A New Method for the Integer Order Approximation of Fractional Order Models

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Abstract This paper is concerned with the finite–dimensional approximation of a fractional–order system represented in state–space form. To this purpose, resort is made to the Oustaloup method for approximating a fractional–order integrator by a rational filter. The dimension of the resulting integer–order model can be reduced using an efficient algorithm for the minimization of the L_2 norm of a weighted equation error. Two numerical examples are worked out to show how the desired approximation accuracy can be ensured.

Keywords Fractional–order models · Approximation · Oustaloup method · Model reduction · Equation error

1 Introduction

Non–integer order systems have been recently considered with increasing attention in the control literature because many plants can be described more satisfactorily by models of this kind [2, 8, 9, 13]. However, such systems are infinite–dimensional and their transfer function is irrational. Therefore, *ad hoc* methods and algorithms are needed to simulate their behaviour. Since the approaches based on the numerical solution of fractional differential equations are, in general, computationally hard, most techniques resort to the approximation, over suitably–defined frequency ranges, of these systems by means of integer–order models (see, e.g., [5, 12, 14]).

This paper considers a general (not necessarily commensurate) fractional–order system given in the state–space form. By applying the integer–order approximation

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of the fractional integrator operator $1/s^\alpha$ ($\alpha \in \mathbb{R}_+$) proposed in [14], a finite–dimensional state–space model with block companion state matrix is obtained. The sparsity of this matrix simplifies simulations. However, since the order of this model tends to be high, it has been suggested to approximate it using a method developed for finite–dimensional systems. For example, the model reduction method based on the Singular Value Decomposition has been used in [7] and the method based on the minimization of the unweighted L_2 norm of the impulse–response error has been used in [17]. Recently, the present authors have suggested to apply the iterative–interpolation algorithm for L_2 model reduction presented in [4]. In this paper, to reduce the dimensionality of the integer–order model, the more efficient weighted equation–error approach [3] is applied instead.

The rest of the paper is organized as follows. Section 2 briefly presents the formal description of non–integer order linear time–invariant (LTI) systems. Some recent approaches to the rational approximation of fractional operators and to model simplification are outlined and discussed in Sect. 3. The suggested approximation method is presented in Sect. 4. Two meaningful examples taken from the literature are worked out in Sect. 5 to show the performance of the suggested approximation. Some concluding remarks are drawn in Sect. 6.

2 Non–Integer Order Linear Systems Recap

Fractional–order calculus is a generalization of integer–order differentiation and integration. Many definitions of fractional–order differentiation and integration operators have been proposed. Especially successful have been those of Grünwald–Letnikov, Riemann–Liouville and Caputo [11]. The last one is most commonly used in engineering applications.

The Laplace transform of the fractional Caputo derivative $D^\alpha x(t)$ is

$$\mathscr{L}\{D^\alpha x(t)\} = s^\alpha \mathscr{L}\{x(t)\} - \sum_{i=0}^{[\alpha]-1} s^{\alpha-i-1} \frac{d^i x}{dt^i}(0), \tag{1}$$

where $[\alpha]$ denotes the integer part of α.
Consider a scalar[1] LTI fractional–order system described by the differential equation

$$y(t) + \sum_{i=1}^{n} a_i D^{\alpha_i} y(t) = \sum_{i=1}^{m} b_i D^{\beta_i} u(t), \tag{2}$$

where $a_i, b_i \in \mathbb{R}$, $D^\lambda = \dfrac{d^\lambda}{dt^\lambda}$, $\alpha_i, \beta_i \in \mathbb{R}_+$.

[1]This assumption is made to simplify the exposition. The case of MIMO systems can be treated in a similar way.

