Complexity of Uniform Membership of Context-Free Tree Grammars

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Abstract. We show that the uniform membership problem of context-free tree grammars is PSPACE-complete. The proof of the upper bound is by construction of an equivalent pushdown tree automaton representable in polynomial space. With this technique, we also give an alternative proof that the respective non-uniform membership problem is in NP. A corollary for uniform membership of ε-free indexed grammars is obtained.

1 Introduction

Context-free tree grammars (cftg) [3,10] generalize the concept of context-free rewriting to the realm of tree languages. They have been studied, among others, for their close connection to indexed grammars: their yield languages are precisely the indexed languages [1,10]. Recently, there has been renewed interest in cftg within the area of natural language processing, as they – and related formalisms such as tree adjoining grammars – allow modelling particular linguistic phenomena.

In this paper, we investigate the computational complexity of the uniform membership problem of cftg. In Section 5, this problem is shown to be PSPACE-complete. In order to prove containment in PSPACE, an equivalent pushdown tree automaton (pta) $M^\dagger$ [4] is constructed from G in a succession of intermediate steps (Section 4). We demonstrate that $M^\dagger$ can be implemented in polynomial space. The idea behind $M^\dagger$ is taken from Aho's proof that the indexed languages are context-sensitive [1, Sec. 5]. Note that in [1], the construction is given directly by means of a rather complex Turing machine, and without proof of correctness. In contrast, by employing pta, we can provide a formal proof; moreover, we think that this presentation is easier to understand. As a corollary, we establish the PSPACE-completeness of uniform membership of ε-free indexed grammars.

To show that the constructed pta $M^\dagger$ is also of potential interest besides the paper's main theorem, we use $M^\dagger$ in Section 6 for an alternative proof of the fact that the non-uniform membership problem of cftg is in NP. Note that this result already follows from the containment of the indexed languages in NP, whose proof in [11] rests, however, upon the correctness of the Turing machine mentioned above. In [7], containment in NP was proven for the class of output languages of compositions of macro tree transducers, which contains the context-free tree languages properly.

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Recall that there are two restricted modes of derivation for cftg, the OI and the IO mode. In fact, the OI mode is equivalent to the unrestricted mode used in this paper [3]. For complexity results on cftg under the IO mode cf. [2,14].

2 Preliminaries

The set of natural numbers with zero is denoted by $\mathbb{N}$, and the set $\{1, \ldots, n\}$ by $[n]$, for every $n \in \mathbb{N}$. Note that $[0] = \emptyset$, the empty set. Let A be a set. Given relations $R, S \subseteq A \times A$, their product $R \circ S$ is the relation $\{(a, c) \in A \times A \mid \exists b \in A\colon (a, b) \in R, (b, c) \in S\}$. An alphabet is a finite nonempty set. The set of words over A is A^*, the empty word is ε, and $A^+ = A^* \setminus \{\varepsilon\}$. Let $w = a_1 \ldots a_n$ with $a_1, \ldots, a_n \in A$ for some $n \in \mathbb{N}$. Then $|w| = n$, and $\widetilde{w} = a_n \ldots a_1$, the reversal of w.

An alphabet Σ equipped with a function $\mathrm{rk}_\Sigma\colon \Sigma \to \mathbb{N}$ is a *ranked alphabet.* Let Σ be a ranked alphabet. When Σ is obvious, we write rk instead of rk_Σ. Let $k \in \mathbb{N}$. Then $\Sigma^{(k)} = \mathrm{rk}^{-1}(k)$. We often write $\sigma^{(k)}$ and mean that $\mathrm{rk}(\sigma) = k$. We assume tacitly that there are some $\alpha^{(0)}$ and $\sigma^{(n)} \in \Sigma$ such that $n \geq 2$. Let U be a set and Λ denote $\Sigma \cup U \cup C$, where C is made up of the three symbols '(', ')', and ','. The set $T_\Sigma(U)$ of *trees (over Σ indexed by U)* is the smallest set $T \subseteq \Lambda^*$ such that $U \subseteq T$, and for every $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $\xi_1, \ldots, \xi_k \in T$, we also have that $\sigma(\xi_1, \ldots, \xi_k) \in T$. A tree $\alpha()$ is abbreviated by α, a tree $\gamma(\xi)$ by $\gamma\xi$, and $T_\Sigma(\emptyset)$ by T_Σ. Let $\xi, \zeta \in T_\Sigma(U)$. The set of *positions (Gorn addresses) of* ξ is denoted by $\mathrm{pos}(\xi) \subseteq \mathbb{N}^*$. The *size* $|\xi|$ of ξ is $|\mathrm{pos}(\xi)|$. Denote the *label* of ξ at its position w by $\xi(w)$, and the *subtree* of ξ at w by $\xi|_w$. The result of *replacing* the subtree $\xi|_w$ in ξ by ζ is $\xi[\zeta]_w$. Given $k \in \mathbb{N}$, $\xi, \xi_1, \ldots, \xi_k \in \Lambda^*$, and pairwise different $u_1, \ldots, u_k \in U$, denote by $\xi[u_1/\xi_1, \ldots, u_k/\xi_k]$ the result of *substituting* every occurrence of u_i in ξ with ξ_i, where $i \in [k]$. If $\xi, \xi_1, \ldots, \xi_k$ are trees in $T_\Sigma(U)$, then so is $\xi[u_1/\xi_1, \ldots, u_k/\xi_k]$. We will use the sets of *variables* $X = \{x_1, x_2, \ldots\}$ and $Y = \{y\}$. For each $k \in \mathbb{N}$, let $X_k = \{x_i \mid i \in [k]\}$. Unless specified, Σ and N denote arbitrary ranked alphabets, and Γ an arbitrary alphabet.

We presuppose the basic definitions and results from computational complexity theory, cf. e.g. [8]. In particular, we will use the same concept of reduction as in [8]; i.e., many-one reductions that are computable by a deterministic multi-tape Turing machine with work tape space in $\mathcal{O}(\log n)$. Functions that are computable in this manner are *logspace-computable.* Assuming a reasonable encoding, operations on trees such as determining the j-th subtree of a node, or substitution at a given position, are logspace-computable, cf. [6, Lem. 2].

