$$\Sigma_n = \sum_{i=1}^{n} \sigma_i X_i,$$

where $\{\sigma_i,\ i \geq 1\}$ is a deterministic sequence and $\{X_i,\ i \geq 1\}$ an i.i.d. family. The situation is a bit simpler than for integrated random walks because the non-stationary increments of Σ are independent, nevertheless the sequence $\{\sigma_i,\ i \geq 1\}$ has also more generality. Setting μ for the law of X_1 and introducing

$$T_\sigma = \inf\{n \geq 1,\ \Sigma_n > 0\},$$

the following has been obtained in [3] via strong approximation techniques:

Theorem 3.14 (Aurzada-Baumgarten) *Suppose that μ is centered and that its Laplace transform is finite in an open neighborhood of zero. Suppose that $\{\sigma_i,\ i \geq 1\}$ is increasing with $\sigma_n \asymp n^p$ for some $p > 0$. Then*

$$\mathbb{P}[T_\sigma > n] = n^{-(p+1/2)+o(1)}.$$

The results of Aurzada and Baumgarten [3] are more precise and allow other weight functions not necessarily increasing when μ is Gaussian. The general case reduces to the Gaussian one via strong approximation and drift transformations, as in [5]. A universal speed not depending on μ can also be obtained for weight functions growing faster than polynomials, like e^{n^γ} for some $\gamma < 1/4$, and the persistence probability then has stretched exponential decay. However, it is also shown in [3] that the speed does depend on μ for weight functions growing too fast, like e^{n^γ} for some $\gamma \geq 1$. It would be interesting to find σ's critical growth rate for the universality of the speed. The following is also a natural question.

Conjecture 8 Suppose that μ is centered and has finite variance. Suppose that $\{\sigma_i,\ i \geq 1\}$ is increasing with $\sigma_n \asymp n^p$ for some $p > 0$. Then

$$\mathbb{P}[T_\sigma > n] = n^{-(p+1/2)+o(1)}.$$

Let us now consider iterated Lévy processes, which are processes of the type $\{X \circ |Y_t|,\ t \geq 0\}$ with X, Y two independent real Lévy processes starting from zero. If $|Y|$ is a subordinator, then $X \circ |Y|$ is another Lévy process which is called a subordinate Lévy process, a notion introduced by Bochner in the context of harmonic analysis. Iterated Lévy processes were introduced by Burdzy in the Brownian framework and can be viewed as a generalisation of subordinate Lévy processes. They are known to have strong connections with PDE's of higher order, especially through their first passage times. Let us introduce

$$T_x \;=\; \inf\{t > 0,\; X \circ |Y_t| > 1\}$$

for all $x \geq 0$, and notice that as for integrated Lévy processes the law of T_x is difficult to study in general since $X \circ |Y|$ is non-Markov. Among other results, the following was obtained in [12] and subsequently improved by Vysotsky [86]:

Theorem 3.15 (Baumgarten, Vysotsky) *Suppose that the random variables X_1 and Y_1 have finite variance and $\mathbb{E}[X_1] = 0$. Then*

$$\mathbb{P}[T_1 > t] \asymp t^{-1/4} \qquad \textit{for } \mathbb{E}[Y_1] = 0,$$
$$\mathbb{P}[T_1 > t] = t^{-1/2+o(1)} \qquad \textit{for } \mathbb{E}[Y_1] \neq 0,$$

where for the second statement one has to assume that X_1 and Y_1 have some finite stretched exponential moment.

In particular, the persistence exponent of iterated Brownian motion is $1/4$. The strong dichotomy between the situations where Y_1 is centered and non-centered is not surprising since in the former case $|Y_t|$ grows roughly like $\sqrt{t}$ whereas in the latter case $|Y_t|$ grows like t. The persistence exponent of $X \circ |Y|$ is given by the product of the persistence exponent of X and typical growth rate of Y. This can be seen in different situations, too, for example if X is an FBM. As in many above statements, the question of replacing the exponential moment condition by the sole assumption of finite variance remains open.

We conclude this paragraph in mentioning some results for AR(p) processes, that is

$$X_n = a_1 X_{n-1} + \ldots + a_p X_{n-p} + Y_n,$$

where $\{Y_n, n \geq 1\}$ is a sequence of i.i.d. random variables and $X_n = 0$ for $n \leq 0$. The first and already non-trivial question here is to determine for what values of the parameters $a_1, \ldots, a_p$ the persistence probability (a) converges to some positive constant, (b) decays polynomially, or (c) decays faster than any polynomial (in which case it is usually exponentially small). To do so, one first has to notice, that X can be represented as $X_n = \sum_{k=1}^{n} c_{n-k} Y_k$ with some (c_n) which depend on the parameters $a_1, \ldots, a_p$ via the solutions of some characteristic polynomial. The reason for the behavior (a) is that the (c_n) tend to infinity exponentially fast and thus the first few Y_n decide the sign of all X_n. For (c) basically two different main reasons appear: the (c_n) may tend to zero exponentially fast, which means that X is essentially a stationary AR and one obtains exponential decay of the persistence probability; or the modulus of (c_n) may tend to infinity while the sign is alternating, in which case the sign of X_n is determined essentially by the sign of Y_n for all n and one basically obtains an independent sequence again leading to exponential decay.