By applying (1)–(2) and assuming zero initial conditions, the system transfer function turns out to be

$$G(s) = \frac{b(s)}{a(s)} = \frac{\sum_{i=1}^{m} b_i s^{\beta_i}}{1 + \sum_{i=1}^{n} a_i s^{\alpha_i}}. \quad (3)$$

If all fractional orders are multiples of the same real number α (which qualifies the system as a commensurate fractional–order system), (3) can be written as

$$G(s) = \frac{\sum_{i=1}^{m} b_i (s^{\alpha})^i}{1 + \sum_{i=1}^{n} a_i (s^{\alpha})^i}. \quad (4)$$

The state–space model corresponding to (3) is

$$D^{(\alpha)}(x)(t) = Ax(t) + bu(t), \quad (5)$$
$$y(t) = cx(t) + du(t), \quad (6)$$

where

$$D^{(\alpha)}(x) = \left[D^{\alpha_1} x_1, D^{\alpha_2} x_2, \ldots, D^{\alpha_n} x_n \right]^T$$

and $A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^{n \times 1}, c \in \mathbb{R}^{1 \times n}$.
In the commensurate case, Eqs. (5) and (6) become

$$D^{\alpha}(x)(t) = Ax(t) + bu(t), \quad (7)$$
$$y(t) = cx(t) + du(t), \quad (8)$$

where

$$D^{\alpha}(x) = \left[D^{\alpha} x_1, D^{\alpha} x_2, \ldots, D^{\alpha} x_n \right]^T.$$

3 Fractional Order Model Simplification

The analysis of non–integer order models is made difficult by the irrational nature of their transfer function and by the infinite dimensionality of their state–space representations. Therefore, a number of methods have been proposed to simplify such models. Two alternative kinds of methods can be used in this regard:

1. the methods leading to a simpler model hat is still described by an irrational transfer function or an infinite–dimensional state–space representation,
2. the methods that approximate the non–integer order model by means of a finite dimensional one.

The first group of methods is useful for commensurate systems like (4): see, for example, [16, 18]. Indeed, in this case, by setting $s^\alpha = w$, a transfer function that is rational with respect to w is obtained:

$$\hat{G}(w) = \frac{\sum_{i=1}^{m} b_i w^i}{1 + \sum_{i=1}^{n} a_i w^i} \tag{9}$$

to which any order reduction method can then be applied. However, this approach does not guarantee the stability of the resulting model. An even more serious drawback is that this model may not be truly simpler than the original one. To show this, consider the fractional system put forth in [15] whose transfer function is

$$G(s) = \frac{s^{1.56} + 4}{s^{3.46} + 10s^{2.69} + 20s^{1.56} + 4}. \tag{10}$$

Since $\alpha = 1/100$, the rational transfer function (9) corresponding to (10) is of order 346. Even if its order could be reduced to 10, the denominator of this reduced transfer function will consists of 11 terms, whereas the denominator in (10) consists of only 4 terms. Hence the above approach can be successful when almost all coefficients a_i in (4) or in (9) are non–zero, as in the following example considered in [16, 18]:

$$G(s) = \frac{(s^{0.8} + 4)(s^{1.6} + 2s^{0.8} + 4)(s^{1.6} + 3s^{0.8} + 1)}{(s^{0.8} + 1)(s^{0.8} + 3)(s^{1.6} - 2s^{0.8} + 37)(s^{1.6} + 4s^{0.8} + 8)}. \tag{11}$$

The methods of the second type are usually based on the rational approximation of the operator s^α. Among the various approaches of this kind (see, e.g., [5, 9]), the most popular is almost certainly the one due to Oustaloup [10] by which the fractional differentiator operator s^α, $0 \le \alpha \le 1$, is replaced by a rational filter $\mathcal{D}^\alpha(s)$ whose zeros and poles are distributed over a frequency band $[\omega_m, \omega_M]$ centred at

$$\omega_u = \sqrt{\omega_m \omega_M}. \tag{12}$$

Precisely, the approximating filter is formed by the cascade of $2N + 1$ first–order cells:

$$\mathcal{D}^\alpha(s) = K_\alpha \prod_{k=-N}^{N} \frac{1 + \dfrac{s}{\omega_k'}}{1 + \dfrac{s}{\omega_k}}, \tag{13}$$

where ω_k' and ω_k are computed recursively according to

$$\omega_0' = \delta^{-\frac{1}{2}} \omega_u, \quad \omega_0 = \delta^{\frac{1}{2}} \omega_u,$$

$$\frac{\omega'_{k+1}}{\omega'_k} = \frac{\omega_{k+1}}{\omega_k} = \delta\,\eta > 1,$$

$$\frac{\omega_k}{\omega'_k} = \delta > 0, \quad \frac{\omega'_{k+1}}{\omega_k} = \eta > 0,$$

$$\omega'_{-N} = \eta^{\frac{1}{2}}\omega_m, \quad \omega'_N = \eta^{-\frac{1}{2}}\omega_M,$$

with [12]

$$\delta = \left(\frac{\omega_M}{\omega_m}\right)^{\frac{\alpha}{2N+1}}, \quad \eta = \left(\frac{\omega_M}{\omega_m}\right)^{\frac{1-\alpha}{2N+1}}.$$

The gain K_α is chosen so as to ensure that $\mathcal{D}^\alpha(s)$ has the same magnitude as s^α at ω_u. The number of filter cells is clearly related to the goodness of the approximation.