3 Context-Free Tree Grammars and Pushdown Automata

A *context-free tree grammar (cftg)* over Σ is a tuple $G = (N, \Sigma, S, P)$ such that Σ and N are disjoint ranked alphabets (of *terminal* resp. *nonterminal symbols*), $S \in N^{(0)}$, and P is a finite set of *productions* of the form $A(x_1, \ldots, x_k) \to \xi$ for some $k \in \mathbb{N}$, $A \in N^{(k)}$, and $\xi \in T_{N \cup \Sigma}(X_k)$. Let $G = (N, \Sigma, S, P)$ be a cftg. Given $\zeta_1, \zeta_2 \in T_{N \cup \Sigma}$, we write $\zeta_1 \Rightarrow_G \zeta_2$ if there are $(A(x_1, \ldots, x_k) \to \xi) \in P$ and $w \in \mathrm{pos}(\zeta_1)$ such that $\zeta_1(w) = A$ and $\zeta_2 = \zeta_1[\xi[x_1/\zeta_1|_{w1}, \ldots, x_k/\zeta_1|_{wk}]]_w$.

Table 1. Membership problems of cftg (over Σ)

	Uniform Membership	**Membership**
Input	cftg G over Σ, $\xi \in T_\Sigma$	$\xi \in T_\Sigma$
Question	Is $\xi \in L(G)$?	Is $\xi \in L(G)$ for a fixed cftg G over Σ?

The *tree language of* G, denoted by $L(G)$, is the set $\{\xi \in T_\Sigma \mid S \Rightarrow_G^* \xi\}$. In this situation, we call $L(G)$ a *context-free tree language.* The *size* of G, denoted by $|G|$, is $|N| + \sum_{(l \to r) \in P}(|l| + |r|)$.

In this work, we will investigate the *uniform membership problem*, as well as the *(non-uniform) membership problem* of cftg (over Σ), as defined in Tab. 1.

A *pushdown tree system (pts)* is a tuple $M = (Q, \Sigma, \Gamma, q_0, R)$ such that Q is an alphabet (of *states*), Σ is a ranked alphabet, Γ is a nonempty set, $q_0 \in Q$, and R is a set of *rules* of the following three forms:

$$\textit{(i)}\ q(y) \to \sigma(p_1(y), \ldots, p_k(y)), \qquad \textit{(ii)}\ q(y) \to p(\gamma y), \qquad \textit{(iii)}\ q(\gamma y) \to p(y),$$

where $y \in Y$, $\sigma \in \Sigma^{(k)}$ for some $k \in \mathbb{N}$, q, p, p_1, $\ldots$, $p_k \in Q$, and $\gamma \in \Gamma$. We call M a *pushdown tree automaton (pta)* when Γ (and thus R) is finite.[1] The set of rules from R of form *(i)* (resp. *(ii)*, *(iii)*) is denoted by R_Σ (resp. by $R_\uparrow$, $R_\downarrow$), and their elements are called *stay, push,* and *pop rules.*

Given a pts $M = (Q, \Sigma, \Gamma, q_0, R)$, let $\mathcal{C}_M = \{q(\eta) \mid q \in Q,\ \eta \in \Gamma^*\}$ and $\mathcal{DF}_M = T_\Sigma(\mathcal{C}_M)$, the sets of *configurations* and *derivation forms* of M. Given a rule $\rho \in R$ of the form $l \to r$, and ζ_1, $\zeta_2 \in \mathcal{DF}_M$, we write $\zeta_1 \Rightarrow_M^\rho \zeta_2$ if there are some $w \in \mathrm{pos}(\zeta_1)$ and $\eta \in \Gamma^*$ such that $\zeta_1|_w = l[y/\eta]$, $\zeta_2 = \zeta_1\big[r[y/\eta]\big]_w$, and w is the leftmost position in ζ_1 which is labeled by an element of Q. We let $\Rightarrow_M = \bigcup_{\rho \in R} \Rightarrow_M^\rho$. Let ζ, $\zeta' \in \mathcal{DF}_M$. A *derivation of* ζ' *from* ζ *in* M is a sequence $\rho_1 \ldots \rho_m \in R^*$ such that there are ζ_0, $\ldots$, $\zeta_m \in \mathcal{DF}_M$ where $\zeta = \zeta_0$, $\zeta_{i-1} \Rightarrow_M^{\rho_i} \zeta_i$ for every $i \in [m]$, and $\zeta' = \zeta_m$. If d is of this form, we write $\zeta_0 \Rightarrow_M^d \zeta_m$. The set of all derivations of ζ' from ζ in M is denoted by $\mathcal{D}_M(\zeta, \zeta')$. We let $\mathcal{D}_M = \bigcup_{\zeta \in \mathcal{C}_M, \xi \in T_\Sigma} \mathcal{D}_M(\zeta, \xi)$. The *tree language of* M, denoted by $L(M)$, is the set $\{\xi \in T_\Sigma \mid q_0(\varepsilon) \Rightarrow_M^* \xi\}$, and the *size of* M, which is denoted by $|M|$, is $|Q| + |\Gamma| + \sum_{(l \to r) \in R}(|l| + |r|)$.

Let Z_1 be a cftg or a pts, and Z_2 be a cftg or a pts. Then Z_1 and Z_2 are *equivalent* if $L(Z_1) = L(Z_2)$. If not specified otherwise, G will denote an arbitrary cftg (N, Σ, S, P) in the sequel, and M an arbitrary pta $(Q, \Sigma, \Gamma, q_0, R)$.

As proven in [4, Thm. 1], pta accept exactly the context-free tree languages. A close inspection of the proof shows that all constructions are logspace-computable.

[1] In fact, pta as given here are in two ways a special case of the *restricted pushdown tree automata (rpta)* of [4]. First, the pushdowns of rpta are monadic trees from $T_\Gamma(\{Z\})$ for some distinct nullary symbol Z, while pta use words over Γ. Both approaches are clearly equivalent. Second, the rules of pta are more restricted than those of rpta. However, in the construction of an equivalent rpta from a cftg in [4, Thm. 3], only rules of type *(i)*–*(iii)* are created, so the restriction has no impact.

Lemma 1 **([4]).** *Let $L \subseteq T_\Sigma$. There is a cftg G such that $L = L(G)$ iff there is a pta M such that $L = L(M)$. Also, M is logspace-computable from G, and vice versa.*

4 Compact pts and Finite Representations

Both in our analysis of uniform and non-uniform membership of cftg, the proof of containment in the respective complexity class rests on the successive application of certain transformations to the pta M that is obtained from an input cftg G by using Lem. 1.

We introduce these transformations in the following subsections, drawing upon ideas Aho used in the construction of the linear bounded automaton that demonstrates the context-sensitivity of the indexed languages [1, Sec. 5]. We apply these ideas directly at the level of pta instead of building a complex Turing machine, and provide formal proofs of correctness.

