The Rise of "the Mathematicals": Placing Maths into the Hands of Practitioners—The Invention and Popularization of Sectors and Scales

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Abstract Following John Napier's invention of logarithms in 1614, the remainder of the sixteenth century saw an explosion of interest in the art of mathematics as a practical and worldly activity. Mathematics was no longer the exclusive realm of scholars, mathematicians, astronomers, and occasional gentlemen. Teachers of mathematics, instrument makers, chart makers, printers, booksellers, and authors of pamphlets, manuals, and books developed new audiences for the study of mathematics and changed the public's perception of the status and aims of mathematics itself. The inventions of mathematical instrument makers facilitated the rapid expansion of sophisticated mathematical problem solving among craftsmen and practitioners in areas as diverse as navigation, surveying, cartography, military engineering, astronomy, and the design of sundials.

1 Introduction

During the period between Johann Müller (Regiomontanus) (1436–1476) and Bartolomeo Pitiscus (1561–1613) the entire science of planar and spherical trigonometry was reimagined and systematized. The development of new functions, new tables, new instruments, new applications and new audiences flourished. Increasingly, the geometry of regular shapes was seen to have real-world applications of importance beyond the realm of scholars, and beyond its traditional area of application: astronomy (Finck 1583; Pitiscus 1600; Regiomontanus 1967; Rheticus and Otho 1596; Viete 1579).

Subsequent to this restructuring of trigonometry, a family of instruments that came to be called "sectors" were developed in England and on the continent that greatly broadened the community of practitioners who would come to employ this new trigonometry in areas of more practical concern (Galilei 1606; Gunter 1623;

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Hood 1598). In a very real sense, the mathematics of the heavens was brought down to the realm of earthly concerns and endeavors.

Immediately following this period a second revolution in trigonometric applications was ushered in by the invention of logarithms by Napier (1614, 1618) and the rapid development of tables of the logarithms of both natural numbers and trigonometric values by Napier and Briggs (1619) and others. A second family of mathematical instruments was promptly developed and refined which enabled the easy manipulation of these new logarithmic or artificial values. These instruments were called rules or scales, often referred to as Gunter's Rules or simply Gunters in England (Gunter 1623). Originally engraved on the arms of a cross-staff, they became instruments in their own right, engraved on a two-foot long oblong shaped piece of wood, bone, ivory, or brass, with a multiplicity of scales (most of which were logarithmic scales) which were manipulated with the aid of a pair of dividers.

Due to limitations of time and space, this paper limits its discussion to sectors and scales of English origin. English instruments were not only among the very first to be developed, but it is the English tradition that led to the most highly complex and sophisticated of these instruments, and among those whose use was most widespread and long lasting. The comparison between English instruments and their continental counterparts in Italy, France, and Germany will be reserved for another time and place.

2 The Importance of Proportional Analogies in General Problem Solving

An analogy is a comparison between two things which have some features that are similar and other features which differ. In mathematics, we may wish to compare two lines or two circles or two spheres, but the two similar objects may have different lengths, areas, volumes. This comparison of differences is expressed as a ratio.

A proportional analogy may be understood as an analogy between four things, in which the relationship between A and B is the same as the relationship between C and D, or A:B = C:D, or as in the notation I will use in this paper, A:B :: C:D. In this case A and B are like objects, and C and D are like objects. If the relationship between like objects is expressed as a ratio of some characteristic which differs between the like objects, then a proportional analogy states that, the ratio of the differing characteristic of similar objects A and B is the same as the ratio of the differing characteristic between similar objects C and D, for some characteristic of A and B and some (possibly different) characteristic of C and D.

Finding such a relationship was generally the primary approach to framing and solving many problems for which we today would use arithmetic or algebra. In modern times we generally rely on the algebraic solution of equations or systems of equations. The earlier approach is closely tied to the geometry of similar figures since the ratios of corresponding or homologous sides of two similar figures always have the same ratio. This is perhaps best illustrated by way of an example.

Here is a problem drawn from William Kempe's English translation (Ramus and Kempe 1592, p. 50) of the arithmetic portion of Pierre de la Ramée's 1569 book on arithmetic and geometry (Ramus 1569, Book II, Chap. 7, p.29).

