Chapter 3
Fuzzy Calculus

This chapter treats two types of fuzzy calculus: one for fuzzy-set-valued functions and other for fuzzy bunches of functions. Section 3.1 reviews definitions of fuzzy Aumann, Henstock, and Riemann integrals and the Hukuhara derivative and its generalizations. It also provides some theorems, including a Fundamental Theorem of Calculus. All these definitions and results were previously presented in the literature. Section 3.2 introduces derivative and integral for fuzzy bunches of functions and results concerning them, some of which never published before. Examples illustrate some of the concepts and theorems, especially in the last section, where new results provide comparisons between the different approaches.

3.1 Fuzzy Calculus for Fuzzy-Set-Valued Functions

This section reviews some known approaches of integrals (Aumann, Riemann, and Henstock integrals) and derivatives (Hukuhara and generalized derivatives) for fuzzy-set-valued functions. It also presents results connecting these fuzzy integrals and derivatives. The reader interested in other proposals may refer to (e.g., [11–13, 16]).

3.1.1 Integrals

The first integral proposed for fuzzy-number-valued functions is based on Aumann integral for multivalued functions [2] and was defined in [21] and [23].

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Denote by $S(G)$ the subset of all integrable selections of a set-valued function $G : I \to \mathscr{P}(\mathbb{R}^n)$, i.e.,

$$S(G) = \{g : I \to \mathbb{R}^n : g \text{ is integrable and } g(t) \in G(t), \forall t \in I\}. \tag{3.1}$$

Definition 3.1 ([21, 23]). The Aumann integral of a fuzzy-set-valued function $F : [a, b] \to \mathscr{F}_{\mathscr{C}}(\mathbb{R}^n)$ over $[a, b]$ is defined levelwise

$$\left[(A)\int_a^b F(x)\,dx\right]_\alpha = \int_a^b [F]_\alpha\,dx \tag{3.2}$$

$$= \left\{\int_a^b g(x)\,dx : g \in S([F(x)]_\alpha)\right\} \tag{3.3}$$

for all $\alpha \in [0, 1]$.

The function $F : [a, b] \to \mathscr{F}_{\mathscr{C}}(\mathbb{R}^n)$ is said to be Aumann integrable over $[a, b]$ if $(A)\int_a^b F(x)\,dx \in \mathscr{F}_{\mathscr{C}}(\mathbb{R}^n)$.

The following integrals have been defined for functions $F : [a, b] \to \mathscr{F}_{\mathscr{C}}(\mathbb{R})$.

Definition 3.2 ([15, 26]). The Riemann integral of a fuzzy-number-valued function $F : [a, b] \to \mathscr{F}_{\mathscr{C}}(\mathbb{R})$ over $[a, b]$ is the fuzzy number A such that for every $\epsilon > 0$ there exist $\delta > 0$ such that for any division $d : a = x_0 < x_1 < \ldots < x_n = b$ with $x_i - x_{i-1} < \delta$, $i = 1, \ldots, n$, and $\xi_i \in [x_i - x_{i-1}]$

$$d_\infty\left(\sum_{i=1}^{n-1} F(\xi_i)(x_i - x_{i-1}), A\right) < \epsilon. \tag{3.4}$$

The function $F : [a, b] \to \mathscr{F}_{\mathscr{C}}(\mathbb{R})$ is said to be Riemann integrable over $[a, b]$ if $A \in \mathscr{F}_{\mathscr{C}}(\mathbb{R})$. We denote $(R)\int_a^b F(x)\,dx = A$

Definition 3.3 ([7, 26]). Consider $\delta_n : a = x_0 < x_1 < \ldots < x_n = b$ a partition of the interval $[a, b]$, $\xi_i \in [x_i - x_{i-1}]$, $i = 1, \ldots, n$, a sequence ξ in δ_n and $\delta(x) > 0$ a real-valued function over $[a, b]$. The division $P(\delta_n, \xi)$ is considered to be δ-fine if

$$[x_{i-1}, x_i] \subseteq (\xi_{i-1} - \delta(\xi_{i-1}), \xi_{i-1} + \delta(\xi_{i-1})) \tag{3.5}$$

The Henstock integral of a fuzzy-number-valued function $F : [a, b] \to \mathscr{F}_{\mathscr{C}}(\mathbb{R})$ over $[a, b]$ is the fuzzy number A such that for every $\epsilon > 0$ there exist a real-valued function δ such that for any δ-fine division $P(\delta_n, \xi)$,

$$d_\infty\left(\sum_{i=1}^{n-1} F(\xi_i)(x_i - x_{i-1}), A\right) < \epsilon. \tag{3.6}$$

The function $F : [a,b] \to \mathscr{F}_{\mathscr{C}}(\mathbb{R})$ is said to be Henstock integrable over $[a,b]$ if $A \in \mathscr{F}_{\mathscr{C}}(\mathbb{R})$. We denote $(H) \int_a^b F(x)\, dx = A$.

Henstock integral is more general than Riemann, i.e., whenever a function is Riemann integrable, it is Henstock integrable as well.

Remark 3.1. Writing that a function is *integrable*, without specifying whether it is Aumann, Riemann, or Henstock, means it is integrable in all these three senses.

Corollary 3.1 ([5, 21, 26]). *If a function $F : [a,b] \to \mathscr{F}_{\mathscr{C}}(\mathbb{R})$ is continuous, then it is integrable. Moreover,*

$$\left[\int F\right]_\alpha = \left[\int f_\alpha^-, \int f_\alpha^+\right] \tag{3.7}$$

for all $\alpha \in [0,1]$.

Theorem 3.1 ([5, 21, 26]). *Let $F : [a,b] \to \mathscr{F}_{\mathscr{C}}(\mathbb{R})$ be integrable and $a \le x_1 \le x_2 \le x_3 \le b$. Then*

$$\int_{x_1}^{x_3} F = \int_{x_1}^{x_2} F + \int_{x_2}^{x_3} F. \tag{3.8}$$

Theorem 3.2 ([5, 21, 26]). *Let $F, G : [a,b] \to \mathscr{F}_{\mathscr{C}}(\mathbb{R})$ be integrable, then*

(i) $\int (F+G) = \int F + \int G$;
(ii) $\int (\lambda F) = \lambda \int F$, *for any* $\lambda \in \mathbb{R}$;
(iii) $d_\infty(F,G)$ *is integrable;*
(iv) $d_\infty(\int F, \int G) \le \int d_\infty(F,G)$.

3.1.2 Derivatives

The Hukuhara differentiability for fuzzy functions is based on the concept of Hukuhara differentiability for interval-valued functions [20].

Definition 3.4 ([22]). Let $F : (a,b) \to \mathscr{F}_{\mathscr{C}}(\mathbb{R}^n)$. If the limits

$$\lim_{h \to 0^+} \frac{F(x_0+h) \ominus_H F(x_0)}{h} \quad \text{and} \quad \lim_{h \to 0^+} \frac{F(x_0) \ominus_H F(x_0-h)}{h} \tag{3.9}$$

exist and equal some element $F'_H(x_0) \in \mathscr{F}_{\mathscr{C}}(\mathbb{R}^n)$, then F is Hukuhara differentiable (H-differentiable for short) at x_0 and $F'_H(x_0)$ is its Hukuhara derivative (H-derivative for short) at x_0.

Example 3.1. The fuzzy-number-valued function of Example 2.6, $F(x) = Ax$ with $A = (-1; 0; 1)$, is an H-differentiable function for $x \ge 0$ and

$$F'_H(x) = A. \tag{3.10}$$

For $x < 0$, F is not H-differentiable since $F(x+h) \ominus_H F(x)$ is not defined. Considering $x > 0$, F is a particular case of Example 8.30 in [5], which shows that any function $G(x) = Bg(x)$ with $g(x) > 0$, $g'(x) > 0$ and B a fuzzy number is H-differentiable. Moreover,

$$G'_H(x) = Bg'(x). \tag{3.11}$$

An H-differentiable fuzzy function has H-differentiable α-cuts (that is, its α-cuts are interval-valued H-differentiable functions). The converse, however, is not true, unless its α-cuts are uniformly H-differentiable (see [21]).

Definition 3.5 ([24]). Let $F : [a,b] \to \mathscr{F}_{\mathscr{C}}(\mathbb{R})$. If

$$[(f_\alpha^-)'(x_0), (f_\alpha^+)'(x_0)] \tag{3.12}$$

exists for all $\alpha \in [0,1]$ and defines the α-cuts of a fuzzy number $F'_S(x_0)$, then F is Seikkala differentiable at x_0 and $F'_S(x_0)$ is the Seikkala derivative of F at x_0.

If $F : [a,b] \to \mathscr{F}_{\mathscr{C}}(\mathbb{R})$ is H-differentiable, then $f_\alpha^-(x)$ and $f_\alpha^+(x)$ are differentiable and

$$[F'(x_0)]_\alpha = [(f_\alpha^-)'(x_0), (f_\alpha^+)'(x_0)], \tag{3.13}$$

that is, if F is H-differentiable, it is Seikkala differentiable and the derivatives are the same [21].