4.1 Augmented pta

Later on, we will require for our proofs that the amount of steps in a derivation of a tree $\xi \in T_\Sigma$ in the pta M is bounded. However, for an arbitrary pta, it seems difficult to find the respective bound. This is due to the presence of *unnecessary turns* of M. Take, e.g., the following derivation in some pta M:

$$q(\gamma) \Rightarrow_M q_1(\delta\gamma) \Rightarrow_M q_2(\tau\delta\gamma) \Rightarrow_M q_3(\delta\gamma) \Rightarrow_M q_4(\gamma) \Rightarrow_M p(\varepsilon)\,.$$

Clearly, the turn $q(\gamma) \Rightarrow_M^* q_4(\gamma)$ was in some sense unnecessary, and we could have avoided it if there was already some rule $q(\gamma y) \to p(y)$ in R, because then $q(\gamma) \Rightarrow_M p(\varepsilon)$. The existence of such rules to avoid unnecessary turns is exactly what constitutes an *augmented* pta. Formally, a pta M is augmented if for every q_1, q_2, $q_3 \in Q$ and $\gamma \in \Gamma$ such that $q_1(\varepsilon) \Rightarrow_M q_2(\gamma) \Rightarrow_M q_3(\varepsilon)$, and for every rule $q_3(l) \to r$ in R, the rule $q_1(l) \to r$ is also in R.

Lemma 2. *For every pta M, an equivalent augmented pta M' is constructible in polynomial time.*

Proof. Given a pta M, define the pta $M' = (Q, \Sigma, \Gamma, q_0, R')$, where R' results from the following fixed-point iteration. Initially, let $R' = R$. Then, while there are q_1, q_2, $q_3 \in Q$, $\gamma \in \Gamma$, and $(q_3(l) \to r) \in R'$ such that $q_1(\varepsilon) \Rightarrow_{M'} q_2(\gamma) \Rightarrow_{M'} q_3(\varepsilon)$ and $(q_1(l) \to r) \notin R'$, insert $(q_1(l) \to r)$ into R'.

It is easy to see that every iteration respects the invariant $L(M') = L(M)$. Moreover, after termination of the algorithm, M' is obviously augmented.

Observe that the maximal number of rules of a pta over the terminal alphabet Σ is in $\mathcal{O}(|Q|^2 \cdot |\Gamma| + |\Sigma| \cdot |Q|^{m+1})$, where m is the maximal rank of a symbol from Σ. As a rule is added in every iteration, the algorithm terminates eventually. Since Σ is fixed, the number of iterations is polynomial in the input. $\square$

A derivation with no unneccessary turns is called *succinct*. More precisely, $d \in \mathcal{D}_M$ is *succinct* if there are $e_1 \in R_{\downarrow}^*$, $e_2 \in R_{\uparrow}^*$, $\omega \in R_\Sigma$, $k \in \mathbb{N}$ and d_1, …, $d_k \in \mathcal{D}_M$ such that $d = e_1 e_2 \omega d_1 \dots d_k$ and for every $i \in [k]$, d_i is succinct. The set of succinct derivations of $\xi \in T_\Sigma$ from $q(\eta) \in \mathcal{C}_M$ in M is denoted by $\mathcal{DS}_M(q(\eta), \xi)$, and the set of all succinct elements of $\mathcal{D}_M$ by $\mathcal{DS}_M$. The following lemma means that in an augmented pta M, we need only consider succinct derivations. We omit its proof, which is based on the observation that in a derivation d of M, it is not necessary to apply a push rule ρ_1 right before a pop rule ρ_2, as we can always replace $\rho_1\rho_2$ in d by some other rule ρ' of M.

Lemma 3. *Let M be augmented, $q(\eta) \in \mathcal{C}_M$, and $\xi \in T_\Sigma$. If $q(\eta) \Rightarrow_M^* \xi$, then there is also a succinct derivation $d \in \mathcal{DS}_M(q(\eta), \xi)$.*

4.2 Compact pts

Besides avoiding unnecessary turns of M, there is still one problem to solve. We might refer to it as M being too verbose in its pushdowns. E.g., in the derivation

$$q(\varepsilon) \Rightarrow_M q'(\gamma) \Rightarrow_M q''(\delta\gamma) \Rightarrow_M \sigma\big(u(\delta\gamma), p(\delta\gamma)\big) \Rightarrow_M^2 \sigma\big(\alpha, p(\gamma)\big) \Rightarrow_M \sigma\big(\alpha, p(\varepsilon)\big)$$

one could save time and space – i.e., derivation steps and pushdown cells – if there was some pushdown symbol $[\delta\gamma]$ such that

$$q(\varepsilon) \Rightarrow_M q''([\delta\gamma]) \Rightarrow_M \sigma\big(u([\delta\gamma]), p([\delta\gamma])\big) \Rightarrow_M^2 \sigma\big(\alpha, p(\varepsilon)\big)\,.$$

We will construct a pts $M^\sharp$ with such pushdown symbols, i.e., with all symbols of the form $[\eta]$, where $\eta \in \Gamma^+$. It is said to be a *compact* pts, since, as will be proved later on, for every tree $\xi \in L(M^\sharp)$, only polynomially many steps in $|\xi|$ are required for a derivation of ξ in $M^\sharp$, and the sizes of the pushdowns in the derivation can also bounded in this manner. Evidently, $M^\sharp$ can be infinite. However, considering $M^\sharp$ makes the following proofs easier, hence we stick with it for now, and deal with the question of a finite representation of $M^\sharp$ later.

The pushdown words of $M^\sharp$ can be understood as *subdivisions* of those of M. Because we must subdivide $M^\sharp$'s pushdown words even further in some proofs, the following definitions are needed. Choose two symbols '[' and ']' not from Γ, and let $\mathcal{S}(\Gamma) = \{[\eta] \mid \eta \in \Gamma^+\}$. Let $\eta = \gamma_1 \dots \gamma_n$ from Γ^+, where $\gamma_i \in \Gamma$ for $i \in [n]$. Given k_0, …, $k_m \in \mathbb{N}$ with $0 = k_0 < \dots < k_m = n$ for some $m > 0$, the $(k_0, \dots, k_m)$*-subdivision of* η is the word $[\gamma_{k_0+1} \dots \gamma_{k_1}] \dots [\gamma_{k_{m-1}+1} \dots \gamma_{k_m}] \in \mathcal{S}(\Gamma)^+$. Moreover, the ε*-subdivision of* ε is ε. An $\eta' \in \mathcal{S}(\Gamma)^*$ is a *subdivision* of an $\eta \in \Gamma^*$, denoted by $\eta' \sqsubseteq \eta$, if η' is an $\boldsymbol{E}$-subdivision of η for some $\boldsymbol{E} \in \mathbb{N}^*$. This $\boldsymbol{E}$ is unique; we denote it by $\boldsymbol{E}(\eta')$. If $\boldsymbol{E}(\eta') = (k_1, \dots, k_m)$, then let $E(\eta') = \{k_1, \dots, k_m\}$. Define $\iota\colon \Gamma^* \to \mathcal{S}(\Gamma)^*$ by $\iota(\varepsilon) = \varepsilon$ and $\iota(\eta) = [\eta]$ for $\eta \in \Gamma^+$. Let now $\eta \in \Gamma^*$ and η', $\eta'' \in \mathcal{S}(\Gamma)^*$ with η', $\eta'' \sqsubseteq \eta$. We write $\eta' \sqsubseteq \eta''$ if $E(\eta') \supseteq E(\eta'')$. We denote the unique $\kappa \sqsubseteq \eta$ with $E(\kappa) = E(\eta') \cup E(\eta'')$ by $\eta' \sqcap \eta''$. Note that $\eta' \sqcap \eta'' \sqsubseteq \eta'$ and $\eta' \sqcap \eta'' \sqsubseteq \eta''$. Regarding the length of $\eta' \sqcap \eta''$ as an element of $\mathcal{S}(\Gamma)^*$,

