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Stefan Domek
Paweł Dworak
Editors

Theoretical Developments and Applications of Non-Integer Order Systems

7th Conference on Non-Integer Order Calculus and Its Applications, Szczecin, Poland

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Editors
Stefan Domek
Faculty of Electrical Engineering
West Pomeranian University of Technology
Szczecin
Poland

Paweł Dworak
Faculty of Electrical Engineering
West Pomeranian University of Technology
Szczecin
Poland

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Preface

"Thus it follows that $d^{\frac{1}{2}}x$ will be equal to $x\sqrt{dx\colon x}$, an apparent paradox, from which one day useful consequences will be drawn." This first remark on the idea of a non-integer order derivative was found in a letter from Gottfried Wilhelm Leibnitz to the Guillaume de l'Hôpital dated 1695.[1] It has become a motivation for future generations of mathematicians to create basics of the non-integer order (fractional) calculus. And since the mid-twentieth century this mathematical apparatus has been used for creation of increasingly better models of simple and very complex physical phenomena systems and processes. As we know from numerous studies, fractional order models can depict the physical plant better than the classical integer order ones. This covers different research fields such as modeling of insulator properties, visco-elastic materials, electrodynamic, electrothermal, electrochemical, economic processes, etc.

Despite a huge increase of research activities and many remarkable theoretical achievements in the area of fractional calculus, we still face theoretical and practical challenges. The complicity of the non-integer order calculus causes most of the work to be theoretically oriented. This can also be seen in the contents of the previous[2,3] and this conference volume. All this shows that we are still in the early stages of development of non-integer order control systems. However, increasing potentialities offered by modern automation equipment gives hope of developing effective control techniques that could be applied and implemented also for non-integer order modeled processes.

[1]Oldham, K.B. and Spanier, J.: The Fractional Calculus. Academic Press, 1974.

[2]Mitkowski, W., Kasprzyk, J., Baranowski, J.: Advances in the Theory and Applications of Non-integer Order Systems, the 5th Conference on Non-integer Order Calculus and Its Applications, Cracow, Poland; Springer, Lecture Notes in Electrical Engineering, vol. 257.

[3]Latawiec, J.K., Łukaniszyn, M., Stanisławski, R.: Advances in Modelling and Control of Non-integer Order Systems, the 6th Conference on Non-integer Order Calculus and Its Applications, Opole, Poland; Springer, Lecture Notes in Electrical Engineering, vol. 320.

Some new ideas and examples of modeling, synthesis, and practical realizations of fractional order systems may be found in this study. This volume contains 24 papers divided into four parts covering: mathematical fundamentals, modeling and approximations, controllability, observability and stability problems, and practical applications of fractional control systems.

Part 1 expands the base of tools and methods of the mathematical basis for non-integer order calculus.

Malgorzata Klimek ("Fractional Sturm-Liouville Problem in Terms of Riesz Derivatives") formulates a regular fractional Sturm-Liouville problem on a bounded domain in terms of Riesz derivatives. The considered case includes vanishing Dirichlet boundary conditions. They prove that its eigenvalues are real, eigenfunctions are continuous, and form orthogonal sets of functions in the respective Hilbert spaces. In addition, boundedness results for eigenvalues are derived and a connection between the discussed fractional Sturm-Liouville equations and Euler-Lagrange equations for the corresponding action functionals is established.

Agnieszka B. Malinowska and Tatiana Odzijewicz ("Multidimensional Discrete-Time Fractional Calculus of Variations") introduce a discrete-time multidimensional fractional calculus of variations. They define fractional operators in the sense of Grünwald–Letnikov. Then they derive necessary optimality conditions and give examples illustrating the use of obtained results.

Dominik Sierociuk, Wiktor Malesza, and Michal Macias ("On a New Symmetric Fractional Variable Order Derivative") present particular definitions of symmetric fractional variable order derivatives. The AD and DA types of the fractional variable order derivatives and their properties are introduced. Additionally, they show the switching order schemes equivalent to these types of definitions. Finally, the theoretical considerations are validated on numerical examples.

Piotr Ostalczyk ("Linearization of the Non-linear Time-Variant Fractional-Order Difference Equation") discusses a linearization procedure of the fractional-order nonlinear time-variant discrete system. Starting from the nonlinear fractional-order difference equation he derives its equivalent state-space form and assuming the knowledge of the nominal trajectory evaluates the linear state-space model. The investigations are supported by numerical examples.

Ewa Girejko, Ewa Pawłuszewicz, and Małgorzata Wyrwas ("The Z-Transform Method for Sequential Fractional Difference Operators") discuss the linear Caputo–type sequential difference fractional-order systems. They use a classical $\mathcal{Z}$-transform method to show the general solutions of sequential systems in the form $\left(\Delta_*^\alpha(\Delta_*^\alpha x)\right)(n)+b\left(\Delta_*^\alpha x\right)(n)+cx(n)=0$, where $b, c \in \mathbb{R}$. In proofs they base on the formula for the image of the discrete Mittag-Leffler function in the $\mathcal{Z}$-transform.

Part 2 focuses on new methods and developments in process modeling and fractional derivative approximations.

Wojciech Mitkowski and Krzysztof Oprzędkiewicz ("An Estimation of Accuracy of Charef Approximation") present a new accuracy estimation method for Charef approximation. Charef approximation allows them to describe fractional-order systems with the use of an integer-order, proper transfer function.

They estimate the accuracy of approximation by comparing step responses of the plant and Charef approximation. The step response of the plant was calculated with the use of an accurate analytical formula and it can be interpreted as a standard. The presented approach can be applied for effective tuning of Charef approximant for a given plant. The use of the proposed method does not require knowing the step response of the modeled plant. The proposed methodology can be easily generalized to other known approximations.

Wieslaw Krajewski and Umberto Viaro ("A New Method for the Integer Order Approximation of Fractional Order Models") concern themself with the finite-dimensional approximation of a fractional-order system represented in state-space form. To this purpose, resort is made to the Oustaloup method for approximating a fractional-order integrator by a rational filter. They reduce the dimension of the resulting integer-order model using an efficient algorithm for minimization of the L_2 norm of a weighted equation error. Two numerical examples are worked out to show how the desired approximation accuracy can be ensured.

Jerzy Baranowski, Waldemar Bauer, and Marta Zagórowska ("Stability Properties of Discrete Time-Domain Oustaloup Approximation") present an analysis of discrete time domain realization of Oustaloup approximation. They present the scheme for realization along with a method of implementation of discretization formulas. The authors analyze also the stability considering influences of sampling frequency, order, and bandwidth. Analysis is illustrated with behavior of spectral radius of the discretized system.

Konrad Andrzej Markowski and Krzysztof Hryniów ("Digraphs Minimal Positive Stable Realisations for Fractional One-Dimensional Systems") present a method of the determination of positive stable realization of the fractional continuous-time positive system. The algorithm finds a complete set of all possible realizations instead of only a few realizations. They show that all realizations in the set are minimal and stable. The method proposed by them uses a parallel computing algorithm based on a digraphs theory, which is used to gain much needed speed and computational power for a numerical solution. The presented procedure is illustrated with a numerical example.

Marek Rydel, Rafał Stanisławski, Grzegorz Bialic, and Krzysztof J. Latawiec ("Modeling of Discrete-Time Fractional-Order State Space Systems Using the Balanced Truncation Method") present a new method of approximation of linear time-invariant discrete-time fractional-order state space systems by means of the Balanced Truncation Method. This reduction method is applied to the rational form of fractional-order system in terms of expanded state equation. As an approximation result the authors obtain rational and relatively low-order state space system. Simulation experiments show very high accuracy of the introduced methodology.

Dominik Sierociuk, Michal Macias, and Pawel Ziubinski ("Experimental Results of Modeling Variable Order System Based on Discrete Fractional Variable Order State-Space Model") present experimental results of modeling fractional variable order system using Discrete Fractional Variable Order State-Space Model. Experimental results given were obtained on the basis of modified multi-order switching analog realization of the constant parameter case introduced in this work. During the identification process two algorithms were used: direct and dual. Finally they present joint estimation results for parameter estimation, in order to verify constant parameters for the proposed analog model.

Part 3 provides a bunch of papers which raise problems of controllability, observability, and stability of non-integer order systems.

Tadeusz Kaczorek ("Positivity and Stability of a Class of Fractional Descriptor Discrete-Time Nonlinear Systems") proposes a method of analysis of the fractional descriptor nonlinear discrete-time systems with regular pencils of linear part. The method is based on the Weierstrass-Kronecker decomposition of the pencils. He establishes necessary and sufficient conditions for the positivity of the nonlinear systems. Then he proposes a procedure for computing the solution to the equations describing the nonlinear systems. Using an extension of the Lyapunov method to positive nonlinear systems, he derives sufficient conditions for the asymptotic stability.

Małgorzata Wyrwas and Dorota Mozyrska ("Stability of Linear Discrete–Time Systems with Fractional Positive Orders") study the problem of the stability of the Grünwald–Letnikov-type linear discrete-time systems with fractional positive orders. The method of reducing the considered systems by transforming them to the multi-order linear systems with the partial orders from the interval (0,1] is presented. For the reduced multi-order systems the authors formulate conditions for stability based on the $\mathcal{Z}$-transform as an effective method for stability analysis of linear systems.

Yassine Boukal, Michel Zasadzinski, Mohamed Darouach, and Nour-Eddine Radhy ("Robust H_∞ Observer-Based Stabilization of Disturbed Uncertain Fractional-Order Systems Using a Two-Step Procedure") consider the problem of robust H_∞ observer-based stabilization for a class of linear Disturbed Uncertain Fractional-Order Systems (DU-FOS) using H_∞-norm optimization. Based on the H_∞-norm analysis for FOS, they establish a new design methodology to stabilize a linear DU-FOS using robust H_∞ Observer-Based Control (OBC). The existence conditions are derived, and using the H_∞-optimization technique, the stability of the estimation error and stabilization of the original system are given in an inequality condition, where all the observer matrices gains and the control law can be computed by solving a single inequality condition in two steps. Finally, the authors give a simulation example to illustrate the validity of their results.

Zbigniew Zaczkiewicz ("Relative Observability for Fractional Differential-Algebraic Delay Systems within Riemann-Liouville Fractional Derivatives") presents the problems of relative R-observability for linear stationary fractional differential-algebraic delay system (FDAD). FDAD system consists of fractional differential equation in the Riemann-Liouville sense and difference equations. He introduces the determining equation systems and their properties. Applying the Laplace transformation he obtains solution representations into series of their determining equation solutions and presents effective parametric rank criteria for relative R-observability. He also formulates a dual controllability result.

Part 4 is devoted to the presentation of different fractional order control applications.

Jerzy Klamka ("Minimum Energy Control of Linear Fractional Systems") considers the minimum energy control problem of infinite-dimensional fractional-discrete time linear systems. He establishes necessary and sufficient conditions for the exact controllability of the system and gives sufficient conditions for the solvability of the minimum energy control of infinite-dimensional fractional discrete-time systems. Finally, he proposes a procedure for computation of the optimal sequence of inputs that minimizes the quadratic performance index.

Adam Makarewicz ("Use of Alpha-Beta Filter to Synchronization of the Chaotic Ikeda Systems of Fractional Order") considers a problem of signal filtering used in synchronization of two-fractional delay Ikeda systems, combined linearly by coupling. The synchronization uses Alpha-Beta filter, which operates on predicting the next value, based on the measured signal in a current point in time. He uses numerical simulations to investigate effects of fractional order and coupling rate on synchronization. Simulations are performed using Ninteger Fractional Control Toolbox for MatLab.

Paweł Dworak ("On Dynamic Decoupling of MIMO Fractional Order Systems") considers problems with a dynamic decoupling of multi-input multi-output fractional order systems. He shows their similarities and differences to integer order decoupling methods. Basing on a few examples he carries out simulations of decoupled fractional order systems to show the applicability of the considered methods. He ends his work with some final remarks on a practical implementation of decoupling methods for fractional order systems.

Łukasz Wach and Wojciech P. Hunek ("Perfect Control for Fractional-Order Multivariable Discrete-Time Systems") analyze the perfect control method for multi-input/multi-output MIMO fractional-order discrete-time systems in state space. The presented simulation example for nonsquare MIMO system carried out in a Matlab/Simulink environment confirms the correctness of the proposed algorithm.

Waldemar Bauer ("Implementation of Non-Integer Order Controller Using Oustaloup Parallel Approximation for Air Heating Process Trainer") presents a new implementation method of non-integer order controller. This controller is designed and analyzed for the model of air heating process trainer system. The author shows that the proposed controller is suitable for control of the inertial system with time-delay and time-varying gain. He also presents a new method of implementation of Oustaloup approximation and shows it usefulness, which allows operation of non-integer order controller in real-time environment.

Bogdan Broel-Plater, Krzysztof Jaroszewski, and Paweł Dworak ("Classical Versus Fractional Order PI Current Controller in Servo Drive") compare the fractional order PI current controller of the servo drive with its classical counterpart. They analyze structures of such a fractional order controller without as well as with antiwindup blocks. They present and discuss results of simulations carried out in Matlab/Simulink environment.

Marta Zagórowska ("Parametric Optimization of Non-Integer Order PD^{μ} Controller for Delayed System") analyzes a new tuning method for PD^{α} controller using approximation with Laguerre functions. She performs the optimization for various sets of parameters and also analyzes the convergence of chosen optimization parameters. The results are tested for a first-order system with delay.

Jerzy Baranowski, Waldemar Bauer, Marta Zagórowska, Aleksandra Kawala-Janik, Tomasz Dziwiński, and Paweł Piątek ("Adaptive Non-Integer Controller for Water Tank System") consider a new method of designing adaptive controller for non-integer order systems. The theoretical approach was verified with a computer simulation of three-tank system.

Stefan Domek ("Model-Plant Mismatch in Fractional Order Model Predictive Control") shows the effect of various plant-model mismatches on the performance of fractional order model predictive control (FOMPC) systems. He presents an algorithm of a FOMPC and describes different types of plant-model mismatches for fractional-order systems. His analysis is illustrated by results obtained from simulation tests.

This volume is a result of fruitful and stimulating discussions during the RRNR'2015, the 7th Conference on Non-integer Order Calculus and Its Applications organized by the Faculty of Electrical Engineering, West Pomeranian University of Technology, Szczecin, Poland. The conference gathered a number of researchers active in the fields of fractional calculus, here those interested in theoretical aspects of mathematical fundamentals, modeling, and approximations and those focused on the practical issues that have to be solved during control system implementation. Such a wide spectrum of interests displayed by the outstanding participants exploded in stimulation of lively discussions across the field and contributed to the success of the conference. We are grateful to the conference participants for sharing their research results and active and inspiring discussion. We would also like to acknowledge the contribution of the anonymous referees, whose comments allowed us to improve the final form of the papers. Finally, we wish to thank Dr. Thomas Ditzinger and Holger Schäpe from Applied Sciences and Engineering at Springer for their assistance and support in this editorial work.

Szczecin — Stefan Domek
Autumn 2015 — Paweł Dworak

Contents

Part I
Mathematical Fundamentals

Fractional Sturm-Liouville Problem in Terms of Riesz Derivatives

Malgorzata Klimek

Abstract In the paper, a regular fractional Sturm-Liouville problem on a bounded domain is formulated in terms of Riesz derivatives. The considered case includes vanishing Dirichlet boundary conditions and we prove that its eigenvalues are real, eigenfunctions are continuous and form orthogonal sets of functions in the respective Hilbert spaces. In addition, a boundedness results for eigenvalues are derived and a connection between the discussed fractional Sturm-Liouville equations and Euler-Lagrange equations for the corresponding action functionals is established.

Keywords Riesz derivative · Fractional Sturm-Liouville problem · Eigenvalue problem · Eigenfunctions and eigenvalues

1 Introduction

The aim of this paper is to formulate a regular fractional Sturm-Liouville problem (FSLP) including Riesz derivative and discuss its fundamental properties. We shall restrict our study to the case with vanishing Dirichlet boundary conditions and to one-term fractional Sturm-Liouville operator (FSLO). It is a new variant of variational construction, introduced in [6, 7]. Our results show that the proposed construction of FSLO leads to real eigenvalues and orthogonal sets of eigenfunctions in the respective Hilbert spaces.

FSLPs were first discussed as fractional deformations of classical problems in papers [1–3, 12, 14, 15]. The formulation based on one-sided fractional derivatives, however, does not lead to orthogonal eigenfunctions. This fact was the main motivation for a variational approach started in [6]. The characteristic feature of the constructed regular and singular FSLPs [6–10, 16, 18] is mixing of the left and the right differential operators appearing in the fractional Sturm-Liouville operator. It

M. Klimek (✉)
Institute of Mathematics, Czestochowa University of Technology,
Armii Krajowej 21, 42-200 Czestochowa, Poland
e-mail: mklimek@im.pcz.pl

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is a known fact that fractional differential equations (FDEs), mixing the left and the right derivatives arise in the fractional calculus of variations (FVC) [5, 13]. FSLPs are important class of such FDEs as linear eigenvalue problems constructed within the framework of FVC. Similar to the classical Sturm-Liouville problems they produce orthogonal bases in the respective function spaces. Thence, we expect that they will be important and useful tools in analytically and/or numerically solving many equations appearing in the mathematical modelling of real-world phenomena (for classical results in integer-order calculus compare [21]). Preliminary results on non-integer order numerical methods can be found in [18–20]. Preliminary applications in partial FDEs theory and in anomalous diffusion theory are enclosed in [7–11].

We begin by recalling definitions, properties and facts from fractional calculus.

2 Preliminaries

First, we recall the definitions and properties of fractional integral and differential operators [4, 17]. Fractional operators can be defined for complex orders but in our study we discuss fractional eigenvalue problems of real order and accordingly we quote the respective definitions and properties.

Definition 1 The left and the right Riemann–Liouville fractional integrals are defined as follows when $\alpha > 0$

$$I^{\alpha}_{a+}f(x) := \frac{1}{\Gamma(\alpha)}\int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}}, \quad x \in (a, b], \tag{1}$$

$$I^{\alpha}_{b-}f(x) := \frac{1}{\Gamma(\alpha)}\int_x^b \frac{f(t)dt}{(t-x)^{1-\alpha}}, \quad x \in [a, b), \tag{2}$$

where $\Gamma(\alpha)$ denotes Euler's gamma function. In the case $b = \infty$, the right Riemann-Liouville integral becomes the right Liouville integral: I^{α}_{-}.

The following results on fractional integrals will be useful in further considerations.

Lemma 1 (cf. Lemma 2.1 [4]) *Riemann-Liouville integrals are bounded operators in the $L^p(a, b)$- space for $p \geq 1$:*

$$||I^{\alpha}_{a+}f||_{L^p} \leq K_{\alpha}||f||_{L^p} \quad ||I^{\alpha}_{b-}f||_{L^p} \leq K_{\alpha}||f||_{L^p}, \tag{3}$$

where constant $K_{\alpha} = \frac{(b-a)^{\alpha}}{\Gamma(\alpha+1)}$.

Lemma 2 (cf. Lemma 2.7 [4]) *Let $f \in L^p(a, b)$, $g \in L^q(a, b)$ and $\frac{1}{p} + \frac{1}{q} \leq 1 + \alpha$ ($p \neq 1$ and $q \neq 1$ in the case, when $\frac{1}{p} + \frac{1}{q} = 1 + \alpha$). Then,*

$$\int_a^b f(x) I_{a+}^{\alpha} g(x) dx = \int_a^b \left(I_{b-}^{\alpha} f(x) \right) g(x) dx. \tag{4}$$

Having defined the fractional integrals we can construct Riemann-Liouville and Caputo fractional derivatives.

Definition 2 The left and the right Riemann–Liouville fractional derivative of order $\alpha \in (0, 1)$ of a function f, denoted by $D_{a+}^{\alpha} f$ and $D_{b-}^{\alpha} f$ respectively, are given by

$$\forall x \in (a, b], \ \ D_{a+}^{\alpha} f(x) := D\, I_{a+}^{1-\alpha} f(x), \tag{5}$$

$$\forall x \in [a, b), \ \ D_{b-}^{\alpha} f(x) := -D\, I_{b-}^{1-\alpha} f(x), \tag{6}$$

where we denoted $D = \dfrac{d}{dx}$. In the case $a = 0, \ b = \infty$ the above derivatives are called Liouville derivatives in $\mathbb{R}_+$.

The left and the right Caputo fractional derivatives of order $\alpha \in (0, 1)$ are given by

$$\forall x \in (a, b], \ \ {}^{c}D_{a+}^{\alpha} f(x) := D_{a+}^{\alpha} \left[f(x) - f(a) \right], \tag{7}$$

$$\forall x \in [a, b), \ \ {}^{c}D_{b-}^{\alpha} f(x) := D_{b-}^{\alpha} \left[f(x) - f(b) \right]. \tag{8}$$

Let us note that for functions vanishing at ends of interval $[a, b]$, we obtain:

$$f(a) = 0 \qquad D_{a+}^{\alpha} f = {}^{c}D_{a+}^{\alpha} f \tag{9}$$

$$f(b) = 0 \qquad D_{b-}^{\alpha} f = {}^{c}D_{b-}^{\alpha} f. \tag{10}$$

For $\alpha \in (0, 1)$ and $f \in AC[a, b]$, the Caputo fractional derivatives satisfy the following relations:

$${}^{c}D_{a+}^{\alpha} f(x) = I_{a+}^{1-\alpha} D f(x) \quad {}^{c}D_{b-}^{\alpha} f(x) = -I_{b-}^{1-\alpha} D f(x). \tag{11}$$

Next, we formulate results on composition rules for derivatives and integrals of non-integer order.

Lemma 3 (cf. Lemma 2.4 and 2.21 [4]) *If $\alpha > 0$ and $f \in L^p(a, b)$, $(1 \leq p \leq \infty)$, then the following composition rules are valid:*

$$D_{a+}^{\alpha} I_{a+}^{\alpha} f(x) = f(x), \quad D_{b-}^{\alpha} I_{b-}^{\alpha} f(x) = f(x), \tag{12}$$

for almost all $x \in [a, b]$. If function f is continuous, then the composition rules hold for all $x \in [a, b]$.

If f is continuous on interval $[a, b]$, then

$${}^{c}D_{a+}^{\alpha} I_{a+}^{\alpha} f(x) = f(x), \quad {}^{c}D_{b-}^{\alpha} I_{b-}^{\alpha} f(x) = f(x). \tag{13}$$

Now, we recall the notion of Riesz potentials.

Definition 3 Riesz potentials in halfaxis $\mathbb{R}_+$ and in $\mathbb{R}$ are defined as follows, when $\gamma \in (0, 1)$:

$$I_0^\gamma f(x) := \gamma_0 \int_0^\infty |x - t|^{\gamma - 1} f(t) dt \tag{14}$$

$$I^\gamma f(x) := \gamma_0 \int_{-\infty}^\infty |x - t|^{\gamma - 1} f(t) dt \tag{15}$$

with coefficient $\gamma_0 := (2\Gamma(\gamma)\cos(\pi\gamma/2))^{-1}$.

The lemma below connects Liouville integrals in $\mathbb{R}_+$ and Riesz potentials.

Lemma 4 (cf. Theorem 12.4 [17]) *Let* $\gamma \in (0, 1)$ *and* $f \in L^p(\mathbb{R}_+)$ *with* $p \in [1, 1/\gamma)$. *Then, the following formula holds:*

$$I_0^\gamma f = I_-^{\gamma/2} I_{0+}^{\gamma/2} f, \tag{16}$$

with Liouville integrals described in Definition 1.

We shall formulate the fractional Sturm-Liouville problem in terms of Riesz derivatives. It is a special case of Riesz-Feller derivative with skewness equal to zero.

Definition 4 Riesz derivative of order $\gamma \in (0, 2)$ is given for suitable functions as

$$D_R^\gamma f(x) := -\mathcal{F}^{-1}\left[|k|^\gamma \hat{f}\right](x), \tag{17}$$

where $\mathcal{F}^{-1}$ denotes the inverse Fourier transform and $\hat{f}$ denotes the Fourier transform of function f.

We define the function space, where we shall study FSLPs with Riesz derivative.

Definition 5 We say that $g \in \Omega_1 \subseteq L^2(\mathbb{R}; \mathbb{C})$ iff it fulfills the following conditions:

$$supp(g) \subseteq [0, b], \quad (b - x)^{1-\alpha} g|_{[0,b]} \in C[0, b]. \tag{18}$$

In addition, $f \in I_{0+}^\alpha(\Omega_1)$ iff there exists function $g \in \Omega_1$ such that $f = I_{0+}^\alpha g$ a.e.

Now, we prove a fundamental result on solutions of the regular FSLPs including the composition of the left and the right fractional derivative. It appears that starting from the assumption that the solution belongs to the $L^2(0, b)$ space we arrive at the fact that it belongs to the subspace $I_{0+}^\alpha(\Omega_1)$.

Proposition 1 *Let* $y \in L^2(0, b)$, *order* $\alpha \in (1/2, 1)$, *functions* $p, q, w \in C([0, b]; \mathbb{R})$ *and* $p > 0$, $w > 0$. *If function* y *solves FSLP:*

$$p(x) D_{b-}^\alpha \, {}^c D_{0+}^\alpha y(x) + q(x) y(x) = \Lambda w(x) y(x) \tag{19}$$

with $y(0) = y(b) = 0$, *then* $y \in I_{0+}^\alpha(\Omega_1)$.

Proof From Eq. (19) we obtain the equality

$$^{c}D^{\alpha}_{0+}y(x) = I^{\alpha}_{b-}\left(\frac{q(x)}{p(x)}y(x) - \Lambda\frac{w(x)}{p(x)}y(x)\right)dx + A(b-x)^{\alpha-1}$$

and we note that as $\alpha \in (1/2, 1)$, then the derivative of the solution of Eq. (19) belongs to the Hilbert space, namely

$$^{c}D^{\alpha}_{0+}y \in L^2(0, b).$$

As $y(0) = 0$ we can rewrite FSLE in the form of

$$D^{\alpha}_{b-}D^{\alpha}_{0+}y(x) + \frac{q(x)}{p(x)}y(x) = \Lambda\frac{w(x)}{p(x)}y(x)$$

and apply the composition rules from Lemma 3:

$$D^{\alpha}_{b-}D^{\alpha}_{0+}\left[y(x) + I^{\alpha}_{0+}I^{\alpha}_{b-}\left(\frac{q(x)}{p(x)}y(x) - \Lambda\frac{w(x)}{p(x)}y(x)\right)\right] = 0.$$

The resulting fractional integral equation is:

$$y(x) + I^{\alpha}_{0+}I^{\alpha}_{b-}\left(\frac{q(x)}{p(x)}y(x) - \Lambda\frac{w(x)}{p(x)}y(x)\right) = A_1x^{\alpha-1} + A_2I^{\alpha}_{0+}(b-x)^{\alpha-1}.$$

By using boundary conditions $y(0) = y(b) = 0$ we get:

$$A_1 = 0 \quad A_2 = \frac{\Gamma(\alpha)(2\alpha-1)}{b^{2\alpha-1}} \cdot I^{\alpha}_{0+}I^{\alpha}_{b-}\left(\frac{q(x)}{p(x)}y(x) - \Lambda\frac{w(x)}{p(x)}y(x)\right)|_{x=b}.$$

Thence, eigenfunction y can be expressed as follows:

$$y(x) = -I^{\alpha}_{0+}I^{\alpha}_{b-}\left(\frac{q(x)}{p(x)}y(x) - \Lambda\frac{w(x)}{p(x)}y(x)\right) + A_2I^{\alpha}_{0+}(b-x)^{\alpha-1}. \tag{20}$$

We note that for functions p, q, w fulfilling assumptions, we have

$$\frac{q}{p}y \in L^2(0, b) \quad \frac{w}{p}y \in L^2(0, b)$$

and for $\alpha \in (1/2, 1)$ also $(b-x)^{\alpha-1} \in L^2(0, b)$. Therefore, $I^{\alpha}_{0+}(b-x)^{\alpha-1} \in C[0, b]$ as well as

$$I^{\alpha}_{b-}\left(\frac{q(x)}{p(x)}y(x) - \Lambda\frac{w(x)}{p(x)}y(x)\right) \in C[0, b]$$

and

$$I_{0+}^{\alpha} I_{b-}^{\alpha} \left(\frac{q(x)}{p(x)} y(x) - \Lambda \frac{w(x)}{p(x)} y(x) \right) \in C[0, b].$$

Taking into account the above integral form of eigenfunction y given by (20), we conclude that assuming $y \in L^2(0, b)$ we arrive at $y \in C[0, b]$ and $y \in I_{0+}^{\alpha}(\Omega_1)$. □

In the lemma below, we show two representations for Riesz derivative in the case when a function belongs to the $I_{0+}^{\alpha}(\Omega_1)$ space.

Lemma 5 *For functions $f \in I_{0+}^{\alpha}(\Omega_1)$, the following relations hold provided $\alpha \in (1/2, 1)$:*

$$-D_R^{2\alpha} f = D_{b-}^{\alpha}\, {}^{c}D_{0+}^{\alpha} f, \tag{21}$$

$$D_R^{2\alpha} f = \frac{-1}{2\cos(\pi\alpha)} D \left(I_{0+}^{2-2\alpha} + I_{b-}^{2-2\alpha} \right) Df. \tag{22}$$

Proof First, we show the connection between FSLO including Riesz potential and the one constructed using the left and right Liouville derivatives:

$$-DI^{2-2\alpha} Df = -DI_0^{2-2\alpha} Df = -DI_-^{1-\alpha} I_{0+}^{1-\alpha} Df$$

which is valid as $2 - 2\alpha \in (0, 1)$. Next, we recall that any function $f \in I_{0+}^{\alpha}(\Omega_1)$ can be represented as $f = I_{0+}^{\alpha} g$ with $g \in \Omega_1$. Therefore, we obtain

$$-DI_-^{1-\alpha} I_{0+}^{1-\alpha} Df = -DI_-^{1-\alpha} DI_{0+}^{1-\alpha} I_{0+}^{\alpha} g$$

$$= -DI_-^{1-\alpha} DI_{0+}^{1} g = -DI_-^{1-\alpha} g = -DI_{b-}^{1-\alpha} g$$

$$= -DI_{b-}^{1-\alpha}\, {}^{c}D_{0+}^{\alpha} f = D_{b-}^{\alpha}\, {}^{c}D_{0+}^{\alpha} f.$$

We check Eq. (21) by calculating the Fourier transform of the fractional differential operator on the right-hand side:

$$\mathcal{F}\left[-DI^{2-2\alpha} Df \right] = k^2 |k|^{2\alpha-2} \hat{f} = |k|^{2\alpha} \hat{f}.$$

Thus, we conclude that for functions from the $I_{0+}^{\alpha}(\Omega_1)$ space the relation from Eq. (21) is valid.
The relation given in (22) follows from the definitions of Riemann-Liouville integrals in $[0, b]$ and Riesz potentials. □

Lemma 6 *Let* $\alpha \in (1/2, 1)$. *For functions* $f, g \in C[0, b]$ *fulfilling boundary conditions:*

$$f(0) = f(b) = 0 \quad g(0) = g(b) = 0 \tag{23}$$

the following formula is valid:

$$\int_0^b f(x) D_R^{2\alpha} g(x)\, dx = \int_0^b \left(D_R^{2\alpha} f(x)\right) g(x)\, dx. \tag{24}$$

Proof The above formula of integration by parts results from formula (22), the assumed boundary conditions and properties of fractional integration given in Lemma 2. □

3 Main Results

In this section, we formulate a one-term fractional Sturm-Liouville problem (FSLP) in terms of Riesz derivatives. Here, we only consider a case with vanishing Dirichlet conditions and describe some of its important properties. The problem can be formulated twofold. In the first version we look for a nontrivial solutions of eigenvalue problem belonging to the $I^{\alpha}_{0+}(\Omega_1)$-space:

$$-p(x)\, D_R^{2\alpha} y(x) + q(x)y(x) = \Lambda w(x)y(x) \tag{25}$$

$$y(0) = 0 \quad y(b) = 0 \tag{26}$$

$$y \in I^{\alpha}_{0+}(\Omega_1). \tag{27}$$

An analogous version is the FSLP in the form of

$$-\, D_R^{2\alpha}\, p(x)y(x) + q(x)y(x) = \Lambda w(x)y(x) \tag{28}$$

$$y(0) = 0 \quad y(b) = 0 \tag{29}$$

$$y \in I^{\alpha}_{0+}(\Omega_1). \tag{30}$$

The proposition below tells that eigenfunctions of the above problems, considered on space $I^{\alpha}_{0+}(\Omega_1)$ are continuous in $[0, b]$. In the above formulation of FSLPs and the following theorems and proofs we assume that

(H1) order $\alpha \in (1/2, 1)$, functions $p, q, w \in C\,([0, b]; \mathbb{R})$ and $p > 0,\ w > 0$.

The first proposition results from Proposition 1 and Lemma 5.

Proposition 2 *Let (H1) be fulfilled. Eigenfunctions of the fractional eigenvalue problem* (25–27) *and* (28–30) *are continuous in* $[0, b]$.

Similar to FSLPs, considered in [6, 7, 9–11], the eigenvalues of problems (25–27) and (28–30) are real. This fact is proved in the two propositions below.

Proposition 3 *Let (H1) be fulfilled and* $q \neq 0$. *Eigenvalues of the fractional eigenvalue problem* (25–27) *are real. The set of eigenvalues is bounded below, namely*

$$\Lambda \geq \frac{\min_{x\in[0,b]} \frac{q(x)}{p(x)}}{\Theta}, \tag{31}$$

where $\Theta = ||\frac{w}{p}||$ *when* $\min_{x\in[0,b]} q(x) < 0$ *and* $\Theta = \min_{x\in[0,b]} \frac{w(x)}{p(x)}$ *otherwise.*

Proof Let us note that p, q and w are real–valued functions. Therefore, if solution y corresponds to eigenvalue Λ then conjugate function $\bar{y}$ is a solution of the problem (25–27) corresponding to eigenvalue $\bar{\Lambda}$. The following equations are fulfilled by functions y and $\bar{y}$ in $[0, b]$:

$$-p(x)\, D_R^{2\alpha} y(x) + q(x)y(x) = \Lambda w(x)y(x) \tag{32}$$

$$-p(x)\, D_R^{2\alpha} \bar{y}(x) + q(x)\bar{y}(x) = \bar{\Lambda} w(x)\bar{y}(x). \tag{33}$$

We multiply the first equation with $\bar{y}$ and the second with y, subtract and obtain the relation

$$\bar{y}(x)D_R^{2\alpha} y(x) - y(x)D_R^{2\alpha}\bar{y}(x) = (\bar{\Lambda} - \Lambda)\frac{w(x)}{p(x)} y(x)\bar{y}(x).$$

We integrate both sides of this equality and obtain

$$\int_0^b \left(\bar{y}(x)D_R^{2\alpha} y(x) - y(x)D_R^{2\alpha}\bar{y}(x)\right) dx = (\bar{\Lambda} - \Lambda) \int_0^b \frac{w(x)}{p(x)} y(x)\bar{y}(x)dx.$$

Now, we apply the property of fractional differentiation from Lemma 6 and arrive at the equality

$$(\bar{\Lambda} - \Lambda)\; ||y||_{w/p} = 0$$

which yields $\bar{\Lambda} - \Lambda = 0$, therefore $\Lambda \in \mathbb{R}$.
To prove estimation (31), we multiply Eq. (25) with $\bar{y}$ and integrate over interval $[0, b]$:

$$\int_0^b \left(-\bar{y}(x)D_R^{2\alpha} y(x) + \frac{q(x)}{p(x)}\bar{y}(x)y(x)\right) dx = \int_0^b \Lambda \frac{w(x)}{p(x)} y(x)\bar{y}(x)\, dx.$$

We obtain the following inequality for eigenvalue Λ:

$$\Lambda = \frac{\int_0^b \left(-\bar{y}(x) D_R^{2\alpha} y(x) + \frac{q(x)}{p(x)} \bar{y}(x) y(x)\right) dx}{\int_0^b \frac{w(x)}{p(x)} y(x)\bar{y}(x)\, dx}$$

$$= \frac{\int_0^b \left(|{}^c D_{0+}^{\alpha} y(x)|^2 + \frac{q(x)}{p(x)} |y(x)|^2\right) dx}{\int_0^b \frac{w(x)}{p(x)} |y(x)|^2\, dx}$$

$$\geq \frac{\int_0^b \frac{q(x)}{p(x)} |y(x)|^2 dx}{\int_0^b \frac{w(x)}{p(x)} |y(x)|^2\, dx} \geq \frac{\min_{x\in[0,b]} \frac{q(x)}{p(x)}}{\Theta}$$

and this ends the proof. □

The proof for FSLP (28–30) is analogous so we only formulate the result.

Proposition 4 *Let (H1) be fulfilled and* $q \neq 0$. *Eigenvalues of the fractional eigenvalue problem* (28–30) *are real. The set of eigenvalues is bounded below, namely*

$$\Lambda \geq \frac{\min_{x\in[0,b]} q(x)p(x)}{\theta}, \tag{34}$$

where $\theta = ||w\, p||$ *when* $\min_{x\in[0,b]} q(x) < 0$ *and* $\theta = \min_{x\in[0,b]} w(x)p(x)$ *otherwise.*

When $q = 0$ FSLPs given in (25–27) or (28–30) can be rewritten in the form including the left and right fractional derivatives and introduced in our previous papers [6, 7, 9–11]. In addition, we prove a general boundedness result.

Proposition 5 *Let (H1) be fulfilled. Eigenvalues of FSLPs:*

$$-p(x)\, D_R^{2\alpha} y(x) = \Lambda w(x) y(x) \tag{35}$$

$$y(0) = 0 \quad y(b) = 0 \tag{36}$$

$$y \in I_{0+}^{\alpha}(\Omega_1) \tag{37}$$

and

$$-D_R^{2\alpha}\, p(x) y(x) = \Lambda w(x) y(x) \tag{38}$$

$$y(0) = 0 \quad y(b) = 0 \tag{39}$$

$$y \in I_{0+}^{\alpha}(\Omega_1) \tag{40}$$

are positive and fulfill the inequality

$$\Lambda \geq \frac{1}{K_\alpha^2 \cdot ||\frac{w}{p}||}, \tag{41}$$

where $||\cdot||$ *denotes the supremum norm in space* $C[0,b]$.

Proof We multiply Eq. (35) with $\bar{y}$, integrate over interval $[0, b]$ and obtain by applying Lemma 5:

$$-\int_0^b \bar{y}(x) D_R^{2\alpha} y(x)dx = \int_0^b \bar{y}(x) D_{b-}^{\alpha}\, {}^cD_{0+}^{\alpha} y(x)dx = \int_0^b \Lambda \frac{w(x)}{p(x)} \bar{y}(x)y(x)dx.$$

Next, we transform the part including fractional derivatives using the fractional and classical integration by parts formulas and boundary conditions:

$$\int_0^b {}^cD_{0+}^{\alpha}\bar{y}(x) \cdot {}^cD_{0+}^{\alpha} y(x)dx = \Lambda \int_0^b \frac{w(x)}{p(x)} |y(x)|^2 dx.$$

Thence, eigenvalue Λ can be expressed and estimated as follows:

$$\Lambda = \frac{\int_0^b \left| {}^cD_{0+}^{\alpha} y(x) \right|^2 dx}{\int_0^b \frac{w(x)}{p(x)} |y(x)|^2 dx} > 0.$$

Let us note that as $y \in C[0, b]$ and $y(0) = 0$, we have

$$\int_0^b \frac{w(x)}{p(x)} |y(x)|^2 dx = \int_0^b \frac{w(x)}{p(x)} |I_{0+}^{\alpha}\, {}^cD_{0+}^{\alpha} y(x)|^2 dx \le K_\alpha^2 \cdot ||\frac{w}{p}|| \cdot ||{}^cD_{0+}^{\alpha} y||_{L^2}^2.$$

From the above inequality we obtain for eigenvalue Λ estimation (41), namely:

$$\Lambda \ge \frac{||{}^cD_{0+}^{\alpha} y||_{L^2}^2}{K_\alpha^2 \cdot ||\frac{w}{p}|| \cdot ||{}^cD_{0+}^{\alpha} y||_{L^2}^2} = \frac{1}{K_\alpha^2 \cdot ||\frac{w}{p}||}.$$

The proof of the second part of the proposition is analogous. □

Next, we show that the FSLPs with the assumed vanishing Dirichlet boundary conditions have orthogonal sets of eigenfunctions.

Proposition 6 *Let (H1) be fulfilled. Eigenfunctions of the fractional eigenvalue problem* (25–27) *corresponding to distinct eigenvalues are orthogonal in the* $L^2_{w/p}(0, b)$*-space.*

Proof Let functions y_1, y_2 be eigenfunctions, corresponding to eigenvalues Λ_1 and Λ_2. Thence, y_1 and $\bar{y}_2$ fulfill equations:

$$-p(x)\, D_R^{2\alpha} y_1(x) + q(x)y_1(x) = \Lambda_1 w(x)y_1(x) \tag{42}$$

$$-p(x)\, D_R^{2\alpha} \bar{y}_2(x) + q(x)\bar{y}_2(x) = \Lambda_2 w(x)\bar{y}_2(x). \tag{43}$$

We multiply the above equations by $\bar{y}_2$ and y_1 respectively and subtract to obtain the following relation

$$-\bar{y}_2(x)D_R^{2\alpha}y_1(x) + y_1(x)D_R^{2\alpha}\bar{y}_2(x) = \frac{w(x)}{p(x)}(\Lambda_1 - \Lambda_2)y_1(x)\bar{y}_2(x).$$

Now, we integrate both sides:

$$\int_0^b \left(-\bar{y}_2(x)D_R^{2\alpha}y_1(x) + y_1(x)D_R^{2\alpha}\bar{y}_2(x)\right) dx = (\Lambda_1 - \Lambda_2)\int_0^b \frac{w(x)}{p(x)}y_1(x)\bar{y}_2(x)dx$$

and by using Lemma 6, for $\Lambda_1 \neq \Lambda_2$, we arrive at

$$\int_0^b \frac{w(x)}{p(x)}y_1(x)\bar{y}_2(x)dx = 0.$$

Therefore, eigenfunctions corresponding to distinct eigenvalues are orthogonal in the $L^2_{w/p}(0, b)$-space. □

We omit the proof of the analogous result.

Proposition 7 *Let (H1) be fulfilled. Eigenfunctions of the fractional eigenvalue problem* (28–30) *corresponding to distinct eigenvalues are orthogonal in the* $L^2_{wp}(0, b)$*-space.*

Finally, by applying Lemma 6, we note that FSLEs (25, 28) are in fact Euler-Lagrange equations for respective actions.

Proposition 8 *Let (H1) be fulfilled. Consider the following action functional:*

$$S_1 = \int_0^b \frac{1}{2}\left[-f(x)D_R^{2\alpha}f(x) + f(x)\left(\frac{q(x)}{p(x)} - \Lambda\frac{w(x)}{p(x)}\right)f(x)\right] dx \tag{44}$$

on the space of real-valued functions with a support bounded by interval $[0, b]$*, continuous in* $[0, b]$ *and fulfilling vanishing Dirichlet boundary conditions. Then, FSLE* (25) *is its Euler-Lagrange equation.*
Consider the following action functional:

$$S_2 = \int_0^b \frac{1}{2}\left[-p(x)f(x)D_R^{2\alpha}p(x)f(x) + f(x)\left(q(x)p(x) - \Lambda w(x)p(x)\right)f(x)\right] dx \tag{45}$$

on the space of real-valued functions with a support bounded by interval $[0, b]$*, continuous in* $[0, b]$ *and fulfilling vanishing Dirichlet boundary conditions. Then, FSLE* (28) *is its Euler-Lagrange equation.*

Proof Let us assume that $\eta \in C[0, b]$ is an arbitrary continuous function obeying $\eta(0) = \eta(b) = 0$. Then

$$(f + \eta)(0) = (f + \eta)(b) = 0$$

and the corresponding variation of the action functional is:

$$\delta S_1(\eta) = -\int_0^b \frac{1}{2}\left[\eta(x)D_R^{2\alpha}f(x) + f(x)D_R^{2\alpha}\eta(x)\right] dx$$

$$+\int_0^b \eta(x)\left(\frac{q(x)}{p(x)} - \Lambda\frac{w(x)}{p(x)}\right)f(x)dx.$$

From Lemma 6 it follows that

$$\delta S_1(\eta) = \int_0^b \eta(x)\left[-D_R^{2\alpha} + \left(\frac{q(x)}{p(x)} - \Lambda\frac{w(x)}{p(x)}\right)\right]f(x)dx.$$

As variation η was arbitrary we infer from the extremal condition $\delta S_1(\eta) = 0$ that:

$$\left[-D_R^{2\alpha} + \frac{q(x)}{p(x)} - \Lambda\frac{w(x)}{p(x)}\right]f(x) = 0, \quad x \in [0, b].$$

Under assumption (H1) the above equation coincides with (25).
To prove the second part of the thesis let us note that the arbitrary variation $\eta \in C[0, b]$ obeying $\eta(0) = \eta(b) = 0$ leads to the following variation of action functional S_2

$$\delta S_2(\eta) = -\int_0^b \frac{1}{2}\left[p(x)\eta(x)D_R^{2\alpha}p(x)f(x) + p(x)f(x)D_R^{2\alpha}p(x)\eta(x)\right] dx$$

$$+\int_0^b \eta(x)\left(q(x)p(x) - \Lambda w(x)p(x)\right)f(x)dx.$$

Applying Lemma 6 we obtain

$$\delta S_2(\eta) = \int_0^b p(x)\eta(x)\left[-D_R^{2\alpha}p(x) + (q(x) - \Lambda w(x))\right]f(x)dx$$

and the principle $\delta S_2(\eta) = 0$ leads to the equation

$$\left[-D_R^{2\alpha}p(x) + q(x) - \Lambda w(x)\right]f(x) = 0, \quad x \in [0, b]$$

which coincides with (28).

4 Conclusions

In the paper, we introduced a simple regular fractional Sturm-Liouville problem on a bounded domain, constructed using the Riesz derivative. We only studied it assuming the vanishing boundary conditions and show that this version of FSLP retains the important properties of classical SLPs and FSLPs formulated previously in [6, 7]. We proved that its eigenvalues are real and eigenfunctions are continuous and form an orthogonal set of functions in the respective L^2-spaces. The results, discussed here, will be extended to the formulation with a generalized version of boundary conditions as well as to many-term FSLPs. The important feature of the classical and fractional SLPs is the existence of a purely discrete, countable spectrum and the eigenfunctions' bases under suitable boundary conditions. We expect that it is also valid in the case of FSLPs (25, 27) and (28, 30) and we shall follow this line of investigation in our subsequent paper.

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Multidimensional Discrete-Time Fractional Calculus of Variations

Agnieszka B. Malinowska and Tatiana Odzijewicz

Abstract In this paper a discrete-time multidimensional fractional calculus of variations is introduced. The fractional operators are defined in the sense of Grünvald–Letnikov. We derive necessary optimality conditions and then give examples illustrating the use of obtained results.

Keywords Backward fractional difference · Forward fractional difference · Grünvald–Letnikov fractional difference, Euler–Lagrange equations

1 Introduction

The continuous-time fractional calculus of variations (CFCV) has been widely developed since 1996 when the seminal paper [1], by Fred Riewe, about this subject was published. The literature on the CFCV is vast and covers problems with different types of fractional integrals and/or derivatives. We refer the reader to [2, 3] for a general treatment of CFCV. Significantly less works is devoted to the discrete-time fractional calculus of variations (DFCV). Bastos et al. [4, 5] published in 2011 papers introducing the DFCV. They proved first and second order necessary optimality conditions for the basic problems of the calculus of variations depending on the right and left Riemann–Liouville fractional differences [6, 7]. The fractional difference considered in this paper is based on the Grünvald–Letnikov fractional derivative

A.B. Malinowska (✉)
Faculty of Computer Science, Bialystok University of Technology,
15-351 Bialystok, Poland
e-mail: a.malinowska@pb.edu.pl

T. Odzijewicz
Department of Mathematics and Mathematical Economics, Warsaw School of Economics,
02-554 Warsaw, Poland

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[8–10] and it is defined in the following way: let function $f : \{0, 1, \ldots, k\} \to \mathbb{R}$, then the fractional α-order (backward) difference on f is given by

$$_0\Delta_k^\alpha f(k) := \sum_{i=0}^{k} (-1)^i \frac{\alpha(\alpha-1)\cdots(\alpha-i+1)}{i!} f(k-i).$$

An interest in the DFCV with the fractional α-order (backward) difference has been shown by Bourdin et al. [11, 12], who discussed in [11] the Gauss Grünvald–Letnikov embedding and the corresponding variational integrators on fractional Lagrangian systems. In this context they defined the forward fractional difference of order α as follows:

$$_k\Delta_N^\alpha f(k) := \sum_{i=0}^{N-k} (-1)^i \frac{\alpha(\alpha-1)\cdots(\alpha-i+1)}{i!} f(k+i),$$

where $f : \{0, 1, \ldots, N\} \to \mathbb{R}$, (note that here we assume that $h = 1$). In [12] a fractional Noether-type theorem without transformation of time was proved. Those two papers coincide with an opinion presented in [13] by Ortigueira: *"most of the articles that appear in the scientific literature, in the framework of the fractional calculus and their applications, the authors use those derivatives (Riemann–Liouville and/or Caputo) but at the end they contrast their model using a numerical approach based in a finite number of terms from the series that define the Grünvald–Letnikov derivative"*. This can be also observed in framework of the CFCV (see, e.g., [14]). Therefore, we consider pertinent to develop the theory of the DFCV with the fractional α-order difference.

The paper is organized as follows. In Sect. 2, we define partial backward and forward fractional differences, and remind results that will be useful in the sequel. Our results are presented in Sect. 3: we prove necessary and sufficient optimality conditions for basic and isoperimetric two-dimensional problems of the DFCV. Clearly, those results can be easily generalized to the high-dimensional case. Finally, in Sect. 4, we illustrate our results through examples.

2 Preliminaries

Let $K_1 = \{0, 1, 2, \ldots, N\}$, $K_2 = \{0, 1, 2, \ldots, M\}$ be two given subsets of $\mathbb{Z}$ and put $D = \{(k_1, k_2) : k_1 \in K_1, k_2 \in K_2\}$, which is a complete matric space with the metric d defined by

$$d((k_1, k_2), (k_1', k_2')) = \sqrt{(k_1' - k_1)^2 + (k_2' - k_2)^2}$$

for (k_1, k_2), $(k_1^{'}, k_2^{'}) \in D$. For a given δ, the δ-neighborhood of $(k_1^{'}, k_2^{'})$ is given by

$$U_\delta(k_1^{'}, k_2^{'}) = \{(k_1, k_2) \in D : d((k_1^{'}, k_2^{'}), (k_1, k_2)) < \delta\}.$$

In what follows $\alpha, \beta \in \mathbb{R}$ and $0 < \alpha, \beta \leq 1$. Moreover, we set

$$a_i^{(\alpha)} = \begin{cases} 1, & \text{if } i = 0 \\ (-1)^i \frac{\alpha(\alpha-1)\cdots(\alpha-i+1)}{i!}, & \text{if } i = 1, 2, \ldots \end{cases}$$

and

$$b_j^{(\beta)} = \begin{cases} 1, & \text{if } j = 0 \\ (-1)^j \frac{\beta(\beta-1)\cdots(\beta-j+1)}{j!}, & \text{if } j = 1, 2, \ldots . \end{cases}$$

The definition of the fractional backward (or forward) difference can be extended to discrete functions of two variables.

Definition 1 The first-order partial backward fractional difference of order α with respect to k_1 of function $f : D \to \mathbb{R}$ is defined by

$${}_0\Delta_{k_1}^{\alpha} f(k_1, k_2) := \sum_{i=0}^{k_1} a_i^{(\alpha)} f(k_1 - i, k_2),$$

while

$${}_0\Delta_{k_2}^{\beta} f(k_1, k_2) := \sum_{j=0}^{k_2} b_j^{(\beta)} f(k_1, k_2 - j)$$

is first-order partial backward fractional difference of order β with respect to k_2 of function f.

Next we define partial forward fractional differences.

Definition 2 The first-order partial forward fractional difference of order α with respect to k_1 of function $f : D \to \mathbb{R}$ is defined by

$${}_{k_1}\Delta_{N}^{\alpha} f(k_1, k_2) := \sum_{i=0}^{N-k_1} a_i^{(\alpha)} f(k_1 + i, k_2),$$

while

$${}_{k_2}\Delta_{M}^{\beta} f(k_1, k_2) := \sum_{j=0}^{M-k_2} b_j^{(\beta)} f(k_1, k_2 + j)$$

is first-order partial forward fractional difference of order β with respect to k_2 of function f.

Example 1 (cf. [10]) Let

$$f(k_1,k_2) = \begin{cases} 0, & \text{if } k_1,k_2 < 0 \\ 1, & \text{if } k_1,k_2 \geq 0. \end{cases}$$

Then

$${}_0\Delta^{\alpha}_{k_1} f(k_1,k_2) = \sum_{i=0}^{k_1} a_i^{(\alpha)}, \quad {}_0\Delta^{\beta}_{k_2} f(k_1,k_2) = \sum_{j=0}^{k_2} b_j^{(\beta)},$$

and

$${}_{k_1}\Delta^{\alpha}_{N} f(k_1,k_2) = \sum_{i=0}^{N-k_1} a_i^{(\alpha)}, \quad {}_{k_2}\Delta^{\beta}_{M} f(k_1,k_2) = \sum_{j=0}^{M-k_2} b_j^{(\beta)}.$$

Fractional backward (or forward) differences are linear operators.

Theorem 1 (cf. [10]) *Let f, g be two real functions defined on D and $a, b \in \mathbb{R}$. Then*

$${}_0\Delta^{\alpha}_{k_1}[af(k_1,k_2) + bg(k_1,k_2)] = a{}_0\Delta^{\alpha}_{k_1} f(k_1,k_2) + b{}_0\Delta^{\alpha}_{k_1} g(k_1,k_2).$$

Similar results hold for ${}_0\Delta^{\beta}_{k_2}$, ${}_{k_1}\Delta^{\alpha}_{N}$ and ${}_{k_2}\Delta^{\beta}_{M}$.

In order to obtain an analogue of the Euler–Lagrange equation for fractional problems we need the following formula of the summation by parts for one dimensional fractional operators.

Lemma 1 (cf. [11]) *Let f, g be two real functions defined on K_1. Then*

$$\sum_{k=0}^{N} g(k){}_0\Delta^{\alpha}_{k} f(k) = \sum_{k=0}^{N} f(k){}_k\Delta^{\alpha}_{N} g(k).$$

If $f(0) = f(N) = 0$ or $g(0) = g(N) = 0$, then

$$\sum_{k=1}^{N} g(k){}_0\Delta^{\alpha}_{k} f(k) = \sum_{k=0}^{N-1} f(k){}_k\Delta^{\alpha}_{N} g(k). \tag{1}$$

3 Main Result

Let a function $L : \mathbb{R}^3 \times D \to \mathbb{R}$, $(x, w, v, k_1, k_2) \mapsto L(x, w, v, k_1, k_2)$ be given. We assume that L has continuous first order partial derivatives with respect to x, w, v, those derivatives we denote by: L_x, L_w, L_v. Function L is called a Lagrangian. The problem under our consideration is to extremize (minimize or maximize) functional

$$\mathcal{L}[x] = \sum_{k_1=1}^{N} \sum_{k_2=1}^{M} L(x(k_1, k_2), {}_0\Delta_{k_1}^{\alpha} x(k_1, k_2), {}_0\Delta_{k_2}^{\beta} x(k_1, k_2), k_1, k_2) \tag{2}$$

on $\mathcal{D} = \{x : D \to \mathbb{R} : x|_{\mathrm{bd}D} = g\}$, where g is a fixed function defined on bd$D$.

Definition 3 A function $\tilde{x} \in \mathcal{D}$ is called a local minimizer (or maximizer) for functional $\mathcal{L}$ on $\mathcal{D}$ provided there exists $\delta > 0$ such that $\mathcal{L}[\tilde{x}] \le \mathcal{L}[x]$ (or $\mathcal{L}[\tilde{x}] \ge \mathcal{L}[x]$) for all $x \in \mathcal{D}$ such that $\|\tilde{x} - x\| < \delta$, where $\|f\| = \max_{(k_1,k_2)\in D} |f(k_1, k_2)|$.

Definition 4 A function $\eta : D \to \mathbb{R}$ is called an admissible variation provided $\eta \neq 0$ and $\eta|_{\mathrm{bd}D} = 0$.

For a fixed function $x \in \mathcal{D}$ and a fixed admissible variation η we define a function $\Phi : \mathbb{R} \to \mathbb{R}$ by

$$\Phi(\varepsilon) = \Phi(\varepsilon; x, \eta) = \mathcal{L}[x + \varepsilon\eta].$$

By assumptions imposed on L, Φ is continuously differentiable. The first variation of the functional $\mathcal{L}$ at x we define by

$$\mathcal{L}_1[x, \eta] = \Phi'(0; x, \eta).$$

It follows that

$$\begin{aligned}\mathcal{L}_1[x, \eta] &= \Phi'(0)\\ &= \sum_{k_1=1}^{N} \sum_{k_2=1}^{M} \Big(L_x(\cdot)\eta(k_1, k_2) + L_w(\cdot){}_0\Delta_{k_1}^{\alpha} \eta(k_1, k_2) + L_v(\cdot){}_0\Delta_{k_2}^{\beta} \eta(k_1, k_2)\Big),\end{aligned} \tag{3}$$

where $(\cdot) = (x(k_1, k_2), {}_0\Delta_{k_1}^{\alpha} x(k_1, k_2), {}_0\Delta_{k_2}^{\beta} x(k_1, k_2), k_1, k_2)$. The next standard theorem serves a necessary optimality condition for local minimizers (or maximizers) of $\mathcal{L}$.

Theorem 2 *If $\tilde{x} \in \mathcal{D}$ is a local minimizer (or maximizer) of $\mathcal{L}$, then $\mathcal{L}_1[\tilde{x}, \eta] = 0$ for all admissible variations.*

Now we can derive a necessary optimality condition of the Euler–Lagrange type.

Theorem 3 *If $\tilde{x} \in \mathcal{D}$ is a local minimizer (or maximizer) of $\mathcal{L}$, then it satisfies the Euler–Lagrange equation of the following form:*

$$L_x(\cdot) + {}_{k_1}\Delta_N^\alpha L_w(\cdot) + {}_{k_2}\Delta_M^\beta L_v(\cdot) = 0 \tag{4}$$

for all $(k_1, k_2) \in \{1, 2, \ldots, N-1\} \times \{1, 2, \ldots, M-1\}$, *where* $(\cdot) = (x(k_1, k_2), {}_0\Delta_{k_1}^\alpha x(k_1, k_2), {}_0\Delta_{k_2}^\beta x(k_1, k_2), k_1, k_2)$.

Proof By Theorem 2 we have $\mathcal{L}_1[\tilde{x}, \eta] = 0$ for all admissible variations. Therefore,

$$0 = \sum_{k_1=1}^{N} \sum_{k_2=1}^{M} \left(L_x(\star)\eta(k_1, k_2) + L_w(\star)_0\Delta_{k_1}^\alpha \eta(k_1, k_2) + L_v(\star)_0\Delta_{k_2}^\beta \eta(k_1, k_2) \right), \tag{5}$$

where $(\star) = (\tilde{x}(k_1, k_2), {}_0\Delta_{k_1}^\alpha \tilde{x}(k_1, k_2), {}_0\Delta_{k_2}^\beta \tilde{x}(k_1, k_2), k_1, k_2)$. Since $\eta|_{\mathrm{bd}D} = 0$, using (1) we have

$$\sum_{k_1=1}^{N} \sum_{k_2=1}^{M} L_w(\star)_0\Delta_{k_1}^\alpha \eta(k_1, k_2) = \sum_{k_1=1}^{N-1} \sum_{k_2=1}^{M} {}_{k_1}\Delta_N^\alpha L_w(\star)\eta(k_1, k_2)$$

and

$$\sum_{k_1=1}^{N} \sum_{k_2=1}^{M} L_v(\star)_0\Delta_{k_2}^\beta \eta(k_1, k_2) = \sum_{k_1=1}^{N} \sum_{k_2=1}^{M-1} {}_{k_2}\Delta_M^\beta L_v(\star)\eta(k_1, k_2).$$

Consequently, remembering that $\eta|_{\mathrm{bd}D} = 0$, we get from (5):

$$0 = \sum_{k_1=1}^{N-1} \sum_{k_2=1}^{M-1} \left(L_x(\star) + {}_{k_1}\Delta_N^\alpha L_w(\star) + {}_{k_2}\Delta_M^\beta L_v(\star) \right) \eta(k_1, k_2).$$

Since the value of η is arbitrary on $\{1, 2, \ldots, N-1\} \times \{1, 2, \ldots, M-1\}$, the Euler–Lagrange equation (4) holds along $\tilde{x}$.

Definition 5 A function $\tilde{x} \in \mathcal{D}$ that is a solution to the Euler–Lagrange equation (4) we call an extremal of $\mathcal{L}$.

The next theorem provides a sufficient condition for an extremal to be a global minimaizer (maximizer).

Theorem 4 *Let function L in (2) be jointly convex (concave) with respect (x, w, v) for all $(k_1, k_2) \in D$. If $\tilde{x} \in \mathcal{D}$ is a solution to the Euler–Lagrange equation (4), then it is a global minimaizer (maximizer) of functional (2) on $\mathcal{D}$.*

Proof Assume that L is jointly convex with respect (x, w, v) for all $(k_1, k_2) \in D$, then for any h such that $\tilde{x} + h \in \mathcal{D}$, we have

$$\mathcal{L}[\tilde{x}+h] - \mathcal{L}[\tilde{x}] = \sum_{k_1=1}^{N}\sum_{k_2=1}^{M}\Big[L(\tilde{x}(k_1,k_2)+h(k_1,k_2), {}_0\Delta_{k_1}^{\alpha}(\tilde{x}(k_1,k_2)+h(k_1,k_2)),$$

$$ {}_0\Delta_{k_2}^{\beta}(\tilde{x}(k_1,k_2),k_1,k_2)+h(k_1,k_2)) - L(\tilde{x}(k_1,k_2), {}_0\Delta_{k_1}^{\alpha}\tilde{x}(k_1,k_2), {}_0\Delta_{k_2}^{\beta}\tilde{x}(k_1,k_2),k_1,k_2)\Big]$$

$$\geq \sum_{k_1=1}^{N}\sum_{k_2=1}^{M}\Big(L_x(\star)h(k_1,k_2) + L_w(\star){}_0\Delta_{k_1}^{\alpha}h(k_1,k_2) + L_v(\star){}_0\Delta_{k_2}^{\beta}h(k_1,k_2)\Big),$$

where $(\star) = (\tilde{x}(k_1,k_2), {}_0\Delta_{k_1}^{\alpha}\tilde{x}(k_1,k_2), {}_0\Delta_{k_2}^{\beta}\tilde{x}(k_1,k_2),k_1,k_2)$. Proceeding as in the proof of Theorem 3, we obtain

$$\mathcal{L}[\tilde{x}+h] - \mathcal{L}[\tilde{x}] \geq \sum_{k_1=1}^{N-1}\sum_{k_2=1}^{M-1}\Big(L_x(\star) + {}_{k_1}\Delta_N^{\alpha}L_w(\star) + {}_{k_2}\Delta_M^{\beta}L_v(\star)\Big)h(k_1,k_2).$$

As $\tilde{x}$ satisfies Eq. (4) we have $\mathcal{L}(\tilde{x}+h) - \mathcal{L}(\tilde{x}) \geq 0$.

Now we shall consider the isoperimetric problem, one of the oldest and interesting class of variational problems with roots in the Queen Dido problem of the calculus of variations. The discrete fractional isoperimetric problem is defined in the following way: extremize (minimize or maximize) functional

$$\mathcal{L}[x] = \sum_{k_1=1}^{N}\sum_{k_2=1}^{M} L(x(k_1,k_2), {}_0\Delta_{k_1}^{\alpha}x(k_1,k_2), {}_0\Delta_{k_2}^{\beta}x(k_1,k_2),k_1,k_2) \tag{6}$$

on $\mathcal{D} = \{x : D \to \mathbb{R} : x|_{\mathrm{bd}D} = g\}$, where g is a fixed function defined on bd$D$ and subject to the isoperimetric constraint

$$\mathcal{I}[x] = \sum_{k_1=1}^{N}\sum_{k_2=1}^{M} G(x(k_1,k_2), {}_0\Delta_{k_1}^{\alpha}x(k_1,k_2), {}_0\Delta_{k_2}^{\beta}x(k_1,k_2),k_1,k_2) = \xi, \tag{7}$$

where $\xi \in \mathbb{R}$ is given, and L, G have continuous first order partial derivatives with respect to x, w, v.

Theorem 5 *If $\tilde{x} \in \mathcal{D}$ is a local minimizer (or maximizer) of* (6) *subject to the isoperimetric constraint* (7)*, then there exist two real constants, λ_0 and λ, not both zero, such that $\tilde{x}$ satisfies the following equation:*

$$H_x(\cdot) + {}_{k_1}\Delta_N^{\alpha}H_w(\cdot) + {}_{k_2}\Delta_M^{\beta}H_v(\cdot) = 0$$

for all $(k_1,k_2) \in \{1,2,\ldots,N-1\} \times \{1,2,\ldots,M-1\}$, where
$(\cdot) = (x(k_1,k_2), {}_0\Delta_{k_1}^{\alpha}x(k_1,k_2), {}_0\Delta_{k_2}^{\beta}x(k_1,k_2),k_1,k_2)$ and $H = \lambda_0 L + \lambda G$.

Proof Can be done by using the abnormal Lagrange multiplier rule [[15], Theorem 4.1.3].

Remark 1 If $\tilde{x}$ is a normal extremizer to the isoperimetric problem, that is, $\tilde{x}$ is not a solution to equation

$$G_x(\cdot) + {}_{k_1}\Delta_N^{\alpha} G_w(\cdot) + {}_{k_2}\Delta_M^{\beta} G_v(\cdot) = 0,$$

where $(\cdot) = (x(k_1,k_2), {}_0\Delta_{k_1}^{\alpha} x(k_1,k_2), {}_0\Delta_{k_2}^{\beta} x(k_1,k_2), k_1, k_2)$, then we can choose $\lambda_0 = 1$ in Theorem 5. For abnormal extremizers, Theorem 5 holds with $\lambda_0 = 0$. The condition $(\lambda_0, \lambda) \neq 0$ guarantees that Theorem 5 is a useful necessary optimality condition.

4 Example

In this section we present three illustrative examples. In the first example the two-dimensional problem is considered. We show that the Lagrangian is invariant under a gauge transformation. Therefore, we can expect that the Euler–Lagrange equations do not uniquely determine a solution to this problem (see, e.g., [16]). The next example shows that the solutions of the fractional problems coincide with the solutions of the corresponding non-fractional variational problems when the order of the discrete derivatives is an integer value [17]. Moreover, we observe that, in this particular case, solutions obtained by us are very similar to those presented in [4], where problems with the Riemann–Liouville fractional differences were considered. In Example 4 the isoperimetric problem is considered. This problem leads us to the discrete fractional Sturm–Liouville eigenvalue problem.

Example 2 Let us consider the following problem: minimize

$$\mathcal{L}[x_1,x_2] = \sum_{k_1=1}^{N} \sum_{k_2=1}^{M} \left({}_0\Delta_{k_1}^{\alpha} x_2(k_1,k_2) - {}_0\Delta_{k_2}^{\beta} x_1(k_1,k_2) \right)^2 \tag{8}$$

on $\mathcal{D} = \{(x_1,x_2) : D \to \mathbb{R}^2 : (x_1,x_2)|_{\mathrm{bd}D} = (g_1,g_2)\}$, where g_1, g_2 are fixed real functions defined on bd$D$. Necessary conditions for a minimizer are as follows:

$$\begin{aligned} {}_{k_1}\Delta_N^{\alpha} \left({}_0\Delta_{k_1}^{\alpha} x_2(k_1,k_2) - {}_0\Delta_{k_2}^{\beta} x_1(k_1,k_2) \right) &= 0, \\ {}_{k_2}\Delta_M^{\beta} \left({}_0\Delta_{k_1}^{\alpha} x_2(k_1,k_2) - {}_0\Delta_{k_2}^{\beta} x_1(k_1,k_2) \right) &= 0 \end{aligned}$$

for all $(k_1,k_2) \in \{1,2,\ldots,N-1\} \times \{1,2,\ldots,M-1\}$. Note that, the Lagrangian is invariant with respect to a gauge transformation:

$$\bar{x}_1(k_1,k_2) = x_1(k_1,k_2) + {}_0\Delta^{\alpha}_{k_1} f(k_1,k_2),$$
$$\bar{x}_2(k_1,k_2) = x_2(k_1,k_2) + {}_0\Delta^{\beta}_{k_2} f(k_1,k_2)$$

where $f : D \to \mathbb{R}$ is an arbitrary function. Indeed,

$$\begin{aligned}
&\left({}_0\Delta^{\alpha}_{k_1}\bar{x}_2(k_1,k_2) - {}_0\Delta^{\beta}_{k_2}\bar{x}_1(k_1,k_2)\right)^2 \\
&\quad = \left({}_0\Delta^{\alpha}_{k_1}(x_2(k_1,k_2) + {}_0\Delta^{\beta}_{k_2} f(k_1,k_2)) - {}_0\Delta^{\beta}_{k_2}(x_1(k_1,k_2) + {}_0\Delta^{\alpha}_{k_1} f(k_1,k_2))\right)^2 \\
&= \left({}_0\Delta^{\alpha}_{k_1} x_2(k_1,k_2) + {}_0\Delta^{\alpha}_{k_1}{}_0\Delta^{\beta}_{k_2} f(k_1,k_2) - {}_0\Delta^{\beta}_{k_2} x_1(k_1,k_2) - {}_0\Delta^{\beta}_{k_2}{}_0\Delta^{\alpha}_{k_1} f(k_1,k_2)\right)^2.
\end{aligned}$$

Since

$${}_0\Delta^{\alpha}_{k_1}{}_0\Delta^{\beta}_{k_2} f(k_1,k_2) = {}_0\Delta^{\beta}_{k_2}{}_0\Delta^{\alpha}_{k_1} f(k_1,k_2),$$

by Theorem 5.2.1 [10], the desired equality holds.

Example 3 Let us consider the following problem: minimize

$$\mathcal{L}[x] = \sum_{k=1}^{N} \left({}_0\Delta^{\alpha}_{k} x(k)\right)^2 \tag{9}$$

subject to $\mathcal{D} = \{x : D \to \mathbb{R} : x(0) = A, x(N) = B\}$, where $D = \{0,\dots,N\}$ and N, A, B are fixed. In this case the Euler–Lagrange equation takes the form

$${}_k\Delta^{\alpha}_{N}{}_0\Delta^{\alpha}_{k} x(k) = 0, \quad k = 1,\dots,N-1.$$

For $N = 2$, the solution to the considered problem is:

$$x(0) = 0, \quad x(1) = \frac{\alpha A + \alpha B + 1/2\alpha^2(\alpha-1)A}{1+\alpha^2}, \quad x(2) = B.$$

Observe that for $\alpha = 1$: $x(1) = \frac{A+B}{2}$, as one can expect. Indeed, for $\alpha = 1$ our problem coincides with a discrete problem: minimize

$$\sum_{k=1}^{2} (x(k) - x(k-1))^2$$

on $\mathcal{D} = \{x : D \to \mathbb{R} : x(0) = A, x(2) = B\}$.
Let us choose now $A = 0$, $B = 1$ and $N = 4$. Table 1 and Fig. 1 present solutions to the considered problem for different values of α.

Table 1 Minimizer values of Example 3 with $A = 0$, $B = 1$, $N = 4$, and different α's: $1/4, 1/2, 3/4, 1$

α	$\tilde{x}(1)$	$\tilde{x}(2)$	$\tilde{x}(3)$	$\mathcal{L}[\tilde{x}]$
0.25	0.007181731484	0.1454300624	0.2668462100	0.9185395467
0.50	0.1293800539	0.3234501348	0.5229110512	0.6788674886
0.75	0.2267883116	0.4544228483	0.68702895690	0.4255906641
1	0.25	0.50	0.75	0.25

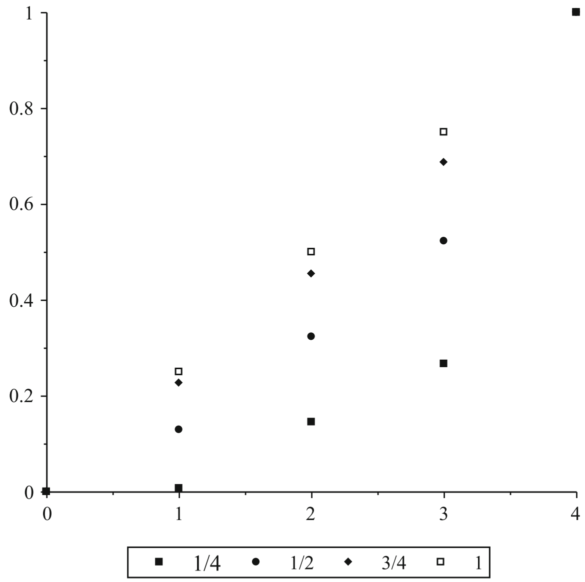

Fig. 1 Minimizer $\tilde{x}$ of Example 3 with $A = 0$, $B = 1$, $N = 4$, and different α's: $1/4, 1/2, 3/4, 1$

Note that the smallest value of $\mathcal{L}$ occurs for $\alpha = 1$ (for the classical non-fractional case). However, the values of function $\tilde{x}$ are respectively the biggest in the case of $\alpha = 1$.

Example 4 As the third example we consider the isoperimetric problem: minimize

$$\mathcal{L}[x] = \sum_{k=1}^{N} \left[p(k) \left({}_0\Delta_k^\alpha x(k) \right)^2 + q(k) x^2(k) \right] \tag{10}$$

on $\mathcal{D} = \{x : D \to \mathbb{R} : x(0) = 0, x(N) = 0\}$, and

$$\sum_{k=1}^{N} r(k) \left(x(k) \right)^2 = 1 \tag{11}$$

where $D = \{0, \dots, N\}$, N is fixed, and $p, r : D \to \mathbb{R}_+$, $q : D \to \mathbb{R}$. By Theorem 5 and Remark 1 every nontrivial solution to this problem has to satisfy the following equation:

$$ {}_k\Delta_N^\alpha \left(p(k) {}_0\Delta_k^\alpha x(k)\right) + q(k)x(k) = \lambda r(k)x(k), \quad k = 1, \dots, N-1, \tag{12} $$

for some λ. It is easily seen that Eq. (12) together with the boundary condition:

$$ x(0) = 0, \quad x(N) = 0, $$

is a kind of the Sturm–Liouville eigenvalue problem (see, e.g., [18–20]). Let us choose $D = \{0, \dots, 3\}$, $p = r = 1$, $q = 0$. Then extremizers for considered isoperimetric problem have to satisfy the following conditions:

$$ {}_k\Delta_3^\alpha {}_0\Delta_k^\alpha x(k) = \lambda x(k), \quad k = 1, 2, \tag{13} $$

$$ \sum_{k=1}^{3} (x(k))^2 = 1, \tag{14} $$

$$ x(0) = 0, \quad x(3) = 0. \tag{15} $$

Note that solutions to the system (13)–(15) depend on λ. Table 2 presents examples of solutions for different values of α and for chosen values of λ.

Table 2 Examples of solutions to to system of Eqs. (13)–(15) for different values of α's: $1/4, 1/2, 3/4, 1$

α	$x(1)$	$x(2)$	λ
0.25	−0.7002167868	−0.7139302847	0.8402894156
0.50	−0.7007659037	−0.71339130098	0.8202427511
0.75	−0.7048172168	−0.7093889560	0.8871928247
1	−0.7071067812	−0.7071067812	1

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On a New Symmetric Fractional Variable Order Derivative

Dominik Sierociuk, Wiktor Malesza and Michal Macias

Abstract The paper presents particular definitions of symmetric fractional variable order derivatives. The $\mathcal{AD}$ and $\mathcal{DA}$ types of the fractional variable order derivatives and their properties have been introduced. Additionally, the switching order schemes equivalent to these types of definitions have been shown. Finally, the theoretical considerations have been validated on numerical examples.