The polynomial behaviour is possible only on the critical line between behavior (a) and (c) and corresponds to (c_n) tending to a positive constant. For $p = 1$, this is particularly easy: part (a) corresponding to $a_1 > 1$, part (b) to $a_1 = 1$, and part (c) to $a_1 < 1$ (since $c_n = a_1^n$, the above-mentioned distinction happens for $a_1 \in (-1, 1)$ and $a_1 < -1$). For $p = 2$ one obtains the critical curve

$$\{a_1 = 0, a_2 > 1\} \cup \{a_1 \in [0, 2], a_1 + a_2 = 1\} \cup \{a_1 > 2, a_1^2 + 4a_2 = 0\}$$

for the parameter values (a_1, a_2) that separates (a) and (c). This question was studied in [71] for $p = 1$ and in [11] for general p with a focus on $p = 2$. Determining the explicit rate of exponential decay on the region (c) is open in all interesting cases (even for $p = 1$) and seems to be challenging. Also see [28, 60].

4 Some Connections with Physics

4.1 Regular Points of Inviscid Burgers Equation with Self-similar Initial Data

The statistical study of the one-dimensional Burgers equation

$$\partial_t u \; + \; u\partial_x u \; = \; \nu \partial_{xx} u \tag{30}$$

with viscosity $\nu > 0$ and an initial condition $u_0(x) := u(0, x) = X_x$ given by a self-similar stochastic process $\{X_x, x \in \mathbb{R}\}$ has been initiated in the papers [79, 83]. Though this equation is accorded to be an unrealistic physical model for turbulence, the competition between the irregularities of X and the irregularities generated by (30) remains an interesting mathematical study. In the inviscid limit $\nu = 0$, the Hopf-Cole solution to (30) is given by

$$u(t, x) \; = \; \frac{x - a(t, x)}{t}$$

for every $t > 0, x \in \mathbb{R}$, where $a(t, x) = \max\{y \in \mathbb{R},\ \dot{C}_t(y) \le xt^{-1}\}$ and $\dot{C}$ is the right-derivative of the convex minorant of the function

$$F_t : y \; \mapsto \; \int_0^y (X_x + xt^{-1})\, \mathrm{d}x.$$

This variational formula is obtained in considering the explicit solution to (30) which can be obtained for $\nu > 0$ and letting $\nu \to 0$—see [14, 53, 79, 83] for details. Notice that $a(t, x)$ is well-defined only if

$$|x|^{-1} X_x \; \to \; 0 \quad \text{a.s. when } x \to \pm\infty. \tag{31}$$

The function $x \mapsto a(t,x)$ is right-continuous but not continuous in general, and the so-called Lagrangian regular points at time $t > 0$ are defined as the set

$$\mathscr{L}_t = \overline{\{a(t,x),\ x \in \mathbb{R} \ \text{ and } \ a(t,x-) = a(t,x)\}}$$

which consists of the points where F_t equals its convex minorant. In physical terms, the set $\mathscr{L}_t$ describes the initial locations of the particles which have not been shocked up to time t. It is easy to see that when X is a self-similar process, the map $t \mapsto \mathscr{L}_t$ has also some self-similarity which makes the a.s. Hausdorff dimension of $\mathscr{L}_t$ independent of $t > 0$. Setting $\mathscr{L} = \mathscr{L}_1$ and "Dim" for "Hausdorff dimension", the following is stated in [79]:

Conjecture 9 (Aurell-Frisch-She) Suppose that X is the fractional Brownian motion with Hurst parameter $H \in (0,1)$. Then *Dim* $\mathscr{L} = H$ a.s.

Notice that in the above, the fractional Brownian motion is defined over the whole $\mathbb{R}$, and coincides with the two-sided Brownian motion when $H = 1/2$. This conjecture remains open in general, but the following has been shown:

Theorem 4.1 (Handa-Sinai, Bertoin) *Suppose that X is the two-sided Brownian motion. Then* Dim $\mathscr{L} = 1/2$ *a.s.*

This result was first stated in [83], although no strict proof is given therein for the lower bound Dim $\mathscr{L} \geq 1/2$ a.s. A simple argument based on integration by parts and Frostman's lemma is presented in [48], which yields the general lower bound Dim $\mathscr{L} \geq H$ a.s. when X is the fractional Brownian motion with Hurst parameter H. In [14], the exact computation of the Hausdorff dimension follows as a simple corollary to the more general result that $x \mapsto a(1,x)$ has stationary and independent increments with explicit Laplace transform. This result extends to Lévy processes with no positive jumps satisfying the growth condition (31). In particular one has the

Theorem 4.2 (Bertoin) *Suppose that X is a two-sided α-stable spectrally negative Lévy process with index $\alpha \in (1,2)$. Then* Dim $\mathscr{L} = 1/\alpha$ *a.s.*

