

Blowing Up Solutions to the Zakharov System for Langmuir Waves

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Abstract Langmuir waves take place in a quasi-neutral plasma and are modeled by the Zakharov system. The phenomenon of collapse, described by blowing up solutions, plays a central role in their dynamics. We present in this article a review of the main mathematical properties of blowing up solutions. They include conditions for blowup in finite or infinite time, description of self-similar singular solutions and lower bounds for the rate of blowup of certain norms associated with the solutions.

1 Introduction

Langmuir waves take place in a non-magnetized or weakly magnetized plasma and are described by the Zakharov system [1]

$$i\partial_t \mathbf{E} - \alpha \nabla \times (\nabla \times \mathbf{E}) + \nabla(\nabla \cdot \mathbf{E}) = n\mathbf{E}, \quad (1)$$

$$\partial_{tt} n - \Delta n = \Delta |\mathbf{E}|^2. \quad (2)$$

Equation (2) originates from the hydrodynamic system

$$n_t + \nabla \cdot \mathbf{v} = 0, \quad (3)$$

$$\mathbf{v}_t + \nabla n = -\nabla |\mathbf{E}|^2, \quad (4)$$

governing ion sound waves. $\mathbf{E}(\mathbf{x}, t)$ is the complex envelope of the electric field oscillations. $n(\mathbf{x}, t)$ denotes the fluctuations of density of ions and $\mathbf{v}(\mathbf{x}, t)$ their velocity, with $\mathbf{x} \in \mathbb{R}^d$, in dimension $d = 2$ or 3 . The parameter α in (1) is defined as the square ratio of the light speed and the electron Fermi velocity and

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is usually large. A simplified system of equations is obtained in the electrostatic limit ($\alpha \rightarrow \infty$) expanding the electric field in the form $\mathbf{E} = \nabla\psi + (1/\alpha)\mathbf{E}_1 + \dots$. Substituting this expansion in (1) and taking the divergence of the equation gives a system describing the interaction of the electrostatic potential with the plasma density [1, 2]

$$\Delta(i\partial_t\psi + \Delta\psi) = \nabla \cdot (n\nabla\psi), \quad (5)$$

$$\partial_{tt}n - \Delta n = \Delta(|\nabla\psi|^2). \quad (6)$$

A further simplification leads to

$$i\psi_t + \Delta\psi = n\psi, \quad (7)$$

$$n_{tt} - \Delta n = \Delta|\psi|^2. \quad (8)$$

with

$$n_t + \nabla \cdot \mathbf{v} = 0, \quad (9)$$

$$\mathbf{v}_t + \nabla n = -\nabla|\psi|^2, \quad (10)$$

usually called the scalar Zakharov system. Introducing the hydrodynamic potential U such that $\mathbf{v} = -\nabla U$, (4) becomes

$$\partial_t U = n + |\mathbf{E}|^2. \quad (11)$$

Heuristic derivations of the Zakharov system can be found in [3, 4]. Viewing the plasma as a two interpenetrating fluids (electrons and ions), the Zakharov system (7)–(9) can be obtained using a multiple-scale modulation analysis [5]. A rigorous derivation of the scalar model is given in [6] using techniques of geometric optics and semi-classical calculus.

Invariance properties of the system by simple transformations lead to several conserved quantities. In particular, if (\mathbf{E}, n) is a smooth solution of (7)–(10), the wave energy $N = |\mathbf{E}|_{L^2}^2$ and the Hamiltonian

$$H = \alpha|\nabla \times \mathbf{E}|_{L^2}^2 + |\nabla \cdot \mathbf{E}|_{L^2}^2 + \frac{1}{2}|n|_{L^2}^2 + \frac{1}{2}|\nabla U|_{L^2}^2 + \int n|\mathbf{E}|^2 \, d\mathbf{x} \quad (12)$$

are conserved. Other invariants are the linear and angular momenta

$$\mathbf{P} = \int \left(\frac{i}{2} \sum_j (E_j \nabla E_j^* - E_j^* \nabla E_j) + n\mathbf{v} \right) d\mathbf{x} \quad (13)$$

and

$$\mathbf{M} = \int (i\mathbf{E} \times \mathbf{E}^* + \mathbf{x} \times \mathbf{P}) \, d\mathbf{x}. \quad (14)$$

Modulational instability leads to the formation of regions where the density of the plasma is very low. In these regions referred to as cavities, high-frequency oscillations of the electric field are trapped. Their nonlinear evolution gives rise to the collapse of the cavities and a strong amplification of the amplitude of the oscillations of the electric field. Heuristic arguments and numerical simulations show that, for large enough initial conditions, solutions blowup in a finite time both in two and three dimensions (see [5] for a review).

In this article, we present an overview of mathematical results and open questions concerning blowing up solutions for the scalar Zakharov model. We also discuss the extension of some of the features of blowup to the Vectorial Zakharov system for which very few rigorous are known apart from local well-posedness and global well-posedness under the assumption of small enough initial conditions.

2 The Scalar Zakharov System

We consider the scalar Zakharov system (7) and (8) with initial conditions

$$\psi(\mathbf{x}, 0) = \psi_0(\mathbf{x}), \quad n(\mathbf{x}, 0) = n_0(\mathbf{x}), \quad n_t(\mathbf{x}, 0) = n_1(\mathbf{x}). \quad (15)$$

The conserved quantities are:

- the wave energy

$$N = |\psi|_{L^2}^2, \quad (16)$$

- the linear momentum

$$\mathbf{P} = \int \left(\frac{i}{2} (\psi \nabla \psi^* - \psi^* \nabla \psi) + n \mathbf{v} \right) d\mathbf{x}, \quad (17)$$

- the angular momentum

$$\mathbf{M} = \int \mathbf{x} \times \mathbf{P} d\mathbf{x}, \quad (18)$$

and

- the Hamiltonian

$$H = \int \left(|\nabla \psi|^2 + n |\psi|^2 + \frac{1}{2} |\mathbf{v}|^2 + \frac{1}{2} n^2 \right) d\mathbf{x}. \quad (19)$$

There is a large literature devoted to the local and global well-posedness of the initial value problem. Earlier works concern smooth solutions, in particular solutions with finite energy (Hamiltonian) [7–13]. More recently, there has been

an interest in solutions with lower regularity assumptions and in particular in solutions with infinite energy [14–19]. Associated to the long time existence theory are the important questions of scattering theory, existence of wave operators [20], and precise decay of solutions for large time. In particular, in three dimensions, it is proved in [21] that, if the initial conditions are small and localized, then $\sup_x |\psi(t)| \leq C|t|^{-7/6-}$, $\sup_x |n(t)| \leq C|t|^{-1}$, and the solution (ψ, n) scatters to a solution to the associated linear problem as $|t| \rightarrow \infty$. Here the notation $7/6 -$ means $7/6 - \varepsilon$, for any $\varepsilon > 0$.