Keywords Fractional calculus · Variable order derivative

1 Introduction

Fractional calculus is a generalization of traditional integer order integration and differentiation actions onto non-integer orders fundamental operator. Nowadays, has been widely used by engineers and researchers in many areas. In [1], fractional calculus was applied to modeling behavior of ultracapacitors more efficiently than in classical way. In similar manner and effect was used to modelling a mechanical system, e.g. results for electrical drive system with flexible shaft [3].

When the order is not constant but depends on time, then the various types of fractional variable order derivatives can be distinguished. In literature plenty of such definitions can be encountered, however, authors put only minor emphasis on their interpretations. In [7], nine different variable order derivative definitions are given and in [2, 18], three general types of variable order definitions can be found but without clear interpretation of them. In papers [4, 11, 12, 14–16] the explanation for two

D. Sierociuk (✉) · W. Malesza · M. Macias
Institute of Control and Industrial Electronics, Warsaw University of Technology, Koszykowa 75, 00-662 Warsaw, Poland
e-mail: dsieroci@ee.pw.edu.pl

W. Malesza
e-mail: wmalesza@ee.pw.edu.pl

M. Macias
e-mail: michal.macias@ee.pw.edu.pl

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main types and two recursive types of derivatives in the form of switching schemes are given. The equivalence between particular types of definitions and appropriate switching strategies are proven by authors. Moreover, based on these strategies, analog models of proper types derivatives were build and validated according to their numerical implementations. Another methods for numerical realization of fractional variable order integrators or differentiators can be found in [8, 17]. Paper [9] shows comparison of control system behavior with fractional variable order PID controller designed according to few types of fractional variable order derivatives.

It is worth to mention that for variable order derivatives, known in literature, the opposite orders composition does not hold for one type of derivative, but is only satisfied between dual derivatives [13]. It can yield some difficulties, e.g., in analysis of variable order differential equations. It rises to the problem: if it is possible to derive alternative definitions of variable order derivatives for which the opposite orders composition hold, that is a composition of two derivatives of the same type, and of opposite orders, gives original function.

In this work, we will solve this problem by defining symmetric fractional variable order derivatives. The duality property between particular types of definitions presented with details in [13] are used to achieve some properties of symmetric operators. Additionally, the switching order schemes equivalent to these types of definitions have been shown.

The paper is organized as follows. In Sect. 2 particular types of fractional variable order derivatives are introduced. Section 3 contains the main contribution of the paper—symmetric variable order definitions, their basic properties and switching schemes. At the end the theoretical considerations are validated on numerical example.

2 Fractional Variable Order Grunwald-Letnikov Type Derivatives

As a base of generalization of the constant fractional order $\alpha \in \mathbb{R}$ derivative onto variable order case, the following definition is taken into consideration:

$$
{}_0\mathrm{D}_t^{\alpha} f(t) = \lim_{h \to 0} \frac{1}{h^{\alpha}} \sum_{r=0}^{n} (-1)^r \binom{\alpha}{r} f(t - rh), \tag{1}
$$

where $h > 0$ is a step time, and $n = \lfloor t/h \rfloor$.

The matrix form of the fractional order derivative is given as follows [5, 6]:

$$
\begin{pmatrix} {}_0\mathrm{D}_0^{\alpha} f(0) \\ {}_0\mathrm{D}_h^{\alpha} f(h) \\ {}_0\mathrm{D}_{2h}^{\alpha} f(2h) \\ \vdots \\ {}_0\mathrm{D}_{kh}^{\alpha} f(kh) \end{pmatrix} = \lim_{h \to 0} W(\alpha, k) \begin{pmatrix} f(0) \\ f(h) \\ f(2h) \\ \vdots \\ f(kh) \end{pmatrix},
$$

where

$$W(\alpha,k)=\begin{pmatrix} h^{-\alpha} & 0 & 0 & \dots & 0\\ w_{\alpha,1} & h^{-\alpha} & 0 & \dots & 0\\ w_{\alpha,2} & w_{\alpha,1} & h^{-\alpha} & \dots & 0\\ w_{\alpha,3} & w_{\alpha,2} & w_{\alpha,1} & \dots & 0\\ \vdots & \vdots & \vdots & \dots & \vdots\\ w_{\alpha,k} & w_{\alpha,k-1} & w_{\alpha,k-2} & \dots & h^{-\alpha} \end{pmatrix},$$

$W(\alpha,k)\in\mathbb{R}^{(k+1)\times(k+1)}$, $w_{\alpha,i}=\frac{(-1)^i\binom{\alpha}{i}}{h^\alpha}$, and $h=t/k$, k is a number of samples.

For the case of order changing with time (variable order case), many different types of definitions can be found in the literature [2, 18]. Among them, we present only two. The first one is obtained by replacing in (1) a constant order α by variable order $\alpha(t)$. In that approach, all coefficients for past samples are obtained for present value of the order, and is given as follows:

Definition 1 The $\mathcal{A}$ -type of fractional variable order derivative is defined as follows:

$$ {}_0^{\mathcal{A}}\mathrm{D}_t^{\alpha(t)}f(t)=\lim_{h\to 0}\frac{1}{h^{\alpha(t)}}\sum_{r=0}^{n}(-1)^r\binom{\alpha(t)}{r}f(t-rh). \tag{2}$$

The matrix form of the 1st type of fractional-variable order derivative [10] is given by

$$\begin{pmatrix} {}_0^{\mathcal{A}}\mathrm{D}_0^{\alpha(t)}f(0)\\ {}_0^{\mathcal{A}}\mathrm{D}_h^{\alpha(t)}f(h)\\ {}_0^{\mathcal{A}}\mathrm{D}_{2h}^{\alpha(t)}f(2h)\\ \vdots\\ {}_0^{\mathcal{A}}\mathrm{D}_{kh}^{\alpha(t)}f(kh) \end{pmatrix}=\lim_{h\to 0}{}^{\mathcal{A}}W(\alpha,k)\begin{pmatrix} f(0)\\ f(h)\\ f(2h)\\ \vdots\\ f(kh)\end{pmatrix}, \tag{3}$$

where

$$ {}^{\mathcal{A}}W(\alpha,k)=\begin{pmatrix} h^{-\alpha(0)} & 0 & 0 & \dots & 0\\ w_{\alpha(h),1} & h^{-\alpha(h)} & 0 & \dots & 0\\ w_{\alpha(2h),2} & w_{\alpha(2h),1} & h^{-\alpha(2h)} & \dots & 0\\ w_{\alpha(3h),3} & w_{\alpha(3h),2} & w_{\alpha(3h),1} & \dots & 0\\ \vdots & \vdots & \vdots & \dots & \vdots\\ w_{\alpha(kh),k} & w_{\alpha(kh),k-1} & w_{\alpha(kh),k-2} & \dots & h^{-\alpha(kh)} \end{pmatrix}. \tag{4}$$

Let us consider the following so-called output-reductive switching scheme [10, 16] presented in Fig. 1 based on the chain of derivatives blocks related by the following switching rule. The switches S_i, $i=1,\dots,N$, take the following positions

$$S_i=\begin{cases} b & \text{for } t_{i-1}\le t<t_i,\\ a & \text{otherwise,}\end{cases}\qquad i=1,\dots,N,$$

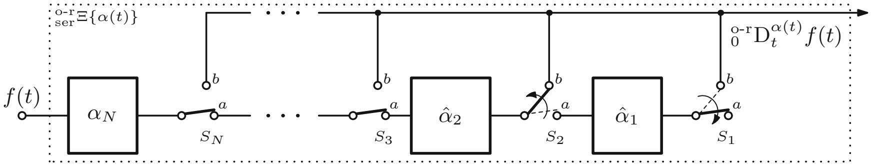

Fig. 1 Structure of output-reductive switching order scheme in serial form ${}^{\text{o-r}}_{\text{ser}}\Xi\{\alpha(t)\}$ (presented configuration: switching from α_1 to α_2)

and

$$\alpha_i = \alpha_{i+1} + \hat{\alpha}_i, \qquad i = 1, \dots, N-1.$$

Theorem 1 ([16]) *The $\mathcal{A}$-type of variable order derivative (given by Definition 1) is equivalent to output-reductive switching order scheme presented in Fig. 1, i.e.,*

$${}^{\mathcal{A}}_{0}\mathrm{D}^{\alpha(t)}_{t} f(t) \equiv {}^{\text{o-r}}_{0}\mathrm{D}^{\alpha(t)}_{t} f(t).$$

Second type of variable order derivative, we will consider, is given by the following definition:

Definition 2 ([16]) The $\mathcal{D}$-type of fractional variable order derivative is defined as follows:

$${}^{\mathcal{D}}_{0}\mathrm{D}^{\alpha(t)}_{t} f(t) = \lim_{h\to 0}\left(\frac{f(t)}{h^{\alpha(t)}} - \sum_{j=1}^{n}(-1)^j\binom{-\alpha(t)}{j}\, {}^{\mathcal{D}}_{0}\mathrm{D}^{\alpha(t)}_{t-jh} f(t)\right). \tag{5}$$

The $\mathcal{D}$-type fractional derivative can be expressed in the following matrix form [16]:

$$\begin{pmatrix} {}^{\mathcal{D}}_{0}\mathrm{D}^{\alpha(t)}_{0} f(0) \\ {}^{\mathcal{D}}_{0}\mathrm{D}^{\alpha(t)}_{h} f(h) \\ \vdots \\ {}^{\mathcal{D}}_{0}\mathrm{D}^{\alpha(t)}_{kh} f(kh) \end{pmatrix} = \lim_{h\to 0} \mathfrak{Q}^{k}_{0}(\alpha) \begin{pmatrix} f(0) \\ f(h) \\ \vdots \\ f(kh) \end{pmatrix}, \tag{6}$$

where

$$\mathfrak{Q}^{k}_{0}(\alpha) = \left(\begin{array}{c|c|c|c|c|c} h^{-\alpha_0} & 0 & 0 & \cdots & 0 & 0 \\ \hline \mathfrak{q}_{2,1} & h^{-\alpha_1} & 0 & \cdots & 0 & 0 \\ \hline \mathfrak{q}_{3,1} & \mathfrak{q}_{3,2} & h^{-\alpha_2} & \cdots & 0 & 0 \\ \hline \vdots & \vdots & \vdots & \ddots & \ddots & \vdots \\ \hline \mathfrak{q}_{k,1} & \mathfrak{q}_{k,2} & \mathfrak{q}_{k+1,3} & \cdots & h^{-\alpha_{k-1}} & 0 \\ \hline \mathfrak{q}_{k+1,1} & \mathfrak{q}_{k+1,2} & \mathfrak{q}_{k+1,3} & \cdots & \mathfrak{q}_{k+1,k} & h^{-\alpha_k} \end{array}\right), \tag{7}$$

where, for $i, j = 1, \dots, k+1$,

$$\mathfrak{q}_{i,j} = \begin{cases} q_{i-1}(\mathfrak{q}_{1,j}, \ldots, \mathfrak{q}_{i-1,j})^T & \text{for } i > j, \\ h^{-\alpha_i} & \text{for } i = j, \\ 0 & \text{for} i < j, \end{cases} \tag{8}$$

and for $r = 1, \ldots, k$

$$q_r = (-v_{-\alpha_r,r}, \ldots, -v_{-\alpha_r,1}) \in \mathbb{R}^{1\times r}, \tag{9}$$

$$v_{-\alpha_r,l} = (-1)^l \binom{-\alpha_r}{l}, \quad l = 1, \ldots, r, \tag{10}$$

that is, the pth element of q_r, $p = 1, \ldots, r$, is

$$(q_r)_p = -v_{-\alpha_r,r-p+1} = (-1)^{r-p+1} \binom{-\alpha_r}{r-p+1}. \tag{11}$$

Let us consider the following so-called input-reductive switching order scheme in the serial form, denoted ${}^{\text{i-r}}_{\text{ser}}\Xi\{\alpha(t)\}$ and presented in Fig. 2. The switches S_i, $i = 1, \ldots, N$, take the following positions

$$S_i = \begin{cases} b & \text{for } t_{i-1} \le t < t_i, \\ a & \text{otherwise}, \end{cases} \qquad i = 1, \ldots, N.$$

and

$$\alpha_i = \alpha_{i+1} + \hat{\alpha}_i, \qquad i = 1, \ldots, N-1.$$

Theorem 2 ([16]) *The $\mathcal{D}$-type of variable order derivative (given by Definition 2) is equivalent to input-reductive switching order scheme presented in Fig. 2, i.e.,*

$${}^{\mathcal{D}}_0\mathrm{D}^{\alpha(t)}_t f(t) \equiv {}^{\text{i-r}}_0\mathrm{D}^{\alpha(t)}_t f(t).$$

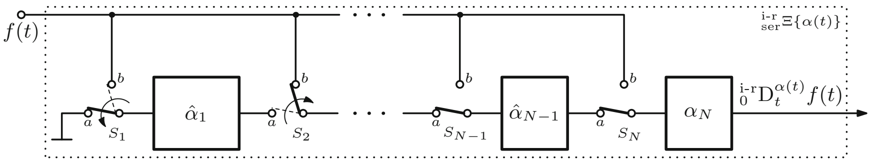

Fig. 2 Structure of input-reductive switching order scheme ${}^{\text{i-r}}_{\text{ser}}\Xi\{\alpha(t)\}$ (presented configuration: switching from α_1 to α_2)

Remark 1 For a fractional constant order $\alpha = \text{const}$, the fractional derivatives given by Definitions 1 and 2 are numerically identical with constant order fractional derivative given by (1).

Remark 2 What is crucial for both $\mathcal{A}$-type and $\mathcal{D}$-type derivative is that they are, in general—for variable order $\alpha(t)$, not symmetric, i.e.,

$$ {}_0^{\mathcal{A}}\mathrm{D}_t^{\alpha(t)} {}_0^{\mathcal{A}}\mathrm{D}_t^{-\alpha(t)} f(t) \neq f(t) $$

and

$$ {}_0^{\mathcal{D}}\mathrm{D}_t^{\alpha(t)} {}_0^{\mathcal{D}}\mathrm{D}_t^{-\alpha(t)} f(t) \neq f(t). $$

However, the following duality properties [13] hold

$$ {}_0^{\mathcal{A}}\mathrm{D}_t^{\alpha(t)} {}_0^{\mathcal{D}}\mathrm{D}_t^{-\alpha(t)} f(t) = f(t) \tag{12} $$

and

$$ {}_0^{\mathcal{D}}\mathrm{D}_t^{\alpha(t)} {}_0^{\mathcal{A}}\mathrm{D}_t^{-\alpha(t)} f(t) = f(t). \tag{13} $$

3 Main Result—symmetric Variable Order Derivatives

This section contains the main contribution of the paper, that is introducing definitions of variable order derivatives possessing fundamental property—opposite orders composition for one type of derivative. The operators satisfying this property will be called symmetric variable order derivatives, and we will introduce two types of them: so-called $\mathcal{AD}$-type and $\mathcal{DA}$-type.

Definition 3 The $\mathcal{AD}$-type and $\mathcal{DA}$-type of symmetric fractional variable order derivative are defined, respectively, as follows:

$$ {}_0^{\mathcal{AD}}\mathrm{D}_t^{\alpha(t)} f(t) = {}_0^{\mathcal{A}}\mathrm{D}_t^{\frac{\alpha(t)}{2}} {}_0^{\mathcal{D}}\mathrm{D}_t^{\frac{\alpha(t)}{2}} f(t) \tag{14} $$

and

$$ {}_0^{\mathcal{DA}}\mathrm{D}_t^{\alpha(t)} f(t) = {}_0^{\mathcal{D}}\mathrm{D}_t^{\frac{\alpha(t)}{2}} {}_0^{\mathcal{A}}\mathrm{D}_t^{\frac{\alpha(t)}{2}} f(t), \tag{15} $$

where ${}_0^{\mathcal{A}}\mathrm{D}_t^{\frac{\alpha(t)}{2}} f(t)$ and ${}_0^{\mathcal{D}}\mathrm{D}_t^{\frac{\alpha(t)}{2}} f(t)$ are given by (2) and (5), respectively.

Remark 3 For a fractional constant order $\alpha = \text{const}$, the $\mathcal{AD}$-type and $\mathcal{DA}$-type symmetric fractional variable order derivatives are numerically identical with constant order α fractional derivative given by (1). Indeed,

$$
\begin{aligned}
{}^{\mathcal{AD}}_{0}\mathrm{D}^{\alpha}_{t} f(t) &= {}^{\mathcal{A}}_{0}\mathrm{D}^{\frac{\alpha}{2}}_{t}\, {}^{\mathcal{D}}_{0}\mathrm{D}^{\frac{\alpha}{2}}_{t} f(t) \\
&= {}_{0}\mathrm{D}^{\frac{\alpha}{2}}_{t}\, {}_{0}\mathrm{D}^{\frac{\alpha}{2}}_{t} f(t) = {}_{0}\mathrm{D}^{\alpha}_{t} f(t),
\end{aligned}
$$

and

$$
\begin{aligned}
{}^{\mathcal{DA}}_{0}\mathrm{D}^{\alpha}_{t} f(t) &= {}^{\mathcal{D}}_{0}\mathrm{D}^{\frac{\alpha}{2}}_{t}\, {}^{\mathcal{A}}_{0}\mathrm{D}^{\frac{\alpha}{2}}_{t} f(t) \\
&= {}_{0}\mathrm{D}^{\frac{\alpha}{2}}_{t}\, {}_{0}\mathrm{D}^{\frac{\alpha}{2}}_{t} f(t) = {}_{0}\mathrm{D}^{\alpha}_{t} f(t).
\end{aligned}
$$

The $\mathcal{AD}$-type and $\mathcal{DA}$-type fractional derivatives can be expressed, respectively, in the following matrix forms:

$$
\begin{pmatrix} {}^{\mathcal{AD}}_{0}\mathrm{D}^{\alpha(t)}_{0} f(0) \\ {}^{\mathcal{AD}}_{0}\mathrm{D}^{\alpha(t)}_{h} f(h) \\ \vdots \\ {}^{\mathcal{AD}}_{0}\mathrm{D}^{\alpha(t)}_{kh} f(kh) \end{pmatrix} = \lim_{h\to 0} {}^{\mathcal{AD}}\mathfrak{W}^{k}_{0}(\alpha) \begin{pmatrix} f(0) \\ f(h) \\ \vdots \\ f(kh) \end{pmatrix}, \tag{16}
$$

where

$$
{}^{\mathcal{AD}}\mathfrak{W}^{k}_{0}(\alpha) = {}^{\mathcal{A}}W\left(\frac{\alpha}{2}\right) {}^{\mathcal{D}}\mathfrak{Q}^{k}_{0}\left(\frac{\alpha}{2}\right);
$$

and

$$
\begin{pmatrix} {}^{\mathcal{DA}}_{0}\mathrm{D}^{\alpha(t)}_{0} f(0) \\ {}^{\mathcal{DA}}_{0}\mathrm{D}^{\alpha(t)}_{h} f(h) \\ \vdots \\ {}^{\mathcal{DA}}_{0}\mathrm{D}^{\alpha(t)}_{kh} f(kh) \end{pmatrix} = \lim_{h\to 0} {}^{\mathcal{DA}}\mathfrak{W}^{k}_{0}(\alpha) \begin{pmatrix} f(0) \\ f(h) \\ \vdots \\ f(kh) \end{pmatrix}, \tag{17}
$$

where

$$
{}^{\mathcal{DA}}\mathfrak{W}^{k}_{0}(\alpha) = {}^{\mathcal{D}}\mathfrak{Q}^{k}_{0}\left(\frac{\alpha}{2}\right) {}^{\mathcal{A}}W\left(\frac{\alpha}{2}\right),
$$

and ${}^{\mathcal{A}}W$ and ${}^{\mathcal{D}}\mathfrak{Q}^{k}_{0}$ are given by (4) and (7), respectively.

Let us consider the following so-called output-input-reductive and input-output-reductive switching order schemes in the serial forms, denoted ${}^{\text{o-i-r}}_{\text{ser}}\Xi\{\alpha(t)\}$ and ${}^{\text{i-o-r}}_{\text{ser}}\Xi\{\alpha(t)\}$, presented in Figs. 3 and 4, respectively.

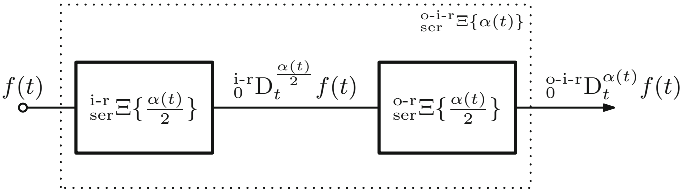

Fig. 3 Structure of output-input-reductive switching order scheme $^{\text{o-i-r}}_{\text{ser}}\Xi\{\alpha(t)\}$

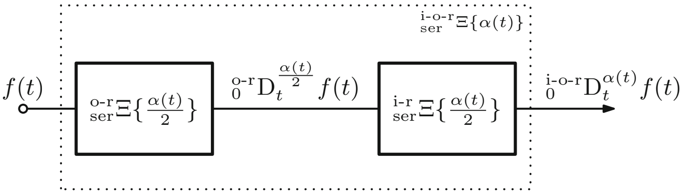

Fig. 4 Structure of input-output-reductive switching order scheme $^{\text{i-o-r}}_{\text{ser}}\Xi\{\alpha(t)\}$

Proposition 1 *The $\mathcal{AD}$-type and $\mathcal{DA}$-type derivatives are equivalent, respectively, to* $^{\text{o-i-r}}_{\text{ser}}\Xi\{\alpha(t)\}$ *and* $^{\text{i-o-r}}_{\text{ser}}\Xi\{\alpha(t)\}$ *switching schemes, i.e.,*

$$^{\mathcal{AD}}_{0}\mathrm{D}_t^{\alpha(t)} f(t) \equiv {}^{\text{o-i-r}}_{0}\mathrm{D}_t^{\alpha(t)} f(t)$$

and

$$^{\mathcal{DA}}_{0}\mathrm{D}_t^{\alpha(t)} f(t) \equiv {}^{\text{i-o-r}}_{0}\mathrm{D}_t^{\alpha(t)} f(t).$$

Proof It follows directly from the fact that the switching schemes $^{\text{o-r}}_{\text{ser}}\Xi\{\alpha(t)\}$ and $^{\text{i-r}}_{\text{ser}}\Xi\{\alpha(t)\}$ are equivalent, respectively, to $\mathcal{A}$-type and $\mathcal{D}$-type derivatives.

Proposition 2 *The $\mathcal{AD}$-type and $\mathcal{DA}$-type derivatives are symmetric, i.e.,*

$$^{\mathcal{AD}}_{0}\mathrm{D}_t^{\alpha(t)} {}^{\mathcal{AD}}_{0}\mathrm{D}_t^{-\alpha(t)} f(t) = f(t)$$
$$^{\mathcal{DA}}_{0}\mathrm{D}_t^{\alpha(t)} {}^{\mathcal{DA}}_{0}\mathrm{D}_t^{-\alpha(t)} f(t) = f(t)$$

Proof We have

$$\begin{aligned} ^{\mathcal{AD}}_{0}\mathrm{D}_t^{\alpha(t)} {}^{\mathcal{AD}}_{0}\mathrm{D}_t^{-\alpha(t)} f(t) &= {}^{\mathcal{A}}_{0}\mathrm{D}_t^{\frac{\alpha(t)}{2}} {}^{\mathcal{D}}_{0}\mathrm{D}_t^{\frac{\alpha(t)}{2}} {}^{\mathcal{A}}_{0}\mathrm{D}_t^{\frac{-\alpha(t)}{2}} {}^{\mathcal{D}}_{0}\mathrm{D}_t^{\frac{-\alpha(t)}{2}} f(t) \\ &= {}^{\mathcal{A}}_{0}\mathrm{D}_t^{\frac{\alpha(t)}{2}} {}^{\mathcal{D}}_{0}\mathrm{D}_t^{\frac{-\alpha(t)}{2}} f(t) \\ &= f(t) \end{aligned}$$

using the duality properties, first (12), and then (13).

Similarly,

$$
\begin{aligned}
{}^{\mathcal{DA}}_{0}\mathrm{D}_{t}^{\alpha(t)}\,{}^{\mathcal{DA}}_{0}\mathrm{D}_{t}^{-\alpha(t)}f(t) &= {}^{\mathcal{D}}_{0}\mathrm{D}_{t}^{\frac{\alpha(t)}{2}}\,{}^{\mathcal{A}}_{0}\mathrm{D}_{t}^{\frac{\alpha(t)}{2}}\,{}^{\mathcal{D}}_{0}\mathrm{D}_{t}^{\frac{-\alpha(t)}{2}}\,{}^{\mathcal{A}}_{0}\mathrm{D}_{t}^{\frac{-\alpha(t)}{2}}f(t) \\
&= {}^{\mathcal{D}}_{0}\mathrm{D}_{t}^{\frac{\alpha(t)}{2}}\,{}^{\mathcal{A}}_{0}\mathrm{D}_{t}^{\frac{-\alpha(t)}{2}}f(t) \\
&= f(t)
\end{aligned}
$$

using the duality properties, first (13), and then (12).

Corollary 1 *There is no duality property between $\mathcal{AD}$-type and $\mathcal{DA}$-type derivatives, i.e.,*

$$
\begin{aligned}
&{}^{\mathcal{AD}}_{0}\mathrm{D}_{t}^{\alpha(t)}\,{}^{\mathcal{DA}}_{0}\mathrm{D}_{t}^{-\alpha(t)}f(t) \neq f(t) \\
&{}^{\mathcal{DA}}_{0}\mathrm{D}_{t}^{\alpha(t)}\,{}^{\mathcal{AD}}_{0}\mathrm{D}_{t}^{-\alpha(t)}f(t) \neq f(t).
\end{aligned}
$$

4 Numerical Example

Below we present an example of plots of $\mathcal{AD}$-type and $\mathcal{DA}$-type symmetric variable order derivatives compared with results obtained from equivalent switching schemes.

Let us consider the following fractional variable order integrals: ${}^{\mathcal{AD}}_{0}\mathrm{D}_{t}^{\alpha(t)}f(t)$ and ${}^{\mathcal{DA}}_{0}\mathrm{D}_{t}^{\alpha(t)}f(t)$, for Heaviside step function $f(t) = H(t)$ and

$$
\alpha(t) = \begin{cases} -\frac{1}{8} & \text{for } t \in [0, 0.3) \\ -\frac{3}{4} & \text{for } t \in [0.3, 0.6) \\ -1 & \text{for } t \in [0.6, 0.8) \\ -\frac{1}{4} & \text{for } t \in [0.8, 1). \end{cases}
$$

The graphs of integration are depicted in Fig. 5. From the plots in Fig. 5, we can see that the behavior of $\mathcal{AD}$-type and $\mathcal{DA}$-type of derivative is similar to $\mathcal{A}$-type and $\mathcal{D}$-type derivative, respectively.

The graphs of $\mathcal{AD}$-type and $\mathcal{DA}$-type derivatives composition properties are depicted in Fig. 6. From the plots in Fig. 6, we can see that the compositions ${}^{\mathcal{AD}}_{0}\mathrm{D}_{t}^{\alpha(t)}\,{}^{\mathcal{AD}}_{0}\mathrm{D}_{t}^{-\alpha(t)}H(t)\cdot t$ and ${}^{\mathcal{DA}}_{0}\mathrm{D}_{t}^{\alpha(t)}\,{}^{\mathcal{DA}}_{0}\mathrm{D}_{t}^{-\alpha(t)}H(t)\cdot t$ yields the original functions $H(t)\cdot t$, i.e., the symmetry property holds, and the duality between $\mathcal{AD}$- and $\mathcal{DA}$-type derivatives does not hold, i.e., ${}^{\mathcal{AD}}_{0}\mathrm{D}_{t}^{\alpha(t)}\,{}^{\mathcal{DA}}_{0}\mathrm{D}_{t}^{-\alpha(t)}H(t)\cdot t \neq H(t)\cdot t$.

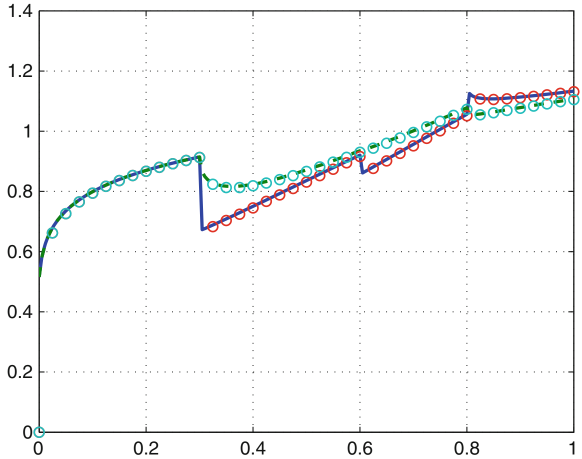

Fig. 5 Integration plots of ${}_0^{\mathcal{AD}}\mathrm{D}_t^{\alpha(t)}H(t)$ (*solid line*) and ${}_0^{\mathcal{DA}}\mathrm{D}_t^{\alpha(t)}H(t)$ (*dashed line*) compared with integration plots realized, respectively, by means of switching schemes ${}^{\text{o-i-r}}_{\text{ser}}\Xi\{\alpha(t)\}$ and ${}^{\text{i-o-r}}_{\text{ser}}\Xi\{\alpha(t)\}$ (*circles lines*)

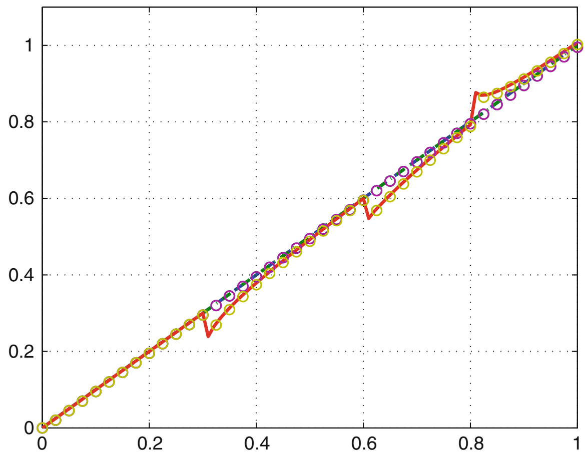

Fig. 6 Numerical solution plots of ${}_0^{\mathcal{AD}}\mathrm{D}_t^{\alpha(t)}\,{}_0^{\mathcal{AD}}\mathrm{D}_t^{-\alpha(t)}(H(t)\cdot t)$ (*dashed-dotted line*), ${}_0^{\mathcal{DA}}\mathrm{D}_t^{\alpha(t)}\,{}_0^{\mathcal{DA}}\mathrm{D}_t^{-\alpha(t)}(H(t)\cdot t)$ (*dashed line*), and ${}_0^{\mathcal{AD}}\mathrm{D}_t^{\alpha(t)}\,{}_0^{\mathcal{DA}}\mathrm{D}_t^{-\alpha(t)}(H(t)\cdot t)$ (*solid line*) compared with plots realized by means of corresponding switching schemes compositions (*circles lines*)

5 Conclusions

In the paper, the $\mathcal{AD}$-type and $\mathcal{DA}$-type of symmetric variable order derivatives were given. The switching order schemes equivalent to these type of definitions were presented as the output-input reductive and input-output reductive structures as well. Then, the numerical example comparing particular definitions with their switching schemes were shown. It was also proven that, there is no duality property between $\mathcal{AD}$-type and $\mathcal{DA}$-type derivatives.

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Linearization of the Non-linear Time-Variant Fractional-Order Difference Equation

Piotr Ostalczyk

Abstract In this paper a linearization procedure of the fractional-order non-linear time-variant discrete system is discussed. Starting from the non-linear fractional-order difference equation one derives its equivalent state-space form. Then assuming a knowledge of the nominal trajectory one evaluates the linear stat-space model. The investigations are supported by numerical example.

Keywords Discrete fractional calculus · Fractional difference equation · Discrete fractional state-space model · Linearization

1 Introduction

The fractional calculus [7, 10, 12] has become a widely used mathematical tool in control theory and technical applications. Currently, the research of dynamic system control is concentrated upon the fractional-order mathematical models [9, 11, 13, 15]. In the classical approach the non-linear models are linearized to simplify the closed—loop system synthesis [1, 3]. In the time-variant dynamic system the steady-state solution is generalized to the so called nominal solution. The Taylor series expansion leads to the linear model. The numerical example shows that this procedure is useful for specified parameter ranges.

The paper structure is as follows. First fundamental definitions of the fractional-order (FO) backward difference (BD) in the Grunwald-Letnikov (GL) related to the FO form the left GL derivative are given. A simple approximation of the FO non-linear differential equation by the FO difference equation (DE) is introduced.

P. Ostalczyk (✉)
Institute of Applied Computer Science, Lodz University of Technology,
ul. Stefanowskiego 18/22, Lodz, Poland
e-mail: postalcz@p.lodz.pl

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2 Mathematical Preliminaries

Definition 1 (*Grünwald-Letnikov form of the FOBD)* For a discrete-variable, bounded function $f(k)$ the Grunwald-Letnikov FOBD is defined as a sum

$$
\begin{aligned}
{}_{k_0}^{GL}\Delta_k^{(\nu)} f(k) &= \sum_{i=k_0}^{k} a^{(\nu)}(i-k_0) f(k+k_0-i) \\
&= \sum_{i=0}^{k-k_0} a^{(\nu)}(i) f(k-i),
\end{aligned}
\tag{1}
$$

where a function $a^{(\nu)}(k)$ is defined as follows

$$
a^{(\nu)}(k) = \begin{cases} 0 & \text{for} \quad k<0 \\ 1 & \text{for} \quad k=0 \\ (-1)^k \frac{\nu(\nu-1)\ldots(\nu-k+1)}{k!} & \text{for } k=1,2,3,\ldots \end{cases}
\tag{2}
$$

and $[k_0, k]$ is the FOBD calculation range, $\nu \in \mathbb{R}_+$ is an order. A superscript "GL" stands for the Grünwald—Letnikov form. The Grünwald—Letnikov FO left-derivative (LD) is related to the FOBD as follows.

Definition 2 (*Grünwald-Letnikov form of the FOLD*) For a continuous-variable, bounded function $f(t)$ the Grünwald-Letnikov FOLD of order $\nu \in \mathbb{R}$ is defined as a limit

$$
\begin{aligned}
{}_{t_0}^{GL}D_t^{(\nu)} f(t) &= \lim_{\substack{h \to 0^+ \\ kh = t-t_0}} \frac{{}_{k_0}^{GL}\Delta_k^{(\nu)} f(kh)}{h^\nu} \\
&= \lim_{\substack{h \to 0^+ \\ kh = t-t_0}} \frac{\sum_{i=0}^{k} a^{(\nu)}(i) f\left[(k-i)h\right]}{h^\nu}
\end{aligned}
\tag{3}
$$

It is worth to note that for $\nu = n = 1$ Definitions 1 and 2 denotes first-order backward difference [14] $\Delta f(k)$ and first-order left derivative $\frac{df(t)}{dt}$, respectively. Classical non-linear time-variant pth order differential equation [8]

$$
F\left[\frac{d^p y(t)}{dt^p}, \frac{d^{p-1} y(t)}{dt^{p-1}}, \ldots, \frac{y(t)}{dt}, y(t), \frac{d^q u(t)}{dt^q}, \frac{d^{q-1} u(t)}{dt^{q-1}}, \ldots, \frac{u(t)}{dt}, u(t), t\right] = 0,
\tag{4}
$$

where $p \geqslant q$ can be generalized to the FO differential equation

$$F\left[{}_{t_0}^{GL}D_t^{\nu_p}y(t), {}_{t_0}^{GL}D_t^{\nu_{p-1}}y(t), \dots, {}_{t_0}^{GL}D_t^{\nu_1}y(t), y(t), \times \right.$$
$$\left. \times\, {}_{t_0}^{GL}D_t^{\mu_q}u(t) \dots, {}_{t_0}^{GL}D_t^{\mu_1}u(t), u(t), t\right] = 0, \tag{5}$$

where

$$\nu_p > \nu_{p-1} > \cdots > \nu_1 > 0,\ \mu_p > \mu_{p-1} > \cdots > \mu_1 > 0. \tag{6}$$

where $\nu_p \geqslant \nu_q$. Performing a substitution of the Grüwald-Letnikov fractional left derivative by the FOBD divided by a sampling time (which should be relatively small) in (3)

$${}_{t_0}^{GL}D_t^{(\nu_i)}y(t) \approx \frac{{}_{k_0h}^{GL}\Delta_{kh}^{(\nu_i)}y(kh)}{h^{\nu_i}},\ {}_{t_0}^{GL}D_t^{(\mu_j)}u(t) \approx \frac{{}_{k_0h}^{GL}\Delta_{kh}^{(\nu_j)}u(kh)}{h^{\mu_j}}, \tag{7}$$

for $i = 1, 2, \dots, p, j = 1, 2, \dots, q$ one obtains the FO difference equation (FODE)

$$F\left[{}_{k_0h}^{GL}\Delta_{kh}^{(\nu_p)}y(kh), \dots, {}_{k_0h}^{GL}\Delta_{kh}^{(\nu_1)}y(kh), y(kh), \times \right.$$
$$\left. \times\, {}_{k_0h}^{GL}\Delta_{kh}^{(\mu_q)}u(kh), \dots, {}_{k_0h}^{GL}\Delta_{kh}^{(\mu_1)}u(kh), u(kh), kh\right] = 0. \tag{8}$$

For a known and constant sampling time $h =$ const there be used a simplified notation omitting h. Hence the FODE (5) takes a form

$$F\left[{}_{k_0}^{GL}\Delta_k^{(\nu_p)}y(k), \dots, {}_{k_0}^{GL}\Delta_k^{(\nu_1)}y(k), y(k), {}_{k_0}^{GL}\Delta_k^{(\mu_q)}u(k), \dots, {}_{k_0}^{GL}\Delta_k^{(\mu_1)}u(k), u(k), k\right] = 0 \tag{9}$$

Further one considers only the commensurate systems, i.e., systems described by FODE with

$$\nu_i = \frac{n_i}{d} = n_i\nu \text{ for } i = 1, 2, \dots, p, \text{ and } e_i, d_i \in \mathbb{Z}_+ \tag{10}$$

$$\mu_i = \frac{m_i}{d} = m_i\nu \text{ for } j = 1, 2, \dots, q, \text{ and } g_i, f_i \in \mathbb{Z}_+ \tag{11}$$

where

$$\nu = \frac{1}{d}. \tag{12}$$

and d is the least common denominator of all FOs. Then (9) takes the form

$$F\left[{}_{k_0}^{GL}\Delta_k^{(n_p\nu)}y(k), {}_{k_0}^{GL}\Delta_k^{(n_{p-1}\nu)}y(k), \dots, y(k), \times \right.$$
$$\left. \times {}_{k_0}^{GL}\Delta_k^{(m_q\nu)}u(k), \dots, {}_{k_0}^{GL}\Delta_k^{(m_{q-1}\nu)}u(k), \dots, u(k), k\right] = 0. \tag{13}$$

In a wide class of the FODE one can extract from (13) the FOBE of the highest order. Then an equivalent form is as follows

$${}_{k_0}^{GL}\Delta_k^{(n_p\nu)}y(k) = f\left[{}_{k_0}^{GL}\Delta_k^{(n_{p-1}\nu)}y(k), {}_{k_0}^{GL}\Delta_k^{(n_{p-2}\nu)}y(k), \dots, y(k), \times \right.$$
$$\left. \times {}_{k_0}^{GL}\Delta_k^{(m_q\nu)}u(k), \dots, {}_{k_0}^{GL}\Delta_k^{(m_{q-1}\nu)}u(k), \dots, u(k), k\right]. \tag{14}$$

Defining a set of equations

$$\begin{aligned} x_1(k) &= y(k), \\ x_2(k) &= {}_{k_0}^{GL}\Delta_k^{(\nu)}x_1(k) = {}_{k_0}^{GL}\Delta_k^{(\nu)}y(k), \\ x_3(k) &= {}_{k_0}^{GL}\Delta_k^{(\nu)}x_2(k) = {}_{k_0}^{GL}\Delta_k^{(2\nu)}y(k), \\ &\vdots \\ x_{n_p}(k) &= {}_{k_0}^{GL}\Delta_k^{(\nu)}x_{n_p-1}(k) = {}_{k_0}^{GL}\Delta_k^{((p-1)\nu)}y(k). \end{aligned} \tag{15}$$

one obtains the so-called state-space equation of the non-linear FOS

$$\begin{bmatrix} {}_{k_0}^{GL}\Delta_k^{(\nu)}x_1(k) \\ {}_{k_0}^{GL}\Delta_k^{(\nu)}x_2(k) \\ \vdots \\ {}_{k_0}^{GL}\Delta_k^{(\nu)}x_{p-1}(k) \\ {}_{k_0}^{GL}\Delta_k^{(\nu)}x_p(k) \end{bmatrix} = {}_{k_0}^{GL}\Delta_k^{(\nu)}\begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n_p-1}(k) \\ x_{n_p}(k) \end{bmatrix}$$
$$= \begin{bmatrix} x_2(k) \\ x_3(k) \\ \vdots \\ x_{n_p}(k) \\ f_p\left[x_{n_p}(k), x_{n_p-1}(k), \dots, x_2(k), x_1(k), {}_{k_0}^{GL}\Delta_k^{(m_q\nu)}u(k), \dots, u(k), k\right] \end{bmatrix}. \tag{16}$$

The first equation in (15) denoting the FODE solution remains unchanged but expressed in opposite order

$$y(k) = \left[x_1(k)\right]. \tag{17}$$

The non-linear SSE can be generalized to more general form

$$ {}_{k_0}^{GL}\Delta_k^{(\nu)}\mathbf{x}(k) = \mathbf{f}\left[\mathbf{x}(k), \mathbf{u}(k), k\right], \tag{18} $$

and

$$ \mathbf{y}(k) = \mathbf{g}\left[\mathbf{x}(k), \mathbf{u}(k), k\right], \tag{19} $$

where

$$ \mathbf{x}(k) = \begin{bmatrix} x_1(k) \\ x_2(k) \\ \vdots \\ x_{n_p-1}(k) \\ x_{n_p}(k) \end{bmatrix}, \ \mathbf{u}(k) = \begin{bmatrix} u_1(k) \\ u_2(k) \\ \vdots \\ u_{m_q-1}(k) \\ u_{m_q}(k) \end{bmatrix}, \ \mathbf{y}(k) = \begin{bmatrix} y_1(k) \\ y_2(k) \\ \vdots \\ y_{r-1}(k) \\ y_r(k) \end{bmatrix} \tag{20} $$

$$ \mathbf{f}\left[\mathbf{x}(k), \mathbf{u}(k), k\right] = \begin{bmatrix} f_1\left[\mathbf{x}(k), \mathbf{u}(k), k\right] \\ f_2\left[\mathbf{x}(k), \mathbf{u}(k), k\right] \\ \vdots \\ f_{n_p-1}\left[\mathbf{x}(k), \mathbf{u}(k), k\right] \\ f_{n_p}\left[\mathbf{x}(k), \mathbf{u}(k), k\right] \end{bmatrix} $$

$$ = \begin{bmatrix} f_1\left[x_{n_p}(k), \ldots, x_1(k), u_{m_q}(k) \ldots, u_1(k), k\right] \\ f_2\left[x_{n_p}(k), \ldots, x_1(k), u_{m_q}(k) \ldots, u_1(k), k\right] \\ \vdots \\ f_{n_p-1}\left[x_{n_p}(k), \ldots, x_1(k), u_{m_q}(k) \ldots, u_1(k), k\right] \\ f_{n_p}\left[x_{n_p}(k), \ldots, x_1(k), u_{m_q}(k) \ldots, u_1(k), k\right] \end{bmatrix} \tag{21} $$

$$ \mathbf{g}\left[\mathbf{x}(k), \mathbf{u}(k), k\right] = \begin{bmatrix} g_1\left[\mathbf{x}(k), \mathbf{u}(k), k\right] \\ g_2\left[\mathbf{x}(k), \mathbf{u}(k), k\right] \\ \vdots \\ g_{r-1}\left[\mathbf{x}(k), \mathbf{u}(k), k\right] \\ g_r\left[\mathbf{x}(k), \mathbf{u}(k), k\right] \end{bmatrix} $$

$$ = \begin{bmatrix} g_1\left[x_{n_p}(k), \ldots, x_1(k), u_q(k) \ldots, u_1(k), k\right] \\ g_2\left[x_{n_p}(k), \ldots, x_1(k), u_q(k) \ldots, u_1(k), k\right] \\ \vdots \\ g_{r-1}\left[x_{n_p}(k), \ldots, x_1(k), u_q(k) \ldots, u_1(k), k\right] \\ g_r\left[x_{n_p}(k), \ldots, x_1(k), u_q(k) \ldots, u_1(k), k\right] \end{bmatrix} \tag{22} $$

3 Linearization of the Non-linear Time-Variant FOS

Suppose that $\mathbf{u}_s(k)$ is a given input (nominal input) to a system described by (18) and (19) and $\mathbf{x}_s(k)$ is a nominal trajectory satisfying

$$ {}_{k_0}^{GL}\Delta_k^{(\nu)}\mathbf{x}_s(k) = \mathbf{f}\left[\mathbf{x}_s(k), \mathbf{u}_s(k), k\right], \tag{23}$$

Then form (19) one gets

$$ \mathbf{y}_s(k) = \mathbf{g}\left[\mathbf{x}_s(k), \mathbf{u}_s(k), k\right], \tag{24}$$

The linearisation procedure of the non-linear FODE assumes a "small" change (a small perturbation) around the known input function $\mathbf{u}_s(k)$. This perturbation will be denoted as

$$ \mathbf{u}(\mathbf{k}) = \mathbf{u}_s(k) + \delta\mathbf{u}(k) \tag{25}$$

Here $\delta\mathbf{u}(k)$ denotes a relatively "small" perturbation (positive or negative) around the steady-state value of the input signal $\mathbf{u}(k)$. A response of Eq. (18) is "slightly" changed due to (25) and will be denoted as

$$ \mathbf{x}(k) = \mathbf{x}_s(k) + \delta x(k),\ \ \mathbf{y}(k) = \mathbf{y}_s + \delta\mathbf{y}(k). \tag{26}$$

The expressions $\delta\mathbf{x}(k)$ and $\delta y(k)$ represent induced "small" changes around the nominal solutions $\mathbf{x}_s(k)$ and $y_s(k)$ caused by a change of (25). Then by (18)

$$ {}_{k_0}^{GL}\Delta_k^{(\nu)}\mathbf{x}(k) = {}_{k_0}^{GL}\ \Delta_k^{(\nu)}\left[\mathbf{x}_s(k) + \delta\mathbf{x}(k)\right] = {}_{k_0}^{GL}\ \Delta_k^{(\nu)}\mathbf{x}_s(k) + {}_{k_0}^{GL}\ \Delta_k^{(\nu)}\delta\mathbf{x}(k) \tag{27}$$

and

$$ {}_{k_0}^{GL}\Delta_k^{(\nu)}\mathbf{x}(k) = \mathbf{f}\left[\mathbf{x}_s(k) + \delta\mathbf{x}(k), \mathbf{u}_s(k) + \delta\mathbf{u}(k)\right], \tag{28}$$

$$ \mathbf{y}_s(k) + \delta\mathbf{y}(k) = \mathbf{g}\left[\mathbf{x}_s(k) + \delta\mathbf{x}(k), \mathbf{u}_s(k) + \delta\mathbf{u}(k)\right]. \tag{29}$$

Making a Taylor expansion of $\mathbf{f}\left[\mathbf{x}_s(k) + \delta\mathbf{x}(k), \mathbf{u}_s(k) + \delta\mathbf{u}(k), k\right]$ one obtains

$$ \begin{aligned} \mathbf{f}\left[\mathbf{x}_s(k) + \delta\mathbf{x}(k), \mathbf{u}_s(k) + \delta\mathbf{u}(k)\right] &= \mathbf{f}\left[\mathbf{x}_s(k), \mathbf{u}_s(k), k\right] \\ + \mathbf{J}_{\mathbf{x}}\left[\mathbf{x}_s(k), \mathbf{u}_s(k), k\right]\delta\mathbf{x}(k) &+ \mathbf{J}_{\mathbf{u}}\left[\mathbf{x}_s(k), \mathbf{u}_s(k), k\right]\delta\mathbf{u}(k) + \mathbf{h}(k) \end{aligned} \tag{30}$$

where $\mathbf{J_x}\left[\mathbf{x}_s(k), \mathbf{u}_s(k), k\right]$ and $\mathbf{J_u}\left[\mathbf{x}_s(k), \mathbf{u}_s(k), k\right]$ are the Jacobian matrices of the function $\mathbf{f}\left[\mathbf{x}(k), \mathbf{u}(k)\right]$ with respect to $\mathbf{x}(k)$ and $\mathbf{u}(k)$, respectively, that is,

$$\mathbf{J_x}\left[\mathbf{x}_s(k), \mathbf{u}_s(k), k\right] = \mathbf{J_x}\left[\mathbf{x}(k), \mathbf{u}(k), k\right]\Big|_{\substack{\mathbf{x}(k) = \mathbf{x}_s(k) \\ \mathbf{u}(k) = \mathbf{u}_s(k)}} = \mathbf{A}(k) = \begin{bmatrix} a_{11}(k) & a_{12}(k) & \dots & a_{1,n_p}(k) \\ a_{21}(k) & a_{22}(k) & \dots & a_{2,n_p}(k) \\ \vdots & \vdots & & \vdots \\ a_{n_p,1}(k) & a_{n_p,2}(k) & \dots & a_{n_p,n_p}(k) \end{bmatrix} \quad (31)$$

where

$$a_{ij}(k) = \left.\frac{\partial f_i\left[\mathbf{x}(k), \mathbf{u}(k), k\right]}{\partial x_j}\right|_{\substack{\mathbf{x}(k) = \mathbf{x}_s(k) \\ \mathbf{u}(k) = \mathbf{u}_s(k)}}, \quad (32)$$

and

$$\mathbf{J_u}\left[\mathbf{x}_s(k), \mathbf{u}_s(k), k\right] = \mathbf{J_u}\left[\mathbf{x}(k), \mathbf{u}(k), k\right]\Big|_{\substack{\mathbf{x}(k) = \mathbf{x}_s(k) \\ \mathbf{u}(k) = \mathbf{u}_s(k)}} = \mathbf{B}(k) = \begin{bmatrix} b_{11}(k) & b_{12}(k) & \dots & b_{1,m_q}(k) \\ b_{21}(k) & b_{22}(k) & \dots & b_{2,m_q}(k) \\ \vdots & \vdots & & \vdots \\ b_{n_p,1}(k) & b_{n_p,2}(k) & \dots & b_{n_p,m_q}(k) \end{bmatrix} \quad (33)$$

where

$$b_{ij}(k) = \left.\frac{\partial f_i\left[\mathbf{x}(k), \mathbf{u}(k), k\right]}{\partial u_j}\right|_{\substack{\mathbf{x}(k) = \mathbf{x}_s(k) \\ \mathbf{u}(k) = \mathbf{u}_s(k)}}, \quad (34)$$

The term $\mathbf{h}(k)$ is an expression that is supposed "very small" with respect to $\delta\mathbf{x}(k)$ and $\delta\mathbf{x}(k)$. Neglecting $\mathbf{h}(k)$ and substituting (30), (28) into (23) yields

$${}_{k_0}^{GL}\Delta_k^{(\nu)}\delta\mathbf{x}(k) = \mathbf{A}(k)\delta\mathbf{x}(k) + \mathbf{B}(k)\delta\mathbf{u}(k), \quad (35)$$

Analogous procedure performed on (24) gives

$$\delta\mathbf{y}(k) = \mathbf{C}(k)\delta\mathbf{x}(k) + \mathbf{D}(k)\delta\mathbf{u}(k), \quad (36)$$

with

$$c_{ij}(k) = \left. \frac{\partial g_i\left[\mathbf{x}(k), \mathbf{u}(k), k\right]}{\partial x_j} \right|_{\substack{\mathbf{x}(k) = \mathbf{x}_s(k) \\ \mathbf{u}(k) = \mathbf{u}_s(k)}},$$

$$d_{ij}(k) = \left. \frac{\partial f_i\left[\mathbf{x}(k), \mathbf{u}(k), k\right]}{\partial u_j} \right|_{\substack{\mathbf{x}(k) = \mathbf{x}_s(k) \\ \mathbf{u}(k) = \mathbf{u}_s(k)}}. \tag{37}$$

4 Numerical Example

Consider the non-linear time-invariant discretized Verhulst's equation [4]

$$y(k' + 1) = (1 - r)y(k') - ry^2(k'). \tag{38}$$

This equation models the population restricted growth. The quantity r is called the growth rate [2]. Substituting $k = k' - 1$ on gets

$$y(k) = (1 - r)y(k - 1) - ry^2(k - 1). \tag{39}$$

and after elementary transformations one obtains

$$y(k) - y(k - 1) = {}^{GL}_{k_0}\Delta^{(1)}_k y(k) = -ry(k - 1)\left[1 + y(k - 1)\right]. \tag{40}$$

This is the first-order homogenous non-linear DE. To find its unique solution one should have an initial state (the initial population) y_{-1}. The initial conditions [5] should be recounted to the discrete-time equation. The growth modeling adequacy can be improved by admitting FO $\nu \in [\nu_{min}, \nu_{max}]$, $\nu_{min} < 1, \nu_{max} > 1, r_{min}, r_{max} > 0$ and time-varying growth rate $r(k) \in [r_min, r_max]$. This lead to the FODE

$$ {}^{GL}_{k_0}\Delta^{(\nu)}_k y(k) = -r(k)y(k - 1)\left[1 + y(k - 1)\right]. \tag{41}$$

4.1 Non-linear Time-Invariant FODE of the Restricted Growth

In further investigations one assumes initial time $k_0 = 0$. First, one validates the simulation tools by simulating the solution of (41) for $\nu = 1$, and $r \in \{0.1, 1.8, 2.5, 2.8\}$. The related plots are given in Figs. 1, 2, 3, 4, respectively.

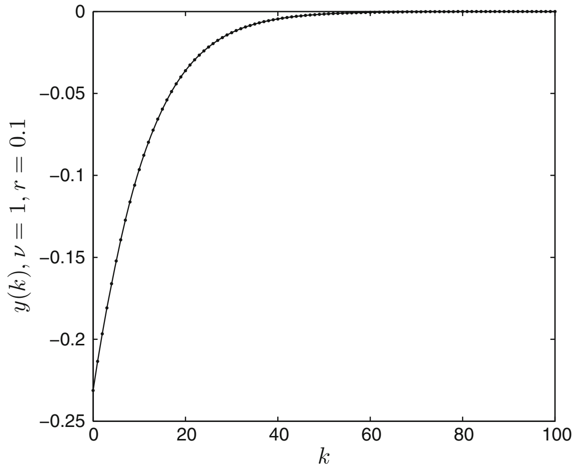


Fig. 1 Solution of (41) for $\nu = 1$, and $r = 0.1$ and $y_{-1} = -0.25$

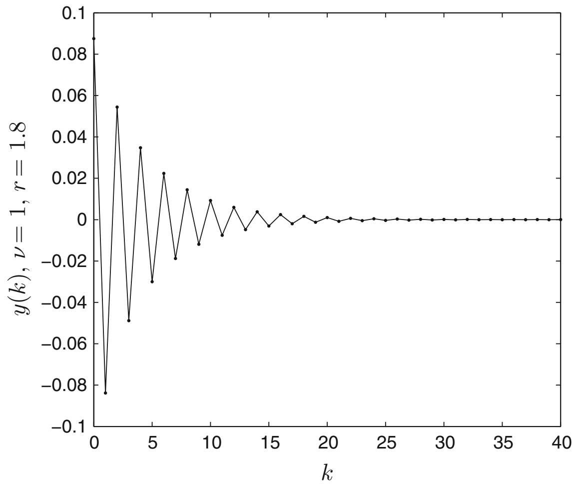


Fig. 2 Solution of (41) for $\nu = 1$, and $r = 1.8$ and $y_{-1} = -0.25$

Remark 1 The simulation results confirm the correctness of the software because for $r \in (0, 2]$ one gets a stable growth, for $r \in (2, 2.57]$ there is a cyclic growth and for $r \in (2.57, 3]$ the growth is chaotic [2].

Now one shows the FO transient behavior of (41) for two FOs $\nu \in \{0.9, 1.1\}$. The orders are close to the 1. In Figs. 5 and 6 related plots are presented, respectively.

Fig. 3 Solution of (41) for $\nu = 1$, and $r = 2.5$ and $y_{-1} = -0.25$

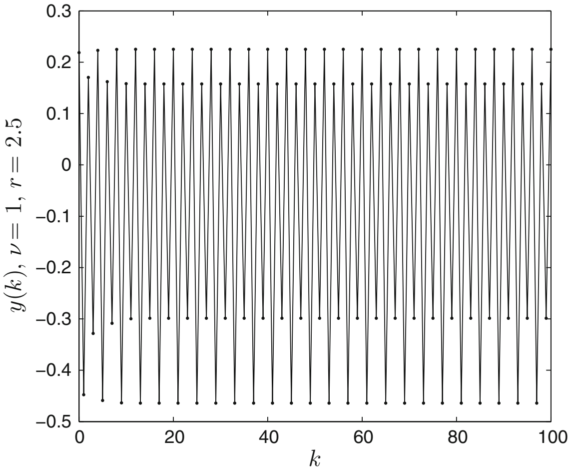
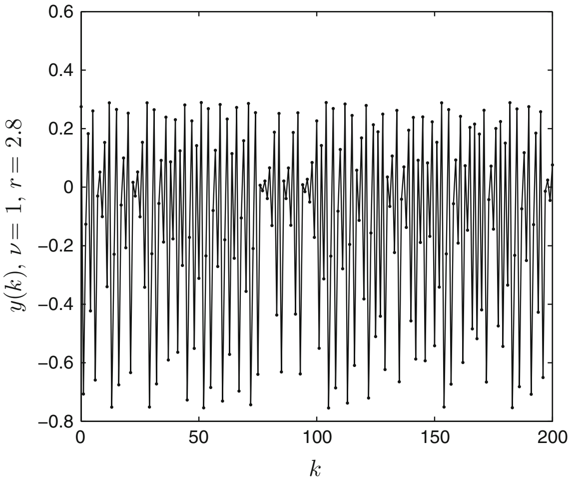


Fig. 4 Solution of (41) for $\nu = 1$, and $r = 2.8$ and $y_{-1} = -0.25$



Remark 2 According to the FOs adjacency the related plots are also close to the integer order DE solution. One can realize very small overshoot for $\nu = 1.1$ and monotonic growth for $\nu = 0.9$. Such transient behavior is in accordance with the FO dynamics [6].

The FODE describing restricted growth have also stable oscillations and chaotic properties. These are presented in Figs. 7 and 8, respectively.

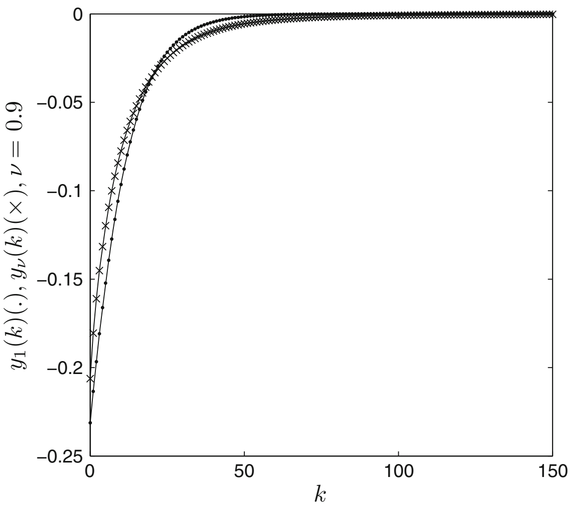


Fig. 5 Solution of (41) for $\nu = 0.9$, and $r = 0.1$ and $y_{-1} = -0.25$

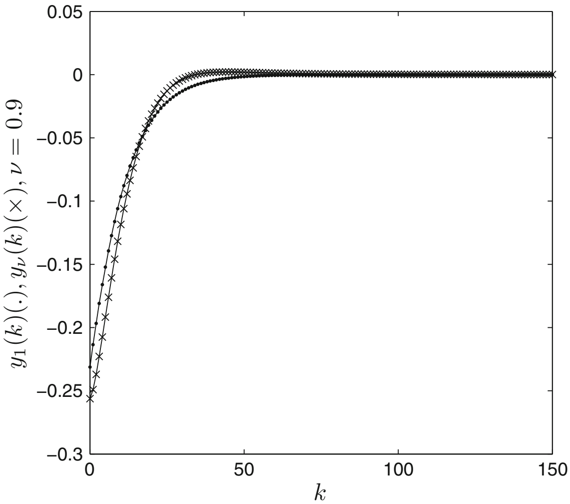


Fig. 6 Solution of (41) for $\nu = 1.1$, and $r = 0.1$ and $y_{-1} = -0.25$

4.2 Linearized Time-Variant FODE of the Restricted Growth

By way of confirmation of the linarization procedure one performs necessary steps on the FO restricted growth Eq. (41) to get the linear model. Related to (41) state-space equations of a non-linear model are trivial

$$ {}_{k_0}^{GL}\Delta_k^{(\nu)}\left[x_{n,1}(k)\right] = -r(k)x_{n,1}(k-1)\left[1 + x_{n,1}(k-1)\right],\ y(k) = x_{n,1}(k), \tag{42} $$

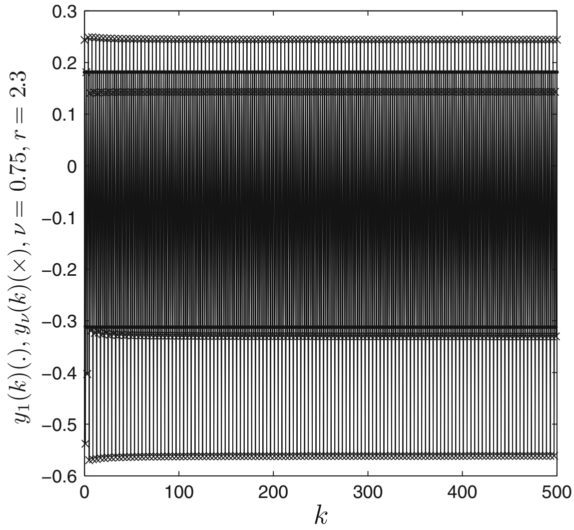


Fig. 7 Solution of (41) for $\nu = 0.75$, and $r = 2.3$ and $y_{-1} = -0.25$

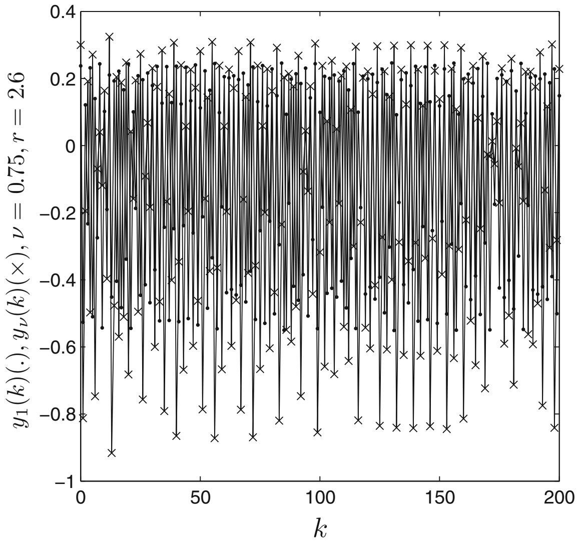


Fig. 8 Solution of (41) for $\nu = 0.75$, and $r = 2.6$ and $y_{-1} = -0.25$

with

$$f_1\left[x_1(k)\right] = -r(k)x_1(k-1)\left[1 + x_1(k-1)\right]. \tag{43}$$

hence, according to (32)

$$a_{11}(k) = \frac{\partial f_1\left[x_1(k)\right]}{\partial x_1(k)} = -r(k) - 2r(k)x_1(k-1) = -r(k)\left[1 + 2x_1(k)\right]. \tag{44}$$

To evaluate matrix (31) one should calculate the nominal solution. For a given initial state $y_1 = x_{1,-1}$ one gets a series representing a nominal solution values $x_{s,1}(k)$. Hence the linearized model is as follows

$$ {}^{GL}_{k_0}\Delta^{(\nu)}_{k}\left[\delta x_{l,1}(k)\right] = \left[-r(k) - 2r(k)x_{s,1}(k-1)\right]\delta x_{l,1}(k). \tag{45} $$

This is linear time-variant state-space model. Numerical simulations effects of the non-linear and linear approximation for $r(k) = r = \text{const}$ are given in Figs. 9 and 10, respectively.

Remark 3 Simulation results indicate that the non-linear system possessing stable oscillations and chaotic response properties cannot be approximated by the linear system, even by the time-variant one.

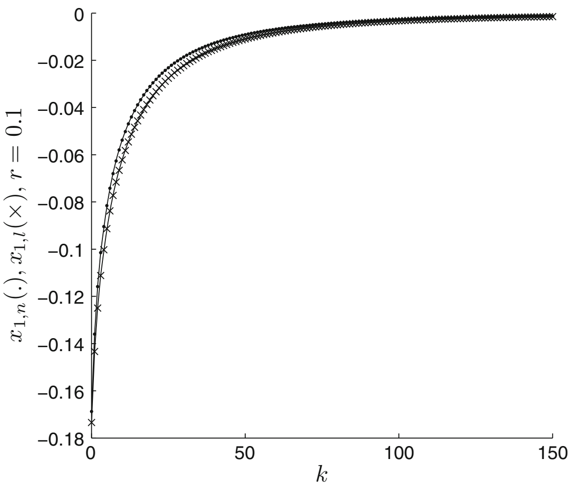


Fig. 9 Solution of (45) for $\nu = 0.75$, and $r = 0.1$ and $y_{-1} = -0.25$

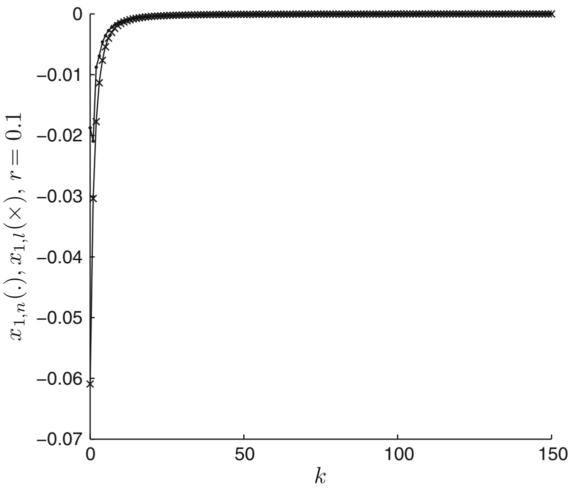


Fig. 10 Solution of (45) for $\nu = 0.75$, and $r = 0.9$ and $y_{-1} = -0.25$

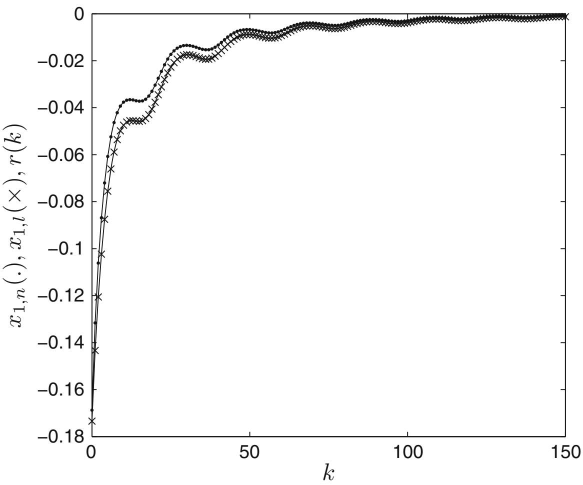


Fig. 11 Solution of (45) for $\nu = 0.75$, and $r = 0.1$ and $y_{-1} = -0.25$

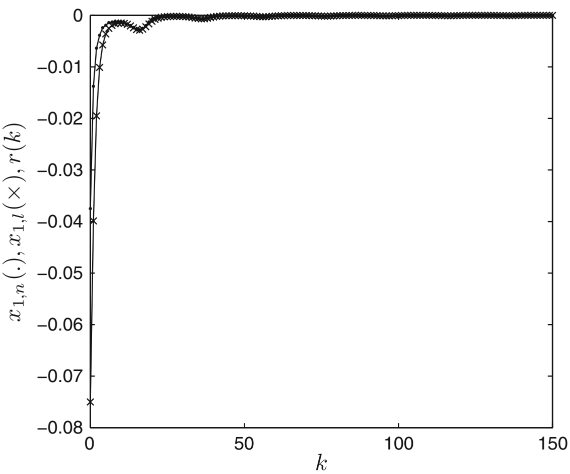


Fig. 12 Solution of (45) for $\nu = 0.75$, and $r = 0.8$ and $y_{-1} = -0.25$

4.3 Non-linear Time-Variant FODE of the Restricted Growth

In practice, the growth rate r is not a constant. It is a time function $r(k)$, so one can simulate the response of the non-linear time-variant FODE. Assuming that $r(k) = r\,[1 + \alpha \sin(\beta k)]$ one performs the linearization procedure on the time variant FODE. For $\alpha = 1, \beta = \frac{\pi}{10}$, $y_1 = 0.25$ and two selected values $r = 0.1$ and $r = 0.8$ one gets the non-linear and linearized solutions, which are presented in Figs. 11 and 12, respectively.

5 Final Conclusions

For the FODE one may apply the classical linearization procedure though the effects are satisfactory only for selected parameter ranges. Linear model cannot cover typical non-linear system transient behavior as a statical nonlinearity or chaos. Keeping in mind these limitations, however, the linearization procedure can be used for nonlinear systems characterized by monotonic responses. The model simplicity compensates inaccuracies of the system response.

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The Z-Transform Method for Sequential Fractional Difference Operators

Ewa Girejko, Ewa Pawłuszewicz and Małgorzata Wyrwas

Abstract The linear Caputo–type sequential difference fractional-order systems are discussed. The classical $\mathcal{Z}$-transform method is used to show the general solutions of sequential systems in the form $\left(\Delta_*^\alpha(\Delta_*^\alpha x)\right)(n) + b\left(\Delta_*^\alpha x\right)(n) + cx(n) = 0$, where $b, c \in \mathbb{R}$. In proofs we base on the formula for the image of the discrete Mittag-Leffler function in the $\mathcal{Z}$-transform.

1 Introduction

Roots of the fractional calculus are so old as the classical calculus is. The concept of the classical derivative is traditionally associated to an integer, i.e. one can differentiate a given function an integer times. The fact that Leibnitz was interested in the possibility of differentiating the given function a real number of times gives the beginning for the theory of fractional calculus. In the recent years the fractional calculus is viewed as a more adequate and better tool in descriptions of real system' behaviors, see for example [1, 6, 7, 10, 13, 18]. In modeling the real phenomena authors emphatically use generalizations of n-th order differences to their fractional forms. For this goal the study problems concerning solving methods for fractional differential and difference equations are important task. It can be noticed that many solution methods have been transferred from differential and integral equations theory.

The goal of the present work is to develop and discuss an approach to the solution of sequential difference equations with constant coefficients of a fractional orders. The idea of sequential fractional order differentiation came from [11, 17]. During

E. Girejko (✉) · E. Pawłuszewicz · M. Wyrwas
Bialystok University of Technology, Wiejska 45A, 15-351 Białystok, Poland
e-mail: e.girejko@pb.edu.pl

E. Pawłuszewicz
e-mail: e.pawluszewicz@pb.edu.pl

M. Wyrwas
e-mail: m.wyrwas@pb.edu.pl

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last years the solutions for a class of sequential differential equations of fractional order are under strong consideration, see for example [5, 8, 9] and the references within. In [12] the method of approximation of this type of differential equation was given. Due to our knowledge, the study of methods of solving sequential difference equations of fractional order are still not enough developed.

On the basis of the classical theory of difference equations (see for example [3]) we investigate and discuss the existence and form of solutions for sequential difference systems with the Caputo-type fractional discrete operator. We assume that in the given equation coefficients are constant. For this purpose the $\mathcal{Z}$-transform method is used. Some properties of $\mathcal{Z}$-transform for Caputo-type fractional difference are given in Sect. 2, more can be found in [16]. In Sect. 3 the solution of considered sequential equation is discussed. Once the solutions are delivered they can be used in further research: in formulating conditions providing existence of viable solutions to difference equations, also based on discretization of systems with continuous time.

2 Preliminaries

The necessary definitions and technical propositions that are used in the sequel of the paper are recalled.

Let $a \in \mathbb{R}$ and $\mathbb{N}_a := \{a, a+1, a+2, \ldots\}$. For a function $x : \mathbb{N}_a \to \mathbb{R}$ the forward difference operator is defined as (see [4]) $(\Delta x)(t) = x(t+1) - x(t)$, where $t \in \mathbb{N}_a$ and $(\Delta^0 x)(t) := x(t)$. Let $q \in \mathbb{N}_0$ and $\Delta^q := \Delta \circ \cdots \circ \Delta$ is q-fold application of operator Δ. Then

$$(\Delta^q x)(t) = \sum_{k=0}^{q} (-1)^{q-k} \binom{q}{k} x(t+k).$$

Let us introduce the family of binomial functions on $\mathbb{Z}$ parameterized by $\mu > 0$ and given by the values:

$$\widetilde{\varphi}_\mu(n) := \begin{cases} \binom{n+\mu-1}{n}, & \text{for } n \in \mathbb{N}_0 \\ 0, & \text{for } n < 0. \end{cases} \tag{1}$$

Definition 1 For a function $x : \mathbb{N}_a \to \mathbb{R}$ the *fractional sum of order* $\alpha > 0$ is given by

$$\left({}_a\Delta^{-\alpha} x\right)(t) := \left(\widetilde{\varphi}_\alpha * \overline{x}\right)(s),$$

where $t = a + s$, $\overline{x}(s) := x(a+s)$, $s \in \mathbb{N}_0$ and "$*$" denotes a convolution operator, i.e.

$$\left(\widetilde{\varphi}_\alpha * \overline{x}\right)(s) := \sum_{k=0}^{s} \binom{s-k+\alpha-1}{s-k} \overline{x}(k).$$

Additionally, we define $\left({}_a\Delta^0 x\right)(t) := x(t)$.

For $a = 0$ we will write shortly $\Delta^{-\alpha}$ instead of ${}_0\Delta^{-\alpha}$. Note that ${}_a\Delta^{-\alpha}x : \mathbb{N}_a \to \mathbb{R}$.

Let us recall that the $\mathcal{Z}$-transform of sequence $\{y(n)\}_{n\in\mathbb{N}_0}$ is a complex function given by

$$Y(z) := \mathcal{Z}[y](z) = \sum_{k=0}^{\infty} y(k)z^{-k},$$

where $z \in \mathbb{C}$ is a complex number for which the series $\sum_{k=0}^{\infty} y(k)z^{-k}$ converges absolutely. Note that since $\binom{k+\alpha-1}{k} = (-1)^k\binom{-\alpha}{k}$, then for $|z| > 1$ we have

$$\mathcal{Z}\left[\widetilde{\varphi}_\alpha\right](z) = \sum_{k=0}^{\infty} \binom{k+\alpha-1}{k} z^{-k} = \left(\frac{z}{z-1}\right)^{\alpha}. \tag{2}$$

Proposition 1 ([16]) *For $t = a + s \in (\mathbb{Z})_a$ let us define $y(s) := \left({}_a\Delta^{-\alpha}x\right)(t)$ and $\overline{x}(s) = x(a + s)$. Then*

$$\mathcal{Z}\left[y\right](z) = \left(\frac{z}{z-1}\right)^{\alpha} X(z), \tag{3}$$

where $X(z) := \mathcal{Z}\left[\overline{x}\right](z)$.