$$|\eta' \sqcap \eta''| = |E(\eta' \sqcap \eta'')| - 1 \leq |E(\eta')| + |E(\eta'')| - 3 = |\eta'| + |\eta''| - 1 \qquad (1)$$

whenever $\eta \in \Gamma^+$, and if $\eta = \varepsilon$, then obviously $|\eta' \sqcap \eta''| = |\eta'| + |\eta''| = 0$. Finally, if $\eta' \in \mathcal{S}(\Gamma)^*$ is the $(k_1, \ldots, k_m)$-subdivision of η, then let $\widetilde{\eta}'$ denote the $(|\eta| - k_m, \ldots, |\eta| - k_1)$-subdivision of $\widetilde{\eta}$.

Now we define the *compact pts* $M^\sharp = (Q, \Sigma, \Gamma_\sharp, q_0, R_\sharp)$ *of* M, where $\Gamma_\sharp = \mathcal{S}(\Gamma)$, and $R_\sharp$ contains the rules *(i)* $q_1(y) \to q_2([\eta]y)$ for every $\eta \in \Gamma^+$ such that $q_1(\varepsilon) \Rightarrow_M^{r_1 \ldots r_k} q_2(\eta)$ with $r_1, \ldots, r_k \in R_\uparrow$, denote the resulting rule by $[r_1 \ldots r_k]$; *(ii)* $q_1([\eta]y) \to q_2(y)$ for every $\eta \in \Gamma^+$ such that $q_1(\eta) \Rightarrow_M^{r_1 \ldots r_k} q_2(\varepsilon)$ with $r_1, \ldots, r_k \in R_\downarrow$, denote the resulting rule by $[r_1 \ldots r_k]$; *(iii)* and for every rule $\omega \in R_\Sigma$, the rule $[\omega]$, which is identical to ω.

Obviously, $L(M^\sharp) = L(M)$. By the notation for the rules of $M^\sharp$, we have $R_\sharp \subseteq \mathcal{S}(R)$. The notion of subdivision, of the relation $\sqsubseteq$, and of the operation $\sqcap$, carries over to derivations of $M^\sharp$ in a straightforward manner. In a derivation d in $M^\sharp$, a subdivision η' of a pushdown η determines a corresponding subdivision d' of d, and vice versa. The following lemma circumstantiates this observation.

Lemma 4. *Let q, $p \in Q$, $\eta \in \Gamma^*$, and $d \in R^*$. Moreover, let $d' \sqsubseteq d$ and $\eta' \sqsubseteq \eta$.*

(i) *If $d \in R_\downarrow^*$ with $q(\eta) \Rightarrow_M^d p(\varepsilon)$, then $q(\eta') \Rightarrow_{M^\sharp}^{d'} p(\varepsilon)$ iff $\boldsymbol{E}(\eta') = \boldsymbol{E}(d')$.*

(ii) *If $d \in R_\uparrow^*$ with $q(\varepsilon) \Rightarrow_M^d p(\eta)$, then $q(\varepsilon) \Rightarrow_{M^\sharp}^{d'} p(\eta')$ iff $\boldsymbol{E}(\widetilde{\eta}') = \boldsymbol{E}(d')$.*

The following restricted mode of derivation is important as well. Let $\mu \in \mathbb{N}$ and $\zeta \in \mathcal{DF}_M$. We say that ζ *has μ-bounded pushdowns* if there is no infix $\kappa \in \Gamma^*$ of ζ with $|\kappa| > \mu$. Thus the size of every pushdown occurring in ζ is at most μ. Let moreover ζ_1, $\zeta_2 \in \mathcal{DF}_M$. We write $\zeta_1 \overset{(\mu)}{\Longrightarrow}{}_M^\rho \zeta_2$ if $\zeta_1 \Rightarrow_M^\rho \zeta_2$ and both ζ_1 and ζ_2 have μ-bounded pushdowns. The relations $\overset{(\mu)}{\Longrightarrow}{}_M$ and $\overset{(\mu)}{\Longrightarrow}{}_M^d$, for some $d \in R^*$, are defined analogously. In the latter case, *all* intermediate derivation forms of d are required to have μ-bounded pushdowns.

In the following lemmas, we establish polynomial bounds for the lengths of successful derivations in $M^\sharp$, and for the sizes of the pushdowns "along the way."

Lemma 5. *Let M be augmented, and let $q(\eta) \in \mathcal{C}_M$, $\eta' \sqsubseteq \eta$, $d \in \mathcal{DS}_M$, $d' \sqsubseteq d$, $\xi \in T_\Sigma$, and $\mu \in \mathbb{N}$ with $q(\eta) \Rightarrow_M^d \xi$ and $q(\eta') \overset{(\mu)}{\Longrightarrow}{}_{M^\sharp}^{d'} \xi$. For every $\eta'' \sqsubseteq \eta'$, there is a $d'' \sqsubseteq d'$ such that $q(\eta'') \overset{(\mu')}{\Longrightarrow}{}_{M^\sharp}^{d''} \xi$, and $\mu' = \mu + |\eta''| - |\eta'|$.*