Theorem 3.3 ([21]). *Let $F : [a,b] \to \mathscr{F}_{\mathscr{C}}(\mathbb{R}^n)$ be an H-differentiable function. Then it is continuous.*

Theorem 3.4 ([21]). *Let $F, G : [a,b] \to \mathscr{F}_{\mathscr{C}}(\mathbb{R}^n)$ be H-differentiable functions and $\lambda \in \mathbb{R}$. Then $(F+G)'_H = F'_H + G'_H$ and $(\lambda F)'_H = \lambda F'_H$.*

If F is Seikkala (or Hukuhara) differentiable, $(f_\alpha^-)'(x) \leq (f_\alpha^+)'(x)$, hence the function diam $[F(x)]_\alpha = f_\alpha^+(x) - f_\alpha^-(x)$ is nondecreasing on $[a,b]$. It means that the function has nondecreasing fuzziness. As will be clear in Chap. 4, this is considered a shortcoming since an H-differentiable function cannot represent a function with decreasing fuzziness or periodicity. In order to overcome this, the generalized differentiability concepts were created. They generalize the H-differentiability, that is, they are defined for more cases of fuzzy-number-valued functions and whenever the H-derivative of a function exists, its generalization exists and has the same value.

Definition 3.6 ([6, 8]). Let $F : (a,b) \to \mathscr{F}_{\mathscr{C}}(\mathbb{R})$. If the limits of some pair

(i) $\lim\limits_{h\to 0^+} \dfrac{F(x_0+h) \ominus_H F(x_0)}{h}$ and $\lim\limits_{h\to 0^+} \dfrac{F(x_0) \ominus_H F(x_0-h)}{h}$ or

(ii) $\lim\limits_{h\to 0^+} \dfrac{F(x_0) \ominus_H F(x_0+h)}{-h}$ and $\lim\limits_{h\to 0^+} \dfrac{F(x_0-h) \ominus_H F(x_0)}{-h}$ or

(iii) $\lim_{h\to 0^+} \frac{F(x_0+h) \ominus_H F(x_0)}{h}$ and $\lim_{h\to 0^+} \frac{F(x_0-h) \ominus_H F(x_0)}{-h}$ or

(iv) $\lim_{h\to 0^+} \frac{F(x_0) \ominus_H F(x_0+h)}{-h}$ and $\lim_{h\to 0^+} \frac{F(x_0) \ominus_H F(x_0-h)}{h}$

exist and are equal to some element $F'_G(x_0)$ of $\mathscr{F}_{\mathscr{C}}(\mathbb{R})$, then F is strongly generalized differentiable (or GH-differentiable) at x_0 and $F'_G(x_0)$ is the strongly generalized derivative (GH-derivative for short) of F at x_0.

An (i)-strongly generalized differentiable function presents nondecreasing diameter, since it is the definition of the H-differentiability. (ii)-strongly generalized differentiability (we call (ii)-differentiability, for short), on the other hand, implies in nonincreasing diameter. The (iii) and (iv)-differentiability cases correspond to points where the function changes its behavior with respect to the diameter. It means that a strongly differentiable non-crisp function may present periodical behavior, as well as convergence to a single point.

In case F is defined on a closed interval, that is, $F : [a,b] \to \mathscr{F}_{\mathscr{C}}(\mathbb{R})$, we define the derivative at a using the limit from the right and at b using the limit from the left.

Example 3.2. The fuzzy-number-valued function of Example 2.6, $F(x) = Ax$ with $A = (-1; 0; 1)$, is a GH-differentiable function for $x \in \mathbb{R}$ and

$$F'_{gH}(x) = A. \tag{3.14}$$

Different from the H-derivative case, the GH-derivative of F is defined for $x < 0$. According to Example 8.35 in [5], any function $G(x) = Bg(x)$ with B a fuzzy number and $g : (a,b) \to \mathbb{R}$ differentiable with at most a finite number of roots in (a,b) is GH-differentiable. Moreover,

$$G'_H(x) = Bg'(x). \tag{3.15}$$

Example 3.2 illustrates that, different from the H-derivative, GH-differentiable functions can have decreasing diameter.

Definition 3.7 ([8]). Let $F : (a,b) \to \mathscr{F}_{\mathscr{C}}(\mathbb{R})$ and $x_0 \in (a,b)$. For a nonincreasing sequence $h_n \to 0$ and $n_0 \in \mathbb{N}$ we denote

$$A^{(1)}_{n_0} = \left\{n \geq n_0; \exists E^{(1)}_n := F(x_0+h_n) \ominus_H F(x_0)\right\}, \tag{3.16}$$

$$A^{(2)}_{n_0} = \left\{n \geq n_0; \exists E^{(2)}_n := F(x_0) \ominus_H F(x_0+h_n)\right\}, \tag{3.17}$$

$$A^{(3)}_{n_0} = \left\{n \geq n_0; \exists E^{(3)}_n := F(x_0) \ominus_H F(x_0-h_n)\right\}, \tag{3.18}$$

$$A^{(4)}_{n_0} = \left\{n \geq n_0; \exists E^{(4)}_n := F(x_0-h_n) \ominus_H F(x_0)\right\}. \tag{3.19}$$

The function F is said to be weakly generalized differentiable at x_0 if for any nonincreasing sequence $h_n \to 0$ there exists $n_0 \in \mathbb{N}$, such that

$$A_{n_0}^{(1)} \cup A_{n_0}^{(2)} \cup A_{n_0}^{(3)} \cup A_{n_0}^{(4)} = \{n \in \mathbb{N}; n \geq n_0\} \tag{3.20}$$

and moreover, there exists an element in $\mathscr{F}_C(\mathbb{R})$, such that if for some $j \in \{1, 2, 3, 4\}$ we have card $(A_{n_0}^{(j)}) = +\infty$, then

$$\lim_{h_n \searrow 0, n \to \infty, n \in A_{n_0}^{(j)}} d_\infty \left(\frac{E_n^{(j)}}{(-1)^{j+1} h_n}, F'(x_0) \right) = 0. \tag{3.21}$$

Definition 3.7 is more general than Definition 3.6, that is, it is defined for more cases of fuzzy-number-valued functions and whenever the latter exists, the former also exists and has the same value.

The next definition is equivalent to Definition 3.7 (see [10]).

Definition 3.8 ([10, 25]). Let $F : (a, b) \to \mathscr{F}_{\mathscr{C}}(\mathbb{R})$. If the limit

$$\lim_{h \to 0} \frac{F(x_0 + h) \ominus_{gH} F(x_0)}{h} \tag{3.22}$$

exists and belongs to $\mathscr{F}_{\mathscr{C}}(\mathbb{R})$, then F is generalized Hukuhara differentiable (gH-differentiable for short) at x_0 and $F'_{gH}(x_0)$ is the generalized Hukuhara derivative (gH-derivative for short) of F at x_0.

Theorem 3.5 ([10]). *Let $F : [a, b] \to \mathscr{F}_{\mathscr{C}}(\mathbb{R}^n)$ be a gH-differentiable function at x_0. Then it is levelwise continuous at x_0.*

Theorem 3.6 ([10]). *Let $F : [a, b] \to \mathscr{F}_{\mathscr{C}}(\mathbb{R})$ be such that the functions $f_\alpha^-(x)$ and $f_\alpha^+(x)$ are real-valued functions, differentiable with respect to x, uniformly in $\alpha \in [0, 1]$. Then the function $F(x)$ is gH-differentiable at a fixed $x \in [a, b]$ if and only if one of the following two cases holds:*

(a) $\left(f_\alpha^-\right)'(x)$ *is increasing,* $\left(f_\alpha^+\right)'(x)$ *is decreasing as functions of* α*, and* $\left(f_1^-\right)'(x) \leq \left(f_1^+\right)'(x)$*, or*

(b) $\left(f_\alpha^-\right)'(x)$ *is decreasing,* $\left(f_\alpha^+\right)'(x)$ *is increasing as functions of* α*, and* $\left(f_1^+\right)'(x) \leq \left(f_1^-\right)'(x)$.

Moreover,

$$\left[F'_{gH}(x)\right]_\alpha = [\min\{\left(f_\alpha^-\right)'(x), \left(f_\alpha^+\right)'(x)\}, \max\{\left(f_\alpha^-\right)'(x), \left(f_\alpha^+\right)'(x)\}], \tag{3.23}$$

for all $\alpha \in [0, 1]$.

The next concept further extends the gH-differentiability.

Definition 3.9 ([25]). Let $F : (a, b) \to \mathscr{F}_{\mathscr{C}}(\mathbb{R})$. If the limit

$$\lim_{h \to 0} \frac{F(x_0 + h) \ominus_g F(x_0)}{h} \tag{3.24}$$

exists and belongs to $\mathscr{F}_{\mathscr{C}}(\mathbb{R})$, then F is generalized differentiable (g-differentiable for short) at x_0 and $F'_g(x_0)$ is the fuzzy generalized derivative (g-derivative for short) of F at x_0.

Example 3.3. Recall the fuzzy-number-valued function of Example 2.13, $F : [0, 0.5] \to \mathscr{F}_{\mathscr{C}}(\mathbb{R})$ with α-cuts

$$[F(x)]_\alpha = \begin{cases} \left[x^2 - 3 + \alpha, (1 - 2\alpha)x^2 - 2\alpha + 2\right], \text{ if } & 0 \leq \alpha \leq 0.5 \\ \left[x^2 - 3 + \alpha, (2\alpha - 1)x^2 - 6\alpha + 4\right], \text{ if } & 0.5 < \alpha \leq 1 \end{cases}. \quad (3.25)$$

The aim is to calculate the gH and the g-derivative of F.