The definition of the Caputo–type fractional h-difference operator can be found, for example, in [2]. Here we use operators that do not change the domain of a function.

Definition 2 Let $\alpha \in (q-1, q]$, $q \in \mathbb{N}_1$ and $a \in \mathbb{R}$. The *Caputo–type fractional difference operator* ${}_a\Delta^{\alpha}$ of order α for a function $x : \mathbb{N}_a \to \mathbb{R}$ is defined by

$$\left({}_a\Delta_*^{\alpha}x\right)(t) = \left({}_a\Delta^{-(q-\alpha)}\left(\Delta^q x\right)\right)(t), \tag{4}$$

where $t \in \mathbb{N}_a$.

Moreover, for $\alpha = q \in \mathbb{N}_1$ we have $\left({}_a\Delta_*^{q}x\right)(t) = \left(\Delta^q x\right)(t)$. In the case $a = 0$ we write: $\Delta_*^{\alpha} :=_0 \Delta_*^{\alpha}$. Note that for $\alpha \in (0, 1]$ the Caputo–type fractional difference operator is as follows

$$\left({}_a\Delta_*^{\alpha}x\right)(t) = \left({}_a\Delta^{-(1-\alpha)}\left(\Delta x\right)\right)(t),$$

where $t \in \mathbb{N}_a$.

In general case we get the following result that is proven in [15].

Proposition 2 ([15]) *For $a \in \mathbb{R}$, $\alpha \in (q-1, q]$, $q \in \mathbb{N}_1$ let us define $y(s) := \left({}_a\Delta_*^{\alpha}x\right)(t)$, where $t \in \mathbb{N}_a$ and $t = a + s$, $s \in \mathbb{N}_0$. Then*

$$\mathcal{Z}\left[y\right](z) = z^q\left(\frac{z}{z-1}\right)^{-\alpha}\left(X(z) - \frac{z}{z-1}\sum_{k=0}^{q-1}(z-1)^{-k}\left(\Delta^k x\right)(a)\right), \tag{5}$$

where $X(z) = \mathcal{Z}[\overline{x}](z)$ and $\overline{x}(s) := x(a + s)$.

From Proposition 2 it follows that for $\alpha \in (0, 1]$ we get the following result:

Corollary 1 *For* $a \in \mathbb{R}$, $\alpha \in (0, 1]$ *let us define* $y(s) := \left({}_a\Delta_*^\alpha x\right)(t)$, *where* $t \in \mathbb{N}_a$ *and* $t = a + s$, $s \in \mathbb{N}_0$. *Then*

$$\mathcal{Z}\left[y\right](z) = \left(\frac{z}{z-1}\right)^{1-\alpha}\left((z-1)X(z) - zx(a)\right), \tag{6}$$

where $X(z) = \mathcal{Z}[\overline{x}](z)$ *and* $\overline{x}(s) := x(a+s)$.

We restrict our consideration to sequential fractional operators with order $\alpha \in (0, 1]$, so in this case $q = 1$. Applying the $\mathcal{Z}$-transform to the composition of two Caputo-type fractional operators with orders $\alpha, \beta \in (0, 1]$ we get:

Proposition 3 *For* $a \in \mathbb{R}$, $\alpha, \beta \in (0, 1]$ *let us define* $w(s) := \left({}_a\Delta_*^\beta\left({}_a\Delta_*^\alpha x\right)\right)(t)$, *where* $t \in \mathbb{N}_a$ *and* $t = a + s$, $s \in \mathbb{N}_0$. *Then*

$$\mathcal{Z}\left[w\right](z) = z^2\left(\frac{z-1}{z}\right)^{\alpha+\beta}X(z) - z\left(\frac{z-1}{z}\right)^{\beta-1}\left[z\left(\frac{z-1}{z}\right)^{\alpha}x(a) + \left(\Delta_*^\alpha x\right)(0)\right],$$

where $X(z) = \mathcal{Z}[\overline{x}](z)$ *and* $\overline{x}(s) := x(a+s)$.

Proof By (6) we get

$$\begin{aligned}
\mathcal{Z}\left[w\right](z) &= z\left(\frac{z-1}{z}\right)^{\beta}\mathcal{Z}\left[\Delta_*^\alpha x\right](z) - z\left(\frac{z-1}{z}\right)^{\beta-1}\left(\Delta_*^\alpha x\right)(0)\\
&= z\left(\frac{z-1}{z}\right)^{\beta}\left\{z\left(\frac{z-1}{z}\right)^{\alpha}X(z) - z\left(\frac{z-1}{z}\right)^{\alpha-1}x(a)\right\}\\
&\quad - z\left(\frac{z-1}{z}\right)^{\beta-1}\left(\Delta_*^\alpha x\right)(0)\\
&= z^2\left(\frac{z-1}{z}\right)^{\alpha+\beta}X(z) - z^2\left(\frac{z-1}{z}\right)^{\beta+\alpha-1}x(a)\\
&\quad - z\left(\frac{z-1}{z}\right)^{\beta-1}\left(\Delta_*^\alpha x\right)(0).
\end{aligned}$$

Therefore the thesis holds.

Observe that for $\alpha = \beta = 1$ we have $\left(\Delta_*^\alpha x\right)(0) = x(1) - x(0)$ and consequently, $\mathcal{Z}\left[w\right](z) = (z-1)^2 X(z) - z^2x(0) - zx(1)$ agrees with the transform of difference Δ^2 of $\overline{x}$.

Now, for $q = 1$ let us define the *discrete Mittag-Leffler two-parameter function* (similarly as in [14]) as follows:

$$E_{(\alpha,\beta)}(\lambda, n) := \sum_{k=0}^{\infty} \lambda^k \widetilde{\varphi}_{k\alpha+\beta}(n-k) = \sum_{k=0}^{n} \lambda^k \widetilde{\varphi}_{k\alpha+\beta}(n-k), \tag{7}$$

where $\widetilde{\varphi}_{k\alpha+\beta}$ is given by (1) and the second equality only claims that for $n < k$ we have values of $\widetilde{\varphi}_{k\alpha+\beta}(n-k) = 0$. In particular cases we have

$$E_{(\alpha,\alpha)}(\lambda, n) = \sum_{k=0}^{\infty} \lambda^k \widetilde{\varphi}_{k\alpha+\alpha}(n-k) = \sum_{k=0}^{\infty} \lambda^k \binom{n-k+(k+1)\alpha-1}{n-k}, \tag{8}$$

$$E_{(\alpha)}(\lambda, n) := E_{(\alpha,1)}(\lambda, n) = \sum_{k=0}^{\infty} \lambda^k \widetilde{\varphi}_{k\alpha+1}(n-k) = \sum_{k=0}^{\infty} \lambda^k \binom{n-k+k\alpha}{n-k}, \tag{9}$$

$$E_{(\alpha,0)}(\lambda, n) = \sum_{k=0}^{\infty} \lambda^k \widetilde{\varphi}_{k\alpha}(n-k) = \sum_{k=0}^{\infty} \lambda^k \binom{n-k+k\alpha-1}{n-k}. \tag{10}$$

Functions $E_{(\alpha,\alpha)}, E_{(\alpha)}$ and $E_{(\alpha,0)}$ are important in the solutions of sequential fractional difference equations that are considered in the next section.

Based on (2), for family of functions $\widetilde{\varphi}_{k\alpha+\beta}$ we can state the following result for discrete Mittag-Leffler function.

Proposition 4 ([14]) *Let* $\alpha \in (0, 1]$ *and* $\nu = \alpha - 1$. *Then*

1. $E_{(\alpha,\beta)}(\lambda, 0) = 1$.
2. *For* z *such that* $|z| > 1$ *we have*

$$\mathcal{Z}\left[E_{(\alpha,\beta)}(\lambda, \cdot)\right](z) = \left(\frac{z}{z-1}\right)^{\beta} \left(1 - \frac{\lambda}{z}\left(\frac{z}{z-1}\right)^{\alpha}\right)^{-1},$$

where $|z| > 1$ *and* $|z-1|^{\alpha}|z|^{1-\alpha} > |\lambda|$.

Observe that for $\alpha \in (0, 1]$, by Corollary 1 and Proposition 4, we get

$$\left(\Delta_*^{\alpha} E_{(\alpha)}(\lambda, \cdot)\right)(n) = \lambda E_{(\alpha)}(\lambda, n). \tag{11}$$

3 Solutions of Sequential Fractional Difference Equations Derived by $\mathcal{Z}$-transform method

Let us consider for $\alpha \in (0,1]$ the following sequential fractional difference equation:

$$\left(\Delta_*^\alpha(\Delta_*^\alpha x)\right)(n) + b\left(\Delta_*^\alpha x\right)(n) + cx(n) = 0\,, \tag{12}$$

where $b, c \in \mathbb{R}$.

Our goal is to find the function $x : \mathbb{N}_0 \to \mathbb{R}$ that is a solution to (12).

Let $\left(\Delta_*^{\alpha,\alpha} x\right) := \left(\Delta_*^\alpha(\Delta_*^\alpha x)\right)$. Then by (11) for $x(n) = E_{(\alpha)}(\lambda, n)$ we get

$$(\Delta_*^\alpha x)(n) = \lambda E_{(\alpha)}(\lambda, n)$$
$$\left(\Delta_*^{\alpha,\alpha} x\right)(n) = \lambda^2 E_{(\alpha)}(\lambda, n).$$

Hence the solution $x(\cdot) = E_{(\alpha)}(\lambda, \cdot)$ of (12) that we search for satisfies the following equality

$$\lambda^2 E_{(\alpha)}(\lambda, n) + b\lambda E_{(\alpha)}(\lambda, n) + cE_{(\alpha)}(\lambda, n) = 0. \tag{13}$$

Let us assume that $E_{(\alpha)}(\lambda, n) \neq 0$. Then we get the following form of characteristic equation for (12)

$$\lambda^2 + b\lambda + c = 0. \tag{14}$$

We will examine three cases for square roots of (14).

First, let $b^2 - 4c > 0$, then solutions to (14) are $\lambda_1 = \frac{-b-\sqrt{b^2-4c}}{2}$, $\lambda_2 = \frac{-b+\sqrt{b^2-4c}}{2} \in \mathbb{R}$ and we get the general solution of (12) as follows

$$x(n) = C_1 E_{(\alpha)}(\lambda_1, n) + C_2 E_{(\alpha)}(\lambda_2, n) = C_1 x_1(n) + C_2 x_2(n).$$

Since $\lambda_1 \neq \lambda_2$, the solutions $x_1(n) = E_{(\alpha)}(\lambda_1, n)$ and $x_2(n) = E_{(\alpha)}(\lambda_2, n)$ are linearly independent.

Now, let $b^2 - 4c = 0$. Then Eq. (12) can be rewritten as follows

$$\left[\left(\Delta_*^\alpha - \lambda\right)^2 x\right](n) = 0\,, \tag{15}$$

where $\lambda = -\frac{b}{2} \in \mathbb{R}$. By Corollary 1 we get

$$\begin{aligned}\mathcal{Z}\left[\Delta_*^{\alpha,\alpha} x\right](z) &= z\left(\frac{z-1}{z}\right)^\alpha \mathcal{Z}\left[\Delta_*^\alpha x\right](z) - z\left(\frac{z-1}{z}\right)^{\alpha-1} y(0)\\ &= z\left(\frac{z-1}{z}\right)^\alpha \left\{z\left(\frac{z-1}{z}\right)^\alpha X(z) - z\left(\frac{z-1}{z}\right)^{\alpha-1} x(0)\right\}\end{aligned}$$

$$- z\left(\frac{z-1}{z}\right)^{\alpha-1} y(0)$$
$$= z^2\left(\frac{z-1}{z}\right)^{2\alpha} X(z) - B_1(z)$$

where $y(0) := \left(\Delta_*^\alpha x\right)(0)$ and $B_1(z) = z^2\left(\frac{z-1}{z}\right)^{2\alpha-1} x(0) + z\left(\frac{z-1}{z}\right)^{\alpha-1} y(0)$. Moreover, we have

$$\begin{aligned}\mathcal{Z}\left[\left(\Delta_*^\alpha - \lambda\right)^2 x\right](z) &= \mathcal{Z}\left[\left(\Delta_*^{\alpha,\alpha} - 2\lambda\Delta_*^\alpha + \lambda^2\right)x\right](z)\\ &= z^2\left(\frac{z-1}{z}\right)^{2\alpha} X(z) - B_1(z) - 2\lambda z\left(\frac{z-1}{z}\right)^{\alpha} X(z)\\ &\quad + 2\lambda z\left(\frac{z-1}{z}\right)^{\alpha-1} x(0) + \lambda^2 X(z).\end{aligned}$$

After taking the $\mathcal{Z}$-transform of both sides of (15) we get

$$z^2\left(\frac{z-1}{z}\right)^{2\alpha}\left[1 - \frac{\lambda}{z}\left(\frac{z}{z-1}\right)^{\alpha}\right]^2 X(z) = B(z)\,,$$

where

$$\begin{aligned}B(z) &= B_1(z) - 2\lambda z\left(\frac{z-1}{z}\right)^{\alpha-1} x(0)\\ &= z^2\left(\frac{z-1}{z}\right)^{2\alpha-1} x(0) - 2\lambda z\left(\frac{z-1}{z}\right)^{\alpha-1} x(0) + z\left(\frac{z-1}{z}\right)^{\alpha-1} y(0).\end{aligned}$$

Consequently,

$$\begin{aligned}X(z) &= \frac{1}{z^2}\left(\frac{z-1}{z}\right)^{-2\alpha}\left[1 - \frac{\lambda}{z}\left(\frac{z}{z-1}\right)^{\alpha}\right]^{-2} B(z)\\ &= \frac{z}{z-1}\left[1 - \frac{\lambda}{z}\left(\frac{z}{z-1}\right)^{\alpha}\right]^{-1}\left[1 - \frac{\lambda}{z}\left(\frac{z}{z-1}\right)^{\alpha}\right]^{-1} x(0)\\ &\quad - 2\frac{\lambda}{z}\left(\frac{z}{z-1}\right)\left[1 - \frac{\lambda}{z}\left(\frac{z}{z-1}\right)^{\alpha}\right]^{-1} x(0)\left(\frac{z}{z-1}\right)^{\alpha}\left[1 - \frac{\lambda}{z}\left(\frac{z}{z-1}\right)^{\alpha}\right]^{-1}\\ &\quad + \frac{1}{z}\left(\frac{z}{z-1}\right)\left[1 - \frac{\lambda}{z}\left(\frac{z}{z-1}\right)^{\alpha}\right]^{-1}\left(\frac{z}{z-1}\right)^{\alpha}\left[1 - \frac{\lambda}{z}\left(\frac{z}{z-1}\right)^{\alpha}\right]^{-1} y(0)\end{aligned}$$

and by Proposition 4 we get

$$x(n) = E_{(\alpha)}(\lambda, n) * \left[E_{(\alpha,0)}(\lambda, n)x(0) - 2\lambda E^{\rho}_{(\alpha,\alpha)}(\lambda, n)x(0) + E^{\rho}_{(\alpha,\alpha)}(\lambda, n)y(0) \right] ,$$

where $E^{\rho}_{(\alpha,\alpha)}(\lambda, n) := E_{(\alpha,\alpha)}(\lambda, n-1)$. Therefore the general solution of (12) is as follows:

$$x(n) = C_1 \left(E_{(\alpha)}(\lambda, \cdot) * E_{(\alpha,0)}(\lambda, \cdot) \right)(n) + C_2 \left(E_{(\alpha)}(\lambda, \cdot) * E^{\rho}_{(\alpha,\alpha)}(\lambda, \cdot) \right)(n) ,$$

where $\lambda = -\frac{b}{2}$.

Finally, let $b^2 - 4c < 0$. Then $c > 0$ and $\lambda_{1,2} = -\frac{b}{2} \pm \frac{\sqrt{4c-b^2}}{2} i$. Hence $|\lambda_{1,2}| = \sqrt{c}$. Additionally, for $\lambda = -\frac{b}{2} + \frac{\sqrt{4c-b^2}}{2} i = \sqrt{c} e^{i\zeta}$ we have

$$\begin{aligned}
\overline{E_{(\alpha)}}(\lambda, n) &:= E_{(\alpha)}\left(-\frac{b}{2} \pm \frac{\sqrt{4c-b^2}}{2} i, n \right) = E_{(\alpha)}(|\lambda| e^{i\zeta}, n) \\
&= \sum_{k=0}^{\infty} (\sqrt{c})^k e^{ik\zeta} \widetilde{\varphi}_{k\alpha+1}(n-k) \\
&= \sum_{k=0}^{\infty} (\sqrt{c})^k \cos(k\zeta) \widetilde{\varphi}_{k\alpha+1}(n-k) + i \sum_{k=0}^{\infty} (\sqrt{c})^k \sin(k\zeta) \widetilde{\varphi}_{k\alpha+1}(n-k)
\end{aligned}$$

Consequently, the general solution of (12) is as follows

$$x(n) = C_1 \sum_{k=0}^{\infty} (\sqrt{c})^k \cos(k\zeta) \widetilde{\varphi}_{k\alpha+1}(n-k) + C_2 \sum_{k=0}^{\infty} (\sqrt{c})^k \sin(k\zeta) \widetilde{\varphi}_{k\alpha+1}(n-k) ,$$

where $\zeta = \arg\left(\frac{-b + i\sqrt{4c-b^2}}{2} \right)$.

Remark 1 Observe that in the particular case when $b = 0$ we get the following sequential fractional difference equation:

$$\left(\Delta^{\alpha}_{*}(\Delta^{\alpha}_{*} x) \right)(n) + cx(n) = 0 , \tag{16}$$

where $c \in \mathbb{R} \setminus \{0\}$. Then the general solution of (16) for $c < 0$ has the following form:

$$x(n) = C_1 E_{(\alpha)}(-\sqrt{-c}, n) + C_2 E_{(\alpha)}(\sqrt{-c}, n) = C_1 x_1(n) + C_2 x_2(n)$$

and similarly as in the general case we get $-\sqrt{-c} \neq \sqrt{-c}$, the solutions $x_1(n) = E_{(\alpha)}(-\sqrt{-c}, n)$ and $x_2(n) = E_{(\alpha)}(\sqrt{-c}, n)$ are linearly independent. In the case when $c > 0$ we get that the general solution of (16) is:

$$x(n) = C_1 \sum_{k=0}^{\infty} (\sqrt{c})^k \cos(k\frac{\pi}{2})\widetilde{\varphi}_{k\alpha+1}(n-k) + C_2 \sum_{k=0}^{\infty} (\sqrt{c})^k \sin(k\frac{\pi}{2})\widetilde{\varphi}_{k\alpha+1}(n-k).$$

Now, let us discuss the case when $\lambda = 0$ is the root of (14). Then we have two possibilities, namely $b \neq 0$, $c = 0$ and $b = c = 0$

Remark 2 Let us observe that in the situation when $b \neq 0$ and $c = 0$ we have the following sequential fractional difference equation:

$$\left(\Delta_*^\alpha(\Delta_*^\alpha x)\right)(n) + b\left(\Delta_*^\alpha x\right)(n) = 0\,, \tag{17}$$

and its characteristic equation is of the form

$$\lambda^2 + b\lambda = 0.$$

Note that $x_1(n) = const$ satisfies (17) because for all $n \in \mathbb{N}_0$ we get $(\Delta x_1)(n) = 0$ and consequently, $\left(\Delta_*^\alpha x_1\right)(n) = 0$ and $\left(\Delta_*^\alpha(\Delta_*^\alpha x)\right)(n) = 0$. Hence the general solution to Eq. (17) takes the following form

$$x(n) = C_1 + C_2 E_{(\alpha)}(-b, n)\,,$$

what after applying formula (7) can be rewritten as

$$x(n) = C_1 + C_2 \sum_{k=0}^{\infty} (-b)^k \widetilde{\varphi}_{k\alpha+1}(n-k).$$

Remark 3 Note that in the situation when $b = c = 0$ we have the following sequential fractional difference equation:

$$\left(\Delta_*^\alpha(\Delta_*^\alpha x)\right)(n) = 0. \tag{18}$$

After taking the $\mathcal{Z}$-transform of both sides of (18) we get

$$z^2\left(\frac{z-1}{z}\right)^{2\alpha} X(z) = z^2\left(\frac{z-1}{z}\right)^{2\alpha-1} x(0) + z\left(\frac{z-1}{z}\right)^{\alpha-1} y(0)\,,$$

where $y(0) := \left(\Delta_*^\alpha x\right)(0)$. Consequently,

$$X(z) = \frac{1}{z^2}\left(\frac{z-1}{z}\right)^{-2\alpha}\left[z^2\left(\frac{z-1}{z}\right)^{2\alpha-1}x(0) + z\left(\frac{z-1}{z}\right)^{\alpha-1}y(0)\right]$$
$$= \frac{z}{z-1}x(0) + \frac{1}{z}\left(\frac{z}{z-1}\right)^{\alpha+1}y(0)$$

and

$$x(n) = x(0) + \widetilde{\varphi}_{\alpha+1}(n-1)y(0).$$

Therefore the general solution of (18) is of the form:

$$x(n) = C_1 + C_2\widetilde{\varphi}_{\alpha+1}(n-1).$$

4 Conclusions

The problem of finding solutions of the linear sequential difference fractional order systems with Caputo–type operator was discussed. The obtained formulas for solution to the given problem extends the class of solutions of standard difference equations with forward time difference to the fractional case. To this aim the $\mathcal{Z}$-transform method was used.

This approach and results are the first step in the way on investigation on solutions of s, $s \geq 2$, linear sequential difference fractional order nonhomogeneous systems. It is also a good starting point in formulating conditions providing existence of viable solutions to difference equations, also based on discretization of systems with continuous time.

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Part II
Modeling and Approximations

An Estimation of Accuracy of Charef Approximation

Wojciech Mitkowski and Krzysztof Oprzędkiewicz

Abstract In the paper a new accuracy estimation method for Charef approximation was presented. Charef approximation allows us to describe fractional-order systems with the use of integer-order, proper transfer function. The accuracy of approximation can be estimated with the use of comparison step responses of plant and Charef approximation. The step response of the plant was calculated with the use of an accurate analytical formula and it can be interpreted as a standard. Approach presented in the paper can be applied to effective tuning of Charef approximant for given plant. The use of proposed method does not require to know a step response of the modeled plant. The proposed methodology can be easily generalized to another known approximations. Results are by simulations illustrated.

Keywords Fractional order transfer function · Charef approximation

1 An Introduction

Fractional order models are able to properly and accurate describe a number of physical phenomena from area of electrotechnics, heat transfer, diffusion etc. Fractional—order approach can be interpreted as generalization of known integer-order models. Fractional order systems has been presented by many Authors ([2, 3, 7, 10]), an example of identification fractional order system can be found in [4, 5] the proposition of generalization the Strejc transfer function model into fractional area was given in [8].

W. Mitkowski · K. Oprzędkiewicz (✉)
Department of Automatics and Biomedical Engineering, Faculty of Electrotechnics, Automatics, Informatics and Biomedical Engineering, AGH University of Science and Technology, Kraków, Poland
e-mail: kop@agh.edu.pl

W. Mitkowski
e-mail: wojciech.mitkowski@agh.edu.pl

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A modeling of fractional—order transfer function in MATLAB/SIMULINK requires us to apply integer order, finite dimensional, proper approximations. An important problem is to assign parameters of approximation and estimating its accuracy. The most known approximations presented by Oustaloup or Charef (see for example [1, 2, 9, 11, 12]) base onto frequency approach. This is caused by a fact, that for fractional order systems the Bode magnitude plot can be drawn exactly and its parameters can be applied to approximants calculation.

Additionally, for elementary fractional-order elements an analytical form of step and impulse responses is known (see [11]). These responses can be applied as reference to estimate a correctness of built approximant.

However models obtained with the use of Charef approximation not always are fully satisfying. This is caused by the fact, that their accuracy is determined by proper selecting maximal permissible error and order of approximation with respect to considered, certain values of modeled transfer function.

The goal of this paper is to present a new, simple method of accuracy estimation for Charef approximation. The presented method uses analytical formulas of step response of fractional order system, proposed by authors cost functions and numerical calculations done with the use of MATLAB. The approach shown in the paper can be applied to effective selecting parameters of the Charef approximation during modeling real plants, described with the use of fractional-order models. Additionally it does not require the use a step response of modeled plant.

Particularly, in the paper the following problems will be presented:

- Transfer function of fractional order plant and analytical form of step response,
- The Charef approximation,
- Cost functions describing the accuracy of approximation,
- An example.

2 A Transfer Function of Fractional Order Plant

Let us consider a an elementary fractional—order inertial plant described with the use of transfer function (1). This transfer function can be applied to modeling of high-order inertial plants, for example heat plant described in [5, 6].

$$G_\alpha(s) = \frac{1}{(T_\alpha s + 1)^\alpha} \tag{1}$$

In (1) $T_\alpha > 0$ is a time constant, $\alpha \in (0;1)$ is a fractional order of the plant.

The analytical form of the step response $y_\alpha(t)$ for plant described with the use of (1) is described as follows (see [2, pp. 8, 9]):

$$y_a(t) = L^{-1}\left\{\frac{1}{s}G_\alpha(s)\right\} = \frac{1}{(T_\alpha)^\alpha} \cdot \frac{\Gamma\left(\alpha, \frac{t}{T_\alpha}\right)}{\Gamma(\alpha)} \tag{2}$$

where *Γ(..)* denotes incomplete and complete Gamma functions:

$$\Gamma\left(\alpha, \frac{t}{T_\alpha}\right) = \int_0^{\frac{t}{T_\alpha}} e^{-x}x^{\alpha-1}dx \tag{3}$$

$$\Gamma(\alpha) = \int_0^\infty e^{-x}x^{\alpha-1}dx \tag{4}$$

Let us assume, that the step response described by (2)–(4) is the accurate response. This implies, that it can be applied as a standard to estimate the accuracy of approximation.

3 The Charef Approximation

The Charef approximation allows us to approximate fractional order transfer function, described by (1) with the use of integer order transfer function $G_{char}(s)$ described as underneath (see [1]):

$$G_{char}(s) = \frac{\prod_{i=0}^{N-1}\left(1 + \frac{s}{z_i}\right)}{\prod_{i=0}^{N}\left(1 + \frac{s}{p_i}\right)} \tag{5}$$

In (5) N denotes order of approximation, z_i and p_i denote zeros and poles of approximation. They can be calculated with the use of transfer function (1) pole p_α and fractional order α:

$$\begin{aligned} p_\alpha &= \frac{1}{T_\alpha} \\ p_0 &= p_\alpha\sqrt{b} \\ p_i &= p_0(ab)^i, \quad i = 1..N \\ z_i &= ap_0(ab)^i \quad i = 1..N-1 \end{aligned} \tag{6}$$

where:

$$a = 10^{\frac{\Delta}{10(1-\alpha)}} \\ b = 10^{\frac{\Delta}{10\alpha}} \tag{7}$$

In (7) Δ denotes a maximal permissible error of approximation, defined as the maximal difference between Bode magnitude plots model and plant, expressed in [dB].

The order N of Charef approximation is assigned to minimize the assumed, maximal approximation error Δ (See [1]):

$$N = \left[\frac{\log\left(\frac{\omega_{\max}}{p_0}\right)}{\log(ab)} \right] + 1 \tag{8}$$

In (8) $\omega_{\max}$ denotes the pulsace, for which the maximal error is achieved. If the value of N with respect to (8) is non-integer, it should be rounded to nearest integer.

Denote the step response of approximation (5) by $y_{char}(t)$. It can be written as underneath:

$$y_{char}(t) = L^{-1}\left\{ \frac{1}{s} G_{char}(s) \right\} \tag{9}$$

The general form of the step response (9) is determined by poles and zeros of transfer function $G_{char}(s)$ described by (6). They are real and different. This implies, that the general form of (9) can be easily expressed as follows:

$$y_{char}(t) = k \cdot 1(t) + \sum_{i=1}^{N} c_i e^{-p_i t} \tag{10}$$

In (10) k denotes the steady-state gain of the approximation, c_i denote coefficients of transfer function (5) factorization.

The step response (9) or (10) can be evaluated numerically with the use of MATLAB/SIMULINK.

4 Cost Functions Describing the Accuracy of Approximation

Let us assume, that the step response $y_a(t)$ described by (2)–(4) is the accurate response. Next, $y_{char}(t)$ is the step response of approximation, described by (9) and (10). Then the approximation error $e(t)$ can be defined as follows:

$$e_a(t) = y_a(t) - y_{char}(t) \tag{11}$$

Furthermore, let us introduce the following cost functions, describing the accuracy of approximation:

$$\begin{aligned} I_{\max}(\Delta, N) &= \max_t |e_a(t))| \\ I_2(\Delta, N) &= \int_0^\infty e_a^2(t)dt \end{aligned} \tag{12}$$

In (12) $e_a(t)$ denotes the approximation error described by (11).

The both cost functions (12) for given plant (described by T_α and α) are functions of approximation parameters: order N and maximal permissive error of approximation Δ expressed in [dB]. The N order is associated to Δ by (8), but in a number of situations it can be set independently. It can be expected, that increasing N for constant Δ should increase an approximant quality, described by cost functions (12). However, results of simulations point, that too high value of N can cause bad conditioning of a model and consequently, make it useless.

The fastest method to check proper setting the approximation parameters N and Δ is to calculate the both proposed cost functions (12). An example of proper tuning a Charef approximant with the use of simulations will be shown in the next section.

5 An Example

As an example let us consider the application of Charef approximation to modeling the fractional-order transfer function described by (13):

$$G_\alpha(s) = \frac{1}{(25s+1)^\alpha} \tag{13}$$

Values of both cost functions (12) for different Δ, N and α are given in Table 1. The following values of model parameters were tested: $\Delta = 0.5$[dB], 1.0[dB], $N = 5$, 10, 15, 25, $\alpha = 0.2$, 0.5, 0.9.

Let $e_a(t)$ be an approximation error described by (11). Exemplary diagrams $e_a(t)$ as a function of a time t for selected values of parameters: α, Δ, and N are shown in Figs. 1, 2, 3, 4 and 5.

From Table 1 and Figs. 1, 2, 3, 4 and 5 we can conclude at once, that the good approximant can be obtained with the use of lower orders N and the smaller permissible error Δ requires higher value of order N to assure the good performance of approximation.

Table 1 Values of cost functions (12) for different α, Δ, and N

Δ [dB]	N	Fractional order α					
		0.2		0.5		0.9	
		I_{max}	I_2	I_{max}	I_2	I_{max}	I_2
1.0	5	0.1398	0.0651	0.1466	0.2550	0.0825	0.0854
	10	0.1410	0.0659	0.1505	0.2683	0.0825	0.0854
	15	0.1410	0.0659	0.1505	0.2684	0.5750	83.7787
	25	0.1399	0.0602	0.1505	0.2684	NaN	NaN
0.5	5	0.2599	0.0597	0.0952	0.0952	0.1662	0.4288
	10	0.0810	0.0600	0.0904	0.1647	0.1662	0.4292
	15	0.0816	0.0608	0.0933	0.1748	0.1662	0.4292
	25	0.0816	0.0608	0.0936	0.1759	0.1662	0.4283

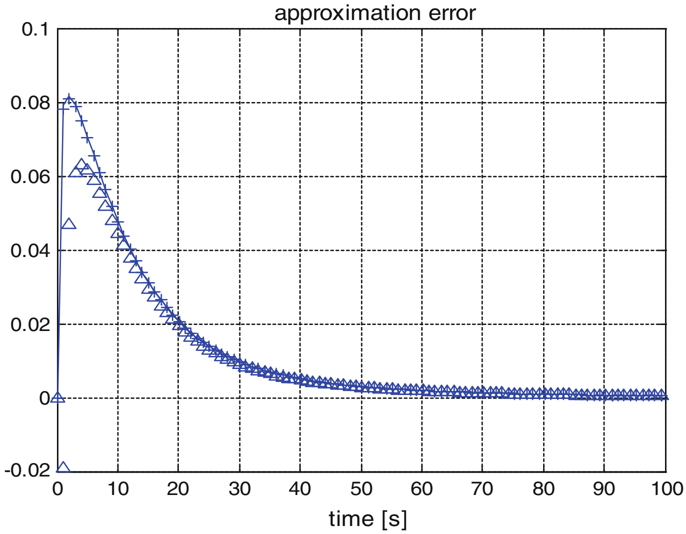

Fig. 1 Approximation error $e_a(t)$ described by (11) for: $\alpha = 0.2$, $\Delta = 0.5$[dB], and $N = 5$ (^), $N = 10(+)$, $N = 15(-)$

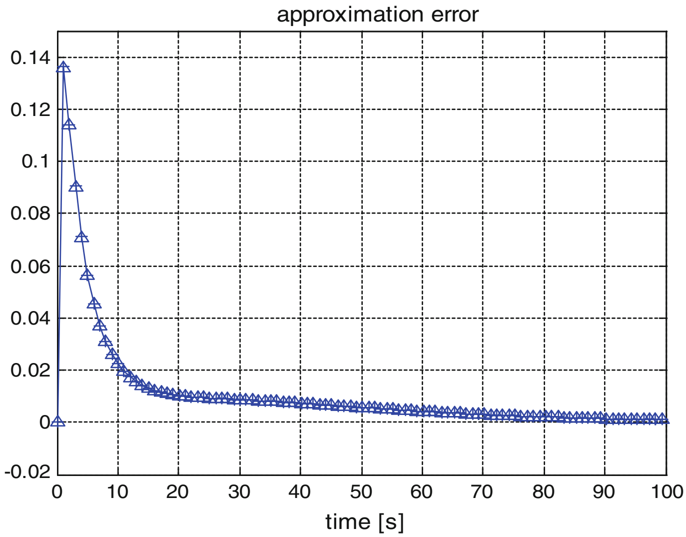

Fig. 2 Approximation error $e_a(t)$ described by (11) for: $\alpha = 0.2$, $\Delta = 1.0$[dB], and $N = 5$ (^), $N = 10(+)$, $N = 15(-)$

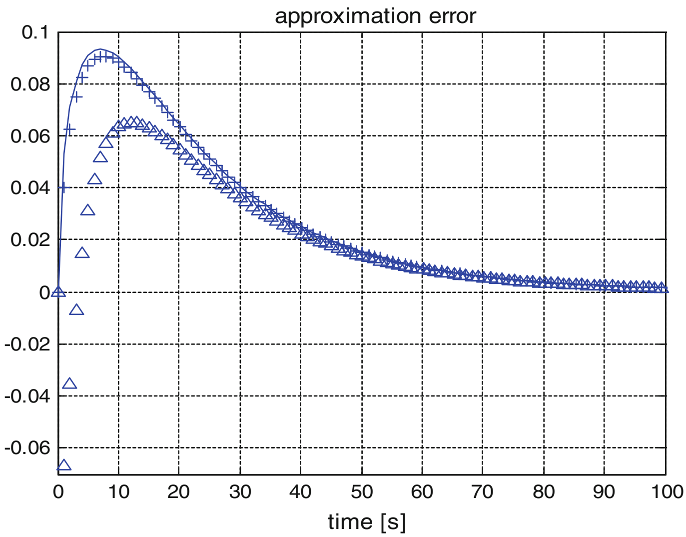

Fig. 3 Approximation error $e_a(t)$ described by (11) for: $\alpha = 0.5$, $\Delta = 0.5$[dB], and $N = 5$ (^), $N = 10(+)$, $N = 15(-)$

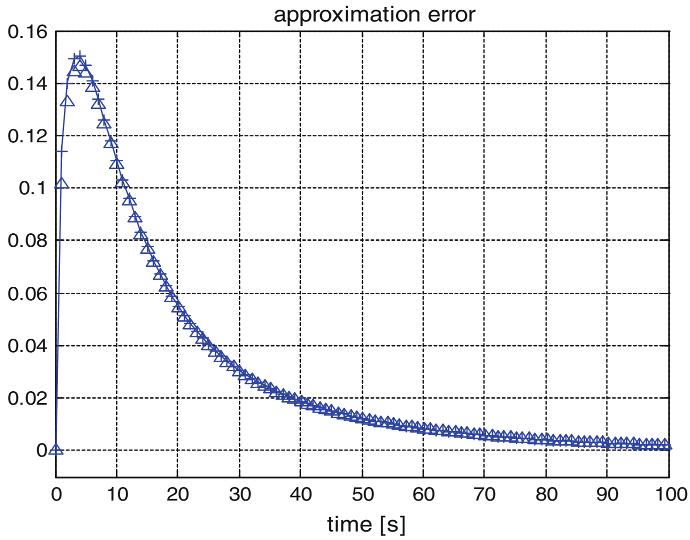

Fig. 4 Approximation error $e_a(t)$ described by (11) for: $\alpha = 0.5$, $\Delta = 1.0$[dB], and $N = 5$ (^), $N = 10(+)$, $N = 15(-)$

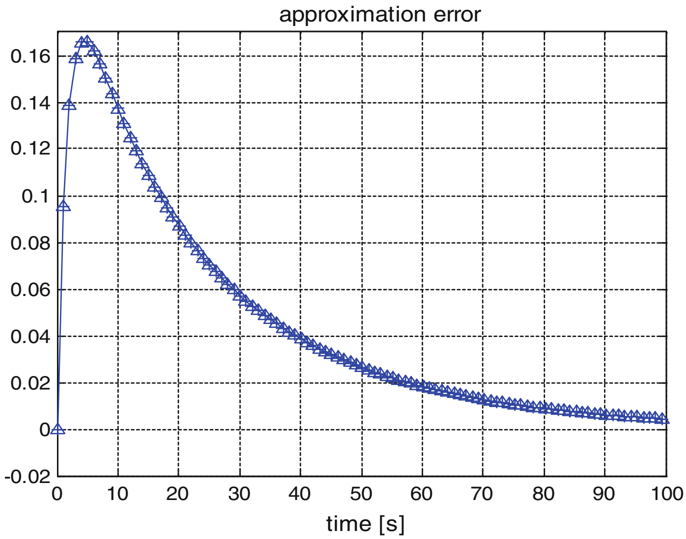

Fig. 5 Approximation error $e_a(t)$ described by (11) for: $\alpha = 0.9$, $\Delta = 0.5$[dB], and $N = 5$ (^), $N = 10(+)$, $N = 15(-)$

6 Final Conclusions

The final conclusions for the paper can be formulated as follows:

- In the paper the analysis of accuracy the Charef approximation as a function of its parameters (order N and maximal permissive error Δ) was presented. Different fractional orders were also tested.
- The accuracy of Charef approximation is stronger dependent on maximal permissive error Δ, than approximation order, described by N.
- The low order of approximation N, equal 5 assures the good accuracy of approximation. The improving of this order does not improve this accuracy.
- Too high order of approximation causes numerical errors (see Table 1—results described as "Nan"—this result is returned by MATLAB if the result is not mathematically defined, for example 0/0). Causes of this phenomenon are not known, but they were observed many times during modeling of fractional order systems at MATLAB/SIMULNK platform.
- Decreasing of maximal permissive error Δ improves the accuracy of approximation in the sense of considered cost functions.
- Method presented in the paper can be applied to effective tuning of Charef approximant for given plant, described by time constant and fractional order. Notice, that the use of the proposed method does not require to know the step response of a modeled plant.
- The approach presented in this paper can be applied to another typical approximations, both continuous (ORA approximation) and discrete (PSE and CFE approximations) used to modeling fractional order elements. This problem is going to be considered by authors.
- As an another area of further investigations will be formulating analytical conditions directly associating the cost functions (12) with plant parameters T_α and α and approximation parameters N and Δ.

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A New Method for the Integer Order Approximation of Fractional Order Models

Wieslaw Krajewski and Umberto Viaro

Abstract This paper is concerned with the finite–dimensional approximation of a fractional–order system represented in state–space form. To this purpose, resort is made to the Oustaloup method for approximating a fractional–order integrator by a rational filter. The dimension of the resulting integer–order model can be reduced using an efficient algorithm for the minimization of the L_2 norm of a weighted equation error. Two numerical examples are worked out to show how the desired approximation accuracy can be ensured.

Keywords Fractional–order models · Approximation · Oustaloup method · Model reduction · Equation error

1 Introduction

Non–integer order systems have been recently considered with increasing attention in the control literature because many plants can be described more satisfactorily by models of this kind [2, 8, 9, 13]. However, such systems are infinite–dimensional and their transfer function is irrational. Therefore, *ad hoc* methods and algorithms are needed to simulate their behaviour. Since the approaches based on the numerical solution of fractional differential equations are, in general, computationally hard, most techniques resort to the approximation, over suitably–defined frequency ranges, of these systems by means of integer–order models (see, e.g., [5, 12, 14]).

This paper considers a general (not necessarily commensurate) fractional–order system given in the state–space form. By applying the integer–order approximation

W. Krajewski (✉)
Systems Research Institute, Polish Academy of Sciences,
ul. Newelska 6, 01–447 Warsaw, Poland
e-mail: krajewsk@ibspan.waw.pl

U. Viaro
Department of Electrical, Management and Mechanical Engineering,
University of Udine, via delle Scienze 206, 33100 Udine, Italy
e-mail: umberto.viaro@uniud.it

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of the fractional integrator operator $1/s^{\alpha}$ ($\alpha \in \mathbb{R}_+$) proposed in [14], a finite–dimensional state–space model with block companion state matrix is obtained. The sparsity of this matrix simplifies simulations. However, since the order of this model tends to be high, it has been suggested to approximate it using a method developed for finite–dimensional systems. For example, the model reduction method based on the Singular Value Decomposition has been used in [7] and the method based on the minimization of the unweighted L_2 norm of the impulse–response error has been used in [17]. Recently, the present authors have suggested to apply the iterative–interpolation algorithm for L_2 model reduction presented in [4]. In this paper, to reduce the dimensionality of the integer–order model, the more efficient weighted equation–error approach [3] is applied instead.

The rest of the paper is organized as follows. Section 2 briefly presents the formal description of non–integer order linear time–invariant (LTI) systems. Some recent approaches to the rational approximation of fractional operators and to model simplification are outlined and discussed in Sect. 3. The suggested approximation method is presented in Sect. 4. Two meaningful examples taken from the literature are worked out in Sect. 5 to show the performance of the suggested approximation. Some concluding remarks are drawn in Sect. 6.

2 Non–Integer Order Linear Systems Recap

Fractional–order calculus is a generalization of integer–order differentiation and integration. Many definitions of fractional–order differentiation and integration operators have been proposed. Especially successful have been those of Grünwald–Letnikov, Riemann–Liouville and Caputo [11]. The last one is most commonly used in engineering applications.

The Laplace transform of the fractional Caputo derivative $D^{\alpha}x(t)$ is

$$\mathscr{L}\{D^{\alpha}x(t)\} = s^{\alpha}\mathscr{L}\{x(t)\} - \sum_{i=0}^{[\alpha]-1} s^{\alpha-i-1}\frac{d^i x}{dt^i}(0), \tag{1}$$

where $[\alpha]$ denotes the integer part of α.
Consider a scalar[1] LTI fractional–order system described by the differential equation

$$y(t) + \sum_{i=1}^{n} a_i D^{\alpha_i} y(t) = \sum_{i=1}^{m} b_i D^{\beta_i} u(t), \tag{2}$$

where $a_i, b_i \in \mathbb{R}$, $D^{\lambda} = \dfrac{d^{\lambda}}{dt^{\lambda}}$, $\alpha_i, \beta_i \in \mathbb{R}_+$.

[1]This assumption is made to simplify the exposition. The case of MIMO systems can be treated in a similar way.

By applying (1)–(2) and assuming zero initial conditions, the system transfer function turns out to be

$$G(s) = \frac{b(s)}{a(s)} = \frac{\sum_{i=1}^{m} b_i s^{\beta_i}}{1 + \sum_{i=1}^{n} a_i s^{\alpha_i}}. \tag{3}$$

If all fractional orders are multiples of the same real number α (which qualifies the system as a commensurate fractional–order system), (3) can be written as

$$G(s) = \frac{\sum_{i=1}^{m} b_i (s^{\alpha})^i}{1 + \sum_{i=1}^{n} a_i (s^{\alpha})^i}. \tag{4}$$

The state–space model corresponding to (3) is

$$D^{(\alpha)}(x)(t) = Ax(t) + bu(t), \tag{5}$$
$$y(t) = cx(t) + du(t), \tag{6}$$

where

$$D^{(\alpha)}(x) = \left[D^{\alpha_1} x_1, D^{\alpha_2} x_2, \dots, D^{\alpha_n} x_n \right]^T$$

and $A \in \mathbb{R}^{n \times n}, b \in \mathbb{R}^{n \times 1}, c \in \mathbb{R}^{1 \times n}$.
In the commensurate case, Eqs. (5) and (6) become

$$D^{\alpha}(x)(t) = Ax(t) + bu(t), \tag{7}$$
$$y(t) = cx(t) + du(t), \tag{8}$$

where

$$D^{\alpha}(x) = \left[D^{\alpha} x_1, D^{\alpha} x_2, \dots, D^{\alpha} x_n \right]^T.$$

3 Fractional Order Model Simplification

The analysis of non–integer order models is made difficult by the irrational nature of their transfer function and by the infinite dimensionality of their state–space representations. Therefore, a number of methods have been proposed to simplify such models. Two alternative kinds of methods can be used in this regard:

1. the methods leading to a simpler model hat is still described by an irrational transfer function or an infinite–dimensional state–space representation,
2. the methods that approximate the non–integer order model by means of a finite dimensional one.

The first group of methods is useful for commensurate systems like (4): see, for example, [16, 18]. Indeed, in this case, by setting $s^\alpha = w$, a transfer function that is rational with respect to w is obtained:

$$\hat{G}(w) = \frac{\sum_{i=1}^{m} b_i w^i}{1 + \sum_{i=1}^{n} a_i w^i} \tag{9}$$

to which any order reduction method can then be applied. However, this approach does not guarantee the stability of the resulting model. An even more serious drawback is that this model may not be truly simpler than the original one. To show this, consider the fractional system put forth in [15] whose transfer function is

$$G(s) = \frac{s^{1.56} + 4}{s^{3.46} + 10s^{2.69} + 20s^{1.56} + 4}. \tag{10}$$

Since $\alpha = 1/100$, the rational transfer function (9) corresponding to (10) is of order 346. Even if its order could be reduced to 10, the denominator of this reduced transfer function will consists of 11 terms, whereas the denominator in (10) consists of only 4 terms. Hence the above approach can be successful when almost all coefficients a_i in (4) or in (9) are non–zero, as in the following example considered in [16, 18]:

$$G(s) = \frac{(s^{0.8} + 4)(s^{1.6} + 2s^{0.8} + 4)(s^{1.6} + 3s^{0.8} + 1)}{(s^{0.8} + 1)(s^{0.8} + 3)(s^{1.6} - 2s^{0.8} + 37)(s^{1.6} + 4s^{0.8} + 8)}. \tag{11}$$

The methods of the second type are usually based on the rational approximation of the operator s^α. Among the various approaches of this kind (see, e.g., [5, 9]), the most popular is almost certainly the one due to Oustaloup [10] by which the fractional differentiator operator s^α, $0 \le \alpha \le 1$, is replaced by a rational filter $\mathcal{D}^\alpha(s)$ whose zeros and poles are distributed over a frequency band $[\omega_m, \omega_M]$ centred at

$$\omega_u = \sqrt{\omega_m \omega_M}. \tag{12}$$

Precisely, the approximating filter is formed by the cascade of $2N + 1$ first–order cells:

$$\mathcal{D}^\alpha(s) = K_\alpha \prod_{k=-N}^{N} \frac{1 + \dfrac{s}{\omega_k'}}{1 + \dfrac{s}{\omega_k}}, \tag{13}$$

where ω_k' and ω_k are computed recursively according to

$$\omega_0' = \delta^{-\frac{1}{2}} \omega_u, \quad \omega_0 = \delta^{\frac{1}{2}} \omega_u,$$

$$\frac{\omega'_{k+1}}{\omega'_k} = \frac{\omega_{k+1}}{\omega_k} = \delta\,\eta > 1,$$

$$\frac{\omega_k}{\omega'_k} = \delta > 0, \quad \frac{\omega'_{k+1}}{\omega_k} = \eta > 0,$$

$$\omega'_{-N} = \eta^{\frac{1}{2}}\omega_m, \quad \omega'_N = \eta^{-\frac{1}{2}}\omega_M,$$

with [12]

$$\delta = \left(\frac{\omega_M}{\omega_m}\right)^{\frac{\alpha}{2N+1}}, \quad \eta = \left(\frac{\omega_M}{\omega_m}\right)^{\frac{1-\alpha}{2N+1}}.$$

The gain K_α is chosen so as to ensure that $\mathcal{D}^\alpha(s)$ has the same magnitude as s^α at ω_u. The number of filter cells is clearly related to the goodness of the approximation.

The fractional–order integrator operator $1/s^\alpha$ can be approximated in a way consistent with that adopted for the differentiator operator. Precisely, the approximation of the fractional integrator operator can be chosen [14] as

$$\mathcal{I}^\alpha(s) = \frac{K_\alpha}{s}\prod_{k=-N}^{N}\frac{1+\dfrac{s}{\omega'_k}}{1+\dfrac{s}{\omega_k}}, \tag{14}$$

which behaves (almost) like $1/s^\alpha$ in an interval $[\omega_m, \omega_M]$.

Functions $\mathcal{D}^\alpha(s)$ and $\mathcal{I}^\alpha(s)$ allow us to find rational models of practically any fractional system. However, the direct application of these operators often leads to high–dimensional models. Consider again the fractional transfer function (10). By setting $\omega_m = 10^{-3}$, $\omega_M = 10^3$, $N = 10$, and applying (13) to $s^{0.56}$, $s^{0.46}$ and $s^{0.69}$, the order of the integer–order approximating transfer function turns out to be 87.

Also, high–order transfer–function models tend to be ill–conditioned: for example, the ratio of the largest to the smallest values of the transfer function coefficients obtained according to the above procedure may be even higher than 10^{80}. Therefore, numerical difficulties are encountered with almost all order reduction algorithms. These difficulties may be avoided if LTI state–space models (of both fractional and integer order) are considered. Examples of such an approach can be found in [4, 6, 7, 14, 15, 18]. The integer order approximation of the non integer order model (5)–(6) obtained according to the procedure in [4] is briefly outlined next. For details see [4].

Consider the state equations (5) and let the fractional–order integrators $1/s^{\alpha_k}$ be approximated according to (14) as

$$\mathcal{I}^{\gamma_k}(s) = \frac{\sum_{j=0}^{m} f_{k,j} s^j}{s \sum_{j=0}^{m} g_{k,j} s^j}, \tag{15}$$

where $m = 2N + 1$. Then, define the matrices

$$A_0 = -\text{diag}\{f_{1,0}, f_{2,0}, \dots, f_{\ell,0}\} A,$$

$$A_k = \text{diag}\{g_{1,k-1}, g_{2,k-1}, \dots, g_{\ell,k-1}\} - \text{diag}\{f_{1,k}, f_{2,k}, \dots, f_{\ell,k}\} A,$$

for $k = 1, \dots, m$, and

$$B_k = \text{diag}\{f_{1,k}, f_{2,k}, \dots, f_{\ell,k}\} b,$$

for $k = 0, \dots, m$. The following state–space integer order model approximating (5)–(6) is obtained:

$$\dot{\hat{x}}(t) = \hat{A}\hat{x}(t) + \hat{B}u(t), \tag{16}$$

$$\hat{y}(t) = \hat{C}\hat{x}(t) + du(t), \tag{17}$$

where $\hat{x} \in \mathbb{R}^{(2N+2)\ell}$, and matrices $\bar{A} \in \mathbb{R}^{(2N+2)\ell\times(2N+2)\ell}$, $\bar{B} \in \mathbb{R}^{(2N+2)\ell\times 1}$ and $\bar{C} \in \mathbb{R}^{1\times(2N+2)\ell}$ are given by

$$\hat{A} = \begin{bmatrix} 0 & 0 & \dots & 0 & -A_0 \\ I & 0 & \dots & 0 & -A_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & -A_{m-1} \\ 0 & 0 & \dots & I & -A_m \end{bmatrix}, \quad \hat{B} = \begin{bmatrix} B_0 \\ B_1 \\ \vdots \\ B_{m-1} \\ B_m \end{bmatrix}, \quad \hat{C} = \begin{bmatrix} 0 & 0 & \dots & 0 & c \end{bmatrix}, \tag{18}$$

with c and d as in the original representation (5).

4 Model Reduction

The only way to ensure a more accurate integer–order approximation of a given fractional–order system is to increase the value of N. This, however, leads to high–dimensional models that require the design of complex and expensive controllers. To overcome this problem, resort can be made to the following two–step procedure. First, a high–dimensional integer–order model corresponding to a large value of N is determined, thus ensuring the desired accuracy. Then, a reduced–order model is found from this high–order model by applying a suitable reduction algorithm.

The methods suggested in the literature to find L_2-optimal reduced–order models red (see, e.g., [4, 17]) are difficult to implement or depend crucially on the initial conditions. To avoid these difficulties, resort can be made to a slightly different approach that refers to the L_2 norm of the so–called equation error [3].

Let the triple (A_r, B_r, C_r), where $A_r \in \mathbb{R}^{q\times q}$, $B_r \in \mathbb{R}^{q\times 1}$, $C_r \in \mathbb{R}^{1\times q}$, represent the low order model. The aforementioned procedure involves the determination of two projection matrices L_r and T_r such that

$$A_r = L_r\hat{A}T_r, \;\; B_r = L_r\hat{B}, \;\; C_r = \hat{C}T_r. \tag{19}$$

Next, define

$$\mathcal{O}_{[-k:q-k-1]} \doteq \begin{bmatrix} \hat{C}\hat{A}^{-k} \\ \vdots \\ \hat{C} \\ \vdots \\ \hat{C}\hat{A}^{q-k-1} \end{bmatrix}. \tag{20}$$

and assume that W_c is the controllability Gramian, which is the solution of the Lyapunov equation:

$$\hat{A}W_c + W_c\hat{A}^T + \hat{B}\hat{B}^T = 0\,. \tag{21}$$

The projection matrices in (19) may be determined in such a way that L_r^T spans the range of $\mathcal{O}^T_{[-k:q-k-1]}$ and $T_r = W_c L_r^T (L_r W_c L_r^T)^{-1}$. It can be shown [3] that, in this way, model (A_r, B_r, C_r) retains:

(i) the k time moments $\hat{C}\hat{A}^{-i}\hat{B},\; i = 1, \ldots, k$,
(ii) the $q-k-1$ Markov parameters $\hat{C}\hat{A}^{i}\hat{B},\; i = 0, \ldots, q-k-1$,
(iii) the k low–frequency power moments $\hat{C}\hat{A}^{-i}W_c(\hat{A}^T)^{-i}\hat{C}^T,\; i = 1, \ldots, k$, and
(iv) the $q-k-1$ high–frequency power moments $\hat{C}\hat{A}^{i}W_c(\hat{A}^T)^{i}\hat{C}^T,\; i = 1, \ldots, q-k-1$.

Matrix L_r^T can conveniently be determined using the Arnoldi algorithm, which allows to construct an orthonormal basis for the Krylov space $\mathcal{K}(F, X, n) = \text{Im}\,[X, FX, \ldots, F^{n-1}X]$ generated by matrices F and X. In the present context, the columns of L_r^T are determined so as to form an orthonormal basis for the Krylov space $\mathcal{K}(A^T, (CA^{-k})^T, q)$.

The accuracy of the proposed approximation strongly depends on the selection of parameters N, ω_m, ω_M and on the order reduction method used in the second step. It has been proved [12] that for N sufficiently large the frequency responses of $\mathcal{D}^\alpha(s)$, $\mathcal{I}^\alpha(s)$ tend to the ideal ones in the range between ω_m and ω_M. The order reduction method used in the second step of the suggested procedure ensures that the L_2 norm of the difference between left and right hand sides of the input–output high order model is minimal when the original response is replaced by the response of the reduced model, which qualifies the method as an *equation error* method [3]. Moreover, as already observed, this method guarantees that a number of first–order and second–order information indices (e.g., Markov parameters and power moments) are retained by the reduced–order model.

5 Examples

In the following, the advantages of the procedure proposed in Sect. 4 are demonstrated by means of two examples taken from [15, 16, 18]. The step responses as well as frequency responses (Bode plots) of the original non–integer order model and its low order approximation are compared to show the desired accuracy is ensured. The same examples have been considered in [4] where the L_2–optimal model reduction method is used.

Example 1 Consider the system given in the frequency domain by the transfer function (10). Its state–space equations are

$$\begin{bmatrix} D^{1.56}x_1(t) \\ D^{1.13}x_2(t) \\ D^{0.77}x_3(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -4 & -20 & -10 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t), \tag{22}$$

$$y(t) = \begin{bmatrix} 4 & 1 & 0 \end{bmatrix} x(t). \tag{23}$$

Choosing $N = 10$, $\omega_m = 10^{-3}$ and $\omega_M = 10^3$, the procedure outlined in Sect. 3 leads to a 66th order model. Next, this model has been reduced to a 5th order one by means of the procedure outlined in Sect. 4 with $k = 2$, so that 2 time moments and low–

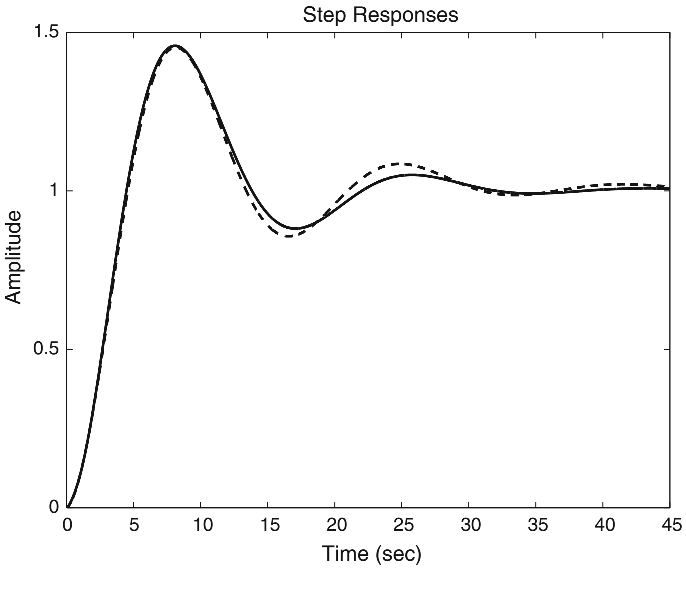

Fig. 1 Step responses for the original model (22)–(23) (*solid line*) and its 5–th order approximation (*dashed line*) with $k = 2$

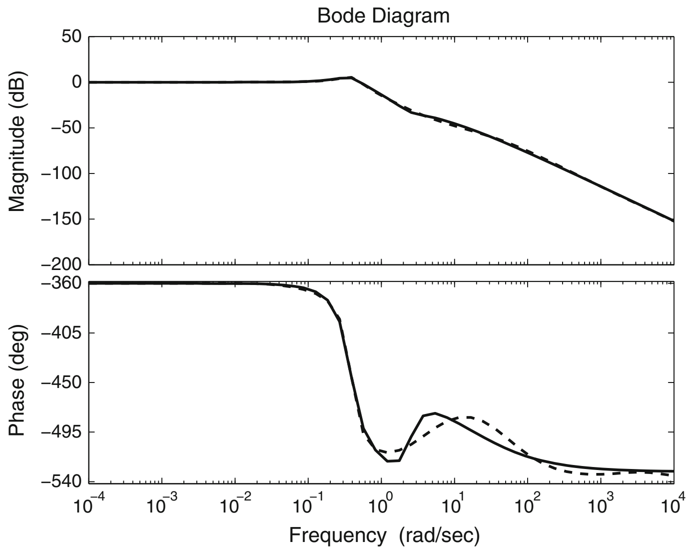

Fig. 2 Comparison of the Bode plots for the original model (22)–(23) (*solid line*) and its 5–th order approximation (*dashed line*) with $k = 2$

frequency power moments as well as three Markov parameters and high–frequency power moments are retained.

The step response of the 5th order model is compared in Fig. 1 with the original step response computed according to the Matlab code described in [1] whereas the Bode plots are compared in Fig. 2. Since these responses practically coincide, the 5th order model can safely be used for controller design. The step response and Bode plots for the high order model are almost exactly equal to those of the original non–integer order model and, therefore, are not shown.

Example 2 The suggested approximation procedure has also been applied to the state–space model:

$$\begin{bmatrix} D^{0.8}x_1(t) \\ D^{0.8}x_2(t) \\ D^{0.8}x_3(t) \\ D^{0.8}x_4(t) \\ D^{0.8}x_5(t) \\ D^{0.8}x_6(t) \end{bmatrix} = \begin{bmatrix} -6 & -6 & -4.4688 & -7.3047 & -6.1719 & -3.4688 \\ 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} x(t)$$

$$+ \begin{bmatrix} 2 \; 0 \; 0 \; 0 \; 0 \; 0 \end{bmatrix}^T u(t), \tag{24}$$

$$y(t) = \begin{bmatrix} 0.5 & 0.5625 & 0.2422 & 0.2266 & 0.1172 & 0.0313 \end{bmatrix} x(t), \tag{25}$$

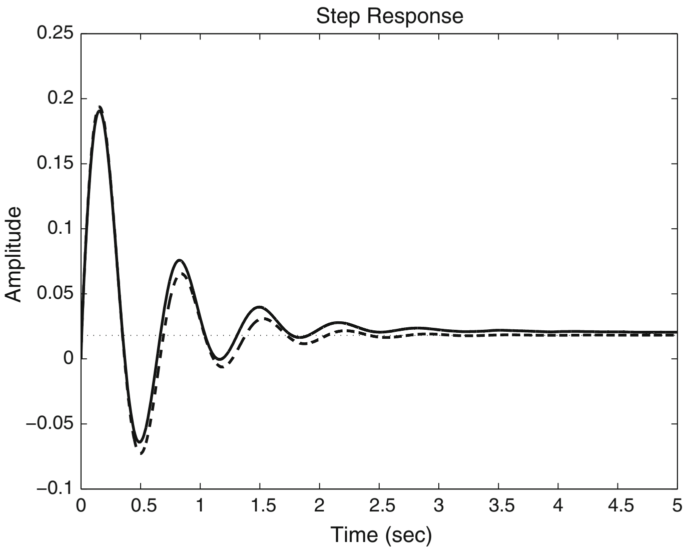

Fig. 3 Step responses for the original model (24)–(25) (*solid line*) and its 7–th order approximation (*dashed line*)

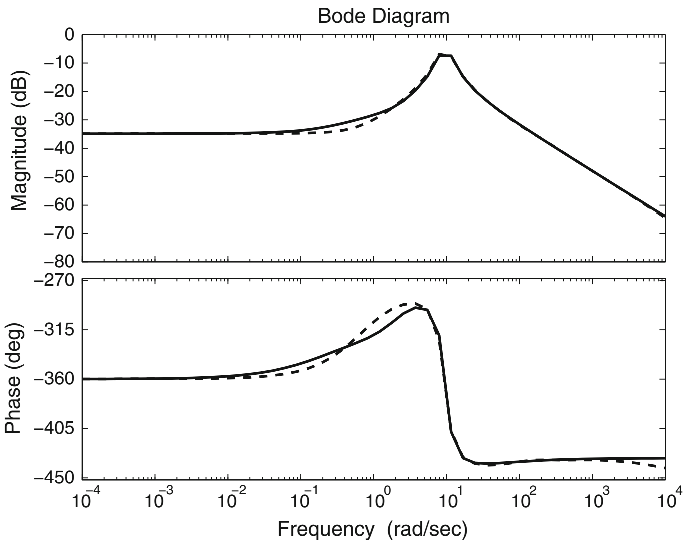

Fig. 4 Comparison of the Bode plots for the original model (24)–(25) (*solid line*) and its 7–th order approximation (*dashed line*)

corresponding to the transfer function (11). Choosing $\omega_m = 10^{-3}$, $\omega_M = 1000$ and $N = 10$ leads to a 132nd order model. Next, this model has been reduced to a 7–th order one by means of the procedure outlined in Sect. 4 with $k = 2$, so that 2 time moments and low–frequency power moments as well as 7 Markov parameters and high–frequency power moments are retained.

The step response of the 7th order model is compared in Fig. 3 with the original step response computed according to the Matlab code described in [1]. The Bode plots are compared in Fig. 4. The responses practically coincide, so that the 7th order model can be used safely for controller design. The step response and Bode plots for the high order model are practically equal to those obtained for the original non–integer order model and are not shown.

6 Conclusions

An efficient and easily implementable procedure to find integer–order models approximating a fractional order system represented in the state-space form has been presented. It consists of two stages. First, a high order model whose state matrix exhibits a sparse block–companion structure is determined. Next, an equation error method is adopted to find a reduced model that retains a number of Markov parameters and time moments as well as some low– and high–frequency power moments of the integer–order model obtained in the first step. Simulations have shown that the procedure leads to approximating models whose responses match well those of the original system.

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Stability Properties of Discrete Time-Domain Oustaloup Approximation

Jerzy Baranowski, Waldemar Bauer and Marta Zagórowska

Abstract The paper presents an analysis of discrete time domain realization of Oustaloup approximation. The scheme for realization is presented along with method of implementation of discretization formulas. Discussion for need for such realization is also presented. Finally the stability analysis is given, considering influences of sampling frequency, order and bandwidth. Analysis is illustrated with behavior of spectral radius of the discretized system.

1 Introduction

Non-integer (fractional) order systems are becoming a staple of modern control theory applications. They are very attractive for possible application allowing design of controllers with high robustness and great design flexibility. Unfortunately there are certain problems with efficient realization of these controllers. Non-integer order controllers cannot be realized directly with use of non-integer differentiators and integrators because of infinite memory requirements. That is why implementation needs efficient approximation. The standard approach is to use method by Oustaloup, which relies on approximating the non-integer order system in the frequency domain for the selected band. It replaces the non-integer order system of order α with a one of high integer order N, with accuracy increasing with rising N. Unfortunately both increasing the order of approximation and frequency band leads to transfer functions that cannot be practically realized in digital environment, because of stability problems [21].

J. Baranowski · W. Bauer (✉) · M. Zagórowska
AGH University of Science and Technology, Department of Automatics
and Biomedical Engineering, Al. Mickiewicza 30, 30-059 Kraków, Poland
e-mail: bauer@agh.edu.pl

J. Baranowski
e-mail: jb@agh.edu.pl

M. Zagórowska
e-mail: zagor@agh.edu.pl

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The goal of this paper is to develop a new method of using the Oustaloup approximation. Our method is based in the time domain in order to avoid the problems with realization, especially to preserve the stability. The results are continuation of author earlier research [2]

General results concerning theory of non-integer order systems can be found in [8, 13, 22]. Oustaloup method was presented in [18] and is analyzed among the others in [17, 19]. This approximation can be efficiently used in simulations [6, 7, 10] and with appropriate care experiments [9]. Its sensitivity and stability problems during discretization were discussed in [21]. Different methods based on definitions of non-integer order also exhibit sensitivity, especially towards quantisation [20]. Different method of approximation is based on Laguerre functions and does not poses this sensitivity [1, 4, 25] however it is much more adequate for filters than for the controllers.

Other method of approximation of non-integer order system in time domain were described in [15, 24] and [23]. In [15, 24] the approximation arising from so called 'distributed' frequency model was considered. In that case a numerical integration of arising improper integral is realized on a logarithmically spaced grid. This results in formulas for Oustaloup approximation. In considered papers transfer function realization is based on simple fractions transformation of product of first order rational transfer functions into a sum. This approach however is very sensitive numerically, significantly impacting the frequency response of the system (however stability is kept). In [23] the Oustaloup transfer function is realized with Frobenius matrix form. This approach does not address the problems with rounding errors of coefficients.

The approach presented in this paper does not exhibit sensitivity to discretization. Moreover it allows approximations of very high order and very large bands without any instability.

The rest of the paper is organized as follows. The classical method of Oustaloup is presented with brief discussion of its properties. Then typical discretization schemes are discussed with justification of their shortcommings. Then the method of time domain realization is presented and its merits are discussed. Two schemes allowing stable discretization are presented. Stability of the approximation is analyzed on an example of a non-integer order differentiator analyzing the influence of sampling period, approximation order and approximation bandwidth. Finally the limits of the method are discussed and conclusions are drawn.

2 Oustaloup Method

Oustaloup filter approximation to a fractional-order differentiator $G(s) = s^{\alpha}$ is widely used in applications [17]. An Oustaloup filter can be designed as

$$G_t(s) = K \prod_{i=1}^{N} \frac{s + \omega_i'}{s + \omega_i} \tag{1}$$

where:

$$\begin{aligned} \omega_i' &= \omega_b \omega_u^{(2i-1-\alpha)/N} \\ \omega_i &= \omega_b \omega_u^{(2i-1+\alpha)/N} \\ K &= \omega_h^{\alpha} \\ \omega_u &= \sqrt{\frac{\omega_h}{\omega_b}} \end{aligned} \tag{2}$$

Approximation is designed for frequencies $\omega \in [\omega_b, \omega_h]$ and N is the order of the approximation. As it can be seen its representation takes form of a product of a series of stable first order linear systems. As one can observe choosing a wide band of approximation results in large ω_u and high order N result in spacing of poles spacing from close to $-\omega_h$ to those very close to $-\omega_b$. This spacing is logarithmic with a grouping near $-\omega_b$ and causes problems in discretization. Wide band of approximation is on other hand desirable, because approximation behaves the best in the interior of the interval and not at its boundary, so certain margins need to be kept.

3 Discretization of Transfer Function

In applications, especially when using Matlab/Simulink environment for real time control two methods of implementing continuous time controllers are established:

1. Use them as continuous blocks in simulink, then use a fixed step explicit solver (for example RK4) to evaluate blocks in real time.
2. Create discrete transfer functions using some discretization scheme—usually Tustin because it should preserve stability.

The problem with first approach, that is using explicit solvers, for Oustaloup filters comes from the fastest pole of the transfer function. If the band of approximation is wide, and includes high frequencies this pole can be located very far to the right. It can be observed [12] that Runge-Kutta type algorithms preserve stability of linear differential equation for such eigenvalues λ and discretization step T that stability function of the algorithm $R(T\lambda)$ is less than one. For classical Runge-Kutta algorithm the stability function is

$$R(z) = 1 + z + \frac{1}{2}z^2 + \frac{1}{6}z^3 + \frac{1}{24}z^4 \tag{3}$$

That means, that for real λ, $T\lambda$ must belong to interval $(-2.785293..., 0)$. In considered case and wide bands the sampling frequency must be relatively close to the upper end of the approximation band, which often cannot be practically realized (usually for PC controlled plants the limit is 1 KHz).

Problems with the seconds approach come from different reasons. Poles of continuous transfer function have a tendency to group near 0 (especially when lower frequency is small). Those poles will be mapped close to 1 during discretisation. In that case, discretizing every pole separately, denominator of of the entire system will include a group of discrete poles with the following form

$$(z - 1 + \varepsilon_i)(z - 1 + \varepsilon_{i+1})(z - 1 + \varepsilon_{i+2}) \dots$$

where $\varepsilon_i > 0$ are distances of the pole from stability boundary, and they are usually very small (orders of magnitude from 10^{-4} to 10^{-9} are not uncommon). In that case final denominator will include numbers that will be products of ε_i with each other, resulting in numbers close to, or below 2.22×10^{-16} which is a smallest number that can be added to another in Matlab (in different computation systems the number is of similar magnitude). That results in automatic rounding error, which with high sensitivity of polynomial roots to coefficient values leads to automatic instability. These rounding errors are unavoidable, even when substituting to symbolic computation results.

4 Time Domain Approximation

The proposed approach is to realize every block of the transfer function (1) in form of a state space system. Those first order systems will be then collected in a single matrix resulting in full matrix realization. This continuous system of differential equations will be then discretized.

4.1 Realization

One can easily observe that for zero initial condition

$$\frac{s + \omega'_k}{s + \omega_K} \Longleftrightarrow \begin{cases} \dot{x}_k = A_k x k + B_k u_k \\ y_k = x_k + u_k \end{cases}$$

where

$$\begin{aligned} A_k &= -\omega_k \\ B_k &= \omega'_k - \omega_k \\ C_i &= 1 \\ D_i &= 1 \end{aligned}$$

This can be written in vector matrix notation

$$
\begin{aligned}
\dot{\mathbf{x}} &= \begin{bmatrix} A_1 & 0 & 0 & \dots & 0 \\ B_2 & A_2 & 0 & \dots & 0 \\ B_3 & B_3 & A_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_N & B_N & \dots & B_N & A_N \end{bmatrix} \mathbf{x} + \begin{bmatrix} KB_1 \\ KB_2 \\ KB_3 \\ \vdots \\ KB_N \end{bmatrix} u \\
y &= \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \end{bmatrix} \mathbf{x} + Ku
\end{aligned}
\tag{4}
$$

or in brief

$$
\begin{aligned}
\dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\
y &= \mathbf{C}\mathbf{x} + Du
\end{aligned}
\tag{5}
$$

What can be immediately observed is that the matrix **A** is lower triangular. This is an extremely important in the case of this problem, as all its eigenvalues (poles of transfer function (1) are on its diagonal, so there is no need for eigenvalue products, which would lead to rounding errors. That is why discretization of (4) should have a structure preserving property.

4.2 Structure Preserving Discretization

Choice of discretization is important, in order to both preserve structure and stability. Explicit methods generally preserve structure (Euler forward, RK4) but they are still susceptible to problems with fast eigenvalues. That is why implicit methods have to be used. For state space discretization the most natural was to use backward Euler and Tustin approximations. Both these methods preserve structure and should preserve stability. Euler is the most stable method of all—it transform left open complex half plane into a subset of unit circle, and Tustin has a very good frequency domain behavior. However while both these methods are very easy to implement in transfer functions through appropriate substitutions of $s \leftarrow \frac{z-1}{Tz}$ and $s \leftarrow \frac{2(z-1)}{T(z+1)}$ respectively their time domain application is more complicated. Essentially these methods do not preserve natural state variables but introduce new ones that include both original state and input signal [16]. Fortunately, the input output mapping is unchanged, which is satisfactory for our purposes.

Discretized system takes form (for simplicity e.g. $u(Tk)$ is written as $u(k)$)

$$
\begin{aligned}
\mathbf{w}(k+1) &= \mathbf{\Phi}\mathbf{w}(k) + \mathbf{\Gamma}u(k) \\
y(k) &= \mathbf{H}\mathbf{w}(k) + Ju(k)
\end{aligned}
\tag{6}
$$

where for Euler approximation

$$
\begin{aligned}
\mathbf{w} &= (\mathbf{I} - \mathbf{A}T)\mathbf{x} - T\mathbf{B}u \\
\mathbf{\Phi} &= (\mathbf{I} - \mathbf{A}T)^{-1} \\
\mathbf{\Gamma} &= (\mathbf{I} - \mathbf{A}T)^{-1}\mathbf{B}T \\
\mathbf{H} &= \mathbf{C}(\mathbf{I} - \mathbf{A}T)^{-1} \\
J &= D + \mathbf{C}(\mathbf{I} - \mathbf{A}T)^{-1}\mathbf{B}T
\end{aligned}
$$

and for Tustin approximation

$$
\begin{aligned}
\sqrt{T}\mathbf{w} &= (\mathbf{I} - \mathbf{A}\frac{T}{2})\mathbf{x} - \frac{T}{2}\mathbf{B}u \\
\mathbf{\Phi} &= (\mathbf{I} + \mathbf{A}\frac{T}{2})(\mathbf{I} - \mathbf{A}\frac{T}{2})^{-1} \\
\mathbf{\Gamma} &= (\mathbf{I} - \mathbf{A}\frac{T}{2})^{-1}\mathbf{B}\sqrt{T} \\
\mathbf{H} &= \sqrt{T}\mathbf{C}(\mathbf{I} - \mathbf{A}\frac{T}{2})^{-1} \\
J &= D + \mathbf{C}(\mathbf{I} - \mathbf{A}\frac{T}{2})^{-1}\mathbf{B}\frac{T}{2}
\end{aligned}
$$

As it can be observed in both methods matrix $\mathbf{\Phi}$ consists of either inverse of lower triangular matrix which is lower triangular itself or product of two lower triangular matrices, which also preserves structure. Because of that diagonal consists of discrete system eigenvalues which are discretized individually, so the field for rounding errors is severely reduced. Also because the method is implicit, the fastest eigenvalue is mapped well into unit circle.