We now briefly describe the link between an upper bound for Dim $\mathscr{L}$ and the computation of certain persistence exponents, in the self-similar framework. This is the original argument of Sinai [83] for Brownian motion and it extends to fractional Brownian motion [70] or α-stable Lévy processes [81]. Specifically, setting

$$\hat{\mathscr{L}} = \left\{ a \in \mathbb{R},\ \int_0^y (X_x + x)\mathrm{d}x \right.$$
$$\left. \geq \int_0^a (X_x + x)\,\mathrm{d}x + (y-a)(X_a + a),\ \forall y \in \mathbb{R} \right\},$$

Sinai's remarkable and simple observation is that up to some countable set, one has

$$\mathscr{L} \subseteq \hat{\mathscr{L}} \qquad \text{a.s.}$$

On the other hand, the Borel-Cantelli lemma and some elementary inequalities—see Lemma 1 in [70] and [81, pp. 742–745] for all details—show that if

$$\mathbb{P}\left[\int_0^y (X_x + x)\,\mathrm{d}x \;\geq\; -\delta^{1+H},\;\; \forall y \in [-1,1]\right] \;\leq\; \delta^{1-K+o(1)} \tag{32}$$

as $\delta \to 0$, where H is X's self-similarity index and $K \in [0,1]$, then $\mathrm{Dim}\,\bar{\mathscr{L}} \leq K$ a.s. Hence, the "two-sided" persistence probability evaluation (32) entails the upper bound $\mathrm{Dim}\,\mathscr{L} \leq H$ a.s. The upper asymptotic inequality (32) is shown in [82] for Brownian motion and the above method is used in [83] to obtain the upper bound in Theorem 4.1.

If $X = B^H$ is the fractional Brownian motion, then the drift appearing in (32) can be removed by quasi-invariance and some analysis, and one sees by symmetry and self-similarity that the required estimate to get the upper bound in Conjecture 9 is

$$\mathbb{P}\left[\int_0^y B_x^H\,\mathrm{d}x \;\leq\; 1,\;\; \forall y \in [-t,t]\right] \;\leq\; t^{1-H+o(1)}, \quad t \to +\infty.$$

The above estimate is formulated as a conjecture in [68, 70], with an equality instead of the inequality. Notice that this latter problem is independent of Conjecture 7 since the increments of B^H are correlated. Actually, even the sole existence of the persistence exponent for integrated double-sided fractional Brownian motion has not yet been established.

If X is a two-sided α-stable Lévy process with $\alpha \in (1,2)$, there is no quasi-invariance argument. But the fact that $\alpha > 1$ and the bare-hand analysis performed in [81] make believe that the drift appearing in (32) can also be removed. By self-similarity and independence of the positive and negative increments of Z the inequality (32) would then amount to

$$\mathbb{P}[\hat{T}_1 > t] \;\leq\; t^{(1-K)/2+o(1)}$$

at infinity, where $\hat{T}_1$ is the first-passage time at 1 of the integral of $\hat{Z} = -Z$. In particular, Theorem 3.9 would lead to

$$\mathrm{Dim}\,\mathscr{L} \;\leq\; ((\rho(\alpha+2)-1)/(1+\alpha\rho))_+ \quad \text{a.s.}$$

where $\rho = \mathbb{P}[Z_1 > 0]$ is the positivity parameter of Z—recall that $\rho \in [1-1/\alpha, 1/\alpha]$ for $\alpha \in (1,2)$. It is also natural to believe that the above inequality is actually an equality.

Conjecture 10 Suppose that X is a two-sided α-stable Lévy process with $\alpha \in (1,2)$ and positivity parameter $\rho = \mathbb{P}[Z_1 > 0]$. Then, with the above notation,

$$\mathrm{Dim}\,\mathscr{L} \;=\; ((\rho(\alpha+2)-1)/(1+\alpha\rho))_+ \quad \text{a.s.}$$

If this were true, then one should have $\operatorname{Dim} \mathscr{L} = 0$ a.s. for all $\rho \le 1/(2+\alpha)$ (which can appear only if $\alpha \le \sqrt{2}$). In any case, we believe that this Hausdorff dimension should depend on the positivity parameter and not only on the self-similarity parameter. The value $1/\alpha$ had been conjectured in [53] through multifractal analysis, and the invalidity of this conjecture when α is close to 1 had been proved in [81] with the help of Sinai's approach.