Equation (2.53) provides easy means to calculate (3.22). For $\alpha \in [0, 0.5]$ one obtains

$$[F(x + h) \ominus_{gH} F(x)]_\alpha = [(1 - 2\alpha)(2xh + h^2), 2xh + h^2]. \quad (3.26)$$

Thus

$$\lim_{h\to 0} \frac{[F(x + h) \ominus_{gH} F(x)]_\alpha}{h} = [(1 - 2\alpha)2x, 2x] \quad (3.27)$$

and as consequence

$$\lim_{h\to 0} \frac{[F(x + h) \ominus_{gH} F(x)]_0}{h} = \{2x\} \quad (3.28)$$

and

$$\lim_{h\to 0} \frac{[F(x + h) \ominus_{gH} F(x)]_{0.25}}{h} = [x, 2x]. \quad (3.29)$$

The condition

$$\alpha < \beta \Rightarrow \lim_{h\to 0} \frac{[F(x + h) \ominus_{gH} F(x)]_\beta}{h} \subset \lim_{h\to 0} \frac{[F(x + h) \ominus_{gH} F(x)]_\alpha}{h} \quad (3.30)$$

does not hold, hence $\lim_{h\to 0} \frac{[F(x+h)\ominus_{gH} F(x)]_\alpha}{h}$ cannot be a fuzzy number and the gH-derivative is not defined for this function.

Equation (2.54) can be used in this case to find (3.24), for all $\alpha \in [0, 1]$. Since $f_\beta^-(x+h) - f_\beta^-(x) = 2xh + h^2$ and $f_\beta^+(x+h) - f_\beta^+(x) = (1-2\alpha)2xh + h^2$ for $\beta \leq 0.5$ and $f_\beta^-(x + h) - f_\beta^-(x) = 2xh + h^2$ and $f_\beta^+(x + h) - f_\beta^+(x) = (2\alpha - 1)2xh + h^2$ for $\beta > 0.5$, we obtain for $\alpha > 0.5$:

$$\lim_{h\to 0}\frac{[F(x+h)\ominus_g F(x)]_\alpha}{h} = \text{cl}\bigcup_{\beta\geq\alpha>0.5}[(2\beta-1)2x,2x] = [(2\alpha-1)2x,2x]. \tag{3.31}$$

For $\alpha \leq 0.5$, the levelwise limit becomes

$$\text{cl}\left(\bigcup_{0.5\geq\beta\geq\alpha\geq 0}[(1-2\beta)2x,2x]\right)\bigcup\left(\bigcup_{\beta>0.5}[(2\beta-1)2x,2x]\right) = [0,2x]. \tag{3.32}$$

The result is the fuzzy number $F_g' : [0,0.5] \to \mathscr{F}_{\mathscr{C}}(\mathbb{R})$ with α-cuts

$$[F_g'(x)]_\alpha = \begin{cases} [0,2x]\,, \text{ if } & 0\leq\alpha\leq 0.5 \\ [(2\alpha-1)2x,2x]\,, \text{ if } & 0.5<\alpha\leq 1 \end{cases}. \tag{3.33}$$

as the g-derivative.

The g-difference is not defined for all pairs of fuzzy numbers, as we showed in Example 2.3. The same happens to the g-derivative, that is, it is not always well-defined (see also [17]).

Example 3.4. The definition of the g-derivative of the fuzzy-number-valued function of Example 2.12 leads to

$$[F_g'(x)]_\alpha = \begin{cases} \{20x\}\bigcup\{0\}, \text{ if } & 0\leq\alpha\leq 0.5 \\ \{0\}, \text{ if } & 0.5<\alpha\leq 1 \end{cases}. \tag{3.34}$$

That is, it is not a fuzzy-number-valued function. Hence F is not g-differentiable.

The function F in Example 3.4 has $f_\alpha^-(x)$ and $f_\alpha^+(x)$ differentiable real-valued functions with respect to x, uniformly with respect to $\alpha \in [0,1]$, but it is not g-differentiable. In the case a function is g-differentiable and satisfy the just mentioned hypothesis, it has a formula that has been proved by [10].

Theorem 3.7. *Let $F : [a,b] \to \mathbb{R}_{\mathscr{F}}$ with $f_\alpha^-(x)$ and $f_\alpha^+(x)$ differentiable real-valued functions with respect to x, uniformly with respect to $\alpha \in [0,1]$. Then*

$$\left[F_g'(x)\right]_\alpha \tag{3.35}$$

$$= \left[\inf_{\beta\geq\alpha}\min\{\left(f_\beta^-\right)'(x),\left(f_\beta^+\right)'(x)\},\sup_{\beta\geq\alpha}\max\{\left(f_\beta^-\right)',\left(f_\beta^+\right)'(x)\}\right] \tag{3.36}$$

whenever F is g-differentiable.

Proof. See [10].

Summary of the derivatives for fuzzy-number-valued functions:

- **The GH-, gH-, and g-derivatives generalize the H-derivative.** An H-differentiable function is always GH-, gH-, and g-differentiable.
- **The gH- and g-derivatives generalize the GH-derivative.** A GH-differentiable function is always gH- and g-differentiable.
- **The g-derivative generalizes the gH-derivative.** A gH-differentiable function is always g-differentiable.

3.1.3 Fundamental Theorem of Calculus

Fundamental Theorems of Calculus provide connections between derivatives and integrals, showing that they are inverses of one another.

Theorem 3.8 ([21]). *Let* $F : [a, b] \to \mathscr{F}_{\mathscr{C}}(\mathbb{R}^n)$ *be continuous, then* $G(x) = \int_a^x F(s)ds$ *is H-differentiable and*

$$G'_H(x) = F(x). \tag{3.37}$$

Theorem 3.9 ([21]). *Let* $F : [a, b] \to \mathscr{F}_{\mathscr{C}}(\mathbb{R}^n)$ *be H-differentiability and the H-derivative* F'_H *be integrable over* $[a, b]$*. Then*

$$F(x) = F(a) + \int_a^x F'_H(s)ds, \tag{3.38}$$

for each $x \in [a, b]$.

The H-differentiable is equivalent to strongly generalized differentiability (i) in Definition 3.6. For the case (ii) in the same definition, Bede and Gal have proved the following theorem.

Theorem 3.10 ([9]). *Let* $F : [a, b] \to \mathscr{F}_{\mathscr{C}}(\mathbb{R})$ *be (ii)-differentiable. Then the derivative* F'_G *is integrable over* $[a, b]$ *and*

$$F(x) = F(b) - \int_x^b F'_G(s)ds, \tag{3.39}$$

for each $x \in [a, b]$.

3.2 Fuzzy Calculus for Fuzzy Bunches of Functions

The fuzzy calculus for fuzzy bunches of functions, based on the definitions of derivative and integral via extension of the correspondent classical operators, was recently elaborated in [4, 18, 19]. This theory is reviewed and further developed in the present section.

3.2.1 Integral

The integral operator will be represented by $\int$, i.e.,

$$\begin{aligned} \int :\ L^1([a,b];\mathbb{R}^n) &\to \mathscr{A}C([a,b];\mathbb{R}^n) \\ f &\mapsto \textstyle\int_a^t f \end{aligned} \tag{3.40}$$

$t \in [a,b]$ (see Appendix for definitions of spaces of functions).

Definition 3.10 ([3, 18]). Let $F \in \mathscr{F}(L^1([a,b];\mathbb{R}^n))$. The integral of F is given by $\hat{\int} F$, whose membership function is

$$\mu_{\hat{\int} F}(y) = \begin{cases} \sup_{f \in \int^{-1} y} \mu_F(f), & \text{if } \int^{-1} y \neq \emptyset \\ 0, & \text{if } \int^{-1} y = \emptyset \end{cases}, \tag{3.41}$$

for all $y \in \mathscr{A}C([a,b];\mathbb{R}^n)$. In words, $\hat{\int}$ is the extension of the operator $\int$.

The next theorem is a consequence of Theorem 2.6.

Theorem 3.11. *If* $F \in \mathscr{F}(L^1([a,b];\mathbb{R}^n))$,

$$\begin{aligned} \left[\hat{\int} F\right]_\alpha &= \int [F]_\alpha \\ &= \left\{ \int f : f \in [F]_\alpha \subset L^1([a,b];\mathbb{R}^n) \right\}, \end{aligned} \tag{3.42}$$

for all $\alpha \in [0,1]$.

Proof. Since the integral is a continuous operator, the result follows directly from Theorem 2.6.