It should be noted, that Tustin method is implemented in Matlab `c2d` function, however this implementation is not optimized for detecting matrix structure and resulting matrix is not fully triangular (there are nonzero elements with order of magnitude 10^{-12} above diagonal). Implementation using efficient matrix inversion algorithms with structure detection solved the problem.

4.3 Method of Analysis

In order to quantize the stability properties of considered approximations two quantities will be considered:

- For systems where loss of stability is expected it will be a spectral radius that is $\rho = \max |z_i|$, where z_i are poles of approximating system.
- For introduced time domain based approximation loss of stability was not observed - that is why a *stability margin* is introduced. It is defined as $\eta = 1 - \rho$.

4.4 Numerical Experiments

Method was tested for multiple non-integer orders, orders of approximation, sampling periods and bandwidths. Here because of space reasons only analysis of a single non integer order case will be presented. The considered system is a differentiator of order $\alpha = 0.7$. The following tests were conducted:

1. For bandwidth $[10^{-6}, 10^{6}]$ and approximation order of 6 the sampling frequency was being increased with logarithmic spacing from 100 Hz to 100 kHz.
2. For bandwidth $[10^{-6}, 10^{6}]$ and sampling frequency of 1 kHz the approximation order was increased from 1 to 20.
3. For sampling frequency of 1 kHz and approximation order of 6 the bandwidth was being increased. The compared approximation had symmetric bandwidth centered at 1 Hz and it was increased by decade per experiment.

Results of tests are presented in Figs. 1, 2 and 3 respectively. In all of the figures for time domain approximations vertical scale is logarithmic. the reason for such presentation, is that the differences between points are of small magnitude but varying orders. They would be indistinguishable in linear scale. In the Fig. 1 illustrates influence of sampling frequency on the stability. Frequency is presented logarithmically on the x-axis. In the Fig. 1a one can observe very interesting effect of irregular changing of spectral radius with sampling frequency. It can be connected with the highly nonlinear effect of rounding errors on polynomial root (and in consequence pole) locations. Because of structural properties of proposed time domain realization the influence of sampling frequency on stability is much more regular. As it can be observed in the Fig. 1b the stability margin reduces with increased frequency, and this reduction is linear on logarithmic scale.

In the Fig. 2 one can observe that increasing order also influences stability, however rounding errors in the frequency domain approach lead to quick destabilization, the time domain approach asymptotically tends to the stability boundary. Similar

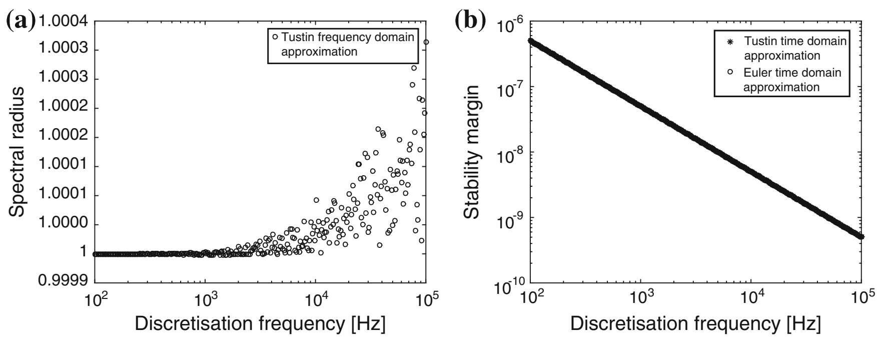

Fig. 1 Comparison of approximations for different sampling frequencies. **a** Frequency domain approximation **b** Time domain approximations

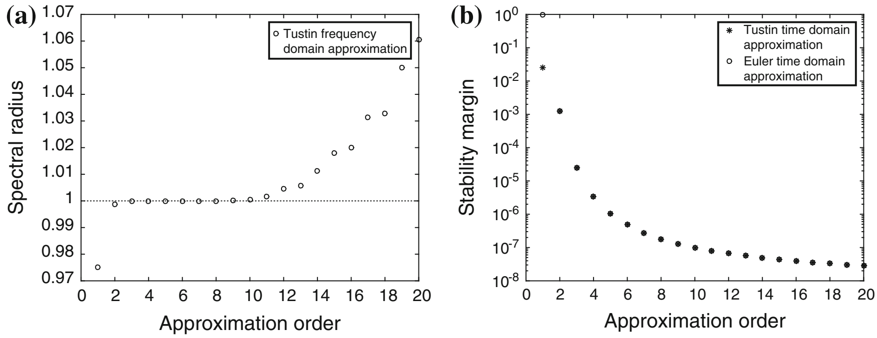

Fig. 2 Comparison of approximations for different orders of approximation. **a** Frequency domain approximation **b** Time domain approximations

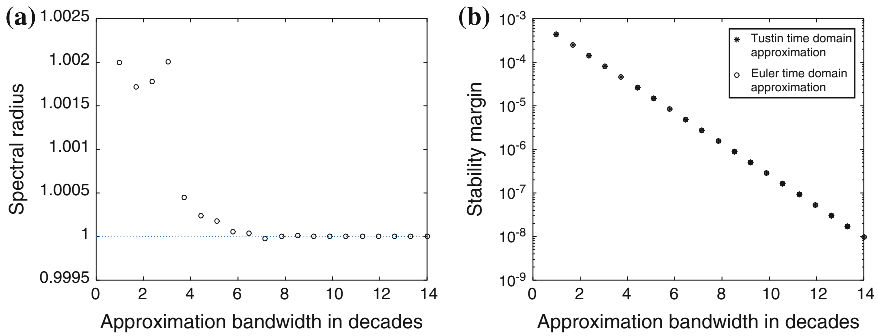

Fig. 3 Comparison of approximations for different bandwidths. **a** Frequency domain approximation **b** Time domain approximations

tendency is observed in Fig. 3. Increasing the bandwidth leads to problems with stability for the transfer function, but proposed state space realization is immune.

4.5 Method Limitations

Certain main limitations of the method has been discovered. One of them is the number of coefficients needed for implementation. In case of transfer function it is $O(N)$ and time domain method requires $O(N^2)$. It can be troublesome for very high orders of approximations. Also the implementation requires matrix multiplication or multiple difference equations, so number of operations is also increased. Second area of limitations is present in frequency characteristics near the Nyqvist frequency, it is especially visible for Euler discretization. We think that these limitations can be worked around in order to create efficient controllers.

5 Conclusion

Presented method is very promising for future applications. It was already tested in real-time control environment and behaved correctly. Method was also tested for very high orders, like $N = 20$ and still resulted in stable and well behaving discrete approximation. What was not presented here, directly discretized transfer function for lower orders can keep stability, but frequency response is not consistent with continuous one.

There is a broad field for improvement of the method, as the proposed realization is basic. There are results in realization theory that at least require analysis. Especially in the context of possible implementation in embedded systems. Further experiments in real time control environment are necessary. Also comparison with other robustification method developed by the authors are in order—namely 'frequency separation method'. Also discretization schemes are worth revisiting in order to improve behavior of the method. Natural directions are Al-Alaoui method and implicit Runge-Kutta methods.

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Digraphs Minimal Positive Stable Realisations for Fractional One-Dimensional Systems

Konrad Andrzej Markowski and Krzysztof Hryniów

Abstract This paper presents a method of the determination of positive stable realisation of the fractional continuous-time positive system. The algorithm finds a complete set of all possible realisations instead of only a few realisations. In addition, all realisations in the set are minimal and stable. The proposed method uses a parallel computing algorithm based on a digraphs theory which is used to gain much needed speed and computational power for a numeric solution. The presented procedure has been illustrated with a numerical example.

Keywords Stable realisation · Minimal realisation · Fractional system · Positive · Digraphs · Algorithm

1 Introduction

In the recent years many researchers have been interested in positive linear systems [3, 6, 15, 16, 20]. In positive systems inputs, state variables and outputs take only non-negative values [4]. Positive linear systems are defined on cones and not on linear spaces, therefore the theory of positive systems is more complicated than standard systems [2, 3, 6, 15, 17, 18]. The realisation problem is a very difficult task. In many research studies, we can find a canonical form of the system, i.e. constant matrix form, which satisfies the system described by the transfer function. With the use of this form we are able to write only one realisation of the system, while there

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K.A. Markowski (✉) · K. Hryniów
Warsaw University of Technology, Faculty of Electrical Engineering,
Institute of Control and Industrial Electronics, Koszykowa 75, 00-662 Warsaw, Poland
e-mail: Konrad.Markowski@ee.pw.edu.pl
url: http://nas.isep.pw.edu.pl/repo

K. Hryniów
e-mail: Krzysztof.Hryniow@ee.pw.edu.pl

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exists a set of solutions. The state of the art in positive systems theory is given in the monographs [6, 16, 20], while in [10–12] a solution for finding a set of possible realisations of the characteristic polynomial was proposed, that allows for finding many sets of matrices which fit into a system transfer function.

The first definition of the fractional derivative was introduced by Liouville and Riemann at the end of the 19th century [22]. Mathematical fundamentals of fractional calculus are given in the monographs [5, 19, 21–24].

In this paper, a new method of determination of positive minimal realisation for the fractional continuous one-dimensional system will be proposed and the procedure for computation of the minimal realisation will be given. The procedure will be illustrated with a numerical example.

This work has been organised as follows: Sect. 2 presents some notations and basic definitions of digraph theory. Section 3 presents basics of fractional order systems theory. In this section is defined fractional continuous-time system as the state-space representation. In Sect. 4, we construct algorithm for the determination of positive minimal realisation of the fractional continuous-time system. Finally we demonstrate the workings of the algorithm on two numerical examples in Sect. 5 and at the end we present some concluding remarks, open problems and bibliography positions.

2 Notion and Digraph Basic Definitions

2.1 Notion

In this paper the following notion will be used. The matrices will be denoted by the bold font (for example **A**, **B**, ...), the sets by the double line (for example $\mathbb{A}$, $\mathbb{B}$, ...), lower/upper indices and polynomial coefficients will be written as a small font (for example a, b, ...) and digraph will bee denoted using a mathfrak font $\mathfrak{D}$.

The set $n \times m$ of real matrices will be denoted by $\mathbb{R}^{n\times m}$ and $\mathbb{R}^n = \mathbb{R}^{n\times 1}$. If $\mathbf{G} = [g_{ij}]$ is a matrix, we write $\mathbf{G} \gg 0$ (matrix **G** is called strictly positive), if $g_{ij} > 0$ for all i, j; $\mathbf{G} > 0$ (matrix **G** is called positive), if $g_{ij} > 0$ for all i, j; $\mathbf{G} \geqslant 0$ (matrix **G** is called non-negative), if $g_{ij} \geqslant 0$ for all i, j. The set of $n \times m$ real matrices with non-negative entries will be denoted by $\mathbb{R}_+^{n\times m}$ and $\mathbb{R}_+^n = \mathbb{R}_+^{n\times 1}$. The $n \times n$ identity matrix will be denoted by $\mathbf{I}_n$. The set $n \times n$ of Metzler matrix will be denoted by $\mathbf{M}^{n\times n}$. For more information about the matrix theory, an interested reader may be referred to, for instance: [4, 9].

2.2 Digraph

A directed graph (or just digraph) $\mathfrak{D}$ consists of a non-empty finite set $\mathbb{V}(\mathfrak{D})$ of elements called vertices and a finite set $\mathbb{A}(\mathfrak{D})$ of ordered pairs of distinct vertices

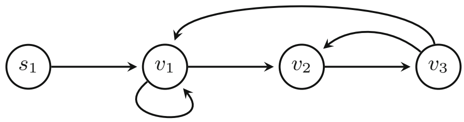

Fig. 1 One-dimensional digraphs

called arcs [1]. We call $\mathbb{V}(\mathfrak{D})$ the vertex set and $\mathbb{A}(\mathfrak{D})$ the arc set of digraph $\mathfrak{D}$. We will often write $\mathfrak{D} = (\mathbb{V}, \mathbb{A})$ which means that $\mathbb{V}$ and $\mathbb{A}$ are the vertex set and arc set of $\mathfrak{D}$, respectively. The order of $\mathfrak{D}$ is the number of vertices in $\mathfrak{D}$. The size of $\mathfrak{D}$ is the number of arcs in $\mathfrak{D}$. For an arc (v_1, v_2) the first vertex v_1 is its tail and the second vertex v_2 is its head.

There exists $\mathfrak{A}$-arc from vertex v_j to vertex v_i if and only if the (i, j)th entry of the matrix $\mathbf{A}$ is nonzero. There exists $\mathfrak{B}$-arc from source s to vertex v_j if and only if the lth entry of the matrix $\mathbf{B}$ is nonzero.

Example 1 Let be given the positive system single input described by the following matrices

$$\underbrace{\begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{\mathbf{A}}, \underbrace{\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}}_{\mathbf{B}} \tag{1}$$

we can draw one-dimensional digraph $\mathfrak{D}^{(1)}$ consisting of vertices v_1, v_2, v_3, v_4 and source s_1. One-dimensional digraph corresponding to system (1) is presented in Fig. 1.

We present below some basic notions from graph theory which are used in further considerations. A walk in a digraph $\mathfrak{D}^{(1)}$ is a finite sequence of arcs in which every two vertices v_i and v_j are adjacent or identical. A walk in which all of the arcs are distinct is called a path. The path that goes through all vertices is called a finite path. If the initial and the terminal vertices of the path are the same, then the path is called a cycle.

More information about use digraph theory in positive system is given in [7, 8, 12, 13].

3 Fractional Order System

3.1 Model and Representation

The equation for a continuous-time dynamic system of the fractional-order can be written as follows:

$$H(\mathfrak{D}^{\alpha_0\alpha_1\cdots\alpha_m})(y_1, y_2, \ldots, y_l) = G(\mathfrak{D}^{\beta_0\beta_1\cdots\beta_n})(u_1, u_2, \ldots, y_k) \quad (2)$$

where y_i, u_i are the functions of time and $H(\cdot)$, $G(\cdot)$ are the combinations of the fractional-order derivative operator. If we describe the linear time-invariant single-variable case we obtain the following equation:

$$H(\mathfrak{D}^{\alpha_0\alpha_1\cdots\alpha_m})(y_t) = G(\mathfrak{D}^{\beta_0\beta_1\cdots\beta_n})(u_t) \quad (3)$$

with

$$H(\mathfrak{D}^{\alpha_0\alpha_1\cdots\alpha_n}) = \sum_{k=0}^{n} a_k \mathfrak{D}^{\alpha_k}, \quad a_k \in \mathbb{R} \quad (4)$$
$$G(\mathfrak{D}^{\beta_0\beta_1\cdots\beta_m}) = \sum_{k=0}^{m} b_k \mathfrak{D}^{\beta_k}, \quad b_k \in \mathbb{R}.$$

or

$$a_n \mathfrak{D}^{\alpha_n} y(t) + a_{n-1}\mathfrak{D}^{\alpha_{n-1}} y(t) + \cdots + a_0 \mathfrak{D}^{\alpha_0} y(t) = \quad (5)$$
$$= b_m \mathfrak{D}^{\beta_m} u(t) + b_{m-1}\mathfrak{D}^{\beta_{m-1}} u(t) + \cdots + b_0 \mathfrak{D}^{\beta_0} u(t)$$

Applying the Laplace transform to (5) with zero initial conditions, the input-output representation of fractional-order system can be obtained. The fractional-order system as the transfer function have the following form:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{b_m s^{\beta_m} + b_{m-1}s^{\beta_{m-1}} + \cdots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1}s^{\alpha_{n-1}} + \cdots + a_0 s^{\alpha_0}}. \quad (6)$$

In the case of a commensurate-order system, the continuous-time transfer function (6) is given in the following form

$$G(s) = \frac{\sum_{k=0}^{m} b_k (s^\alpha)^k}{\sum_{k=0}^{n} a_k (s^\alpha)^k}. \quad (7)$$

The transfer function (7) can be considered as a pseudo-rational function $H(\lambda)$ of the variable $\lambda = s^\alpha$ in the form:

$$H(\lambda) = \frac{\sum_{k=0}^{m} b_k \lambda^k}{\sum_{k=0}^{n} a_k \lambda^k} = \frac{b_0 + b_1\lambda + b_2\lambda^2 + \cdots + b_{k-1}\lambda^{k-1} + b^k\lambda^k}{a_0 + a_1\lambda + a_2\lambda^2 + \cdots + a_{k-1}\lambda^{k-1} + a^k\lambda^k}. \quad (8)$$

3.2 State-Space Representation

Let us consider the continuous-time fractional linear system described by state-space equations:

$$
\begin{aligned}
{}_0\mathfrak{D}_t^{\alpha} x(t) &= \mathbf{A}x(t) + \mathbf{B}u(t), \quad 0 < \alpha \leqslant 1 \\
y(t) &= \mathbf{C}x(t) + \mathbf{D}u(t)
\end{aligned}
\tag{9}
$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$, $y(t) \in \mathbb{R}^p$ are the state, input and output vectors respectively and $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$, $\mathbf{C} \in \mathbb{R}^{p\times n}$ and $\mathbf{D} \in \mathbb{R}^{p\times m}$. The following Caputo definition of the fractional derivative will be used:

$$
{}_a^C\mathfrak{D}_t^{\alpha} = \frac{d^{\alpha}}{dt^{\alpha}} = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau \tag{10}
$$

where $\alpha \in \mathbb{R}$ is the order of fractional derivative, $f^{(n)}(\tau) = \frac{d^n f(\tau)}{d\tau^n}$ and $\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt$ is the gamma function.

Theorem 1 *The Laplace transform of the derivative-integral* (10) *has the form*

$$
\mathcal{L}\left[{}_0^C\mathfrak{D}_t^{\alpha}\right] = s^{\alpha}F(s) - \sum_{k=1}^{n} s^{\alpha-k} f^{(k-1)}(0^+) \tag{11}
$$

The proof of the Theorem 1 is given in [19].

Definition 1 The fractional system (9) is called the internally positive fractional system if and only if $x(t) \in \mathbb{R}_+^n$ and $y(t) \in \mathbb{R}_+^p$ for $t \geqslant 0$ for any initial conditions $x_0 \in \mathbb{R}_+^n$ and all inputs $u(t) \in \mathbb{R}_+^m$ for $t \geqslant 0$.

Definition 2 A square real matrix $\mathbf{A} = \left[a_{ij}\right]$ is called the Metzler matrix if its off-diagonal entries are non-negative, i.e. $a_{ij} \geqslant 0$ for $i \neq j$.

Definition 3 The fractional system (9) is positive if and only if

$$
\mathbf{A} \in \mathbb{M}^{n\times n},\ \mathbf{B} \in \mathbb{R}_+^{n\times m},\ \mathbf{C} \in \mathbb{R}_+^{p\times n},\ \mathbf{D} \in \mathbb{R}_+^{p\times m} \tag{12}
$$

In [19] the basic definitions of the positive realisation problem has been presented in detail.

Using the Laplace transform to (9), Theorem 1 and taking into account

$$
X(s) = \mathcal{L}\left[x(t)\right] = \int_0^{\infty} x(t)e^{-st} dt, \tag{13a}
$$

$$
\mathcal{L}\left[\mathfrak{D}^{\alpha}x(t)\right] = s^{\alpha}X(s) - s^{\alpha-1}x_0 \tag{13b}
$$

we obtain:

$$\begin{aligned} X(s) &= \left[\mathbf{I}_n s^\alpha - \mathbf{A}\right]^{-1}\left[s^{\alpha-1}x_0 + \mathbf{B}U(s)\right] \\ Y(s) &= \mathbf{C}X(s) + \mathbf{D}U(s) \\ U(s) &= \mathcal{L}\left[u(t)\right]. \end{aligned} \tag{14}$$

Using (14) we can determine transfer matrix of the system in the following form:

$$\mathbf{T}(s) = \mathbf{C}\left[\mathbf{I}_n s^\alpha - \mathbf{A}\right]^{-1}\mathbf{B} + \mathbf{D}. \tag{15}$$

We can present the transfer matrix (15) in the form of (8). In this case the transfer matrix is the function of the operator $\lambda = s^\alpha$ and for multi-input multi-output system the transfer function has the following form:

$$\mathbf{T}(\lambda) = \mathbf{C}\left[\mathbf{I}_n\lambda - \mathbf{A}\right]^{-1}\mathbf{B} + \mathbf{D} = \begin{bmatrix} \mathbf{T}_{11}(\lambda) & \dots & \mathbf{T}_{1,m}(\lambda) \\ \vdots & \ddots & \vdots \\ \mathbf{T}_{p,1}(\lambda) & \dots & \mathbf{T}_{p,m}(\lambda) \end{bmatrix} \tag{16}$$

where:

$$\mathbf{T}_{i,j}(\lambda) = \frac{n_{i,j}(\lambda)}{d_{i,j}(\lambda)} = \frac{b_0^{i,j} + b_1^{i,j}\lambda + \cdots + b_{n-1}^{i,j}\lambda^{n-1} + b_n^{i,j}\lambda^n}{a_0^{i,j} + a_1^{i,j}\lambda + \cdots + a_{n-1}^{i,j}\lambda^{n-1} + \lambda^n} \tag{17}$$

The matrix **D** can be found by the use of the formula

$$\lim_{\lambda\to\infty}\mathbf{T}(\lambda) = \mathbf{D} \in \mathbb{R}^{p\times m} = \begin{bmatrix} b_0^{11}/a_0^{11} & \dots & b_0^{1,m}/a_0^{1,m} \\ \vdots & \ddots & \vdots \\ b_0^{p,1}/a_0^{p,1} & \dots & b_0^{p,m}/a_0^{p,m} \end{bmatrix}. \tag{18}$$

Using (18) and transfer matrix (16) we can determine strictly proper transfer matrix in the following form:

$$\mathbf{T}_{sp}(\lambda) = \mathbf{T}(\lambda) - \mathbf{D} = \begin{bmatrix} \widetilde{\mathbf{T}}_{11}(\lambda) & \dots & \widetilde{\mathbf{T}}_{1,m}(\lambda) \\ \vdots & \ddots & \vdots \\ \widetilde{\mathbf{T}}_{p,1}(\lambda) & \dots & \widetilde{\mathbf{T}}_{p,m}(\lambda) \end{bmatrix} \tag{19}$$

where:

$$\widetilde{\mathbf{T}}_{i,j}(\lambda) = \frac{\widetilde{n}_{i,j}(\lambda)}{d_{i,j}(\lambda)} = \frac{\widetilde{b}_1^{i,j}\lambda + \cdots + \widetilde{b}_{n-1}^{i,j}\lambda^{n-1} + \widetilde{b}_n^{i,j}\lambda^n}{a_0^{i,j} + a_1^{i,j}\lambda + \cdots + a_{n-1}^{i,j}\lambda^{n-1} + \lambda^n} \tag{20}$$

Matrices (12) are called positive realisation of the transfer matrix if they satisfy the equality (16). The realisation is called minimal if the dimension of the state matrix **A** is minimal among all possible realisation of $T(\lambda)$.

3.3 Problem Formulation

For given transfer matrix (15) determine minimal stable positive realisation of the system (9) using one-dimensional $\mathfrak{D}^{(1)}$ digraph theory. The dimension of the system must be the minimal among possible.

4 Problem Solution

By multiplying nominator and denominator of (19) by λ^{-n} we obtain:

$$\mathbf{T}_{sp}(\lambda^{-1}) = \mathbf{C}\left[\mathbf{I}_n - \mathbf{A}\lambda^{-1}\right]^{-1}\mathbf{B} = \begin{bmatrix} \widetilde{\mathbf{T}}_{11}(\lambda^{-1}) & \dots & \widetilde{\mathbf{T}}_{1,m}(\lambda^{-1}) \\ \vdots & \ddots & \vdots \\ \widetilde{\mathbf{T}}_{p,1}(\lambda^{-1}) & \dots & \widetilde{\mathbf{T}}_{p,m}(\lambda^{-1}) \end{bmatrix} \tag{21}$$

where:

$$\widetilde{\mathbf{T}}_{i,j}(\lambda^{-1}) = \frac{\widetilde{n}_{i,j}(\lambda^{-1})}{d_{i,j}(\lambda^{-1})} = \frac{\widetilde{b}_1^{i,j}\lambda^{1-n} + \cdots + \widetilde{b}_{n-1}^{i,j}\lambda^{-1} + \widetilde{b}_n^{i,j}}{a_0^{i,j} + a_1^{i,j}\lambda + \cdots + a_{n-1}^{i,j}\lambda^{n-1} + \lambda^n}. \tag{22}$$

In the first step the proposed method finds state matrix **A** using decomposition of the characteristic polynomial $d_{i,j}(\lambda^{-1})$ into a set of simple monomials. For each simple monomial we create a digraph representation. Then we can determine all possible characteristic polynomial realisations using all combinations of the digraph monomial representations.

Theorem 2 *There exists positive state matrix* **A** *of the fractional continuous-time linear system* (9) *corresponding to the characteristic polynomial* $d(\lambda^{-1})$ *if:*

- ***(C1)*** *the* $\mathbb{D}_1$ *and* $\mathbb{D}_2$ *sets corresponding to two one-dimensional digraphs representing the monomial are not disjoint;*
- ***(C2)*** *the obtained digraph does not have additional cycles;*
- ***(C3)*** *the poles of the characteristic polynomial are distinct real and negative.*

Proof Condition (C1): The sets $\mathbb{D}_1$ and $\mathbb{D}_2$ are disjoint if $\mathbb{D}_1 \cap \mathbb{D}_2 = \emptyset$. Then we have a digraph whose vertices can be divided into two disjoint sets (bipartite digraph). It means that we obtain an additional simple monomial in a characteristic polynomial $d(\lambda^{-1})$. In this situation, we obtain a new polynomial. *Condition (C2):* Each monomial is represented by one cycle. If after combining all digraphs (each corresponding to one monomial) we obtain an additional cycle, this means that in the polynomial additional simple monomial appears. *Condition (C3):* If the state matrix $\mathbf{A}$ have the following structure $\mathbf{A} = diag\left[\mathbf{I}_{n_1}\lambda_1^{-1}\ \mathbf{I}_{n_2}\lambda_2^{-1}\ \dots\ \mathbf{I}_{n_n}\lambda_n^{-1}\right]$ then $\lambda_1^{-1}, \lambda_2^{-1}, \dots, \lambda_n^{-1}$ are real negative and the matrix $\mathbf{A}$ is stable and is a Metzler matrix. □

The Algorithm 1 consist from three parts: the first (lines 2–6), in which we construct all possible digraph representations for every monomial from a characteristic polynomial; the second (lines 7–19), in which we create all possible digraph structures using all combinations of the monomial digraph representations; the third part (lines 20–31), in which we determine minimal realisation of the system and check the positivity.

Let assume thet the matrix $\mathbf{B}$ and matrix $\mathbf{C}$ have the following form:

$$\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} & \dots & b_{1,m} \\ b_{21} & b_{22} & \dots & b_{2,m} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n,1} & b_{n,2} & \dots & b_{n,m} \end{bmatrix} \in \mathbb{R}_+^{n\times 1}, \quad \mathbf{C} = \begin{bmatrix} c_{11} & c_{12} & \dots & c_{1,n} \\ c_{21} & c_{22} & \dots & c_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ c_{p,1} & c_{p,2} & \dots & c_{p,n} \end{bmatrix} \in \mathbb{R}_+^{p\times n}. \tag{23}$$

After determining state matrix $\mathbf{A}$ from the Algorithm 1 and inserting matrices (23) to the equation $\mathbf{T}(\lambda^{-1}) = \mathbf{C}\left[\mathbf{I}_n - \mathbf{A}\lambda^{-1}\right]^{-1}\mathbf{B} + \mathbf{D}$ we obtain the polynomial matrix $\widetilde{\mathbf{N}}(\lambda^{-1})$. Comparing variables with the same power of λ polynomials $\widetilde{n}_{i,j}(\lambda^{-1}) = n_{i,j}(\lambda^{-1})$ we receive the set of equations. After solving the equation we obtain matrices (12).

4.1 Complexity

For polynomials consisting only of 1D monomials $(m-1)$ operations are needed for the first part of the algorithm (creating monomial digraphs) making in solvable in linear time and creating polynomial digraphs is executed on

$$\prod_{1}^{n} \frac{1}{n}\left(\frac{(m-1)!}{x_1!x_2!\dots x_c!} * \prod_{i=0}^{m}(V-i)\right) \tag{24}$$

kernels and takes

$$V(n(m+V)) + V^2 + m\log m + V\log V^n \tag{25}$$

operations on each of them. Overall, the algorithms has linearithmic complexity that can be denoted as $\mathbf{T}(\mathbf{V}) = \mathbf{O}(V\log V^n)$ in big $\mathbf{O}$ notation. Analysis of algorithms complexity in detail is presented in [13] and [14].

Algorithm 1 *DetermineMinimalRealisation*()

```
monomial = 1;
Determine number of cycles in characteristic polynomial;
for monomial = 1 to cycles do
    Determine digraph 𝔇^(1) for all monomial;
    MonomialRealisation(monomial);
end for
for monomial = 1 to cycles do
    Determine digraph as a combination of the digraph monomial representation
    PolynomialRealisation(monomial);
    if PolynomialRealisation ! = cycles then
        Digraph contains additional cycles or digraph contains disjoint union
        BREAK
    else if PolynomialRealisation == cycles then
        Digraph satisfies characteristic polynomial;
        Determine weights of the arcs in digraph;
        Write state matrix A;
        return (PolynomialRealisation, A);
    end if
end for
for PolynomialRealisation = 1 to j do
    Input - state matrix A;
    Determine polynomial ñ(λ^-1);
    Compare variables with the same power of the n(λ^-1);
    Solve non-linear set of the equations;
    if Matrix B ⩾ 0 AND matrix C ⩾ 0 then
        return (MinimalRealisation, A, B, C);
    else if Matrix B < 0 OR matrix C < 0 then
        Realisation is not positive;
        BREAK
    end if
end for
```

5 Numerical Examples

5.1 Example I

Find a positive minimal realisation of the strictly proper transfer function

$$T_{sp}(s) = \frac{3.138s^{1.2} + 24.86s^{0.6} + 20.33}{s^{1.8} + 11.5s^{1.2} + 15.5s^{0.6} + 5}. \quad (26)$$

The transfer function can be considered as a pseudo-rational function $T(\lambda)$ of the variable $\lambda = s^{0.6}$ in the form:

$$T_{sp}(\lambda) = \frac{3.138\lambda^2 + 24.86\lambda + 20.33}{\lambda^3 + 11.5\lambda^2 + 15.5\lambda + 5}. \quad (27)$$

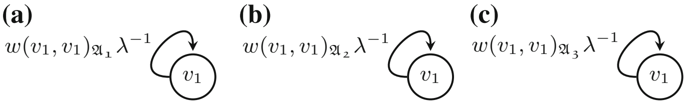

Fig. 2 One-dimensional digraphs corresponding to characteristic polynomials (29)

Multiplying the nominator and denominator of the transfer function (27) by λ^{-3} we obtain:

$$T_{sp}(\lambda^{-1}) = \frac{3.138\lambda^{-1} + 24.86\lambda^{-2} + 20.33\lambda^{-3}}{1 + 11.5\lambda^{-1} + 15.5\lambda^{-2} + 5\lambda^{-3}} = \frac{n(\lambda^{-1})}{d(\lambda^{-1})} \tag{28}$$

where

$$d(\lambda^{-1}) = (1 + 0.5\lambda^{-1})(1 + 10\lambda^{-1})(1 + \lambda^{-1}) \tag{29}$$

is the characteristic polynomial. The characteristic polynomial can be presented in the following form $d(\lambda^{-1}) = d_1(\lambda^{-1})d_2(\lambda^{-1})d_3(\lambda^{-1})$. In this case, our task can be divided into three subtasks.

Consider the polynomial $d_1(\lambda^{-1}) = 1 + 0.5\lambda^{-1}$. In the first step we write the following initial conditions: number of colours in digraphs: *colour* = 1; monomial $M_1 = -0.5\lambda^{-1}$.

In this example we have only one realisation of the characteristic polynomial $d_1(\lambda^{-1})$ presented in Fig. 2a. The condition (C1) and (C2) of the Theorem 2 are satisfied. Finally we must verify the third condition. In considered polynomial $d_1(\lambda^{-1})$ the pole equal to $\lambda_1 = -0.5$ is real and negative. Described realisation satisfies Condition (C3). **The realisation does satisfy all conditions and is correct**. In the same way we determine realisations of the characteristic polynomial $d_2(\lambda^{-1})$ presented in Fig. 2b and $d_3(\lambda^{-1})$ presented in Fig. 2b.

From the obtained digraphs, we can write state matrix **A** in the form:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_1 & 0 & 0 \\ 0 & \mathbf{A}_2 & 0 \\ 0 & 0 & \mathbf{A}_3 \end{bmatrix} = \begin{bmatrix} w(v_1, v_1)_{\mathfrak{A}_1} & 0 & 0 \\ 0 & w(v_1, v_1)_{\mathfrak{A}_2} & 0 \\ 0 & 0 & w(v_1, v_1)_{\mathfrak{A}_3} \end{bmatrix} = \tag{30}$$
$$= \begin{bmatrix} -0.5 & 0 & 0 \\ 0 & -10 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

Inserting matrices (23) and (30) to the equation (19) we obtain the polynomial $\widetilde{n}(\lambda^{-1})$. After comparison of the coefficients of the same power λ polynomials $\widetilde{n}(\lambda^{-1}) = n(\lambda^{-1})$ we receive the set of the equations. Solving them, we obtain the following matrices:

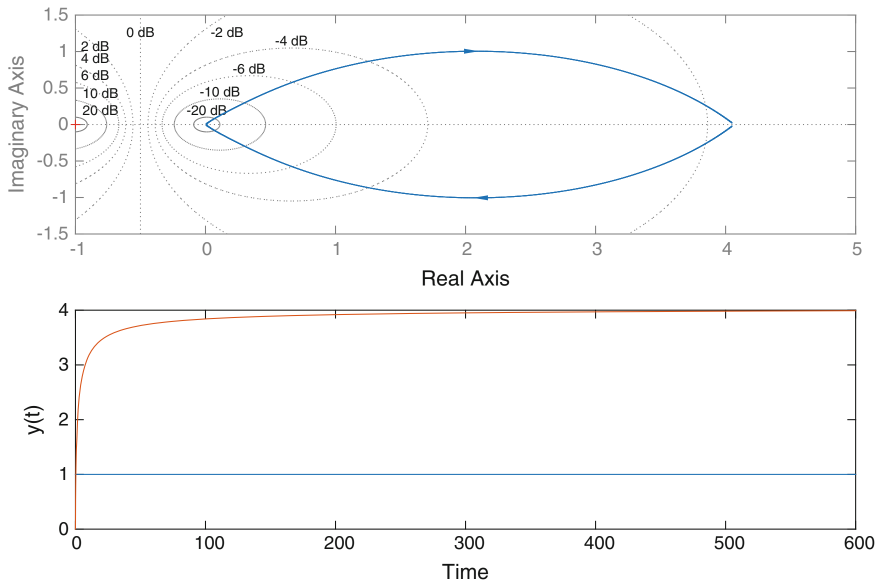

Fig. 3 Step response and Nyquist characteristic of the system (26)

$$\mathbf{B} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} 1.8283 \\ 1.0004 \\ 0.3093 \end{bmatrix}, \qquad \mathbf{C} = \begin{bmatrix} c_1 \; c_2 \; c_3 \end{bmatrix} = \begin{bmatrix} 1 \; 1 \; 1 . \end{bmatrix} \tag{31}$$

The desired positive realisation of the (26) is given by (30) and (31). The obtained realisation is stable, as can be seen on Fig. 3, showing the step response of the system and the Nyquist characteristic.

5.2 *Example II*

Find a positive minimal realisation of the strictly proper transfer matrix

$$\mathbf{T}_{sp}(\lambda^{-1}) = \begin{bmatrix} \frac{2\lambda^{-1}}{1+0.1\lambda^{-1}} & \frac{\lambda^{-1}}{1+0.2\lambda^{-1}} \\ \frac{\lambda^{-1}}{1+0.2\lambda^{-1}} & \frac{\lambda^{-1}}{1+0.3\lambda^{-1}} \end{bmatrix} \tag{32}$$

with the variable $\lambda = s^{\alpha}$, $0 < \alpha < 1$. The problem of determination of the minimal characteristic polynomial realisation can be divided into four smaller sub-problems. In the first step we write the following initial conditions:

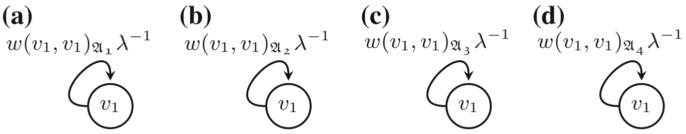

Fig. 4 One-dimensional digraphs corresponding to characteristic polynomials (33)

- number of colours in digraphs: *colour* = 1;
- characteristic polynomials:

$$d_1(\lambda^{-1}) = 1 + 0.1\lambda^{-1}; \quad d_2(\lambda^{-1}) = d_3(\lambda^{-1}) = 1 + 0.2\lambda^{-1}; \\ d_4(\lambda^{-1}) = 1 + 0.3\lambda^{-1}. \tag{33}$$

It should be noted that characteristic polynomial $d_1(\lambda^{-1})$, consists of one monomial M_1. Since the polynomial consists of one monomial there is only one possible realisation. Digraph realisation for the characteristic polynomial $d_1(\lambda^{-1})$ is presented in Fig. 4a. Te realisation meets conditions (C1) and (C2) of the Theorem 2. Finally we must verify the third condition. In considered characteristic polynomial the poles are real and negative. Described realisation satisfies Condition (C3). **The realisation does satisfy all conditions and is correct**. In the same way we determine realisations of the characteristic polynomial $d_2(\lambda^{-1})$ presented in Fig. 4b, $d_3(\lambda^{-1})$ presented in Fig. 4c and polynomial $d_4(\lambda^{-1})$ presented in Fig. 4d.

From the obtained digraphs, we can write state matrix **A** in the form:

$$\mathbf{A} = diag\left(\mathbf{A}_{i|_{i=1,\dots 4}}\right) = diag\left(w(v_1,v_1)_{\mathfrak{A}_{i|_{i=1,\dots 4}}}\right) = \\ = \begin{bmatrix} -0.1 & 0 & 0 & 0 \\ 0 & -02 & 0 & 0 \\ 0 & 0 & -0.2 & 0 \\ 0 & 0 & 0 & -0.3 \end{bmatrix} \tag{34}$$

Inserting matrices (23) and (34) to the equation (19) we obtain the polynomial matrix $\widetilde{\mathbf{N}}(\lambda^{-1})$. After comparison of the coefficients of the same power λ in polynomials $\widetilde{n}_{p,m}(\lambda^{-1}) = n_{p,m}(\lambda^{-1})$ we receive the set of equations. Solving them, we obtain the following matrices:

$$\mathbf{B} = \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}, \ \mathbf{C} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \tag{35}$$

The desired positive realisation of the (32) is given by (34) and (35).

6 Concluding Remarks

The paper presents a method, based on one-dimensional digraph theory and parallel computing, for finding the complete set of one-dimensional characteristic polynomial realisations, that can be used to solve the minimal positive realisation problem of one-dimensional continuous-time fractional system which includes single-input and single-output (SISO). The difference between the proposed in this paper algorithm and currently used methods based on canonical forms of the system (i.e. constant matrix forms) is the creation of not one (or few) minimal realisations, but a set of every possible minimal realisation.

Further work includes extension of the algorithm to find all possible solutions (and not only all possible minimal solutions), solving the realisation problem, reachability and controllability of systems using the fast graph-based method. There is also very difficult open problem of the analysis of systems dynamics for realisations on a different number of nodes in digraphs. Currently, the method of determining a positive polynomial realisation using GPU units and digraphs methods is optimized and in the next step will be implemented in the memory-efficient way.

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Modeling of Discrete-Time Fractional-Order State Space Systems Using the Balanced Truncation Method

Marek Rydel, Rafał Stanisławski, Grzegorz Bialic and Krzysztof J. Latawiec

Abstract This paper presents a new method of approximation of linear time-invariant (LTI) discrete-time fractional-order state space systems by means of the Balanced Truncation Method. This reduction method is applied to the rational form of fractional-order system in terms of expanded state equation. As an approximation result we obtain rational and relatively low-order state space system. Simulation experiments show very high accuracy of the introduced methodology.

1 Introduction

Fractional-order dynamic models have attracted a considerable research interest. It is due to specific properties of fractional-order system that can make them more adequate in modeling of selected industrial systems. However, discrete-time fractional-order difference (FD) may lead to computational explosion. Therefore, a number of various concepts have been developed to modeling discrete-time fractional difference systems. Those concepts are mainly based on two types of applications, where (1) approximators are used to modeling of discrete-time fractional-order difference involved in fractional-order system and (2) the whole fractional order has been modeled by rational, integer-order approximator. In the first case, the solution has led to e.g. least-squares (LS) fit of an impulse/step response of a discrete-time integer-order IIR filter [3, 4, 23]. On the other hand, a fit of a FIR filter to FD has been analyzed in the frequency and time domains [6, 7, 21], also in terms of time-varying filters [17]. Another approach has been the employment of an approximating filter incorporating discrete-Laguerre filters [9, 15, 19]. In the second case, applications are based on orthonormal basis functions (OBF) methods [2, 8, 16, 18], and a number of other applications [5, 12, 13].

M. Rydel · R. Stanislawski (✉) · G. Bialic · K.J. Latawiec
Department of Electrical, Control and Computer Engineering,
Opole University of Technology, Opole, Poland
e-mail: r.stanislawski@po.opole.pl

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The main problem of approximation of a discrete-time fractional-order system is that it leads to very high order of rational, 'integer-order' approximators. Therefore, accurate description of dynamic behavior of fractional dynamical systems causes higher complexity of the mathematical models. Despite increasing computational speed of computers, simulation, optimization or controller designing for large–scale systems is difficult because of system requirements, long time simulation and numerical errors. For this reason, an ability to properly reduce the model complexity without the loss of its dominant dynamic behavior becomes highly significant [20]. There are several techniques for complex model reduction [1, 11]. Among reduction methods, a great attention has been given to the SVD-based methods, which use the balanced model realization theory and the Krylov-based approximation methods, based on moment matching of the impulse response [1, 14]. In this paper, an SVD-based technique for complex model reduction is applied in modeling of discrete-time fractional-order state space systems.

The paper is organized as follows. Having introduced the approximation problem for discrete-time fractional-order systems in Sect. 1, the discrete-time fractional-order state space systems involving FD and its approximations, are recalled in Sect. 2. An application of the Balanced Truncation Approximation (BTA) method for fractional-order system is presented in Sect. 3. A simulation example which confirms the effectiveness of the introduced methodology is shown in Sect. 4 and conclusions of Sect. 5 complete the paper.

2 System Representation

Consider fractional-order discrete-time state space system

$$\begin{aligned} \Delta^{\alpha} x(t+1) &= A_f x(t) + Bu(t), \quad x_0 \\ y(t) &= Cx(t) + Du(t) \end{aligned} \tag{1}$$

where $x(t) \in \mathfrak{R}^n$ is the state vector, $u(t) \in \mathfrak{R}^{n_u}$ and $y(t) \in \mathfrak{R}^{n_y}$ are input and output vectors, respectively, $A_f \in \mathfrak{R}^{n \times n}$, $B \in \mathfrak{R}^{n \times n_u}$, $C \in \mathfrak{R}^{n_y \times n}$ and $D \in \mathfrak{R}^{n_y \times n_u}$. Fractional order difference Δ^{α} is represented as the well known Grünwald-Letnikov frational-order difference (FD)

$$\Delta^{\alpha} x(t+1) = x(t+1) + \sum_{j=1}^{t} \binom{\alpha}{j} x(t-j+1) \tag{2}$$

with

$$\binom{\alpha}{j} = \begin{cases} 1 & j = 0 \\ \frac{\alpha(\alpha-1)\ldots(\alpha-j+1)}{j!} & j > 0 \end{cases}$$

The main problem encountered in application of the fractional-order difference is that the sum is calculated from 0 to t, so each incoming sample increases a complication of equation (3). Therefore, in practical applications finite approximation of the fractional-order difference, called Finite Fractional Difference (FFD), have been used , where the sum is limited to the upper bound $\overline{J}$

$$\Delta_F^{\alpha} FDx(t) = x(t) + \sum_{j=1}^{J} P_j(\alpha)x(t)q^{-j} \tag{3}$$

where $J = \min(t, \overline{J})$.

2.1 Expanded State Model of a Fractional-Order State Space System

The state space system (1) for the FFD approximation (3) can be described as a regular (integer-order) system in the expanded state space form as follows

$$\begin{aligned} x(t+1) &= \tilde{A}x(t) + \tilde{B}u(t) \\ y(t) &= \tilde{C}x(t) + Du(t) \end{aligned} \tag{4}$$

with

$$\tilde{A} = \begin{bmatrix} \underline{A} & -\beta_2 & -\beta_3 & - \dots & -\beta_{\overline{J}-1} & -\beta_{\overline{J}} \\ I & 0 & 0 & \dots & 0 & 0 \\ 0 & I & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & I & 0 \end{bmatrix} \tag{5}$$

$$\tilde{B} = \begin{bmatrix} B^T & 0^2_{n_u \times n} & \dots & 0^{\overline{J}}_{n_u \times n} \end{bmatrix}^T \tag{6}$$

$$\tilde{C} = \begin{bmatrix} C & 0^2_{n_y \times n} & \dots & 0^{\overline{J}}_{n_y \times n} \end{bmatrix} \tag{7}$$

where $\underline{A} = A_f + \alpha I$, $\beta_j = P_j(\alpha)I, j = 2, ...\overline{J}$, and the null matrices in (5) are $n \times n$. Note that the dimensions of the vectors are as follows $x(t) \in \Re^{n\overline{J}}$, $u(t) \in \Re^{n_u}$ and $y(t) \in \Re^{n_y}$ and matrices $\tilde{A} \in \Re^{n\overline{J} \times n\overline{J}}$, $\tilde{B} \in \Re^{n\overline{J} \times n_u}$, $\tilde{C} \in \Re^{n_y \times n\overline{J}}$ and $D \in \Re^{n_y \times n_u}$.

Approximation of the fractional order system by implementation of FFD-based systems results in a complex model of high order and is not useful from the computational viewpoint. Despite increasing computational speed of computers, simulation, optimization or controller designing for large-scale systems is difficult because of system requirements, long time and numerical errors. However, the expanded space equation results in representation of fractional-order system as regular (integer-order) state space form. Therefore, we can use classical, Gramians-based methods to reduction of the expanded state model.

3 Model Order Reduction

The task of reduction of linear discrete-time model can be presented as follows: for the model of order $n\bar{J}$ (4), a reduced model of order k should be determined, where $k << n\bar{J}$, such as the determined approximation error norm $\|y(t) - y_r(t)\|$ reaches the minimum value:

$$\begin{aligned} x_r(t+1) &= \tilde{A}_r x_r(t) + \tilde{B}_r u(t) \\ y_r(t) &= \tilde{C}_r x_r(t) + D_r u(t) \end{aligned} \tag{8}$$

where: $\tilde{A}_r \in \Re^{k \times k}$, $\tilde{B}_r \in \Re^{k \times n_u}$, $\tilde{C}_r \in \Re^{n_y \times k}$, $D_r \in \Re^{n_y \times n_u}$.

There are several techniques for complex model reduction. Among them high popularity was obtained by SVD-methods, which were introduced in Moore's works [10]. The concept of balanced model realization is an easy way to determine the dominating part of the model and reduction by 'cutting' the matrices, describing dynamics of the model in state space (*Balanced Truncation Approximation*). An important consideration in model reduction is the ability to classify states according to their degree of reachability and observability. It can be achieved with the help of controllability and obervability Gramians of the system.

If the discrete-time system is stable, the Gramians of the system are defined for $t \to \infty$:

$$\begin{aligned} P = SS^T &= \sum_{k=0}^{\infty} \tilde{A}^k \tilde{B} \tilde{B}^T \left(\tilde{A}^T\right)^k \\ Q = RR^T &= \sum_{k=0}^{\infty} \left(\tilde{A}^T\right)^k \tilde{C}^T \tilde{C} \tilde{A}^k \end{aligned} \tag{9}$$

The states that are difficult to reach (i.e. require a large amount of energy to reach) and difficult to observe (i.e. yield small amounts of observation energy) correspond to small eigenvalues of controllability and observability Gramians. On this basis reduced-order model can be obtained by eliminating states which are difficult to reach and are simultaneously difficult to observe. In this purpose, the following transformation T of Gramians are necessary:

$$TPT^T = \left(T^T\right)^{-1} QT^{-1} = \Sigma = diag\left(\sigma_i\right) \tag{10}$$

where σ_i are called the Hankel singular values of the system.

Table 1 Balancing-free square root algorithm

1. Compute a Cholesky factorization of controllability and observability Gramians (9)
2. Compute the SVD-decompositions of SR^* as:
$SR^* = \begin{bmatrix} U_1 & U_2 \end{bmatrix} \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \end{bmatrix}$
where: $\Sigma_1 = diag\left(\sigma_1, \sigma_2, \cdots, \sigma_k\right)$, $\Sigma_2 = diag\left(\sigma_{k+1}, \sigma_{k+2}, \cdots, \sigma_n\right)$,
$\sigma_1 \geq \cdots \geq \sigma_k > \sigma_{k+1} \geq \cdots \geq \sigma_n > 0$
3. Compute the QR-decompositions of $S^* U_1$ and $R^* V_1$ as:
$S^* U_1 = \begin{bmatrix} W_1 \\ W_2 \end{bmatrix} X \quad R^* V_1 = \begin{bmatrix} Z_1 \\ Z_2 \end{bmatrix} Y$
4. Compute the SVD-decompositions of $Z_1^* W_1^*$ as:
$Z_1^* W_1 = U_E \Sigma_E V_E^*$
5. Create transformation matrices:
$T = \Sigma_E^{-\frac{1}{2}} U_E^* Z_1^* \quad L = W_1 V_E \Sigma_E^{-\frac{1}{2}}$

For balanced system ($\overline{A} = T\tilde{A}T^{-1}$, $\overline{B} = T\tilde{B}$, $\overline{C} = \tilde{C}T^{-1}$), for which Gramians are equal to a diagonal matrix, it is possible to partition the system as follows:

$$\begin{bmatrix} x_1(t+1) \\ x_2(t+1) \end{bmatrix} = \begin{bmatrix} \overline{A}_{11} & \overline{A}_{12} \\ \overline{A}_{21} & \overline{A}_{22} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} \overline{B}_1 \\ \overline{B}_2 \end{bmatrix} u(t)$$
$$y = \begin{bmatrix} \overline{C}_1 & \overline{C}_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + D\,u(t) \tag{11}$$

The reduction system can be obtained through cutting matrices:

$$\tilde{A}_r = \overline{A}_{11} \quad \tilde{B}_r = \overline{B}_1 \quad \tilde{C}_r = \overline{C}_1 \quad D_r = D \tag{12}$$

Transformation of Gramians is not a unique operation and there are many different algorithms for determining the transformation matrix T and its inverse T^{-1} (size $n\overline{J} \times n\overline{J}$) described in the literature [1]. For complex models of high order, the transformation matrix often has properties similar to a singular matrix, which results in large numerical errors for inverse transformation matrix. Therefore, robust numerical algorithms calculate simultaneously two rectangular transformation matrix (T and L) with dimensions respectively $k \times n\overline{J}$ and $n\overline{J} \times k$. This allows to balance and reduce high-order model at the same time (e.g. *Balancing-free square root algorithm*) [22] (Table 1).

Reduced-order system obtained by balancing truncation has the following properties:

- reduced model has poles in the closed unit circle (reduction preserves stability),
- if $\sigma_{min}\left(\Sigma_1\right) > \sigma_{max}\left(\Sigma_2\right)$ the reduced system is reachable an observable,

- h_∞-norm of difference between full and reduced-order models is upper bounded by twice the sum of the neglected Hankel singular values:

$$\left\|\Sigma - \Sigma_1\right\|_{h_\infty} \leq 2\left(\sigma_{k+1} + \cdots + \sigma_n\right) \tag{13}$$

- $\mathcal{L}_1$-norm of the impulse response of the error system can be derived as follows:

$$\left\|h - h_k\right\|_{\mathcal{L}_1} \leq 2\left(4\sum_{\ell=k+1}^{n} \ell\sigma_\ell - 3\sum_{\ell=k+1}^{n}\sigma_\ell\right) \tag{14}$$

4 Simulation Example

Consider the discrete-time fractional-order state space system $\left(A_f, B, C, D\right)$ of order $\alpha = 0,9$ with:

$$\left[\begin{array}{cccc|c} 2.37 & -4.3849 & 2.602023 & -0.5886251 & 1 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \\ \hline 1 & -1.8 & 0.9 & 0 & 0 \end{array}\right]$$

The system is modeled by use (1) FFD-based expanded state-space model with $J = 200$ and reduced models obtained by using BTA reduction method of order $k = 2$, $k = 4$ and $k = 6$, applied to the FFD-based expanded state space model with 1000 parameters.

The frequency responses of the FD-based system and FFD-based fractional model and BTA-based models are presented in Fig. 1.

Table 2 shows the values of the approximation errors for the reduced models.

The abbreviations in Table 2 are as follows:

- *DCE*—steady state error of reduced model,
- *RMSAE*—relative mean square approximation error for the model adequacy range $\omega \in (10^{-3} - \pi)$,
- *MSE*—the mean square error of step response.

The resuls of Table 2 show that the BTA-based model of order $k = 2$ gives better time-domain results than FFD-based counterpart of order 800. The BTA-based model of order $k = 6$ gives a very good approximation accuracy both in frequency and time domains. This model gives similar time-domain results to the FFD-based model of order 4000. Good results of the FFD-based model in terms of *RMSAE* come from good high-frequency approximation accuracy, at the cost of low-frequency performance.

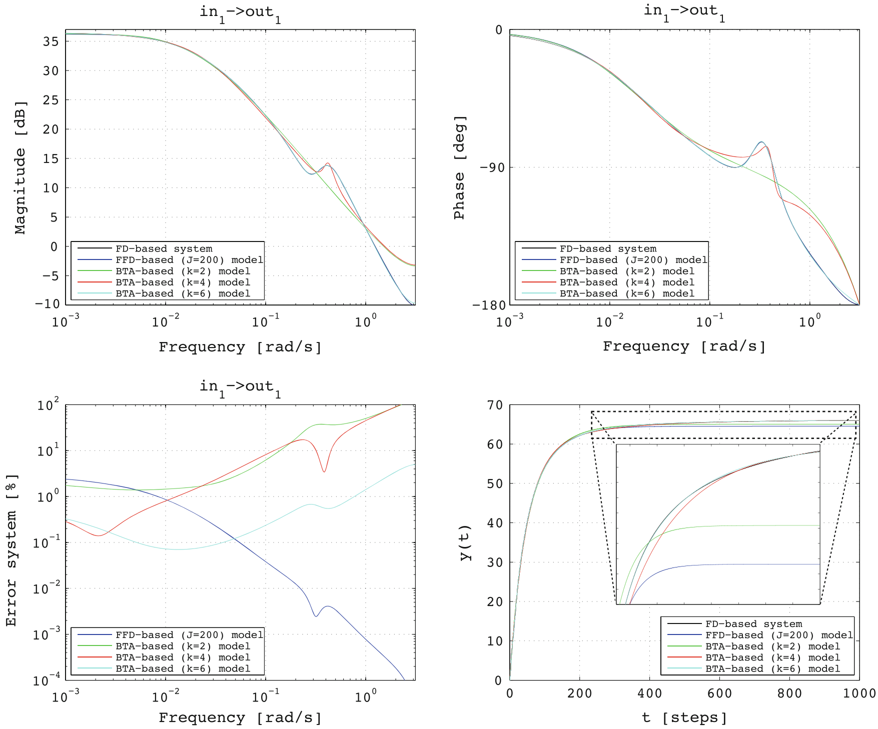

Fig. 1 Frequency responses, step responses and approximation errors for the reduced models

Table 2 Model reduction results

	DCE	*RMSAE*	*MSE*
FFD-based ($J = 200$) model	2.0458	0.0098	0.8731
BTA-based ($J = 1000, k = 2$) model	1.5375	0.3864	0.3288
BTA-based ($J = 1000, k = 4$) model	0.4882	0.3475	0.0212
BTA-based ($J = 1000, k = 6$) model	0.4797	0.0132	1.0462e-04

5 Conclusion

This paper has presented a method for approximation of discrete-time fractional-order state space system by use of the SVD-based reduction method in terms of Balanced Truncation Approximation. As a result we obtain a regular, relatively low-order state space system. A simulation example presents that the introduced methodology can yield a very good approximation performance.

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Experimental Results of Modeling Variable Order System Based on Discrete Fractional Variable Order State-Space Model

Dominik Sierociuk, Michal Macias and Pawel Ziubinski

Abstract The paper presents experimental results of modelling fractional variable order system with using Discrete Fractional Variable Order State-Space Model. Experimental results were obtained based on modified multi-order switching analog realization for a case of constant parameter case, that is also introduced in this paper. During identification process two algorithms were used: direct and dual. Additionally, joint estimation results for parameter estimation were presented, in order to verification of constant parameter for proposed analog model.

Keywords Fractional calculus · Variable order derivative · Analog model · Identification

1 Introduction

Fractional calculus is a generalization of traditional integer order integration and differentiation actions onto non-integer order. The idea of such a generalization has been mentioned in 1695 by Leibniz and L'Hospital. In the end of 19th century, Liouville and Riemann introduced first definition of fractional derivative. However, only just in late 60' of the 20th century, this idea drew attention of engineers. Fractional calculus was found a very useful tool for modelling behavior of many materials and systems, especially those based on the diffusion processes. The description and experimental results of modelling heat transfer processes were presented in [1, 2].

D. Sierociuk (✉) · M. Macias · P. Ziubinski
Institute of Control and Industrial Electronics, Warsaw University of Technology, Koszykowa 75, 00-662 Warsaw, Poland
e-mail: dsieroci@ee.pw.edu.pl

M. Macias
e-mail: michal.macias@ee.pw.edu.pl

P. Ziubinski
e-mail: pawel.ziubinski@ee.pw.edu.pl

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When the fractional order of derivative is not constant but depends on time, then the various types of fractional variable-order derivatives can be distinguished. In [3], nine different variable-order derivative definitions are given and in [4, 5], three general types of variable order definitions can be found but without clear interpretation of them. In papers [6–10] the explanation for two main types and two recursive types of derivatives in the form of switching schemes are given. The equivalence between particular types of definitions and appropriate switching strategies are proven by authors. Moreover, based on these strategies, analog models of proper types derivatives were build and validated according to their numerical implementations.Clear interpretation of some definitions in the form of switching schemes was presented in [6, 11]. Based on those papers, it is possible to categorize fractional order derivatives into three switching strategies. Moreover, analog models, based on proposed switching schemes, and its experimental validation have been presented in [6, 7, 11]. The experimental results show high accuracy for modelling the appropriate types of variable-order definitions.

In [12] analog realization of variable order derivative for multiple-switching order case was introduced, however, presented model gives non-stationary (variable parameter) system. In this paper, modification of this analog realization for constant parameter case is introduced. Obtained analog model will be experimentally validated based on Discrete Fractional Variable Order State-Space Model (DFVOSS), due to confirm accuracy of obtained analog model, and ability of DFVOSS to model real plant. Additionally, experimental results of parameter estimation will be presented, in order to emphasize constant parameter of introduced analog model.

The paper is organized as follows. In Sect. 2 particular types of fractional variable-order derivatives are introduced. Section 3 introduces analog realization of variable fractional order derivative with constant parameter. In Sect. 4 basic properties of Discrete Fractional Variable Order State-Space Model are recalled. Finally, in Sect. 5 experimental results of modelling—the main contribution of this paper—are presented.

2 Fractional Variable-Order Grunwald-Letnikov Type Derivatives

As a base of generalization of the constant fractional order $\alpha \in \mathbb{R}$ difference onto variable-order case, the following definition is taken into consideration:

$$ {}_0\Delta_k^{\alpha} f_k = \frac{1}{h^{\alpha}} \sum_{r=0}^{k} (-1)^r \binom{\alpha}{r} f_{k-r}, \tag{1} $$

where $h > 0$ is a step time.

For the case of order changing with time (variable order case), many different types of definitions can be found in the literature [4, 5]. Among them, we present only two. The first one is obtained by replacing in (1) a constant-order α by variable order $\alpha(t)$. In that approach, all coefficients for past samples are obtained for present value of the order, and is given as follows:

Definition 1 The $\mathcal{A}$-type of fractional variable order difference is defined as follows:

$$ {}_{0}^{A}\Delta_{k}^{\alpha_k} f_k = \frac{1}{h^{\alpha_k}} \sum_{r=0}^{k} (-1)^r \binom{\alpha_k}{r} f_{k-r}. \tag{2} $$

Dual type of variable order derivative, that we will consider, is given by the following definition:

Definition 2 ([8]) The $\mathcal{D}$-type of fractional variable order difference is defined as follows:

$$ {}_{0}^{D}\Delta_{k}^{\alpha_k} f_k = \left(\frac{f_k}{h^{\alpha_k}} - \sum_{j=1}^{k} (-1)^j \binom{-\alpha_k}{j} {}_{0}^{D}\Delta_{k-j}^{\alpha_{k-j}} f_{k-j} \right). \tag{3} $$

Remark 1 For a fractional constant-order $\alpha = \text{const}$, the fractional derivatives given by Definition 1 and 2 are numerically identical with constant-order fractional derivative given by (1).

3 Analog Model

In experimental setup an analog model equivalent to $\mathcal{D}$-type fractional variable order definition presented in Fig. 1 were used. The analog model in such configuration allows to keep the constant value of system parameter. Then, based on this the fractional variable order inertial system shown in Fig. 2 were design. All measurement data for both fractional variable order systems were gathered with time sample equal to 0.001 s.

3.1 Realization of the Multi-switching Integral System

Multi-switching analog model, designed according to $\mathcal{D}$-type fractional variable order integral is presented in Fig. 1.

Passive elements such as: R_1, R_2 and C_1, C_2 were used to build a half-order impedance according to the algorithm meticulously described in [13]. Structure used in experimental setup consists of 200 elements.

It is worth to notice that the presented realizations are the multi-switching analog models. Position of switches (S_1, S_2 and S_3) can be change in any time t. To eliminate the variable system parameter the S_3 switch were applied. Depends on order of system the switch can be directly connected to resistor R_a or R_b to keep the same value of system parameter.

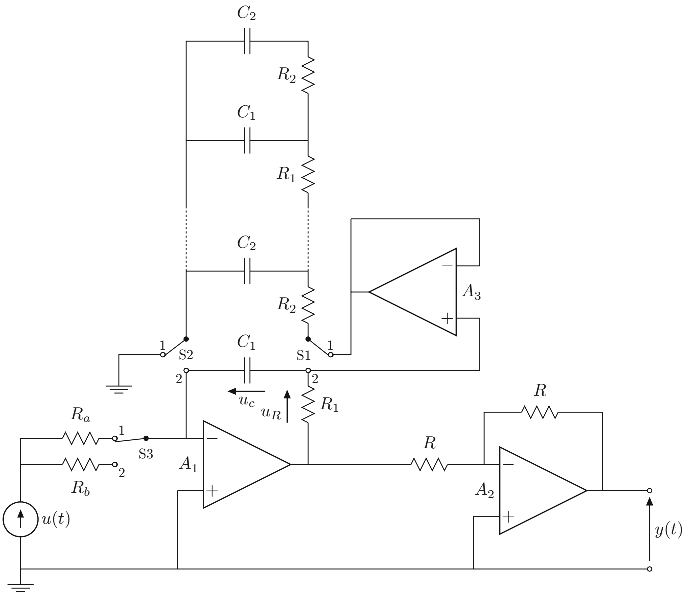

Fig. 1 Multi-switching analog realization of the $\mathcal{D}$-type fractional variable-order integral

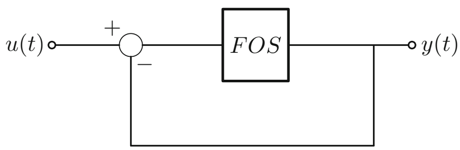

Fig. 2 Realization of the fractional variable order inertial system based on fractional variable order system presented in Fig. 1

Depending on switches position marked as S_i, $(i = 1, 2, 3)$ in Fig. 1, the circuit can be described by fractional order integral system ($\alpha = -0.5$) or traditional integral system ($\alpha = -1$).

1. For a case, when all S_i switches are connected to terminals marked as 1, the following fractional order derivative function has been obtained:

$$y_1(t) = \frac{1}{T} {}_0\mathrm{D}_t^{-1} u(t), \tag{4}$$

where T is a time constant.

2. For a case, when all S_i switches are connected to terminals marked as 2, the following transfer function has been obtained:

$$y_2(t) = \frac{1}{T} {}_0\mathrm{D}_t^{-0.5} u(t), \tag{5}$$

where T is a time constant of the half order integral

To invert polarization of output signal, the A_2 operational amplifier has been used. Based on switches position, the system can be switched in two ways:

1. switching from terminals 1 to 2: In this case, the system described by the first order integral $y_1(t)$ is switched to system of order half described by function $y_2(t)$. To keep the behavior of the $\mathcal{D}$-type definition, it is necessary to maintain a continuous voltage of capacitors in the rest branches, between terminals marked as 1. The voltages of capacitors are set to the value of voltage on the capacitor in the first order integral system (the first capacitor).
2. switching from terminals 2 to 1: In this case the system described by function $y_2(t)$ (half-order) is switched to system described by the function $y_1(t)$ (first order). In this configuration, branches in closed-loop are connected to terminals marked as 2, and after switching, the terminals are changed to 1.

3.2 Realization of the Fractional Variable-Order Inertial System

Realization of the fractional variable order inertial system based on fractional variable order integral system presented in Fig. 1 has been shown in Fig. 2. The order of the system depends on position of S_1 and S_2 switches presented in Fig. 1. Switch S_3 is used to keep the constant parameter of fractional variable-order inertial system. When switches do not changes their positions during experiment the system is considering as a fractional constant order inertial system.

4 Discrete Variable Fractional Order State-Space System

Let us consider a linear discrete fractional variable order state-space (DFVOSS) $\mathcal{A}$-type system

$$ {}_0^A\Delta_{k+1}^{\alpha_{k+1}} x_{k+1} = Ax_k + Bu_k \,, \tag{6} $$

$$ \begin{aligned} x_{k+1} &= {}_0^A\Delta_{k+1}^{\alpha_{k+1}} x_{k+1} \\ &\quad - h^{\alpha_{k+1}} \sum_{j=1}^{k+1} (-1)^j \binom{\alpha_{k+1}}{j} x_{k-j+1} \,, \end{aligned} \tag{7} $$

$$ y_k = Cx_k \,, \tag{8} $$

where $\alpha_k \in \mathbb{R}$ is the fractional variable-order of the system, $u_k \in \mathbb{R}^d$ is a system input, $y_k \in \mathbb{R}^p$ is a system output, $A \in \mathbb{R}^{N\times N}$, $B \in \mathbb{R}^{N\times d}$ and $C \in \mathbb{R}^{p\times N}$ are the state system, input, and output matrices, respectively, $x_k \in \mathbb{R}^N$ is a state vector, N is a number of state equations, and h is a time sampling.

Basic properties of the constant order DFOSS can be found in [14–16].

Let us consider the following DFVOSS system

$$ {}_0^A\Delta_{k+1}^{\alpha_{k+1}} x_{k+1} = u_k \,. \tag{9} $$

This can be expanded into (assuming $h = 1$)

$$ \sum_{j=0}^{k+1} (-1)^j \binom{\alpha_{k+1}}{j} x_{k-j+1} = u_k \tag{10} $$

and rewritten as

$$ x_{k+1} = u_k - \sum_{j=1}^{k+1} (-1)^j \binom{\alpha_{k+1}}{j} x_{k-j+1} \,. \tag{11} $$

The solution of the system given by the $\mathcal{A}$-type definition has the structure of $\mathcal{D}$-type definition, namely

$$ {}_0^D\Delta_{k+1}^{\beta_k} w_k = w_k - \sum_{j=1}^{k} (-1)^j \binom{-\beta_k}{j} {}_0^D\Delta_{k+1}^{\beta_{k-j}} u_{k-j} \,. \tag{12} $$

Comparison of these two relations, along with substitutions

$$ w_{k+1} = u_k \,, \quad -\alpha_{k+1} = \beta_{k+1} \,, $$

and

$$ x_{k+1} = {}_0^D\Delta_{k+1}^{\alpha_{k+1}} w_{k+1} $$

yields

$$ x_{k+1} = {}_0^D\Delta_{k+1}^{-\alpha_{k+1}} u_k \,. $$

This leads us to the conclusion (as it was presented in [17]), that in order to model system build using $\mathcal{D}$-type integrals a DFVOSS based on $\mathcal{A}$-type definition is needed.

4.1 Parametric Identification

For simplicity, let us consider single state variable system

$$ {}_0^A\Delta_{k+1}^{\alpha_{k+1}} x_{k+1} = a x_k + b u_k \,. $$

The parameter b can be retrieved from the following equation

$$\underbrace{\begin{bmatrix} {}_0^A\Delta_{k+1}^{\alpha_{k+1}} x_{k+1} \\ {}_0^A\Delta_{k}^{\alpha_{k}} x_{k} \\ \vdots \\ {}_0^A\Delta_{1}^{\alpha_{1}} x_{1} \end{bmatrix}}_{\equiv \Delta X} = \underbrace{\begin{bmatrix} x_k & u_k \\ x_{k-1} & u_{k-1} \\ \vdots & \\ x_0 & u_0 \end{bmatrix}}_{\equiv U} \begin{bmatrix} a \\ b \end{bmatrix}. \tag{13}$$

Exploiting duality relations discussed in [17], and taking into account (13), one obtains

$$\underbrace{\begin{bmatrix} x_{k+1} \\ x_k \\ \vdots \\ x_1 \end{bmatrix}}_{\equiv X} = \underbrace{\begin{bmatrix} {}_0^D\Delta_{k+1}^{-\alpha_{k+1}} x_k & {}_0^D\Delta_{k+1}^{-\alpha_{k+1}} u_k \\ {}_0^D\Delta_{k}^{-\alpha_{k}} x_{k-1} & {}_0^D\Delta_{k}^{-\alpha_{k}} u_{k-1} \\ \vdots & \\ {}_0^D\Delta_{1}^{-\alpha_{1}} x_0 & {}_0^D\Delta_{1}^{-\alpha_{1}} u_0 \end{bmatrix}}_{\equiv \Delta U} \begin{bmatrix} a \\ b \end{bmatrix}. \tag{14}$$

5 Experimental Results

In this section comparison of data collected from the analog model for the fractional order inertial system and the results of identification will be presented. Parameter identification was made for the two cases, for direct method case based on formula 13 and dual method case based on formula 14.

Figure 3 presents the way in which the order was switched. For $\mathcal{A}$-type definition in direct equation order is switched between 0.5 and 1, and for $\mathcal{D}$-type definition dual equation order was switched between −0.5 and −1.

Numerical results, presented in this section, were obtained in Matlab/Simulink environment using the FSST Toolkit [18].

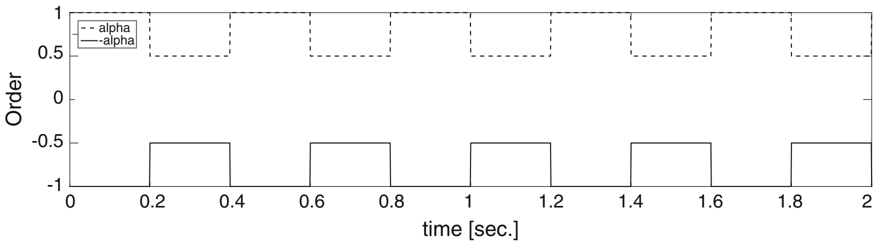

Fig. 3 Variable system order used in experiments

5.1 Identification Results

Results of parameters identification based on formula 13 were obtained by pseudoinversion of matrix U in equation $p = \mathrm{pinv}(U)\Delta X$, where $p = \begin{bmatrix} a \\ b \end{bmatrix}$ is a vector of parameters. The solution error e was obtained in the following way $e = \Delta X - Up$. Identified parameters are $a = -2.9839$ and $b = 2.9951$. For parameters identification based on dual formula (given by Eq. 14), pseudoinversion of matrix ΔU in equation $p = \mathrm{pinv}(\Delta U)X$ was necessary. The solution error of identification equation was defined as follows $e = X - \Delta Up$. Identified parameters are $a = -3.0457$ and $b = 3.02207$. Figure 4a, b presents results of solution error for Eqs. 13 and 14.

As it can be seen in Fig. 4 the solution error for dual method is much smaller than for direct method (this conclusion is similar to this presented in [17]).

Example of comparison between analog model and numerical implementation is presented in Figs. 5 and 6 for identification methods given in Eqs. 13 and 14.

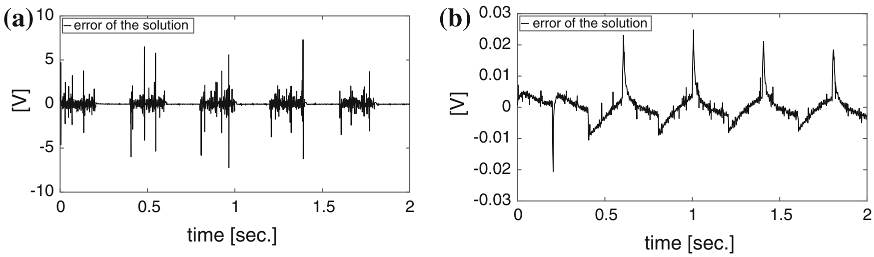

Fig. 4 Solution error for direct method: direct (based on Eq. 13) and dual (based on Eq. 14). **a** Solution error for direct method. **b** Solution error for dual method

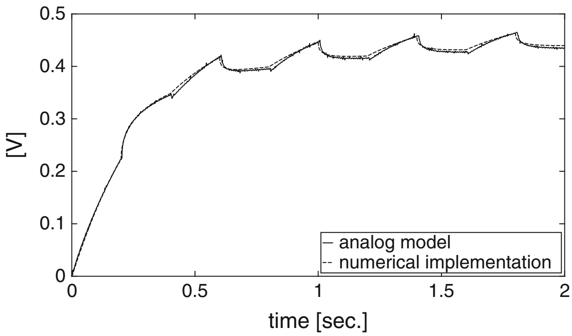

Fig. 5 Comparison between analog model and numerical implementation for direct method based on formula 13

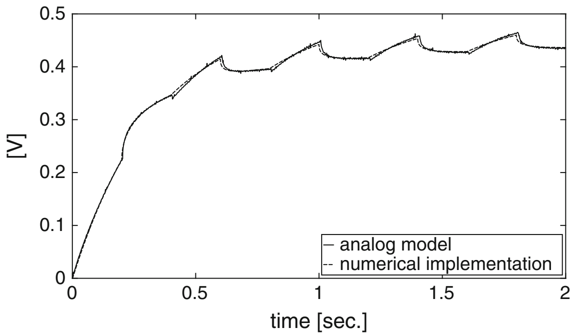

Fig. 6 Comparison between analog model and numerical implementation for dual method based on formula 14

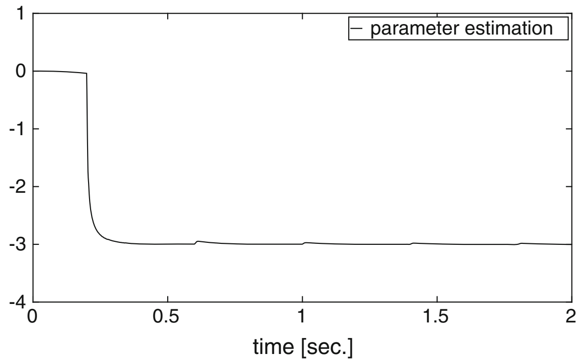

Fig. 7 Results for joint estimation of parameter *a*

5.2 Parameter Estimation Results

In this Section estimation results obtained accordingly to the joint estimation with Extended Fractional Kalman Filter algorithm is presented. The joint estimation algorithm assumed that the state vector is extended with the parameter a which is also estimated as another state variable (Fig. 7).