Theorem 4.3 (Simon) *For every $c < 1$ there exists $\alpha_0 > 1$ such that for every $\alpha \in (1, \alpha_0)$ and every $\rho \in [1 - 1/\alpha, c \wedge 1/\alpha]$, if X is a two-sided α-stable Lévy process with positivity parameter ρ, then* $\operatorname{Dim} \mathscr{L} < 1/\alpha$ *a.s.*

4.2 Positivity of Random Polynomials and Diffusion Equation

A classical question dating back to the beginning of probability theory is to understand the distribution of the roots of random polynomials. Consider

$$P_n(X) = \sum_{i=0}^{n-1} \xi_i X^i$$

with large even degree where $\{\xi_i,\ i \ge 0\}$ is some i.i.d. sequence and X the deterministic variable, and set N_n for the number of its real roots. Among other results, the following was recently obtained in [30]:

Theorem 4.4 (Dembo-Poonen-Shao-Zeitouni) *Suppose that ξ_1 is centered and has polynomial moments of all order. Then*

$$\mathbb{P}\left[N_{2n+1} = 0\right] = n^{-4b+o(1)} \tag{33}$$

where $b = -\lim_{t\to\infty} t^{-1} \log \mathbb{P}\left[Y_s < 0,\ \forall\, s \in [0,t]\right]$, *with Y the centered* GSP *with correlation* $\mathbb{E}[Y_0 Y_t] = 1/\cosh(t/2)$.

In the above, the exact value of b is unknown and numerical simulations suggest $4b = 0.79 \pm 0.03$—see [30]. It is remarkable that this constant b relates to n times integrated Brownian motion. We saw indeed in Sect. 3.3 that $b = \lim_{n\to+\infty} \theta(n)$ where $\theta(n)$ is the persistence exponent of the process

$$t \mapsto \frac{1}{n!}\int_0^t (t-s)^n \, \mathrm{d}B_s, \quad t \ge 0.$$

The problem of computing $\theta(n)$ for $n > 1$ is believed to be very challenging. Numerical simulations [64] suggest $\theta(2) = 0.231 \pm 0.01$.

Let us give some insight on the proof of the above result. The hard part is to show (33) when $\xi_1 \sim \mathscr{N}(0,1)$. The general case follows by strong approximation, whence the assumption made on the moments, but notice that it is also conjectured in [30] that (33) should hold under the sole condition that ξ_1 is centered and has finite variance. When $\xi_1 \sim \mathcal{N}(0,1)$, the process $x \to P_n(x)$, which is the so-called Kac's polynomial, is centered Gaussian with covariance $\mathbb{E}[P_n(x)P_n(y)] = 1+\cdots+(xy)^{n-1}$. Its correlation function is given by

$$\left|\frac{(xy)^n - 1}{xy - 1}\right| \sqrt{\left|\frac{(x^2-1)(y^2-1)}{(x^{2n}-1)(y^{2n}-1)}\right|}$$

for every $x, y \neq \pm 1$. This function is invariant under the transformations $(x,y) \mapsto (-x,-y)$ and $(x,y) \mapsto (1/x, 1/y)$, and an involved argument based on Slepian's lemma shows that

$$\mathbb{P}\left[N_{2n+1} = 0\right] \;=\; (\mathbb{P}[N_{2n+1}^{[0,1]} = 0])^{4+o(1)}$$

where $N_n^{[0,1]}$ is the number of roots of P_n on $[0,1]$. The link between $N_{2n+1}^{[0,1]}$ for large n and the zero-crossings of Y is established after changing the variable $x = e^{-t}$ and isolating the contributions for small t. The latter is the crucial step, since it follows from the singularities of the correlation function that the density of the real roots of P_n around ± 1 is very big for large n. Notice that the link between Y and N_n is rather easily understood at the expectation level: a classical formula due to Kac—see [30] for details—yields

$$\mathbb{E}[N_{2n+1}^{[0,1]}] \;\sim\; \frac{1}{2\pi}\log n,$$

whereas the Itô-Rice's formula shows that if N_t is the number of zero-crossings of Y on $[0,t]$, then $\mathbb{E}[N_t] \sim t/(2\pi)$. The problem of evaluating $\mathbb{P}[N_{2n+1}^{[0,1]} = 0]$ is however much more intricate than the sole estimation of $\mathbb{E}[N_{2n+1}^{[0,1]}]$, in analogy with what happens for the zero-crossings of Gaussian stationary processes.

It is also a remarkable fact that the above constant b appears as the persistence exponent of another, seemingly disconnected random evolution phenomenon, which is studied in [63, 77, 78]. Consider the heat equation on $\mathbb{R}^d$

$$\frac{\partial u_d}{\partial t} \;=\; \Delta u_d \tag{34}$$

with random initial condition $u_d(x,0) = \dot{W}(x)$ a d-dimensional white noise. Integrating along the heat kernel, it is easy to see by linearity that for every $x \in \mathbb{R}^d$ the solution $t \mapsto u_d(x,t)$ to (34) is a $(-d/4)$-self-similar centered Gaussian process with covariance function

$$\mathbb{E}[u_d(t,x)u_d(s,x)] \;=\; \frac{1}{(\pi(t+s))^{d/2}}. \tag{35}$$

In particular the law of $\{u_d(t,x),\ t>0\}$ does not depend on x, which is also clear from the white noise initial condition. The Lamperti transformation $t \mapsto (2\pi)^{1/4}e^{t/4}u_d(x,e^t)$ is a centered GSP with correlation function $1/(\cosh(t/2))^{d/2}$, and this GSP coincides with Y for $d=2$. For every $d \geq 1$, the standard subadditivity argument and Slepian's lemma yield the existence of $\lambda_d > 0$ such that