We denote by $H^k(\mathbb{R}^d)$ the Sobolev space of functions f such that f and its derivatives of order p , $|p| \leq k$, are bounded in the L^2 -space. It is also convenient to define the product space

$$H_k = H^k(\mathbb{R}^d) \times H^{k-1}(\mathbb{R}^d) \times H^{k-2}(\mathbb{R}^d). \quad (20)$$

The energy space corresponds to H_1 .

2.1 Blowup in Finite or Infinite Time

A central tool in the theory of blowup for the Nonlinear Schrödinger (NLS) equation

$$i\partial_t \psi + \Delta \psi + |\psi|^{2\sigma} \psi = 0, \quad \psi(\mathbf{x}, t) = \psi_0(\mathbf{x}) \quad (21)$$

is the variance identity

$$\frac{d^2}{dt^2} \int |\mathbf{x}|^2 |\psi|^2 \, d\mathbf{x} = 8H_{\text{NLS}} - 4 \frac{d\sigma - 2}{\sigma + 1} \int |\psi|^{2\sigma+2} \, d\mathbf{x},$$

where $H_{\text{NLS}} = \int (|\nabla \psi|^2 - (\sigma + 1)|\psi|^{2\sigma+2}) \, d\mathbf{x}$ is the NLS Hamiltonian. Under the assumption that the initial condition ψ_0 is in $H^1(\mathbb{R}^d)$ has finite variance and $H_{\text{NLS}}(\psi_0) < 0$, the solution of (21) blows up in a finite time if $\sigma d \geq 2$. For the Zakharov system, the usual variance $\int |\mathbf{x}|^2 |\psi|^2 \, d\mathbf{x}$ can be replaced by the quantity

$$\mathcal{V}(t) = \frac{1}{4} \int |\mathbf{x}|^2 |\psi|^2 \, d\mathbf{x} + \int_0^t \int (\mathbf{x} \cdot \mathbf{v}) n \, d\mathbf{x} \, dt \quad (22)$$

which is well defined for functions in the space

$$\Sigma' = \left\{ (\psi, n, \mathbf{v}) \in H_1, \int (|\mathbf{x}|^2 |\psi|^2 + |\mathbf{x}|(|n|^2 + |\mathbf{v}|^2)) \, d\mathbf{x} < \infty \right\}. \quad (23)$$

The function $\mathcal{V}(t)$ and satisfies

$$\frac{d^2 \mathcal{V}}{dt^2}(t) = dH - (d-2)|\nabla \psi|_{L^2}^2 - (d-1)|\mathbf{v}|_{L^2}^2, \quad (24)$$

where H is the Hamiltonian defined in (19). In dimension $d \geq 2$, one has $(d^2/dt^2)\mathcal{V} < 0$ if the initial conditions are such that $H < 0$. However, one cannot conclude on existence of blowup solutions because, unlike the NLS case, \mathcal{V} does not have a fixed sign. In particular, in dimension 2, Merle [22] proved that it tends to $-\infty$ as the singularity is approached. Nevertheless, one can get partial results, under the assumption of radial symmetry. Indeed in this case, there is a useful result referred to as the Strauss Lemma [23] which gives an upper bound of the sup norm of a function in terms of its H^1 -norm far from the origin. Namely, if f is a radially symmetric function in $H^1(\mathbb{R}^d)$ with $d \geq 2$, then, for any $R > 0$,

$$|f|_{L^\infty(|\mathbf{x}|>R)}^2 \leq CR^{-d+1} |\nabla f|_{L^2(|\mathbf{x}|>R)} |f|_{L^2(|\mathbf{x}|>R)}. \quad (25)$$

The radial assumption has been useful in other contexts such as in existence and scattering theory, where it allows a larger range of parameters for linear estimates of Strichartz type [24].

The method consists in modifying the quadratic weight $|\mathbf{x}|^2$ in \mathcal{V} defined in (22) by a smooth function $p(\mathbf{x})$ that behaves like $|\mathbf{x}|^2$ near the origin and like $|\mathbf{x}|$ at infinity, and by considering the time derivative $y(t) = -(d/dt)\mathcal{U}$ of the modified variance

$$\mathcal{U}(t) = \frac{1}{2} \int p(\mathbf{x}) |\psi|^2 \, d\mathbf{x} + \int_0^t \int (\nabla p \cdot \mathbf{v}) n \, d\mathbf{x} \, dt, \quad (26)$$

as in the case of solutions of the NLS equation with infinite variance. However, the modification of the weight induces additional terms in the time evolution of the function $y(t)$ that need to be estimated. For this purpose, one uses a sequence of rescaled weights $p_m(|\mathbf{x}|) = m^2 p(\mathbf{x}/m)$, and proves that the additional contributions are controlled for m sufficiently large. More precisely, one proves that

$$y_m(t) = -\Im \int ((\nabla p_m \cdot \nabla \psi) \psi^* - (\nabla p_m \cdot \mathbf{v}) n) \, d\mathbf{x} \quad (27)$$

satisfies, for m sufficiently large

$$y_m(t) \geq \frac{d}{2} |H| t. \quad (28)$$

On the other hand, the function $y_m(t)$ is controlled by the norm of the solution in the energy space, namely

$$|y_m(t)| \leq C(|\psi_0|_{L^2}^2 + |\nabla \psi|_{L^2}^2 + |\mathbf{v}|_{L^2}^2 + |n|_{L^2}^2). \quad (29)$$

We have the following result proved by Merle in [25]:

Theorem 1. *Consider the Zakharov system (7), (9), (10) in dimension $d = 2$ or $d = 3$ with initial conditions in the space Σ' . Assume that there exists a*

smooth solution (ψ, n, \mathbf{v}) during an interval of time $[0, t_0]$. In particular, its mass and Hamiltonian are conserved and its variance \mathcal{U} is well defined. Assume in addition that the solution is radially symmetric and its Hamiltonian $H < 0$. Then, either $|\psi|_{H^1} + |n|_{L^2} + |\mathbf{v}|_{L^2} \rightarrow \infty$ as $t \rightarrow t^*$ with t^* finite, or (ψ, n, \mathbf{v}) exists for all time and $|\psi|_{H^1} + |n|_{L^2} + |\mathbf{v}|_{L^2} \rightarrow \infty$ as $t \rightarrow \infty$.