Proof. Presume η, η', d, d', q, ξ, and μ as above, and let $\eta'' \sqsubseteq \eta'$ and $\mu' = \mu + |\eta''| - |\eta'|$. The proof is by structural induction on ξ, hence suppose $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$ and ξ_1, …, $\xi_k \in T_\Sigma$ such that $\xi = \sigma(\xi_1, \ldots, \xi_k)$. As $d \in \mathcal{DS}_M$, there are $e_1 \in R_\downarrow^*$, $e_2 \in R_\uparrow^*$, $\omega \in R_\Sigma$, u, $p \in Q$, $p_i(\theta) \in \mathcal{C}_M$ and $d_i \in \mathcal{DS}_M(p_i(\theta), \xi_i)$ for every $i \in [k]$ such that $d = e_1 e_2 \omega d_1 \ldots d_k$ and

$$q(\eta_1\eta_2) \Rightarrow_M^{e_1} u(\eta_2) \Rightarrow_M^{e_2} p(\eta_3\eta_2) \Rightarrow_M^{\omega} \sigma(p_1(\theta), \ldots, p_k(\theta)) \Rightarrow_M^{d_1} \cdots \Rightarrow_M^{d_k} \xi\,,$$

for some η_1, η_2, $\eta_3 \in \Gamma^*$ with $\eta = \eta_1\eta_2$ and $\theta = \eta_3\eta_2$.

By definition of $M^\sharp$, we have $d' = e_1' e_2' [\omega] d_1' \ldots d_k'$ for some $e_1' \sqsubseteq e_1$, $e_2' \sqsubseteq e_2$, and $d_i' \sqsubseteq d_i$, for every $i \in [k]$. Furthermore, $q(\eta') \overset{(\mu)}{\Longrightarrow}{}_{M^\sharp}^{e_1' e_2'} p(\theta')$, where $\eta' = \eta_1' \eta_2'$,

$\theta' = \eta'_3\eta'_2$, and $\eta'_i \sqsubseteq \eta_i$ for every $i \in [3]$. Observe that $|\theta'| \leq \mu$. As $\eta'' \sqsubseteq \eta'$, there must be $\eta''_1 \sqsubseteq \eta'_1$ and $\eta''_2 \sqsubseteq \eta'_2$ such that $\eta'' = \eta''_1\eta''_2$. Note that $|\eta''_1| \geq |\eta'_1|$. Let e''_1 be the $\boldsymbol{E}(\eta''_1)$-subdivision of η_1, then $q(\eta''_1\eta''_2) \Rightarrow^{e''_1}_{M^\sharp} u(\eta''_2)$. In fact,

$$|\eta''| = |\eta'| + |\eta''| - |\eta'| \leq \mu + |\eta''| - |\eta'|\,,$$

hence $q(\eta''_1\eta''_2) \xRightarrow{(\mu')}{}^{e''_1}_{M^\sharp} u(\eta''_2)$. Moreover, $u(\eta''_2) \Rightarrow^{e'_2}_{M^\sharp} p(\eta'_3\eta''_2)$. Let $\theta'' = \eta'_3\eta''_2$, then

$$\begin{aligned}|\theta''| &= |\eta'_3| + |\eta'_2| + |\eta''_2| - |\eta'_2| = |\theta'| + |\eta'_1| + |\eta''_2| - (|\eta'_1| + |\eta'_2|)\\ &\leq \mu + |\eta''_1| + |\eta''_2| - (|\eta'_1| + |\eta'_2|) = \mu + |\eta''| - |\eta'|\,,\end{aligned}$$

and thus $u(\eta''_2) \xRightarrow{(\mu')}{}^{e'_2}_{M^\sharp} p(\eta'_3\eta''_2)$. Since $\theta'' \sqsubseteq \theta'$, the induction hypothesis implies that for every $i \in [k]$, there are some $d''_i \sqsubseteq d'_i$ such that $p_i(\theta'') \xRightarrow{(\mu'')}{}^{d''_i}_{M^\sharp} \xi_i$ and $\mu'' = \mu + |\theta''| - |\theta'|$. We have

$$\begin{aligned}\mu'' &= \mu + |\theta''| - |\theta'| = \mu + |\eta'_3| + |\eta''_2| - |\eta'_3| - |\eta'_2|\\ &\leq \mu + |\eta''_1| + |\eta''_2| - |\eta'_1| - |\eta'_2| = \mu + |\eta''| - |\eta'| = \mu'\,.\end{aligned}$$

The inequation holds because $|\eta''_1| \geq |\eta'_1|$. Thus for each $i \in [k]$, $p_i(\theta'') \xRightarrow{(\mu')}{}^{d''_i}_{M^\sharp} \xi_i$. We set $d'' = e''_1e'_2[\omega]d''_1 \dots d''_k$, yielding $q(\eta'') \xRightarrow{(\mu')}{}^{d''}_{M^\sharp} \xi$. □

In the following, we denote the number $2 \cdot |\xi|$ by $\mu(\xi)$, for every tree $\xi \in T_\Sigma$.

Lemma 6. *Suppose that M is augmented. For every $q(\eta) \in \mathcal{C}_M$, $\xi \in T_\Sigma$ and $d \in \mathcal{DS}_M(q(\eta), \xi)$, there are $\eta' \sqsubseteq \eta$ and $d' \sqsubseteq d$ such that $q(\eta') \xRightarrow{(\mu(\xi))}{}^{d'}_{M^\sharp} \xi$.*

Proof. Assume $q(\eta)$, ξ and d as given above. The proof is by structural induction on ξ, therefore let $\xi = \sigma(\xi_1, \dots, \xi_k)$ for some $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$ and $\xi_1, \dots, \xi_k \in T_\Sigma$. Moreover, let $d = e_1e_2\omega d_1 \dots d_k$ such that $e_1 \in R^*_\downarrow$, $e_2 \in R^*_\uparrow$, $\omega \in R_\Sigma$, and d_1, $\dots$, $d_k \in \mathcal{DS}_M$. Thus there are η_1, η_2, η_3, and $\theta \in \Gamma^*$ with $\eta = \eta_1\eta_2$ and $\theta = \eta_3\eta_2$, as well as u, p, p_1, $\dots$, $p_k \in Q$, satisfying

$$q(\eta_1\eta_2) \Rightarrow^{e_1}_M u(\eta_2) \Rightarrow^{e_2}_M p(\eta_3\eta_2) \Rightarrow^{\omega}_M \sigma(p_1(\theta), \dots, p_k(\theta)) \Rightarrow^{d_1}_M \cdots \Rightarrow^{d_k}_M \xi\,.$$