$$\mathbb{P}[u_d(s,x) < 0,\ \forall s \in [1,t]] \;=\; t^{-\lambda_d + o(1)}$$

and one has $b = \lambda_2$. In [63], an empirical approach using independent interval approximation is described, proposing λ_d as the first zero on the negative axis of the function

$$x \;\mapsto\; 1 \;+\; \sqrt{\frac{2}{d}}\left(\pi x - 2x^2 \int_0^\infty e^{-xt}\sin^{-1}[1/(\cosh(t/2))^{d/2}]\,\mathrm{d}t\right),$$

which yields the numerical values $\lambda_1 = 0.1207$, $\lambda_2 = 0.1862$ and $\lambda_3 = 0.2358$. In the above formula, the function $d \mapsto \lambda_d$ is increasing, which is somehow in heuristic accordance with the fact that in the first-order expansion

$$1/(\cosh(t/2))^{d/2} \;=\; 1 \;-\; \frac{dt^2}{16} \;+\; o(t^2),$$

the coefficient $d \mapsto d/16$ also increases. In [78] it is argued that $\lambda_d \sim c\sqrt{d}$ at infinity, for some constant $c > 0$. The paper [78] also establishes for every $d \geq 1$ a general connection between the survival analysis of Eq. (34) and the positivity of a family of random polynomials defined as

$$P_n^d(X) \;=\; \xi_0 \;+\; \sum_{i=1}^{n-1} i^{(d-2)/4}\xi_i X^i \tag{36}$$

where $\{\xi_i,\ i \geq 0\}$ is a i.i.d. sequence of $\mathcal{N}(0,1)$ random variables and X is the deterministic variable. Setting N_n^d for the number of its real roots, it is argued in [78] that

$$\mathbb{P}\left[N_{2n+1}^d = 0\right] \;=\; n^{-2(\lambda_2+\lambda_d)+o(1)}. \tag{37}$$

The results are justified in the recent paper [29], where the connection between the persistence of the Gaussian process u_d defined in (35) and the random polynomials (36) for Gaussian ξ_i is proved rigorously. Further, the notion of a solution to the heat equation (34) is made rigorous. It would be interesting to obtain again a universality result similar to Theorem 4.4 in the sense that (37) holds independent of the law of the ξ_i. Further, Dembo and Mukherjee [29] establishes connections of other types of random polynomials and the corresponding Gaussian processes (also considered in [78]). Also here it would be interesting to obtain a universality result.

4.3 Wetting Models with Laplacian Interactions

Let f be a bounded and everywhere positive probability density over $\mathbb{R}$, centered and having finite variance. Introduce the Hamiltonian $\mathscr{H}_{[a,b]}(\varphi)$, defined for $a, b \in \mathbb{Z}$ with $b - a \geq 2$ and for $\varphi : \{a, \ldots, b\} \to \mathbb{R}$ by

$$\mathscr{H}_{[a,b]}(\varphi) \;=\; \sum_{n=a+1}^{b-1} V(\Delta\varphi_n)$$

where $V = -\log(f)$ is the potential and

$$\Delta\varphi_n \;=\; (\varphi_{n+1} - \varphi_n) - (\varphi_n - \varphi_{n-1}) \;=\; \varphi_{n+1} - 2\varphi_n + \varphi_{n-1}$$

is the discrete Laplacian on $\mathbb{Z}$. The free pinning model with Laplacian interaction is the probability measure on $\mathbb{R}^{N-1}$ defined by

$$\mathbb{P}^p_{0,N}(\,\mathrm{d}\varphi_1, \ldots, \,\mathrm{d}\varphi_{N-1}) \;=\; \frac{\exp(-\mathscr{H}_{[-1,N+1]}(\varphi))}{\mathscr{Z}^p_{0,N}}\,\mathrm{d}\varphi_1 \ldots \,\mathrm{d}\varphi_{N-1}$$

where $\mathscr{Z}^p_{0,N}$ is the normalization constant which is called the partition function, and where the boundary conditions are given by $\varphi_{-1} = \varphi_0 = \varphi_N = \varphi_{N+1} = 0$. This probability measure models a certain $(1+1)$-dimensional field (viz. a linear chain $\{(n, \varphi_n),\, n = 0 \ldots N\}$) with zero boundary conditions and whose interacting structure is described by the discrete Laplacian and the potential V. This chain can be viewed as an example of a discrete random polymer in $(1+1)$-dimension.

The free pinning model with gradient interaction, where Δ is replaced by the discrete gradient $\nabla\varphi_n = \varphi_{n+1} - \varphi_n$ and where the boundary conditions are $\varphi_0 = \varphi_N = 0$, has been well studied in the literature and has a natural interpretation in terms of random bridges with increment density given by f. The model with Laplacian interaction has exactly the same interpretation in terms of integrated random bridges. Specifically, one can easily show—see Sect. 2 in [24] for details—that $\mathbb{P}^p_{0,N}$ is, with the notations of Sect. 3.1, the law of an integrated random walk $\{A_n = S_1 + \cdots + S_n,\, n = 1 \ldots N-1\}$ conditioned on $A_N = A_{N+1} = 0$, with increment density given by f. The partition function $\mathscr{Z}^p_{0,N}$ is then the value at $(0,0)$ of the density of (S_{N+1}, A_{N+1}). Notice that both above free pinning models have natural counterparts in continuous time in the context of semiflexible polymers. The gradient interacting case corresponds to directed polymers, whereas the Laplacian interacting case corresponds to polymers with non-zero bending energy—see [23] for details.