Remark 1. An open question is the extension of this analysis to solutions that are not radially symmetric, and furthermore, to solutions to the full vector Zakharov system. Based on numerical observations, it is believed that blowup does indeed occur in a finite time for general initial conditions with negative Hamiltonian.

2.2 Self-similar Blowing Up Solutions

2.2.1 Dimension $d = 2$

Unlike for the NLS equation, there is no conformal mapping for the two-dimensional Zakharov system. Nevertheless one can construct exact self-similar blowing up solutions that have the form in the form [26]

$$\begin{aligned} \psi(x, t) &= \frac{1}{a(t^* - t)} P\left(\frac{|x|}{a(t^* - t)}\right) \\ &\quad \times \exp\left(i\left(\theta + \frac{1}{a^2(t^* - t)} - \frac{|x|^2}{4(t^* - t)}\right)\right), \end{aligned} \quad (30)$$

$$n(x, t) = \frac{1}{a^2(t^* - t)^2} N\left(\frac{|x|}{a(t^* - t)}\right), \quad (31)$$

where (P, N) are real functions satisfying the system of ODEs

$$\Delta P - P - NP = 0, \quad (32)$$

$$a^2(\eta^2 N_{\eta\eta} + 6\eta N_\eta + 6N) - \Delta N = \Delta P^2, \quad (33)$$

with η being the rescaled independent variable and $a > 0$ a free parameter. Gnanogbo and Merle have rigorously studied the system (32) and (33) in [11, 12]. We summarize below the most important properties. When $a = 0$, $N = -P^2$ and (32) becomes

$$\Delta P - P + P^3 = 0. \quad (34)$$

It is known that (34) has an infinite number of radially symmetric solutions that decay exponentially at infinity, only one of them, denoted R and called the NLS the ground state, (also known as the Townes soliton) is strictly positive and monotone decreasing (see, for example, [27, 28]). It plays a central role in the study of NLS equations.

If the coefficient a in (33) is sufficiently small, there exists a solution (P_a, N_a) in $H^1 \times L^2$ with $P_a > 0$. This solution is in fact C^∞ and its derivatives of order k satisfy the decay properties

$$|P^{(k)}(\eta)| \leq c_k e^{-\delta\eta}, \quad |N^{(k)}(\eta)| \leq \frac{c_k}{|\eta|^{k+3}}$$

for large η . When the parameter a is small, the solution (P_a, N_a) is constructed by a continuation method from the solution $(R, -R^2)$ corresponding to $a = 0$.

Furthermore, for any value c strictly larger than the L^2 -norm of the NLS ground state R , there exists a_c such that for any $a < a_c$, there is a unique solution (P_a, N_a) in $H^1 \times L^2$ with $P_a > 0$ and $|P_a|_{L^2} < c$.

Numerical simulations show that for a large class of radially symmetric initial conditions having a strictly negative Hamiltonian, the solutions display a self-similar collapse as $t \rightarrow t^*$ as described by (30) and (31) [29, 30]. The coefficient a in the equation for the limiting profiles (P, N) depends on the initial conditions. When considering a sequence of initial conditions with an initial L^2 -norm of ψ_0 decreasing to $|R|_{L^2}^2$, where R is the NLS ground state, it was observed that the computed value of the coefficient a tends to zero. In this limit, the self-similar profile becomes (strongly) subsonic and tends to the NLS ground state R . This limit is delicate, since solutions of the scalar Zakharov equation with critical norm $|\psi_0|_{L^2}^2 = |R|_{L^2}^2$ remain smooth for all time [11, 12]. Indeed, unlike the NLS equation, there are no minimal blowing up solutions to the 2D Zakharov system. For initial conditions in the energy space such that $|\psi_0|_{L^2} \leq |R|_{L^2}$ solutions remain in the energy space for all times. The case $|\psi_0|_{L^2} < |R|_{L^2}$ is straightforward and follows the NLS analysis [8, 31]. When $|\psi_0|_{L^2} = |R|_{L^2}$, the global well-posedness property is very specific to the Zakharov system.

Finally, when numerical simulations are performed with anisotropic initial conditions with negative Hamiltonian, it was observed that the solutions become isotropic near collapse with the same limiting profiles as those obtained with isotropic initial conditions [30].

2.2.2 Dimension $d = 3$

In three dimensions, there are no known explicit blowing up solutions. Self-similar solutions exist only asymptotically close to collapse and have the universal form [26, 32]

$$\psi(x, t) \sim \frac{1}{(t^* - t)} P\left(\frac{|x|}{\sqrt{3}(t^* - t)^{2/3}}\right) \exp(i(t^* - t)^{-1/3}), \quad (35)$$

$$n(x, t) \sim \frac{1}{3(t^* - t)^{4/3}} N\left(\frac{|x|}{\sqrt{3}(t^* - t)^{2/3}}\right), \quad (36)$$

where $P(\eta)$ and $N(\eta)$ are radially symmetric scalar functions satisfying the coupled system of ODEs

$$\Delta P - P - NP = 0, \quad (37)$$

$$\frac{2}{9}(2\eta^2 N_{\eta\eta} + 13\eta N_\eta + 14N) = \Delta P^2. \quad (38)$$

This type of blowup is referred to as supersonic collapse, because, when substituting the expressions (35) and (36) into the Zakharov system, the pressure term Δn is of lower order than $\partial_{tt}n$.

Note that, unlike the 2D case, there is no free parameter in the system. As discussed in [26] and proved more recently in [33], there exists an infinite number of solutions (P_k, N_k) to (37) and (38) such that, for all k ,

$$0 < P_k(\eta) < P_{k+1}(\eta) \quad \text{and} \quad N_k(\eta) < 0. \quad (39)$$

The profiles P_k decay exponentially $|P_k(\eta)| \leq C_k e^{-\delta\eta}$ for η large, while the $|N_k| \leq C_k/(1 + \eta^2)$ decay algebraically.