We next define a linear structure in $\mathscr{F}(L^1([a,b];\mathbb{R}^n))$. Given two fuzzy bunches of functions F and G and $\lambda \in \mathbb{R}$,

$$\mu_{F+G}(h) = \sup_{f+g=h} \min\{\mu_F(f), \mu_G(g)\}, \tag{3.43}$$

$$\mu_{\lambda F}(f) = \begin{cases} \mu_F(h/\lambda) & \text{if } \lambda \neq 0 \\ \chi_0(f) & \text{if } \lambda = 0 \end{cases}. \tag{3.44}$$

Since these operations are extensions of addition and multiplication by scalar, which are continuous, Theorem 2.6 assures that given $F, G \in \mathscr{F}_{\mathscr{K}}(L^1([a,b];\mathbb{R}^n))$ and $\lambda \in \mathbb{R}$,

$$F + G \in \mathscr{F}_{\mathscr{K}}(L^1([a,b];\mathbb{R}^n)) \quad \text{and} \quad [F+G]_\alpha = [F]_\alpha + [G]_\alpha \tag{3.45}$$

and

$$\lambda F \in \mathscr{F}_{\mathscr{K}}(L^1([a,b];\mathbb{R}^n)) \quad \text{and} \quad [\lambda F]_\alpha = \lambda [F]_\alpha \tag{3.46}$$

for all $\alpha \in [0,1]$.

Theorem 3.12. *Let $F, G \in \mathscr{F}_{\mathscr{K}}(L^1([a,b];\mathbb{R}^n))$, then*

(i) $\hat{\int}(F+G) = \hat{\int} F + \hat{\int} G$;
(ii) $\hat{\int}\lambda F = \lambda \hat{\int} F$, *for any* $\lambda \in \mathbb{R}$.

Proof. From Theorem 2.6 and the linearity of the integral operator,

$$\begin{aligned}
[\hat{\textstyle\int}(F+G)]_\alpha &= \textstyle\int [F+G]_\alpha \\
&= \textstyle\int \{h : h = f + g, f \in [F]_\alpha, g \in [G]_\alpha\} \\
&= \{\textstyle\int (f+g), f \in [F]_\alpha, g \in [G]_\alpha\} \\
&= \{\textstyle\int f + \int g, f \in [F]_\alpha, g \in [G]_\alpha\} \\
&= \{\textstyle\int f, f \in [F]_\alpha\} + \{\int g, g \in [G]_\alpha\} \\
&= \textstyle\int [F]_\alpha + \int [G]_\alpha \\
&= [\hat{\textstyle\int} F]_\alpha + [\hat{\textstyle\int} G]_\alpha
\end{aligned} \tag{3.47}$$

and

$$\begin{aligned}
[\hat{\textstyle\int} \lambda F]_\alpha &= \textstyle\int [\lambda F]_\alpha \\
&= \{\textstyle\int \lambda f : f \in [F]_\alpha\} \\
&= \{\lambda \textstyle\int f : f \in [F]_\alpha\} \\
&= \lambda \{\textstyle\int f : f \in [F]_\alpha\} \\
&= \lambda \textstyle\int [F]_\alpha \\
&= \lambda [\hat{\textstyle\int} F]_\alpha
\end{aligned} \tag{3.48}$$

for all $\alpha \in [0,1]$.

Example 3.5. Let A be the symmetrical triangular fuzzy number with support $[-a, a]$, $a > 0$. The fuzzy function $F(\cdot) \in \mathscr{F}(L^1([0,T];\mathbb{R}))$ such that

$$[F(\cdot)]_\alpha = \{f(\cdot) : f(t) = \gamma t, \gamma \in [A]^\alpha\} \tag{3.49}$$

where $f(\cdot) : [0, T] \to \mathbb{R}$, for each $\alpha \in [0, 1]$, has attainable sets

$$F(t) = At. \tag{3.50}$$

To determine the integral of F using Definition 3.10, one needs to explicit the membership function of A and F:

$$\mu_A(\gamma) = \begin{cases} \dfrac{\gamma}{a} + 1, & \text{if } -a \leq \gamma < 0 \\ -\dfrac{\gamma}{a} + 1, & \text{if } 0 \leq \gamma < a \\ 0, & \text{otherwise} \end{cases} \tag{3.51}$$

and

$$\mu_F(f) = \begin{cases} \dfrac{\gamma}{a} + 1, & \text{if } f(t) = \gamma t \text{ with } -a \leq \gamma < 0 \\ -\dfrac{\gamma}{a} + 1, & \text{if } f(t) = \gamma t \text{ with } 0 \leq \gamma < a \\ 0, & \text{otherwise} \end{cases} \tag{3.52}$$

Formula (3.41) states that $\mu_{\hat{\int} F}(y) \neq 0$ only if there exists f such that $\int f = y$ and $\mu_F(f) \neq 0$. In this example, it happens only if $f(t) = \gamma t$ with $\gamma \in [A]^\alpha$, that is, $y = \gamma t^2/2$.

$$\begin{aligned} \mu_{\hat{\int} F}(\gamma t^2/2) &= \sup_{\int f = \gamma t^2/2} \mu_F(f) \\ &= \sup_{\int (\gamma t) = \gamma t^2/2} \mu_F(\gamma t) \\ &= \mu_F(\gamma t) \\ &= \begin{cases} \frac{\gamma}{a} + 1, & \text{if } -a \leq \gamma < 0 \\ -\frac{\gamma}{a} + 1, & \text{if } 0 \leq \gamma < a \\ 0, & \text{otherwise} \end{cases} \\ &= \mu_A(\gamma). \end{aligned} \tag{3.53}$$

Hence

$$\mu_{\hat{\int} F}(f) = \begin{cases} \dfrac{\gamma}{a} + 1, & \text{if } f(t) = \gamma t^2/2 \text{ with } -a \leq \gamma < 0 \\ -\dfrac{\gamma}{a} + 1, & \text{if } f(t) = \gamma t^2/2 \text{ with } 0 \leq \gamma < a \\ 0, & \text{otherwise} \end{cases} \tag{3.54}$$

or

$$[F(\cdot)]_\alpha = \{f(\cdot) : f(t) = \gamma t^2/2, \gamma \in [A]_\alpha\}. \tag{3.55}$$

For each $\alpha \in [0, 1]$, its attainable sets are

$$F(t) = At^2/2. \tag{3.56}$$

The Aumann integral of (3.50) can be calculated levelwise and we obtain the same attainable sets as obtained with $\hat{\int}$:

$$\begin{aligned}[\textstyle\int F(t)]_\alpha &= [\textstyle\int f_\alpha^-, \int f_\alpha^-] \\ &= [-at^2/2, at^2/2] \\ &= [A]_\alpha t^2/2.\end{aligned} \tag{3.57}$$

The next section introduces the derivative operator for fuzzy bunches of functions. It is defined for more restricted spaces than the integral since they are extensions of the classical case. Also, different from the integral case, we explore the derivative on different spaces (Example 3.9) due to the fact that it is not a continuous operator (in general). We are more interested, though, in differentiating fuzzy bunches of the space of absolutely continuous functions (see Appendix), since we can differentiate more elements in this space than in the space of differentiable functions. Furthermore, it is used and has been explored in the differential inclusions theory, which, as already mentioned, has important connections with the theory we propose to develop.

3.2.2 Derivative

The derivative operator in the sense of distributions (see [1]) will be represented by D, that is,

$$\begin{aligned} D : \mathscr{A}C([a,b];\mathbb{R}^n) &\to L^1([a,b];\mathbb{R}^n) \\ f &\mapsto Df \end{aligned} \tag{3.58}$$

Thus, there exists $Df(t)$ *a.e.*, in $[a, b]$.

Definition 3.11. Let $F \in \mathscr{F}(\mathscr{A}C([a,b];\mathbb{R}^n))$. The derivative of F is given by $\hat{D}F$, whose membership function is

$$\mu_{\hat{D}F}(y) = \begin{cases} \sup_{f \in D^{-1}y} \mu_F(f), & \text{if } D^{-1}y \neq \emptyset \\ 0, & \text{if } D^{-1}y = \emptyset \end{cases}. \tag{3.59}$$

for all $y \in L^1([a,b];\mathbb{R}^n)$. In words, $\hat{D}$ is the extension of operator D.

Example 3.6. Let $F(\cdot)$ be the same fuzzy bunch as in Example 3.5. We note that $F(\cdot) \in \mathscr{F}(\mathscr{A}C([a,b];\mathbb{R}))$.

Following the same reasoning as Example 3.5,

$$\begin{aligned}\mu_{\hat{D}F}(\gamma) &= \sup_{Df=\gamma} \mu_F(f) \\ &= \sup_{D(\gamma t)=\gamma} \mu_F(\gamma t) \\ &= \mu_F(\gamma t) \\ &= \begin{cases} \frac{\gamma}{a}+1, & \text{if } -a \le \gamma < 0 \\ -\frac{\gamma}{a}+1, & \text{if } 0 \le \gamma < a \\ 0, & \text{otherwise} \end{cases} \\ &= \mu_A(\gamma).\end{aligned} \tag{3.60}$$

It means that the support of $\hat{D}F(\cdot)$ is composed of constant functions such that, at each instant t, the derivative of $F(\cdot)$ is always the fuzzy number A.

Lemma 3.1. *For D defined as above, the preimage $D^{-1}g$ is a closed nonempty subset in the space of functions $\mathscr{A}C([a,b];\mathbb{R}^n)$ with respect to the uniform norm for each $g \in L^1([a,b];\mathbb{R}^n)$.*

Proof. $D^{-1}g$ is a finite dimensional subspace of $\mathscr{A}C([a,b];\mathbb{R}^n)$ since $D^{-1}g = \{f + k : k \in \mathbb{R}^n\}$ for $f \in \mathscr{A}C([a,b];\mathbb{R}^n)$ such that $f = \int_a^x g$. Hence $D^{-1}g$ is closed.