The connection with persistence of integrated random bridges is made in considering the corresponding wetting model with Laplacian interactions, which is the probability measure on $\mathbb{R}^{N-1}$ defined by

$$\mathbb{P}^w_{0,N}(\,\mathrm{d}\varphi_1,\ldots,\,\mathrm{d}\varphi_{N-1}) = \mathbb{P}^p_{0,N}(\,\mathrm{d}\varphi_1,\ldots,\,\mathrm{d}\varphi_{N-1}\mid\varphi_1\geq 0,\ldots,\varphi_{N-1}\geq 0)$$
$$= \frac{\exp(-\mathscr{H}_{[-1,N+1]}(\varphi))}{\mathscr{Z}^w_{0,N}}\mathbf{1}_{\{\varphi_1\geq 0,\ldots,\varphi_{N-1}\geq 0\}}\,\mathrm{d}\varphi_1\ldots\,\mathrm{d}\varphi_{N-1}$$

where $\mathscr{Z}^w_{0,N}$ is the normalization constant, and with the same boundary conditions $\varphi_{-1}=\varphi_0=\varphi_N=\varphi_{N+1}=0$. In this model, the discrete random polymer is in the presence of a one-dimensional hard wall at zero which forces it to stay non negative. From the very definition, one sees that $\mathbb{P}^W_{0,N}$ is the law of the above integrated random walk $\{A_n,\ n=1\ldots N-1\}$ whose increment has the density f, and conditioned on $\Omega^+_{N-1}\cap\{A_N=A_{N+1}=0\}$ with the notation $\Omega^+_{N-1}=\{A_1\geq 0,\ldots,A_{N-1}\geq 0\}$. It is then easily shown that the partition function is given by

$$\mathscr{Z}^w_{0,N} \;=\; \mathbb{P}[\Omega^+_{N-1}\,|\,A_N=A_{N+1}=0]f_N(0,0)$$

where $f_N(0,0)$ is the value at $(0,0)$ of the density of (S_{N+1},A_{N+1}). As a consequence of a local limit theorem—see Sect. 2 in [24] for details—it can be shown that $f_N(0,0)\sim cN^{-2}$ at infinity for some explicit constant $c>0$. Hence the behavior of $\mathscr{Z}^w_{0,N}$ for N large, which has some importance in physics, is specified by the persistence probability

$$\mathbb{P}[\Omega^+_{N-1}\,|\,A_N=A_{N+1}=0].$$

The latter quantity has also some independent interest as a question about entropic repulsion—see all the references listed in [24] for more on this subject, and the following is stated in [24]:

Conjecture 11 (Caravenna-Deuschel) With the above notations, one has

$$\mathbb{P}[\Omega^+_{N-1}\,|\,A_N=A_{N+1}=0] \;\asymp\; N^{-1/2}$$

for every centered increment law μ having finite variance.

This conjecture is related to integrated random walks considered in Sect. 3.1 since the event $\{\Omega^+_{N-1}\,|\,A_N=A_{N+1}=0\}$ can be decomposed into $\{A_1\geq 0,\ldots,A_{N/2}\geq 0\,|\,A_N=A_{N+1}=0\}\cap\{A_{N/2+1}\geq 0,\ldots,A_{N-1}\geq 0\,|\,A_N=A_{N+1}=0\}$, the intersection of two roughly independent events with roughly the same probability $\mathbb{P}[A_1\geq 0,\ldots,A_{N/2}\geq 0]$, a quantity which should behave like $N^{-1/4}$. Notice that in the context of semiflexible polymers, a continuous counterpart of Ω^+_{N-1} in the case when μ is Gaussian was investigated (without conditioning) in [23], where the estimate (22) is proved. In [24], the following weak bounds are obtained

$$\frac{c}{N^{c-}} \;\leq\; \mathbb{P}[\Omega^+_{N-1}\,|\,A_N=A_{N+1}=0] \;\leq\; \frac{C}{(\log N)^{c+}}$$

for some constants $c, C, c_- > 0$ and $c_+ > 1$. The lower bound entails that the free energy vanishes:

$$\lim_{N\to+\infty} \frac{1}{N} \log \mathscr{Z}^w_{0,N} \;=\; 0,$$

whereas the fact that $c_+ > 1$ in the upper bound is crucial to show that the phase transition of the wetting model with reward, which is simply the value of the positive parameter ε after which the free energy of the probability measure

$$\frac{\exp(-\mathscr{H}_{[-1,N+1]}(\varphi))}{\mathscr{Z}^w_{\varepsilon,N}} \prod_{n=1}^{N-1} (\varepsilon\delta_0(\,\mathrm{d}\varphi_n) + \mathbf{1}_{\{\varphi_n\geq 0\}}\,\mathrm{d}\varphi_n)$$

becomes positive, is of first order. In [24], to which we again refer for more details, it is mentioned that Conjecture 11 would yield some further path results for the wetting model with reward at criticality.