6 Conclusions

In this paper experimental results of modelling fractional variable order system with using Discrete Fractional Variable Order State-Space Model were presented. For modelling the fractional variable order system a modified multi-order switching analog realization was used. The main feature of proposed realization was a constant parameter case. For identification of parameters two algorithms were used. The first based directly on difference equation, and the second based on dual definition difference equation. Obtained results presents ability of Discrete Fractional Variable Order State-Space Model to describe variable-order dynamics, and confirm also efficiency of used identification algorithms. Additionally, joint estimation results for parameter estimation were presented. Obtained results confirm constancy of parameter for proposed analog realization.

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Part III
Controllability, Observability and Stability

Positivity and Stability of a Class of Fractional Descriptor Discrete-Time Nonlinear Systems

Tadeusz Kaczorek

Abstract A method of analysis of the fractional descriptor nonlinear discrete-time systems with regular pencils of linear part is proposed. The method is based on the Weierstrass-Kronecker decomposition of the pencils. Necessary and sufficient conditions for the positivity of the nonlinear systems are established. A procedure for computing the solution to the equations describing the nonlinear systems are proposed. Using an extension of the Lyapunov method to positive nonlinear systems, sufficient conditions for the asymptotic stability are derived.

Keywords Fractional · Descriptor · Nonlinear · System · Weierstrass-Kronecker decomposition · Positivity · Lyapunov method · Stability

1 Introduction

Descriptor (singular) linear systems have been considered in many papers and books [1–17]. The eigenvalues and invariants assignment by state and output feedbacks have been investigated in [4, 15, 18] and the minimum energy control of descriptor linear systems in [19–21]. The computation of Kronecker's canonical form of singular pencil has been analyzed in [16]. The positive linear systems with different fractional orders have been addressed in [20]. Selected problems in theory of fractional linear systems has been given in monograph [13].

A dynamical system is called positive if its trajectory starting from any nonnegative initial state remains forever in the positive orthant for all nonnegative inputs. An overview of state of the art in positive theory is given in [22]. Variety of models having positive behavior can be found in engineering, economics, social sciences, biology and medicine, etc.

T. Kaczorek (✉)
Faculty of Electrical Engineering, Bialystok University of Technology, Bialystok, Poland
e-mail: kaczorek@isep.pw.edu.pl

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Descriptor standard positive linear systems by the use of Drazin inverse has been addressed in [1–4, 13, 14, 23]. The shuffle algorithm has been applied to checking the positivity of descriptor linear systems in [24]. The stability of positive descriptor systems has been investigated in [17]. Reduction and decomposition of descriptor fractional discrete-time linear systems have been considered in [11]. A new class of descriptor fractional linear discrete-time systems has been introduced in [12]. The standard and positive descriptor discrete-time nonlinear systems have been addressed in [10].

In this paper a method of analysis of the fractional descriptor standard and positive nonlinear discrete-time systems with regular pencils will be proposed. The method is based on the Weierstrass-Kronecker decomposition of the pencil of the linear part of the equation describing the nonlinear system and on an extension of the Lyapunov method to positive nonlinear systems.

The paper is organized as follows. In Sect. 2 the Weierstrass-Kronecker decomposition is applied to analysis of the descriptor nonlinear systems. Necessary and sufficient conditions for the positivity of the nonlinear systems are established in Sect. 3. In Sect. 4 the stability of positive nonlinear systems by the use of extended Lyapunov method is analyzed. Concluding remarks are given in Sect. 5.

The following notation will be used: $\mathfrak{R}$—the set of real numbers, $\mathfrak{R}^{n\times m}$—the set of $n\times m$—real matrices, Z_+—the set of nonnegative integers, $\mathfrak{R}_+^{n\times m}$—the set of $n\times m$ matrices with nonnegative entries and $\mathfrak{R}_+^n=\mathfrak{R}_+^{n\times 1}$, I_n—the $n\times n$ identity matrix.

2 Fractional Descriptor Discrete-Time Nonlinear Systems and Their Solution

Consider the fractional descriptor discrete-time nonlinear system

$$E\Delta^\alpha x_{i+1}=Ax_i+f(x_i,u_i), i\in Z_+=\{0,1,\ldots\}, 0<\alpha<1 \tag{2.1a}$$

$$y_i=g(x_i,u_i), \tag{2.1b}$$

where $x_i\in\mathfrak{R}^n$, $u_i\in\mathfrak{R}^m$, $y_i\in\mathfrak{R}^p$, $i\in Z_+$ are the state, input and output vectors, $f(x_i,u_i)\in\mathfrak{R}^n$, $g(x_i,u_i)\in\mathfrak{R}^p$ are continuous and bounded vector functions of x_i and u_i satisfying the conditions $f(0,0)=0$, $g(0,0)=0$ and $E,A\in\mathfrak{R}^{n\times n}$ and

$$\Delta^\alpha x_i=\sum_{j=0}^{i}(-1)^j\binom{\alpha}{j}x_{i-j} \tag{2.1c}$$

$$\binom{\alpha}{j}=\begin{cases}1 & \text{for} \quad j=0\\ \dfrac{\alpha(\alpha-1)\ldots(\alpha-j+1)}{j!} & \text{for} \quad j=1,2,\ldots\end{cases}. \tag{2.1d}$$

is the fractional $\alpha\in\mathfrak{R}$ order difference of x_i.

It is assumed that $\det E = 0$ and the

$$\det[Ez - A] \neq 0 \text{ for some } z \in \mathrm{C} (\text{the field of complex numbers}). \tag{2.2}$$

Substituting (2.1c) into (2.1a) we obtain

$$Ex_{i+1} = A_\alpha x_i + \sum_{j=2}^{i+1} c_j E x_{i-j+1} + f(x_i, u_i) \tag{2.3a}$$

where

$$A_\alpha = A + E\alpha, \quad c_j = (-1)^{j+1} \binom{\alpha}{j}. \tag{2.3b}$$

It is well-known [14] that if (2.2) holds then there exist nonsingular matrices $P, Q \in \mathfrak{R}^{n \times n}$ such that

$$P[Ez - A_\alpha]Q = \begin{bmatrix} I_{n_1} z - A_{1\alpha} & 0 \\ 0 & Nz - I_{n_2} \end{bmatrix}, A_{1\alpha} \in \mathfrak{R}^{n_1 \times n_1}, N \in \mathfrak{R}^{n_2 \times n_2} \tag{2.4}$$

where $n_1 = {}^\circ\{\det[Ez - A_\alpha]\}$, $n_2 = n - n_1$ and N is the nilpotent matrix with the index μ, i.e. $N^{\mu-1} \neq 0$, $N^\mu = 0$.

The matrices P and Q can be computed using procedures given in [14, 16].

Premultiplying (2.3a) by the matrix P and introducing the new state vector

$$\bar{x}_i = \begin{bmatrix} \bar{x}_{1,i} \\ \bar{x}_{2,i} \end{bmatrix} = Q^{-1} x_i, \bar{x}_{1,i} \in \mathfrak{R}^{n_1}, \bar{x}_{2,i} \in \mathfrak{R}^{n_2}. \tag{2.5}$$

From (2.3a) and (2.5) we obtain

$$PEQQ^{-1}x_{i+1} = PA_\alpha QQ^{-1}x_i + \sum_{j=2}^{i+1} c_j PEQQ^{-1} x_{i-j+1} + Pf(Q\bar{x}_i, u_i), \tag{2.6}$$

and

$$\bar{x}_{1,i+1} = A_{1\alpha}\bar{x}_{1,i} + \sum_{j=2}^{i+1} c_j \bar{x}_{1,i-j+1} + \bar{f}_1(\bar{x}_i, u_i), \tag{2.7a}$$

$$N\bar{x}_{2,i+1} = \bar{x}_{2,i} + \sum_{j=2}^{i+1} c_j N \bar{x}_{2,i-j+1} - \bar{f}_2(\bar{x}_i, u_i), \tag{2.7b}$$

where

$$\begin{bmatrix} \bar{f}_1(\bar{x}_i, u_i) \\ -\bar{f}_2(\bar{x}_i, u_i) \end{bmatrix} = Pf(Q\bar{x}_i, u_i). \tag{2.7c}$$

Note that if $0 < \alpha < 1$ then

$$c_j = (-1)^{j+1} \binom{\alpha}{j} > 0 \text{ for } j = 1, 2, \ldots, i+1. \tag{2.8}$$

To simplify the notation it is assumed that the nilpotent matrix contains only one block, i.e.

$$N = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & 0 \end{bmatrix} \in \mathfrak{R}^{n_2 \times n_2}. \tag{2.9}$$

In this case the solution to the Eqs. (2.1a), (2.1b), (2.1c), (2.1d) for given initial conditions $x_0 \in \mathfrak{R}^n$ and input $u_i \in \mathfrak{R}^m$ for i = 0,1,… can be computed iteratively as follows.

From (2.7b) and (2.9) for i = 0 we have

$$\begin{aligned} \bar{x}_{22,1} &= \bar{x}_{21,0} - f_{21}(\bar{x}_0, u_0) \\ \bar{x}_{23,1} &= \bar{x}_{22,0} - f_{22}(\bar{x}_0, u_0) \\ &\vdots \\ \bar{x}_{2n_2,1} &= \bar{x}_{2n_2-1,0} - f_{2n_2-1}(\bar{x}_0, u_0) \end{aligned} \tag{2.10a}$$

$$\bar{x}_{2n_2,0} = f_{2n_2}(\bar{x}_0, u_0) \tag{2.10b}$$

where

$$\bar{x}_{2,i} = [\bar{x}_{21,i} \quad \bar{x}_{22,i} \quad \ldots \quad \bar{x}_{2n_2,i}]^T \tag{2.10c}$$

$$\bar{f}_2(\bar{x}_0, u_0) = [f_{21}(\bar{x}_0, u_0) \quad f_{22}(\bar{x}_0, u_0) \quad \ldots \quad f_{2n_2}(\bar{x}_0, u_0)]^T. \tag{2.10d}$$

From (2.10a) and (2.10c) it follows that $\bar{x}_{21,1}$ can be chosen arbitrary and $\bar{x}_{2n_2,0}$ should satisfy the condition (2.10b).

Next using (2.7a) for i = 0 we have

$$\bar{x}_{1,1} = \begin{bmatrix} \bar{x}_{11,1} \\ \bar{x}_{12,1} \\ \vdots \\ \bar{x}_{1n_1,1} \end{bmatrix} = A_{1\alpha}\bar{x}_{1,0} + \bar{f}_1(\bar{x}_0, u_0).$$

Knowing $\bar{x}_1$ we can compute from (2.7b) for $i = 1$

$$\begin{aligned}&\bar{x}_{22,2} = \bar{x}_{21,1} + c_2\bar{x}_{22,0} - f_{21}(\bar{x}_1, u_1)\\&\bar{x}_{23,2} = \bar{x}_{22,1} + c_2\bar{x}_{23,0} - f_{22}(\bar{x}_1, u_1)\\&\vdots\\&\bar{x}_{2n_2,2} = \bar{x}_{2n_2-1,1} + c_2\bar{x}_{2n_2,0} - f_{2n_2-1}(\bar{x}_1, u_1)\end{aligned} \tag{2.12a}$$

$$\bar{x}_{2n_2,1} = f_{2n_2}(\bar{x}_1, u_1) \tag{2.12b}$$

and next from (2.7a)

$$\bar{x}_{1,2} = \begin{bmatrix}\bar{x}_{11,1}\\ \bar{x}_{12,1}\\ \vdots\\ \bar{x}_{1n_1,1}\end{bmatrix} = A_{1\alpha}\bar{x}_{1,1} + c_2\bar{x}_{1,0} + f_1(\bar{x}_1, u_1) \tag{2.14}$$

where $c_2 = \frac{\alpha(1-\alpha)}{2}$.

Repeating the procedure we may compute the state vector $\bar{x}_i$ for $i = 1,2,\ldots$ and next from the equality

$$x_i = Q\bar{x}_i \tag{2.12}$$

the desired solution x_i of the Eq. (2.1a).

3 Positive Fractional Descriptor Nonlinear Systems

Consider the descriptor discrete-time nonlinear system (2.1a), (2.1b), (2.1c), (2.1d).

Definition 3.1 The fractional descriptor discrete-time nonlinear system (2.1a), (2.1b), (2.1c), (2.1d) is called positive if $x_i \in \mathfrak{R}_+^n$, $y_i \in \mathfrak{R}_+^p$, $i \in Z_+$ for any consistent initial conditions $x_0 \in X_0 \in \mathfrak{R}_+^n$ and all admissible inputs $u_i \in U_a \in \mathfrak{R}_+^m$.

Note that for positive systems (2.1a), (2.1b), (2.1c), (2.1d) $\bar{x}_i = Q^{-1}x_i \in \mathfrak{R}_+^n$ if and only if the matrix $Q \in \mathfrak{R}_+^{n\times n}$ is monomial. In this case $Q^{-1} \in \mathfrak{R}_+^{n\times n}$.

Note that for fractional positive systems (2.7a) $\bar{x}_i = Q^{-1}x_i \in \mathfrak{R}_+^n$ for $i \in Z_+$ if and only if

$$A_{1\alpha} \in \mathfrak{R}_+^{n_1\times n_1} \text{ and } \bar{f}_1(\bar{x}_i, u_i) \in \mathfrak{R}_+^{n_1} \text{ for all } \bar{x}_i \in \mathfrak{R}_+^n \text{ and, } u_i \in \mathfrak{R}_+^m\ i \in Z_+. \tag{3.1}$$

From the structure of the matrix (2.9) and the Eq. (2.7b) it follows that $\bar{x}_{2,i} \in \mathfrak{R}_+^{n_2}$, $i \in Z_+$ if and only if

$$\bar{f}_2(\bar{x}_i, u_i) \in \Re_+^{n_2} \text{ for all } \bar{x}_i \in \Re_+^n \text{ and } u_i \in \Re_+^m, i \in Z_+. \tag{3.2}$$

The solution of the Eqs. (2.7a), (2.7b), (2.7c) $\bar{x}_i \in \Re_+^n$ if and only if the conditions (3.1) and (3.2) are satisfied.

Therefore, the following theorem of the positivity of the system (2.1a), (2.1b), (2.1c), (2.1d) has been proved.

Theorem 3.1 *The fractional descriptor nonlinear system* (2.1a), (2.1b), (2.1c), (2.1d) *is positive if and only if the conditions* (3.1) *and* (3.2) *are satisfied, the matrix* $Q \in \Re_+^{n \times n}$ *is monomial and*

$$g(x_i, u_i) \in \Re_+^p \text{ for } x_i \in \Re_+^n \text{ and } u_i \in \Re_+^m, i \in Z_+. \tag{3.3}$$

Remark 3.1 If the nilpotent matrix N consist of q block then the condition (2.10b) should be substituted by suitable q conditions of each for the blocks.

Remark 3.2 If the nilpotent matrix N consists of q blocks then for each of the blocks one state variable can be chosen arbitrarily.

Example 3.1 Consider the fractional descriptor nonlinear system (2.1a), (2.1b), (2.1c), (2.1d) with $\alpha = 0.5$ and

$$E = \begin{bmatrix} 0 & 0 & 0.5 & -0.5 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.2 & 0 & 0 & 0 \end{bmatrix}, \quad A = \begin{bmatrix} 0.5 & -0.5 & -0.25 & 0.25 \\ 0.6 & 0 & 0.4 & -0.2 \\ 0.5 & 0.5 & -0.25 & -0.25 \\ 0.3 & 0 & 0.2 & 0.4 \end{bmatrix},$$
$$f(x_i, u_i) = \begin{bmatrix} 0.5x_{3,i}^2 - x_{2,i}^2 + e^{-i} - 0.5 \\ 0.2x_{1,i}^2 + 0.2e^{-i} + 0.4(1+i^2) \\ x_{2,i}^2 + 0.5x_{3,i}^2 + 0.5 \\ 0.2(1+i^2) - 0.4x_{1,i}^2 - 0.4e^{-i} \end{bmatrix}, \tag{3.4a}$$

with the initial conditions

$$x_0 = \begin{bmatrix} x_{1,0} \\ x_{2,0} \\ x_{3,0} \\ x_{4,0} \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}. \tag{3.4b}$$

The assumption (2.2) is satisfied since

$$\det E = \begin{vmatrix} 0 & 0 & 0.5 & -0.5 \\ 0.4 & 0 & 0 & 0 \\ 0 & 0 & 0.5 & 0.5 \\ 0.2 & 0 & 0 & 0 \end{vmatrix} = 0 \tag{3.5}$$

and

$$\det[Ez - A_\alpha] = \begin{vmatrix} -0.5 & 0.5 & 0.5z & -0.5z \\ 0.4z - 0.8 & 0 & -0.4 & 0.2 \\ -0.5 & -0.5 & 0.5z & 0.5z \\ 0.2z - 0.4 & 0 & -0.2 & -0.4 \end{vmatrix} = 0.1z^2 - 0.2z - 0.1 \neq 0. \tag{3.6}$$

In this case

$$P = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 2 \end{bmatrix}, \quad Q = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \tag{3.7}$$

Using (2.4), (2.7a), (2.7b), (2.7c) and (3.7) we obtain

$$P[Ez - A_\alpha]Q = \begin{bmatrix} I_{n_1}z - A_{1\alpha} & 0 \\ 0 & Nz - I_{n_2} \end{bmatrix}, A_{1\alpha} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix}, N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, n_1 = n_2 = 2 \tag{3.8}$$

$$\bar{x}_i = \begin{bmatrix} \bar{x}_{1,i} \\ \bar{x}_{2,i} \\ \bar{x}_{3,i} \\ \bar{x}_{4,i} \end{bmatrix} = Q^{-1}x_i = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{1,i} \\ x_{2,i} \\ x_{3,i} \\ x_{4,i} \end{bmatrix} = \begin{bmatrix} x_{3,i} \\ x_{1,i} \\ x_{2,i} \\ x_{4,i} \end{bmatrix}, \tag{3.9}$$

$$Pf(\bar{x}_i, u_i) = \begin{bmatrix} \bar{f}_1(\bar{x}_i, u_i) \\ -\bar{f}_2(\bar{x}_i, u_i) \end{bmatrix} = \begin{bmatrix} \bar{x}_{1,i}^2 + e^{-i} \\ 1 + i^2 \\ 2\bar{x}_{3,i}^2 - e^{-i} + 1 \\ \bar{x}_{2,i}^2 + e^{-i} \end{bmatrix} \tag{3.10}$$

and

$$\begin{bmatrix} \bar{x}_{1,i+1} \\ \bar{x}_{2,i+1} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} \bar{x}_{1,i} \\ \bar{x}_{2,i} \end{bmatrix} + \sum_{j=2}^{i+1} c_j \begin{bmatrix} \bar{x}_{1,i-j+1} \\ \bar{x}_{2,i-j+1} \end{bmatrix} + \begin{bmatrix} \bar{x}_{1,i}^2 + e^{-i} \\ 1 + i^2 \end{bmatrix}, \tag{3.11}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \bar{x}_{3,i+1} \\ \bar{x}_{4,i+1} \end{bmatrix} = \begin{bmatrix} \bar{x}_{3,i} \\ \bar{x}_{4,i} \end{bmatrix} + \sum_{j=2}^{i+1} c_j \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \bar{x}_{3,i-j+1} \\ \bar{x}_{4,i-j+1} \end{bmatrix} + \begin{bmatrix} 2\bar{x}_{3,i}^2 - e^{-i} + 1 \\ -\bar{x}_{2,i}^2 - e^{-i} \end{bmatrix} \tag{3.12}$$

with the initial conditions

$$\bar{x}_0 = Q^{-1}x_0 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 2 \end{bmatrix}. \tag{3.13}$$

The fractional descriptor system (2.1a), (2.1b), (2.1c), (2.1d) with (3.4a), (3.4b) is positive since the conditions (3.1) and (3.2) are satisfied and the matrix Q defined by (3.7) is monomial.

Using the procedure presented in Sect. 3 we obtain the following:

From (3.12) for $i = 0$ we have

$$\bar{x}_{4,1} = \bar{x}_{3,0} + 2\bar{x}_{3,0}^2 - e^0 + 1 = 0, \tag{3.14a}$$

and the condition (3.2) is satisfied since

$$\bar{x}_{4,0} = \bar{x}_{2,0}^2 + 1 = 2. \tag{3.14b}$$

Using (3.11) for $i = 0$ and (3.13) we obtain

$$\begin{aligned} \bar{x}_{1,1} &= \bar{x}_{2,0} + \bar{x}_{1,0}^2 + e^0 = 3, \\ \bar{x}_{2,1} &= \bar{x}_{1,0} + 2\bar{x}_{2,0} + 1 = 4 \end{aligned} \tag{3.15}$$

and from (3.12) for $i = 1$

$$\begin{aligned} \bar{x}_{4,2} &= \bar{x}_{3,1} + 0.125\bar{x}_{4,1} + 2\bar{x}_{3,1}^2 - e^{-1} + 1, \\ \bar{x}_{4,1} &= \bar{x}_{2,1}^2 + e^{-1} \end{aligned} \tag{3.16}$$

for arbitrary $\bar{x}_{3,1} \geq 0$.

From (3.11) for $i = 1$ we have

$$\begin{aligned} \bar{x}_{1,2} &= \bar{x}_{2,1} + 0.125\bar{x}_{1,0} + \bar{x}_{1,1}^2 + e^{-1}, \\ \bar{x}_{2,2} &= \bar{x}_{1,1} + 2\bar{x}_{2,1} + 0.125\bar{x}_{2,0} + 2 \end{aligned} \tag{3.17}$$

and from (3.12) for $i = 2$

$$\begin{aligned}\bar{x}_{4,3} &= \bar{x}_{3,2} + 0.125\bar{x}_{4,1} + 0.0625\bar{x}_{4,0} + 2\bar{x}_{3,2}^2 - e^{-2} + 1,\\ \bar{x}_{4,2} &= \bar{x}_{2,2}^2 + e^{-2}\end{aligned} \tag{3.18}$$

for arbitrary $\bar{x}_{3,2} \geq 0$.

Continuing the procedure we may compute the solution $\bar{x}_i$ of the Eqs. (3.11) and (3.12) and next the solution

$$x_i = Q\bar{x}_i = \begin{bmatrix} \bar{x}_{2,i} \\ \bar{x}_{3,i} \\ \bar{x}_{1,i} \\ \bar{x}_{4,i} \end{bmatrix} \tag{3.19}$$

of the Eqs. (2.1a), (2.1b), (2.1c), (2.1d) with (3.4a), (3.4b).

4 Stability of Positive Nonlinear Systems

Consider the fractional descriptor nonlinear system (2.1a) decomposed into two nonlinear subsystems (2.7a) and (2.7b) of the form

$$\bar{x}_{1,i+1} = A_{1\alpha}\bar{x}_{1,i} + \sum_{j=2}^{i+1} c_j\bar{x}_{1,i-j+1} + \bar{f}_1(\bar{x}_{1,i}), \tag{4.1}$$

$$N\bar{x}_{2,i+1} = \bar{x}_{2,i} + \sum_{j=2}^{i+1} c_j N\bar{x}_{2,i-j+1} - \bar{f}_2(\bar{x}_{1,i}, \bar{x}_{2,i}), \tag{4.2}$$

where $\bar{f}_1(\bar{x}_{1,i}) = \bar{f}_1(\bar{x}_{1,i}, 0)$ and $\bar{f}_2(\bar{x}_{1,i}, \bar{x}_{2,i}) = \bar{f}_2(\bar{x}_{1,i}, \bar{x}_{2,i}, 0)$.

Definition 4.1 The positive nonlinear system (2.1a) is called asymptotically stable in the region D if $x_i \in \mathfrak{R}_+^n$, $i \in Z_+$ satisfies the condition

$$\lim_{i\to\infty} x_i = 0 \text{ for any finite } x_0 \in \mathfrak{R}_+^n. \tag{4.3}$$

Note that the positive nonlinear system (2.1a) is asymptotically stable if and only if the positive nonlinear subsystem (4.1) and (4.2) are asymptotically stable, since from (2.5) for the monomial matrix Q we have

$$\lim_{i\to\infty} x_i = 0 \text{ if and only if } \lim_{i\to\infty} \bar{x}_i = 0. \tag{4.4}$$

Remark 4.1 The positive nonlinear system (2.1a) is asymptotically stable only if the linear part of the system (4.1), (4.2) (for $\bar{f}_1(\bar{x}_{1,i}) = 0$, $\bar{f}_2(\bar{x}_{1,i}, \bar{x}_{2,i}) = 0$) is asymptotically stable. The asymptotic stability of this positive linear system

$$\bar{x}_{1,i+1} = A_{1\alpha}\bar{x}_{1,i} + \sum_{j=2}^{i+1} c_j \bar{x}_{1,i-j+1}, \tag{4.5a}$$

$$N\bar{x}_{2,i+1} = \bar{x}_{2,i} + \sum_{j=2}^{i+1} c_j N\bar{x}_{2,i-j+1}, \tag{4.5b}$$

can be verified using the tests presented in [13].

The subsystem (4.5b) is asymptotically stable since its solution $\bar{x}_{2,i} = 0$ for $i = 1, 2, \ldots$.

To investigate the asymptotic stability of the positive nonlinear subsystem (4.1) we will apply the Lyapunov method. As a candidate of the Lyapunov function for the subsystem (4.5a) we choose

$$V(\bar{x}_{1,i}) = d\bar{x}_{1,i} > 0 \text{ for } \bar{x}_{1,i} \in \Re_+^{n_1}, i \in Z_+, \tag{4.6}$$

where $d = [d_1 \quad \ldots \quad d_{n_1}]^T$ is a strictly positive vector with $d_k > 0$ for $k = 1, 2, \ldots, n_1$.

Using (4.6) and (4.1) we obtain

$$\Delta V(\bar{x}_{1,i}) = V(\bar{x}_{1,i+1}) - V(\bar{x}_{1,i}) = d\left[(A_{1\alpha} - I_{n_1})\bar{x}_{1,i} + \sum_{j=2}^{i+1} c_j \bar{x}_{1,i-j+1} + \bar{f}_1(\bar{x}_{1,i})\right]. \tag{4.7}$$

From (4.7) it follows that $\Delta V(\bar{x}_{1,i}) < 0$ if

$$(A_{1\alpha} - I_{n_1})\bar{x}_{1,i} + \sum_{j=2}^{i+1} c_j \bar{x}_{1,i-j+1} + \bar{f}_1(\bar{x}_{1,i}) < 0 \text{ for } \bar{x}_{1,i} \in D \in \Re_+^{n_1} \text{ and } i \in Z_+ \tag{4.8}$$

since d is a strictly positive vector.

Therefore, the following theorem has been proved.

Theorem 4.1 The positive nonlinear system (4.1) is asymptotically stable in the region D if the condition (4.8) is satisfied.

Theorem 4.2 *The positive nonlinear system* (4.2) *is asymptotically stable if the positive nonlinear system* (4.1) *is asymptotically stable and*

$$\lim_{i \to \infty} \bar{f}_1(\bar{x}_{1,i}) = 0. \tag{4.9}$$

Proof If the positive nonlinear subsystem (4.1) is asymptotically stable then $\lim_{i \to \infty} \bar{x}_{1,i} = 0$ and from (4.2) it follows that $\lim_{i \to \infty} \bar{x}_{2,i} = 0$ if the condition (4.9) is satisfied. □

Example 4.1 Consider the fractional nonlinear subsystems for $\alpha = 0.5$

$$\begin{bmatrix} x_{1,i+1} \\ x_{2,i+1} \end{bmatrix} = \begin{bmatrix} 0.1 & 0.3 \\ 0 & 0.2 \end{bmatrix} \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix} + \sum_{j=2}^{i+1} c_j \begin{bmatrix} x_{1,i-j+1} \\ x_{2,i-j+1} \end{bmatrix} + \begin{bmatrix} x_{1,i}^2 \\ x_{2,i}x_{2,i} \end{bmatrix}, \tag{4.10a}$$

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{3,i+1} \\ x_{4,i+1} \end{bmatrix} = \begin{bmatrix} x_{3,i} \\ x_{4,i} \end{bmatrix} + \sum_{j=2}^{i+1} c_j \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x_{3,i-j+1} \\ x_{4,i-j+1} \end{bmatrix} - \begin{bmatrix} 2x_{4,i}^2 \\ x_{2,i}^2 \end{bmatrix}. \tag{4.10b}$$

From comparison of (4.10a), (4.10b) and (4.1) it follows that

$$\bar{x}_{1,i} = \begin{bmatrix} x_{1,i} \\ x_{2,i} \end{bmatrix}, \quad A_{1\alpha} = \begin{bmatrix} 0.1 & 0.3 \\ 0 & 0.2 \end{bmatrix}, \quad \bar{f}_1(\bar{x}_i) = \begin{bmatrix} x_{1,i}^2 \\ x_{2,i}x_{2,i} \end{bmatrix}, \tag{4.11a}$$

and

$$\bar{x}_{2,i} = \begin{bmatrix} x_{3,i} \\ x_{4,i} \end{bmatrix}, \quad N = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad \bar{f}_2(\bar{x}_i) = \begin{bmatrix} 2x_{4,i}^2 \\ x_{2,i}^2 \end{bmatrix}. \tag{4.11b}$$

The nonlinear system (4.10a), (4.10b) is positive since $A_{1\alpha} \in \Re_+^{2\times 2}, \bar{f}_1(\bar{x}_i) \in \Re_+^2$, $\bar{f}_2(\bar{x}_i) \in \Re_+^2$ for $\bar{x}_i \in \Re_+^2$, $u_i \in \Re_+$, $i \in Z_+$ and the conditions of Theorem 3.1 are satisfied.

Note that the linear part of the nonlinear system (4.11a) is asymptotically stable since the eigenvalues of the matrix $A_{1\alpha}$ are $z_1 = -0.1$, $z_2 = -0.2$.

The nonlinear subsystem (4.11b) is also asymptotically stable since the condition (4.9) is satisfied i.e.

$$\lim_{i\to\infty} \bar{f}_2(\bar{x}_i) = \lim_{i\to\infty} \begin{bmatrix} 2x_{4,i}^2 \\ x_{2,i}^2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{4.12}$$

5 Concluding Remarks

A method of analysis of the fractional descriptor nonlinear discrete-time systems described by the Eqs. (2.1a), (2.1b), (2.1c), (2.1d) with regular pencils (2.2) based on the Weierstrass-Kronecker decomposition of the pencil has been proposed. Necessary and sufficient conditions for the positivity of the nonlinear systems have been established (Theorem 3.1). A procedure for computing the solution to the Eqs. (2.1a), (2.1b), (2.1c), (2.1d) with given initial conditions and input sequences has been proposed and illustrated by numerical example. Using an extension of the Lyapunov method to positive nonlinear systems sufficient conditions for the asymptotic stability have been derived (Theorems 4.1, 4.2). The proposed method can be applied for example to analysis of descriptor nonlinear discrete-time electrical circuits. The considerations can be extended to fractional descriptor nonlinear discrete-time systems.

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Stability of Linear Discrete–Time Systems with Fractional Positive Orders

Małgorzata Wyrwas and Dorota Mozyrska

Abstract The problem of the stability of the Grünwald–Letnikov–type linear discrete-time systems with fractional positive orders is studied. The method of reducing the considered systems by transforming them to the multi-order linear systems with the partial orders from the interval $(0, 1]$ is presented. For the reduced multi-order systems the conditions for the stability are formulated based on the $\mathcal{Z}$-transform as an effective method for stability analysis of linear systems.

Keywords Grünwald–Letnikov–type difference operator, Discrete-time fractional-order systems, Stability

1 Introduction

The applications of fractional calculus can be found in different fields of science and engineering, see for example [2–5, 7, 10, 11] and references therein. Basic information on fractional calculus, ideas can be found for example in [6, 7, 10]. Recently, systems with fractional difference operators are discussed in many papers but usually their properties are presented for fractional orders from the interval $(0, 1]$. In the paper we study the stability of linear discrete-time systems with the Grünwald–Letnikov–type difference operator for any order $\alpha > 0$. Similarly as in the classical theory the $\mathcal{Z}$-transform can be used as an effective method for the stability analysis of linear fractional order difference systems, see for instance [1, 9, 12, 13]. In this paper we take into account fractional orders that are greater than one. We show that one can reduce the order of the considered systems by transforming them to the systems with the partial orders from the interval $(0, 1]$. Then the results given for linear difference systems with fractional order $\alpha \in (0, 1]$ can be used for the reduced

M. Wyrwas (✉) · D. Mozyrska
Faculty of Computer Science, Białystok University of Technology,
Wiejska 45A, 15-351 Białystok, Poland
e-mail: m.wyrwas@pb.edu.pl

D. Mozyrska
e-mail: mozyrska@pb.edu.pl

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multi-order systems and consequently, we get the conditions for stability of linear Grünwald–Letnikov difference systems with fractional orders $\alpha > 0$.

The paper is organized as follows. In Sect. 2 we gather some definitions, notations and results needed in the sequel. Section 3 contains the method of reduction of linear systems with positive higher orders to the multi-order systems with partial orders from the interval $(0, 1]$ and the stability analysis of linear Grünwald–Letnikov difference systems with positive fractional orders. Additionally, the examples illustrate our results. Finally, the conclusions are drawn.

2 Preliminaries

Let us recall that the $\mathcal{Z}$-transform of a sequence $\{y(n)\}_{n\in\mathbb{N}_0}$ is a complex function given by $Y(z) := \mathcal{Z}[y](z) = \sum_{k=0}^{\infty} y(k)z^{-k}$, where $z \in \mathbb{C}$ is a complex number for which the series $\sum_{k=0}^{\infty} y(k)z^{-k}$ converges absolutely. Note that since $\binom{k+\alpha-1}{k} = (-1)^k\binom{-\alpha}{k}$, then for $|z| > 1$ we have

$$\mathcal{Z}\left[\tilde{\varphi}_\alpha\right](z) = \sum_{k=0}^{\infty}\binom{k+\alpha-1}{k}z^{-k} = \left(\frac{z}{z-1}\right)^{\alpha}. \tag{1}$$

Definition 1 Let $\alpha \in \mathbb{R}$. The *Grünwald–Letnikov–type difference operator* ${}_a\Delta^\alpha$ *of order* α *for a function* $x : \mathbb{N}_a \to \mathbb{R}$ is defined by

$$\left({}_a\Delta^\alpha x\right)(t) := \sum_{k=0}^{t-a} c_k^{(\alpha)} x(t-k),$$

where $c_k^{(\alpha)} = (-1)^k\binom{\alpha}{k}$ with $\binom{\alpha}{k} = \begin{cases} 1 & \text{for} k = 0 \\ \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!} & \text{for} k \in \mathbb{N}. \end{cases}$

For $a = 0$ we write: $\Delta^\alpha :={}_0\Delta^\alpha$. It is important to observe that for $\alpha = 1$ we have that $\left({}_a\Delta^1 x\right)(t) := x(t)-x(t-1)$, that agrees with the classical nabla operator. In order to have the classical difference operator $\triangle$ the nabla operator should be composed with the shift operator δ, in the sense that $\left(\triangle x\right)(t) := x(t+1) - x(t) = (\delta\circ\nabla x)(t)$.

Proposition 1 ([9]) *For* $a \in \mathbb{R}$, $\alpha \in (0, 1]$ *let us define* $y(s) := \left({}_a\Delta^\alpha x\right)(t)$, *where* $t \in \mathbb{N}_a$ *and* $t = a + s$, $s \in \mathbb{N}_0$. *Then*

$$\mathcal{Z}\left[y\right](z) = \left(\frac{z}{z-1}\right)^{-\alpha} X(z), \tag{2}$$

where $X(z) = \mathcal{Z}[\overline{x}](z)$ *and* $\overline{x}(s) := x(a+s)$.

Usually there are considered the following fractional order systems of order $\alpha \in (0, 1]$ with the Grünwald-Letnikov–type difference operator:

$$(\Delta^{\alpha}x)(t+1) = Ax(t), \quad t \in \mathbb{N}_0, \tag{3}$$

with initial condition $x(0) = x_0 \in \mathbb{R}^n$, where $x = (x_1, \dots, x_n)^{\mathrm{T}} : \mathbb{N}_0 \to \mathbb{R}^n$ is a vector function and $A \in \mathbb{R}^{n \times n}$.

We consider also systems with multi-order $(\alpha) = (\alpha_1, \dots, \alpha_n)$, $\alpha_i \in (0, 1]$ in the following form:

$$\left(\Delta^{\alpha_i}x_i\right)(t+1) = \sum_{j=1}^{n} A_{ij}x_j(t), \quad t \in \mathbb{N}_0, \tag{4}$$

where $i = 1, \dots, n$, $n \in \mathbb{N}_1$ and with initial condition $(x_1(0), \dots, x_n(0))^{\mathrm{T}} = x_0 \in \mathbb{R}^n$. Taking into account Proposition 1 we can write the general result.

Proposition 2 ([9]) *Let* $\alpha \in (0, 1]$. *Then*

$$\mathcal{Z}\left[\left(\delta \circ \Delta^{\alpha_i}x_i\right)\right](z) = z\left(\left(\frac{z}{z-1}\right)^{-\alpha_i} X_i(z) - x_i(0)\right), \tag{5}$$

where $(\delta \circ f)(t) := f(t+1)$, $i = 1, \dots, n$, $n \in \mathbb{N}_1$, $X_i(z) = \mathcal{Z}[x_i](z)$.

Proposition 3 *Let* $\alpha_i \in (0, 1]$, $(\alpha) = (\alpha_1, \dots, \alpha_n)$ *and* Φ *be the fundamental matrix for system (4) such that* $x(t) = \Phi(t)x_0$ *is the solution of* (4) *with the initial condition* $x_0 \in \mathbb{R}^n$. *Then*

$$\mathcal{Z}[\Phi](z) = \left(\Lambda_{(\alpha)} - \frac{1}{z}A\right)^{-1}, \tag{6}$$

where $\Lambda_{(\alpha)} = diag\{\left(1 - \frac{1}{z}\right)^{\alpha_1}, \dots, \left(1 - \frac{1}{z}\right)^{\alpha_n}\}$.

Proof Taking the $\mathcal{Z}$–transform of each equation of system (4) and using formula (5) we get the following system of algebraic equations:

$$z\left(\left(\frac{z}{z-1}\right)^{-\alpha_i} X_i(z) - x_i(0)\right) = \sum_{j=1}^{n} A_{ij}X_j(z),$$

where $i = 1, \dots, n$ and $n \in \mathbb{N}_1$. Gathering the equations and writing in the matrix form we have

$$\left(zI_n - \Lambda_{(-\alpha)}A\right)X(z) = z\Lambda_{(-\alpha)}x_0,$$

where $\Lambda_{(\alpha)} = diag\{\left(1 - \frac{1}{z}\right)^{\alpha_1}, \dots, \left(1 - \frac{1}{z}\right)^{\alpha_n}\}$, $X(z) = \left[X_1(z) \dots X_n(z)\right]^{\mathrm{T}}$, $A = (A_{ij}) \in \mathbb{R}^{n \times n}$. Then

$$X(z) = \left(\Lambda_{(\alpha)} - \frac{1}{z}A\right)^{-1} x_0 = \mathcal{Z}\left[\Phi\right](z)x_0$$

for arbitrary initial condition $x_0 \in \mathbb{R}^n$. Hence we get the thesis.

3 Stability of Systems with Positive Order

We say that the constant vector $x^{\mathrm{eq}} = \left(x_1^{\mathrm{eq}}, \ldots, x_n^{\mathrm{eq}}\right)^{\mathrm{T}}$ is an *equilibrium point* of fractional difference system (4) if and only if

$$\left(\Delta^{\alpha_i} x_i^{\mathrm{eq}}\right)(t+1) = \sum_{j=1}^{n} A_{ij} x_j^{\mathrm{eq}},$$

where $i = 1, \ldots, n$ and $t \in \mathbb{N}_0$. Note that the trivial solution $x \equiv 0$ is an equilibrium point of system (4). Of course, if the determinant of the matrix $A = \left(A_{ij}\right) \in \mathbb{R}^{n\times n}$, is nonzero, then the systems (4) have only one equilibrium point $x^{\mathrm{eq}} = 0$.

Definition 2 The equilibrium point $x^{\mathrm{eq}} = 0$ of (4) is said to be

(a) *stable* if, for each $\epsilon > 0$, there exists $\delta = \delta(\epsilon) > 0$ such that $\|x(0)\| < \delta$ implies $\|x(t)\| < \epsilon$, for all $t \in \mathbb{N}_0$.
(b) *attractive* if there exists $\delta > 0$ such that $\|x(0)\| < \delta$ implies

$$\lim_{s\to\infty} x(s) = 0\,.$$

(iv) *asymptotically stable* if it is stable and attractive.

The fractional difference system (4) is called *stable/asymptotically stable* if their equilibrium points $x^{\mathrm{eq}} = 0$ are stable/asymptotically stable.

Proposition 4 *Let R be the set of all roots of the equation*

$$det\left(\Lambda_{(\alpha)} - \frac{1}{z}A\right) = 0\,, \tag{7}$$

where $\Lambda_{(\alpha)} = diag\{\left(1-\frac{1}{z}\right)^{\alpha_1}, \ldots, \left(1-\frac{1}{z}\right)^{\alpha_n}\}$. *Then the following items are satisfied.*

(a) If all elements from R are strictly inside the unit circle, then system (4) is asymptotically stable.
(b) If there is $z \in R$ such that $|z| > 1$, then system (4) is not stable.

Proof The proof is similar to those presented in [9]. Here we need to base the proof on the formula of $\mathcal{Z}$–transform of function $\Phi(\cdot)$ from Proposition 3.

The following comparison has been proven in [8]:

Proposition 5 *Let* $\alpha \in (0, 1]$. *Then* $\left(\nabla\left(\Delta^{-(1-\alpha)}x\right)\right)(s) = \left(\Delta^{\alpha}x\right)(s)$, *where* $s \in \mathbb{N}_0$ *and* $(\nabla x)(s) := x(s) - x(s-1)$.

Proposition 6 *Let $\alpha \in (q-1, q]$ and $\beta = \alpha - q + 1$. Then*

$$(\Delta^{\alpha} x)(t) = \left(\nabla^{q-1}\left(\Delta^{\beta} x\right)\right)(t), \tag{8}$$

where $t \in \mathbb{N}_0$.

Proof It is easily to show that for any positive order $\alpha > 0$ we have that

$$(\nabla(\Delta^{\alpha} x))(t) = \left(\Delta^{\alpha+1} x\right)(t).$$

This follows from the fact that $\binom{\alpha}{k} + \binom{\alpha}{k-1} = \binom{\alpha+1}{k}$. Hence for $\beta = \alpha - q + 1 \in (0, 1]$ we have that $(\Delta^{\alpha} x)(t) = \left(\nabla^{q-1}\left(\Delta^{\beta} x\right)\right)(t)$.

Observe that to have the uniqueness of the solution to the system with higher orders we need to introduce the following problem. Let $\alpha \in (q-1, q]$, $q \in \mathbb{N}_1$, then we need q initial conditions and we use the shift operator of order q:

$$(\Delta^{\alpha} x)(t+q) = Ax(t), \quad t \in \mathbb{N}_0, \tag{9}$$

with initial condition $x(0), x(1), \ldots, x(q-1) \in \mathbb{R}^n$, $A \in \mathbb{R}^{n \times n}$. Note that for natural values $\alpha = q$ we have that the Grünwald–Letnikov–type difference operator agree with classical q-fold application of difference operator $\triangle$, i.e.

$$(\Delta^{q} x)(t+q) = (\triangle^{q} x)(t) := \sum_{k=0}^{q} (-1)^{q-k} \binom{q}{k} x(t+k),$$

where $\triangle^{q} := \triangle \circ \cdots \circ \triangle$ is q-fold application of operator $\triangle$. Therefore we get $\delta^{q} \circ \Delta^{q} x = \triangle^{q} x$, where $q \in \mathbb{N}_1$ and $(\delta^{q} \circ \Delta^{q} x)(t) = (\Delta^{q} x)(t+q)$. Observe that $\left(\Delta^{1} x\right)(t+1) = (\triangle x)(t) = (\nabla x)(t+1)$.

We construct the condition for stability of linear discrete–time fractional systems with order $\alpha \in (q-1, q]$, where $q \in \mathbb{N}_1$. Our goal is to present given systems with positive orders and state $x \in \mathbb{R}^n$ into systems with multi–order that are partial orders from $(0, 1]$ and new state $y \in \mathbb{R}^{qn}$. Now, let us consider the following systems with positive order $\alpha \in (q-1, q]$:

$$(\Delta^{\alpha} x)(t+q) = A_1 x(t) + \sum_{k=1}^{q-1} A_{k+1} \left(\nabla^{k-1}\left(\Delta^{\beta} x\right)\right)(t+k), \tag{10}$$

where $\beta = \alpha - q + 1 \in (0, 1]$ and the operator Δ^{α} is defined by Definition 1. Additionally, for $q = 1$ we treat that in (10) the part with summation is equal to zero. Moreover, let $y_1(t) := x(t) \in \mathbb{R}^n$, $y_2(t) := \left(\Delta^{\beta} y_1\right)(t+1)$, and for $k = 2, \ldots, q-1$ let us define $y_{k+1} := \delta \circ \Delta^{1} y_k$. Then we write system (10) as system with multi–order $(\tilde{\alpha}) = (\beta, 1, \ldots, 1)$:

$$\begin{cases} \left(\Delta^{\beta} y_1\right)(t+1) &= y_2(t) \\ \left(\Delta^{1} y_2\right)(t+1) &= y_3(t) \\ & \vdots \\ \left(\Delta^{1} y_{q-1}\right)(t+1) &= y_q(t) \\ \left(\Delta^{1} y_q\right)(t+1) &= A_1 y_1(t) + \sum_{k=1}^{q-1} A_{k+1} y_{k+1}(t) \end{cases} \tag{11}$$

or equivalently in the matrix form

$$\left(\Delta^{\alpha_i} y_i\right)(t+1) = \sum_{j=1}^{n} \tilde{A}_{ij} y_j(t)\,, \quad t \in \mathbb{N}_0\,, \tag{12}$$

where $i = 1, \ldots, q$, $q \in \mathbb{N}_1$, $\alpha_1 = \beta$, $\alpha_k = 1$, $2 \leq k \leq q$ and then $\left(\Delta^{\alpha_i} y_i\right)(t+1) = \left(\nabla y_i\right)(t+1)$ and with initial condition $y_0 := (y_1(0), \ldots, y_q(0)) \in \mathbb{R}^{qn}$ and

$$\tilde{A} = \left(\tilde{A}_{ij}\right) = \begin{bmatrix} 0 & I_n & 0 & \ldots & 0 & 0 \\ 0 & 0 & I_n & \ldots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & I_n & 0 \\ 0 & 0 & 0 & \ldots & 0 & I_n \\ A_1 & A_2 & A_3 & \ldots & A_{q-1} & A_q \end{bmatrix}. \tag{13}$$

From Proposition 4 the following facts for systems with operators with higher orders follow immediately.

Proposition 7 *Let R be the set of all roots of the equation*

$$det\left(\Lambda_{(\tilde{\alpha})} I_{qn} - \frac{1}{z}\tilde{A}\right) = 0\,, \tag{14}$$

where $\tilde{A}$ is given by Eq. (13) and

$$\Lambda_{(\tilde{\alpha})} = diagblocks\left\{\left(1-\frac{1}{z}\right)^{\beta} I_n, \left(1-\frac{1}{z}\right) I_n, \ldots, \left(1-\frac{1}{z}\right) I_n\right\}.$$

Then the following items are satisfied.

(a) If all elements from R are strictly inside the unit circle, then system (10) is asymptotically stable.
(b) If there is $z \in R$ such that $|z| > 1$, then system (10) is not stable.

Additionally, we have:

Proposition 8 *For $\alpha \in (q-1, q]$, where $q \in \mathbb{N}_1$, $A_1 = A$ and $A_i = \mathbf{0}$ for $i = 2, \ldots, q$ and $q \geq 2$, let R be the set of all roots of the equation*

$$det\left(I_{nq}-\frac{1}{z}\Lambda_{(\tilde{\alpha})}\tilde{A}\right)=det\left(I_n-\frac{1}{z^q}\left(\frac{z}{z-1}\right)^{\alpha}A\right). \tag{15}$$

where $\tilde{A}$ is given by Eq. (13) and

$$\Lambda_{(\tilde{\alpha})}=diagblocks\left\{\left(1-\frac{1}{z}\right)^{\beta}I_n,\left(1-\frac{1}{z}\right)I_n,\dots,\left(1-\frac{1}{z}\right)I_n\right\}.$$

Then the following items are satisfied.

(a) If all elements from R are strictly inside the unit circle, then system (10) is asymptotically stable.
(b) If there is $z\in R$ such that $|z|>1$, then system (10) is not stable.

Proof For $\alpha\in(q-1,q]$, where $q\in\mathbb{N}_1$, $A_1=A$ and $A_i=\mathbf{0}$ for $i=2,\dots,q$ and $q\geq 2$, the Eq. (14) takes the following form:

$$det\begin{bmatrix} I_n & -\frac{1}{z}\frac{z}{z-1}I_n & \mathbf{0} & \dots & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & I_n & -\frac{1}{z}\frac{z}{z-1}I_n & \dots & \mathbf{0} & \mathbf{0}\\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \dots & I_n & -\frac{1}{z}\frac{z}{z-1}I_n\\ -\frac{1}{z}\left(\frac{z}{z-1}\right)^{\beta}A & \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} & I_n \end{bmatrix}=0. \tag{16}$$

Moreover, using also the operations on block matrices we receive that:

$$det\left(I_{nq}-\frac{1}{z}\Lambda_{(\tilde{\alpha})}\tilde{A}\right)=det\left(I_n-\frac{1}{z^q}\left(\frac{z}{z-1}\right)^{\alpha}A\right). \tag{17}$$

In the particular case when $(\Delta^{\alpha}x)(t+2)=\lambda x(t)$ the conditions for nonstability of this system are presented in the following example.

Example 1 Let us consider the case with $n=1, q=2$, so $\alpha\in(1,2]$ and with $A_1=\lambda$ and $A_2=0$. Then the condition (14) takes the form

$$1-\frac{1}{z^2}\left(\frac{z}{z-1}\right)^{\alpha}\lambda=0.$$

Moreover, then $\lambda=z^2\left(1-\frac{1}{z}\right)^{\alpha}$. Hence we can easily conclude that if

$$|\lambda|>2^{\alpha},$$

then the system is not stable.

Now, let us present a numerical example that illustrates our results.

Example 2 Let us take $n = 1$, $q = 2$ and $\alpha \in (1, 2]$, and consider system $(\Delta^{\alpha} x)(t + 2) = \lambda x(t)$. Then $\beta = \alpha - 1$, hence $\beta \in (0, 1]$. Let $y_1(t) := x(t) \in \mathbb{R}$, and $y_2(t) := \left(\Delta^{\beta} y_1\right)(t + 1)$. Then the considered system can be written as a system with multi–order $(\tilde{\alpha}) = (\beta, 1)$:

$$\begin{cases} \left(\Delta^{\beta} y_1\right)(t+1) = y_2(t) \\ \left(\Delta^{1} y_2\right)(t+1) = \lambda y_1(t) \end{cases} \tag{18}$$

with initial condition

$$y(0) = \begin{bmatrix} x(0) \\ \left(\Delta^{\beta} x\right)(0) \end{bmatrix} \in \mathbb{R}^2 .$$

By Proposition 7 we get that the stability of the considered systems depends on order α. Figures 1, 2 and 3 present the phase trajectories for the considered systems with different fractional positive order α. Solving system (18) we use the recurrence representation of it

$$\begin{cases} y_1(t+1) = y_2(t) + \sum_{i=0}^{t} (-1)^{t-i} \binom{\beta}{t+1-i} y_1(i) \\ y_2(t+1) = y_2(t) + \lambda y_1(t) \end{cases} \tag{19}$$

for $t \in \mathbb{N}_0$. For $\lambda = -0.5$, the numerically checked highest value of α that confirms stability is $\alpha = 1.48$. And oppositely the biggest $|\lambda|$ that preserves stability for $\alpha = 1.5$ is 0.48, where at Fig. 2 we use $\lambda = -0.47$ to have no doubts that we see that solution is stable.

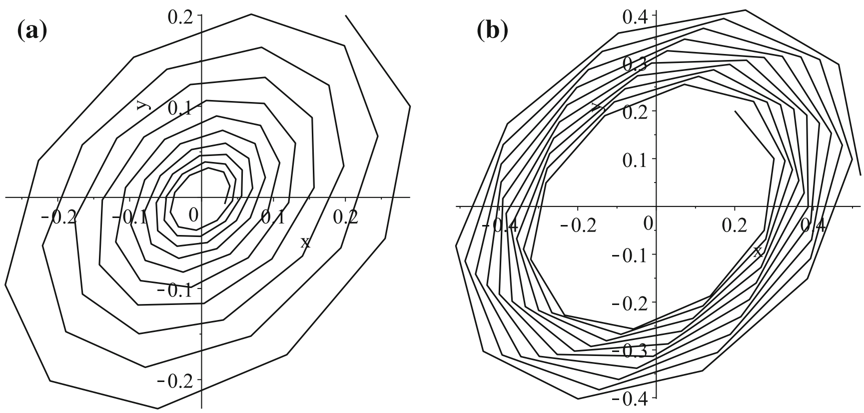

Fig. 1 The solution of the initial value problem for the system (18) with initial condition $x(0) = 0.2$, $y_2(0) = 0.2$, and $\lambda = -0.5$. **a** Phase trajectory—stable, $\alpha = 1.45$. **b** Phase trajectory—unstable, $\alpha = 1.5$

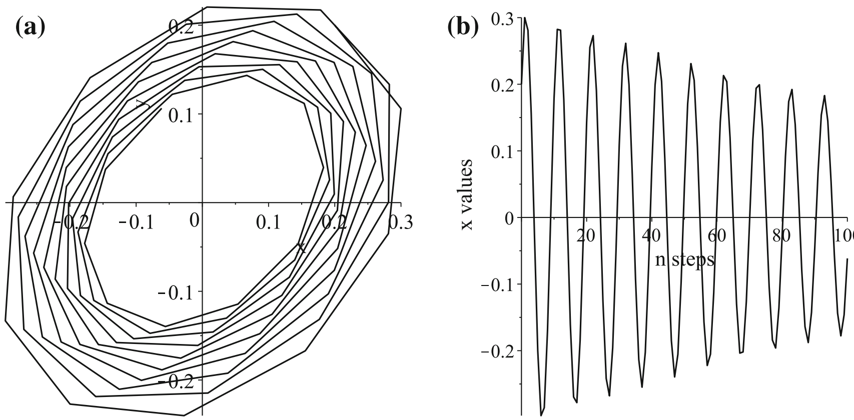

Fig. 2 The solution of the initial value problem for the system (18) with initial condition $x(0) = 0.2$, $y_2(0) = 0.2$, and $\lambda = -0.47$. **a** Phase trajectory—stable, $\alpha = 1.5$. **b** The graph of x for $n = 100$ steps, $\alpha = 1.5$

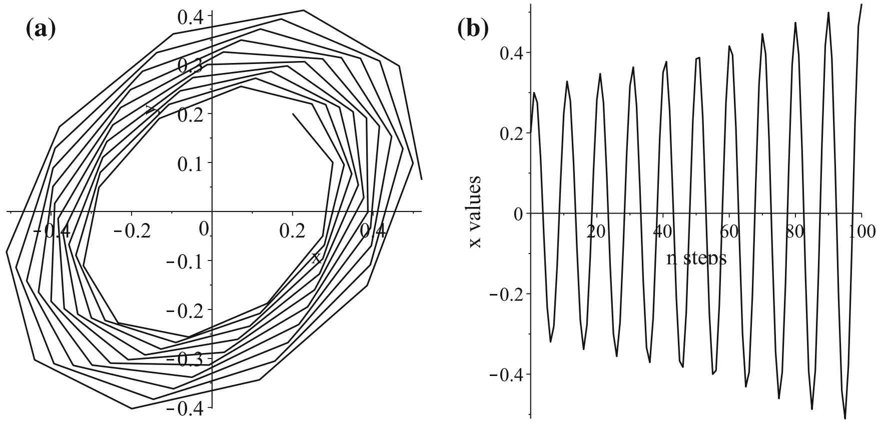

Fig. 3 The solution of the initial value problem for the system (18) with initial condition $x(0) = 0.2$, $y_2(0) = 0.2$, and $\lambda = -0.5$. **a** Phase trajectory—unstable, $\alpha = 1.5$. **b** The graph of x for $n = 100$ steps, $\alpha = 1.5$

4 Conclusions

The stability of the Grünwald–Letnikov–type linear discrete-time systems with fractional positive orders is discussed. An effective method for stability analysis of linear discrete-time systems is the $\mathcal{Z}$-transform, so in the case of considered fractional systems we also use this method. We present the method of transforming the considered systems with arbitrary positive orders to the systems with multi-orders where the partial orders are from the interval $(0, 1]$. It is possible to extend the presented results for linear fractional order systems with various step h. Additionally, one can compare the stability of systems with continuous–time fractional operators with their discretization.

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Robust H_∞ Observer-Based Stabilization of Disturbed Uncertain Fractional-Order Systems Using a Two-Step Procedure

Yassine Boukal, Michel Zasadzinski, Mohamed Darouach and Nour-Eddine Radhy

Abstract The main objective of this work is the problem of robust H_∞ observer-based stabilization for a class of linear Disturbed Uncertain Fractional-Order Systems (DU-FOS) by using H_∞-norm optimization. Based on the H_∞-norm analysis for FOS, a new design methodology is established to stabilize a linear DU-FOS by using robust H_∞ Observer-Based Control (OBC). The existence conditions are derived, and by using the H_∞-optimization technique, the stability of the estimation error and stabilization of the original system are given in an inequality condition, where all the observer matrices gains and the control law can be computed by solving a single inequality condition in two step. Finally, a simulation example is given to illustrate the validity of the results.

Keywords Fractional-order system · Observer-based control · H_∞-norm, H_∞ Observer design

1 Introduction

For past centuries, fractional calculus has been a very interesting topic, but just for mathematicians, to make the subject well understandable for engineers or scientists point of view [1–4]. Only in the last decades, fractional calculus have been caught much attention, because it has been shown that non-integer models can be both theoretically challenging and pertinent for many fields of science and technology such

Y. Boukal · M. Zasadzinski (✉) · M. Darouach
Centre de Recherche En Automatique de Nancy (CRAN - UMR 7039, CNRS),
Université de Lorraine, IUT de Longwy, 186 rue de Lorraine, 54400
Cosnes et Romain, France
e-mail: Michel.zasadzinski@univ-lorraine.fr

Y. Boukal · N.-E. Radhy
Faculté des Sciences Ain-Chock Laboratoire Physique et Matériaux Microélectronique
Automatique et Thermique (LPMMAT), Université Hassan II, BP: 5366 Maarif, 20100
Casablanca, Morocco

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as chemistry [5–7], biology [8, 9], economics [10, 11], psychology [12, 13], mass diffusion, heat conduction, physical and engineering applications [14, 15] etc.

In some real models, output feedback control may not guarantee the stability of the closed loop, because the system output measurements do not provide a complete information on the internal state of the system. For this reason, observer design for estimating the states of a system has received considerable attention in the past [16–18]. Observer-based controllers are generally used to stabilize unstable systems or to improve the system performances. Recently, the research activities for the OBC have been developed for FOS [19–21].

The H_∞ theory was generally restricted to integer-order systems. In the recent years, some works about the extension of the H_∞-norm computation to FOS have been appeared [22], where the authors defined the pseudo Hamiltonian matrix of a fractional order system, and they proposed two methods to compute FOS H_∞-norm based on this pseudo Hamiltonian matrix. The first one was a dichotomy algorithm and the second one used LMI formalism. Based on these analysis results, methods to design H_∞ state feedback controllers and H_∞ observer were proposed in [23–26].

The main idea of our study is to design robust H_∞ observer-based controllers for DU-FOS, affected by disturbances that are supposed to have finite energy. After stating OBCs design objectives, our results are given in matrix inequalities. Firstly, the existence conditions of the OBC of such systems are given. In the second section, by using the H_∞-optimization technique, the stability of the estimation error and stabilization of the original system are given in inequality condition, where all the observer gains and the control law can be computed by solving this inequality condition in two step. The method proposed have two objectives, the first one, satisfies the H_∞ performance index, and the second one is the stabilization of the DU-FOS.

Notation: $\mathbb{R}^n$ and $\mathbb{R}^{n\times m}$ denote the n dimensional Euclidean space and the set of all $n \times m$ real matrices, respectively; A^T and A^* denote the transpose and the conjugate transpose of matrix A, respectively; matrix A is symmetric positive definite if and only if $A^T = A$ and $A > 0$; A^+ means the generalized inverse of matrix A which satisfies $AA^+A = A$; $\|\|_\infty$ is the H_∞ norm; I and 0 denote the identity matrix and zero matrix, respectively, of appropriate dimension. $Sym\{X\}$ is used to denote $X^* + X$. The notation $(*)$ is the conjugate transpose of the off-diagonal part.

2 Preliminaries

The fractional-order derivative definition introduced by Caputo for a function $f(t)$ can be given as [27]

$$ {}_a^C D_t^\alpha f(t) = \frac{1}{\Gamma(\alpha - n)} \int_a^t \frac{f^n(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau \quad , \qquad (n-1) < \alpha < n \tag{1} $$

with $n \in \mathbb{N}^*$ and $\alpha \in \mathbb{R}^+$, where $\Gamma(.)$ is the Gamma function.

Consider the following linear fractional-order system

$$\begin{cases} D^\alpha x(t) = A_0 x(t) + B_0 u(t) \\ y(t) = C_0 x(t) + D_0 u(t) \end{cases}, \qquad 0 < \alpha < 2 \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the control input vector, $y(t) \in \mathbb{R}^p$ is the measured output and D^α is used to denote the fractional derivative of order α. A_0, B_0, C_0 and D_0 are known constant matrices and with appropriate dimensions.

Two tools of H_∞ norm computation for FOS were firstly proposed in [22], and the calculation of L_2-gain for FOS was initially presented in [22] and it was extended to give a new formulation in [28]. The following lemmas show these computations.

Lemma 1 ([22]) *H_∞-norm of a fractional-order system $G = (A, B, C, D, \alpha)$ is bounded by a real positive number γ if and only if the eigenvalues of matrix A_γ lie in the stable domain defined by $s \in C : |arg(s)| > \alpha\frac{\pi}{2}$.*

$$A_\gamma = \begin{pmatrix} (A + BRD^T C) & \Xi BRB^T \\ C^T(I + DRD^T)C & \Xi(A^T + C^T DRB^T) \end{pmatrix} \tag{3}$$

where

$$\Xi = e^{-\alpha j\pi} \quad \text{and} \quad R = (\gamma^2 I - D^T D)^{-1}$$

Lemma 2 ([28]) *For a LTI-FOS $G = (A_0, B_0, C_0, D_0, \alpha)$, the L_2-gain is bounded by γ, if there exists a positive definite Hermitian matrix P, such that*

$$\Gamma_1 = \begin{bmatrix} \Xi_1 PA_0 + \Xi_1^* A_0^T P & PB_0 & \Xi_1^* C_0^T \\ B_0^T P & -\gamma^2 I & D_0^T \\ \Xi_1 C_0 & D_0 & -I \end{bmatrix} < 0 \tag{4}$$

with

$$\Xi_1 = e^{(1-\alpha)j\pi/2} \text{and} \Xi_1^* = e^{-(1-\alpha)j\pi/2}$$

Lemma 3 ([29]) *Let D, E and F be real matrices of appropriate dimensions and F satisfies $F^T F \leqslant I$. Then for any scalar $\epsilon > 0$ and vectors $x, y \in \mathbb{R}^n$, we have*

$$2x^T DFEy \leqslant \epsilon^{-1} x^T DD^T x + \epsilon y^T E^T Ey \tag{5}$$

3 Main Results

3.1 H_∞-norm Computation for FOS with Uncertainties

Let us consider the following linear systems with uncertainties and disturbances:

$$\begin{cases} D^\alpha x(t) = (A + \Delta A)x(t) + (B_w + \Delta B_w)w(t) \\ y(t) = (C + \Delta C)x(t) + (D_w + \Delta D_w)w(t) \end{cases}, \qquad 0 < \alpha < 2 \tag{6}$$

The terms ΔA, ΔB_w, ΔC, and ΔD_w are unknown matrices representing time-varying parameter uncertainties.

Assumption 1 In this paper, we will consider the following structure for the uncertainties

$$\begin{bmatrix} \Delta A & \Delta B_w \\ \Delta C & \Delta D_w \end{bmatrix} = MF(t) \begin{bmatrix} N_A & N_{B_w} \\ N_C & N_{D_w} \end{bmatrix} \tag{7}$$

where M, N_A, N_{B_w}, N_C, and N_{D_w} are known real constant matrices and of appropriate dimensions and $F(t)$ is an unknown real-valued time-varying matrix satisfying

$$F^T(t)F(t) \leq I \tag{8}$$

when the elements of $F(t)$ are Lebesgue measurable.

Lemma 4 *The L_2-gain of an uncertain LTI-FOS (6) is bounded by γ, if there exists a positive definite Hermitian matrix P and two positive scalars μ_1 and μ_2 such that*

$$\begin{bmatrix} \overbrace{\begin{bmatrix} Sym\{\Xi_1 PA\} & PM & PB_w + \frac{1}{\mu_1} N_A^T N_{B_w} \\ * & -\mu_1 I & 0 \\ * & * & -\gamma^2 I \end{bmatrix}}^{\Theta_{11}} & \overbrace{\begin{bmatrix} N_A^T & N_C^T & 0 & 0 & \Xi_1^* I^T & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & N_{B_w} & N_{D_w} & D_w & 0 \end{bmatrix}}^{\Theta_{12}} \\ * & \overbrace{\begin{bmatrix} \frac{1}{\mu_1} I & 0 & 0 & 0 & 0 & 0 \\ * & -\frac{1}{\mu_2} I & 0 & 0 & 0 & 0 \\ * & * & -\frac{1}{\mu_1} I & 0 & 0 & 0 \\ * & * & * & -\frac{1}{\mu_2} I & 0 & 0 \\ * & * & * & * & -I & 0 \\ * & * & * & * & * & -\mu_2 I \end{bmatrix}}^{\Theta_{22}} \end{bmatrix} < 0 \tag{9}$$

with

$$\Xi_1 = e^{(1-\alpha)j\pi/2}, \Xi_1^* = e^{-(1-\alpha)j\pi/2}, \Xi_1 \times \Xi_1^* = 1$$

Proof By letting $A_0 = A + \Delta A$, $B_0 = B_w + \Delta B_w$, $C_0 = C + \Delta C$ and $D_0 = D_w + \Delta D_w$, the inequality (4) can be written as

$$\Gamma_1 = \Omega_1 + Sym\{X_1 F(t) Y_1 + X_2 F(t) Y_2\} < 0 \tag{10}$$

with

$$\Omega_1 = \begin{bmatrix} \Xi_1 PA + \Xi_1^* A^T P & PB & \Xi_1^* C^T \\ B^T P & -\gamma^2 I & D^T \\ \Xi_1 C & D & -I \end{bmatrix} \quad X_1 = \begin{bmatrix} PM \\ 0 \\ 0 \end{bmatrix} \quad X_2 = \begin{bmatrix} 0 \\ 0 \\ M \end{bmatrix}$$

$$Y_1 = \begin{bmatrix} \Xi_1 N_A & N_{B_w} & 0 \end{bmatrix} \qquad Y_2 = \begin{bmatrix} \Xi_1 N_C & N_{D_w} & 0 \end{bmatrix}$$

According to Lemma 3, we obtain the following inequality

$$\Gamma_1 < \hat{\Gamma}_1 = \Omega_1 + \mu_1 X_1 X_1^T + \mu_1^{-1} Y_1^T Y_1 + \mu_2 X_2 X_2^T + \mu_2^{-1} Y_2^T Y_2 \tag{11}$$

One can see from the above results that the inequality (10) is verified, if there exists two positive scalars μ_1 and μ_2 such that

$$\Omega_1 + \mu_1 X_1 X_1^T + \mu_1^{-1} Y_1^T Y_1 + \mu_2 X_2 X_2^T + \mu_2^{-1} Y_2^T Y_2 < 0 \tag{12}$$

Then, the inequality (12) is equivalent to

$$\hat{\Gamma}_1 = \Omega_1 + \begin{bmatrix} \frac{1}{\mu_1} N_A^T N_A + \frac{1}{\mu_2} N_C^T N_C + \mu_1 PM(PM)^T & \frac{1}{\mu_1} N_A^T N_{B_w} + \frac{1}{\mu_2} N_C^T N_{D_w} & 0 \\ * & \frac{1}{\mu_1} N_{B_w}^T N_{B_w} + \frac{1}{\mu_2} N_{D_w}^T N_{D_w} & 0 \\ * & * & \mu_2 MM^T \end{bmatrix} < 0 \tag{13}$$

According to Schur complement, one can see that the equivalence between the above inequality (13) and the the following

$$\begin{bmatrix} Sym\{\Xi_1 PA\} & N_A^T & N_C^T & PM & PB_w + \frac{1}{\mu_1} N_A^T N_{B_w} + \frac{1}{\mu_2} N_C^T N_{D_w} & 0 & 0 & \Xi_1^* C^T & 0 \\ * & -\frac{1}{\mu_1} I & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & -\frac{1}{\mu_2} I & 0 & 0 & 0 & 0 & 0 & 0 \\ * & * & * & -\mu_1 I & 0 & 0 & 0 & 0 & 0 \\ * & * & * & * & -\gamma^2 I & N_{B_w}^T & N_{D_w}^T & D_w^T & 0 \\ * & * & * & * & * & -\frac{1}{\mu_1} I & 0 & 0 & 0 \\ * & * & * & * & * & * & -\frac{1}{\mu_2} I & 0 & 0 \\ * & * & * & * & * & * & * & -I & M \\ * & * & * & * & * & * & * & * & -\mu_2 I \end{bmatrix} < 0 \tag{14}$$

We can easily deduce by permuting some rows and columns, the equivalence between the inequalities (9) and (14).

One can see from the above results that a sufficient condition for L_2-gain of the LTI-FOS (6) to be bounded by γ is that if there exist a positive definite Hermitian matrix P and two positive scalars μ_1 and μ_2 such that LMI (9) must hold, which completes the proof. □

3.2 Robust H_∞ Observer-Based Controller Parametrization

Without loss of generality, we consider the DU-FOS represented by the following form:

$$\begin{cases} D^\alpha x(t) = (A + \Delta A)x(t) + B_u u(t) + B_w w(t) \\ y(t) = Cx(t) \end{cases} \tag{15}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $u(t) \in \mathbb{R}^m$ is the input vector, $y(t) \in \mathbb{R}^q$ the measurement output vector and $w(t) \in \mathbb{R}^p$ is the disturbance input vector. A, B_u, B_w and C are known matrices of appropriate dimensions. The term ΔA is an unknown matrix uncertainty, assumed to respect the Assumption 1.

Assumption 2 We assume that $w(t) \in L_2$, where the L_2-norm is defined as

$$||w||_{L_2} = \left(\int_0^{\infty} w(t)^T w(t) \mathrm{d}t \right)^{\frac{1}{2}} \tag{16}$$

To allow the stabilization of system using measurement feedback, we need to reconstruct the state variable. For this, we consider a robust H_∞ observer with a linear feedback control law of the form

$$\begin{cases} D^\alpha \eta(t) = N\eta(t) + Hu(t) + Jy(t) \\ \quad \hat{x}(t) = \eta(t) + Ey(t) \\ \quad u(t) = K_u \hat{x}(t) \end{cases} \tag{17}$$

where $\eta(t) \in \mathbb{R}^n$ is the state vector of robust H_∞ observer, $\hat{x}(t) \in \mathbb{R}^n$ is the estimate of $x(t)$ and $u(t) \in \mathbb{R}^m$ is the feedback control law. The matrices N, J, H and E are observer unknown matrices of appropriate dimensions which must be determined, such that $\hat{x}(t)$ converges asymptotically to $x(t)$ for $w(t) = 0$ and $\frac{||e||_2}{||w||_2} < \gamma$, for $w(t) \neq 0$. The matrix $K_u \in \mathbb{R}^{m \times n}$ is the controller gain to be determined.

Before designing the robust H_∞ observer, the estimation error is defined as

$$e(t) = x(t) - \hat{x}(t) \tag{18}$$

and has the fractional-order dynamic

$$D^\alpha e(t) = D^\alpha x(t) - D^\alpha \hat{x}(t) \tag{19a}$$

or equivalently

$$\begin{aligned} D^\alpha e(t) = Ne(t) + (RA - NR - JC + R\Delta A)x(t) \\ + (RB_u - H)u(t) + RB_w w(t) \end{aligned} \tag{19b}$$

where $R = I_n - EC$.

We can easily deduce that the input control law stabilizing the system (15) has the following form

$$u(t) = K_u x(t) - K_u e(t) \tag{20}$$

The system (15) can be rewritten as

$$D^\alpha x(t) = (A + B_u K_u + \Delta A)x(t) - B_u K_u e(t) + B_w w(t) \tag{21}$$

System (15) and estimation error (19b) can be combined in the following augmented system

$$D^\alpha \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = \left(\begin{bmatrix} A + B_u K_u & -B_u K_u \\ \Theta_1 & N \end{bmatrix} + \begin{bmatrix} \Delta A & 0 \\ R\Delta A & 0 \end{bmatrix} \right) \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} B_w \\ RB_w \end{bmatrix} w(t) + \begin{bmatrix} 0 \\ \Theta_2 \end{bmatrix} u(t) \tag{22}$$

where

$$\Theta_1 = NR + JC - RA$$
$$\Theta_2 = H - RB_u$$

Problem 1 Get, if possible, an OBC (17), i.e. determine all the observer and controller gain matrices N, J, H, E and K_u of appropriate dimensions such that the uncertain system (15) is stabilized for all initial states values.

Proposition 1 *System* (17) *is a robust H_∞ observer-based controller of the system* (15)*, with respect to the Assumption* 2 *and for any finite* $x(0)$ *and* $\hat{x}(0)$ *if*

(i) *The L_2-gain of the augmented system* (22) *is bounded by* $\gamma > 0$.
(ii) $NR + JC - RA = 0$
(iii) $H = RB_u$

where $R = I_n - EC$.

By using the definition of R, the expression of Θ_1 can be rewritten as

$$N + ECA + KC = A \tag{23a}$$

where $K = J - NE$.

Now, equation (23a) can be written as

$$\begin{bmatrix} N & K & E \end{bmatrix} \mathcal{M}_1 = \mathcal{M}_2 \tag{24}$$

where

$$\mathcal{M}_1 = \begin{bmatrix} I_n \\ C \\ CA \end{bmatrix} \text{ and } \mathcal{M}_2 = \begin{bmatrix} A \end{bmatrix} \tag{25}$$

The necessary and sufficient condition for the existence of the solution of (24) can be given by the following lemma.

Lemma 5 ([30]) *There exists a solution to* (24) *if and only if*

$$rank \begin{bmatrix} \mathcal{M}_1 \\ \mathcal{M}_2 \end{bmatrix} = rank \begin{bmatrix} \mathcal{M}_1 \end{bmatrix} \ . \tag{26}$$

If (26) is satisfied, the general solution of (24) is given by

$$\begin{bmatrix} N \ K \ E \end{bmatrix} = \mathcal{M}_2 \mathcal{M}_1^+ - Z(I - \mathcal{M}_1 \mathcal{M}_1^+) \tag{27}$$

where $\mathcal{M}_1^+$ is a generalized inverse matrix of $\mathcal{M}_1$ [30] (i.e. $\mathcal{M}_1 = \mathcal{M}_1 \mathcal{M}_1^+ \mathcal{M}_1$) and Z is an arbitrary matrix of appropriate dimension.