Theorem 3.13 ([4]). *Let $F \in \mathscr{F}_{\mathscr{K}}(\mathscr{A}C([a,b];\mathbb{R}^n))$. Then*

$$[\hat{D}F]_\alpha = D[F]_\alpha. \tag{3.61}$$

Proof. This proof will make use of the result: $[F]_0 \cap D^{-1}(g)$ is compact. It is true since the subset $D^{-1}(g)$ is nonempty and it is closed (from Lemma 3.1). Also, $[F]_0 \cap D^{-1}(g)$ is a closed subset of the compact set $[F]_0$, hence it is compact.

We show inclusion $[\hat{D}(F)]_\alpha \subset D([F]_\alpha)$ considering two cases: $\alpha \in (0,1]$ and later $\alpha = 0$.

(i) For $\alpha \in (0,1]$, let $g \in [\hat{D}(F)]_\alpha$, then

$$\alpha \le \hat{D}(F)(g) = \sup_{h \in D^{-1}(g)} F(h) = \sup_{h \in [F]_0 \cap D^{-1}(g)} F(h) = F(f)$$

for some f, since F is an upper semicontinuous function (that is, the membership of F is usc) and $[F]_0 \cap D^{-1}(g)$ is compact. So, $F(f) \ge \alpha$. That is, $f \in [F]_\alpha \cap D^{-1}(g)$. Hence $g \in D([F]_\alpha)$.

(ii) For $\alpha = 0$,

$$\cup_{\alpha \in (0,1]} [\hat{D}(F)]_\alpha \subset \cup_{\alpha \in (0,1]} D([F]_\alpha) \subseteq D([F]_0).$$

Consequently,

$$[\hat{D}(F)]_0 = \overline{\cup_{\alpha \in (0,1]} [\hat{D}(F)]_\alpha} \subset \overline{\cup_{\alpha \in (0,1]} D([F]_\alpha)} \subseteq \overline{D([F]_0)} = D([F]_0).$$

The last equality holds because D is a closed operator.

Now we prove the inclusion $D([F]_\alpha) \subset [\hat{D}(F)]_\alpha$. If $g \in D([F]_\alpha)$, there exists $f \in [F]_\alpha$ such that $D(f) = g$. Thus,

$$\hat{D}(F)(g) = \sup_{h \in D^{-1}(g)} F(h) \geq F(f) \geq \alpha \Rightarrow g \in [\hat{D}(F)]_\alpha$$

for all $\alpha \in [0, 1]$.

We have proved that $[\hat{D}(F)]_\alpha \subset D([F]_\alpha)$ and $D([F]_\alpha) \subset [\hat{D}(F)]_\alpha$, for all $\alpha \in [0, 1]$, then (3.61) holds.

Example 3.7. Consider $g : [a, b] \to \mathbb{R}$ a differentiable and positive function, $A = (c; d; e)$ a triangular fuzzy number and the fuzzy-number-valued function

$$F(x) = Ag(x). \tag{3.62}$$

We have

$$[F(x)]_\alpha = [f_\alpha^-(x), f_\alpha^+(x)] \tag{3.63}$$

with

$$f_\alpha^-(x) = [a + \alpha(b - a)]g(x) \quad \text{and} \quad f_\alpha^+(x) = [e - \alpha(e - d)]g(x) \tag{3.64}$$

differentiable with respect to x and continuous with respect to α.

The continuity in α means that $F(x) \in \mathscr{F}_C^0(\mathbb{R})$. It will be proved in Theorem 3.17 that the representative bunch of first kind of this function has compact α-cuts in $\mathscr{A}C([a, b]; \mathbb{R})$, since it satisfies the hypotheses of the theorem.

The derivative of the representative bunch of first kind has α-cuts

$$\begin{aligned}[\hat{D}\tilde{F}]_\alpha &= \bigcup_{\beta \geq \alpha} \bigcup_{0 \leq \lambda \leq 1} (f_\beta^\lambda)' \\ &= (1 - \lambda)[a + \alpha(b - a)]g' + \lambda[e - \alpha(e - d)]g' \\ &= \{a \cdot g', a \in [A]_\alpha\}\end{aligned} \tag{3.65}$$

for all $\alpha \in [0, 1]$, that is,

$$\hat{D}\tilde{F} = Ag'. \tag{3.66}$$

It is a similar result as in Example 3.2 for GH-derivative, in terms of attainable sets.

Example 3.8. Let

$$f(x) = Be^{cx} \tag{3.67}$$

be a fuzzy-set-valued function where c is a real constant and B is a fuzzy subset in $\mathbb{R}$ such that $B(1) = 1$, $B(0.5) = 0.5$ and $B(x) = 0$ everywhere else. Hence $f(x)$ is not differentiable using Hukuhara or any generalized derivatives since it is not a fuzzy-number-valued function. On the other hand, the fuzzy bunch of functions with α-levels

$$[\tilde{f}(\cdot)]^\alpha = \begin{cases} \{y_1(\cdot), y_2(\cdot)\}, & \text{if } 0 \leq \alpha \leq 0.5 \\ \{y_1(\cdot)\}, & \text{if } 0.5 < \alpha \leq 1 \end{cases},$$

where $y_1(x) = e^{cx}$ and $y_2(x) = 0.5e^{cx}$, has (3.67) as attainable fuzzy sets and is $\hat{D}$-differentiable. Since this α-levels are compact subsets of $\mathscr{A}C([a,b];\mathbb{R})$, we apply Theorem 3.13 and obtain

$$[\hat{D}f(\cdot)]^\alpha = \begin{cases} \{z_1(\cdot), z_2(\cdot)\}, & \text{if } 0 \leq \alpha \leq 0.5 \\ \{z_1(\cdot)\}, & \text{if } 0.5 < \alpha \leq 1 \end{cases},$$

where $z_1(x) = ce^{cx}$ and $z_2(x) = 0.5ce^{cx}$. Its attainable sets are

$$\hat{D}f(x) = cBe^{cx}. \tag{3.68}$$

Remark 3.2. The Hukuhara or the generalized derivatives cannot be used to differentiate fuzzy-set-valued functions whose images are not fuzzy numbers, as the function in Example 3.8. On the other hand, one can use the $\hat{D}$ on correspondent fuzzy bunches of functions and regard its attainable fuzzy sets as derivative.

Example 3.9 ([4]). The operator $\hat{D} : \mathscr{F}_{\mathscr{K}}(C^1([a,b];\mathbb{R}^n)) \to \mathscr{F}_{\mathscr{K}}(C([a,b];\mathbb{R}^n))$ is well defined and for each $F \in \mathscr{F}_{\mathscr{K}}(C^1([a,b];\mathbb{R}^n))$ we have

$$[\hat{D}F]^\alpha = D[F]^\alpha \tag{3.69}$$

for all $\alpha \in [0,1]$, if $C^1([a,b];\mathbb{R}^n)$ is endowed with the norm $\| x \|_1 = \sup_{0\leq t\leq T}\{|x(t)| + |x'(t)|\}$ and $C([a,b];\mathbb{R}^n)$ is endowed with the usual supremum norm. The result follows from Theorem 2.6 since D is a continuous function for these spaces.

Another possibility of D being a continuous operator is as follows:

Theorem 3.14 ([4]). *Consider the subset in* $\mathscr{A}C([0,T];\mathbb{R}^n)$*:*

$$Z_T(\mathbb{R}^n) = \{x(\cdot) \in C([0,T];\mathbb{R}^n) : \exists\, x'(\cdot) \in L^\infty([0,T];\mathbb{R}^n)\}, \tag{3.70}$$

with $Z_T(\mathbb{R}^n)$ *having the uniform norm topology and* $L^\infty([0,T];\mathbb{R}^n)$ *with the weak*-topology. Thus,*

$$\hat{D} : \mathscr{F}_{\mathscr{K}}(Z_T(\mathbb{R}^n)) \to \mathscr{F}_{\mathscr{K}}(L^\infty([0,T];\mathbb{R}^n)), \tag{3.71}$$

where $\hat{D}$ is the extension of the derivative D, is well defined, that is, for each $F \in \mathscr{F}_{\mathscr{K}}(Z_T(\mathbb{R}^n))$, the α-level $[\hat{D}F]^\alpha$ is a compact subset in $L^\infty([0,T];\ \mathbb{R}^n)$ and $[\hat{D}F]^\alpha = D[F]^\alpha$.

Proof. The result follows from the Theorem 2.6 because

$$D : Z_T(\mathbb{R}^n) \to L^\infty([0,T];\mathbb{R}^n) \tag{3.72}$$

is a continuous linear operator (see [1, p. 104]).

Theorem 3.15. *Let $F, G \in \mathscr{F}_{\mathscr{K}}(\mathscr{A}C([a,b];\mathbb{R}^n))$, then*

(i) $\hat{D}(F+G) = \hat{D}F + \hat{D}G$*;*
(ii) $\hat{D}\lambda F = \lambda \hat{D}F$*, for any* $\lambda \in \mathbb{R}$*.*

Proof. This proof is completely analogous to the one of Theorem 3.15, due to the linearity of the derivative operator.

3.2.3 Fundamental Theorem of Calculus

A result connects the concepts of derivative and integral for fuzzy bunches of functions as in the classical case and in the fuzzy-set-valued function case.