By the induction hypothesis, for every $i \in [k]$, there are a $\theta'_i \sqsubseteq \theta$ and a $d'_i \sqsubseteq d_i$ such that $|\theta'_i| \leq \mu(\xi_i)$ and $p_i(\theta'_i) \xRightarrow{(\mu(\xi_i))}{}^{d'_i}_{M^\sharp} \xi_i$. Set $\theta' = \theta'_1 \sqcap \dots \sqcap \theta'_k \sqcap (\iota(\eta_3)\iota(\eta_2))$. Note that if $k = 0$, then $\theta' = \iota(\eta_3)\iota(\eta_2)$. By applying (1) k times,

$$|\theta'| \leq \Big(\sum_{i\in[k]} |\theta'_i|\Big) + 2 - k \leq \Big(\sum_{i\in[k]} \mu(\xi_i)\Big) + 2 - k = \mu(\xi) - k \leq \mu(\xi)\,. \tag{2}$$

Thus $p(\theta') \xRightarrow{(\mu(\xi))}{}^{[\omega]}_{M^\sharp} \sigma(p_1(\theta'), \dots, p_k(\theta'))$. Let $j \in [k]$. Because $\theta' \sqsubseteq \theta'_j$, by Lem. 5, there is some $d''_j \sqsubseteq d'_j$ such that $p_j(\theta') \xRightarrow{(\mu')}{}^{d''_j}_{M^\sharp} \xi_j$, and where

$$\mu' = \mu(\xi_j) + |\theta'| - |\theta'_j| \leq \mu(\xi_j) + \Big(\sum_{i\in[k]} |\theta'_i|\Big) + 2 - |\theta'_j| \leq \Big(\sum_{i\in[k]} \mu(\xi_i)\Big) + 2 = \mu(\xi)\,.$$

Thus also $p_j(\theta') \xRightarrow[M^\sharp]{(\mu(\xi))\, d''_j} \xi_j$. By definition of θ', there must be some $\eta'_2 \sqsubseteq \eta_2$ and $\eta'_3 \sqsubseteq \eta_3$ such that $\theta' = \eta'_3\eta'_2$. Set $\eta' = \iota(\eta_1)\eta'_2$. If $k = 0$, then clearly $|\eta'_2| = 1 < \mu(\xi)$. If $k > 0$, then by (2), $|\eta'_2| \leq |\theta'| < \mu(\xi)$. Thus in both cases $|\eta'| \leq \mu(\xi)$. Hence $q(\eta') \xRightarrow[M^\sharp]{(\mu(\xi))\, \iota(e_1)} u(\eta'_2)$. Moreover, as $\eta'_3 \sqsubseteq \iota(\eta_3)$, by Lem. 4, there is some $e'_2 \sqsubseteq e_2$ with $u(\eta'_2) \xRightarrow[M^\sharp]{(\mu(\xi))\, e'_2} p(\eta'_3\eta'_2)$. Set $d' = \iota(e_1)e'_2[\omega]d''_1 \dots d''_k$, then $q(\eta') \xRightarrow[M^\sharp]{(\mu(\xi))\, d'} \xi$, and the proof is concluded. □

Lemma 7. *Let M be augmented. For every $\xi \in L(M)$, there is a derivation $d' \in \mathcal{D}_{M^\sharp}(q_0(\varepsilon), \xi)$ with $|d'| \leq \mu(\xi)^2 + \mu(\xi)$.*

Proof. Let $\xi \in L(M)$, let $d \in \mathcal{DS}_M(q_0(\varepsilon), \xi)$, and consider the derivation d' as constructed in Lem. 6. Let $w \in \mathrm{pos}(\xi)$, and let d'' be an infix of d' such that $d'' \in \mathcal{D}_{M^\sharp}(q(\eta'), \xi|_w)$, for some $q(\eta') \in \mathcal{C}_{M^\sharp}$. We prove that $|d''| \leq (\mu(\xi)+1) \cdot \mu(\xi|_w)$ by well-founded induction using the relation "is child node of" on $\mathrm{pos}(\xi)$. For this purpose, let $\xi|_w = \sigma(\xi_1, \dots, \xi_k)$ for some $k \in \mathbb{N}$, $\sigma \in \Sigma^{(k)}$, and $\xi_1, \dots, \xi_k \in T_\Sigma$. Observe that d'' is of the form $e_1e_2[\omega]d'_1 \dots d'_k$ for some $e_1 \in (R_\sharp)^*_\downarrow$, $e_2 \in (R_\sharp)^*_\uparrow$, $\omega \in R_\Sigma$, $u, p_1, \dots, p_k \in Q$, $\kappa', \theta' \in \Gamma_\sharp^*$, and $d'_i \in \mathcal{D}_{M^\sharp}(p_i(\theta'), \xi_i)$, for $i \in [k]$, and

$$q(\eta') \xRightarrow[M^\sharp]{(\mu(\xi))\, e_1} u(\kappa') \xRightarrow[M^\sharp]{(\mu(\xi))\, e_2} p(\theta') \xRightarrow[M^\sharp]{(\mu(\xi))\, [\omega]} \sigma(p_1(\theta'), \dots, p_k(\theta'))\,.$$

As the pushdowns η' and θ' are bounded in their size by $\mu(\xi)$, we must have $|e_1e_2[\omega]| \leq 2 \cdot \mu(\xi) + 1$. By the induction hypothesis, $|d'_i| \leq (\mu(\xi)+1) \cdot \mu(\xi_i)$, thus

$$|d''| \leq 2 \cdot (\mu(\xi) + 1) + \sum_{i \in [k]} \big((\mu(\xi) + 1) \cdot \mu(\xi_i)\big) = (\mu(\xi) + 1) \cdot \mu(\xi|_w)\,.$$

The lemma follows with $w = \varepsilon$ and $d'' = d'$. □

4.3 Representing $M^\sharp$ by a Finite Object

Finally we show how to construct from M a finite representation $M^\dagger$ of $M^\sharp$. Let $\Gamma_\dagger = \mathcal{P}(Q \times Q)$ and define a mapping $h\colon \Gamma \to \Gamma_\dagger$ such that, for every $\gamma \in \Gamma$, $h(\gamma) = \big\{(q,p) \mid q(\gamma y) \to p(y) \text{ in } R\big\}$. We set $M^\dagger = (Q, \Sigma, \Gamma_\dagger, q_0, R_\dagger)$, where $R_\dagger$ is the smallest set R' such that *(i)* $R_\Sigma \subseteq R'$, *(ii)* for every rule $q(y) \to p(\gamma y)$ in R, the rule $q(y) \to p\big(h(\gamma)y\big)$ is in R', *(iii)* whenever $q(y) \to p(Uy)$ and $p(y) \to u(Vy)$ are in R', then also $q(y) \to u\big((V \circ U)y\big)$ is in R', *(iv)* for every $U \in \Gamma_\dagger$ and $(q,p) \in U$, the rule $q(Uy) \to p(y)$ is in R'. Note that $R_\dagger$ is given effectively by these conditions. The size of $M^\dagger$ is in general exponential in $|M|$.