From (28a), we obtain

$$N = \mathbb{A}_N - Z\mathbb{B}_N \tag{28a}$$
$$K = \mathbb{A}_K - Z\mathbb{B}_K \tag{28b}$$
$$E = \mathbb{A}_E - Z\mathbb{B}_E \tag{28c}$$

where

$$\mathbb{A}_N = (\mathcal{M}_2 \mathcal{M}_1^+) \begin{bmatrix} I \ 0 \ 0 \end{bmatrix}^T \qquad \mathbb{B}_N = (I - \mathcal{M}_1 \mathcal{M}_1^+) \begin{bmatrix} I \ 0 \ 0 \end{bmatrix}^T$$
$$\mathbb{A}_K = (\mathcal{M}_2 \mathcal{M}_1^+) \begin{bmatrix} 0 \ I \ 0 \end{bmatrix}^T \qquad \mathbb{B}_K = (I - \mathcal{M}_1 \mathcal{M}_1^+) \begin{bmatrix} 0 \ I \ 0 \end{bmatrix}^T$$
$$\mathbb{A}_E = (\mathcal{M}_2 \mathcal{M}_1^+) \begin{bmatrix} 0 \ 0 \ I \end{bmatrix}^T \qquad \mathbb{B}_E = (I - \mathcal{M}_1 \mathcal{M}_1^+) \begin{bmatrix} 0 \ 0 \ I \end{bmatrix}^T$$

Matrices J and H are obtained from

$$\begin{cases} J = K + NE \\ H = (I_n - EC)B \end{cases} \tag{29}$$

By using this results, all the parameters of the Robust H_∞ fractional-order observer (17) can be computed if matrix parameter Z is known.

3.3 Robust H_∞ Observer-Based Controller Design

Now, if conditions (ii and iii) in Proposition 1 are satisfied, then the augmented system (22) can be expressed as

$$D^\alpha \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} = \left(\begin{bmatrix} A + B_u K_u & -B_u K_u \\ 0 & N \end{bmatrix} + \begin{bmatrix} \Delta A & 0 \\ R\Delta A & 0 \end{bmatrix} \right) \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} + \begin{bmatrix} B_w \\ RB_w \end{bmatrix} w(t) \tag{30}$$

or equivalently

$$D^\alpha \widetilde{X}(t) = \left(\widetilde{A} + \Delta\widetilde{A}\right)\widetilde{X}(t) + \widetilde{B}w(t) \tag{31}$$

with

$$\widetilde{X}(t) = \begin{bmatrix} x(t) \\ e(t) \end{bmatrix} \qquad \widetilde{A} = \begin{bmatrix} A + B_u K_u & -B_u K_u \\ 0 & N \end{bmatrix} \qquad \widetilde{B} = \begin{bmatrix} B_w \\ RB_w \end{bmatrix}$$

$$\Delta\widetilde{A} = \widetilde{M}F(t)\widetilde{N}_A \qquad \widetilde{M} = \begin{bmatrix} M \\ RM \end{bmatrix} \qquad \widetilde{N}_A = \begin{bmatrix} N_A & 0 \end{bmatrix}$$

Lemma 6 *System (17) is a robust H_∞ observer-based controller of the uncertain disturbed system (15) with disturbance attenuation given by γ, if there exist two positive definite hermitian matrices P_1 and P_2, two matrices X_1 and X_2, and a positive scalar μ_1, such that the following matrix inequality holds*

$$\begin{bmatrix} \Omega_{11} & \Omega_{12} \\ * & \Omega_{22} \end{bmatrix} < 0 \tag{32}$$

where

$$\Omega_{11} = \begin{bmatrix} Sym\{\Xi_1(AP_1 + B_u X_1)\} & -B_u K_u \\ * & Sym\{\Xi_1(P_2\mathbb{A}_N - X_2\mathbb{B}_N)\} \end{bmatrix}$$

$$\Omega_{12} = \begin{bmatrix} \begin{bmatrix} M \\ P_2\mathbb{A}_R M + X_2\mathbb{B}_R M \end{bmatrix} & \begin{bmatrix} P_1^{-1}N_A^T \\ 0 \end{bmatrix} & \begin{bmatrix} B_w \\ P_2\mathbb{A}_R B_w + X_2\mathbb{B}_R B_w \end{bmatrix} & \Xi_1^* I \end{bmatrix}$$

$$\Omega_{22} = \begin{bmatrix} -\mu_1 I & 0 & 0 & 0 \\ * & -\frac{1}{\mu_1}I & 0 & 0 \\ * & * & -\gamma^2 I & 0 \\ * & * & * & -I \end{bmatrix}$$

with

$$\mathbb{A}_R = I_n - \mathbb{A}_2 C \qquad X_1 = K_u P_1^{-1}$$

$$\mathbb{B}_R = \mathbb{B}_2 C \qquad X_2 = P_2 Z$$

Proof The detailed proof of this result is omitted for space limitation. However, the proof can obtained by replacing all system matrices A, B_w, N_A and M in the inequality (9) by their expression given in the augmented system (30) $\widetilde{A}$, $\widetilde{B}_w$, $\widetilde{N}_A$ and $\widetilde{M}$, respectively. Then, the inequality (32) is obtained by pre and post multiplication of the above inequality with the following matrices, respectively $\begin{bmatrix} P_{01}^{-1} & 0 \\ 0 & I \end{bmatrix}$ and $\begin{bmatrix} P_{01}^{-1} & 0 \\ 0 & I \end{bmatrix}^T$, with $P_{01}^{-1} = P_1$. Which complete the proof. □

We can remark that the robust H_∞ observer based control problem given by the inequality (32) is a non convex problem. The product between the two decision matrices P_1 and K_u, and the presence of matrices P_1 and its inverse P_1^{-1} leading to a bilinear matrix inequality (BMI) structure. Then, the inequality (32) can not be solved for $(P_1, P_2, X_1, X_2, K_u)$ in the same time. According the Schur lemma, all diagonal component must satisfy the inequality. Therefore, we propose to resolve this problem in two step. Firstly, we start by solving the first component in Ω_{11}. After obtaining P_1 and X_1, replacing them into the inequality (32) by their value leads to a feasible LMI.

4 Numerical Example

In this section, the performance of the proposed robust H_∞ observer-based stabilizing controller is presented via a numerical example.

Consider the disturbed uncertain fractional-order system described by

$$\begin{cases} D^{1.5}x(t) = \left(\begin{bmatrix} 0 & 10 \\ 15 & -20 \end{bmatrix} + \Delta A\right) x(t) + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} u(t) + \begin{bmatrix} 1 \\ 0.25 \end{bmatrix} w(t) \\ y(t) = \begin{bmatrix} 1 & 0 \end{bmatrix} x(t) \end{cases} \tag{33}$$

The uncertainty matrix ΔA is given as

$$\Delta A = MF(t)N_A = \begin{bmatrix} 0.1 & -0.5 \\ 0.25 & 0.4 \end{bmatrix} F(t) \begin{bmatrix} -0.1 & 0 \\ 0 & 0.5 \end{bmatrix} \tag{34}$$

with $F(t) = \text{diag}(0.15\sin(25t), 0.15\sin(25t))$.

The inequality condition (32) can be solved by using a two-step procedure. The obtained results for $\gamma = 0.41$ are given by

$$P_1 = \begin{bmatrix} 375.02 & -27.3 \\ -27.3 & 18.9 \end{bmatrix}, X_1 = \begin{bmatrix} -12342 & 970.71 \end{bmatrix} P_2 = \begin{bmatrix} 20449 & -0.025 \\ -0.025 & 26.4 \end{bmatrix},$$
$$X_2 = \begin{bmatrix} 14460 & 2024.2 & -14460 & -202.37 \\ 198.19 & 568.19 & -198.18 & -56.54 \end{bmatrix}$$

Finally, the dynamic of the estimate $\hat{x}(t)$ and the controller law are given by the following observer

$$\begin{cases} D^{1.5}\eta(t) = \begin{bmatrix} -0.7 & -3e-06 \\ -0.01 & -21.7 \end{bmatrix} \eta(t) + \begin{bmatrix} 0 \\ 0.5 \end{bmatrix} u(t) \\ \qquad + \begin{bmatrix} -7e-07 \\ 11.2 \end{bmatrix} y(t) \\ \hat{x}(t) = \eta(t) + \begin{bmatrix} 1 & 0.2 \end{bmatrix} y(t) \\ u(t) = \begin{bmatrix} -32.6 & 4.3 \end{bmatrix} \hat{x}(t) \end{cases} \tag{35}$$

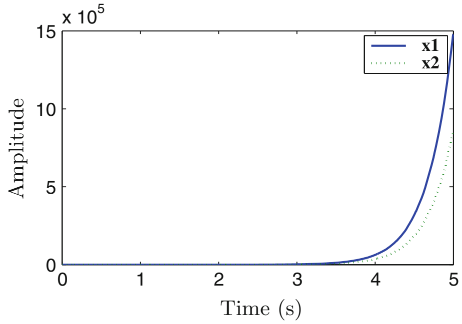

Fig. 1 States trajectories $x_1(t)$ and $x_2(t)$ of the open loop unstable systems (33)

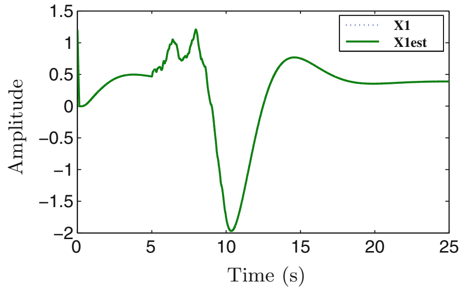

Fig. 2 Evolution of the vector $x_1(t)$ and its estimate $\hat{x}_1(t)$

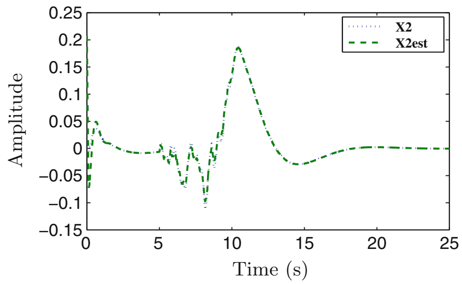

Fig. 3 Evolution of the vector $x_2(t)$ and its estimate $\hat{x}_2(t)$

Figure 1 show the open loop response of the unstable system. In addition, (Figs. 2, 3, 4 and 5) show the performances in the time domain of the proposed robust H_∞ OBC to ensure the stability of the DU-FOS in the closed loop. The actual states are shown with their estimates, and the estimation errors obtained by using the proposed method. It is clear that the estimate state $\hat{x}(t)$ converges to the actual state $x(t)$.

One can see in (Figs. 2, 3, 4 and 5) that disturbance $w(t)$ is activated in the time interval between five and ten seconds. In this time interval, the estimate $\hat{x}(t)$ tracks the actual state $x(t)$ with a small error. This is in agreement with the small value of the disturbance attenuation criterion given by $\gamma = 0.41$.

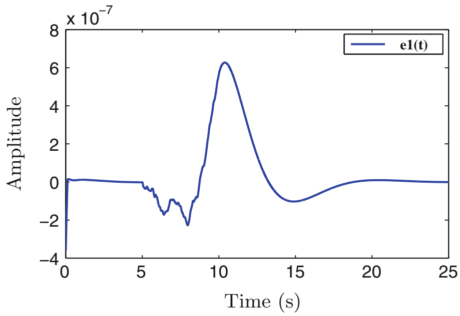

Fig. 4 Evolution of the estimation error for state $x_1(t)$

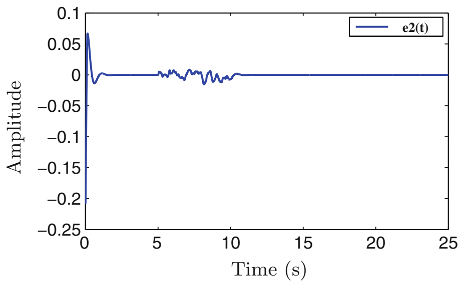

Fig. 5 Evolution of the estimation error for state $x_2(t)$

5 Conclusion

In this paper it has been shown how that a robust H_∞ observer can be used to design a controller such that the robust stabilization of the DU-FOS is ensured. The existence conditions of an observer and the robust stabilization satisfying the performance requirement of closed loop in the presence of uncertainties and disturbances are investigated. When the system is subject to uncertainties and bounded disturbances, the robust H_∞ OBC must ensure the stabilization of the closed loop, and minimize the effect of disturbances on the estimation error and the system. By adopting an H_∞-norm approach for FOS, and on the basis of the algebraic constraints derived from the analysis of the estimation error dynamics, sufficient conditions formed in inequality are given to satisfy the two above requirements. Finally, a numerical example is provided to show the effectiveness of the proposed method.

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Relative Observability for Fractional Differential-Algebraic Delay Systems Within Riemann-Liouville Fractional Derivatives

Zbigniew Zaczkiewicz

Abstract The paper presents the problems of relative R-observability for linear stationary fractional differential-algebraic delay system (FDAD). FDAD system consists of fractional differential equation in the Riemann-Liouville sense and difference equations. We introduce the determining equation systems and their properties. Applying the Laplace transformation we obtain solutions representations into series of their determining equation solutions and present effective parametric rank criteria for relative R-observability. A dual controllability result is also formulated.

Keywords Observability · Fractional differential equations · Determining equations · Differential-algebraic systems

1 Introduction

In the paper, we consider related observability for fractional differential-algebraic delay systems. Observability concept was introduced by R. Kalman in 1960. This is qualitative property of observation systems and is of great importance in control theory. The basic concepts of observability play an important role in dynamical systems analysis.

Recently, a huge research attention has been paid to fractional control systems (see the monograph [1–7] and the papers [8–14] and [15–17]).

The paper deals with linear fractional differential-algebraic delay systems within Riemann-Liouville fractional derivatives (FDAD). FDAD systems consist of some equations being fractional differential in the Riemann-Liouville sense, the other-difference and discrete. We introduce the determining equations the same as for differential-algebraic systems (for example see [18] or [19]). To obtain solutions representations we apply fractional differential calculus especially dealing with the

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Z. Zaczkiewicz (✉)
Faculty of Computer Science, Bialystok University of Technology,
15-351 Bialystok, Poland
e-mail: z.zaczkiewicz@pb.edu.pl

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Laplace transform. By this result we obtain effective parametric rank criterions for $\mathbb{R}$-observability with respect to x_1. The relative $\mathbb{R}$-observability is dual to relatively controllability with respect to x_1 (see [17] for more details). Our results can be considered as a generalization of the known corresponding results for the integer order case [20] and the fractional order in the Caputo sense [15].

The paper is organized as follows. In Sect. 2 the state equations of FDAD systems is presented. The deterministic equations and representation of solutions into series of determining equations solutions are introduced in Sect. 3. The observability problem is analyzed in Sect. 4. The duality result is given in Sect. 4. Finally, Sect. 5 contains an example.

2 Preliminaries

Let us introduce the following notation:

D_t^α is the left-sided Riemann-Liouville fractional derivatives of order α defined by

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)}\frac{d}{dt}\int_0^t \frac{f(\tau)}{(t-\tau)^\alpha}d\tau,$$

where $0 < \alpha < 1$, $\alpha \in \mathbb{R}$ and $\Gamma(t) = \int_0^\infty e^{-\tau}\tau^{t-1}d\tau$ is the Euler gamma function (see [5] for more details). $T_t = \lim_{\epsilon\to+0}\left[\frac{t-\epsilon}{h}\right]$, where the symbol $[z]$ means entire part of the number z; I_n is the identity n by n matrix.

In this paper, we concentrate on the stationary FDAD system in the following form:

$$D_t^\alpha x_1(t) = A_{11}x_1(t) + A_{12}x_2(t),\ t > 0, \tag{1a}$$
$$x_2(t) = A_{21}x_1(t) + A_{22}x_2(t-h),\ t \geq 0, \tag{1b}$$
$$y(t) = B_1x_1(t) + B_2x_2(t), \tag{1c}$$

where $x_1(t) \in \mathbb{R}^{n_1}$, $x_2(t) \in \mathbb{R}^{n_2}$, $u(t) \in \mathbb{R}^r$, $A_{11} \in \mathbb{R}^{n_1\times n_1}$, $A_{12} \in \mathbb{R}^{n_1\times n_2}$, $A_{21} \in \mathbb{R}^{n_2\times n_1}$, $A_{22} \in \mathbb{R}^{n_2\times n_2}$, $B_1 \in \mathbb{R}^{r\times n_1}$, $B_2 \in \mathbb{R}^{r\times n_2}$ are constant (real) matrices, $0 < h$ is a constant delay. We regard an absolute continuous n_1-vector function $x_1(\cdot)$ and a piecewise continuous n_2-vector function $x_2(\cdot)$ as a solution of System (1) if they satisfy the equation (1a) for almost all $t > 0$ and (1b) for all $t \geq 0$.

System (1) should be completed with finite initial conditions:

$$[D_t^{\alpha-1}x_1(t)]_{t=0} = x_{10},\ x_2(\tau) = \psi(\tau),\ \tau \in [-h, 0), \tag{2}$$

where $x_{10} \in \mathbb{R}^{n_1}$; $\psi \in PC([-h, 0), \mathbb{R}^{n_2})$ and $PC([-h, 0), \mathbb{R}^{n_2})$ denotes the set of piecewise continuous n_2-vector-functions in $[-h, 0]$. Observe that $x_2(t)$ at $t = 0$ is determined from Eq. (1b).

3 Representation of Solutions into Series of Determining Equations Solutions

Let us introduce the determining equations of System (1) (see [19] for more details).

$$X_{1,k}(t) = A_{11}X_{1,k-1}(t) + A_{12}X_{2,k-1}(t) + U_{k-1}(t), \tag{3a}$$
$$X_{2,k}(t) = A_{21}X_{1,k}(t) + A_{22}X_{2,k}(t-h),\ k = 0, 1, \dots; \tag{3b}$$
$$Y_k(t) = B_1X_{1,k}(t) + B_2X_{2,k}(t), \tag{3c}$$

with initial conditions

$$X_{1,k}(t) = 0, X_{2,k}(t) = 0,\ Y_k(t) = 0 \text{ for } t < 0 \text{ or } k \le 0;$$
$$U_0(0) = I_{n_1},\ U_k(t) = 0 \text{ for } t^2 + k^2 \ne 0. \tag{3d}$$

Here, we establish some algebraic properties of $X_{1,k},\ X_{2,k}$.

Proposition 1 ([16]) *The following identities hold:*

$$\left(A_{11} + A_{12}(I_{n_2} - \omega A_{22})^{-1}A_{12}\right)^k = \sum_{j=0}^{+\infty} X_{1,k+1}(jh)\omega^j,\ k = 1, 2, \dots;$$

$$(I_{n_2} - \omega A_{22})^{-1}A_{21}\left(A_{11} + A_{12}(I_{n_2} - \omega A_{22})^{-1}A_{12}\right)^k = \sum_{j=0}^{+\infty} X_{2,k+1}(jh)\omega^j, k = 1, 2, \dots;$$

where $|\omega| < \omega_1$ *and* ω_1 *is a sufficiently small real number.*

Theorem 2 *A solution to System (1) with finite initial conditions (2) for* $t \ge 0$ *exists, is unique and can be represented in the form of a series in power of solutions to determining systems (3), in the following form:*

$$x_1(t, x_{10}, \psi) = \sum_{k=0}^{+\infty} \sum_{\substack{j \\ t-jh>0}} \frac{(t-jh)^{\alpha(k+1)-1}}{\Gamma(\alpha(k+1))} X_{1,k+1}(jh)x_{10} +$$
$$\sum_{k=0}^{+\infty} \sum_{\substack{i,j \\ t-(i+j)h>0}} X_{1,k+1}(ih)A_{12}(A_{22})^{i+1} \int_0^{t-(i+j)h} \frac{(t-\tau-(i+j)h)^{\alpha(k+1)-1}}{\Gamma(\alpha(k+1))} \psi(\tau-h)d\tau,$$

$$x_2(t, x_{10}, \psi) =$$
$$\sum_{k=0}^{+\infty} \sum_{\substack{j \\ t-jh>0}} \frac{(t-jh)^{\alpha(k+1)-1}}{\Gamma(\alpha(k+1))} X_{2,k+1}(jh) x_{10} + \sum_{i=0}^{+\infty} (A_{22})^{i+1} \psi(t-(i+1)h) +$$
$$\sum_{k=0}^{+\infty} \sum_{\substack{i,j \\ t-(i+j)h>0}} X_{2,k+1}(ih) A_{12} (A_{22})^{i+1} \int_0^{t-(i+j)h} \frac{(t-\tau-(i+j)h)^{\alpha(k+1)-1}}{\Gamma(\alpha(k+1))} \psi(\tau-h) d\tau$$

where $\psi(\tau) \equiv 0$ *for* $\tau \notin [-h, 0)$.

Proof First we use the classical formula for the Laplace transformation of the fractional derivative of Eq. (1a)

$$\int_0^{\infty} e^{-pt} D_t^{\alpha} x_1(t) dt = p^{\alpha} \check{x}_1(p) - [D_t^{\alpha-1} x_1(t)]_{t=0} = p^{\alpha} \check{x}_1(p) - x_{10}.$$

We apply the Laplace transform to System (1)

$$p^{\alpha} \check{x}_1(p) - x_{10} = A_{11} \check{x}_1(p) + A_{12} \check{x}_2(p), \tag{4}$$
$$\check{x}_2(p) = A_{21} \check{x}_1(p) + A_{22} e^{-ph} \check{x}_2(t) + A_{22} e^{-ph} \int_{-h}^{0} e^{-p\tau} \psi(\tau) d\tau, \tag{5}$$

where $\check{x}_1(p)$, $\check{x}_2(p)$ are Laplace transforms of functions $x_1(t)$, $x_2(t)$ respectively. Solving (5), we obtain

$$\check{x}_2(p) = \left(I_{n_2} - A_{22} e^{-ph}\right)^{-1} A_{21} \check{x}_1(p) + \left(I_{n_2} - A_{22} e^{-ph}\right)^{-1} A_{22} e^{-ph} \int_{-h}^{0} e^{-p\tau} \psi(\tau) d\tau$$
$$\check{x}_1(p) = \left(p^{\alpha} I_{n_1} - A_{11} - A_{12} \left(I_{n_2} - A_{22} e^{-ph}\right)^{-1} A_{21}\right)^{-1} \times$$
$$\left[A_{12} \left(I_{n_2} - A_{22} e^{-ph}\right)^{-1} A_{22} e^{-ph} \int_{-h}^{0} e^{-p\tau} \psi(\tau) d\tau + x_{10}\right] =$$
$$\sum_{k=0}^{+\infty} \frac{1}{(p^{\alpha})^{k+1}} \left(A_{11} + A_{12} \left(I_{n_2} - A_{22} e^{-ph}\right)^{-1} A_{21}\right)^{k} \times$$
$$\left[A_{12} \left(I_{n_2} - A_{22} e^{-ph}\right)^{-1} A_{22} e^{-ph} \int_{-h}^{0} e^{-p\tau} \psi(\tau) d\tau + x_{10}\right].$$

Applying Propositions 1 we obtain

$$\check{x}_1(s) = \sum_{k=0}^{+\infty} \frac{1}{s^{\alpha k+\alpha}} \sum_{j=0}^{+\infty} e^{-jsh} X_{1,k+1}(jh) A_{12} (I_{n_2} - A_{22}\omega)^{-1} A_{22} e^{-sh}$$
$$\times \int_{-h}^{0} e^{-s\tau} \psi(\tau) d\tau + \sum_{k=0}^{+\infty} \frac{1}{s^{\alpha k+\alpha}} \sum_{j=0}^{+\infty} e^{-jsh} X_{1,k+1}(jh) x_{10},$$
$$\check{x}_2(s) = \sum_{k=0}^{+\infty} \frac{1}{s^{\alpha k+\alpha}} \sum_{j=0}^{+\infty} e^{-jsh} X_{2,k+1}(jh) A_{12} (I_{n_2} - A_{22}\omega)^{-1} A_{22} e^{-sh}$$
$$\times \int_{-h}^{0} e^{-s\tau} \psi(\tau) d\tau + \sum_{k=0}^{+\infty} \frac{1}{s^{\alpha k+\alpha}} \sum_{j=0}^{+\infty} e^{-jsh} X_{2,k+1}(jh) x_{10}$$
$$+ (I_{n_2} - A_{22}\omega)^{-1} A_{22} e^{-sh} \int_{-h}^{0} e^{-s\tau} \psi(\tau) d\tau.$$

By applying inverse Laplace transform the proof is complete. □

4 Observability

Here, by [18] we establish some algebraic properties of $Y_k(t)$.

Proposition 3 *The solutions $Y_k(t), t \geq 0$, of the determining equation (3c) satisfy the condition*

$$Y_k(lh) = -\sum_{j=1}^{\Theta_l} r_{0j} Y_k((l-j)h) - \sum_{i=0}^{n_1} \sum_{j=0}^{\Theta_l} r_{ij} Y_{k-i}((l-j)h)$$

for $l = 0, 1, \ldots$, where $\Theta_l = \min\{l, n_1 n_2\}$ and $k = n_1 + 1, n_1 + 2, \ldots$.

Proposition 4 *Solutions $Y_k(lh)$, $k \geq 1, l \geq 0$, of determining equation (3c) satisfy the following conditions:*

$$Y_k(lh) = -\sum_{j=1}^{\widetilde{\theta}_k} p_{0j} Y_{k-j}(lh) - \sum_{i=1}^{n_2} \sum_{j=0}^{\widetilde{\theta}_k} p_{ij} Y_{k-j}((l..i)h),$$

where $k = 1, 2, \ldots, l = n_2 + 1, n_2 + 2, \ldots$, and $\widetilde{\theta}_k = \min\{k-1, n_1(n_2)^2\}$.

Let $x_1(t; x_{10}, \psi), x_2(t; x_{10}, \psi)$ be the solution at time $t > 0$ of System (1) corresponding to finite initial conditions (2). Similarly, $y(t) = y(t; x_{10}, \psi), \tilde{y}(t) = \tilde{y}(t; \tilde{x}_{10}, \psi)$ denote the observing function corresponding to the solutions $x_1(t) = x_1(t; x_{10}, \psi), x_2(t) = x_2(t; x_{10}, \psi)$ and $\tilde{x}_1(t) = \tilde{x}_1(t; \tilde{x}_{10}, \psi)$, $\tilde{x}_2(t) = \tilde{x}_2(t; \tilde{x}_{10}, \psi)$, respectively.

Definition 5 System (1) is said to be $\mathbb{R}^{n_1}$-observable with respect to x_1 if for every x_{10}, $\tilde{x}_{10} \in \mathbb{R}^{n_1}$, the condition $y(t; x_{10}, \psi) \equiv \tilde{y}(t; \tilde{x}_{10}, \psi)$; for every $\psi \in PC([-h; 0); n_2)$ and for $t \geq 0$ implies that $x_{10} = \tilde{x}_{10}$.

For the sequel, we need the following result:

Proposition 6 ([15]) *Functions* $f_{kj}(t) = \frac{(t-jh)^{\alpha k+\alpha-1}}{\Gamma(\alpha(k+1))}$ *for* $t - jh > 0$ *and* $f_{kj}(t) = 0$ *for* $t - jh \leq 0$, *where* $k = 0, 1, \dots; j = 0, 1, \dots$, *are linearly independent for* $t > 0$.

Now we may formulate the main result:

Theorem 7 *System (1) is* $\mathbb{R}^{n_1}$*-observable with respect to* x_1 *if and only if*

$$rank \begin{bmatrix} Y_{k+1}(lh), \\ k = 0, 1, \dots, n_1;\ l = 0, 1, \dots, n_2 \end{bmatrix} = n_1. \tag{6}$$

Proof By Theorem 2 and (1c), $y(t; x_{10}, \psi) \equiv \tilde{y}(t; \tilde{x}_{10}, \psi)$ is equivalent to the following:

$$B_1 \sum_{k=0}^{+\infty} \sum_{\substack{j \\ t-jh>0}} \frac{(t-jh)^{\alpha(k+1)-1}}{\Gamma(\alpha(k+1))} X_{1,k+1}(jh)x_{10}$$
$$+B_2 \sum_{k=0}^{+\infty} \sum_{\substack{j \\ t-jh>0}} \frac{(t-jh)^{\alpha(k+1)-1}}{\Gamma(\alpha(k+1))} X_{2,k+1}(ih)x_{10}$$
$$= B_1 \sum_{k=0}^{+\infty} \sum_{\substack{j \\ t-jh>0}} \frac{(t-jh)^{\alpha(k+1)-1}}{\Gamma(\alpha(k+1))} X_{1,k+1}(jh)\tilde{x}_{10}$$
$$+B_2 \sum_{k=0}^{+\infty} \sum_{\substack{j \\ t-jh>0}} \frac{(t-jh)^{\alpha(k+1)-1}}{\Gamma(\alpha(k+1))} X_{2,k+1}(ih)\tilde{x}_{10}.$$

It follows from here that

$$\sum_{k=0}^{+\infty} \sum_{\substack{j \\ t-jh>0}} \frac{(t-jh)^{\alpha(k+1)-1}}{\Gamma(\alpha(k+1))} [B_1, B_2] \begin{bmatrix} X_{1,k+1}(jh) \\ X_{2,k+1}(jh) \end{bmatrix} (x_{10} - \tilde{x}_{10}) =$$
$$\sum_{k=0}^{+\infty} \sum_{\substack{j \\ t-jh>0}} \frac{(t-jh)^{\alpha(k+1)-1}}{\Gamma(\alpha(k+1))} Y_{k+1}(jh)(x_{10} - \tilde{x}_{10}) = 0.$$

By Lemma 4, $\mathbb{R}^{n_1}$-observable with respect to x_1 is equivalent to

$$\begin{bmatrix} Y_{k+1}(lh), \\ k = 0, 1, \dots;\ i = 0, 1, \dots \end{bmatrix} (x_0^* - \tilde{x}_0^*) = 0 \Rightarrow (x_0^* - \tilde{x}_0^*) = 0.$$

Thus, we have

$$rank \begin{bmatrix} Y_{k+1}(lh), \\ k = 0, 1, \dots ; \ i = 0, 1, \dots \end{bmatrix} = n_1.$$

Taking into account Propositions 3 and 4, we may claim that the property of $\mathbb{R}^{n_1}$-observability with respect to x_1 is saturated and this completes the proof. □

5 Duality

Let us introduce the stationary FDAD Control System (dual to System (1)) in the following form:

$$\begin{aligned} D_t^{\alpha} x_1^*(t) &= A_{11}^T x_1(t) + A_{21}^T x_2^*(t) + B_1^T u(t), \ t > 0, \\ x_2^*(t) &= A_{12}^T x_1^*(t) + A_{22} x_2^*(t-h) + B_1^T u(t), \ t \geq 0, \end{aligned} \tag{7}$$

with finite initial conditions

$$[D_t^{\alpha-1} x_1^*(t)]_{t=0} = x_0^*, \ x_2^*(\tau) = \psi^*(\tau), \ \tau \in [-h, 0),$$

where $x_1^*(t) \in \mathbb{R}^{n_1}, \ x_2^*(t) \in \mathbb{R}^{n_2}, \ u(t) \in \mathbb{R}^r, \ x_0^* \in \mathbb{R}^{n_1}; \ \psi \in PC([-h, 0), \mathbb{R}^{n_2})$.

Let us consider determining equations

$$\begin{aligned} X_{1,k}^*(t) &= A_{11}^T X_{1,k-1}(t) + A_{21}^T X_{2,k-1}^*(t) + B_1^T U_{k-1}(t), \\ X_{2,k}^*(t) &= A_{12}^T X_{1,k}^*(t) + A_{22}^T X_{2,k}^*(t-h) + B_2^T U_{k-1}(t), \ k = 0, 1, \dots ; \end{aligned} \tag{8}$$

with initial conditions

$$\begin{aligned} &X_{1,k}^*(t) = 0, X_{2,k}^*(t) = 0 \text{ for } t < 0 \ \text{ or } \ k \leq 0; \\ &U_0^*(0) = I_n, \ U_k^*(t) = 0 \ \text{ for } \ t^2 + k^2 \neq 0. \end{aligned}$$

Definition 8 ([15]) Control System (7) is called relatively controllable with respect to x_1 if for any initial data x_0^*, ϕ^* and any $x_*^* \in \mathbb{R}^{n_2}$ there exist a time moment $t_* > 0$ and a piecewise continuous control $u(\cdot)$, such that for the corresponding solution $x_1^*(t) = x_1^*(t, x_0^*, \phi^*, u), t > 0$ the condition $x_1^*(t_*) = x_*^*$ is valid.

The following two statements hold [15].

Proposition 9 *The solution $X_{1,k}^*(t), \ t \geq 0$ of the determining equations (8) satisfy the following equations:*

$$(A_{11}^T + A_{21}^T(I_{n_2} - A_{22}^T\omega)^{-1}A_{12}^T)^i(B_1^T + A_{21}^T(I_{n_2} - A_{22}^T\omega)^{-1}B_2^T) \tag{9}$$

$$\equiv \sum_{j=0}^{+\infty} X_{1,i+1}^*(jh)\omega^j, \tag{10}$$

where $|\omega| < \omega_1$; ω_1 *is a sufficiently small real number.*

Proposition 10 *Control System (7) is relatively controllable with respect to* x_1 *if and only if*

$$\text{rank}\left[X_{1,\eta}^*(\xi h),\ \xi = 0, \dots, n_2;\ \eta = 0, \dots, n_1\right] = n_1, \tag{11}$$

where by the symbol $\left[X_{1,\eta}^*(\xi h),\ \xi = 0, \dots, n_2;\ \eta = 0, \dots, n_1\right]$ *we denote a block matrix of columns* $X_{1,\eta}^*(\xi h)$, for $\xi = 0, \dots, n_2$; $\eta = 0, \dots, n_1$.

Now, we can state the duality result.

Theorem 11 *System (1) is* $\mathbb{R}^n$*-observable with respect to* x_1 *if and only if Control System (7) is relatively controllable with respect to* x_1^*.

Proof Transposing (10), we have:

$$(B_1 + B_2(I_{n_2} - A_{22}\omega)^{-1}A_{21})(A_{11} + A_{12}(I_{n_2} - A_{22}\omega)^{-1}A_{21})^k$$
$$= \sum_{j=0}^{+\infty} X_{1,k+1}^{*T}(jh)\omega^j,\ k = 0, 1, \dots; \tag{12}$$

By (3c) and Proposition 1 we obtain:

$$\sum_{l=0}^{+\infty} Y_{i+1}(lh)\omega^l \equiv$$
$$(B_1 + B_2(I_{n_2} - A_{22}\omega)^{-1}A_{21})(A_{11} + A_{12}(I_{n_2} - A_{22}\omega)^{-1}A_{21})^i,\ i = 0, 1, \dots, \tag{13}$$

Then, comparing coefficients of the same power of ω in (12) and (13) we have:

$$X_{1,k+1}^{*T}(jh) = Y_{k+1}(jh).$$

It follows that

$$\begin{bmatrix} Y_{k+1}(lh), \\ k = 0, 1, \dots, n_1;\ i = 0, 1, \dots, n_2 \end{bmatrix}$$
$$= \left[X_{1,\eta+1}^*(\xi h), \xi = 0, \dots, n_2; \eta = 0, \dots, n_1\right]^T.$$

This proves the theorem. □

6 Example

Let us consider the following system:

$$D_t^\alpha x_1(t) = [1]x_1(t) + \begin{bmatrix}0 & -1\end{bmatrix} x_2(t),\ t > 0,$$
$$x_2(t) = \begin{bmatrix}0\\1\end{bmatrix} x_1(t) + \begin{bmatrix}0 & 2\\0 & 0\end{bmatrix} x_2(t-5),\ t \geq 0.$$
$$y(t) = \begin{bmatrix}1\end{bmatrix} x_1(t) + \begin{bmatrix}1 & 0\end{bmatrix} x_2(t). \tag{14}$$

System (14) should be completed with final initial conditions:

$$x_{10} = x_{10}, \psi(\tau) = \begin{bmatrix}\psi_1(\tau)\\ \psi_2(\tau)\end{bmatrix},\ \tau \in [-h, 0), \tag{15}$$

First we present the determining equations of System (14):

$$X_{1,k}(t) = [1]X_{1,k-1}(t) + \begin{bmatrix}0 & -1\end{bmatrix} X_{2,k-1}(t) + U_{k-1}(t),$$
$$X_{2,k}(t) = \begin{bmatrix}0\\1\end{bmatrix} X_{1,k}(t) + \begin{bmatrix}0 & 2\\0 & 0\end{bmatrix} X_{2,k}(t-5),$$
$$Y_k(t) = [1]X_{1,k}(t) + \begin{bmatrix}1 & 0\end{bmatrix} X_{2,k}(t),$$

for $k = 0, 1, \dots;\ t \geq 0$ with initial conditions

$$X_{1,k}(t) = 0, X_{2,k}(t) = 0,\ Y_k(t) = 0 \text{ for } t < 0 \text{ or } k \leq 0;$$
$$U_0(0) = I_{n_1}, U_k(t) = 0 \text{ for } t^2 + k^2 \neq 0.$$

Now we compute the solutions of the determining system:

$$\begin{bmatrix}X_{1,1}(0)\\X_{2,1}(0)\end{bmatrix} = \begin{bmatrix}1\\0\\1\end{bmatrix};\ \begin{bmatrix}X_{1,k}(0)\\X_{2,k}(0)\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix},\ k \geq 2;\ \begin{bmatrix}X_{1,1}(5)\\X_{2,1}(5)\end{bmatrix} = \begin{bmatrix}0\\2\\0\end{bmatrix};$$
$$\begin{bmatrix}X_{1,k}(5)\\X_{2,k}(5)\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix},\ k \geq 2;\ \begin{bmatrix}X_{1,i}(5j)\\X_{2,i}(5j)\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}, i \geq 0,\ j \geq 2;$$
$$Y_1(0) = [1],\ Y_1(5j) = [0],\ j = 1, 2 \dots;$$
$$Y_i(5j) = [0],\ i = 2, 3, \dots;\ j = 0, 1, \dots.$$

By Theorem 2

$$\text{for } 0 \le t < 5 \text{ we have}$$
$$x_1(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} x_{10},$$
$$x_2(t) = \begin{bmatrix} \psi_2(t-h) \\ \frac{t^{\alpha-1}}{\Gamma(\alpha)} x_{10} \end{bmatrix};$$
$$\text{for } 5 \le t \text{ we have}$$
$$x_1(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} x_{10},$$
$$x_2(t) = \begin{bmatrix} 2\frac{(t-h)^{\alpha-1}}{\Gamma(\alpha)} x_{10} \\ \frac{t^{\alpha-1}}{\Gamma(\alpha)} x_{10} \end{bmatrix}.$$

Let us compute (6):

$$rank \begin{bmatrix} Y_k(lh), \\ k = 0, 1, \ldots, n_1;\ i = 0, 1, \ldots, n_2 \end{bmatrix} = n_1.$$

$$rank \begin{bmatrix} Y_1(0) \\ Y_2(0) \\ Y_1(5) \\ Y_2(5) \\ Y_1(10) \\ Y_1(10) \end{bmatrix} = rank \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = 1 = n_1.$$

Thus System (14) is $\mathbb{R}^n$-observable with respect to x_1.
Let us consider the dual system (see [17] for more details):

$$\begin{aligned} D_t^{\alpha} x_1^*(t) &= [1] x_1^*(t) + \begin{bmatrix} 0 & 1 \end{bmatrix} x_2^*(t) + [1] u(t),\ t > 0, \\ x_2(t) &= \begin{bmatrix} 0 \\ -1 \end{bmatrix} x_1^*(t) + \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} x_2^*(t-5) + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u(t),\ t \ge 0. \\ y(t) &= [1] x_1(t) + x_2^*(t). \end{aligned} \tag{16}$$

System (16) should be completed with final initial conditions:

$$x_{10}^* = x_{10}^*, \psi^*(\tau) = \begin{bmatrix} \psi_1^*(\tau) \\ \psi_2^*(\tau) \end{bmatrix},\ \tau \in [-h, 0),$$

First we present the determining equations of System (16):

$$X^*_{1,k}(t) = [1]X^*_{1,k-1}(t) + \begin{bmatrix}0 & 1\end{bmatrix} X^*_{2,k-1}(t) + [1]U^*_{k-1}(t),$$
$$X^*_{2,k}(t) = \begin{bmatrix}0\\-1\end{bmatrix} X^*_{1,k}(t) + \begin{bmatrix}0 & 0\\2 & 0\end{bmatrix} X^*_{2,k}(t-5) + \begin{bmatrix}1\\0\end{bmatrix} U^*_k(t),$$

for $k = 0, 1, ...;\ t \geq 0$ with initial conditions

$$X^*_{1,k}(t) = 0, X^*_{2,k}(t) = 0,\ \text{ for } t < 0 \text{ or } k < 0;$$
$$U^*_0(0) = I_{n_1}, U^*_k(t) = 0 \text{ for } t^2 + k^2 \neq 0.$$

Next we present the determining equations of homogenous System (16):

$$\tilde{X}^*_{1,k}(t) = [1]\tilde{X}^*_{1,k-1}(t) + \begin{bmatrix}0 & 1\end{bmatrix} \tilde{X}^*_{2,k-1}(t),$$
$$\tilde{X}^*_{2,k}(t) = \begin{bmatrix}0\\-1\end{bmatrix} \tilde{X}^*_{1,k}(t) + \begin{bmatrix}0 & 0\\2 & 0\end{bmatrix} \tilde{X}^*_{2,k}(t-5),$$

for $k = 0, 1, \ldots;\ t \geq 0$ with initial conditions

$$\tilde{X}^*_{1,k}(t) = 0, \tilde{X}^*_{2,k}(t) = 0,\ \text{ for } t < 0 \text{ or } k \leq 0;$$
$$\tilde{X}^*_{1,1}(0) = I_{n_1}, \tilde{X}^*_{1,1}(\tau) = 0 \text{ if } \tau \neq 0.$$

Now we compute the solutions of the above determining systems:

$$\begin{bmatrix}X^*_{1,0}(0)\\X^*_{2,0}(0)\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix};\ \begin{bmatrix}X^*_{1,0}(5)\\X^*_{2,0}(5)\end{bmatrix} = \begin{bmatrix}0\\0\\2\end{bmatrix};\ \begin{bmatrix}X^*_{1,0}(5j)\\X^*_{2,0}(5j)\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}; j \geq 2;$$
$$\begin{bmatrix}X^*_{1,1}(0)\\X^*_{2,1}(0)\end{bmatrix} = \begin{bmatrix}1\\0\\-1\end{bmatrix};\ \begin{bmatrix}X^*_{1,1}(5)\\X^*_{2,1}(5)\end{bmatrix} = \begin{bmatrix}2\\0\\-2\end{bmatrix};\ \begin{bmatrix}X^*_{1,1}(5j)\\X^*_{2,1}(5j)\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix},\ j \geq 2;$$
$$\begin{bmatrix}X^*_{1,k}(5j)\\X^*_{2,k}(5j)\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}; k \geq 2,\ j \geq 0;\ \begin{bmatrix}\tilde{X}^*_{1,1}(0)\\\tilde{X}^*_{2,1}(0)\end{bmatrix} = \begin{bmatrix}1\\0\\-1\end{bmatrix},$$
$$\begin{bmatrix}\tilde{X}^*_{1,1}(5j)\\\tilde{X}^*_{2,1}(5j)\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}, j \geq 1;\ \begin{bmatrix}\tilde{X}^*_{1,k}(5j)\\\tilde{X}^*_{2,k}(5j)\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}, k \geq 2,\ j \geq 0.$$

By [17] Theorem 3

for $0 \le t < 5$ we have

$$x_1(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} x_{10}^* + \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(u(\tau) + 2\psi_1^*(\tau-5)\right) d\tau,$$

$$x_2(t) = \begin{bmatrix} u(t) \\ 2\psi_1^*(t-5) - \frac{t^{\alpha-1}}{\Gamma(\alpha)} x_{10}^* - \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \left(u(\tau) + 2\psi_1^*(\tau-5)\right) d\tau \end{bmatrix};$$

for $5 \le t < 10$ we have

$$x_1(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} x_{10}^* + 2\int_0^{t-5} \frac{(t-\tau-5)^{\alpha-1}}{\Gamma(\alpha)} u(\tau) d\tau + \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} u(\tau) d\tau + 2\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \psi_1^*(\tau-5) d\tau,$$

$$x_2(t) = \begin{bmatrix} u(t) \\ -\frac{t^{\alpha-1}}{\Gamma(\alpha)} x_{10}^* - 2\int_0^{t-5} \frac{(t-\tau-5)^{\alpha-1}}{\Gamma(\alpha)} u(\tau) d\tau - \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} u(\tau) d\tau \\ +0 \\ +2u(t-5) - 2\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} \psi_1^*(\tau-5) d\tau \end{bmatrix}$$

for $10 \le t$ we have

$$x_1(t) = \frac{t^{\alpha-1}}{\Gamma(\alpha)} x_{10}^* + 2\int_0^{t-5} \frac{(t-\tau-5)^{\alpha-1}}{\Gamma(\alpha)} u(\tau) d\tau + \int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} u(\tau) d\tau,$$

$$x_2(t) = \begin{bmatrix} u(t) \\ -\frac{t^{\alpha-1}}{\Gamma(\alpha)} x_{10}^* - 2\int_0^{t-5} \frac{(t-\tau-5)^{\alpha-1}}{\Gamma(\alpha)} u(\tau) d\tau \\ 0 \\ -\int_0^t \frac{(t-\tau)^{\alpha-1}}{\Gamma(\alpha)} u(\tau) d\tau + 2u(t-5) \end{bmatrix}.$$

Let us compute (11):

$$rank \left[X_{1,k}(ih), k = 0, 1, \dots, n_1;\ i = 0, 1, \dots, n_2\right] = [0, 0, 1, 2] = 1 = n_1;$$

Thus System (16) is relatively t_1-controllable with respect to x_1 and the duality holds.

7 Conclusions

Representations of solutions for for linear fractional differential-algebraic systems with delay (FDAD) has been presented (Theorem 2). Effective parametric rank criteria for relative observability has been established (Theorem 7). The dual controllability result has been also formulated (Theorem 11). These considerations can be extended to systems with many delays.

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Part IV
Control and Applications

Minimum Energy Control of Linear Fractional Systems

Jerzy Klamka

Abstract The minimum energy control problem of infinite-dimensional fractional-discrete time linear systems is addressed. Necessary and sufficient conditions for the exact controllability of the system are established. Sufficient conditions for the solvability of the minimum energy control of the infinite-dimensional fractional discrete-time systems are given. A procedure for computation of the optimal sequence of inputs minimizing the quadratic performance index is proposed.

Keywords Fractional · Infinite-dimensional discrete-time · Linear system · Minimum energy control · Controllability

1 Introduction

Controllability plays an essential role in the development of the modern mathematical control theory. There are important relationships between controllability, stability and stabilizability of linear control systems. Controllability is also strongly connected with the theory of minimal realization of linear time-invariant control systems. Moreover, it should be pointed out that there exists a formal duality between the concepts of controllability and observability.

Moreover, controllability is strongly connected with so-called minimum energy control problem [9]. It should be pointed out that in the literature there are many results concerning controllability and minimum energy control, which depend on the type of dynamical control system [9].

The reachability, controllability and minimum energy control of positive linear discrete-time systems with time-delays have been considered in [1] and [27]. The

J. Klamka (✉)
Department of Control Engineering, Silesian University of Technology, Gliwice, Poland
e-mail: jerzy.klamka@polsl.pl

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realization problem for cone systems has been addressed in [3]. The reachability and controllability to zero of positive fractional linear systems have been investigated in [2], [4–7], [12, 13].

Mathematical fundamentals of fractional calculus are given in the monographs and papers [14–17], [19], [21–23]. The fractional order controllers have been developed in [19].

A generalization of the Kalman filter for fractional order systems has been proposed in [24]. Some other applications of fractional order systems can be found in [18], [20, 21], [25, 26]. The minimum energy control problem has been solved for different classes of linear systems in [8–11].

In this paper the minimum energy control problem will be addressed for infinite-dimensional fractional discrete-time linear systems.

The paper is divided into five sections and organized as follows. In Sect. 2 the solution of the difference state equation the infinite-dimensional fractional systems is recalled. Necessary and sufficient conditions for the exact controllability of the infinite-dimensional fractional systems are established in Sect. 3. The main result of the paper is presented in Sect. 4, in which the minimum energy control problem is formulated and solved. Concluding remarks are given in Sect. 5.

To the best knowledge of the author the minimum energy control problem for the infinite-dimensional fractional discrete-time linear systems have not been considered yet.

2 Fractional Systems

The set of nonnegative integers will be denoted by Z_+. Let X and U be the separable generally infinite-dimensional Hilbert spaces and $x_k \in X$, $u_k \in U$, $k \in Z_+$. In finite-dimensional case $X = R^n$ and $U = R^m$.

In this paper the fractional difference operator [5] is extended for infinite-dimensional operators and presented in the following form

$$\Delta^\alpha x_k = \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} x_{k-j}, \tag{1}$$

$$n-1 < \alpha < n \in N = \{1, 2, \ldots\}, \; k \in Z_+$$

where similarly as in finite-dimensional case $\alpha \in R$ is the order of the fractional difference and

$$\binom{\alpha}{j} = \begin{cases} 1 & \text{for } j = 0 \\ \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!} & \text{for } j = 1, 2, \ldots \end{cases} \tag{2}$$

Consider the fractional discrete linear system, described by the infinite-dimensional state-space equations

$$\Delta^{\alpha} x_{k+1} = Ax_k + Bu_k, \quad k \in Z_+ \tag{3}$$

where $x_k \in X$, $u_k \in U$ are the state and input and $A\colon X \to X$, $B\colon U \to X$ are given linear and bounded operators. In finite dimensional case A and B are $n \times n$ and $n \times m$ constant matrices, respectively.

Using definition (1) we may write the Eqs. (3) in the equivalent form

$$x_{k+1} + \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} x_{k-j+1} = Ax_k + Bu_k, \quad k \in Z_+ \tag{4}$$

Lemma 1 [4]. *The solution of Eq. (4) with initial condition $x_0 \in X$ is given by*

$$x_k = \Phi_k x_0 + \sum_{i=0}^{k-1} \Phi_{k-i-1} Bu_i \tag{5}$$

where linear and bounded operators $\Phi_k\colon X \to X$ are determined by the equation

$$\Phi_{k+1} = (A + I_n\alpha)\Phi_k + \sum_{i=2}^{k+1} (-1)^{i+1} \binom{\alpha}{i} \Phi_{k-i+1} \tag{6}$$

with initial condition $\Phi_0 = I$, where I is the identity operator.

Remark 1 It should be pointed out that in finite-dimensional case operators $\Phi_k\colon R^n \to R^n$ are constant $n \times n$ dimensional matrices.

3 Controllability

First of all, in order to define controllability concepts let us introduce the notion of reachable set in q steps for infinite-dimensional discrete-time fractional control system (4).

Definition 1 For fractional system (4) reachable set in q steps from $x_0 = 0$ is defined as follows

$$K_q = \{x \in X\colon x \text{ is a solution of equation (4) for } k = q \text{ and for sequence of controls } u_0, u_1, \ldots u_k, \ldots, u_{q-1}\} \tag{7}$$

Remark 2 It should be pointed out, that since there are not any constraints on the admissible controls, than reachable set is a linear subspace in the state space. Moreover, in infinite-dimensional case it is necessary to distinguish between exact and approximate controllability. It follows from the fact, that in infinite-dimensional cases there are linear subspaces which are not closed.

Definition 2 The fractional system (4) is exactly controllable in q-steps if

$$K_q = X \tag{8}$$

Definition 3 The fractional system (4) is approximately controllable in q-steps if

$$cl(K_q) = X \tag{9}$$

where $cl(K_q)$ means the closure of the set K_q.

Theorem 1 *The fractional infinite-dimensional system (4) is exactly controllable in q steps if and only if the image* ImR_q *of controllability operator*

$$R_q := [B,\ \Phi_1 B,\ \ldots, \Phi_{q-1} B] \tag{10}$$

is the whole space X.

Corollary 1 *The fractional system (4) is exactly controllable in q steps if and only* $R_q R_q^*$ *is invertible operator, i.e. there exist linear and bounded operator*

$$\left(R_q R_q^*\right)^{-1}. \tag{11}$$

Corollary 2 *The fractional system (4) is approximately controllable in q steps if and only if* $cl(ImR_q)$ *of controllability operator (10) is the whole state space X, or equivalently if and only if the reachable set in q steps* K_q *is dense in the Hilbert space X.*

Since for finite-dimensional case $X = R^n$ *approximate controllability in q-steps and exact controllability in q-steps coincide, we say shortly controllability in q-steps. Therefore, taking into account Theorem 1 we have the following Corollary.*

Corollary 3 *The fractional finite-dimensional system (4) is controllable in q steps if and only if* $n \times nm$ *dimensional controllability matrix*

$$R_q := [B,\ \Phi_1 B,\ \ldots, \Phi_{q-1} B] \tag{12}$$

has full row rank n.

It contains two linearly independent monomial columns and is nonsingular. Therefore, the fractional system is reachable in two steps.

4 Minimum Energy Control

As it was mentioned in the introduction minimum energy control problem is strongly connected with controllability concept. Moreover, it should be stressed, that controllability operator plays an essential role in the mathematical solution of minimum energy control problem.

Consider the fractional infinite-dimensional system given by difference state Eq. (4). If the system is exactly controllable in q steps then generally there exist many different input sequences that steer the initial state of the system from $x_0 = 0$ to the final state $x_f \in X$.

Among these input sequences we are looking for the sequence ui∈U, $i = 0, 1, \ldots, q-1,\ i \in Z_+$ that minimizes the quadratic performance index

$$I(u) = \sum_{j=0}^{q-1} u_j^* Q u_j \tag{13}$$

where $Q\colon U \to U$ is a self-adjoint positive define operator, q is a given number of steps in which the state of the system is transferred from $x_0 = 0$ to $x_f \in X$ and $u^* \in U$ denotes adjoint element, which in finite dimensional case denotes vector transposition.

The minimum energy control problem for the infinite-dimensional fractional system (4) can be stated as follows. For a given linear bounded operators A, B and the order α of the fractional system (4), the number of steps q, final state $x_f \in X$ and the self-adjoint operator Q of the performance index (14), find a sequence of admissible inputs ui∈U, $i = 0, 1, \ldots, q-1,$ that steers the state of the system from given initial state $x_0 = 0$ to given final state $x_f \in X$ and minimizes the given quadratic performance index (14).

In order to solve the minimum energy problem we define selfa-adjoint operator

$$W(q, Q) = R_q \bar{Q} R_q^* \tag{14}$$

where R_q is controllability operator defined by (10) and self-adjoint operator

$$\bar{Q}\colon \underbrace{U \times U \times \ldots \times U}_{q-times} \to \underbrace{U \times U \times \ldots \times U}_{q-times}$$

is defined as follows

$$\bar{Q} = blockdiag\left[Q^{-1}, Q^{-1}, \ldots, Q^{-1}\right] \tag{15}$$

From (16) it follows that operator *W(q,Q)* is invertible if and only if $R_q R_q^*$ is invertible operator, i.e. there exist linear and bounded operator $\left(R_q R_q^*\right)^{-1}$ and therefore, fractional system (4) is exactly controllable in q steps.

For the case when the condition of Theorem 1 is met then the system is exactly controllable in q steps. In this case we may define for a given $x_f \in X$ the following sequence of admissible inputs

$$\widehat{u}_{0q} = \begin{bmatrix} \widehat{u}_{q-1} \\ \widehat{u}_{q-2} \\ \vdots \\ \widehat{u}_0 \end{bmatrix} = \overleftarrow{Q} R_q^* W^{-1}(q, Q) x_f \tag{16}$$

Theorem 2 *Let the fractional system (4) be exactly controllable in q steps. Moreover let* $\bar{u}_i \in U$, $i = 0, 1, \ldots, q-1$ *be a sequence of inputs that steers the state of the system from* $x_0 = 0$ *to* $x_f \in X$. *Then the sequence of inputs* $\widehat{u}_i \in U$, $i = 0, 1, \ldots, q-1$ *defined by (17) also steers the state of the system from* $x_0 = 0$ *to* $x_f \in X$ *and minimizes the performance index (14), i.e.*

$$I(\hat{u}) \le I(\bar{u}) \tag{17}$$

The minimal value of performance index (14) for the minimum energy control (17) is given by

$$I(\widehat{u}) = x_f^T W^{-1}(q, Q) x_f \tag{18}$$

Proof If the fractional system (4) is exactly controllable in q steps, then for $x_f \in X$ we shall show that the sequence of controls given by equality (17) steers the state of the system (4) from initial state $x_0 = 0$ to final state $x_f \in X$. □

Using equality (5) for $k = q$, $x_0 = 0$ and (13), (17) we obtain

$$x_q = R_q \widehat{u}_{0q} = R_q \bar{Q} R_q^* W^{-1}(q, Q) x_f = x_f \tag{19}$$

since

$$R_q \bar{Q} R_q^* W^{-1}(q, Q) = I$$

The both sequences of inputs $\bar{u}_{0q}$ and $\hat{u}_{0q}$ steer the state of the system from $x_0 = 0$ to the same final state x_f. Hence $x_f = R_q \hat{u}_{0q} = R_q \bar{u}_{0q}$ and

$$R_q[\hat{u}_{0q} - \bar{u}_{0q}] = 0 \tag{20}$$

Using (21) we shall show that

$$[\hat{u}_{0q} - \bar{u}_{0q}]^T \hat{Q}\hat{u}_{0q} = 0 \tag{21}$$

where $\hat{Q} = \text{block diag}\,[Q, \ldots, Q]$.
Therefore, (22) yields

$$\left(\widehat{u}_{0q} - \bar{u}_{oq}\right)^T R_q^* = 0 \tag{22}$$

Multiplying the above equality by $W^{-1}(q, Q)x_f$ we obtain

$$\left(\widehat{u}_{0q} - \bar{u}_{0q}\right)^T R_q^* W^{-1}(q, Q)x_f = 0 \tag{23}$$

Using (17) and (24) we obtain (25) since

$$\begin{aligned}
&\left(\widehat{u}_{0q} - \bar{u}_{0q}\right)^T \widehat{Q}\widehat{u}_{0q} \\
&= \left(\widehat{u}_{0q} - \bar{u}_{0q}\right)^T \widehat{Q}\bar{Q}R_q^* W^{-1}(q, Q)x_f \\
&= \left(\widehat{u}_{0q} - \bar{u}_{0q}\right)^T R_q^* W^{-1}(q, Q)x_f = 0
\end{aligned}$$

and $\widehat{Q}\bar{Q} = I$
It is easy to verify that

$$\bar{u}_{0q}^T \bar{Q}\hat{u}_{0q} = \hat{u}_{0q}^T \hat{Q}\hat{u}_{0q} + [\bar{u}_{0q} - \hat{u}_{0q}]^T \hat{Q}[\bar{u}_{0q} - \hat{u}_{0q}] \tag{24}$$

From (25) it follows that the inequality (22) holds, since

$$[\bar{u}_{0q} - \hat{u}_{0q}]^T \hat{Q}[\bar{u}_{0q} - \hat{u}_{0q}] \geq 0$$

In order to find the minimal value of the performance index we substitute (17) into (19) and next we use (16). Then we obtain

$$\begin{aligned}
I(\widehat{u}) &= \widehat{u}_{0q}^T \widehat{Q}\widehat{u}_{0q} \\
&= \left(\bar{Q}R_q^T W^{-1}(q, Q)x_f\right)^T \widehat{Q}\left(\bar{Q}R_q^* W^{-1}(q, Q)x_f\right) \\
&= x_f^T W^{-1}(q, Q)R_q\bar{Q}R_q^* W^{-1}(q, Q)x_f \\
&= x_f^T W^{-1}(q, Q)x_f
\end{aligned}$$

since $\widehat{Q}\bar{Q} = I$ and $W^{-1}(q, Q)R_q\bar{Q}R_q^* = I$.

Example Given finite dimensional fractional system (4)

$$A = \begin{bmatrix} -\alpha & 0 \\ 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad (n = 2)$$

for $0 < \alpha < 1$ with (5).

Find an optimal sequence of inputs that steers the state of the system from $x_0 = 0$ to $x_f = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ in two steps $(q = 2)$ and minimizes the performance index (13) for Q = [4].

It was shown [7] that the system is reachable in two steps. It is easy to see that the conditions of Theorem 2 are met. Using Procedure presented above we obtain the following.

Step 1. In this case

$$R_2 = [B, \ \Phi_1 B] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and

$$Q = diag[Q^{-1}, Q^{-1}] = \tfrac{1}{4}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Step 2. Using (14) we obtain

$$W = R_q \bar{Q} R_q^T = \bar{Q} = \frac{1}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$W(q, Q) = W(2, 4) = R_2 \bar{Q} R_2^T = \bar{Q}$$

Step 3. Using (16) we obtain

$$\hat{u}_{02} = \begin{bmatrix} \hat{u}_1 \\ 0 \end{bmatrix} = \bar{Q} R_2^T W^{-1} x_f = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \tag{25}$$

It is easy to verify that the sequence (25) steers the state of the system in two steps from $x_0 = 0$ to $x_f = [1 \quad 1]^T$.

Step 4. The minimal value of the performance index in this case is equal to

$$I(\hat{u}) = x_f^T W^{-1} x_f = [1 \quad 1]\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = 4$$

5 Concluding Remarks

The minimum energy control problem of infinite-dimensional fractional discrete linear systems has been addressed. Moreover, necessary and sufficient conditions for the exact controllability in q steps of the systems have been established.

Under assumption on exact controllability in q steps solvability of the minimum energy control of the infinite-dimensional fractional discrete-time linear systems have been given and a procedure for computation of the optimal sequence of inputs minimizing the quadratic performance index has been proposed.

Finally, it should be mentioned, that the considerations can be extended for infinite-dimensional fractional discrete-time linear systems with delays both in control and state variables and for infinite-dimensional fractional continuous-time linear systems with constant parameters.

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Use of Alpha-Beta Filter to Synchronization of the Chaotic Ikeda Systems of Fractional Order

Adam Makarewicz

Abstract The paper considers a problem of signal filtering used in synchronization of two fractional delay Ikeda systems, combined linearly by coupling. Synchronization used Alpha-Beta filter, which operates on predicting the next value, based on measured signal in a current point in time. Using numerical simulations effects of fractional order and coupling rate on synchronization, is investigated. Simulations are performed using Ninteger Fractional Control Toolbox for MatLab.

Keywords Alpha-Beta filter · Fractional · Stability · Synchronization

1 Introduction

The paper will present a filtration of chaotic signals, using Alpha-Beta filter, that operates on a prediction of next value, based on a measured signal in the current point in time, on the basis of measurements from a previous time instant [2, 14]. The Alpha-Beta filter is characterized by its simple implementation, it does not require an estimation equation of state, as in case of the Kalman filter [13]. Filter will be presented on example of a chaotic system synchronization Ikeda [1, 3–13, 14–20]. Use of Alpha-Beta filter in a feedback loop, will reduce synchronization of two coupled, identical systems, working at a different initial conditions.

In this paper, using numerical simulations, we considered a synchronization problem of two coupled—fractionally ordered—Ikeda chaotic systems. Simulations were performed using Ninteger Fractional Control Toolbox for MatLab [18].

A. Makarewicz (✉)
Faculty of Electrical Engineering, Białystok University of Technology, Białystok, Poland
e-mail: adammakarewicz@interia.pl

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2 Preliminaries and Problem Formulation

The Alpha-Beta filtration is based on assumption, that the motion of an object, can be expressed by means of two state components where one component is derived from another. State vector, consisting of position and velocity component. Filter equations have the form of:

$$\hat{y}(t+1\,|\,t) = \hat{y}(t\,|\,t) + \Delta T\hat{v}(t\,|\,t) \tag{1}$$

$$\widehat{v}(t+1\,|\,t) = \hat{v}(t\,|\,t) \tag{2}$$

$$\hat{y}(t\,|\,t) = \hat{y}(t\,|\,t-1) + \alpha_F(y(t) - \hat{y}(t\,|\,t-1)) \tag{3}$$

$$\hat{y}(t\,|\,t) = \hat{v}(t\,|\,t-1) + \frac{\beta_F}{\Delta T}(y(t) - \hat{y}(t\,|\,t-1)) \tag{4}$$

where y represents a measured position, $\hat{y}$ is an estimation of a location and $\hat{v}$ is an estimate speed. Time step between consecutive measurements was determined by ΔT. Description $(t\,|\,t)$ means the value of current point in time, based on measurement of current point in time; $(t+1|t)$ means the value in next point in time, based on measurements of current point in time and $(t\,|\,t-1)$ means a value of current point in time, based on measurements of previous point in time.

Filter α_F and β_F parameters, are used to tune a filter. Their values change from 0 to 1. Values of these parameters, are typically selected experimentally. It is assumed, that higher values α_F and β_F result in faster response to changes in finishing tracking, while smaller values result in reducing the level of noise interference. Main goal of this paper is to investigate the impact of Alfa-Beta filter, in order to synchronize chaotic Ikeda system, fraction an order configuration with combined transmitter/receiver. Single Ikeda equation have a form:

$$_0D_t^\alpha = -a\mathbf{x}(t) + b\sin x(t-h), \tag{5}$$

where $_0D_t^\alpha x(t)$ denotes the Caputo fractional derivative, of a fraction al order α satisfying $0 < \alpha \leq 1$ and $h > 0$ is a constant delay.

In the paper, fractional Ikeda model will be used, which introduces Caputo derivative $_0D_t^\alpha x(t)$ defined by:

$$_0D_t^\alpha x(t) = \frac{1}{\Gamma(p-\alpha)}\int_0^t \frac{x^{(p)}(\tau)d\tau}{(t-\tau)^{\alpha+1-p}},\ p-1 \leq \alpha \leq p, \tag{6}$$

where $x^{(p)}(t) = d^p x(t)/dt^p$; p is a positive integer and $\Gamma(\alpha)$ is the Euler gamma function:

$$\Gamma(\alpha) = \int_0^\infty e^{-t} t^{\alpha-1} dt \tag{7}$$

In addition, it is assumed that $0 < \alpha < 2$ from formula (6) for $p = 1$ and $p = 2$ we have respectively:

$${}_0D_t^\alpha x(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{x^{(1)}(\tau)}{(t-\tau)^\alpha} d\tau, \; 0 < \alpha < 1, \tag{8}$$

$${}_0D_t^\alpha x(t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t \frac{x^{(2)}(\tau) d\tau}{(t-\tau)^{\alpha-1}}, \; 0 < \alpha < 2. \tag{9}$$

Dynamics of this system (5) has been studied in [8] for fractional order between $0 < \alpha < 1$ and of $0 < \alpha < 2$ [2, 3].

In this paper we assumed, that coefficients of the system (5) are a following values:

$$a = 1, \; b = 5, \; h = 1.5. \tag{10}$$

Use of numerical simulations as shown in [3], proves that the system (5) with a values (10), have a chaotic behavior for all values of α equation, fractional value in the range of 0.1–1.9 in steps of $\Delta\alpha = 0.1$.

Examples of trajectories for fractional order: $\alpha = 0.9$ and $\alpha = 1.5$ with the initial condition $x_0(\tau) = 0.1$ for time constant $h = -1.5$. Those results are reported in Figs. 1 and 2.

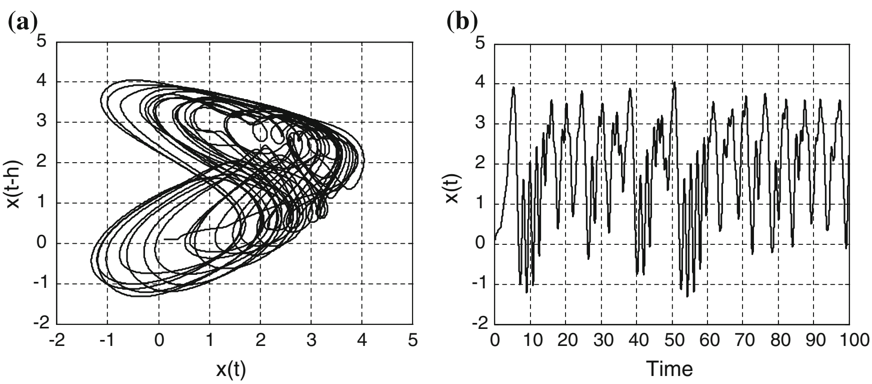

Fig. 1 Plots for $\alpha = 0.9$: **a** chaotic trajectory; **b** plot of $x(t)$ [4]

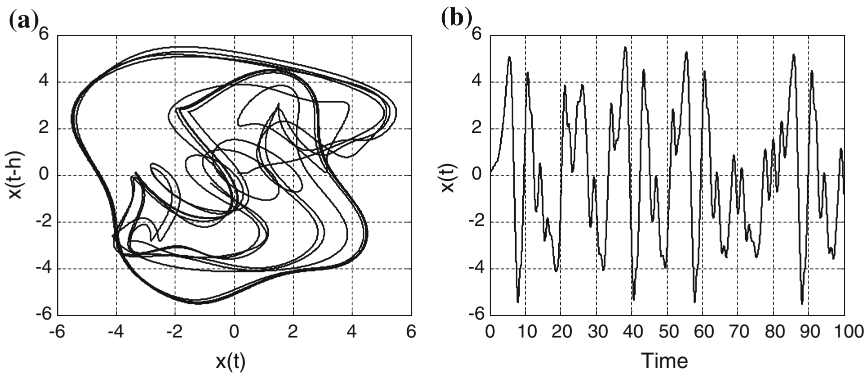

Fig. 2 Plots for α = 1.5: **a** chaotic trajectory; **b** plot of $x(t)$ [4]

In reference [4], a study of chaotic synchronization, during a connection of two Ikeda systems, combined as a transmitter/receiver according to equation:

$$ {}_0D_t^{\alpha}\bar{x}(t) = -a\bar{\mathbf{x}}(t) + b\sin\bar{x}(t-h) + k(x(t) - \bar{x}(t)) \tag{11} $$

where parameter k is a gain in feedback loop. Such a combined system, examined deviation value $e(t) = x(t) - \bar{x}(t)$ during synchronization, based on the fractional order of functional dependence, described by equation:

$$ {}_0D_t^{\alpha}e(t) = -a\,e(t) + b[\sin x(t-h) - \sin\bar{x}(t-h)] + ke(t) \tag{12} $$

Chaotic system is considered to be synchronized, if deviation is set to a minimum:

$$ \min|e(t)| \leq 0.01 \tag{13} $$

3 Main Results

In a simulation study, we investigated a synchronization of two chaotic systems, connected to Ikeda feedback loop. At the output of feedback loop, there was a measured deviation of $e(t) = x(t) - \bar{x}(t)$, for the assessment of entire system synchronization. In order to shorten a time synchronization on the output loop, use an Alpha-Beta filter. Simulation studies examined, at which of the parameter values α_F, β_F time synchronization is the shortest. Tests were performed by measuring system environment using Matlab/Simulink, as shown on Fig. 3.

For a simulations in Matlab/Simulink, toolbox was used Ninteger fractional the order control circuits by block called nid [20]. Filter parameters as α_F, β_F are experimentally chosen on the basis of Eqs. (14–17). During the calculations, a value

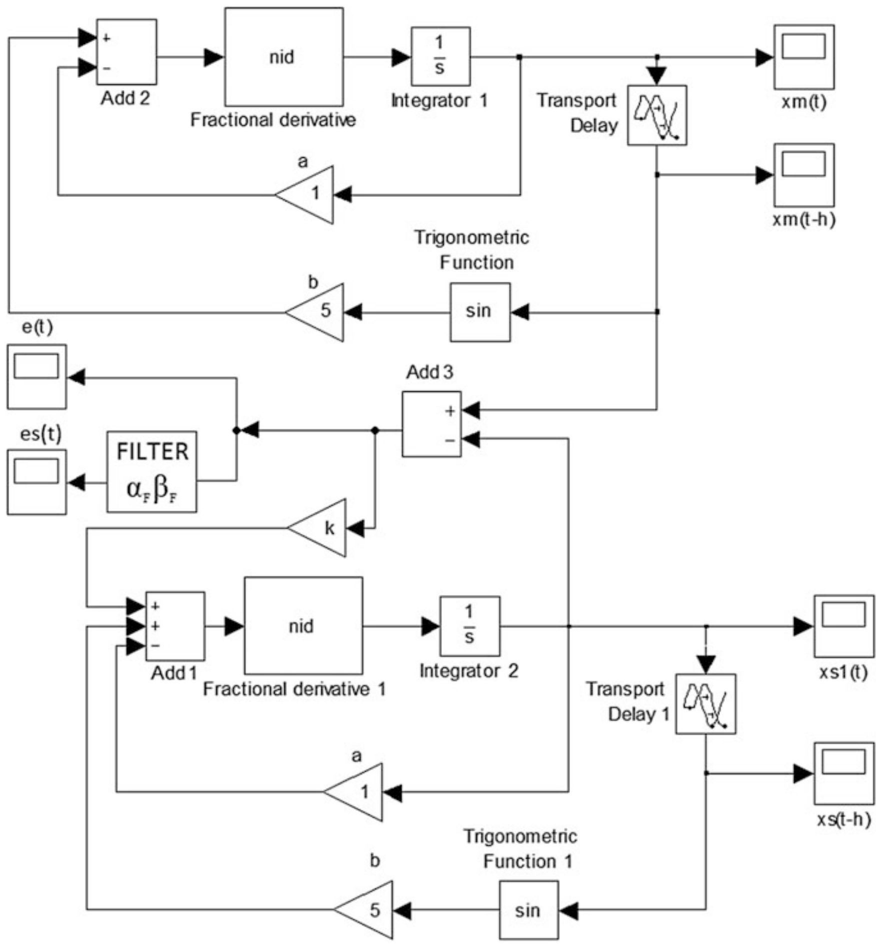

Fig. 3 The model system in Matlab/Simulink based on basis of Eqs. (5), (11), (12) of chaotic Ikeda systems, combined in feedback loop

of prediction error at 99 %, noise variance $\sigma_w{}^2$ process variance $\sigma_v{}^2$ and a value of tracking λ at level of fractional values:

$$\lambda = \frac{\sigma_w T^2}{\sigma_v} \tag{14}$$

$$r = \frac{4 + \lambda - \sqrt{8\lambda + \lambda^2}}{4} \tag{15}$$

$$\alpha_F = 1 - r^2 \tag{16}$$

$$\beta_F = 2(2 - \alpha_F) - 4\sqrt{1 - \alpha_F} \quad (17)$$

The aim of the simulation is to explore how the same measured signal *x(t)* affected by Alpha-Beta filter in which the changed parameter α_F. For example experimentally checked how changes to quality of filtration for the three case, where parameter β_F it was calculated according to the formula (17):

(a) $\alpha_F = 5 \times 10^{-4}$; $\beta_F = 1.25 \times 10^{-7}$;
(b) $\alpha_F = 3 \times 10^{-4}$; $\beta_F = 4.5 \times 10^{-8}$; (c) $\alpha_F = 1 \times 10^{-4}$; $\beta_F = 5 \times 10^{-9}$;

The estimated signal after filtration marked xs(t). The best result was obtained for case (c) where the parameter filtration: $\alpha_F = 1 \times 10^{-4}$; $\beta_F = 5 \times 10^{-9}$; Simulation results are given in Figs. 4, 5 and 6. On the figure adopted designation: x(t) signal before filtration; xs(t) signal after filtration.