The values of P and N at the origin satisfy $N_k(0) = 9P_k(0)^2/(14 - 9P_k^2(0))$. We have also $P'(0) = N'(0) = 0$ due to the radial symmetry. The pair (P_k, N_k) is thus characterized by the value $P_k(0)$. It is proved in [33] that there exists a sequence $\alpha_k = \frac{1}{3}\sqrt{2k(4k+3)} > 0$ such that the values $P_k(0)$ are ordered as $\alpha_k < P_k(0) < \alpha_{k+1}$. It is of interest to see how the values α_k arise in the analysis. They appear when one writes the Taylor series expansion of P and N near the origin:

$$P(\eta) = \sum_{i=0}^{\infty} a_i \eta^{2i}, \quad N(\eta) = \sum_{i=0}^{\infty} b_i \eta^{2i}. \quad (40)$$

The series have only even powers because P and N are radially symmetric. From the substitution of the Taylor series into the system (37) and (38), one gets two relations between the coefficients a_i, b_i . The values α_i appear when solving the equation for the coefficient a_i :

$$(\alpha_i^2 - a_0^2)a_i = F(a_0, \dots, a_{i-1}, b_0, \dots, b_{i-1}) \quad (41)$$

when solving for the coefficients. In order to have well-defined coefficients and an analytic solution, $P(0)$ which identifies to a_0 should be different from the α_i . In [33], it is proved that there is at least one solution P_k with initial value $P_k(0) \in (\alpha_k, \alpha_{k+1})$ which is strictly positive and decays to 0 at infinity. Numerically, we found (at least for the first few that we computed) that there is only one. Figure 1 shows the first four pairs of solutions computed numerically by a shooting method (with the shooting parameter being $P_k(0)$). The values $P_k(0)$ (for $k = 1, \dots, 4$) are:

$$P_1(0) \approx 1.38, \quad P_2(0) \approx 2.43, \quad P_3(0) \approx 3.42, \quad P_4(0) \approx 4.40. \quad (42)$$

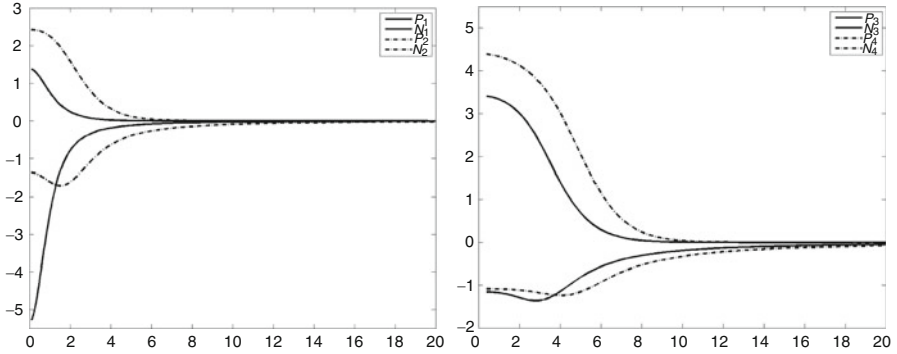


Fig. 1 Solutions (P_k, N_k) of (37) and (38) for $k = 1, \dots, 4$. Left: solid line— (P_1, N_1) , dashed line— (P_2, N_2) corresponding to initial values $P_1(0)$ and $P_2(0)$ in (42), respectively. Right: solid line— (P_3, N_3) , dashed line— (P_4, N_4) corresponding to initial values $P_3(0)$ and $P_4(0)$, respectively

Like in the 2D case, the dynamical stability of the asymptotically self-similar solutions to the 3D Zakharov system for both radially symmetric and anisotropic initial conditions was studied numerically in [30]. It was observed that for a large class of data, blowup solutions asymptotically display a self-similar collapse described by the above solutions. The profiles identify to the first mode (P_1, N_1) solution of (37) and (38) that has the lowest value at the origin, and for which N_1 is monotone increasing.

Remark 2. There is no rigorous proof of dynamic stability of the (2D) self-similar or (3D) asymptotically self-similar solutions even for well-prepared initial conditions (with or without radial symmetry) chosen close to the profiles (P, N) solutions of the ODE systems (32) and (33) or (37) and (38).

2.3 Lower Bounds for Rate of Blowup

2.3.1 Scale Invariance, Criticality, and Local Well-Posedness

An important aspect in the analysis of dispersive equations is the notion of criticality. It is closely related to the invariance properties of the equation. For example, the NLS equation (21) is invariant under the scaling transformation $\psi(\mathbf{x}, t) \rightarrow \psi_\lambda(\mathbf{x}, t) = \lambda^{1/\sigma} \psi(\lambda \mathbf{x}, \lambda^2 t)$. It is said to be \dot{H}^s -critical if the (homogeneous) H^s -norm is unchanged under the above scaling transformation. The corresponding critical Sobolev exponent for NLS is thus $s_c = d/2 - 1/\sigma$. The notion of criticality is not straightforward for the Zakharov system because the Schrödinger equation and the wave equation have different scale invariances. In [14], criticality is defined by considering the scaling

$$\psi \rightarrow \psi_\lambda = \lambda^{3/2}\psi(\lambda\mathbf{x}, \lambda^2t), \quad n \rightarrow n_\lambda = \lambda^2n(\lambda\mathbf{x}, \lambda^2t) \tag{43}$$

that would leave the Zakharov system invariant in the absence of the term Δn . This is indeed the relevant scaling to study blowing up solutions of the three-dimensional Zakharov system as we have seen in the previous section.

In relation to the initial value problem, the Sobolev space with critical exponent often corresponds to the space with minimal regularity in which the problem is locally well-posed. For the Zakharov system, the critical values for the initial value problem in $H^k \times H^l \times H^{l-1}$ are $k = d/2 - \frac{3}{2}$ and $l = d/2 - 2$. Note that $k - l = \frac{1}{2}$, while one would have $k - l = 1$ in the classical setting of the energy space H_1 . We now summarize the well-posedness results from the works of Ginibre et al. [14], Colliander et al. [17], Bejenaru et al. [18], and Bejenaru and Herr [19]. For ill-posedness results in dimension one, see [34].