The fractional–order integrator operator $1/s^\alpha$ can be approximated in a way consistent with that adopted for the differentiator operator. Precisely, the approximation of the fractional integrator operator can be chosen [14] as

$$\mathcal{I}^\alpha(s) = \frac{K_\alpha}{s}\prod_{k=-N}^{N}\frac{1+\frac{s}{\omega'_k}}{1+\frac{s}{\omega_k}}, \tag{14}$$

which behaves (almost) like $1/s^\alpha$ in an interval $[\omega_m, \omega_M]$.

Functions $\mathcal{D}^\alpha(s)$ and $\mathcal{I}^\alpha(s)$ allow us to find rational models of practically any fractional system. However, the direct application of these operators often leads to high–dimensional models. Consider again the fractional transfer function (10). By setting $\omega_m = 10^{-3}$, $\omega_M = 10^3$, $N = 10$, and applying (13) to $s^{0.56}$, $s^{0.46}$ and $s^{0.69}$, the order of the integer–order approximating transfer function turns out to be 87.

Also, high–order transfer–function models tend to be ill–conditioned: for example, the ratio of the largest to the smallest values of the transfer function coefficients obtained according to the above procedure may be even higher than 10^{80}. Therefore, numerical difficulties are encountered with almost all order reduction algorithms. These difficulties may be avoided if LTI state–space models (of both fractional and integer order) are considered. Examples of such an approach can be found in [4, 6, 7, 14, 15, 18]. The integer order approximation of the non integer order model (5)–(6) obtained according to the procedure in [4] is briefly outlined next. For details see [4].

Consider the state equations (5) and let the fractional–order integrators $1/s^{\alpha_k}$ be approximated according to (14) as

$$\mathcal{I}^{\gamma_k}(s) = \frac{\sum_{j=0}^{m} f_{k,j} s^j}{s \sum_{j=0}^{m} g_{k,j} s^j}, \tag{15}$$

where $m = 2N + 1$. Then, define the matrices

$$A_0 = -\mathrm{diag}\{f_{1,0}, f_{2,0}, \dots, f_{\ell,0}\} A,$$

$$A_k = \mathrm{diag}\{g_{1,k-1}, g_{2,k-1}, \dots, g_{\ell,k-1}\} - \mathrm{diag}\{f_{1,k}, f_{2,k}, \dots, f_{\ell,k}\} A,$$

for $k = 1, \dots, m$, and

$$B_k = \mathrm{diag}\{f_{1,k}, f_{2,k}, \dots, f_{\ell,k}\} b,$$

for $k = 0, \dots, m$. The following state–space integer order model approximating (5)–(6) is obtained:

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t), \tag{16}$$

$$\hat{y}(t) = \hat{C}\hat{x}(t) + du(t), \tag{17}$$

where $\hat{x} \in \mathbb{R}^{(2N+2)\ell}$, and matrices $\bar{A} \in \mathbb{R}^{(2N+2)\ell \times (2N+2)\ell}$, $\bar{B} \in \mathbb{R}^{(2N+2)\ell \times 1}$ and $\bar{C} \in \mathbb{R}^{1 \times (2N+2)\ell}$ are given by

$$\hat{A} = \begin{bmatrix} 0 & 0 & \dots & 0 & -A_0 \\ I & 0 & \dots & 0 & -A_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -A_{m-1} \\ 0 & 0 & \dots & I & -A_m \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B_0 \\ B_1 \\ \vdots \\ B_{m-1} \\ B_m \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} 0 & 0 & \dots & 0 & c \end{bmatrix}, \tag{18}$$

with c and d as in the original representation (5).