> A post goeth from Plimmouth to London in 5 dayes, another commeth from London to Plimmouth more speedily in 4 dayes, admit that they begin their journey both on Monday at [3 of the clock and 20 minutes] in the morning, when and where shall they meete?

A modern student might note that since distance traveled is equal to the product of the rate and duration of travel, it follows that the speed of the slower coach, R_{slower} is equal to the unknown distance between Plymouth and London (let us call it D), divided by 5 days, while the speed of the faster coach R_{faster} would be D divided by 4 days. Rearranging the equation $D = R \times T$ we have $T = \frac{D}{R}$.

Since the two coaches started their journey at the same moment, when the coaches meet, they will each have traveled the same amount of time, but will have covered different distances, say x and y, where $x + y = D$. The time that the faster coach traveled is thus $\frac{x}{D/5}$ and must equal the time that the slower coach traveled, $\frac{y}{D/4}$. Simplifying the equation we have $5x = 4y$. Substituting $y = D - x$ in the previous equation yields $5x = 4(D - x)$ or $x = \frac{4}{9}D$ and $y = \frac{5}{9}D$. The time elapsed will be equal to $\frac{5x}{D}$ or $\frac{4y}{D}$, each of which equals $\frac{20}{9}$ of a day, or $2\frac{2}{9}$.

Compare this with the solution given by Kempe.

> Set downe the former propositions thus: the first endeth his journey in 5 dayes, therefore in 1 day he will ende $\frac{1}{5}$ of the journey: the second endeth his journey in 4 dayes, therefore in 1 day, $\frac{1}{4}$: these parts, added together [$\frac{1}{4} + \frac{1}{5}$], are $\frac{9}{20}$ of the journey. Whereupon conclude, seeing [that] $\frac{9}{20}$ of the journey is gone in 1 day, [it follows that], $\frac{20}{20}$[th of the journey], that is [the entirety of the journey, or] 1, is gone in $\frac{20}{9}$, that is $2\frac{2}{9}$ of a day. This is the time of their meeting, to wit, Wednesday at 8 of the clocke and 40 minutes in the fore noone. Then say, the first [post] in 5 dayes goeth 1 [that is, the entire distance from Plymouth to London], therefore in $\frac{20}{9}$ dayes he will go $\frac{4}{9}$ of the journey, which is the place of meeting, and then the second hath gone the rest of the journey, to wit, $\frac{5}{9}$ [of the journey].

Since most problems were framed as proportional analogies, instruments which aided in visualizing and solving them were a useful aid to the mathematical practitioner. Three such instruments are examined in this paper: sectors, plain scales, and Gunter's scales. There are many seventeenth- and eighteenth-century works devoted to these instruments, but they are frequently incomplete, relying on the reader's knowledge of conventions that frequently go unexplained, or even unmentioned—conventions that are unknown to the modern reader. Readers of that era had access to physical instruments that are difficult to obtain today, as well as access to tutors, teachers, instrument makers, and shopkeepers anxious to give hands on instruction in their application and use. Such mentors are no longer available to those desirous of understanding these instruments. I have relied primarily upon the writings of Galilei (1606), Hood (1598), Gunter (1623)—the inventors of these instruments—together with manuals written by their contemporaries and close study and experimentation with antique instruments that I have purchased to help me understand how the instruments were used.

3 Description and Use of the Lines of the Sector

The construction of the sector is modeled after the demonstration by Euclid that the corresponding sides of similar triangles are proportional.

> In equiangular triangles the sides about the equal angles are proportional, and those are corresponding sides which subtend the equal angles [Euclid, Book VI, Prop. 4].

In practice, nearly all sectors (excepting those of Thomas Hood) employ pairs of triangles that are not only equiangular (i.e., similar), but also isosceles. The sectors are constructed by joining two legs at a pivot point. The legs are inscribed with pairs of identical scales which originate at the pivot point. A pair of dividers can be used to measure the distance from the pivot of the sector to a number engraved on one of these paired scales on the leg of a sector. We call this a *parallel distance.* The dividers can also be used to measure or set the distance between like numbers engraved on some pair of scales on the two legs of the sector. This we call a *transverse distance*. The pairs of scales are offset in such a way that the transverse distance between paired scales at the same parallel distance from the pivot are the same for each pair of scales. The various pair of scales are designed that certain scales can be used by performing some manipulations on one pair of scales, followed by a manipulation on a second (and different) pair of scales, as we will explain below.