Theorem 3.16. *Let $F \in \mathscr{F}_{\mathscr{K}}(L^1([a,b];\mathbb{R}^n))$. Hence*

$$\hat{D}\left(\hat{\int} F\right) = F, \tag{3.73}$$

that is,

$$\left[\hat{D}\left(\hat{\int} F\right)\right]^\alpha = [F]^\alpha. \tag{3.74}$$

for all $\alpha \in [0,1]$.

Proof. Since Theorem 3.11 holds,

$$\begin{aligned}[\hat{\int} F]_\alpha &= \int [F]_\alpha \\ &= \{\int f : f \in [F]_\alpha\}\end{aligned} \tag{3.75}$$

for all $\alpha \in [0,1]$ and $\hat{\int} F \in \mathscr{F}_{\mathscr{K}}(\mathscr{A}C([0,T];\mathbb{R}^n))$. Then Theorem 3.13 holds and,

$$\begin{aligned}[\hat{D}\hat{\int} F]_\alpha &= D[\hat{\int} F]_\alpha \\ &= \{D\int f : f \in [F]_\alpha\} \\ &= [F]_\alpha\end{aligned} \tag{3.76}$$

for all $\alpha \in [0,1]$.

3.3 Comparison

Different fuzzy bunches of functions may present the same attainable fuzzy sets, that is, more than one fuzzy bunch of functions may correspond to one single fuzzy-set-valued function. Choosing the suitable fuzzy bunch may lead to equivalence of $\hat{D}$ with derivatives for fuzzy-set-valued functions and equivalence of $\hat{\int}$ with integrals for fuzzy-set-valued functions (in terms of attainable sets). This section discloses similarities of the proposed theory with other approaches.

The motivation for this comparison and the definition of the two different fuzzy bunches of functions of Definition 2.16 is what happens to the fuzzy-number-valued functions of Examples 3.3 and 3.4. In the former the gH-derivative does not exist whereas the g-derivative does and in the latter both do not exist. We calculate the $\hat{D}$-derivative of the corresponding fuzzy bunches of the fuzzy-valued functions in Examples 3.3 and 3.4 next. The fuzzy-number-valued functions do not meet the conditions of the theorems to be stated, revealing the importance of the hypotheses of these theorems.

Example 3.10. Recall Examples 2.13 and 3.3 where the representative bunch of first kind are given by the α-cuts

$$[\tilde{F}_1(\cdot)]_\alpha = \begin{cases} \left(\bigcup_{\beta\geq 0.5}\bigcup_{0\leq\lambda\leq 1} f_\beta^\lambda\right)\bigcup\left(\bigcup_{\alpha\leq\beta\leq 0.5}\bigcup_{0\leq\lambda\leq 1} g_\beta^\lambda\right), \text{ if } \ 0\leq\alpha\leq 0.5 \\ \bigcup_{\beta\geq\alpha}\bigcup_{0\leq\lambda\leq 1} f_\beta^\lambda, \text{ if } \ 0.5<\alpha\leq 1 \end{cases} \tag{3.77}$$

where

$$\begin{cases} f_\beta^\lambda(\cdot) : f_\beta^\lambda(x) = (1-\lambda)(x^2-3+\beta)+\lambda((2\beta-1)x^2-6\beta+4), \\ g_\beta^\lambda(\cdot) : g_\beta^\lambda(x) = (1-\lambda)(x^2-3+\beta)+\lambda((1-2\beta)x^2-2\beta+2), \end{cases} \tag{3.78}$$

for all $\lambda \in [0, 1]$. Since

$$\begin{cases} (f_\beta^\lambda)'(\cdot) : f_\beta^\lambda(x) = (1-2\lambda+2\beta\lambda)2x, \\ (g_\beta^\lambda)'(\cdot) : g_\beta^\lambda(x) = (1-2\beta\lambda)2x, \end{cases} \tag{3.79}$$

using Theorem 3.13 to calculate $[\hat{D}\tilde{F}_1(\cdot)]_\alpha$ we obtain

$$\begin{cases} \left(\bigcup_{\beta\geq 0.5}\bigcup_{0\leq\lambda\leq 1} (f_\beta^\lambda)'\right)\bigcup\left(\bigcup_{\alpha\leq\beta\leq 0.5}\bigcup_{0\leq\lambda\leq 1} (g_\beta^\lambda)'\right), \text{ if } \ 0\leq\alpha\leq 0.5 \\ \bigcup_{\beta\geq\alpha}\bigcup_{0\leq\lambda\leq 1} (f_\beta^\lambda)', \text{ if } \ 0.5<\alpha\leq 1. \end{cases} \tag{3.80}$$

At $x \in [0, 0.8]$

$$[\hat{D}\tilde{F}_1(x)]_\alpha = [m, M] \tag{3.81}$$

with m as

$$\min\left\{\left(\bigcup_{\beta \geq 0.5}\bigcup_{0\leq\lambda\leq 1}(f_\beta^\lambda)'(x)\right)\bigcup\left(\bigcup_{\alpha\leq\beta\leq 0.5}\bigcup_{0\leq\lambda\leq 1}(g_\beta^\lambda)'(x)\right)\right\} = 0 \tag{3.82}$$

if $0 \leq \alpha \leq 0.5$ and

$$\min\left\{\bigcup_{\beta\geq\alpha}\bigcup_{0\leq\lambda\leq 1}(f_\beta^\lambda)'(x)\right\} = (2\alpha - 1)2x \tag{3.83}$$

if $0.5 < \alpha \leq 1$. And M equals

$$\max\left\{\left(\bigcup_{\beta \geq 0.5}\bigcup_{0\leq\lambda\leq 1}(f_\beta^\lambda)'(x)\right)\bigcup\left(\bigcup_{\alpha\leq\beta\leq 0.5}\bigcup_{0\leq\lambda\leq 1}(g_\beta^\lambda)'(x)\right)\right\} = 2x \tag{3.84}$$

if $0 \leq \alpha \leq 0.5$ and

$$\max\left\{\bigcup_{\beta\geq\alpha}\bigcup_{0\leq\lambda\leq 1}(f_\beta^\lambda)'(x)\right\} = 2x \tag{3.85}$$

if $0.5 < \alpha \leq 1$.

Hence the attainable sets of the $\hat{D}$-derivative are

$$[\hat{D}\tilde{F}_1(x)]_\alpha = \begin{cases} [0, 2x], \text{ if } & 0 \leq \alpha \leq 0.5 \\ [(2\alpha - 1)2x, 2x], \text{ if } & 0.5 < \alpha \leq 1 \end{cases} \tag{3.86}$$

that is, the same as the g-derivative of the fuzzy-number-valued function F.

Example 3.11. Recall Examples 2.12 and 3.4 where the representative bunch of first kind is given by the α-cuts

$$[F(x)]_\alpha = \begin{cases} \left[10x^2 - 12, 10x^2 + 2\right], \text{ if } & 0 \leq \alpha \leq 0.5 \\ [-1, 1], \text{ if } & 0.5 < \alpha \leq 1 \end{cases}. \tag{3.87}$$

and the representative bunch of second kind is defined by

$$[\tilde{F}_1(\cdot)]_\alpha = \begin{cases} \bigcup_{i=1}^{2} \bigcup_{0\le\lambda\le1} y_i^\lambda(\cdot), \text{ if } & 0 \le \alpha \le 0.5 \\ \bigcup_{0\le\lambda\le1} y_1^\lambda(\cdot), \text{ if } & 0.5 < \alpha \le 1 \end{cases} \tag{3.88}$$

where

$$\begin{cases} y_1^\lambda(\cdot) : y_1^\lambda(x) = (1-\lambda)(10x^2-12) + \lambda(10x^2+2), \\ y_2^\lambda(\cdot) : y_2^\lambda(x) = (1-\lambda)(-1) + \lambda, \\ y_3^\lambda(\cdot) : y_3^\lambda(x) = (1-\lambda)(-1) + \lambda(10x^2+2), \\ y_4^\lambda(\cdot) : y_4^\lambda(x) = (1-\lambda)(10x^2-12) + \lambda, \end{cases} \tag{3.89}$$

for all $\lambda \in [0, 1]$.

The derivatives of the representative bunch of first kind is given by the α-cuts

$$[\hat{D}\tilde{F}_1(\cdot)]_\alpha = \begin{cases} \bigcup_{i=1}^{2} \bigcup_{0\le\lambda\le1} (y_i^\lambda)'(\cdot), \text{ if } & 0 \le \alpha \le 0.5 \\ \bigcup_{0\le\lambda\le1} y_1^\lambda(\cdot), \text{ if } & 0.5 < \alpha \le 1 \end{cases} \tag{3.90}$$

and the representative bunch of second kind is defined by

$$[\hat{D}\tilde{F}_2(\cdot)]_\alpha = \begin{cases} \bigcup_{i=1}^{4} \bigcup_{0\le\lambda\le1} (y_i^\lambda)'(\cdot), \text{ if } & 0 \le \alpha \le 0.5 \\ \{(y_1)'(\cdot)\} \bigcup_{0\le\lambda\le1} y_1^\lambda(\cdot), \text{ if } & 0.5 < \alpha \le 1 \end{cases} \tag{3.91}$$

where

$$\begin{cases} (y_1)'(\cdot) : (y_1)'(x) = 20x, \\ (y_2)'(\cdot) : (y_2)'(x) = 0, \\ (y_3)'(\cdot) : (y_3)'(x) = \lambda 20x, \\ (y_4)'(\cdot) : (y_4)'(x) = (1-\lambda)20x, \end{cases} \tag{3.92}$$

for all $\lambda \in [0, 1]$.