4 Model Reduction

The only way to ensure a more accurate integer–order approximation of a given fractional–order system is to increase the value of N. This, however, leads to high–dimensional models that require the design of complex and expensive controllers. To overcome this problem, resort can be made to the following two–step procedure. First, a high–dimensional integer–order model corresponding to a large value of N is determined, thus ensuring the desired accuracy. Then, a reduced–order model is found from this high–order model by applying a suitable reduction algorithm.

The methods suggested in the literature to find L_2-optimal reduced–order models red (see, e.g., [4, 17]) are difficult to implement or depend crucially on the initial conditions. To avoid these difficulties, resort can be made to a slightly different approach that refers to the L_2 norm of the so–called equation error [3].

Let the triple (A_r, B_r, C_r), where $A_r \in \mathbb{R}^{q\times q}$, $B_r \in \mathbb{R}^{q\times 1}$, $C_r \in \mathbb{R}^{1\times q}$, represent the low order model. The aforementioned procedure involves the determination of two projection matrices L_r and T_r such that

$$A_r = L_r\hat{A}T_r,\ \ B_r = L_r\hat{B},\ \ C_r = \hat{C}T_r. \tag{19}$$

Next, define

$$\mathcal{O}_{[-k:q-k-1]} \doteq \begin{bmatrix} \hat{C}\hat{A}^{-k} \\ \vdots \\ \hat{C} \\ \vdots \\ \hat{C}\hat{A}^{q-k-1} \end{bmatrix}. \tag{20}$$

and assume that W_c is the controllability Gramian, which is the solution of the Lyapunov equation:

$$\hat{A}W_c + W_c\hat{A}^T + \hat{B}\hat{B}^T = 0\,. \tag{21}$$

The projection matrices in (19) may be determined in such a way that L_r^T spans the range of $\mathcal{O}^T_{[-k:q-k-1]}$ and $T_r = W_cL_r^T(L_rW_cL_r^T)^{-1}$. It can be shown [3] that, in this way, model (A_r, B_r, C_r) retains:

(i) the k time moments $\hat{C}\hat{A}^{-i}\hat{B},\ i = 1, \dots, k$,
(ii) the $q - k - 1$ Markov parameters $\hat{C}\hat{A}^{i}\hat{B},\ i = 0, \dots, q - k - 1$,
(iii) the k low–frequency power moments $\hat{C}\hat{A}^{-i}W_c(\hat{A}^T)^{-i}\hat{C}^T,\ i = 1, \dots, k$, and
(iv) the $q-k-1$ high–frequency power moments $\hat{C}\hat{A}^{i}W_c(\hat{A}^T)^{i}\hat{C}^T,\ i = 1, \dots, q-k-1$.

Matrix L_r^T can conveniently be determined using the Arnoldi algorithm, which allows to construct an orthonormal basis for the Krylov space $\mathcal{K}(F, X, n) = \mathrm{Im}\,[X, FX, \dots, F^{n-1}X]$ generated by matrices F and X. In the present context, the columns of L_r^T are determined so as to form an orthonormal basis for the Krylov space $\mathcal{K}(A^T, (CA^{-k})^T, q)$.

The accuracy of the proposed approximation strongly depends on the selection of parameters N, ω_m, ω_M and on the order reduction method used in the second step. It has been proved [12] that for N sufficiently large the frequency responses of $\mathcal{D}^\alpha(s)$, $\mathcal{I}^\alpha(s)$ tend to the ideal ones in the range between ω_m and ω_M. The order reduction method used in the second step of the suggested procedure ensures that the L_2 norm of the difference between left and right hand sides of the input–output high order model is minimal when the original response is replaced by the response of the reduced model, which qualifies the method as an *equation error* method [3]. Moreover, as already observed, this method guarantees that a number of first–order and second–order information indices (e.g., Markov parameters and power moments) are retained by the reduced–order model.

5 Examples

In the following, the advantages of the procedure proposed in Sect. 4 are demonstrated by means of two examples taken from [15, 16, 18]. The step responses as well as frequency responses (Bode plots) of the original non–integer order model and its low order approximation are compared to show the desired accuracy is ensured. The same examples have been considered in [4] where the L_2–optimal model reduction method is used.