Theorem 2. *In dimension $d = 1$, the Zakharov system is locally well-posed in $H^k \times H^l \times H^{l-1}$, provided that $-\frac{1}{2} < k - l \leq 1, 2k \geq l + \frac{1}{2} \geq 0$. Furthermore, global well-posedness holds in the largest space in which local well-posedness holds, that is $L^2 \times H^{-1/2} \times H^{-3/2}$.*

In dimension 2, it is locally well-posed in the critical space $L^2 \times H^{-1/2} \times H^{-3/2}$, and in dimension 3, it is locally well-posed in $H^\varepsilon \times H^{-1/2+\varepsilon} \times H^{-3/2+\varepsilon}$ which is also, up to arbitrarily small ε the critical space.

Finally, in dimension $d \geq 4$, the whole range of subcritical values $k > d/2 - \frac{3}{2}$ and $l > d/2 - 2$ is covered by the theorem as long as $l \leq k \leq l + 1$ and $2k - l - 1 > d/2 - 2$.

2.3.2 Finite Energy Solutions: The Two-Dimensional Case

The next theorem is due to Merle [22]. It concerns solutions of the 2D scalar Zakharov system with initial conditions (ψ_0, n_0, n_1) in the energy space H_1 , thus having a finite Hamiltonian. Assume that there exists a finite time t^* such that

$$|\nabla\psi(t)|_{L^2} + |n(t)|_{L^2} + |\mathbf{v}(t)|_{L^2} \rightarrow \infty \quad \text{as } t \rightarrow t^*.$$

The question is to determine at what rate these norms become infinite as t approaches t^* .

Theorem 3. *Assume that the solution (ψ, n) to the 2D scalar Zakharov system blows up in the energy space H_1 at a finite time t^* . Then there exist constants $c_1 > 0$ and $c_2 > 0$ depending only on $|\psi_0|_{L^2}$ such that for t close to t^* ,*

$$|\nabla\psi(t)|_{L^2} \geq \frac{c_1}{t^* - t}, \tag{44}$$

$$|n(t)|_{L^2} \geq \frac{c_2}{t^* - t}. \tag{45}$$

More precisely, the constants c_1 and c_2 scale like $(|\psi_0|_{L^2}^2 - |R|_{L^2}^2)^{-1/2}$ where R is the NLS ground state.

Remark 3. This rate is optimal in the sense that the self-similar solutions (30) and (31) satisfy

$$|\nabla\psi(t)|_{L^2} = \frac{1}{a(t^* - t)} |\nabla P|_{L^2}, \quad |n(t)|_{L^2} = \frac{1}{a(t^* - t)} |N|_{L^2}. \quad (46)$$

and thus blow up exactly at the rate stated in theorem. Notice also that the theorem provides the blowup rate for the two quantities $|\nabla\psi(t)|_{L^2}$ and $|n(t)|_{L^2}$ separately but does not give information on $|\mathbf{v}(t)|_{L^2}$.

The derivation of this result is based on scaling properties and conservation of the Hamiltonian. One defines the rescaled functions $\tilde{\psi}(\mathbf{x}, s)$, $\tilde{n}(\mathbf{x}, s)$, $\tilde{\mathbf{v}}(\mathbf{x}, s)$ (where t is seen as a parameter)

$$\tilde{\psi}(\mathbf{x}, s) = \frac{1}{\lambda(t)} \psi\left(\frac{\mathbf{x}}{\lambda(t)}, t + \frac{s}{\lambda(t)}\right), \quad (47)$$

$$\tilde{n}(\mathbf{x}, s) = \frac{1}{\lambda^2(t)} n\left(\frac{\mathbf{x}}{\lambda(t)}, t + \frac{s}{\lambda(t)}\right), \quad (48)$$

$$\tilde{\mathbf{v}}(\mathbf{x}, s) = \frac{1}{\lambda^2(t)} \mathbf{v}\left(\frac{\mathbf{x}}{\lambda(t)}, t + \frac{s}{\lambda(t)}\right). \quad (49)$$

where the scaling factor

$$\lambda(t) = \int (|\nabla\psi|^2 + \frac{1}{2}n^2 + \frac{1}{2}|\mathbf{v}|^2) \, d\mathbf{x}, \quad (50)$$

is associated with the energy norm. Notice that the scaling of the time variable corresponds to the wave equation rather than to the Schrödinger equation. At $s = 0$,

$$\int (|\nabla\tilde{\psi}(0)|^2 + \frac{1}{2}\tilde{n}(0)^2 + \frac{1}{2}|\tilde{\mathbf{v}}(0)|^2) \, d\mathbf{x} = 1. \quad (51)$$

Under the hypothesis of the theorem, $\lambda(t) \rightarrow \infty$ as t approaches t^* .

The analysis consists in establishing bounds for the individual quantities $|\nabla\tilde{\psi}(0)|$, $\tilde{n}(0)$, $\tilde{\mathbf{v}}(0)$ and estimates of $\tilde{\psi}(s)$, $\tilde{n}(s)$ and $\tilde{\mathbf{v}}(s)$ as $t \rightarrow t^*$. It uses delicate compactness arguments allowing the identification of limiting quantities as t goes to t^* . This approach, initiated in [22] and now known as profile decomposition, has led to many breakthroughs in various fields of dispersive PDEs.

2.3.3 Infinite Energy Solutions

We present in this section another approach for the derivation of a lower bound for the rate of blowing up solutions. It is more general, but less precise than the one presented in the previous section. It applies to the problem in dimensions two or three, and to initial conditions that may or may not have a finite Hamiltonian. The result below was established in 3D in [35]. The 2D result follows the same line of proof.

Theorem 4. *Let the initial data $(\psi(0), n(0), n_t(0))$ be in $\mathcal{H}_\ell := H^{\ell+1/2}(\mathbb{R}^d) \times H^\ell(\mathbb{R}^d) \times H^{\ell-1}(\mathbb{R}^d)$, with the condition $0 \leq \ell \leq \frac{1}{2}$ if $d = 2$ and $0 \leq \ell \leq 1$ if $d = 3$. Assume that the solution (ψ, n, n_t) blows up in a finite time t^* , that is, as t approaches t^* , $\|\psi(t)\|_{H^{\ell+1/2}} + \|n(t)\|_{H^\ell} + \|n_t(t)\|_{H^{\ell-1}} \rightarrow \infty$. Then, the rate of blowup of the Sobolev norms satisfies the lower bound estimate*

$$\|\psi(t)\|_{H^{\ell+1/2}} + \|n(t)\|_{H^\ell} + \|n_t(t)\|_{H^{\ell-1}} > C(t^* - t)^{-\theta_\ell} \quad (52)$$

with $\theta_\ell = \frac{1}{4}(4 - d + 2\ell)^-$, $d = 2$ or $d = 3$.