Then, the filter was tested how it affects filter Alpha-Beta to shorten the time synchronization according *e(t)* for the formula (13). Ikeda synchronization test system using the filter, was performed for a fraction α = 1.9. Simulation results are shown on Figs. 7 and 8. Shortest time of synchronization, was observed for a value of filter parameters, where $\alpha_F = 8 \times 10^{-4}$; $\beta_F = 3.2 \times 10^{-7}$; which was less than 8 ms (Fig. 8).

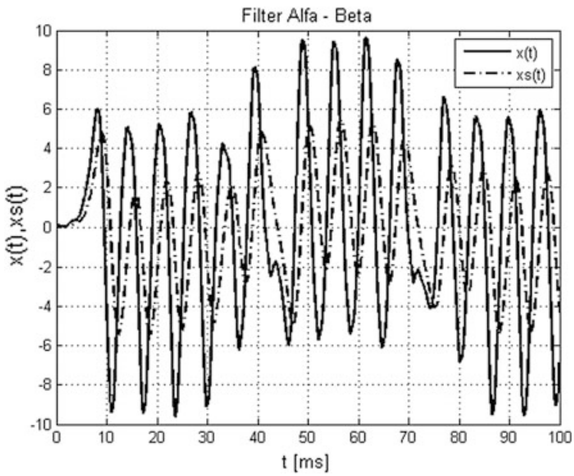

Fig. 4 Time course measured at the output of Ikeda model parameter values for α = 1.9; k = 22; **a** x(t) before filtration **b** xs(t) after filtration for $\alpha_F = 5 \times 10^{-4}$; $\beta_F = 1.25 \times 10^{-7}$

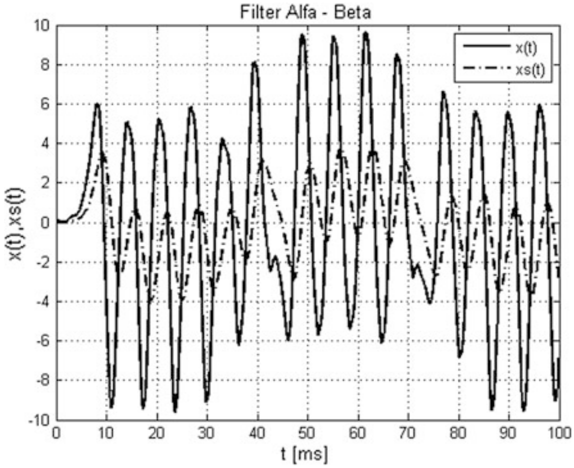

Fig. 5 Time course measured at the output of Ikeda model parameter values for α = 1.9; k = 22; **a** x(t) before filtration; **b** xs(t) after filtration for $\alpha_F = 3 \times 10^{-4}$; $\beta_F = 4.5 \times 10^{-8}$

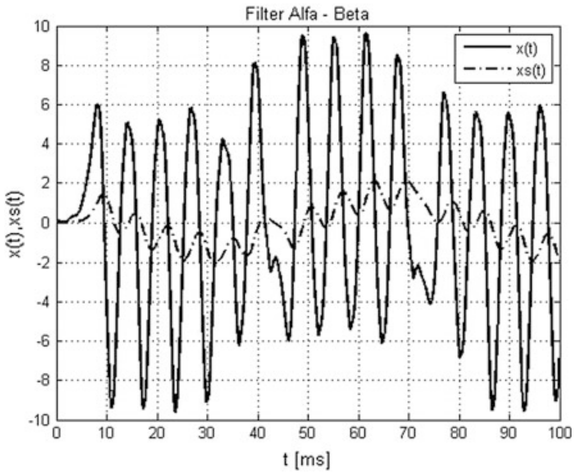

Fig. 6 Time course measured at the output of Ikeda model parameter values, for α = 1.9; k = 22; **a** x(t) before filtration; **b** xs(t) after filtration for $\alpha_F = 1 \times 10^{-4}$; $\beta_F = 5 \times 10^{-9}$

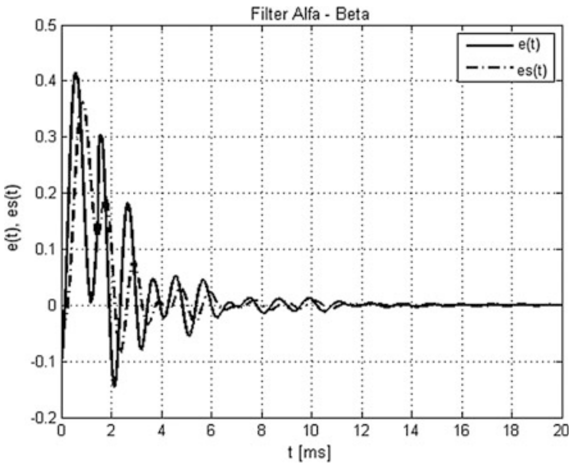

Fig. 7 Time course of cardiovascular Ikeda synchronization for a parameter values of α = 1.9; k = 22; **a** e(t) before filtration; **b** es(t) after filtration for $\alpha_F = 3 \times 10^{-3}$; $\beta_F = 4.5 \times 10^{-6}$;

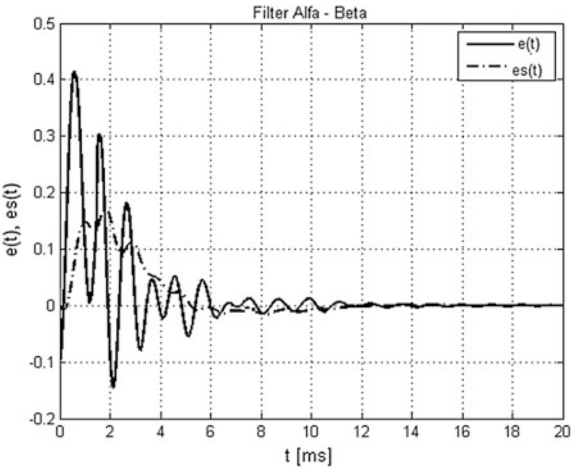

Fig. 8 Time course of cardiovascular Ikeda synchronization for a parameter values of α = 1.9; k = 22; **a** e(t) before filtration; **b** es(t) after filtration for $\alpha_F = 8 \times 10^{-4}$; $\beta_F = 3.2 \times 10^{-7}$

4 Concluding Remarks

For more detailed information on a state of object, there are to be made on the basis of data measurement estimation. This is particularly important, when selecting a values of filter parameters as α_F, β_F and their deviation from a nominal value, chosen experimentally. Parameter values determines a quality of filtering interference, in particular, determine the time synchronization of two chaotic objects. Deviation filter parameters α_F, β_F nominal value, resulting in a lack of filtration and also can create an additional distortions. Sensitivity of the filter can change a parameter values of less than ten thousand, demonstrates sensitivity of an entire filter system. Main result of work, is to obtain a minimum of time synchronization of two coupled chaotic systems in less than 8 ms.

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On Dynamic Decoupling of MIMO Fractional Order Systems

Paweł Dworak

Abstract In the paper problems with a dynamic decoupling of multi-input multi-output MIMO fractional order systems are discussed. Similarities and differences to integer order decoupling methods are shown. Basing on a few examples taken from a literature simulations of decoupled fractional order systems were caried out. Te paper ends with some final remarks on a practical implementation of decoupling methods for fractional order systems.

Keywords Dynamic decoupling · Fractioanl order systems · MIMO

1 Introduction

One of the main problems encountered in the practical application of dynamic decoupling methods is the accuracy of mathematical models describing properties of the controlled plants. Inaccuracy between plant model and its real behavior may make full decoupling impossible or lead directly to the system instability. As we know from numerous studies the fractional order models can depict the physical plant better than the classical integer order ones. This covers different research field such as insulator properties, visco-elastic materials, electrodynamic, electrothermal, electrochemical, economic processes modelling etc. [1, 9, 12, 14, 19–21, 26, 28, 31].

Strong interest in the subject led to the creation of tools for synthesis and simulation of fractional systems, and the most popular are those prepared to work in Matlab environment [3, 4, 24, 33, 34].

Despite the very high interest in a fractional order calculus and its usage in the synthesis of the control system, works directly devoted to control of a fractional MIMO arose relatively few [15, 16, 18, 22, 32]. This may be due to less interest in the specificity of MIMO systems as well as difficulties with the transfer and extension

P. Dworak (✉)
West Pomeranian University of Technology, ul. 26 Kwietnia 10,
71-126 Szczecin, Poland
e-mail: pawel.dworak@zut.edu.pl

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of the methods developed for the integer order systems to fractional ones. Problems of dynamic decoupling of fractional MIMO plants have been raised so far, according to the author's knowledge, only in [16] and related to the transfer of decoupling methods for integer order TITO plants to their factional counterparts.

The paper presents limitations and differences in analysis of fractional MIMO plants in the context of a dynamic decoupling. To do that the paper is organized as follows. In Sect. 2 basic definitions and fractional calculus are presented. Then in Sect. 3 a static decoupling and RGA indices for fractional plants are analyzed. Different examples of dynamic decoupling for fractional TITO and MIMO plants are presented in Sect. 4. A conclusion and final remarks on future research are given in Sect. 5.

2 Basics of Calculus and Modeling of Fractional Order Systems

Fractional order calculus introduces an operator ${}_{t_o}D_t^r$ which is a generalization of the integer order differentiation and integration. t_o and t are the time limits for differentiation and integration and $r \in R$ describes the order of the operator. In general, the rank r may be a complex number [25]. The continuous operator is defined as [3]

$$ {}_{t_o}D_t^r = \begin{cases} \dfrac{d^r}{dt^r} & r > 0 \\ 1 & r = 0 \\ \int\limits_{t_o}^{t} (d\tau)^r & r < 0 \end{cases} \tag{1} $$

There are many definitions of the integro-differential operator. The most popular are the Grunwald-Letnikov, Riemann-Liouville and Caputo [13, 23, 27]. A particular practical importance in the control systems has a definition of Grunwald-Letnikov

$$ {}_{t_o}D_t^r f(t) = \lim_{h \to 0} h^{-r} \sum_{j=0}^{\left[\frac{t-t_o}{h}\right]} (-1)^j \binom{r}{j} f(t - jh) \tag{2} $$

where $[\cdot]$ stands for an integer part.

Assuming a recursive form of the binomial coefficient as

$$ \begin{cases} c_o^r = 1 \\ c_j^r = c_{j-1}^r \left(1 - \frac{r+1}{j}\right), \; j = 1, 2, \ldots \end{cases} \tag{3} $$

the Grunwald-Letnikov integro-differential operator may be written in the following form

$$ {}_{t_o}D_t^r f(t) = \lim_{k\to\infty} \left(\frac{t-t_o}{k}\right)^{-r} \sum_{j=0}^{k} c_j^r f(t - j\frac{t-t_o}{k}) \tag{4} $$

In practice the sum in the above definition must have a finite number of values L (number of samples of $f(t)$) and an estimation error ε can be calculated from

$$ L \geq \left(\frac{M}{\varepsilon|\Gamma(1-r)|}\right)^{1/r} \tag{5} $$

where M denotes maximum value of function $f(t)$ at particular points.

The Laplace transform of the Grunwalda-Letnikova fractional operator (4) can be calculated for $r \in [0, 1]$ only and is given by [13]

$$ \int_0^\infty e^{-st} {}_{t_0}D_t^r f(t) dt = s^r F(s) \tag{6} $$

Dynamics of the modeled control object may be described by fractional differentiation equation [27]

$$ \begin{aligned} &a_n D^{\alpha_n} y(t) + a_{n-1} D^{\alpha_{n-1}} y(t) + \cdots + a_0 D^{\alpha_0} y(t) = \\ &b_m D^{\beta_m} u(t) + b_{m-1} D^{\beta_{m-1}} u(t) + \cdots + b_0 D^{\beta_0} u(t) \end{aligned} \tag{7} $$

where $D^\gamma \equiv {}_0D_t^\gamma$, a_k and α_k $k = 0, 1, \ldots, n$, b_l and β_l $l = 0, 1, \ldots, m$ are real numbers.

A continuous transfer function of a fractional plant may be described by

$$ G(s) = \frac{y(s)}{u(s)} = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \cdots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \cdots + a_0 s^{\alpha_0}} \tag{8} $$

where $y(s)$ and $u(s)$ are the Laplace transforms of the output and input signals respectively.

A fractional linear time-invariant LTI plant may be also depicted by the state and output equations

$$ \begin{aligned} {}_0D_t^r \mathbf{x}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) &= \mathbf{C}\mathbf{x}(t) \end{aligned} \tag{9} $$

with a state vector $\mathbf{x}(t) \in R^n$, $\mathbf{u}(t) \in R^m$ and $\mathbf{y}(t) \in R^l$ vectors of input and output signals respectively.

As it was shown in [22] conversion between models (8) and (9) may be done in the same manner like for the integer order systems. Rank r should be equal for all model states $\mathbf{x}(t) \in R^n$. In general it might be an operator ${}_0D_t^q$ with $q = \{q_1, \ldots, q_n\}$.

A discrete version of a fractional model (8) needs a discrete equivalent of the integro-differential operator ($w(z^{-1})$)—which is a function of the complex variable z or shift operator z^{-1}—takes the form

$$G(z) = \frac{b_m(w(z^{-1}))^{\beta_m} + b_{m-1}(w(z^{-1}))^{\beta_{m-1}} + \cdots + b_0(w(z^{-1}))^{\beta_0}}{a_n(w(z^{-1}))^{\alpha_n} + a_{n-1}(w(z^{-1}))^{\alpha_{n-1}} + \cdots + a_0(w(z^{-1}))^{\alpha_0}} \tag{10}$$

By defining a discrete fractional difference of rank r of a discrete function $f(t)$ as [5, 29]

$$_{t_0}\Delta_t^r f(t) = \sum_{j=0}^{t-t_0} c_j^r f(t-j) \tag{11}$$

with c_j^r defined by (3) one obtain a discrete linear fractional model of the plant in a state space

$$\begin{aligned} \mathbf{x}(t+1) &= \mathbf{A}_d\mathbf{x}(t) + \mathbf{Bu}(t) - \sum_{j=1}^{t+1} c_j^r \mathbf{x}(t+1-j) \\ \mathbf{y}(t) &= \mathbf{Cx}(t) \end{aligned} \tag{12}$$

where $\mathbf{A}_d = \mathbf{A} - \mathbf{I}_n$.

Matrix of discrete transfer functions for a fractional MIMO plant may be then calculated from the formula [5]

$$\mathbf{G}(z^{-1}) = \begin{bmatrix} g_{11}(z^{-1}) & \ldots & g_{1m}(z^{-1}) \\ \ldots & \ldots & \ldots \\ g_{l1}(z^{-1}) & \ldots & g_{lm}(z^{-1}) \end{bmatrix} = \mathbf{C}\left[\mathbf{I}_n \sum_{j=0}^{L} c_j^r z^{-j+1} - \mathbf{A}_d\right]^{-1} \mathbf{B} \tag{13}$$

where L, as in (5), means a finite number of state values.

The presented above methods of modeling of fractional systems seems very like the traditional integer order systems ones. However, it does not mean we can directly use an integer order static and dynamic decoupling methods for fractional plants. The differences arise naturally from another nature of plant dynamic and from the numeric problems with estimation of value of the fractional-order operator (5).

Adopting a fractional model for a controlled plant changes different indicators used in analysis and synthesis of the MIMO control systems. One of them is a very popular Relative Gain Array (RGA) [2]. Authors [16] observe and give a suitable example that RGA value and the way of its change is different for fractional and integer order processes model. Moreover, as we will present later, adopting a fractional plant model and the level of accuracy of the fractional-order operator estimation may also change adopted in the synthesis plant properties and further affect the regulation quality.

3 Static Decoupling of Fractional MIMO Plants

Let us consider a stable discrete fractional order system described by (13). As for integer order systems it can be statically decoupled by a static precompensator $\boldsymbol{G} \in R^{m \times l}$ satisfying the equation

$$\boldsymbol{K}_P \boldsymbol{G} = \boldsymbol{I}_l \tag{14}$$

where $\boldsymbol{K}_P \in R^{l \times m}$ is a gain matrix of the decoupled plant. Its numerical values can be determined for the model (13) from the relation

$$\boldsymbol{K}_P = [\boldsymbol{C}(\mathbf{I}_n \sum_{j=0}^{L} c_j^r z^{-j+1} - \mathbf{A}_d)^{-1} \boldsymbol{B} + \boldsymbol{D}]_{|z=1} = \boldsymbol{C}(\mathbf{I}_n \sum_{j=0}^{L} c_j^r - \mathbf{A}_d)^{-1} \boldsymbol{B} + \boldsymbol{D} \tag{15}$$

From which it follows that the number of elements in the sum L (and model accuracy) affects value of the gain matrix and further value of the precompensator matrix **G**. To illustrate this potential control system synthesis problem let us assume an integer order plant described by state and output equations with matrices

$$\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ -1 & -2 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 2 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \mathbf{C} = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 \end{bmatrix}, \mathbf{D} = \mathbf{0} \tag{16}$$

The plant RGA matrix for a steady state takes the form

$$RGA(0) = \begin{bmatrix} 0.6 & 0.1333 & 0.2667 \\ 0 & 0.5333 & 0.4667 \\ 0.4 & 0.3333 & 0.2667 \end{bmatrix} \tag{17}$$

However assuming the fractional order rank $r = 0.4$ for all plants states the values of RGA matrix parameters will change with the assumed modeling accuracy depending on L as in Fig. 1.

Methods of modeling of the fractional plants, including changes of gain matrices of the modeled plant, may influence a control performance, particularly in the adaptive control systems with controllers switching [30].

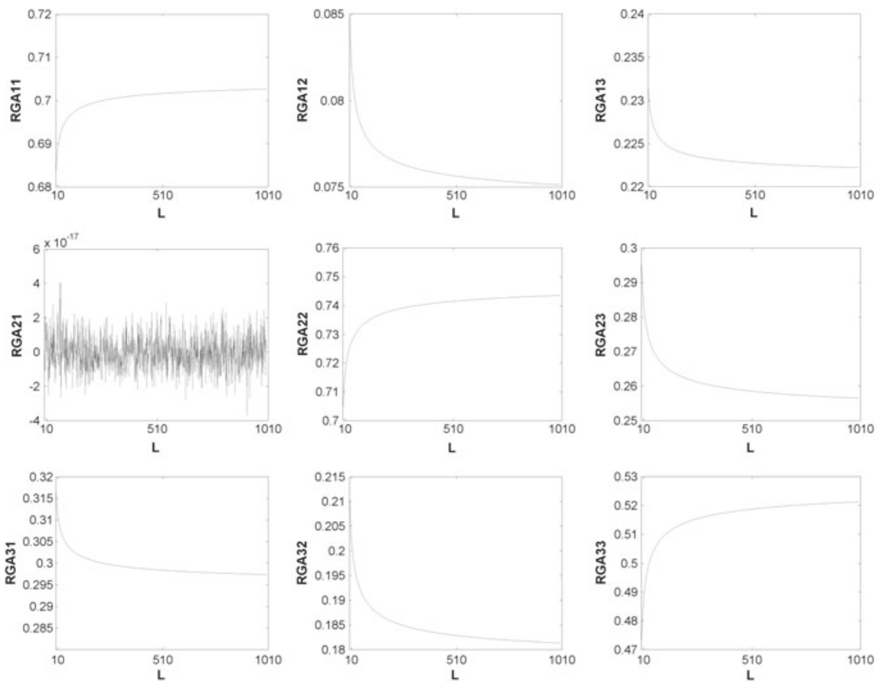

Fig. 1 RGA as the function of L

4 Dynamic Decoupling of Fractional MIMO Plant

4.1 Dynamic Decoupling of TITO Plants

Dynamic decoupling of fractional TITO plants has been analyzed in a [16] only. It was realized by methods used for an integer order TITO processes with a transfer function matrix

$$\mathbf{G}(s) = \begin{bmatrix} g_{11}(s) & g_{12}(s) \\ g_{21}(s) & g_{22}(s) \end{bmatrix} \tag{18}$$

where the particular transfer functions $g_{ij}(s)$ has the form

$$g_{ij}(s) = \frac{K_{ij}}{T_{ij}s^{\alpha_{ij}} + 1}; i, j = 1, 2, \ldots \tag{19}$$

From among three typical decoupling methods [10, 16] for the described by model (18) plant the simples two need to calculate the precompensators

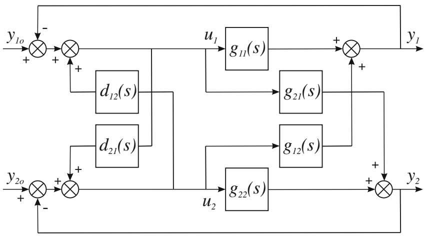

Fig. 2 Scheme of the inverted (*left*) and simplified (*right*) decouplers

$$d_{12}(s) = -\frac{g_{12}(s)}{g_{11}(s)} \text{ and } d_{12}(s) = -\frac{g_{12}(s)}{g_{11}(s)} \tag{20}$$

and their use in structures of *simplified* and *inverted* decoupling (Fig. 2).

Then the closed loop transfer function has the form

$$\mathbf{K}(s) = \begin{bmatrix} g_{11}(s) - \frac{g_{12}(s)g_{21}(s)}{g_{22}(s)} & 0 \\ 0 & g_{22}(s) - \frac{g_{12}(s)g_{21}(s)}{g_{11}(s)} \end{bmatrix} \tag{21}$$

for a simplified decoupling and

$$\mathbf{K}(s) = \begin{bmatrix} g_{11}(s) & 0 \\ 0 & g_{22}(s) \end{bmatrix} \tag{22}$$

for inverted one.

However, just to realize such constructed precompensators the process model (18) has to satisfy the conditions [16]

$$\alpha_{11} \le \alpha_{12} \text{ and } \alpha_{22} \le \alpha_{21} \tag{23}$$

In case of integer order plants such constructed control system allows to fully decouple the plant. These results for fractional plants are difficult (if possible) to achieve.

Example 1—Dynamic Decoupling of the Thermoelectric TITO Plant

After authors of [16] we show an example of decoupling of the thermo-electric temperature plant described as following [17]

$$\mathbf{P}(s) = \begin{bmatrix} \frac{1.2}{2s^{0.5}+1} & \frac{0.6}{3s^{0.7}+1} \\ \frac{0.5}{s^{0.8}+1} & \frac{1.5}{3s^{0.6}+1} \end{bmatrix} \tag{24}$$

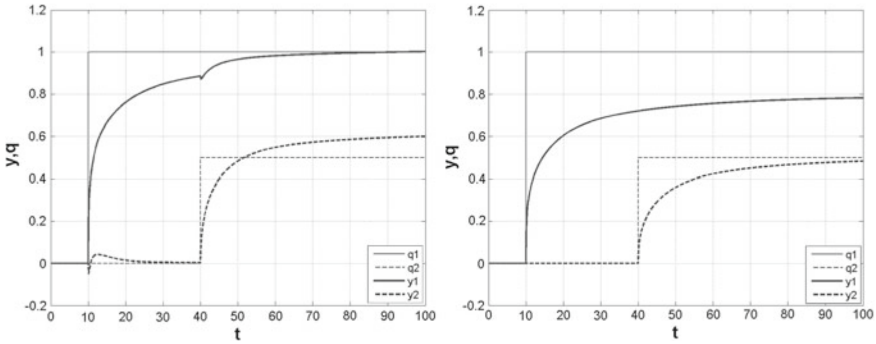

Fig. 3 Result of simulations of the decoupled plant with inverted (*left*) and simplified (*right*) decouplers

The precompensator elements (20) have then the form

$$d_{12}(s) = -\frac{2s^{0.5}+1}{2(3s^{0.7}+1)}, d_{21}(s) = -\frac{3s^{0.6}+1}{3(s^{0.8}+1)} \tag{25}$$

and the simulation results in structures presented in (Fig. 2) are presented in (Fig. 3). The simulations have been carried out in Matlab/Simulink environment with the use of Ninteger toolbox [33, 34]. As we see in both figures in relation to the results shown in [16] the accuracy of modeling the fractional operator may have an influence on incomplete plant decoupling. This is illustrated clearly by another example.

Example 2—Dynamic Decoupling of a Nonsquare MIMO Plant

Let us take under consideration one from the plants analyzed in [11]. It is the integer order plant with four inputs and three outputs described by the following transfer matrix

$$\mathbf{P}(s) = \begin{bmatrix} \frac{-(s+1)}{s-1} & \frac{s+1}{s-1} & \frac{-(s+1)}{s-1} & \frac{-2}{s-1} \\ 0 & 0 & 0 & \frac{s-1}{(s-1)^2} \\ -1 & 1 & \frac{2}{s-1} & \frac{-2}{s+1} \end{bmatrix} \tag{26}$$

The plant has unstable pole $s = 1$, and zeros at $s = 1$ and ∞. According to the theoretical analysis presented in [11] the existing pole zero coincidence is non-structural and one can find a precompensator $\mathbf{C}(s)$, for which the full dynamic decoupling will be possible. For a one of possible precompensators

$$\mathbf{C}(s) = \begin{bmatrix} 0 & 0 & 0 \\ \frac{2}{s+1} & \frac{2}{s+1} & 1 \\ \frac{-(s-1)}{s+1} & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \tag{27}$$

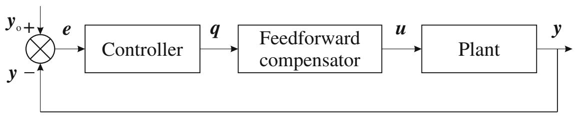

Fig. 4 General scheme of the control system with a dynamically decoupled plant

a diagonal transfer matrix of the decoupled system $\mathbf{D}(s)$, $\mathbf{P}(s)\mathbf{C}(s) = \mathbf{D}(s)$ takes the form

$$\mathbf{D}(s) = \begin{bmatrix} \frac{s+1}{s-1} & 0 & 0 \\ 0 & \frac{s-1}{(s+1)^2} & 0 \\ 0 & 0 & \frac{s+1}{s-1} \end{bmatrix} \tag{28}$$

The decoupled system is unstable and nonminimumphase but its stabilization and control are possible. Example taking three separate PI controllers with the following parameter values: $k_{r1} = 4$, $T_{i1} = 10$ in firs loop, $k_{r2} = 0.1$, $T_{i2} = 10$ in second loop and $k_{r3} = 3$, $T_{i3} = 10$ for third one we obtained a closed loop system (Fig. 4) which simulation results are presented in Fig. 5.

Using a polynomial approach with operator s replaced by $\nu = s^r$ for $r = 0.5$ we obtain matrix of transfer functions $\mathbf{D}(\nu)$ similar to that of integer order (28). The practice of control system synthesis for fractional plants shows that controllers for such plants may be both integer and fractional order. Despite this for the discussed fractional plant neither integer nor fractional PI controller does not ensure closed loop stability (Fig. 6). The controller settings were adopted as for the integer order system. In both cases, it is clear that the methods of decoupling of the integer order plants may not be capable of full application to the fractional plants.

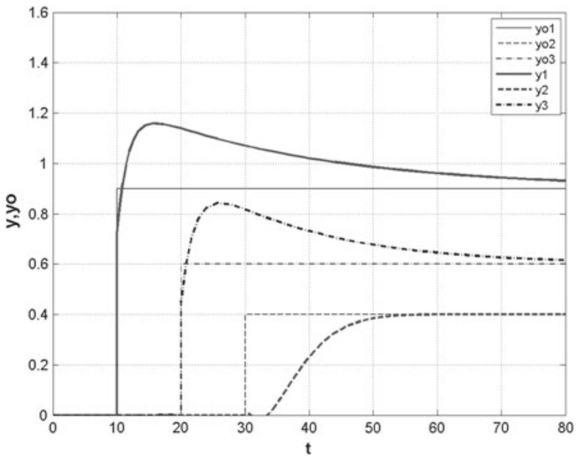

Fig. 5 Result of simulation of the integer order decoupled system

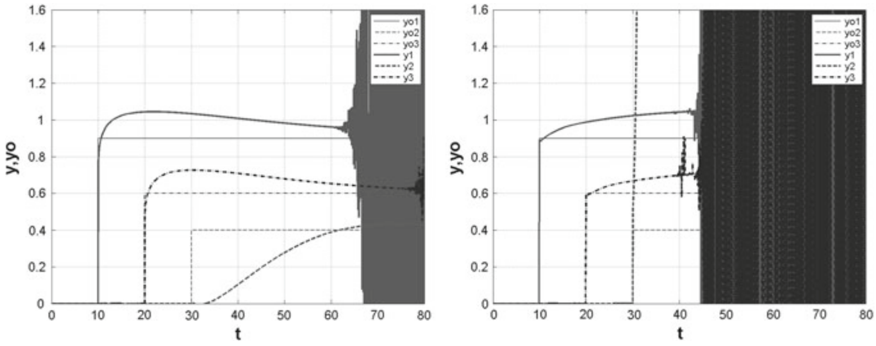

Fig. 6 Result of simulation of the decoupled fractional system with integer order (*left*) and fractional order (*right*) controller

4.2 Dynamic Decoupling of Fractional MIMO Plants with a State Feedback

As the plant may be modeled by a fractional LTI state and output equations (9), it seems to be able to decouple the plant by using a state feedback. It opens a lot of possibilities as the state feedback methods are very popular in decoupling integer order counterparts. One of them is a method presented in [6, 8]. This algorithm may be used for the synthesis of dynamic decouplers for LTI plants which could be unstable, non-minimum phase or both, described by rectangular proper rational full rank transfer matrices. It is a polynomial method where a transfer matrix of the MIMO plant and another system elements are presented in the form of a relatively right or left polynomial matrix fraction description (MFD)

$$\mathbf{T}(\cdot) = \mathbf{C}(\cdot\mathbf{I}_n - \mathbf{A})^{-1}\mathbf{B} + \mathbf{D} = \mathbf{B}_1(\cdot)\mathbf{A}_1{}^{-1}(\cdot) = \mathbf{A}_2{}^{-1}(\cdot)\mathbf{B}_2(\cdot) \tag{29}$$

with polynomial matrices: denominator $\mathbf{A}_1(\cdot) \in \mathbb{R}[\cdot]^{m\times m}$, $\mathbf{A}_2(\cdot) \in \mathbb{R}[\cdot]^{p\times p}$ and numerator $\mathbf{B}_1(\cdot) \in \mathbb{R}[\cdot]^{p\times m}$, $\mathbf{B}_2(\cdot) \in \mathbb{R}[\cdot]^{p\times m}$ here for fractional plants with respect to the $\nu = s^r$ for continuous systems and $\nu = w(z^{-1})$ for discrete-time systems. ($w(z^{-1})$) as in (10) describes a discreet version of a Laplace operator s expressed as a function of the variable z or shift operator z^{-1}. It should be noted that rank r should have the same value for all fractional states. A method of calculating MFD transfer functions for plants with different values $q = \{q_1, \ldots, q_n\}$ of the operator ${}_0D_t^q$ in (9)—according to the author's knowledge—has not been developed. It is quite important limitation of the applicability of integer order decoupling methods for fractional systems and is a major open research problem.

Example 3—Dynamic Decoupling of Fractional MIMO Plants with a State Feedback

Let us assume a plant described by a continuous state space model (9) with matrices as in (16) and ${}_0D_t^q$ with r equal for all model states. The plant is unstable with poles

$\nu_1 = 2$, $\nu_{2,3} = -0.2150 \pm i1.3071$, $\nu_4 = -1$, $\nu_5 = -0.5698$ and non-minimum phase with one transmission zero $\nu_1^o = -2$. Its transfer function matrix $\mathbf{T}(\nu)$ may be described by relatively right polynomial MFD (29) with matrices

$$\mathbf{B}_1(\nu) = \begin{bmatrix} \nu-2 & \nu-8 & 4 \\ 1 & \nu+4 & -1 \\ 0 & 1 & 0 \end{bmatrix}, \mathbf{A}_1(\nu) = \begin{bmatrix} \nu^2-2\nu & -8\nu-1 & 4\nu \\ \nu-2 & \nu^2+\nu-6 & -\nu+3 \\ 0 & 0 & \nu+1 \end{bmatrix} \quad (30)$$

Before starting the synthesis procedure it was assumed that the control system will be diagonalized with the following poles: $\nu_1 = -0.5$ for the first, $\nu_2 = -0.4$ for the second and $\nu_3 = -0.6$, $\nu_4 = -0.4$ for the third block (loop) which gives the denominator matrix $\mathbf{D}(\nu)$ of the decoupled system

$$\mathbf{D}(\nu) = \begin{bmatrix} \nu+0.5 & 0 & 0 \\ 0 & \nu+0.4 & 0 \\ 0 & 0 & \nu^2+\nu+0.24 \end{bmatrix} \quad (31)$$

With the numerator matrix of the decoupled system $\mathbf{N}(\nu) = \mathbf{I}_3$ the method allowed to calculate a static (in general dynamic) precompensator $\mathbf{G}^{-1}(\nu)\mathbf{L}(\nu)$ described by

$$\mathbf{D}_w = \begin{bmatrix} 0.5 & 0 & -0.24 & 1 & 0 & -1 \\ 0 & 0.4 & 0 & 0 & 1 & 0 \\ 0 & 0.4 & 0.24 & 0 & -1 & 1 \end{bmatrix} \quad (32)$$

and feedback matrix $\mathbf{F}$

$$\mathbf{F} = \begin{bmatrix} 0.5 & 1 & 0.5 & 0 & 0 \\ 0 & -0.4 & -1 & 1 & -2.4 \\ 1 & 1.76 & 0 & 0 & 0 \end{bmatrix} \quad (33)$$

Figure 7 shows results of simulations of the diagonalized systems. However, according to the theoretical predictions the above method does not allow for dynamic decoupling in the case of different ranks of operators ${}_0D_t^r$ for the particular states. It is illustrated in Fig. 8 where for the same fractional plant and the decoupled system the fractional rank of the second state was changed to $r_2 = 0.5$—with $r_{1,3,4,5} = 0.9$.

4.3 Interconnection Transmission Zeros in a Dynamic Decoupling of a Fractional MIMO Plant

The above example shows the possibility of using the discussed algorithm to decouple a fractional LTI plant. However, its effectiveness depends also on its ability to meet another requirements, i.e. input/output pairing and grouping, decoupling of left

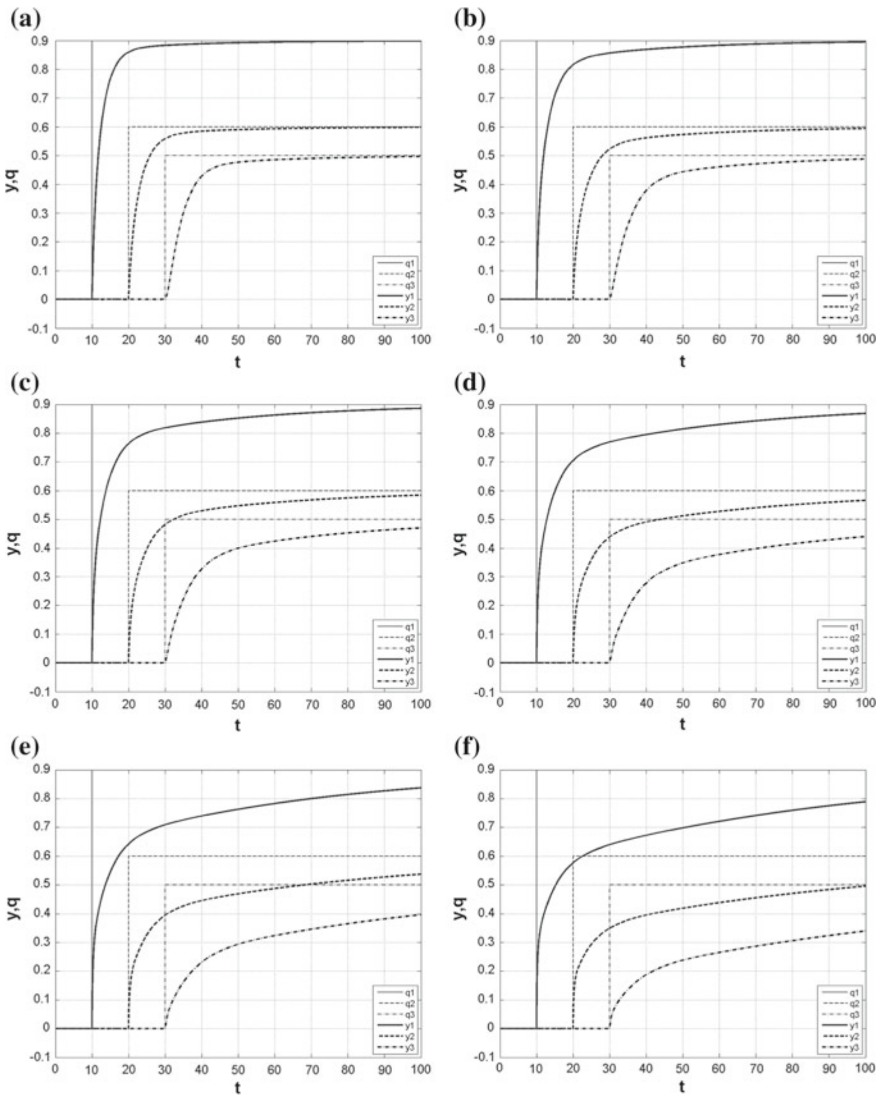

Fig. 7 Result of simulations of the decoupled plant **a** $r = 0.9$, **b** $r = 0.8$, **c** $r = 0.7$, **d** $r = 0.6$, **e** $r = 0.5$, **f** $r = 0.4$

invertible plants and/or dealing with interconnection transmission zeros. It is illustrated by the following example.

Example 4 Let us consider a continuous state space model (9) with matrices as in (16) for which a failure of second input occurred. After crossing out a second column from an input matrix **B** the plant transfer function takes the form

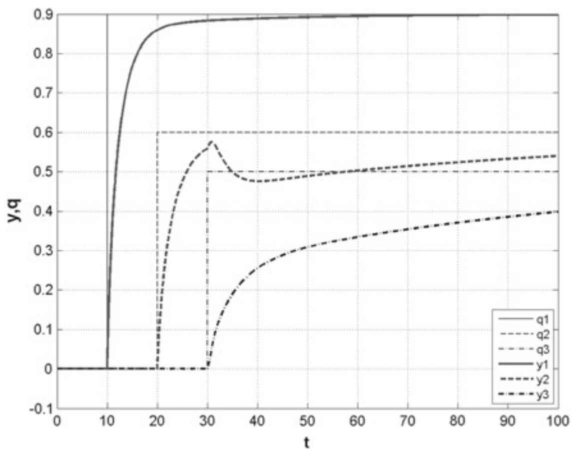

Fig. 8 Result of simulation of the decoupled plant with the rank of the second state $r_2 = 0.5$

$$\mathbf{B}_1(s) = \begin{bmatrix} \nu^2+2 & -\nu-4 \\ -1 & 0 \\ -1 & 1 \end{bmatrix}, \mathbf{A}_1(s) = \begin{bmatrix} \nu^3+\nu^2+2\nu+1 & -2\nu^2-4\nu-1 \\ 0 & \nu^2-\nu-2 \end{bmatrix} \tag{34}$$

As a full (row-by-row) decoupling of the left invertible plant is not possible and also an appropriate theory presented in [7, 35] is not satisfied then to calculate a decoupler we have to cross out any row from matrix $\mathbf{B}_1(s)$. When the second output is omitted then $\mathbf{B}_1(s)$ from (34) takes the form

$$\mathbf{B}_m(s) = \begin{bmatrix} \nu^2+2 & -\nu-4 \\ -1 & 1 \end{bmatrix} \tag{35}$$

with two virtual interconnection squaring down zeros $\nu_1^o = -1$ and $\nu_2^o = 2$ [7]. As ν_2^o has its real part in the right part of the complex plane then the dynamic precompensator cannot be static. Eventually for I/O pairing ($q_1 \rightarrow y_1$; $q_2 \rightarrow y_3$) and the assumed poles $\nu_1 = -0.5$, $\nu_2 = -0.4$, $\nu_3 = -0.6$ for the first and $\nu_4 = -0.4$, $\nu_5 = -1.5$, $\nu_6 = -1.3$, $\nu_7 = -1$ for the second block we obtain a transfer function of the decoupled system $\mathbf{T}_{yq}(\nu) = \mathbf{N}(\nu)\mathbf{D}^{-1}(\nu)$ with matrices

$$\mathbf{N}(\nu) = \begin{bmatrix} \nu^2-\nu-2 & 0 \\ 0 & \nu^2-\nu-2 \end{bmatrix} \tag{36}$$

$$\mathbf{D}(\nu) = \begin{bmatrix} \nu^3+1.5\nu^2+0.74\nu+0.12 & 0 \\ 0 & \nu^4+4.2\nu^3+6.27\nu^2+3.85\nu+0.78 \end{bmatrix}$$

It is realized as in Fig. 9 by a dynamic precompensator described by state and output equations with matrices

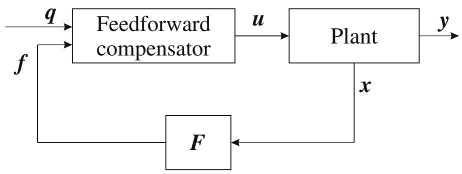

Fig. 9 General scheme of the control system with a dynamically decoupled plant

$$\mathbf{A}_w = \begin{bmatrix} 9.1311 & -0.0768 \\ 5.1810 & -0.8782 \end{bmatrix}, \mathbf{B}_w = \begin{bmatrix} 0 & -0.4171 & 0 & 1.0695 \\ 1.7490 & 7.4747 & -29.1498 & -19.1660 \end{bmatrix},$$
$$\mathbf{C}_w = \begin{bmatrix} -15.8782 & 0.1148 \\ 7.3827 & -0.0559 \end{bmatrix}, \mathbf{D}_w = \begin{bmatrix} -0.0600 & 0.3900 & 1 & -1 \\ 0 & -0.3900 & 0 & 1 \end{bmatrix}$$

and the feedback matrix

$$\mathbf{F} = \begin{bmatrix} 0.2153 & 0.3482 & -0.0506 & 10.4800 & -31.2000 \\ -0.6306 & -1.6865 & -0.9388 & -13.8600 & 41.5800 \end{bmatrix}.$$

As it is shown in Fig. 10, also in this case all of the assumed design objectives are achieved. Change in the value of the first input $q_1(t)$ at $t = 10s$ influences the first $y_1(t)$ and the second (omitted in calculation) $y_2(t)$ output. Similarly input $q_2(t)$ influences outputs $y_2(t)$ and $y_3(t)$ only.

The above simulations have been carried out in Matlab/Simulink with the use of Ninteger toolbox [33, 34]. The condition of obtaining presented here results is the same way to approximate the operator $\nu = s^r$ both for modeled plant and the dynamic precompensator. Different approximations make the system unstable. It seems to be a serious drawback as it demands a very precise plant model identification and particular numeric precision during control of the plant with Interconnection transmission zeros.

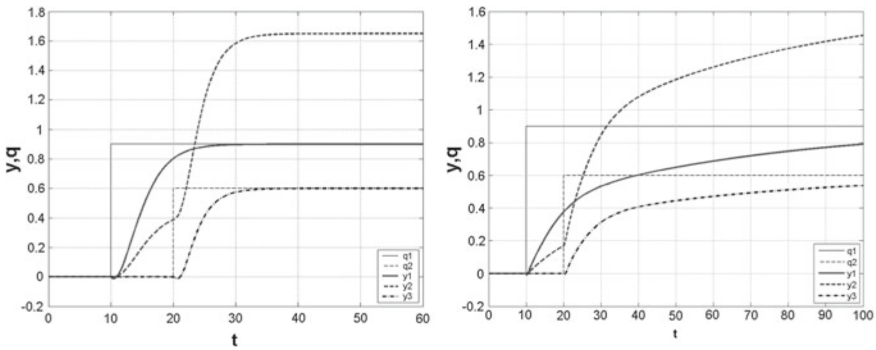

Fig. 10 Result of simulations of the goup decoupled systems; integer order (*left*) and fractional $r = 0.6$ (*right*)

5 Summary

The presented in the paper examples of dynamic decoupling of fractional MIMO plants prove the possibility of using methods developed for integer order systems. It was shown that both state space and output feedback dynamic decoupling methods may be applicable. However, there are some limitations which may make the decoupling difficult or almost impossible to realize. The main problem with that is the numeric approximation of the fractional differentiation and integration which makes it very difficult to deal with plant structural characteristics, e.g. pole-zero coincidence, interconnection transmission zeros. A fractional modeling changes indicators such as RGA used in analysis of the MIMO plants. An open research problem is also to find a method of dynamic decoupling with the use of state feedback for plant models witch different states fractional ranks.

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Perfect Control for Fractional-Order Multivariable Discrete-Time Systems

Łukasz Wach and Wojciech P. Hunek

Abstract In the paper the perfect control for multi-input/multi-output fractional-order discrete-time systems in state space is introduced. A simulation example for nonsquare MIMO system in Matlab/Simulink environment confirms the correctness of the proposed algorithm.

Keywords Perfect control · Fractional-order system · Discrete-time state space system · Nonsquare MIMO system

1 Introduction

In this paper the perfect control algorithm for multivariable fractional-order nonsquare systems, that is systems whose numbers of input and output variables are different, is given. It is emphasized that after some assumption the algorithm reduces to the perfect control of an integer-order discrete-time system in state space framework. The simulation example in Matlab/Simulink environment confirms the correctness of the presented method.

2 Fractional-Order State Space System [9, 10]

Consider an LTI system with n_u-inputs, n_y-outputs and n-state vector in discrete time k described by

$$\begin{cases} \Delta^{\alpha}\mathbf{x}(k+1) = \mathbf{A_d}\mathbf{x}(k) + \mathbf{B}\mathbf{u}(k),\ \mathbf{x}(0) = \mathbf{x_0} \\ \mathbf{y}(k) = \mathbf{C}\mathbf{x}(k) \end{cases} \tag{1}$$

Ł. Wach · W.P. Hunek (✉)
Department of Electrical, Control and Computer Engineering,
Opole University of Technology, Opole, Poland
e-mail: w.hunek@po.opole.pl

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where the discrete-time difference operator Δ^{α} is the Grünwald-Letnikov difference of a fractional order α, with $0 < \alpha < 2$, as follows

$$\Delta^{\alpha}\mathbf{x}(k) = \sum_{j=0}^{k} (-1)^j \binom{\alpha}{j} \mathbf{x}(k-j) \tag{2}$$

where

$$\binom{\alpha}{j} = \begin{cases} 1 & j = 0 \\ \frac{\alpha(\alpha-1)\ldots(\alpha-j+1)}{j!} & j > 0 \end{cases} \tag{3}$$

and $\mathbf{A_d} = \mathbf{A} - \mathbf{I_n}$. Assume also the system is controllable and observable.

3 Fractional-Order Perfect Control

Observe that the Eq. (2) can be rewritten into the form

$$\mathbf{x}(k+1) = \Delta^{\alpha}\mathbf{x}(k+1) - \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} \mathbf{x}(k-j+1). \tag{4}$$

After minimizing the (noise-free) control performance index (for time delay $d = 1$)

$$J(u) = \sum_{k=0}^{\infty} \left\{ \left[\mathbf{y}(k+1) - \mathbf{y_{ref}}(k+1)\right]^{\mathrm{T}} \left[\mathbf{y}(k+1) - \mathbf{y_{ref}}(k+1)\right] \right\} \tag{5}$$

where $\mathbf{y}(k+1) = C\left[\Delta^{\alpha}\mathbf{x}(k+1) - \sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} \mathbf{x}(k-j+1)\right]$ and $\mathbf{y_{ref}}(k+1)$ are the one-step deterministic output predictor and reference/setpoint, respectively, we obtain the perfect control law (for $n_u > n_y$)

$$\mathbf{u}(k) = (\mathbf{CB})^{\mathrm{R}} \left[\mathbf{y_{ref}}(k+1) - \mathbf{CA_d}\mathbf{x}(k) + \mathbf{C}\sum_{j=1}^{k+1} (-1)^j \binom{\alpha}{j} \mathbf{x}(k-j+1)\right] \tag{6}$$

where symbol 'R' denotes any right inverse of **CB** generating the so-called 'control zeros' [1–8].

Remark 1 It should be noted that in our considerations the nonsquare (with $n_u > n_y$) product **CB** of full rank is used (see Sect. 4). Therefore, any nonunique right inverses can be applied in Eq. (6). Of course, for systems with the same numbers of inputs and outputs variables the mentioned inverses reduce to the regular one. Generally, for left-invertible ($n_u < n_y$) or non-full rank **CB** the perfect control does not exist.

Remark 2 For $\alpha = 1$ the Eq. (6) reduces to the perfect control of integer-order systems as follows

$$\mathbf{u}(k) = (\mathbf{CB})^{\mathrm{R}}[\mathbf{y_{ref}}(k+1) - \mathbf{CAx}(k)]. \tag{7}$$

4 Simulation Research

Consider a two-input one-output system with $\mathbf{A_d} = \begin{bmatrix} -1.0 & -0.9 \\ 1.0 & -0.8 \end{bmatrix}$, $\mathbf{C} = \begin{bmatrix} 0 & 2 \end{bmatrix}$, $\mathbf{B} = \begin{bmatrix} -0.025 & -0.085 \\ 0.5 & 0.5 \end{bmatrix}$, $\alpha = 0.5$ and $\mathbf{x_0^T} = \begin{bmatrix} 5 & -8 \end{bmatrix}$. After employing the familiar minimum-norm right inverse of **CB**, for $\mathbf{y_{ref}}(k+1) = 2$, the output remains at the reference/setpoint for $k \geq d = 1$ under the (identical) stabilizing fractional-order perfect controls depicted in Fig. 1. Figure 2 shows the n-state vector $\mathbf{x}(k)$. Note that the fractional difference has been implemented here in form of FIR filter.

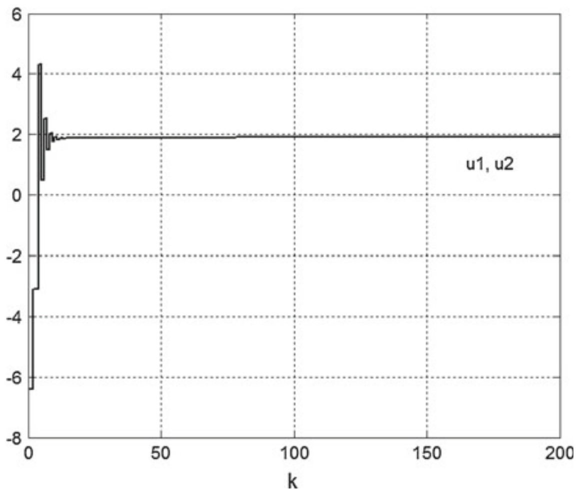

Fig. 1 Fractional-order perfect control: plots of the input signals u_1 and u_2

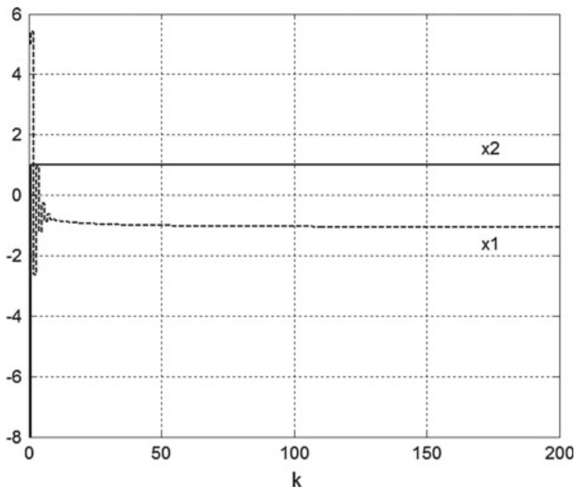

Fig. 2 Fractional-order perfect control: plots of the state signals x_1 and x_2

5 Conclusion

In this paper the perfect control for fractional-order discrete-time MIMO systems in state space has been presented. The objective of future research is synthesis of aforementioned control strategy in terms of its minimum/nonminimum phase behavior and possible control zeros to be obtained.

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Implementation of Non-Integer Order Controller Using Oustaloup Parallel Approximation for Air Heating Process Trainer

Waldemar Bauer

Abstract Nowadays, non-integer controllers are a widely researched problem. One of questions, that is of great importance, is the design of non-integer order controllers and their approximation. In this paper the author presents a new implementation method of non-integer order controller. This controller is designed and analysed for the model of air heating process trainer system belonging to the Department of Automatics and Biomedical Engineering of AGH University and Science and Technology.

1 Introduction

Non-integer controllers are a broadly researched topic. Questions of great importance are the design of non-integer order controllers and their approximation allowing discrete implementation (see [1, 20]).

There are some popular methods of realisation of non-integer order systems in the form of integer order transfer functions. There are however certain issues with their discretisation and subsequent implementation. In this paper, the author proposes a new method of approximation of non-integer order systems based on a method proposed by Oustaloup (see [19]).

Theory of non-integer order systems can be found e.g. in [5, 9, 13, 18, 22]. Oustaloup method was described in [19]. This approximation can be used in simulations [7, 8, 11, 15, 24], filtering [3, 12, 14] and with appropriate care in experiments [10, 17]. Its sensitivity and stability problems during discretization were discussed in [2, 6, 21]. Different method of approximation is based on Laguerre functions [1, 4, 23]. The implementation of the algorithm requires the discretization of the control system designed in a continuous time domain. Earlier results [21] show that transfer

W. Bauer (✉)
AGH University of Science and Technology, Department of Automatics and Biomedical Engineering, Al. Mickiewicza 30, 30-059 Kraków, Poland
e-mail: bauer@agh.edu.pl

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function cannot be directly implemented. That is why a new method of implementation was used.

The rest of the paper is organized as follows. The classical method of Oustaloup is presented with brief discussion of its properties. Then the parallel method of Oustaloup aproximation is presented. Results of experiments are presented and differences between classical and parallel Oustaloup are discussed. Finally the conclusions are drawn.

2 Air Heating Process

Air heating process is a Linear Time Invariant system (LTI) with time delay, and its simplified model can be expressed by transfer function:

$$P(s) = \frac{K}{(T_1 s + 1)(T_2 s + 1)} e^{-s\tau}, \tag{1}$$

where K=18.8, T_1=7.783, T_2=0.0014 and τ=0.5842. The considered model has been precisely identified in work [10].

The considered system is depicted in Fig. 1 and consists of a hollow tube, a temperature sensor, a heater and a fan enforcing the movement of heated air. A photography of air heating process trainer system, which was used in experiments, is shown in Fig. 2.

The designing of control system for a this plant is associated with two major problems. First, the dynamical system of a plant is non-stationary, due to varying gain while the heater and transmission tube are warming-up. Secondly, the study of stability becomes an infinite-dimensional problem, by the presence of continuous time-delay in a dynamical system.

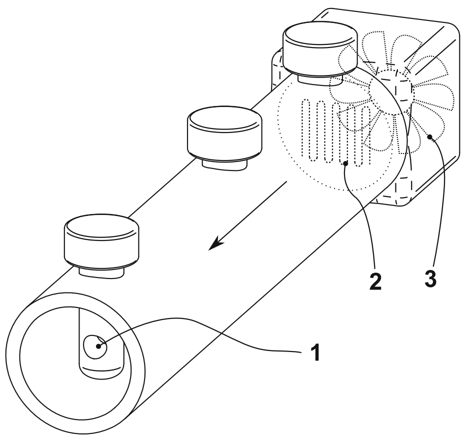

Fig. 1 System schema: *1*—thermoresistive sensor, *2*—heating element, *3*—fan

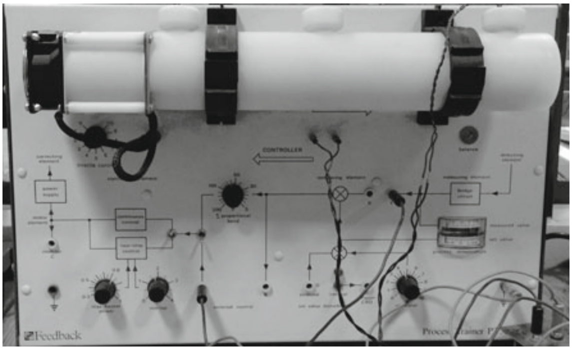

Fig. 2 Process Trainer PT326

3 Design Non-Integer Controller

3.1 Non-Integer Order PID Controller

This section describes a more generalized structure for the classical PID controller. Podlubny proposed a generalization of the PID, namely the $PI^{\lambda}D^{\mu}$ controller, involving an integrator of order λ and a differentiator of order μ. In time domain the equation for the $PI^{\lambda}D^{\mu}$ controller's output has the form (see [22]):

$$u(t) = K_p e(t) + K_i \, {}_0^C\mathrm{D}_t^{-\lambda} e(t) + K_d \, {}_0^C\mathrm{D}_t^{\mu} e(t) \tag{2}$$

where:

- K_p is proportional gain
- K_i is integral gain
- K_d is derivative gain
- $e(t)$ is control deviation in time t
- $\lambda, \mu > 0$

And the transfer function is given by the equation:

$$G(s) = K_p + K_i s^{-\lambda} + K_d s^{\mu} \tag{3}$$

As can be observed, when $\lambda = 1$ and $\mu = 1$ we obtain a classical PID controller, similar when $\lambda = 0$ and $\mu = 1$ give PD, $\lambda = 0$ and $\mu = 0$ give P, $\lambda = 1$ and $\mu = 0$ give PI.

All these classical types of PID are the particular cases of the non-integer $PI^{\lambda}D^{\mu}$. The $PI^{\lambda}D^{\mu}$ is more flexible.

4 Oustaloup Approximation

Oustaloup filter approximation with a fractional-order differentiator $G(s) = s^{\alpha}$ is widely used in applications [16]. An Oustaloup filter can be designed as:

$$G_t(s) = K \prod_{i=1}^{N} \frac{s + \omega_i'}{s + \omega_i} \tag{4}$$

where:

$$\omega_i' = \omega_b \omega_u^{(2i-1-\alpha)/N} \tag{5}$$

$$\omega_i = \omega_b \omega_u^{(2i-1+\alpha)/N} \tag{6}$$

$$K = \omega_h^{\alpha} \tag{7}$$

$$\omega_u = \sqrt{\frac{\omega_h}{\omega_b}} \tag{8}$$

Approximation is designed for frequencies range $\omega \in [\omega_b, \omega_h]$ and N is the order of the approximation. As it can be seen, its representation takes form of a product of a series of stable first order linear systems. As one can observe choosing a wide band of approximation results in large ω_u and high order N result in spacing of poles spacing from close to $-\omega_h$ to those very close to $-\omega_b$. This spacing is not linear (there is a grouping near $-\omega_b$) and causes problems in discretisation process.

5 Oustaloup Parallel Approximation

The method proposed in this paper aims to improve the spacing of poles. Instead of creating a high-order approximation (for $N > 5$) on the entire $[\omega_b, \omega_h]$ interval it is proposed to create a sum of the two approximations: one for lower ($L(s)$) and one for higher ($H(s)$) frequencies, both of order $n = \lfloor N/2 \rfloor$.

$$L(j\omega) \approx j\omega^{\alpha},\ \omega \in [\omega_b, \omega_c] \tag{9}$$

$$H(j\omega) \approx j\omega^{\alpha},\ \omega \in [\omega_c, \omega_h] \tag{10}$$

The division is located in the central frequency ω_c. Both of these approximations should be connected in a series with lowpass and highpass filters respectively (both with cutoff frequency ω_c). Those connections are then connected in parallel. Such construction is presented in Fig. 3.

Because low and high bands are approximated separately such parallel connection is consistent with approximation on the entire band, but with differently spaced poles. In this paper classical first order filters were used.

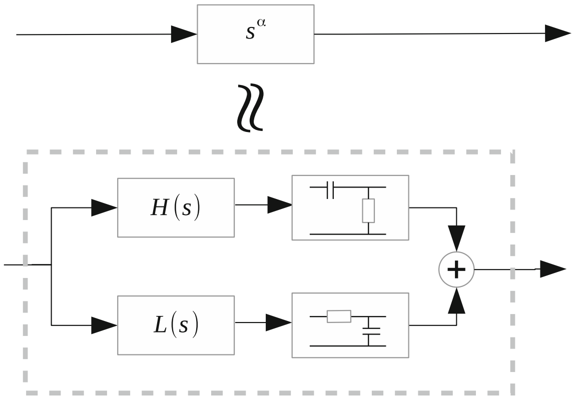

Fig. 3 Oustaloup parallel approximation schema

6 Experiments Results

An experiment has been conducted to compare the performance of classical and parallel Oustaloup approximation of the control for air heating process trainer. Described Oustaloup approximations has been discretaized and implemented in a real-time environment (use RT-DAC and MATLAB/RT-CON). The non-integer order PID settings and Oustaloup approximation parameters are:

- $K_p = 0.0035$
- $K_i = 0.1268$
- $K_d = 0.0206$
- $\alpha = 0.7229$
- $\mu = 0.7307$
- $N = 8$
- $\omega \in [10^{-6}, 10^{6}]$

The considered PID settings has been precisely described in work see [11].

Controllers performance for real plant system has been investigated based on the following experiments. Step response of closed-loop system is shown in Figs. 4 and 5. In the first case we can see that parallel Oustaloup approximation has better numerical performance that classical Oustaloup for the same parameters. This can be seen in figure Fig. 4 in 40th second. This is due to the fact of grouping of poles near $-\omega_b$ in classical Oustaloup method. In parallel implementation this problem does not exist.

The last experiment, presented in Fig. 6 shows the tracking mode of the temperature for the implemented controllers. The comparison of the tracking mode between the basic and author's method shows that parallel implementation can much better control the system in time domain.

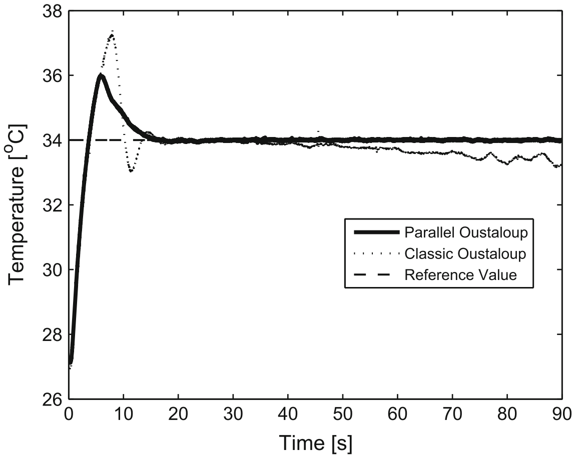

Fig. 4 Comparison of controllers performance for constant reference value

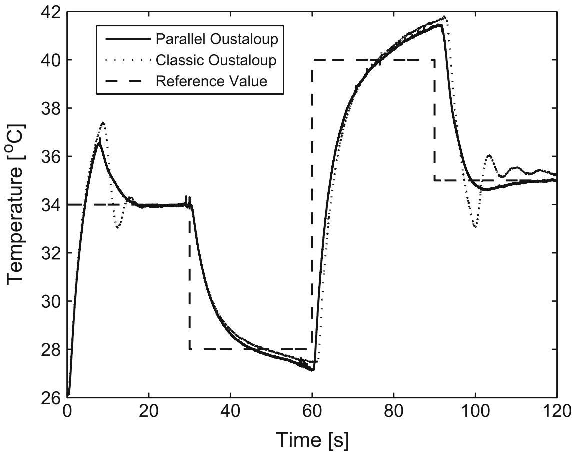

Fig. 5 Comparison of step response

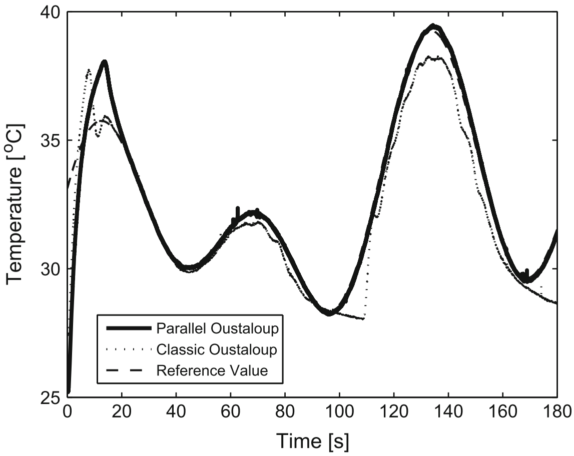

Fig. 6 Comparison of tracking performance

7 Conclusion and Further Research

It has been shown, that $PI^{\alpha}D^{\mu}$ controller is suitable for control of the inertial system with time-delay and time-varying gain.

New method of implementation of Oustaloup approximation was successful and allows operation of non-integer order controller in real time environment. Further work will include implementation of controller in different real time platform and development of methodology for tuning rules of controller.

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Classical Versus Fractional Order PI Current Controller in Servo Drive

Bogdan Broel-Plater, Paweł Dworak and Krzysztof Jaroszewski

Abstract In the paper a fractional order PI current controller of the servo drive is compared with its classical counterpart. The main focus is put on structures of such a fractional order controller without as well as with different antiwindup blocks. Results of simulations carried out in Matlab/Simulink are presented and discussed.

Keywords Servo drive · Fractional order controller · Saturation

1 Introduction

Nowadays servo drives are involved in wide areas of industry. Hence, the DC servo drives have been intensively investigated for many years. For such drives demanding requirements are formulated, especially for these used in robotic and micro machining solutions. For example, servo drives used in machining should fulfil many different expectations, among which the most elementary ones are: high precision of motion in whole spectrum of velocities, loads and operation conditions; high stiffness mechanic characteristic; high ability of start; overload robustness; high dynamics; as well as step-less and smooth motion control in wide range. The highest precision of micro movements are highly desirable also in cases of permanent changes of value of friction due to changes of load mass and torque on the shaft of the drive. Among others, the most essential phenomena which suspends smooth and fast start of motion is dry friction. It also impacts on the drive in case of switching the rotary direction and during operations executed with very small velocities and / or in the case

B. Broel-Plater · P. Dworak · K. Jaroszewski (✉)
West Pomeranian University of Technology, Szczecin, Ul. 26
Kwietnia 10, 71-126 sZCZECIN, Poland
e-mail: kjaroszewski@zut.edu.pl

B. Broel-Plater
e-mail: bbp@zut.edu.pl

P. Dworak
e-mail: pawel.dworak@zut.edu.pl

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of very small movements [5–8]. For drives with mechanical gear drifts elimination by means of modification of construction is rather expensive. Hence, sophisticated control systems have to be used in order to provide expected motion quality at low costs. Moreover, internal values of signals in servo drive control structures are very important. Hence, controller parameters or type of used controller is also very important. On the other hand, the most frequently used is a classical cascade structure with three PI type controllers of: position, velocity and current of the servo drive [5].

Hence, the motivation for authors is to find the best controller and its parameters in order to use it in as many as possible, available on the shelf, servo drive systems with the classical cascade control structure. Many different approaches were used to achieve better servo drive operation efficiency [1, 3, 5, 12, 21]. Proposed in [7] Start-Aid method is one of solutions, especially in case of small motions. Some interesting results may also be obtained by using a fractional order calculus [2, 18]. In [2] authors show positive effects of fractional order controller in the velocity control of a servo system. Following this path in this paper we investigate impact of the fractional order current controller on the control quality. To do that the paper is organized as follows. In Sect. 2 basic definitions and fractional order PID controller are briefly presented. Then in Sect. 3 classical PI controller as well as different structures with fractional order controller are described. Figures presented curves of signals in servo drive control system for all investigated controllers are shown and compared in Sect. 4. A conclusion and final remarks on future research are given in Sect. 5.

2 Basics of Fractional Order Calculus

Fractional order calculus is a generalization of the integer order differentiation and integration. The continuous fractional order operator ${}_{t_o}D_t^r$ is defined as [20]

$$
{}_{t_o}D_t^r = \begin{cases} \frac{d^r}{dt^r} & r > 0 \\ 1 & r = 0 \\ \int_{t_o}^{t} (d\tau)^r & r < 0 \end{cases} \tag{1}
$$

where t_o and t are the time limits for differentiation and integration and $r \in R$ describes the order of the operator. There are many definitions of the integro-differential operator. The most popular are the Grunwald-Letnikov, Riemann-Liouville and Caputo [14, 16, 20].

Utilizing the fractional calculus the dynamics of the modeled control object may be described by fractional differentiation equation [20]

$$\begin{aligned} a_n D^{\alpha_n} y(t) + a_{n-1} D^{\alpha_{n-1}} y(t) + \cdots + a_0 D^{\alpha_0} y(t) = \\ b_m D^{\beta_m} u(t) + b_{m-1} D^{\beta_{m-1}} u(t) + \cdots + b_0 D^{\beta_0} u(t) \end{aligned} \tag{2}$$

where $D^{\gamma} \equiv {}_0D_t^{\gamma}$, a_k and α_k $k = 0, 1, \ldots, n$, b_l and β_l $l = 0, 1, \ldots, m$ are real numbers.

After the Laplace transform of the fractional derivative the continuous transfer function of a plant may be described by

$$G(s) = \frac{y(s)}{u(s)} = \frac{b_m s^{\beta_m} + b_{m-1} s^{\beta_{m-1}} + \cdots + b_0 s^{\beta_0}}{a_n s^{\alpha_n} + a_{n-1} s^{\alpha_{n-1}} + \cdots + a_0 s^{\alpha_0}} \tag{3}$$

where $y(s)$ and $u(s)$ are the Laplace transforms of the output and input signals respectively.

2.1 Fractional Order Controller

As it was shown in many works a more efficient control of the fractional order system may be obtained by the use of fractional order controller [15, 19]. In [19] a generalization of the classic PID controller to the fractional form $PI^{\alpha}D^{\mu}$ was proposed. It involves an integrator of order α and differentiator of order μ. The transfer function of such controller takes then the form

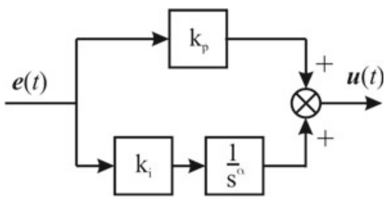
$$G(s) = \frac{u(s)}{e(s)} = k_p + k_i s^{-\alpha} + k_d s^{\mu} \tag{4}$$

where $\alpha, \mu > 0$. This $PI^{\alpha}D^{\mu}$ controller is very flexible than the integer order one (obtained for $\alpha = 1, \mu = 1$) and gives possibility for better plant control.

In many practical application, especially in servo drive systems, are used simpler control algorithm—PI. Scheme of the fractional order PI controller is presented in Fig. 1.

In practice most important issue is minimizing or even eliminating overshooting. One of typical method allowing obtain such effect is using anti-windup. However, such solution may be realized in many different ways [4, 11, 15, 22, 25]. In the paper two of which was investigated in connection with fractional order PI controller. In first one anti-windup covers only integral action what is presented in Fig. 2. In second one anti-windup influences directly on control error—Fig. 3.

Fig. 1 Scheme of the fractional order PI controller



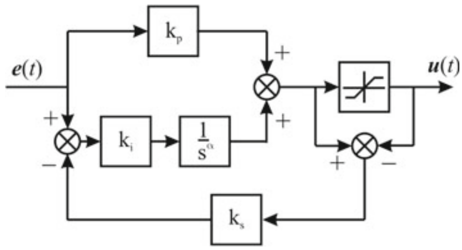


Fig. 2 Scheme of the fractional order PI controller with an anti-windup I

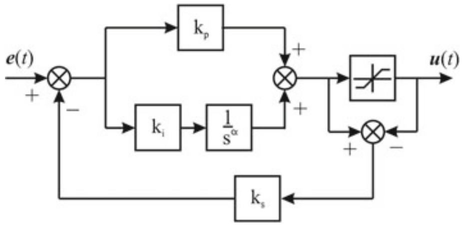


Fig. 3 Scheme of the fractional order PI controller with an anti-windup II

3 Servo Drive

Typical servo drive structure with linear motor is presented in Fig. 4. In that structure there are cascade of three PI controllers responsible for control of: position, velocity and current (torque).

High quality of the control is hard to obtain due to nonlinear friction in servo motor, especially in case of minimal velocity and small motion. In order to take into account above mentioned factors structural model of servo drive motor was used during researches Fig. 5.

Used in simulation model of a DC motor is described by following equations

$$\begin{cases} U(t) = R \cdot I(t) + L \cdot \frac{dI(t)}{dt} + k_e \cdot V(t) \\ J(m) \cdot \frac{dV(t)}{dt} = T_D(I(t)) - T_B(V(t), m) \end{cases} \quad (5)$$

where: m mass of moving detail, T_D and T_B drive and breaking torque, respectively, J moment of inertia, $U(t)$ and $I(t)$ voltage and current of motor supply, respectively,

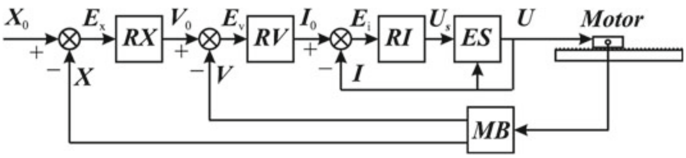


Fig. 4 Scheme of typical servo drive cascade PI controller structure

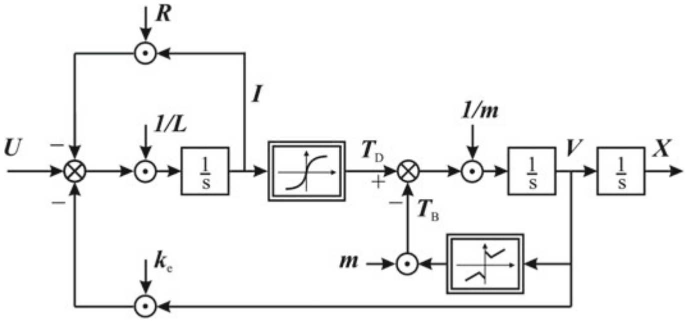


Fig. 5 Scheme of structural model of servo drive motor

R, L, k_e electrical parameters of the motor. At the Fig. 6 the relationship between moment of inertia J and mass m of moving detail is presented.

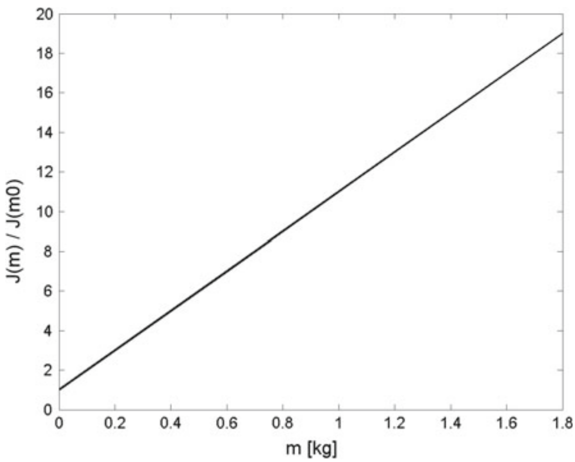
Relationship between breaking torque T_B and velocity V and mass m is described by equation [13]

$$T_B(V, m) = (A_1(m) + (A_2(m) - A_1(m)) \cdot e^{-V^2 \cdot A_3(m)}) \cdot sgn(V) + A_4(m) \cdot V$$

where: $A_1(m), A_2(m), A_3(m)$ and $A_4(m)$ are coefficients dependent of mass of moving detail. Above relationship is presented in Fig. 7.

In simulation nonlinear relationship, presented in the Fig. 8, between drive torque T_D and drive current I was taken into account.