Example 1 Consider the system given in the frequency domain by the transfer function (10). Its state–space equations are

$$\begin{bmatrix} D^{1.56}x_1(t) \\ D^{1.13}x_2(t) \\ D^{0.77}x_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -20 & -10 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t), \quad (22)$$

$$y(t) = \begin{bmatrix} 4 \ 1 \ 0 \end{bmatrix} x(t). \quad (23)$$

Choosing $N = 10$, $\omega_m = 10^{-3}$ and $\omega_M = 10^3$, the procedure outlined in Sect. 3 leads to a 66th order model. Next, this model has been reduced to a 5th order one by means of the procedure outlined in Sect. 4 with $k = 2$, so that 2 time moments and low–

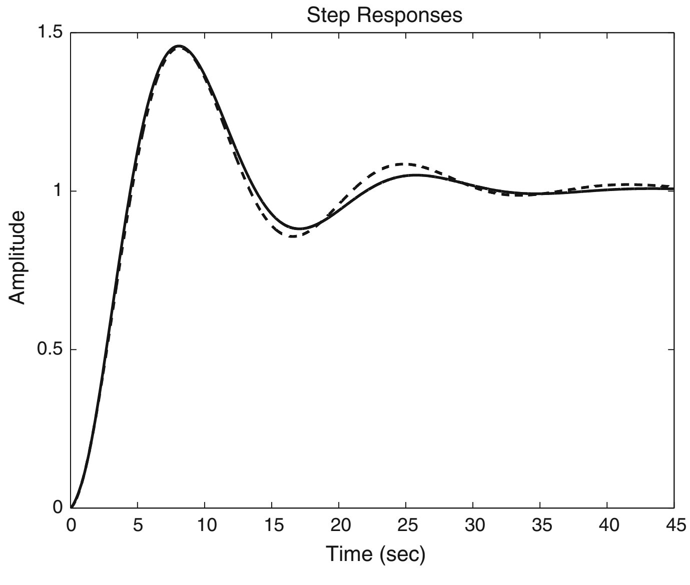

Fig. 1 Step responses for the original model (22)–(23) (*solid line*) and its 5–th order approximation (*dashed line*) with $k = 2$

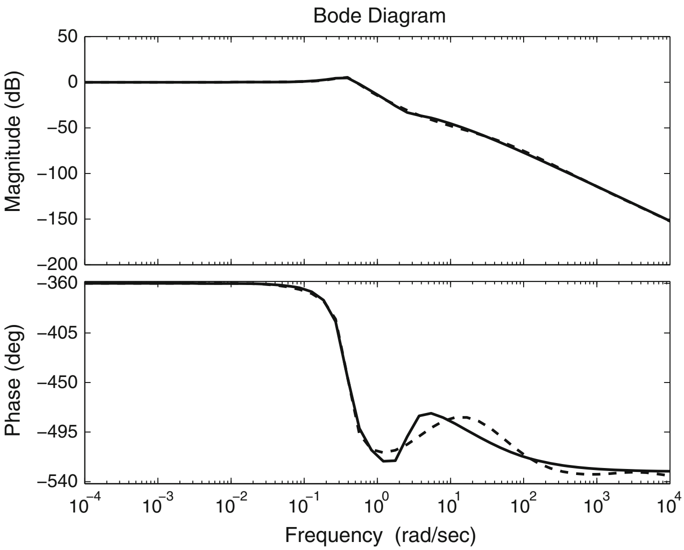

Fig. 2 Comparison of the Bode plots for the original model (22)–(23) (*solid line*) and its 5–th order approximation (*dashed line*) with $k = 2$

frequency power moments as well as three Markov parameters and high–frequency power moments are retained.

The step response of the 5th order model is compared in Fig. 1 with the original step response computed according to the Matlab code described in [1] whereas the Bode plots are compared in Fig. 2. Since these responses practically coincide, the 5th order model can safely be used for controller design. The step response and Bode plots for the high order model are almost exactly equal to those of the original non–integer order model and, therefore, are not shown.