In the above formula, the notation a^- means $(a - \varepsilon)$ for arbitrarily small $\varepsilon > 0$.

Remark 4. Unlike the method of the previous section, this approach provides a lower bound for the sums of the norms of ψ, n, n_t but not for the norms separately.

Remark 5. In 3D, the lower bound (52) is probably not optimal. Indeed, the homogeneous $\dot{H}^{\ell+1/2}$ -norm of ψ and the homogeneous \dot{H}^ℓ -norm of n in the expression of the asymptotic solution (35) and (36) both blowup at a faster rate, namely $\frac{1}{3}(1 + 2\ell)$. Note that these norms blowup at the same rate in 3D, showing that the space $H^{\ell+1/2} \times H^\ell$ is appropriate for the analysis.

Remark 6. In 2D, the norms $\|\psi\|_{\dot{H}^k}$, with $k = \ell + 1$ and $\|n\|_{H^\ell}$ of the exact self-similar solutions (30) and (31) blow up at the same rate $(t^* - t)^{-(\ell+1)}$. Merle's work [22] gives the optimal rate when $k = 1$. Theorem 4 predicts rates of blowup in the space \mathcal{H}_ℓ . It gives almost the optimal rate of blowup for ψ when $\ell = 0$, but it is off then by $\frac{1}{2}$ for n .

Remark 7. In 3D, a particular result about blowup of a space-time norm $L_{x,t}^{q,r}$ of n is given in [33] under the assumption that blowup occurs in the energy space H_1 .

Assume that the solution $u := (\psi, n, n_t)$ exists during a finite time $|t| \leq T$ in the space \mathcal{H}_ℓ . There are two elements in the proof of the theorem above:

1. A local well-posedness estimate for u in the form of the one obtained by Ginibre–Tsutsumi–Velo [14],

$$\|u\|_{X_T} \leq C\|u_0\|_{\mathcal{H}_\ell} + CT^\theta \|u\|_{X_T}^2, \quad \theta > 0, \quad (53)$$

where $\|\cdot\|_{X_T}$ is a space-time norm that will be defined later. For the purpose of local well-posedness, it is not important to determine exactly the power θ , the

only requirement being that it is away from 0. On the other hand, in the blowup analysis, a key element is to maximize the power θ , because it leads to a better estimate for the lower bound of the blowup rate. We find an expression for θ that depends on ℓ , the order of the norm under consideration and increases with ℓ .

2. A classical contradiction argument introduced in [36] for semilinear heat equations and used in [37] for NLS equations that reverses the local well-posedness estimate into a blowup rate estimate.

Step 1 (Local Well-Posedness Estimate). Rewrite the wave equation (8) as two reduced wave equations for

$$w^\pm = n \pm i\omega^{-1}\partial_t n,$$

where $\omega = (-\Delta)^{1/2}$. The Zakharov system then becomes

$$i\partial_t \psi + \Delta \psi = (w^+ + w^-)\psi, \quad (54)$$

$$(i\partial_t \mp \omega)w^\pm = \pm\omega(|\psi|^2). \quad (55)$$

(ψ, w^\pm) solve (54) and (55) with initial data $(\psi_0, w_0^\pm) = (\psi_0, n_0 \pm i\omega^{-1}n_1)$ if and only if (ψ, n) solve (7) and (8) with initial data (ψ_0, n_0, n_1) .

In the analysis, one slightly modifies the above system by replacing the operator $\omega = (-\Delta)^{1/2}$ by $\omega_1 = (1 - \Delta)^{1/2}$ to avoid divergence at low wavenumbers. This leads to an additional term in the wave equation of the form $\langle \nabla \rangle^{-1} \mathfrak{R} w^\pm$, which is linear with a gain in derivatives, thus it is easily controlled (see [35]).

The solution of (54) and (55) is written in its Duhamel formulation. Since the solution is considered in a fixed interval $[-T, T]$, we introduce in addition a cut-off C^∞ function $\varphi(t) = 1$ for $|t| \leq 1$, $\varphi(t) = 0$ for $|t| \geq 2$, $0 \leq \varphi(t) \leq 1$, and define $\varphi_T(t) = \varphi(t/T)$, ($T \leq 1$). The initial value problem (54) and (55) on time interval $[-T, T]$ is equivalent to the system of integral equations

$$\psi(t) = \varphi_1(t)U(t)\psi_0 - i\varphi_T(t) \int_0^t U(t-s)\varphi_{2T}^2(w^+ + w^-)\psi(s) ds, \quad (56)$$

$$w^\pm(t) = \varphi_1(t)W(t)w_0 \pm i\varphi_T(t) \int_0^t W(t-s)\varphi_{2T}^2\omega(|\psi|^2) ds, \quad (57)$$

where $U(t) = e^{it\Delta}$, $W(t) = e^{\mp it\sqrt{-\Delta}}$ are the free Schrödinger and free reduced wave operators, respectively.

The space X_T in (53), in which the analysis performed, is a product of weighted Sobolev spaces, with space-time weights being the Fourier multipliers associated with the linear Schrödinger and linear reduced wave equation [38]. Namely, $X_T = X_S^{\ell+1/2,b} \times X_{W^\pm}^{\ell,b}$, with the norms given by

$$\begin{aligned}\|\psi\|_{X_S^{\ell+1/2,12,b}} &= \|\langle \xi \rangle^{\ell+1/2} \langle \tau + |\xi|^2 \rangle^b \hat{\psi}(\tau, \xi)\|_{L_{\tau,\xi}^2}, \\ \|w\|_{X_{W_{\pm}}^{\ell,b}} &= \|\langle \xi \rangle^{\ell} \langle \tau \pm |\xi| \rangle^b \hat{w}(\tau, \xi)\|_{L_{\tau,\xi}^2},\end{aligned}$$

where $\langle \xi \rangle = (1 + |\xi|^2)^{1/2}$ and $b > \frac{1}{2}$.

There are two distinct elements in the estimates, the linear estimates and the nonlinear ones.