We show that $M^\dagger$ is indeed a faithful representation of $M^\sharp$. Extend h to $\tilde{h}\colon \Gamma^+ \to \Gamma_\dagger$ by $\tilde{h}(\gamma_1 \dots \gamma_k) = h(\gamma_1) \circ \dots \circ h(\gamma_k)$ for $k > 0$ and $\gamma_1, \dots, \gamma_k \in \Gamma$. Further, extend $\tilde{h}$ to $\hat{h}\colon \Gamma_\sharp^* \to \Gamma_\dagger^*$ by $\hat{h}([\eta_1] \dots [\eta_k]) = \tilde{h}(\eta_1) \dots \tilde{h}(\eta_k)$ for every $k \in \mathbb{N}$ and $\eta_1, \dots, \eta_k \in \Gamma^+$. We identify h, $\tilde{h}$, and $\hat{h}$ in the following. There is the following close relation between $M^\sharp$ and $M^\dagger$; the uncomplicated proof is omitted.

Algorithm 1. Nondeterministic decision procedure for uniform membership

Input: pta $M = (Q, \Sigma, \Gamma, q_0, R)$, $\xi \in T_\Sigma$
Output: "Yes" if $\xi \in L(M)$, diverges otherwise

```
ζ ← q₀(ε)
loop
  select leftmost w ∈ pos(ζ) such that ζ(w) = q(η) for some q(η) ∈ C_{M†}
  either
    choose a rule q(y) → σ(p₁(y), ..., p_k(y)) ∈ R
    ζ ← ζ[σ(p₁(η), ..., p_k(η))]_w
  or
    choose a rule q(y) → p(γy) ∈ R and set u ← p, U ← h(γ)
    repeat n times for some n ∈ ℕ
      choose a rule u(y) → v(γy) ∈ R and set u ← v, U ← h(γ) ∘ U
    end repeat
    ζ ← ζ[u(Uη)]_w
  or if η = Uκ for some U ∈ Γ_†, κ ∈ Γ_†*
    choose some (u, p) ∈ U such that u = q
    ζ ← ζ[p(κ)]
  end either
  if ζ = ξ then return "Yes" else if ζ ∈ T_Σ then diverge endif
end loop
```

Lemma 8. *For every n, $\mu \in \mathbb{N}$, $q(\eta) \in \mathcal{C}_{M^\sharp}$, and for every $\xi \in T_\Sigma$, we have that $q(\eta) \overset{(\mu)}{\Longrightarrow}{}^n_{M^\sharp} \xi$ iff $q\big(h(\eta)\big) \overset{(\mu)}{\Longrightarrow}{}^n_{M^\dagger} \xi$.*

Suppose that M is augmented. Then the lemma implies together with Lem. 6 and 7 that while $|M^\dagger|$ may be exponential in $|M|$, we may assume nevertheless that for every $\xi \in L(M)$, there is a derivation d of ξ in $M^\dagger$ such that both the length of d, as well as the size of every configuration in d, are bounded by a polynomial in $|\xi|$.

5 The Uniform Membership Problem

Employing $M^\dagger$, we can now investigate the complexity of the uniform membership problem of cftg. We begin with the upper bound.

Theorem 1. *The uniform membership problem of cftg over Σ is in* PSPACE.

Proof. Let $\xi \in T_\Sigma$ and let G be a cftg over Σ. Construct an augmented pta $M = (Q, \Sigma, \Gamma, q_0, R)$ with $L(M) = L(G)$. By Lem. 1 and 2, this takes time (and thus space) polynomial in $|G|$. Recall the mapping $h\colon \Gamma^+ \to \Gamma_\dagger$ from the definition of $M^\dagger$. Alg. 1 contains a nondeterministic procedure which decides $\xi \in L(M)$ in space restricted to $2 \cdot |\xi|^2 \cdot Q^2$. It works by emulating a derivation d' in the compact pta $M^\dagger$ as constructed above. The construction of d' is "on-the-fly." In each loop, the leftmost configuration $q(\eta)$ in the current derivation form ζ is selected, and a rule ρ is chosen. We may choose ρ to be a stay or pop rule of

$M^\dagger$, it is then applied to $q(\eta)$. Then again, we may choose a nonzero number of push rules of M with compatible states, apply h to the symbols they push, and combine the results by the product of binary relations. Clearly, this procedure can emulate exactly the derivations in $M^\dagger$.

If $\xi \in L(M)$, then there is a succinct derivation $d \in \mathcal{DS}_M(q_0(\eta), \xi)$, and, by Lem. 6 and 8, a derivation $d' \sqsubseteq d$ in $M^\dagger$ that has $(2 \cdot |\xi|)$-bounded pushdowns. Each pushdown symbol that occurs in d' is a subset of $Q \times Q$, and can thus be stored within space $|Q|^2$. As the number of configurations occurring in an intermediate derivation form ζ of d is bounded by $|\xi|$, the space bound of $2 \cdot |\xi|^2 \cdot |Q|^2$ is sufficient to store ζ. By [12, Thm. 1], the procedure is also computable in deterministic space polynomial in $|\xi|$ and $|M|$. □

Theorem 2. *The uniform membership problem of cftg over Σ is* PSPACE*-hard.*

Proof. Recall the following decision problem. Let Δ be an alphabet. The *intersection problem* is specified as follows.

Input: Deterministic finite-state automata $A_1, \ldots, A_k$ over Δ for some $k \in \mathbb{N}$
Question: Is $\bigcap_{i=1}^{k} L(A_i) = \emptyset$?

This problem is PSPACE-complete [5]. We give a reduction of its complement to the uniform membership problem of cftg. Then, as PSPACE = coPSPACE, the latter problem is PSPACE-hard. The reduction's idea is to construct a pta M which guesses some $w \in \Delta^*$ on its pushdown, copies it as often as needed (by stay rules with some symbol σ of at least binary rank), and then simulates the automata $A_1, \ldots, A_k$ on the respective copies. If A_i accepts w, M outputs some symbol α on the i-th branch, else it blocks. The search for $w \in \bigcap_{i=1}^{k} L(A_i)$ is thus reduced to the question $\xi \in L(M)$, for a tree $\xi \in T_\Sigma$ that is independent of w.