Example 2 The suggested approximation procedure has also been applied to the state–space model:

$$\begin{bmatrix} D^{0.8}x_1(t) \\ D^{0.8}x_2(t) \\ D^{0.8}x_3(t) \\ D^{0.8}x_4(t) \\ D^{0.8}x_5(t) \\ D^{0.8}x_6(t) \end{bmatrix} = \begin{bmatrix} -6 & -6 & -4.4688 & -7.3047 & -6.1719 & -3.4688 \\ 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} x(t)$$

$$+ \begin{bmatrix} 2\ 0\ 0\ 0\ 0\ 0 \end{bmatrix}^T u(t), \tag{24}$$

$$y(t) = \begin{bmatrix} 0.5 & 0.5625 & 0.2422 & 0.2266 & 0.1172 & 0.0313 \end{bmatrix} x(t), \tag{25}$$

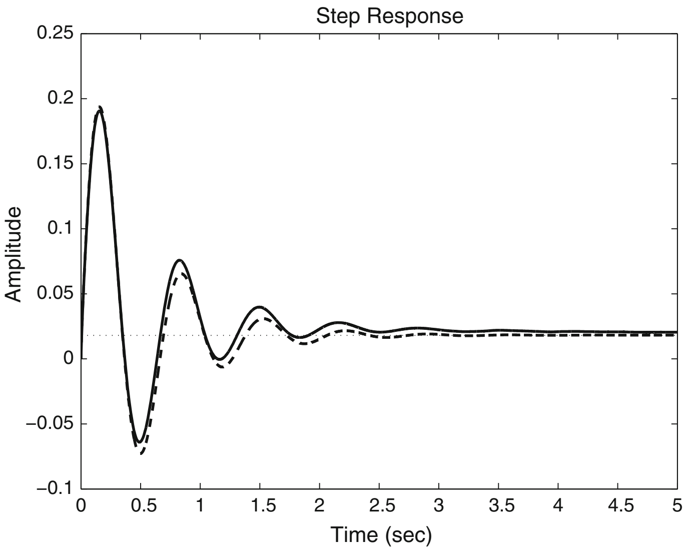

Fig. 3 Step responses for the original model (24)–(25) (*solid line*) and its 7–th order approximation (*dashed line*)

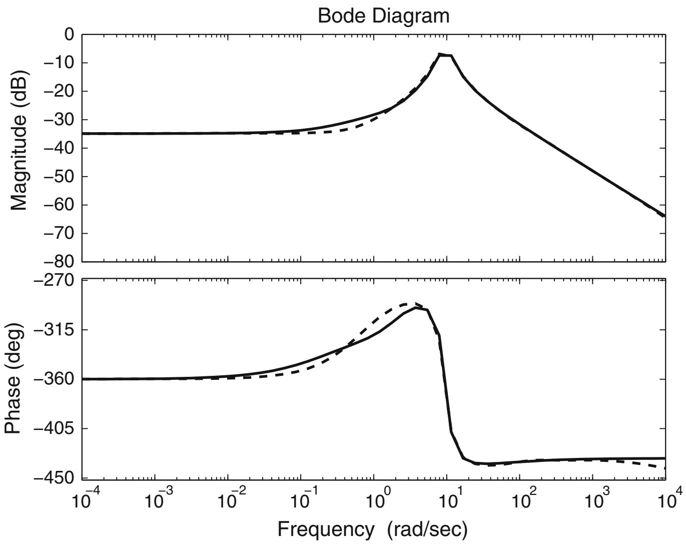

Fig. 4 Comparison of the Bode plots for the original model (24)–(25) (*solid line*) and its 7–th order approximation (*dashed line*)

corresponding to the transfer function (11). Choosing $\omega_m = 10^{-3}$, $\omega_M = 1000$ and $N = 10$ leads to a 132nd order model. Next, this model has been reduced to a 7–th order one by means of the procedure outlined in Sect. 4 with $k = 2$, so that 2 time moments and low–frequency power moments as well as 7 Markov parameters and high–frequency power moments are retained.

The step response of the 7th order model is compared in Fig. 3 with the original step response computed according to the Matlab code described in [1]. The Bode plots are compared in Fig. 4. The responses practically coincide, so that the 7th order model can be used safely for controller design. The step response and Bode plots for the high order model are practically equal to those obtained for the original non–integer order model and are not shown.

6 Conclusions

An efficient and easily implementable procedure to find integer–order models approximating a fractional order system represented in the state-space form has been presented. It consists of two stages. First, a high order model whose state matrix exhibits a sparse block–companion structure is determined. Next, an equation error method is adopted to find a reduced model that retains a number of Markov parameters and time moments as well as some low– and high–frequency power moments of the integer–order model obtained in the first step. Simulations have shown that the procedure leads to approximating models whose responses match well those of the original system.

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