Lemma 1 (Linear Estimates [14, Lemma 2.1]). *Consider the general linear equation*

$$iu_t - \Phi(-i\nabla)u = F \quad \text{on } [0, T] \times \mathbb{R}^d, \quad u(0) = u_0 \in H^s,$$

where Φ is a real valued function. Then for $\frac{1}{2} - \varepsilon < b \leq 1 - \varepsilon$, ($\varepsilon > 0$)

$$\|\varphi_T u\|_{X^{s,b}} \lesssim \|u_0\|_{H^s} + T^\varepsilon \|F\|_{X^{s,b-1+\varepsilon}}, \quad (58)$$

where the norm in $X^{s,b}$ is associated with the linear operator, namely $\|\cdot\|_{X^{s,b}} = \|\langle \xi \rangle^s \langle \tau + \Phi(\xi) \rangle^{b \cdot}(\tau, \xi)\|_{L_{\tau,\xi}^2}$.

An application of (58) to the solution (ψ, w^\pm) gives

$$\|\psi\|_{X_S^{\ell+1/2,b}} \lesssim \|\psi_0\|_{H^{\ell+1/2}} + T^\varepsilon \|\varphi_{2T}^2(w^+ + w^-)\psi\|_{X_S^{\ell+1/2,b-1+\varepsilon}}, \quad (59)$$

and

$$\|w^\pm\|_{X_{W_{\pm}}^{\ell,b}} \lesssim \|w_0\|_{H^\ell} + T^\varepsilon \|\varphi_{2T}^2 \omega |\psi|^2\|_{X_{W_{\pm}}^{\ell,b-1+\varepsilon}}. \quad (60)$$

We now explain how one gets an estimate for the nonlinear terms and produce higher powers of T . The goal is to establish

$$\|\varphi_{2T}^2 w^+ \psi\|_{X_S^{\ell+1/2,b-1+\varepsilon}} \lesssim T^\theta \|\varphi_{2T} w^+\|_{X_{W_+}^{\ell,b}} \|\varphi_{2T} \psi\|_{X_S^{\ell+1/2,b}}, \quad (61)$$

$$\|\varphi_{2T}^2 \omega |\psi|^2\|_{X_{W_+}^{\ell,b-1+\varepsilon}} \lesssim T^\theta \|\varphi_{2T} \psi\|_{X_S^{\ell+1/2,b}}^2, \quad (62)$$

with equivalent estimates for w^- . A classical argument is to consider the nonlinear terms on the Fourier side and use duality. This reduces (61) and (62) to showing the following inequalities

$$|N_1| \lesssim T^\theta \|v\|_2 \|v_1\|_2 \|v_2\|_2 \quad (63)$$

$$|N_2| \lesssim T^\theta \|v\|_2 \|v_1\|_2 \|v_2\|_2, \quad (64)$$

where

$$N_1 = \int \frac{\hat{v}(\xi_1 - \xi_2, \tau_1 - \tau_2) \hat{v}_1(\xi_1, \tau_1) \hat{v}_2(\xi_2, \tau_2) \langle \xi_1 \rangle^k}{\langle \tau + |\xi| \rangle^b \langle \tau_1 + |\xi_1|^2 \rangle^c \langle \tau_2 + |\xi_2|^2 \rangle^b \langle \xi_2 \rangle^k \langle \xi \rangle^l} d\xi_1 d\xi_2 d\tau_1 d\tau_2, \quad (65)$$

$$N_2 = \int \frac{\hat{v}(\xi_1 - \xi_2, \tau_1 - \tau_2) \hat{v}_1(\xi_1, \tau_1) \hat{v}_2(\xi_2, \tau_2) |\xi| \langle \xi \rangle^\ell}{\langle \tau + |\xi| \rangle^c \langle \tau_1 + |\xi_1|^2 \rangle^b \langle \tau_2 + |\xi_2|^2 \rangle^b \langle \xi_1 \rangle^k \langle \xi_2 \rangle^k} d\xi_1 d\xi_2 d\tau_1 d\tau_2. \quad (66)$$

Ginibre–Tsutsumi–Velo [14] showed the above estimates by a repeated application of an inequality obtained from Strichartz estimates and Hölder inequality in time (see [14, Lemmas 3.1–3.4]). Their analysis did not require an optimal power of θ , but needed it to be just large enough, so the final power of T was positive. They find

$$\left(b + 1 - \left(\frac{n}{2} + 1 \right) b_0 - \epsilon \right) \frac{1}{2b}.$$

For the rate of blowup analysis, we seek the optimal power of θ , and obtain estimates (63) and (64) with

$$\theta = b + 1 - \left(\frac{n}{2} + 1 - \ell \right) b_0 - \epsilon. \quad (67)$$

To remove the time cutoff from the right-hand side of (61) and (62), we recall

Lemma 2 ([14]).

$$\|\varphi_T u\|_{X^{s,b}} \leq CT^{-b+1/q} \|u\|_{X^{s,b}},$$

where $s \in \mathbb{R}$, $b \geq 0$, $q \geq 2$ and $bq > 1$.

Applying this twice (since the nonlinearity is quadratic) with $q = 2$ and combining estimates (59) and (60) with (61) and (62) gives the final estimate

$$\begin{aligned} \|\psi\|_{X_S^{\ell+1/2,b}} + \|n\|_{X_{W+}^{\ell,b}} + \|n_t\|_{X_{W+}^{\ell-1,b}} &\leq C(\|\psi_0\|_{H^{\ell+1/2}} + \|n_0\|_{H^\ell} + \|n_1\|_{H^{\ell-1}}) \\ &+ CT^{\theta_\ell} (\|\psi\|_{X_S^{\ell+1/2,b}} + \|n\|_{X_{W+}^{\ell,b}} + \|n_t\|_{X_{W+}^{\ell-1,b}})^2, \end{aligned} \quad (68)$$

with a power of θ_ℓ as stated in Theorem 4.

Step 2 (Contradiction Argument and Lower Bound).