Formally, assume deterministic finite-state automata $A_i = (Q_i, \Delta, q_0^i, F_i, \delta_i)$, defined as usual, for some $k \in \mathbb{N}$ and each $i \in [k]$. We require the state sets Q_i to be pairwise disjoint, and $\alpha \notin \Delta$. By assumption, Σ contains some $\alpha^{(0)}$ and $\sigma^{(n)}$ with $n \geq 2$. Construct the pta $M = (Q, \Sigma, \Delta \cup \{\#\}, q_0, R)$ where $Q = \{q_0\} \cup \{u_0, \ldots, u_k\} \cup \bigcup_{i=1}^{k} Q_i$, with q_0, u_0, …, u_k distinct states, and R contains the rules

$$q_0(y) \to u_k(\#y)\,,\;\; u_k(y) \to u_k(by),\;\; u_i(y) \to \sigma\big(q_0^i(y), u_{i-1}(y), u_0(y), \ldots, u_0(y)\big)$$

for every $i \in [k]$ and $b \in \Delta$. Moreover, for every $i \in [k]$, $b \in \Delta$, $q, p \in Q_i$ such that $\delta_i(q, b) = p$, and $f \in F_i$, the rule set R contains $q(by) \to p(y)$ and $f(\#y) \to \alpha$. Finally, for every $\gamma \in \Delta \cup \{\#\}$, the rule $u_0(\gamma y) \to \alpha$ is in R. Let ξ be the tree $\sigma\big(\alpha, \sigma(\alpha, \cdots \sigma(\alpha, \ldots, \alpha) \cdots, \alpha, \ldots, \alpha), \alpha, \ldots, \alpha\big)$ with exactly k occurrences of σ. Both M and ξ are logspace-computable from the input. It is easy to show that $\xi \in L(M)$ iff there is some $w \in \Delta^*$ such that $w \in \bigcap_{i=1}^{k} L(A_i)$. □

5.1 Uniform Membership of ε-free Indexed Grammars

Following suit to earlier research, established results on cftg can also give new insight on indexed languages.

Let us recall indexed grammars [1]. In the spirit of [11], an *indexed grammar* is a tuple $G = (N, \Sigma, \Gamma, S, P)$, where N, Σ, and Γ are alphabets, $S \in N$, and P is a finite set of productions of the forms *(i)* $A(y) \to B_1(y) \ldots B_k(y)$, *(ii)* $A(y) \to B(\gamma y)$, *(iii)* $A(\gamma y) \to B(y)$, and *(iv)* $A(y) \to a$, for A, B, $B_1, \ldots, B_k \in N$, $k \in \mathbb{N}$, $\gamma \in \Gamma$, and $a \in \Sigma$. We call G *ε-free* if all its productions of form *(i)* satisfy $k \geq 1$.

The similarity of indexed grammars to pta is apparent, and $\Rightarrow_G$, as well as $L(G)$, are defined analogously. This similarity is captured in the "yield theorem" [10, p. 115]. Although its original formulation applies to cftg, we restate it for pta, cf. [4, Prop. 8]. The remark on logspace-computability is easily reexamined.

Lemma 9 ([4,10]). *Let $L \subseteq (\Sigma^{(0)})^*$. Then, there is a pta M over Σ such that $L = \text{yield}(L(M))$ iff there is an ε-free indexed grammar G over $\Sigma^{(0)}$ such that $L = L(G)$. Also, G is logspace-computable from M, and vice versa.*

The following corollary is then a direct consequence of Thms. 1 and 2 together with the yield theorem.

Corollary 1. *The uniform membership problem of ε-free indexed grammars is* PSPACE*-complete.*

In contrast, the uniform membership problem of indexed grammars *with* ε-rules is EXP-complete [13].[2]

6 The Non-Uniform Membership Problem

In this section, we intend to show that the pta $M^\dagger$ may also be useful for other means, by presenting an alternative proof of the NP upper bound of non-uniform membership of cftg. Note that this bound is already known: the class of output languages of compositions of macro tree transducers, a proper superclass of the context-free tree languages, is in NP [7, Thm. 8].

If we regard trees from T_Σ as well-parenthesized words over $\Sigma \cup C$, as defined in the preliminaries, then a context-free tree language can be also understood as an indexed string language. Therefore, the following upper bound is as well a consequence of the containment of the indexed languages in NP. Its proof in [11] rests on the correctness of the Turing machine from [1].

Theorem 3. *The membership problem of cftg over Σ is in* NP.

Proof. Let G be the cftg fixed in the membership problem. Construct an equivalent augmented pta M, as well as $M^\dagger$ as defined above. As G is not part of the input, $M^\dagger$ is constructible in constant time. Consider the nondeterministic decision procedure in Alg. 2. By Lem. 8, $L(M^\dagger) = L(M^\sharp)$, and moreover $L(M^\sharp) = L(M)$. So if the procedure returns "Yes", then there is some

[2] Still, the emptiness problem of ε-free indexed grammars remains EXP-complete. This follows from a small modification of the indexed grammar witnessing EXP-hardness in [13]. Hence, by Lem. 9, the emptiness problem of cftg is EXP-complete, too.

Algorithm 2. Nondeterministic decision procedure for membership of cftg

Input: $\xi \in T_\Sigma$
Output: "Yes" if $\xi \in L(M)$, diverges otherwise
choose some $d \in R_\dagger^*$ with $|d| \leq \mu(\xi)^2 + \mu(\xi)$ and $\mu(\xi)$-bounded pushdowns
if $q_0(\eta) \Rightarrow_{M^\dagger}^{d} \xi$ **then** return "Yes" **else** diverge **endif**

$d \in \mathcal{D}_{M^\dagger}(q_0(\varepsilon), \xi)$, and hence $\xi \in L(M)$. Conversely, if $\xi \in L(M)$, then there must be some $d' \in \mathcal{D}_{M^\sharp}(q_0(\varepsilon), \xi)$, and by Lem. 7, we may assume that $|d'| \leq \mu(\xi)^2 + \mu(\xi)$. By Lem. 8, there is a $d \in \mathcal{D}_{M^\dagger}(q_0(\varepsilon), \xi)$ with equal length bound. Thus the procedure returns "Yes". □

Hardness of the problem can be demonstrated in the same manner as for indexed grammars [11, Prop. 1], by devising a cftg G such that $L(G)$ encodes the set of all satisfiable propositional formulas in 3-conjunctive normal form. For the sake of completeness, we restate the respective theorem.

Theorem 4. *There are a ranked alphabet Σ and a cftg G over Σ such that the membership problem of G is* NP*-hard.*

7 Conclusion

In this paper, the complexity of the uniform membership problem of cftg was proven to be PSPACE-complete. A corollary for uniform membership of indexed grammars was obtained. As a by-product, we could state an alternative proof for the NP-completeness of the non-uniform membership problem of cftg.

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