Recently, there have been two advances in this direction. First, [9] show Conjecture 11 in the case of the simple random walk and integrated simple random walk:

Theorem 4.5 (Aurzada-Dereich-Lifshits) *Let $\{X_i, i \geq 0\}$ be an i.i.d. sequence of Bernoulli random variables, and consider the simple random walk $S_n := \sum_{i=1}^n X_i$ and the integrated random walk $A_n := \sum_{i=1}^n S_i$. Then*

$$\mathbb{P}[A_1 \geq 0, \ldots, A_{4n} \geq 0 | S_{4n} = 0, A_{4n} = 0] \;\asymp\; n^{-1/2}.$$

Further, it is mentioned in [34] that the techniques can be used to show the full Conjecture 11 under the condition of finite $(2+\delta)$ moment for some $\delta > 0$. Presumably, the techniques will be different to [9] and it would be very interesting to study this question and its implications for the wetting models at criticality. Similarly to the above theorem, the case of Gaussian (X_i) was studied very recently by Gao et al. [44] and the corresponding result is shown (up to a log factor in the lower bound).

4.4 Other Physical Applications

4.4.1 Spatial Persistence for Fluctuating Interfaces

A fluctuating interface is a function $h : \mathbb{R}^+ \times \mathbb{R}^d \to \mathbb{R}$ evolving in time, with dynamics governed by a certain random equation. The problem of spatial persistence concerns the probability $p(l)$ that such a fluctuating interface stays above its initial value over a large distance l from a given point in space. One expects a behavior like $p(l) = l^{-\theta+o(1)}$ for a positive number θ independent of the direction, which is called the spatial persistence of the interface. In [64], this question is addressed for the Gaussian interface $h(t,x)$ solution to the equation

$$\frac{\partial h}{\partial t} \;=\; -(-\Delta)^{z/2} h \;+\; \xi \tag{38}$$

where Δ is the d-dimensional Laplacian, ξ a space-time white noise with zero mean, and $z > d$ some fractional parameter, and it is shown with heuristic arguments based on Fourier inversion that if $\rho = (z-d+1)/2$, then the fractional derivative in any direction x_1

$$\frac{\partial^\rho h}{\partial x_1^\rho} \;=\; \frac{\partial}{\partial x_1^{[\rho]+1}}\left(\frac{1}{\Gamma(2-\alpha)}\int_0^{x_1} h(t,y,x_2,\ldots,x_d)(x_1-y)^{1-\alpha}\,\mathrm{d}y\right),$$

where we have decomposed $\rho = [\rho] + \alpha$ into integer and fractional parts, is a one-dimensional white noise. This relates the spatial persistence probability $p(l)$ to the persistence probability of the fractionally integrated Lévy process

$$A_t^\rho \;=\; \frac{1}{\Gamma(\rho)}\int_0^t (t-s)^{\rho-1}\,\mathrm{d}B_s.$$

In [64] two different regimes are considered. The coarsening one, where the reference point is fixed, yields a spatial persistence exponent $\theta = \theta(\rho)$ with the notations of Sect. 3.3. The stationary one, where the reference point is sampled uniformly from the ensemble of steady state configurations, yields from (25) a spatial persistence exponent $\theta = (1-\rho)_+$. In the coarsening regime, this entails that the zero crossings of Gaussian interfaces governed by (38) undergo a morphological transition at $z = d+2$, because then $\rho = 3/2$.

4.4.2 Clustering of Sticky Particles at Critical Time

In this last paragraph we consider a random walk $\{S_n,\, n \geq 1\}$ with positive increments having expectation $\mathbb{E}[S_1] = 1$. If n particles are fixed at the respective positions $i^{-1}S_i, i = 1\ldots n$ with zero initial speed and then move according to the laws of gravitational attraction, these particles end up in sticking together with conservation of mass and momentum, forming new particles called clusters. One is then interested in the number of clusters $K_n(t) \in [1,\ldots,n]$ viz. the total number of particles present at time $t \geq 0$. This is a so-called sticky particle model, which is for n large connected to the inviscid Burgers equation with random initial data (coupled with some scalar transport equation, see [22] for details). This is also an aggregation model having connections with astrophysics, and we refer to the introduction of [84] for a clarification of these relations and the complete dissipation of all possible misunderstanding.