Let us explain the contradiction argument for a general evolution PDE with a quadratic nonlinearity, and an initial data u_0 belonging to some space H . Suppose an a priori estimate of the form

$$\|u\|_{X_T} \leq C\|u_0\|_H + CT^\theta \|u\|_{X_T}^2, \quad \theta > 0 \quad (69)$$

holds, where $\|\cdot\|_{X_T}$ is some appropriate space-time norm. (This is the a priori estimate (68) with $u = (\psi, n, n_t)$ and $H = \mathcal{H}_\ell$.) Let

$$\mathcal{X}(T, M) = \{u : u(0) = u_0, \|u\|_{X_T} \leq M\}.$$

When performing an iteration argument in $\mathcal{X}(T, M)$, we would like to show

$$C\|u_0\|_H + CT^\theta M^2 \leq M \tag{70}$$

to keep the iterates in $\mathcal{X}(T, M)$. Local well-posedness follows if (70) holds (with, for example, $M = 2C\|u_0\|_H$, and T small enough so that $2CT^\theta M < 1$). The relation between the spaces X_T and H is that X_T must be imbedded in $C([-T, T], H)$ meaning that if u belongs to X_T , it must be a continuous function of $t \in [-T, T]$ with values in H .

Let t^* be the maximal time of existence of solutions, that is

$$t^* = \sup\{T : \|u\|_{X_T} < \infty\}.$$

The blowup hypothesis implies that t^* is finite. Returning to (70), let $0 < t < t^*$ and consider $u(t)$ as an initial condition. The following statement must hold:

If there exists some $M > 0$ such that $C\|u(t)\|_H + C(T-t)^\theta M^2 \leq M$, then $T < t^$.*

Or equivalently:

If $T \geq t^$, in particular $T = t^*$, then for all $M > 0$*

$$C\|u(t)\|_H + C(t^* - t)^\theta M^2 > M.$$

We now choose $M = 2C\|u(t)\|_H$, then $\frac{1}{2}M + C(t^* - t)^\theta M^2 > M$ or

$$C(t^* - t)^\theta M^2 > \frac{M}{2} \tag{71}$$

or equivalently

$$\|u(t)\|_H > c(t^* - t)^{-\theta}. \tag{72}$$

Hence, since we cannot continue the time of existence past time t^* , we have a lower bound for the blowup rate of the norm H as given by (72).

Note that the conclusion is about the rate of blowup of the norm H even though the iteration is performed in another norm. One just needs the other norm to embed into $C([-T, T], H)$.

3 The Vectorial Zakharov System

There is no rigorous analysis of blowing up solutions of the full vectorial Zakharov system (1)–(3) although wave collapse is expected when the initial conditions are large enough on the basis of numerical simulations and heuristic arguments (see [5] for review). Here we extend the results of Sect. 2.3.3.

Recalling that for a vector valued function \mathbf{E}

$$\Delta \mathbf{E} = \nabla(\nabla \cdot \mathbf{E}) - \nabla \times (\nabla \times \mathbf{E}),$$

we can write (1)–(3) as

$$i\partial_t \mathbf{E} + \alpha \Delta \mathbf{E} + (1 - \alpha) \nabla(\nabla \cdot \mathbf{E}) = n \mathbf{E}, \quad (73)$$

$$\partial_t n - \Delta n = \Delta |\mathbf{E}|^2. \quad (74)$$

The symbol of the Laplacian is $|\xi|^2$. This together with the one time derivative determines the $(\tau + |\xi|^2)$ weight in the $X^{s,b}$ norm for the NLS equation of the scalar Zakharov system. To determine the weight that should appear in the $X^{s,b}$ norm for the NLS equation of the vectorial Zakharov system, one needs to determine the symbol of the spatial linear operator appearing in the left-hand side of (73). A simple calculation leads to the matrices M_2 in two dimensions and M_3 in three dimensions given by

$$M_2 = (1 - \alpha) \begin{pmatrix} \xi_1^2 & \xi_1 \xi_2 \\ \xi_1 \xi_2 & \xi_2^2 \end{pmatrix} + \alpha |\xi|^2 I_{2 \times 2},$$

and

$$M_3 = (1 - \alpha) \begin{pmatrix} \xi_1^2 & \xi_1 \xi_2 & \xi_1 \xi_3 \\ \xi_1 \xi_2 & \xi_2^2 & \xi_2 \xi_3 \\ \xi_1 \xi_3 & \xi_2 \xi_3 & \xi_3^2 \end{pmatrix} + \alpha |\xi|^2 I_{3 \times 3}.$$

where $I_{d \times d}$ is the $d \times d$ unit matrix. It was observed by Tzvetkov [15] that the symbol of the operator given by matrix M_d is actually equivalent to the symbol of the Laplacian.

Lemma 3 ([15, Proposition 1]). *Let $d = 2, 3$. Then there exists a constant C such that*

$$|\xi|^2 I_{d \times d} \leq M_d \leq C |\xi|^2 I_{d \times d}.$$

Using this lemma, one can obtain a local well-posedness result for (73) and (74) [15] analogous to the scalar case, and a lower bound for the rate of blowup of Sobolev norms.

Theorem 5. Let $d = 2, 3$, and the initial data $(E(0), n(0), n_t(0))$ be in $\mathcal{H}_\ell := (H^{\ell+1/2}(\mathbb{R}^d))^d \times H^\ell(\mathbb{R}^d) \times H^{\ell-1}(\mathbb{R}^d)$, $0 \leq \ell \leq d/2 - \frac{1}{2}$. Assume that the solution (E, n, n_t) blows up in a finite time $t^* < \infty$. Then

$$\|E(t)\|_{H^{\ell+1/2}} + \|n(t)\|_{H^\ell} + \|n_t(t)\|_{H^{\ell-1}} > C(t^* - t)^{-\theta_\ell} \quad (75)$$

with $\theta_\ell = \frac{1}{4}(4 - d + 2\ell)^-$, in dimension $d = 2$ or $d = 3$.

Finally, as for the NLS equation, it is of interest to consider the influence of additional dispersive terms and their effect on blowing up solutions. In [39], Haas and Schukla consider the system

$$i\partial_t E - \alpha \nabla \times (\nabla \times E) + \nabla(\nabla \cdot E) = nE + \Gamma \nabla \Delta(\nabla \cdot E) \quad (76)$$

$$\partial_t n - \Delta n = \Delta |E|^2 - \Gamma \Delta^2 n. \quad (77)$$

which takes into account quantum corrections. The coefficient $\Gamma > 0$ is assumed to be very small. In [40], it is shown rigorously that quantum terms arrest collapse in two and three dimensions, for arbitrarily small values of the parameter Γ .

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