In terms of attainable sets, the derivative of the representative bunch of first kind has attainable sets

$$[\hat{D}\tilde{F}_1(x)]_\alpha = \begin{cases} \{0\} \bigcup \{20x\}, \text{ if } & 0 \le \alpha \le 0.5 \\ \{20x\}, \text{ if } & 0.5 < \alpha \le 1 \end{cases}. \tag{3.93}$$

The derivative of the representative bunch of second kind for $x \in [0, 1]$ has attainable sets

$$[\hat{D}\tilde{F}_2(x)]_\alpha = \begin{cases} [0, 20x], \text{ if } & 0 \le \alpha \le 0.5 \\ \{20x\}, \text{ if } & 0.5 < \alpha \le 1 \end{cases} \tag{3.94}$$

and for $x \in [-1, 0]$,

$$[\hat{D}\tilde{F}_1(x)]_\alpha = \begin{cases} [20x, 0], \text{ if } & 0 \le \alpha \le 0.5 \\ \{20x\}, \text{ if } & 0.5 < \alpha \le 1 \end{cases}. \tag{3.95}$$

Hence the derivative of the representative bunch of first kind at each $x \in [-1, 1]$ does not define fuzzy numbers while the derivative of the representative bunch of second kind does.

Example 3.10 illustrates that the $\hat{D}$-derivative of the fuzzy bunch of first kind of the given fuzzy-number-valued function F exists but its attainable sets are not fuzzy numbers (while the gH-derivative of the fuzzy-number-valued function does not exist). The result that we state next regards the necessary conditions for equivalence between the gH-derivative of a fuzzy-number-valued function and the $\hat{D}$-derivative of the corresponding fuzzy bunch of first kind. The result we state later is connected with Example 3.11, that is, it is necessary that the g-derivative exist for the equivalence with the derivative of the representative bunch of second kind. The $\hat{D}$ derivative in this last case provided a fuzzy-number-valued function, which no derivative for fuzzy-number-valued functions that we presented can do.

Theorem 3.17. *Let $F : [a, b] \to \mathscr{F}^0_{\mathscr{C}}(\mathbb{R})$ be such that the functions $f^-_\alpha(x)$ and $f^+_\alpha(x)$ are real-valued functions, differentiable with respect to x, uniformly in $\alpha \in [0, 1]$. Suppose also that one of the following two cases holds:*

(a) $\left(f^-_\alpha\right)'(x)$ is increasing, $\left(f^+_\alpha\right)'(x)$ is decreasing as functions of α, and

$$\left(f^-_1\right)'(x) \le \left(f^+_1\right)'(x), \tag{3.96}$$

or

(b) $\left(f^-_\alpha\right)'(x)$ is decreasing, $\left(f^+_\alpha\right)'(x)$ is increasing as functions of α, and

$$\left(f^+_1\right)'(x) \le \left(f^-_1\right)'(x). \tag{3.97}$$

Then F generates a representative bunch of first kind $\tilde{F}(\cdot)$ with compact α-levels and whose $\hat{D}$-derivative has attainable sets

$$\left[\hat{D}\tilde{F}(x)\right]_\alpha = [\min\{\left(f_\alpha^-\right)'(x), \left(f_\alpha^+\right)'(x)\}, \max\{\left(f_\alpha^-\right)'(x), \left(f_\alpha^+\right)'(x)\}]. \tag{3.98}$$

In words, the $\hat{D}$-derivative coincides with the gH-derivative at each x.

Proof. We prove that the sets A_α in Definition 2.16 are α-cuts of a fuzzy set in $\mathscr{A}C([a,b];\mathbb{R})$ using the same arguments as in Example 2.11. The only difference is to demonstrate compactness, which we do next. Note that any sequence $(f_{\alpha_i}^{\lambda_i})$ in $\bigcup_{\beta\geq\alpha}\bigcup_{0\leq\lambda\leq 1} f_\beta^\lambda(\cdot)$ has a convergent subsequence whose limit belongs to $\bigcup_{\beta\geq\alpha}\bigcup_{0\leq\lambda\leq 1} f_\beta^\lambda(\cdot)$, due to the continuity of $f_\beta^\lambda(\cdot)$ as function of the real parameters λ and β defined on closed intervals (compact subsets) $[0,1]$ and $[\alpha,1]$, respectively. And since $f_\beta^\pm$ are differentiable, so are f_β^λ. According to [14], the differentiability with respect to x, uniformly in $\alpha \in [0,1]$, assures that if a sequence of functions converges to a function f, the sequence of its derivatives converges to f'. Since f is differentiable, it belongs to $\mathscr{A}C([a,b];\mathbb{R})$. As a result, $\bigcup_{\beta\geq\alpha}\bigcup_{0\leq\lambda\leq 1} f_\beta^\lambda$ is compact in $\mathscr{A}C([a,b];\mathbb{R})$ and it is equal to its closure and hence to A_α.

We next make use of Theorem 3.13 since $\tilde{F} \in \mathscr{F}_{\mathscr{K}}(\mathscr{A}C([a,b];\mathbb{R}))$:

$$\begin{aligned}[\hat{D}\tilde{F}]_\alpha &= D[\tilde{F}]_\alpha \\ &= \bigcup_{\beta\geq\alpha}\bigcup_{0\leq\lambda\leq 1} (f_\beta^\lambda)'\end{aligned} \tag{3.99}$$

for all $\alpha \in [0,1]$. And we observe that for case (a)

$$\bigcup_{0\leq\lambda\leq 1} (f_\beta^\lambda)'(x) = [(f_\beta^-)'(x), (f_\beta^+)'(x)] \tag{3.100}$$

and

$$(f_\alpha^-)'(x) \leq (f_\beta^-)'(x) \leq (f_1^-)'(x) \leq (f_1^+)'(x) \leq (f_\beta^+)'(x) \leq (f_\alpha^+)'(x) \tag{3.101}$$

for $0 \leq \alpha \leq \beta \leq 1$,

$$[(f_\beta^-)'(x), (f_\beta^+)'(x)] \subseteq [(f_\alpha^-)'(x), (f_\alpha^+)'(x)]. \tag{3.102}$$

Hence

$$\begin{aligned}[\hat{D}\tilde{F}(x)]_\alpha &= \bigcup_{\beta\geq\alpha}[(f_\beta^-)'(x), (f_\beta^+)'(x)] \\ &= [(f_\alpha^-)'(x), (f_\alpha^+)'(x)]\end{aligned} \tag{3.103}$$

for all $\alpha \in [0,1]$.

Similarly, case (b) leads to

$$[\hat{D}\tilde{F}(x)]_\alpha = [(f_\alpha^+)'(x), (f_\alpha^-)'(x)]. \tag{3.104}$$

As a result we obtain the desired expression,

$$\left[\hat{D}\tilde{F}(x)\right]_\alpha = \left[\min\{(f_\alpha^-)'(x), (f_\alpha^+)'(x)\}, \max\{(f_\alpha^-)'(x), (f_\alpha^+)'(x)\}\right], \tag{3.105}$$

for all $\alpha \in [0, 1]$, which the same as stated in Theorem 3.6 for the gH-derivative.

A similar result for connecting $\hat{D}$-derivative and g-derivative is presented in what follows.

Theorem 3.18. *Let $F \in [a, b] \to \mathscr{F}_C^0(\mathbb{R})$ be a function such that $f_\alpha^-(x)$ and $f_\alpha^+(x)$ are differentiable real-valued functions with respect to x, uniformly with respect to $\alpha \in [0, 1]$. Then F generates a representative bunch of second kind $\tilde{F}(\cdot)$ with compact α-levels and whose $\hat{D}$-derivative has attainable sets with levels $[\hat{D}\tilde{F}(x)]_\alpha$ given by*

$$\left[\inf_{\beta\geq\alpha} \min\left\{(f_\beta^-)'(x), (f_\beta^+)'(x)\right\}, \sup_{\beta\geq\alpha} \max\left\{(f_\beta^-)'(x), (f_\beta^+)'(x)\right\}.\right] \tag{3.106}$$

It means that the values of the g-derivative of $F(x)$ and the attainable sets of the $\hat{D}$-derivative of $\tilde{F}(\cdot)$ coincide in every $x \in [a, b]$, whenever the g-derivative exists.