Fig. 6 Relative relationship J(m)



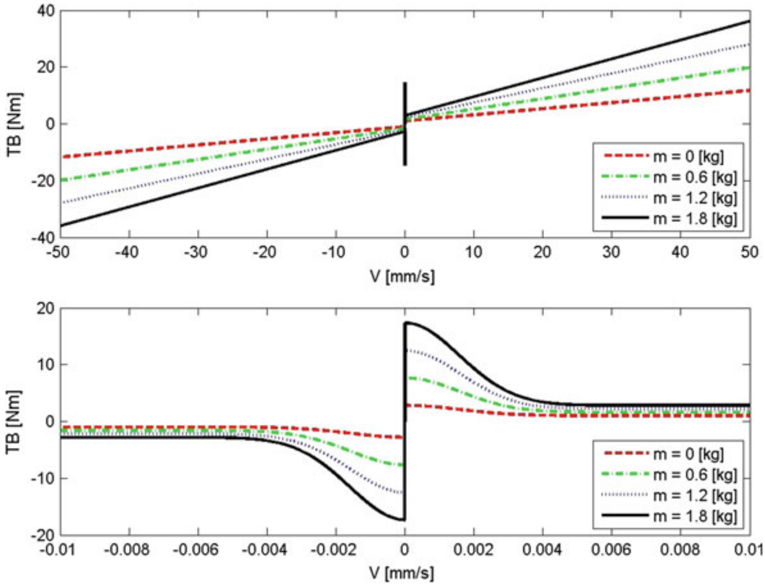
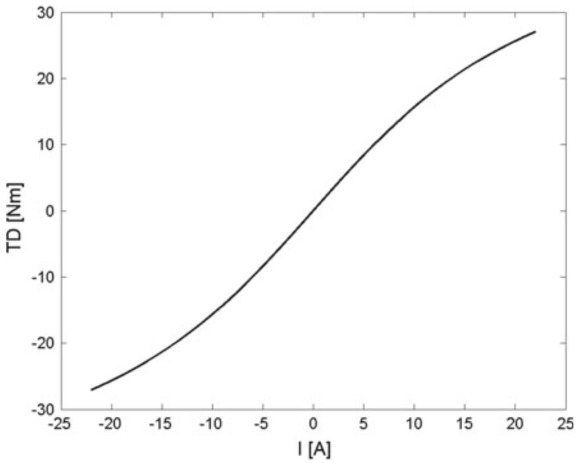


Fig. 7 Relationship between breaking torque T_B and velocity V and mass m

Fig. 8 Relationship T_D(I)



4 Research Assumptions

In typical servo drives control structure classical PI controllers are used. End users of servo drive commonly do not tune current controller. Its parameters are set by servo drive producer. Hence, during researches only current controller were investigated. Parameters of position and velocity controllers were constant as well as its type. The main aim of researches was to compare quality of control obtained by using classical PI controller and fractional order ones.

To compare classical and fractional order controllers in first step classical one were tuned and set parameters were also used in case of fractional one. Such approach were used due to the fact that there are no widely known methods of tuning fractional order controllers. Hence, the only one rest parameters α of fractional controller as well as gain k_s in the anti-windup action were investigated and results of its change were observed taking into consideration the curves of current and position of servo drive obtaind in case of step change (10 mm) of set point position.

5 Results of the Simulations

Strong interest in the fractional order calculus led to the creation of tools for synthesis and simulation of fractional systems, and the most popular are those prepared to work in Matlab environment [9, 10, 17, 23, 24]. The presented below simulations have been carried out in Matlab/Simulink environment with the use of Ninteger toolbox [23, 24].

In Fig. 9 curves obtained in case of use classical (integer order) current controller are presented, that was treated by authors as a reference (K_p = 10, K_i = 1500) for comparison with efficiency of fractional order controllers.

In first step of our investigation best α pramater of fractional order controller were investigated. To do that controller presented in Fig. 1 were used. In Figs. 10, 11, 12 curves of position and current, respectively, for different α (equal 0.1, 0.5 and 0.9, respectively) values are presented.

As the best α was chosen value 0.5 because it ensured luck of position oversooting and accepted current curve. It is observed that too big integral action signal, due to the luck of anti-windup, influences on position—not smooth curve. Ocurring big negative value of current (in time 0.2 s that may be seen on Fig. 11) causes in stopping change of position (maybe seen on Fig. 11). To elimnate this behaviour structures with anti-windup were tested. In such structures the value of $\alpha = 0.5$.

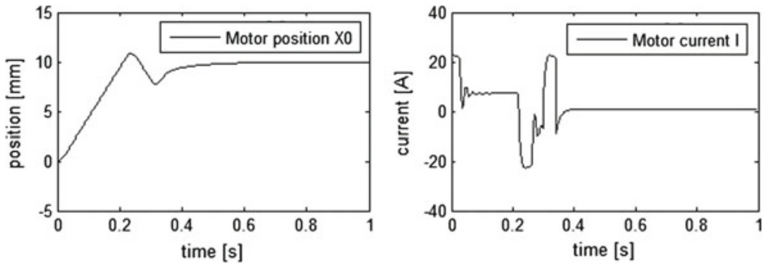


Fig. 9 Curves of servo drive position and current for integer order controller

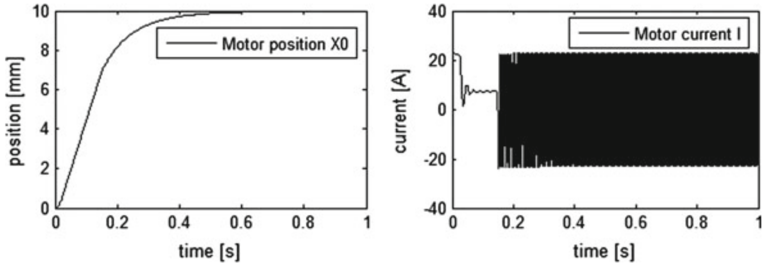


Fig. 10 Curves of servo drive position and current for fractional order controller with $\alpha = 0.1$

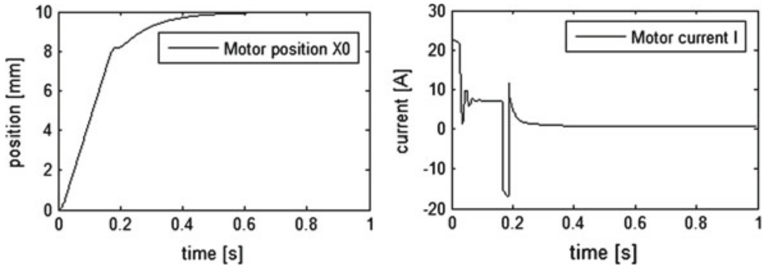


Fig. 11 Curves of servo drive position and current for fractional order controller with $\alpha = 0.5$

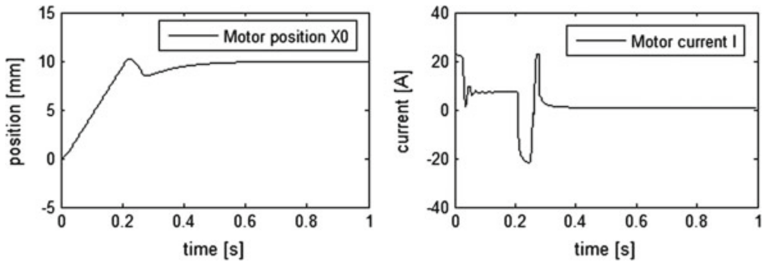


Fig. 12 Curves of servo drive position and current for fractional order controller with $\alpha = 0.9$

The efficiency of anti-windup action was firstly check using controller shown in Fig. 2. In such case the value of k_s coefficient were investigated in the interval [1, …, 1000]. In Fig. 13, 14, 15 curves of position and current, respectively, are presented. It was observed, on the basis of mentioned curves, that examined controllers are characterized by a high effectiveness and ensure high quality of control of position and current of servo drive independently on the value of k_s coefficient.

In the same way controller presented in Fig. 3 were investigated. The curves of position and current, respectively, for such controller are presented on Fig. 16, 17, 18. The quality of control motion of servo drive and current in servo motor are similar to previous system.

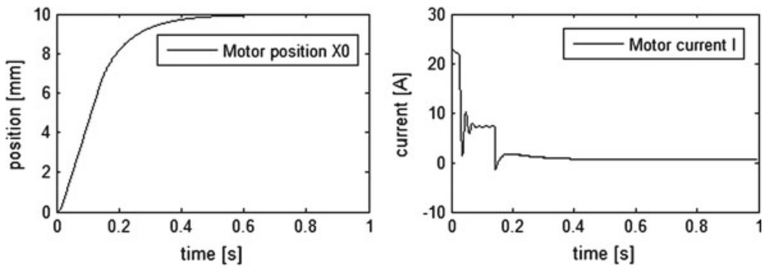


Fig. 13 Curves of servo drive position and current for fractional order controller—anti-windup according to Fig. 2, k_s=1

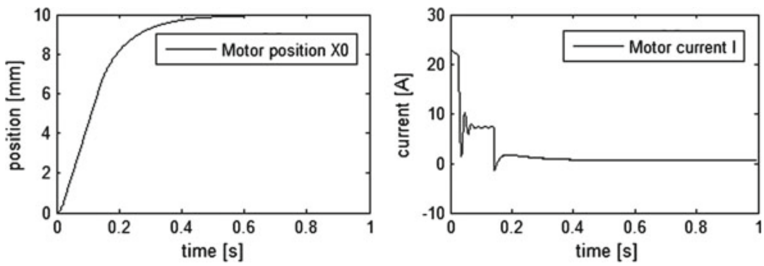


Fig. 14 Curves of servo drive position and current for fractional order controller—anti-windup according to Fig. 2, k_s=10

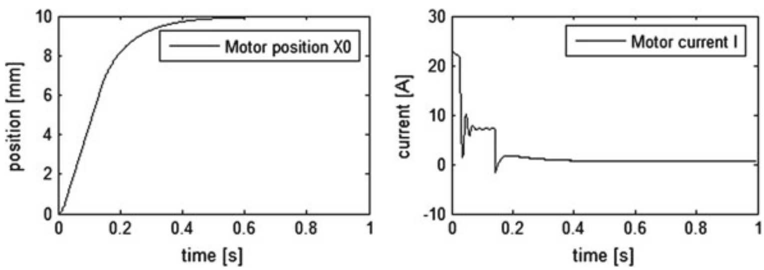


Fig. 15 Curves of servo drive position and current for fractional order controller—anti-windup according to Fig. 2, k_s=100

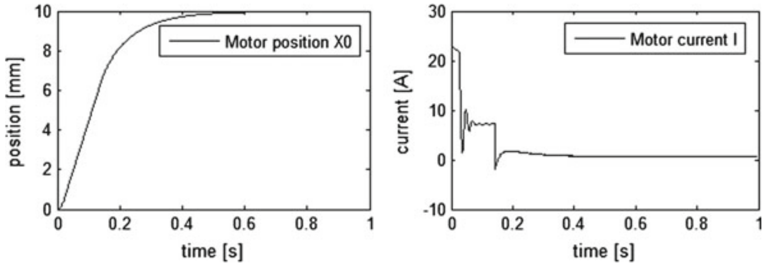


Fig. 16 Curves of servo drive position and current for fractional order controller—anti-windup according to Fig. 3, k_s=1

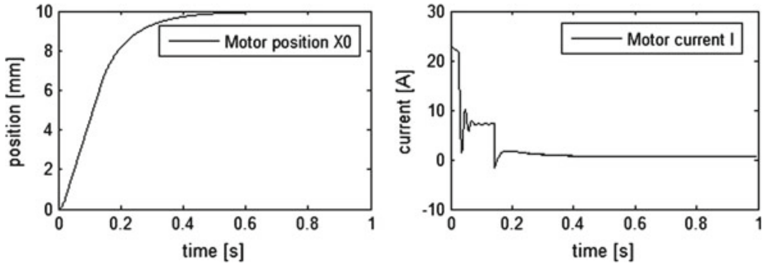


Fig. 17 Curves of servo drive position and current for fractional order controller—anti-windup according to Fig. 3, k_s=10

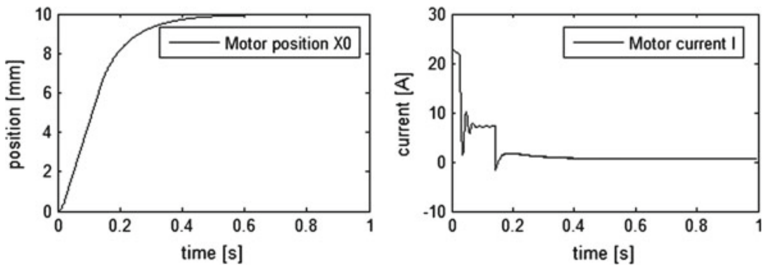


Fig. 18 Curves of servo drive position and current for fractional order controller—anti-windup according to Fig. 3, k_s=100

6 Summary

In the paper it was proved that using fractional order controller makes quality of servo drive control better. The curves of servo drive position are without overshooting and servo motor current are less variant. Such features are higly demanded in many practical applications. Additionally using anti-windup in case of fractional order controller makes servo drive position curves smooth. What is more, simultaneously curves of current become less changeable. Another advantage was luck of sensitivity in wide range of coefficient k_s. It means that fractional order controller parameters selection k_p and k_i may be achieved on the way of using tuning methods of classical (integer order) controller. In practice, it follows from this that using fractional order controller do not require different procedure of tuning in comparison with others controllers in servo drive cascade control structure. Furthermore, end user of servo drive obtain better working servo drive ensuring luck of overshooting.

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Parametric Optimization of Non-Integer Order PD$^{\mu}$ Controller for Delayed System

Marta Zagórowska

Abstract In this paper, we analysed a new tuning method for PD^{α} controller using approximation with Laguerre functions. The optimization was performed for various sets of parameters. We also analysed the convergence of chosen optimization parameters. The results were tested for a first-order system with delay.

1 Introduction

Non-integer (fractional) order systems are one of developing branches of modern control theory. They are widely investigated not only as a direct replacement of classical control systems, but they allow also an extension of existing concepts.

In this article, we investigated a method for optimizing parameters of PD^{α} controllers. We wanted to find a methodology for determining the parameters minimising the $L^2[0, +\infty)$ norm of impulse response of closed loop system. In order to achieve this goal, we used an approximation method consisting of Laguerre functions. For illustration of this methodology two sample systems with delay were proposed.

General works concerning non-integer order systems are e.g. [10, 16, 23, 25]. Most focus is oriented on their properties (see for example [2, 3, 6, 12, 22]) and applications (see for example [4, 5, 8, 11, 24]). Earlier works on parametric optimisation of non-integer order controllers can be found in [7, 9, 11, 17, 21, 25, 27]—both in simulational and in experimental setups. An interesting approach to non-integer fractional controllers was investigated in [18–20], where a concept of robust non-integer controllers is investigated. The delayed systems were analysed i.a. [1, 13, 15]

The main contribution of this paper is the use of previously developed method of approximation [2, 5, 26] in new application allowing efficient controller parameter

M. Zagórowska (✉)
Department of Automatics and Biomedical Engineering, AGH University of Science and Technology, Al. Mickiewicza 30, 30-059 Cracow, Poland
e-mail: zagor@agh.edu.pl

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optimisation. Also, the convergence of the method is investigated and its potential sensitivity.

The remainder of this paper is organised as follows, first the design problem is formulated and chosen test systems are presented. Next, the approximation method is presented along with the method of determining the performance index. Then the optimisation algorithm is given. Algorithm details are illustrated with numerical experiments, showing results of optimisation and analysing convergence behaviour. Finally concluding remarks and plan for future works are given.

2 Problem Description

2.1 Analysed System

We analysed an integer order system from Fig. 1 with

$$G_0(s) = \frac{1}{Ts + 1} \tag{1}$$

It can be seen that the controller was placed in the feedback loop instead of classical position after the summation. In our case it results with the same system as we analyse zero reference value.

The block $G_d(s)$ denotes a delay of value τ with transfer function

$$G_d(s) = \mathrm{e}^{-s\tau} \tag{2}$$

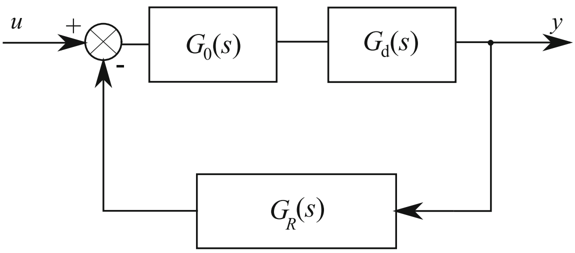
The transfer function $G_R(s)$ denotes the non-integer order PD^α controller

$$G_R(s) = K_P + K_D s^\alpha \tag{3}$$

The open-loop system has the form

$$G_{op}(s) = \frac{\mathrm{e}^{-s\tau}}{Ts + 1} \tag{4}$$

Fig. 1 Closed-loop system



The transfer function of a closed-loop system is

$$G(s) = \frac{e^{-s\tau}}{Ts + 1 + K_D s^\alpha e^{-s\tau} + K_P e^{-s\tau}} \tag{5}$$

We have analyzed two sets of parameters for system (1) and (2) with different behaviour depending ratio between T and τ.

2.2 Optimization Problem

The main goal of this article is to find optimal controller parameters: α, K_D and K_P in order to efficiently bring the system to zero. For this purpose we chose the following objective function

$$J(\alpha, K_D, K_P) = \int_0^\infty g^2(t)\mathrm{d}t \tag{6}$$

where $g(t)$ denotes the impulse response of (5).

3 Approximation Method

Method of impulse approximation can be analysed for the equations of the following type (see [25]):

$$\begin{gathered} {}_0^C\mathrm{D}_t^{\sigma_n} x(t) + \left(\sum_{j=1}^{n-1} p_{n-j} {}_0^C\mathrm{D}_t^{\sigma_{n-j}} x(t) \right) + p_0 x(t) = \\ q_m {}_0^C\mathrm{D}_t^{\gamma_m} u(t) + \left(\sum_{j=1}^{m-1} q_{m-j} {}_0^C\mathrm{D}_t^{\gamma_{m-j}} u(t) \right) + q_0 u(t) \end{gathered} \tag{7}$$

where $j \le \sigma_j \le j+1, j = 1, 2, \ldots, n, j \le \gamma_j \le j+1, j = 1, 2, \ldots, m, p_j$,$q_j \in \mathbb{R}$. The initial conditions are zero. It is also assumed that $|u(t)| \le u_{max}$ for $t \ge 0$ and $u(t) = 0$ for $t < 0$ [2]. Because the initial conditions are assumed to be zero, the differential operator can be either of Riemann-Liouville or Caputo type.

The transfer function of (7) is

$$\hat{g}(s) = \frac{q_m s^{\gamma_m} + q_{m-1} s^{\gamma_{m-1}} + \cdots q_0}{s^{\sigma_n} + p_{n-1} s^{\sigma_{n-1}} + \cdots p_0} \tag{8}$$

and the solution to (7) is given by a convolution

$$x(t) = u * g = \int_0^t u(t-\theta)g(\theta)\mathrm{d}\theta \tag{9}$$

It can be shown that the solution (9) of Eq. (7) can be approximated with a solution of a system of n linear ordinary differential equation [2].

Theorem 1 gives the conditions that must be fulfilled in order to find the approximation with minimal error.

Theorem 1 *If $g \in \mathcal{L}_1(0,\infty) \cup \mathcal{L}_2(0,\infty)$ and $|u(t)| \leq u_{max}$ then:*

1. *The solution of (7) can be approximated with*

$$x_n(t) = \sum_{k=0}^{n} \beta_k \xi_k(t) \tag{10}$$

where functions $\xi_k(t) : [0,\infty) \to \mathbb{R}$ are solution of a system

$$\begin{aligned} \dot{\xi}_k &= -\mu\xi_k - 2\mu\sum_{i=0}^{k-1}\xi_i + \sqrt{2\mu}u \\ \xi_k(0) &= 0,\ k = 0, 1, 2, \dots, n \end{aligned} \tag{11}$$

and

$$\beta_k = \int_0^\infty g(\theta)e_k(\theta,\mu)\mathrm{d}\theta. \tag{12}$$

2. *For every $\varepsilon > 0$ there exists a number n_0 dependant on g, ε and u_{max} that approximation error $\epsilon_n(t) = x(t) - x_n(t)$ fulfils the inequality*

$$|\epsilon_n(t)| < \varepsilon \tag{13}$$

for all $n \geq n_0$ and $t \geq 0$

Proof For the proof see [2].

The functions $e_k(\theta,\mu)$ form orthonormal set of Laguerre functions parametrised by $\mu > 0$ given by

$$e_k(\theta,\mu) = \sqrt{2\mu}L_k(2\mu\theta) \tag{14}$$

where L_k are Laguerre polynomials, and $k = 0, 1, 2, \dots$.

The formula (12) for calculating the coefficients is not convenient for numerical implementation. In [2] the authors presented the recurrence formula allowing efficient computation. This method was used in the following computation.

It is worth noting that the choice of parameter μ is crucial for the quality of approximation. It should be chosen in such way that the objective function

$$I(\mu) = \sum_{k=0}^{n} \beta_k^2(\mu) \tag{15}$$

is maximized with respect to μ (for details see [2]).

Regarding the optimisation of performance index (6) one can observe directly from formula (12) that

$$g(t) = \sum_{i=1}^{n} \beta_i e_k(t, \mu) \tag{16}$$

Taking into account that $e_k(\theta, \mu)$ are orthonormal in $L^2[0, +\infty)$

$$\begin{aligned} \langle e_i,\ e_j \rangle &= 0,\ i \neq j \\ \langle e_i,\ e_j \rangle &= 1,\ i = j \end{aligned} \tag{17}$$

where

$$\langle a,\ b \rangle = \int_0^\infty a(\theta) b(\theta) \mathrm{d}\theta \tag{18}$$

It can be then observed that

$$\begin{aligned} \int_0^\infty g^2(t)\mathrm{d}t &= \int_0^\infty \left(\sum_{i=1}^{n} \beta_i e_i(t, \mu) \right)^2 \mathrm{d}t \\ &= \sum_{i=1}^{n} \beta_i^2 \langle e_i,\ e_i \rangle + \sum_{\substack{i,j=1, \\ i \neq j}}^{n} \beta_i \beta_j \langle e_i,\ e_j \rangle \\ &= \sum_{i=1}^{n} \beta_i^2 \end{aligned} \tag{19}$$

It can be then easily observed that

$$J(\alpha, K_D, K_P) = \sum_{i=1}^{n} \beta_i^2 \tag{20}$$

It can be easily seen that (15) and (20) are equal and that optimisation of impulse response is a minmax problem—we need to maximise the objective function with respect to μ, then we perform minimisation over the parameters of the controller.

4 Optimisation Algorithm

Optimisation can be performed with any type of optimisation algorithm. In our case Matlab/Optimization Toolbox implementation of Nelder-Mead algorithm was used.

1. Choose initial values for α, K_D and K_P.
2. Perform initial maximization of the function (15) with respect to μ.
3. Perform the step of optimisation method.
4. Before evaluating (20) perform maximisation with respect to μ.
5. Update the values of α, K_D and K_P according to chosen optimization method.
6. Repeat the steps 2–5 until the convergence criteria are achieved.

5 Results

We performed the optimization for two cases denoted in Table 1. We have fixed the values of α and wanted to find optimal controller minimizing the objective function (20). We have chosen the following values: 0.6, 0.7, 0.8, 0.9, 1.1, 1.2, 1.3. As one can see, we omitted $\alpha = 1$ as integer order optimization is not the purpose of the analysis. We also fixed the order of approximation $n = 10$. Such value allows sufficient accuracy of approximation and in the same time is small enough not to cause numerical errors concerning factorials. We also fixed the interval where we assumed we can find optimal value of μ—we maximized the quality index over interval $\mu \in [0, 7]$.

In Tables 2 and 3 we have presented the results of optimization for fixed α for two sets of parameters, respectively: $T = 1$, $\tau = 0.1$ and $T = 0.1$, $\tau = 1$.

Table 1 Analyzed values and T/τ ratio

T	τ	Ratio T/τ
1	0.1	10
0.1	1	0.1

Table 2 Result of optimization for $T = 1$, $\tau = 0.1$

α	K_p	K_D	μ	Index value
0.6	7.8332	−0.2213	4.1661	0.1195
0.7	7.5076	−0.0263	4.3263	0.1196
0.8	7.4760	−0.0204	7	0.1197
0.9	7.4527	−0.0162	4.3246	0.1197
1.1	7.4613	−0.0111	4.3227	0.1198
1.2	7.4411	−0.0092	4.3237	0.1197
1.3	634.8787	−0.4181	3.2891	1.1895×10^{-4}

Table 3 Result of optimization for $T = 0.1$, $\tau = 1$

α	K_p	K_D	μ	Index value
0.6	1.081	−0.1034	1.3026	3.0239
0.7	0.6316	−0.0288	3.4079	2.4246
0.8	0.6442	−0.0275	3.4234	2.4204
0.9	−0.1619	−0.0257	5.2270	2.31
1.1	0.6843	−0.0253	6.6106	2.3982
1.2	0.6882	−0.0238	3.4819	2.3871
1.3	8.3654	−0.5903	2.2508	14.7381

In Table 2 it can be seen that the quality index remained almost the same for most values of α. The only exception that is worth notice occurred for $\alpha = 1.3$. In the same time, its value for this α is also the smallest. The worst case happens for $\alpha = 1.1$, where the index value is greater than in most cases and equals 0.1198. This disproportion may be caused by not optimal choice of initial values of parameters for given α. Moreover, also the choice of μ plays a significant role—if performance index (20) is a non-convex function, the optimization algorithm finds sometimes only local optima.

These two cases, for $\alpha = 1.3$ and $\alpha = 1.1$ are depicted in Figs. 2 and 3.

It is also worth notice, that for $\alpha = 0.8$ the parameter μ reaches the maximal preset value $\mu_{max} = 7$. An extension of this interval may cause numerical problems and requires further investigation.

Table 3 presents the results for second set of parameters. $T = 0.1$ and $\tau = 0.1$.

In this case, we do not see any exceptional values (earlier the difference was three orders of magnitude). However, we notice that the best result is obtained for $\alpha = 0.9$ and the worst for $\alpha = 1.3$. Both cases are visible in Figs. 4 and 5.

The figures clearly present the differences between performance indices in every case.

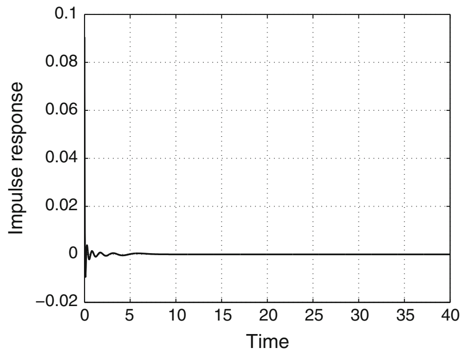


Fig. 2 Impulse response for $\alpha = 1.3$, $T = 1$, $\tau = 0.1$

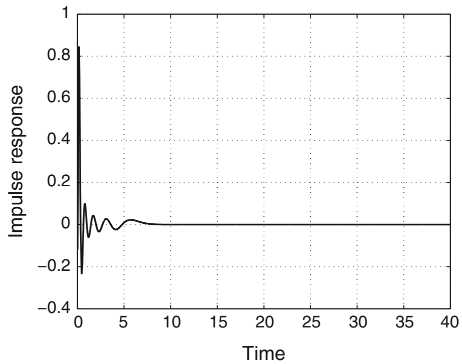


Fig. 3 Impulse response for $\alpha = 1.1$, $T = 1$, $\tau = 0.1$

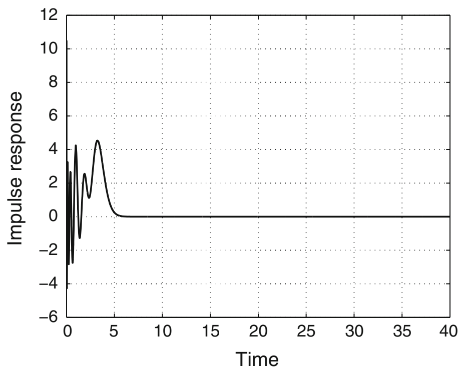


Fig. 4 Impulse response for $\alpha = 0.9$, $T = 0.1$, $\tau = 1$

5.1 Convergence of μ

One of the most interesting aspects in analysis of this method is the behaviour of μ during optimisation. Because effectively we have a saddle-node optimisation, it is difficult to choose appropriate strategy that would allow finding optimal solution with respect to parameters with minimal approximation error (dependent on μ). In

Fig. 5 Impulse response for $\alpha = 1.3$, $T = 0.1$, $\tau = 1$

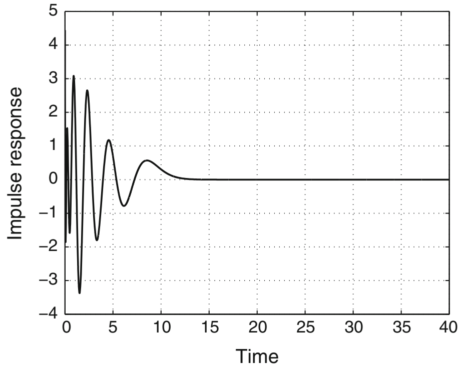


Fig. 6 Optimization of μ for $\alpha = 0.9$, $T = 0.1$, $\tau = 1$

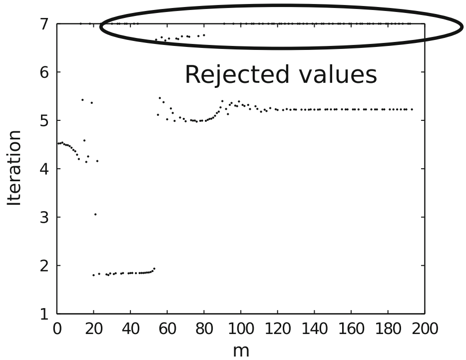
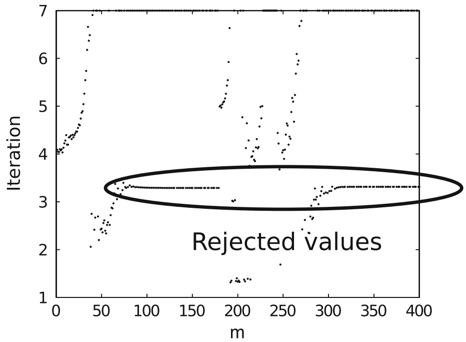


Fig. 7 Optimization of μ for $\alpha = 1.3$, $T = 1$, $\tau = 0.1$



Figs. 6 and 7 one can observe the evolution of μ through iterations of the method. As one can see it converges to the optimal value quickly. Steps of method which significantly modify the impulse response and require modification of μ were subsequently rejected because of no improvement in the values of performance index.

We have chosen the best scenarios for each set of parameter. In both cases, the optimization hit the limit $\mu = 7$. In first case, for $\alpha = 0.9$, it was rejected, whereas in the second, $\mu = 7$ gave the best performance index value.

6 Conclusion and Further Research

In this paper, a new method for tuning non-integer order controllers was analysed. We used the approximation method for finding optimal parameters for a relatively simple system, a first-order system with delay. This method proved to be efficient, however, there are some details, which need further investigation. For example, we want to analyse stability region of optimized system in order both to avoid the numerical problems and to find appropriate initial values for optimization procedures. It is also worth consideration, how different methods of optimization perform in this application.

Also different controller structures and different performance indices will be investigated. Effectively all integral performance criteria with infinite horizon can be used, however, their determination will require operation on the approximated system (10)–(11). Also application of different set of approximating functions can lead to interesting results.

We would also like to compare the results obtained from this optimization with the parameters' range for integer order system (see e.g. [14]). Ensuring global optimization is also a part of further research—in this case, the optimization might have stopped in a local minimum.

One of the areas that require attention is also optimization with respect to α in order to find the best order of the controller. One can suppose that it will require significant numerical calculations, but also calculating analytically the derivatives of transfer function with respect to α seem promising.

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Adaptive Non-Integer Controller for Water Tank System

Jerzy Baranowski, Waldemar Bauer, Marta Zagórowska, Aleksandra Kawala-Janik, Tomasz Dziwiński and Paweł Piątek

Abstract In this article, the authors consider a new method of designing adaptive controller for non-integer order systems. The theoretical approach was verified with computer simulation of three-tank system. Further research will include implementation in a real system.

1 Introduction

General results concerning theory of non-integer order systems can be found in [14, 19, 31]. Oustaloup method was presented in [28] and is analyzed among the others in [27, 29]. This approximation can be efficiently used in simulations [11, 12, 16] and with appropriate care experiments [15]. Its sensitivity and stability problems during discretization were discussed in [30]. Different method of approximation is based on Laguerre functions and does not poses this sensitivity [3, 9, 36] however it is much more adequate for filters than for the controllers.

J. Baranowski · W. Bauer (✉) · M. Zagórowska · T. Dziwiński · P. Piątek
Department of Automatics and Biomedical Engineering, AGH University of Science and Technology, Al. Mickiewicza 30, 30-059 Kraków, Poland
e-mail: bauer@agh.edu.pl

J. Baranowski
e-mail: jb@agh.edu.pl

M. Zagórowska
e-mail: zagor@agh.edu.pl

T. Dziwiński
e-mail: tdz@agh.edu.pl

P. Piątek
e-mail: ppi@agh.edu.pl

A. Kawala-Janik
Faculty of Electrical Engineering, Automatic Control and Informatics, Opole University of Technology,Ul. Proszkowska 76, 45-756 Opole, Poland
e-mail: a.kawala-janik@po.opole.pl

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Stability of non-linear non integer order systems was investigated in [10, 24, 37]. Applications of non-integer order subsystems was investigated among the others in [7, 9, 15]). Works on tuning of non-integer order controllers for linear systems can be found in [12, 15, 20, 26, 31]—both in simulational and in experimental setups. An interesting approach to non-integer fractional controllers was investigated in [21–23], where a concept of robust non-integer controllers is described.

Water tank system is well described in [8] where a survey of literature concerning its investigation is also given.

Adaptive control is a well celebrated area of control theory. Classical results can be found in [2, 33, 35]. Certain interesting results in the theory of non-integer adaptive control can be found in [32, 34]. Results of successful applications can be found in [1, 13, 18, 25].

This paper is organized as follows. In the first part, we present the system of tanks along with its mathematical description. Then the Oustaloup method is introduced, we included also time-domain approach. The last part of theoretical analysis consists of description of adaptive controllers of non-integer order. Then we present the results of simulation of this approach. In the last part, some extensions and further work are proposed.

2 Laboratory Hydraulic Cascaded Three-Tank System

A laboratory hydraulic cascaded three-tank system considered in the paper is depicted on the photograph in the Fig. 1. In the same figure its schematic diagram is presented as well. The system is located in the Department of Automatics and Biomedical Engineering of AGH. The installation consists of three vertically arranged tanks: upper, middle and bottom. A side wall of each container have a different shape:

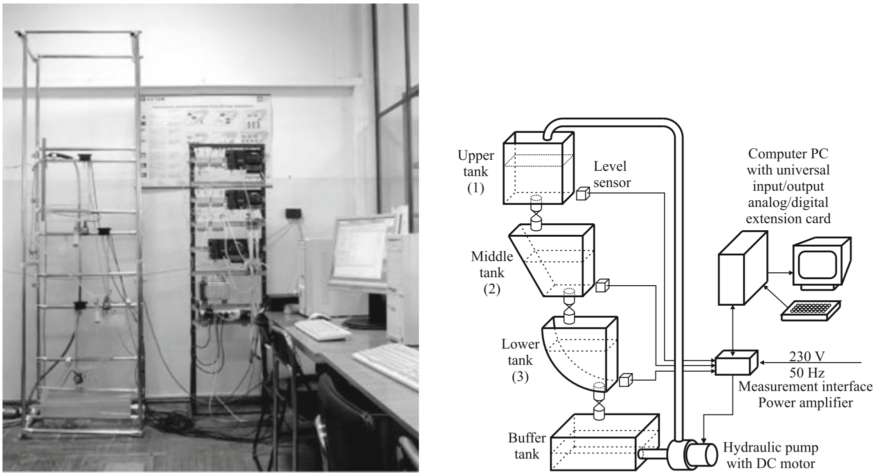

Fig. 1 A photograph and a schematic diagram of the laboratory installation

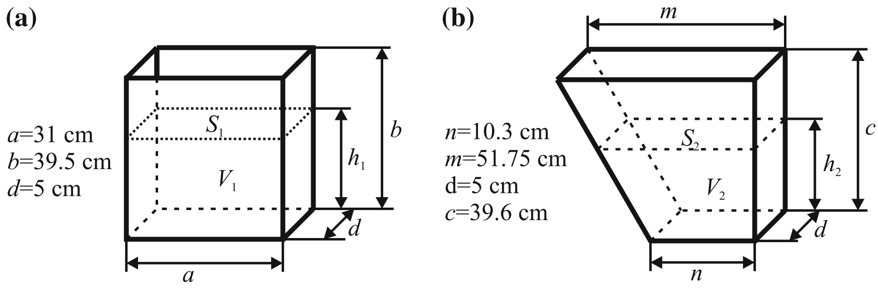

Fig. 2 Shapes and dimensions of three water tanks: **a** upper, **b** middle

rectangular, trapezoidal and quarter circular respectively. Their dimensions are given in the Fig. 2. The area of a water surface in the upper tank is constant but in two remaining tanks it varies with the water level. There is a fourth tank as well, located in the lowest position, serving as a water buffer. A sliding-vane pump driven by a electric DC permanent magnet brush motor pumps water from the buffer to the upper tank. From there the water flows through a constant valve to the middle tank and then to the lower and eventually to buffer tank in a similar fashion. These flows are driven by gravity force and governed by the Torricelli's law. The electric motor is driven by a PWM signal from a power amplifier. Water levels in the three tanks are measured with three pressure sensors and their signals are conditioned in an appropriate electronic interface. There is a PC computer dedicated to the laboratory system, equipped with a universal digital-analog input-output extension card RT DAC 4 PCI which measures analog water level signals and provides digital PWM control signal for the pump DC motor. A MATLAB-Simulink environment with *Real-Time Workshop* (RTW) and *Real-Time Windows Target* (RTWT) toolboxes is used to develop, build and test a real-time application.

2.1 Mathematical Model of the System

A mathematical model of the laboratory installation can be derived from the law of mass conservation which for the three cascaded tanks takes the form of the two following differential equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\varrho V_1\right) = \varrho q_0 - \varrho q_1 \tag{1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\varrho V_2\right) = \varrho q_1 - \varrho q_2 \tag{2}$$

where ϱ is a water density, V_1, V_2 are water volumes in two consecutive tanks, q_0 is a control-dependent pump voluminal flow and q_1, q_2 are level-dependent valves voluminal flows. For a partially filled open tank a derivative of a volume V with respect to time t can be expressed as follows

$$\frac{\mathrm{d}V(h(t))}{\mathrm{d}t} = \left.\frac{\mathrm{d}V(h)}{\mathrm{d}h}\right|_{h=h(t)} \cdot \frac{\mathrm{d}h(t)}{\mathrm{d}t} = S(h(t))\,\dot{h}(t) \tag{3}$$

where S is an area of an open water surface in the tank and h is a water level measured from the bottom. With an additional assumption of a constant water density one can obtain from (1)–(2) and (3) the following state space equations

$$\dot{h}_1(t) = \frac{1}{S_1}\Big(q_0(u) - q_1\big(h_1(t)\big)\Big) \tag{4}$$

$$\dot{h}_2(t) = \frac{1}{S_2\big(h_2(t)\big)}\Big(q_1\big(h_1(t)\big) - q_2\big(h_2(t)\big)\Big) \tag{5}$$

where state variables $h_1(t)$ and $h_2(t)$ are the water levels in cm in upper and middle tank respectively. Symbols S_1, $S_2(h_2)$ are the areas in cm^3 of open water surfaces given by following formulae

$$S_1 = a\,d = \text{const} \tag{6}$$

$$S_2(h_2) = d\left(n + \frac{m-n}{c}\,h_2\right) \tag{7}$$

where c, d, m, n and r are tanks physical dimensions defined on Fig. 2. Function $q_0(u)$ characterizes the relationship between a duty factor $u \in [0, 1]$ of a PWM signal and a voluminal water flow q_0 in cm^3/s produced by the pump. This mapping can be approximated with a polynomial of fifth degree

$$q_0(u) = w_5\,u^5 + w_4\,u^4 + w_3\,u^3 + w_2\,u^2 + w_1\,u + w_0 \tag{8}$$

Functions $q_1(h_1)$, $q_2(h_2)$ and $q_3(h_3)$ describe dependencies between water levels and water flows through valves. According to the slightly modified Torricelli's law these relationships can be approximated as

$$q_i(h_i) = C_i\sqrt{D_i + h_i}, \qquad i \in \{1, 2\} \tag{9}$$

where C_i and D_i are constants which need to be identified and i is the tank index (1 for upper, 2 for middle).

Results of identification are gathered in Table 1 and Table 2.

3 Oustaloup Method

Oustaloup filter approximation to a fractional-order differentiator $G(s) = s^\alpha$ is widely used in applications [27]. An Oustaloup filter can be designed as

Table 1 Results of the identification of the pump—coefficients w_j of the polynomial $q_0(u)$

w_5	1.9287
w_4	−6.5254
w_3	8.9191
w_2	−6.2047
w_1	2.4298
w_0	−0.258

Table 2 Results of the identification of valves—parameters C_i and D_i in relationships $h_i = C_i\sqrt{D_i + h_i}$

i	C_i	D_i
1	29.405	5.9217
2	33.260	4.0083

$$G_t(s) = K\prod_{i=1}^{N}\frac{s+\omega_i'}{s+\omega_i} \tag{10}$$

where:

$$\begin{aligned} \omega_i' &= \omega_b\omega_u^{(2i-1-\alpha)/N} \\ \omega_i &= \omega_b\omega_u^{(2i-1+\alpha)/N} \\ K &= \omega_h^{\alpha} \\ \omega_u &= \sqrt{\frac{\omega_h}{\omega_b}} \end{aligned} \tag{11}$$

Approximation is designed for frequencies $\omega \in [\omega_b, \omega_h]$ and N is the order of the approximation. As it can be seen its representation takes form of a product of a series of stable first order linear systems. As one can observe choosing a wide band of approximation results in large ω_u and high order N result in spacing of poles spacing from close to $-\omega_h$ to those very close to $-\omega_b$. This spacing is logarithmic with a grouping near $-\omega_b$ and causes problems in discretization. Wide band of approximation is on other hand desirable, because approximation behaves the best in the interior of the interval and not at its boundary, so certain margins need to be kept.

4 Time Domain Approximation

The proposed approach is to realize every block of the transfer function (10) in form of a state space system. Those first order systems will be then collected in a single matrix resulting in full matrix realization. This continuous system of differential equations will be then discretized.

4.1 Realization

One can easily observe that for zero initial condition

$$\frac{s+\omega'_k}{s+\omega_K} \iff \begin{cases} \dot{x}_k = A_k xk + B_k u_k \\ y_k = x_k + u_k \end{cases}$$

where

$$\begin{aligned} A_k &= -\omega_k \\ B_k &= \omega'_k - \omega_k \\ C_i &= 1 \\ D_i &= 1 \end{aligned}$$

Full Oustaloup approximation can be then realised as

$$\begin{aligned} \dot{\mathbf{x}} &= \begin{bmatrix} A_1 & 0 & 0 & \dots & 0 \\ B_2 & A_2 & 0 & \dots & 0 \\ B_3 & B_3 & A_3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ B_N & B_N & \dots & B_N & A_N \end{bmatrix} \mathbf{x} + \begin{bmatrix} KB_1 \\ KB_2 \\ KB_3 \\ \vdots \\ KB_N \end{bmatrix} u \\ y &= \begin{bmatrix} 1 & 1 & \dots & 1 & 1 \end{bmatrix} \mathbf{x} + Ku \end{aligned} \tag{12}$$

or in brief

$$\begin{aligned} \dot{\mathbf{x}} &= \mathbf{A}\mathbf{x} + \mathbf{B}u \\ y &= \mathbf{C}\mathbf{x} + Du \end{aligned} \tag{13}$$

What can be immediately observed is that the matrix **A** is lower triangular. This is an extremely important in the case of this problem, as all its eigenvalues (poles of transfer function (10) are on its diagonal, so there is no need for eigenvalue products, which would lead to rounding errors. That is why discretization of (12) has a structure preserving property [5].

5 Adaptive Non-Integer Controller

In this paper a new approach to to adaptive non-integer controllers is considered. In literature most popular approach is to either use the fractional variant of MIT rule (with non-integer derivative for evolution of adapted parameter) or using combining non-integer derivative of deviation with steepest descent as a part of adaptation gain.

We propose different approach—use the classical MIT rule for adapting parameters of non-integer order controller, however the steepest descent is computed using variational equation.

The proposed controller structure is

$$\begin{aligned} e(t) &= r - h_2(t) \\ u(t) &= Ke(t) + I\int_0^t e(t)\mathrm{d}t + \theta {}_0^C\mathrm{D}_t^{\mu} e(t) \end{aligned} \tag{14}$$

where r is the reference value $\mu \in (0, 1)$ is a fixed derivative order and θ is the adapted parameter. This controller structure was chosen because of system specifics. Integer order integration is needed for elimination of steady state errors, while influence of non-integer derivative term can vary depending on the conditions system is in. It was verifie, that strong derivative action is not needed, and $\mu = 0.1$ was completely satisfactory. Gain was chosen using adaptive algorithm.

Adaptation of θ was realized by the celebrated MIT rule [2].

$$\dot{\theta} = -\gamma e(t)\frac{\partial e(t)}{\partial \theta} \tag{15}$$

Because the controlled system is nonlinear, standard application of adaptation formulas is not possible. That is why a variational equation (see for example [4]) is used. With variational equation derivative of deviation with respect to parameters is given by:

$$\frac{\partial e(t)}{\partial \theta} = -\varphi_2(t) \tag{16}$$

where

$$\dot{\boldsymbol{\varphi}} = \mathbf{J}(h_1, h_2, e)\boldsymbol{\varphi} + \theta\mathbf{C}\boldsymbol{\psi} - \theta D\varphi_2 + \mathbf{C}\mathbf{x} + De \tag{17}$$

$$\dot{\boldsymbol{\psi}} = \mathbf{A}\boldsymbol{\psi} - \mathbf{B}\varphi_2 \tag{18}$$

where $\mathbf{x}, \mathbf{A}, \mathbf{B}, \mathbf{C}, D$ are given by (12)–(13). $\mathbf{J}(h_1, h_2, e)$ is the Jacobi matrix of nonlinear system with PI part of the controller.

6 Simulation Result

We implemented the above described method in a simulational analysis of tank system. The adaptive controller was compared with two more controllers—classic, integer order PI and fractional PID controller with non-integer derivative. The reference values was 10 and 11 centimetres and we wanted to verify the performance of each type of control.

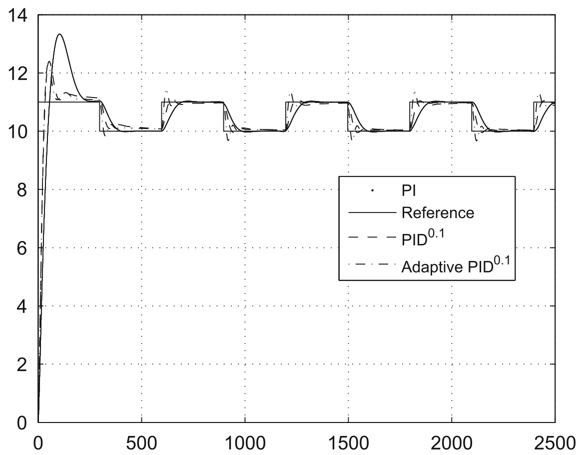

Fig. 3 Simulation result—comparison of three types of controllers

As we can see in Fig. 3, the adaptive controller has the smallest maximal value. However, it introduces visible oscillations. It is also quite effective concerning tracing of a given curve—despite oscillations it reaches the reference values faster than non-integer order PID and classical PI.

7 Conclusions

This article summarizes an early stage of research concerning fractional adaptive control. The proposed approach works fine for this simulation, however, there are some issues that need further consideration. First of all, the choice of adaptation parameter is an still an open question. Moreover, in order to implement this solution in a real-time system and then in existing system of tanks, we need to take into account the numerical part of implementation (see e.g. [5, 6, 17, 30]). Also, one of the main drawbacks that need further analysis is the fact, that this adaptive approach requires use of mathematical model and its linearisation.

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Model-Plant Mismatch in Fractional Order Model Predictive Control

Stefan Domek

Abstract The effectiveness of model predictive control (MPC) depends on the accuracy of the process model, which is utilized directly to compute the manipulated variable. The effect of various model-plant mismatches on the performance of the classic predictive controller is well known. However, many industrial processes exhibit very complex properties, and determining an adequate model for them is not an easy task. In recent years it has been suggested to employ non-integer order models to describe difficult processes. This leads to fractional order model predictive control (FOMPC) systems. Their properties, e.g. stability, robustness, control quality, have become one of the active research topics in control theory and applications. In the paper, the effect of various plant-model mismatches on the performance of FOMPC is illustrated through a simulation experiment.

Keywords Non-integer order systems · Fractional order model predictive control · Model-plant mismatch

1 Introduction

The idea of model predictive control (MPC) is one of the most universal and effective control methods. In MPC the future control actions $u(t+j|t)$ are to be found at each instant $t \in \mathbb{Z}_+$ within the control horizon from $j=0$ to $j=N_u-1$ in order to minimize the differences between the reference values $y^r(t+j|t)$ and the predicted values $y^p(t+j|t)$ within the prediction horizon from $j=N_1$ to $j=N_2$.

S. Domek (✉)
Department of Control Engineering and Robotics, West Pomeranian University of Technology at Szczecin, ul. Sikorskiego 37, 70-313 Szczecin, Poland
e-mail: stefan.domek@zut.edu.pl

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The optimal control sequence is computed in the open loop mode with the use of the plant model. Only the first value of the computed sequence is fed to the input of the real plant, and the whole procedure is repeated at the next discrete time instants (receding horizon principle) [7].

MPC enables various signal constraints and various kinds of disturbances to be taken into account, and may be employed to control processes with practically any number of inputs and outputs. However, especially in the case of processes with particularly complex properties, its effectiveness depends on the accuracy of the process model, which is utilized directly to compute the manipulated variable [2].

From the other side, it has been known for several years that actual properties of many complex phenomena and nonlinear industrial processes can be effectively modeled using the fractional order differential calculus [1, 5, 10]. Thus, taking this circumstance into account at the stage of synthesizing the system may increase naturally the applicability of the controller, which can be regarded as a fractional order model predictive controller (FOMPC) in such a case [2, 11]. Such a controller with a small set of coincidence points was described in [4]. In [3] the possibility of FOMPC to be employed for control of fractional order nonlinear plants was presented. On the other hand, a possibility to include the concept of a fractional order performance index proposed in [12] into the FOMPC algorithm was shown in [11].

In the FOMPC system, like in the classic MPC of integer order, the accuracy of the process model determines the control quality. The effect of various model-plant mismatches on the performance of the classic MPC controller was illustrated in several papers, for example in [8, 14]. In this paper the effect of the model-plant mismatch on the FOMPC performance is studied.

First, the fractional order predictive control algorithm to be considered is presented in Sect. 2. Then, types of model-plant mismatch adopted for the study are described in Sect. 3. The results obtained from simulation tests are displayed in Sect. 4. Finally, the paper is summarized in conclusions.

2 Fractional Order Model Predictive Control

In the predictive control algorithm of integer order the cost function depends on the sum of the weighted squared prediction errors over the prediction horizon and on the sum of the weighted squared control signal increments to be sought within the control horizon

$$J(t) = \sum_{j=N_1}^{N_2} \mu(j)[y^p(t+j|t) - y^r(t+j|t)]^2 + \sum_{j=0}^{N_u-1} \lambda(j)[\Delta u(t+j|t)]^2 \tag{1}$$

with $\Delta u(t+j|t) = 0 \; for \; j \geq N_u$, and $\mu(j) > 0$, $\lambda(j) \geq 0$.

The cost function (1) may be rewritten in vector-matrix form

$$J(t) = \left[Y^p_{\rightarrow}(t) - Y^r_{\rightarrow}(t)\right]^T \boldsymbol{M}\left[Y^p_{\rightarrow}(t) - Y^r_{\rightarrow}(t)\right] + \left[\Delta U_{\rightarrow}(t)\right]^T \boldsymbol{\Lambda} \Delta U_{\rightarrow}(t) \quad (2)$$

where vector $\Delta U_{\rightarrow}(t) \in \mathbb{R}^{mN_u}$ denotes the unknown future increments of the manipulated variable (from $\Delta u(t|t)$ to $\Delta u(t + N_u - 1|t)$) and vectors $Y^{\mathrm{p}}_{\rightarrow}(t) \in \mathbb{R}^{p(N_2 - N_2 + 1)}$, $Y^r_{\rightarrow}(t) \in \mathbb{R}^{p(N_2 - N_2 + 1)}$ stand for future (from the instant $t + N_1$ till the instant $t + N_2$) values of the predicted response of the model and those of the reference trajectory, respectively, and

$$\boldsymbol{M} = \mathrm{diag}[\,\mathrm{diag}(\mu(N_1)) \quad \cdots \quad \mathrm{diag}(\mu(N_2))\,] \in \mathbb{R}^{(N_2 - N_2 + 1)p \,\times\, (N_2 - N_2 + 1)p} \quad (3)$$

$$\boldsymbol{\Lambda} = \mathrm{diag}[\,\mathrm{diag}(\lambda(0)) \quad \cdots \quad \mathrm{diag}(\lambda(N_{u-1}))\,] \in \mathbb{R}^{N_u m \times N_u m} \quad (4)$$

In order to determine the plant output prediction in the FOMPC controller use is made of a non-integer order generalized model of a linear process with different orders of backward differences for individual state variables of the state vector $x(t) \in \mathbb{R}^n$, which may be defined in state space as [6]:

$$\Delta^{\Upsilon} x(t+1) = \boldsymbol{A}_d x(t) + \boldsymbol{B} u(t) \quad (5)$$

$$y(t) = \boldsymbol{C} x(t) \quad (6)$$

The definition (5) may be written in a generalized form by adopting different orders of backward differences for individual state variables of the state vector:

$$\Delta^{\Upsilon} x(t+1) = [\,\Delta^{\alpha_1} x_1(t+1) \quad \cdots \quad \Delta^{\alpha_n} x_n(t+1)\,]^T \quad (7)$$

where

$$\boldsymbol{A}_d = \boldsymbol{A} - \boldsymbol{I}_n \quad (8)$$

and $\boldsymbol{A} \in \mathbb{R}^{n \times n}$—system state matrix, $\boldsymbol{B} \in \mathbb{R}^{n \times m}$—system input matrix, $\boldsymbol{C} \in \mathbb{R}^{p \times n}$—system output matrix, $\boldsymbol{I}_n \in \mathbb{R}^{n \times n}$—identity matrix, with the definition of the real non-integer order $\alpha \in \mathbb{R}$ backward difference for the state vector $x(t)$ based on the Grünwald-Letnikov definition [9]:

$$\Delta^{\alpha} x(t) = \sum_{i=0}^{t} (-1)^i \binom{\alpha}{i} x(t-i), \quad 0 < \alpha \quad (9)$$

$$\binom{\alpha}{i} = \begin{cases} 1 & \text{for} \quad i = 0 \\ \frac{\alpha(\alpha-1)\ldots.(\alpha-i+1)}{i!} & \text{for} \quad i = 1, 2, \ldots \end{cases} \quad (10)$$

or alternatively

$$x(t+1) = \boldsymbol{A}_d x(t) + \boldsymbol{B}u(t) - \sum_{i=1}^{t+1} (-1)^i \boldsymbol{\Upsilon}_i x(t+1-i) \tag{11}$$

$$\boldsymbol{\Upsilon}_j = \mathrm{diag}\left[\begin{pmatrix} \alpha_1 \\ j \end{pmatrix} \quad \cdots \quad \begin{pmatrix} \alpha_n \\ j \end{pmatrix} \right] \tag{12}$$

and

$$y(t) = \boldsymbol{C}x(t) \tag{13}$$

For the model (11)–(13) it can be found the predicted model output for the instant $t+j$ calculated at the current instant t:

$$\begin{aligned} y^p(t+j|t) = \boldsymbol{C}\Bigg[& \sum_{i=0}^{j-1} \sum_{k=0}^{i} \boldsymbol{\Phi}^{\Upsilon}(j-i-1)\boldsymbol{B}\Delta u(t+k) \\ & + \boldsymbol{\Phi}^{\Upsilon}(j)x(t) + \sum_{i=0}^{j-1} \boldsymbol{\Phi}^{\Upsilon}(j-i-1)\boldsymbol{B}u(t-1) \\ & + \sum_{i=2}^{t+1} \sum_{k=-1}^{-i+1} (-1)^{i+1} \boldsymbol{\Phi}^{\Upsilon}(j-k-i)x(t+k) \Bigg] \end{aligned} \tag{14}$$

with the transition matrix

$$\boldsymbol{\Phi}^{\Upsilon}(1) = (\boldsymbol{A}_d + \boldsymbol{\Upsilon}_1), \quad \boldsymbol{\Phi}^{\Upsilon}(0) = I_n \tag{15}$$

$$\begin{aligned} & \boldsymbol{\Phi}^{\Upsilon}(t+1) = (\boldsymbol{A}_d + \boldsymbol{\Upsilon}_1)\boldsymbol{\Phi}^{\Upsilon}(t) + \sum_{j=2}^{t+1} (-1)^{j+1} \boldsymbol{\Upsilon}_j \boldsymbol{\Phi}^{\Upsilon}(t-j+1), \\ & t = 1, 2, \ldots \; j = 1, 2, \ldots, N_2 \end{aligned} \tag{16}$$

From (14) the vector of the predicted response can be expressed as

$$Y^p_{\rightarrow}(t) = \underline{\boldsymbol{E}}\Delta U_{\rightarrow}(t) + Y^0_{\rightarrow}(t) \tag{17}$$

with

$$\underline{\boldsymbol{E}} = \underline{\boldsymbol{B}} E \underline{\boldsymbol{C}} \tag{18}$$

where

$$\begin{aligned} & \underline{\boldsymbol{B}} = \mathrm{diag}(\boldsymbol{B}, \ldots, \boldsymbol{B}) \in R^{N_u n \times N_u m}, \\ & \underline{\boldsymbol{C}} = \mathrm{diag}(\boldsymbol{C}, \ldots, \boldsymbol{C}) \in R^{(N_2 - N_2 + 1)p \,\times\, (N_2 - N_2 + 1)n} \end{aligned} \tag{19}$$

$\boldsymbol{E}$ is the process dynamic matrix of the fractional order model defined by [2]:

$$E = \begin{bmatrix} \sum_{i=0}^{N_1-1} \boldsymbol{\Phi}^{\Upsilon}(i) & \cdots & \cdots & \cdots & 0_n \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \vdots & \cdots & \boldsymbol{A}_d + \boldsymbol{\Upsilon}_1 & I_n & \vdots \\ \sum_{i=0}^{N_u-1} \boldsymbol{\Phi}^{\Upsilon}(i) & \cdots & \cdots & \boldsymbol{A}_d + \boldsymbol{\Upsilon}_1 & I_n \\ \sum_{i=0}^{N_u} \boldsymbol{\Phi}^{\Upsilon}(i) & \cdots & \cdots & \cdots & \boldsymbol{A}_d + \boldsymbol{\Upsilon}_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \sum_{i=0}^{N_2-1} \boldsymbol{\Phi}^{\Upsilon}(i) & \cdots & \cdots & \cdots & \sum_{i=0}^{N_2-N_u} \boldsymbol{\Phi}^{\Upsilon}(i) \end{bmatrix} \tag{20}$$

and $Y^0_{\rightarrow}(t) \in \mathbb{R}^{p(N_2-N_2+1)}$ is the vector of prediction for the natural response of the model output.

By introducing the disturbance vector within the prediction horizon, as in the case of the integer order DMC algorithm [7]

$$D_{\rightarrow}(t) = \begin{bmatrix} y(t) - y^p(t|t-1) \\ y(t) - y^p(t|t-1) \\ \vdots \\ y(t) - y^p(t|t-1) \end{bmatrix} \in \mathbb{R}^{p(N_2-N_2+1)} \tag{21}$$

it is obtained from (14), (17) and (21)

$$\begin{aligned} Y^0(t)_{\rightarrow} = D_{\rightarrow}(t) + \underline{\boldsymbol{C}} \Bigg[& \begin{bmatrix} \boldsymbol{\Phi}^{\Upsilon}(N_1) \\ \vdots \\ \boldsymbol{\Phi}^{\Upsilon}(N_2) \end{bmatrix} x(t) + \begin{bmatrix} \sum_{i=0}^{N_1-1} \boldsymbol{\Phi}^{\Upsilon}(N_1-i-1) \\ \vdots \\ \sum_{i=0}^{N_2-1} \boldsymbol{\Phi}^{\Upsilon}(N_2-i-1) \end{bmatrix} \boldsymbol{B}u(t-1) \\ & + \begin{bmatrix} \sum_{i=1}^{t} (-1)^i \boldsymbol{\Phi}^{\Upsilon}(N_1-i) & \cdots & -\boldsymbol{\Phi}^{\Upsilon}(N_1-1) \\ \vdots & & \vdots \\ \sum_{i=1}^{t} (-1)^i \boldsymbol{\Phi}^{\Upsilon}(N_2-i) & \cdots & -\boldsymbol{\Phi}^{\Upsilon}(N_2-1) \end{bmatrix} X_{\leftarrow}(t) \Bigg] \end{aligned} \tag{22}$$

where $X_{\leftarrow}(t)$ denotes the vector of past values of the plant state vector. Hence, according to the cost function (2), the optimal control is given by

$$\Delta U_{opt\rightarrow}(t) = \left(\underline{\boldsymbol{E}}^T \boldsymbol{M} \underline{\boldsymbol{E}} + \boldsymbol{\Lambda} \right)^{-1} \underline{\boldsymbol{E}}^T \boldsymbol{M} \left[Y^r_{\rightarrow}(t) - Y^0_{\rightarrow}(t) \right] \tag{23}$$

and finally

$$\Delta u(t|t) = [\boldsymbol{I}_m \quad \boldsymbol{0}_m \quad \cdots \quad \boldsymbol{0}_m] \Delta U_{opt\rightarrow}(t) \tag{24}$$

It is pertinent to note that (23) resembles the solution for predictive control of integer order in its structure. This makes it possible to extend easily the proposed algorithm to include the case with signal constraints [2].

3 Model-Plant Mismatch

As already mentioned, in model predictive control, the accuracy of the process model determines in principle the control quality. To investigate the effect of various model-plant mismatches on the effectiveness of FOMPC controller a class of mismatches to be considered is to be adopted.

Let us consider a non-integer order plant (5), (6), (7) with

$$\alpha_i = \alpha \quad \text{for } i \in \{1, \ldots, n\} \tag{25}$$

Such a plant is defined by matrices $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and by a fractional order of the backward difference α, as Eq. (11) indicates in taking the form:

$$x(t+1) = \boldsymbol{A}_d x(t) + \boldsymbol{B} u(t) - \sum_{j=1}^{t+1} (-1)^j \binom{\alpha}{j} x(t+1-j) \tag{26}$$

Therefore, one can investigate, as in the case of integer order predictive control, the effect produced by the plant-model mismatch on dynamic parameters (entries of the plant $\boldsymbol{A}$, $\boldsymbol{B}$, $\boldsymbol{C}$ and model $\boldsymbol{A}_M$, $\boldsymbol{B}_M$, $\boldsymbol{C}_M$ matrices), and, additionally, on the fractional order exhibited by the plant α and the model α_M, i.e.

$$x_M(t+1) = \boldsymbol{A}_{d_M} x_M(t) + \boldsymbol{B}_M u(t) - \sum_{j=1}^{t+1} (-1)^j \binom{\alpha_M}{j} x_M(t+1-j) \tag{27}$$

In the case of fractional order control systems there exists the problem of a practical implementation of the discrete-time fractional order difference (9). There is known a dozen of methods to approximate the difference, for example [4, 10], all of which lead to a reduction of the model memory that is increasing with time (in the sum in Eq. (26)). Therefore, the effect of plant-model mismatch on the control performance can be studied in terms of the adopted method for practical implementation of a fractional order model.

The most common approximation method used in practice adopts the fractional difference model with a finite memory of length L. Equation (26) assumes then the form:

$$x_M(t+1) = \boldsymbol{A}_{d_M} x_M(t) + \boldsymbol{B}_M u(t) - \sum_{j=1}^{\min(t,L)} (-1)^j \binom{\alpha_M}{j} x_M(t+1-j) \quad (28)$$

or the normalized finite memory fractional difference model, in which the steady-state error is minimized [13]:

$$x_M(t+1) = \boldsymbol{A}_{d_M} x_M(t) + \boldsymbol{B}_M u(t) - \frac{1}{N(L)} \sum_{j=1}^{\min(t,L)} (-1)^j \binom{\alpha_M}{j} x_M(t+1-j) \quad (29)$$

where the normalized factor is given by

$$N(L) = 1 / \sum_{j=1}^{L} (-1)^j \binom{\alpha_M}{j} \quad (30)$$

Note, that approximation of the fractional order system by implementating models (28) or (29) results in a complex model of high order, not useful from the computational point of view [1]. However, the expanded state space equation results in representation of fractional order system as an integer order state model. Therefore, the well-known methods for reduction of the model order, e.g. the method based on the Singular Value Decomposition (SVD) and the Frobenius norm [7], can be used here. The extent of the reduction of the model order also may affect the model-plant mismatch.

In the next Section the effect of the mismatch exhibited by the discussed parameters of fractional order models (27)–(29) on the control performance will be demonstrated.

4 Simulation Experiments

Let us consider a dynamic plants

$$\boldsymbol{A} = \begin{bmatrix} 2.7756 & -1.2876 & 0.7985 \\ 2 & 0 & 0 \\ 0 & 0.5 & 0 \end{bmatrix}, \quad \boldsymbol{B} = \begin{bmatrix} 0.0313 \\ 0 \\ 0 \end{bmatrix}, \quad \boldsymbol{C} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \quad (31)$$

with various values of the fractional order

$$\alpha \in \{0.80, \quad 0.85, \quad 0.90, \quad 0.95, \quad 1.00, \quad 1.05, \quad 1.10, \quad 1.15, \quad 1.20\} \quad (32)$$

Figure 1 shows step responses of the plants under study.

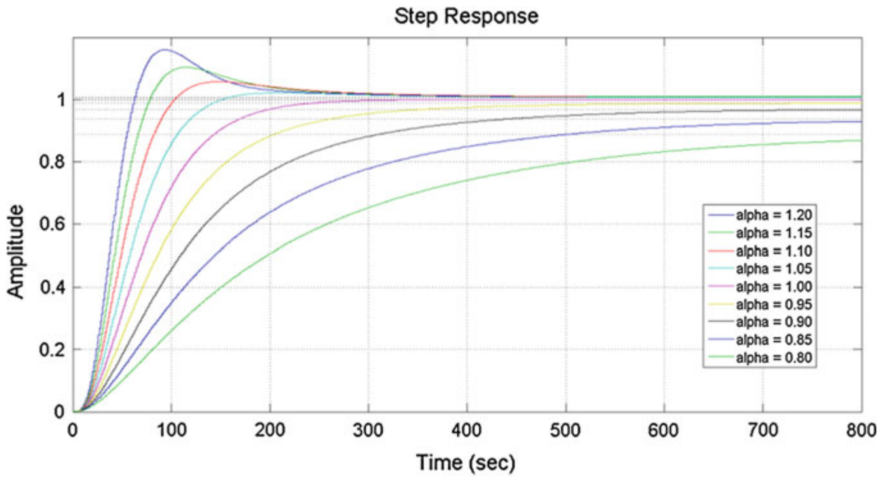

Fig. 1 Step responses of the plants under study

For the set-point tracking the FOMPC controller (23), (24) with the following parameters $N_1 = 1$, $N_2 = 10$, $N_u = 2$, $\mu_i = 0.1$ for $i = 1, 2, \ldots, 10$, $\lambda_i = 1$ for $i = 1, 2$, $T_p = 4s$, has been used.

For each plant the following factors:

- model-plant mismatch in the fractional order,
- model-plant mismatch caused by an improper choice of the finite memory length,
- model-plant mismatch caused by an improper choice of the reduced order of the approximated integer-order model,

have been tested for their effect on the control performance.

In Fig. 2 the effect of the model-plant mismatch $\Delta\alpha = \alpha - \alpha_M$ from -10 to $+10$ % is shown. The ISE and IAE indices have been used here to assess the control performance.

Figure 3 shows examples of step responses in the control system with a selected plant for several values $\Delta\alpha$, and Fig. 4 gives IAE values for the same cases.

Figure 5 exemplifies the effect produced by the model-plant mismatch caused by an improper choice of the finite memory length L on the control performance.

In Fig. 6 it is shown to what extent the reduced order α_r of the approximated integer-order model determined by the SVD method affects the control performance.

As we see in the figures the impact of the particular model-plant mismatch on the control performance is different. It seems that the biggest impact has the mismatch in fractional order and the finite memory length. At the same time we can see that the approximation of the fractional-order model by a reduced integer order one can be effective in practical applications.

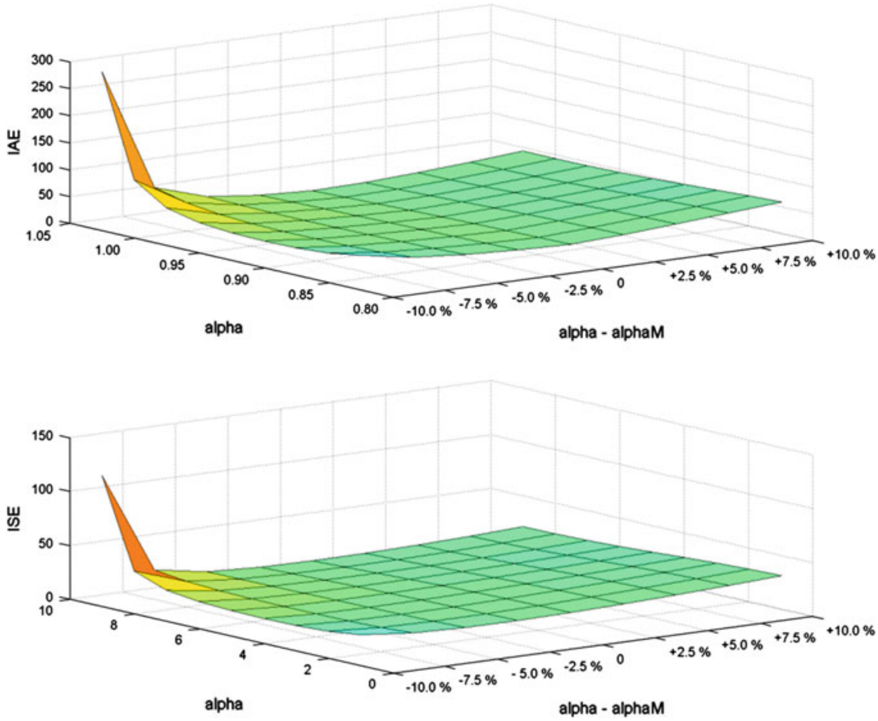

Fig. 2 The effect of the model-plant mismatch $\Delta\alpha = \alpha - \alpha_M$

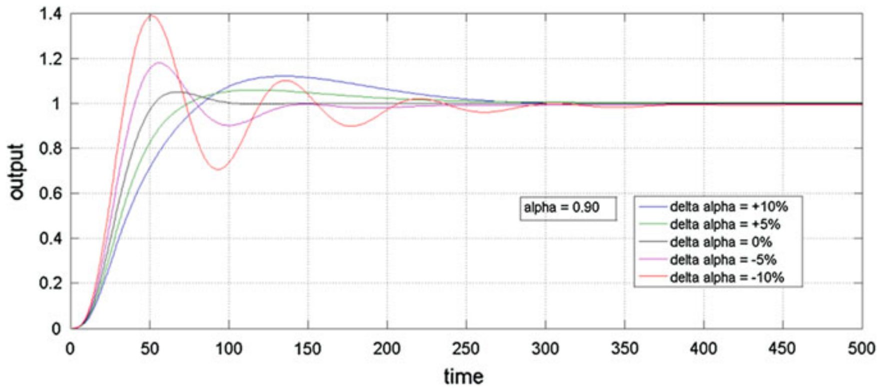

Fig. 3 Step responses in the control system with a selected plant for several values $\Delta\alpha$

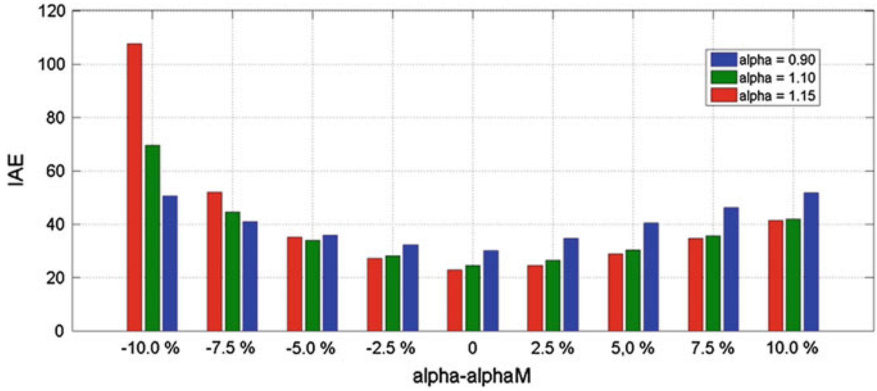

Fig. 4 IAE indices in the control system with a selected plant for several values $\Delta\alpha$

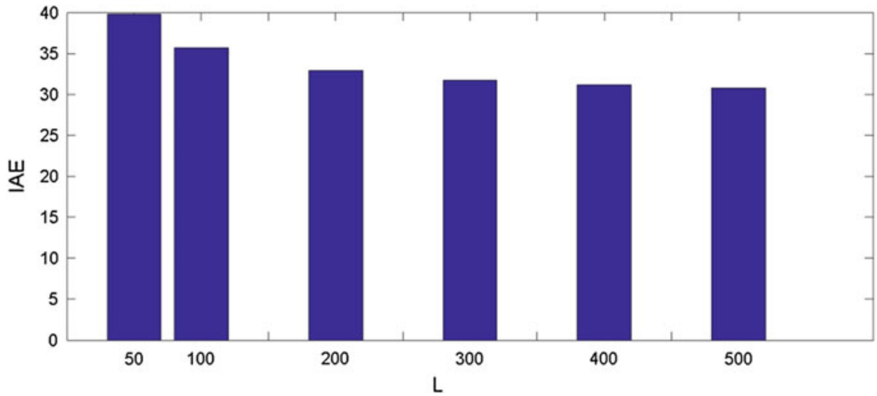

Fig. 5 IAE indices in the control system with a fractional order plant $(\alpha = 0.8)$ for several finite memory length L

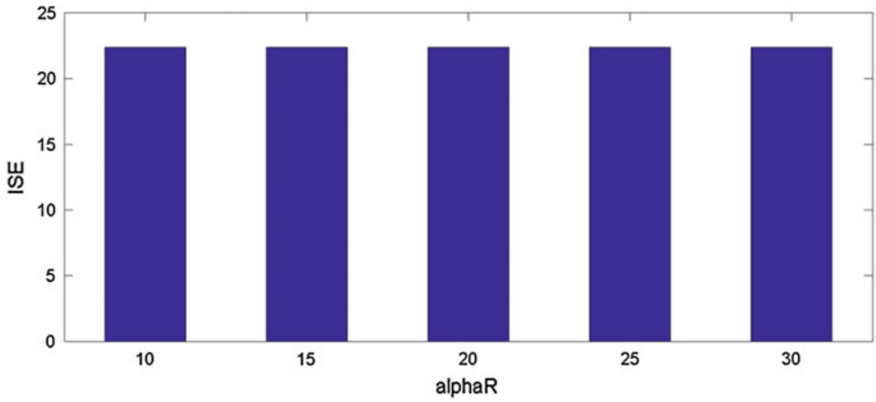

Fig. 6 ISE indices in the control system with a fractional order plant $(\alpha = 0.8)$ for several reduced order α_r of the approximated integer-order model determined by the SVD method

5 Conclusions

In MPC applications, the accuracy of the process model plays a crucial role in providing a satisfactory control performance. Mismatch between the process model and the plant may produce a significant effect on the control quality. The problem is even more important in the case of FOMPC, where there can be much more types of model-plant mismatch. This is corroborated by the results of simulation tests shown as examples. Therefore, it is necessary to conduct further research, including theoretical analysis of FOMPC robustness to plant parameter variations that lead to model-plant mismatch.

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