The normalization $\mathbb{E}[S_1] = 1$ entails that $T_n \to 1$ in probability, where T_n stands for the random terminal time where all particles have aggregated in a single cluster. A more precise result is obtained in [84] in the case when S_1 has uniform or standard Poissonian distribution, namely that the random function

$$\frac{K_n(t) - n(1-t^2)}{\sqrt{n}}$$

converges in law to some Gaussian process on the Skorokhod space $\mathscr{D}[0, 1-\varepsilon]$ for any $\varepsilon > 0$. In particular, the above quantity converges to some Gaussian law at each fixed time $t < 1$. The situation is however different at the critical time $t = 1$, at least when S_1 has a standard Poissonian distribution. In this case it can be proved that $K_n(1)/\sqrt{n}$ does not converge to zero (the only non-negative Gaussian distribution) as could be expected, and this fact is actually a consequence—see [84] for details—of the estimate

$$\mathbb{P}\left[\min_{i=1,\ldots,n} \sum_{j=1}^{i} (\Gamma_j - j) \geq 0\right] \asymp n^{-1/4}$$

where $\{\Gamma_n, n \geq 1\}$ is a random walk with exponential increments—this latter estimate follows from the main result of Dembo et al. [31].

5 Remarks in Press

After this paper was accepted and in addition to those already mentioned in the text, other works falling into the scope of this survey have appeared. We mention them here only briefly, referring to the papers themselves for more detail.

In Sect. 2, the classical framework for first passage times over a constant boundary of random walks and Lévy processes is displayed. A novel approach to this framework has been given in [57, 58], together with new and rather general results. The recent papers [7, 8, 10, 33] deal with the less classical framework of first passage time problems over a moving boundary, and provide some tight estimates on the distribution functions.

The relation between inverse exponential functionals and the persistence probability studied in Sect. 3.3 is based on Molchan's result [67] for continuous-time processes. In [6], this relation between inverse exponential functionals and the persistence probability is studied for discrete-time processes for the first time. The technique is applied to random walks in random sceneries (in any dimension) and to sums of stationary sequences with long-range dependence. We also mention that [35] studies persistence of additive functionals of Sinai's random walk in random environment obtaining the persistence exponent $\theta = (3 - \sqrt{5})/2$; where the value comes from a large deviation rate for the number of sign changes.

Finally, a general question connected to Sect. 3.3 has been studied in [41]: When does a stationary Gaussian process have an exponentially decreasing persistence probability? In the mentioned paper, rather general sufficient conditions are given.

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2. In general SUMMER SCHOOLS and other similar INTENSIVE COURSES are held to present mathematical topics that are close to the frontiers of recent research to an audience at the beginning or intermediate graduate level, who may want to continue with this area of work, for a thesis or later. This makes demands on the didactic aspects of the presentation. Because the subjects of such schools are advanced, there often exists no textbook, and so ideally, the publication resulting from such a school could be a first approximation to such a textbook. Usually several authors are involved in the writing, so it is not always simple to obtain a unified approach to the presentation.
   For prospective publication in LNM, the resulting manuscript should not be just a collection of course notes, each of which has been developed by an individual author with little or no coordination with the others, and with little or no common concept. The subject matter should dictate the structure of the book, and the authorship of each part or chapter should take secondary importance. Of course the choice of authors is crucial to the quality of the material at the school and in the book, and the intention here is not to belittle their impact, but simply to say that the book should be planned to be written by these authors jointly, and not just assembled as a result of what these authors happen to submit.
   This represents considerable preparatory work (as it is imperative to ensure that the authors know these criteria before they invest work on a manuscript), and also considerable editing work afterwards, to get the book into final shape. Still it is the form that holds the most promise of a successful book that will be used by its intended audience, rather than yet another volume of proceedings for the library shelf.

3. Manuscripts should be submitted either online at www.editorialmanager.com/lnm/ to Springer's mathematics editorial, or to one of the series editors. Volume editors are expected to arrange for the refereeing, to the usual scientific standards, of the individual contributions. If the resulting reports can be forwarded to us (series editors or Springer) this is very helpful. If no reports are forwarded or if other questions remain unclear in respect of homogeneity etc, the series editors may wish to consult external referees for an overall evaluation of the volume. A final decision to publish can be made only on the basis of the complete manuscript; however a preliminary decision can be based on a pre-final or incomplete manuscript. The strict minimum amount of material that will be considered should include a detailed outline describing the planned contents of each chapter.
   Volume editors and authors should be aware that incomplete or insufficiently close to final manuscripts almost always result in longer evaluation times. They should also be aware that parallel submission of their manuscript to another publisher while under consideration for LNM will in general lead to immediate rejection.

4. Manuscripts should in general be submitted in English. Final manuscripts should contain at least 100 pages of mathematical text and should always include
   - a general table of contents;
   - an informative introduction, with adequate motivation and perhaps some historical remarks: it should be accessible to a reader not intimately familiar with the topic treated;
   - a global subject index: as a rule this is genuinely helpful for the reader.

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5. Careful preparation of the manuscripts will help keep production time short besides ensuring satisfactory appearance of the finished book in print and online. After acceptance of the manuscript authors will be asked to prepare the final LaTeX source files and also the corresponding dvi-, pdf- or zipped ps-file. The LaTeX source files are essential for producing the full-text online version of the book. For the existing online volumes of LNM see:
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