Proof. Using the same argument of the previous proof, it follows that the resultant B_α in Definition 2.16 are compact sets in $\mathscr{AC}([a, b]; \mathbb{R})$ and are the α-cuts of the representative bunch of second kind of F, $\tilde{F}$. We use Theorem 3.13 and obtain

$$\begin{aligned}[\hat{D}\tilde{F}]_\alpha &= D[\tilde{F}]_\alpha \\ &= \bigcup_{\beta,\gamma\geq\alpha} \bigcup_{0\leq\lambda\leq1} (f_{\beta,\gamma}^\lambda)'\end{aligned} \tag{3.107}$$

We will prove that $L = \inf_{\beta,\gamma\geq\alpha}\left\{(f_{\beta,\gamma}^\lambda)'(x)\right\}$ is attained, that is, that there exists a triple $(\overline{\lambda}, \overline{\beta}, \overline{\gamma})$ such that $(f_{\overline{\beta},\overline{\gamma}}^{\overline{\lambda}})'(x) = L$ with $\overline{\beta}, \overline{\gamma} \in [\alpha, 1]$, $\overline{\lambda} \in [0, 1]$. From the definition of infimum, $y \geq L$ if $y \in \bigcup_{\beta,\gamma\geq\alpha} \bigcup_{0\leq\lambda\leq1} (f_{\beta,\gamma}^\lambda)'(x)$ and there exists a sequence (y_n), $y_n = (f_{\beta_n,\gamma_n}^{\lambda_n})'(x)$ such that

$$(f_{\beta_n,\gamma_n}^{\lambda_n})'(x) \to L, \quad L \leq (f_{\beta_n,\gamma_n}^{\lambda_n})'(x). \tag{3.108}$$

To the sequence (y_n) in $\mathbb{R}$ there corresponds a sequence $(g_n(\cdot))$ of functions such that $g_n(\cdot) = (f_{\beta_n,\gamma_n}^{\lambda_n})'(\cdot)$. This sequence of functions has a convergent subsequence, since the set is sequentially compact (where we use the same result in [14] as previously used). This subsequence of functions defines a subsequence in (y_n), $y_{n_k} = g_{n_k}(x)$. The subsequence (y_{n_k}) also converges to L. The limit of $g_{n_k}(\cdot)$ is attained for some triple $(\overline{\lambda}, \overline{\beta}, \overline{\gamma})$ and its value in x is

$$(f_{\overline{\beta},\overline{\gamma}}^{\overline{\lambda}})'(x) = \lim g_{n_k}(x) = \lim y_{n_k} = L. \tag{3.109}$$

Similarly we prove that the supremum M is also attained. Now we prove that

$$L = \inf_{\beta \geq \alpha} \min \left\{ (f_\beta^-)'(x), (f_\beta^+)'(x) \right\}. \tag{3.110}$$

For any $(f_{\beta,\gamma}^\lambda)'(x)$, we have

$$(f_\beta^-)' \leq (f_{\beta,\gamma}^\lambda)'(x) \leq (f_\gamma^+)' \quad \text{or} \quad (f_\gamma^+)' \leq (f_{\beta,\gamma}^\lambda)'(x) \leq (f_\beta^+)'. \tag{3.111}$$

Hence

$$\inf_{\beta \geq \alpha} \min \left\{ (f_\beta^-)'(x), (f_\beta^+)'(x) \right\} \leq \inf_{\beta,\gamma \geq \alpha} \left\{ (f_{\beta,\gamma}^\lambda)'(x) \right\}. \tag{3.112}$$

Since

$$\bigcup_{\beta \geq \alpha} \left\{ (f_\beta^-)'(x), (f_\beta^+)'(x) \right\} \subset \bigcup_{\beta,\gamma \geq \alpha} \left\{ (f_{\beta,\gamma}^\lambda)'(x) \right\} \tag{3.113}$$

the equality of the infimum holds.

Hence the value $L = \inf_{\beta \geq \alpha} \min \left\{ (f_\beta^-)'(x), (f_\beta^+)'(x) \right\}$ is attained by $(f_\beta^-)'(x)$ or $(f_\beta^+)'(x)$, for some $\beta \geq \alpha$. The same happens to $M = \sup_{\beta \geq \alpha} \max \left\{ (f_\beta^-)'(x), (f_\beta^+)'(x) \right\}$. As a consequence, there are four possible cases:

(1) $L = (f_{\beta_1}^-)'(x)$ and $M = (f_{\beta_2}^+)'(x)$ and any value between L and M is attained by $(f_{\beta_1,\beta_2}^\lambda)'(x)$ for some $\lambda \in [0, 1]$;
(2) $L = (f_{\beta_1}^+)'(x)$ and $M = (f_{\beta_2}^-)'(x)$ and any value between L and M is attained by $(f_{\beta_2,\beta_1}^\lambda)'(x)$ for some $\lambda \in [0, 1]$;
(3) $L = (f_{\beta_1}^-)'(x)$ and $M = (f_{\beta_2}^-)'(x)$ and any value between L and M is attained by $(f_{\beta_1,\beta_1}^\lambda)'(x)$ or $(f_{\beta_2,\beta_1}^\lambda)'(x)$ for some $\lambda \in [0, 1]$.
(4) $L = (f_{\beta_1}^+)'(x)$ and $M = (f_{\beta_2}^+)'(x)$ and any value between L and M is attained by $(f_{\beta_1,\beta_1}^\lambda)'(x)$ or $(f_{\beta_1,\beta_2}^\lambda)'(x)$ for some $\lambda \in [0, 1]$.

It proves that all values in

$$\left[\inf_{\beta \geq \alpha} \min \left\{ (f_\beta^-)'(x), (f_\beta^+)'(x) \right\}, \sup_{\beta \geq \alpha} \max \left\{ (f_\beta^-)'(x), (f_\beta^+)'(x) \right\} \right] \tag{3.114}$$

are attained.

Then the same expression as in Theorem 3.7 for g-differentiable functions is found and the desired result is proved.

The attainable sets of the $\hat{\int}$-integral of certain bunches of functions also coincide with integrals for fuzzy-set-valued functions, as it will be stated in Theorem 3.19.

Theorem 3.19. *Let $F : [a, b] \to \mathscr{F}^0_{\mathscr{C}}(\mathbb{R})$ be continuous. Then the $\hat{\int}$-integral of the representative bunch of first kind has attainable fuzzy sets*

$$\left[\hat{\int}_a^x \tilde{F} \right]_\alpha = \left[\int_a^x f_\alpha^-, \int_a^x f_\alpha^+ \right] \tag{3.115}$$

for all $\alpha \in [0, 1]$.

In words, the $\hat{\int}$-integral coincides with the integrals for fuzzy-set-valued functions at each x.

Proof. It is not hard to prove the compacity of A_α (Definition 2.16) in $L^1([a, b]; \mathbb{R})$. This is assured by the arguments previously used in proving compacity in $\mathscr{A}C([a, b]; \mathbb{R})$. Following the reasoning of the previous results one demonstrate that A_α are the α-cuts of a fuzzy subset in $L^1([a, b]; \mathbb{R})$.

We observe that $\int_a^x f_\beta^\lambda$ is well defined and that

$$\int_a^x f_\alpha^- \leq \int_a^x f_\beta^\lambda \quad \text{and} \quad \int_a^x f_\beta^\lambda \leq \int_a^x f_\alpha^+ \tag{3.116}$$

for all $\lambda \in [0, 1]$ and $0 \leq \alpha \leq \beta \leq 1$. Hence we obtain, for all $\alpha \in [0, 1]$,

$$\begin{aligned} [\hat{\int} \tilde{F}]_\alpha &= \bigcup_{\beta \geq \alpha} \bigcup_{\lambda \in [0,1]} \int_a^x f_\beta^\lambda \\ &= [\int f_\alpha^-, \int f_\alpha^+] \end{aligned} \tag{3.117}$$

where the last identity holds due to the continuity of $\int_a^x f_\beta^\lambda(x)$ on λ, β, and x. Thus, we have proved that the attainable sets of the $\hat{\int}$-integral of $\tilde{F}$ have the same expression of the integrals for fuzzy-set-valued functions at each x.

Summary of the comparison of derivatives and integrals:

- **Equivalence between gH- and $\hat{D}$-derivatives.** The gH-derivative of a certain class of fuzzy-number-valued functions coincides with the attainable sets of the $\hat{D}$-derivative (using the representative bunch of first kind).
- **Equivalence between g- and $\hat{D}$-derivatives.** The g-derivative of a certain class of fuzzy-number-valued functions coincides with the attainable sets of the $\hat{D}$-derivative (using the representative bunch of second kind).
- **Equivalence among integrals.** The Aumann, Riemann, and Henstock integrals of a certain class of fuzzy-number-valued functions coincide with the attainable sets of the $\hat{\int}$-integral (using the representative bunch of first kind).

3.4 Summary

This chapter reviewed fuzzy calculus for fuzzy-set-valued functions and presented the new fuzzy calculus using fuzzy bunches of functions. The concepts and results here displayed are essential for the development of the various approaches of FDEs, to be presented in the next chapter. They are summarized next:

- The Hukuhara derivative is defined for a class of fuzzy-set-valued functions and uses the concept of Hukuhara difference. The strongly generalized Hukuhara derivative, weakly generalized Hukuhara derivative, generalized Hukuhara derivative, and the fuzzy generalized derivative generalize the Hukuhara derivative and are defined for wider classes of fuzzy-number-valued functions.
- The Aumann, Riemann, and Henstock integrals are defined for fuzzy-set-valued functions.
- The derivative and the integral via extension of the derivative and integral operators, denoted by $\hat{D}$ and $\hat{\int}$, are defined for fuzzy bunches of functions.
- The $\hat{D}$-derivative of a class of fuzzy bunches of functions coincides with the generalized derivatives in terms of attainable sets.
- The $\hat{\int}$-integral of a class of fuzzy bunches of functions coincides with the integrals for fuzzy-number-valued functions in terms of attainable sets.

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