Early sectors contained a pair of scales of equal parts (sometimes called arithmetic lines), a pair of scales of surfaces (geometric lines) and a pair of scales of solids (stereometric lines). There were also polygonal lines, tetragonal lines, and lines of metals. A quadrant was often attached with which to protract or to measure angles (Fig. 1).

3.1 Line of Lines

The purpose of these lines is to solve proportional analogies of the form A:B :: C:D where A, B, and C are known, and D is desired. Euclid VI.4 proves that corresponding parts of similar triangles are proportional. The form of the sector (comprising a pivot point, two lateral or lengthwise lines and two transverse or parallel lines) creates similar triangles. If we open the sector so that the transverse distance between A and A on the lines of lines is equal to C, then the transverse distance measured from B to B will be the desired value of D. See Fig. 2 for a numerical example.

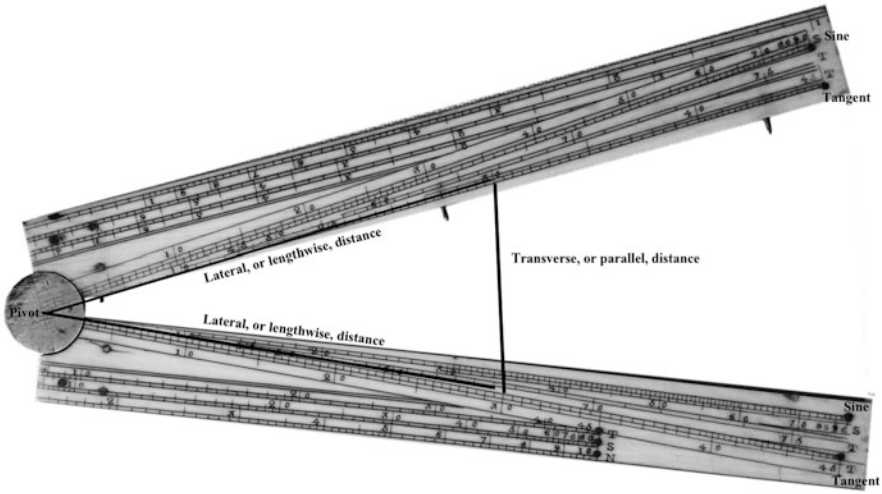

Fig. 1 Each leg of the sector has a scale marked Sine, labeled from 0° to 90°. The lowermost of the scales on each leg are marked Tangent, and are marked from 0° to 45°. The distance from the pivot or center point of the sector to any point on the leg is proportional to the sine or tangent of the angle marked at that point. The right-hand ends of each of these four scales lie on the circumference at a circle centered at the pivot. The length of the chord on this circle connecting the two 90° marks on the Lines of Sines is the same as the length of the chord connecting the two 45° marks on the Line of Tangents. That common value is the radius upon which all trigonometric values are based, and can be set by the user by widening or narrowing the opening of the sector, any transverse or parallel distances changing accordingly

Similarly if two transverse distances and one parallel distance are known the problem may be solved as follows. Open the sector until the transverse distance from A to A is equal to the lateral distance from the pivot point to B. Then open the dividers to a transverse distance equal to C and move the transverse along the legs of the sector until it extends between numerically identical labels on the lateral scales, which will indicate the value of D.

3.2 *Line of Superficies*

The purpose of these lines is to allow the solution of proportional analogies that relate the ratio of the areas of similar figures to the ratio of the lengths of corresponding sides. The lengths of corresponding sides are measured on the Line of Lines, whereas the areas of the similar figures are measured on the Line of Superficies. These lines may also be used to determine the ratio of areas in similar figures whose ratio of corresponding sides are known, or to determine the ratio by which the side of a given figure must be increased or decreased in order to enlarge or reduce the area according to a desired ratio.

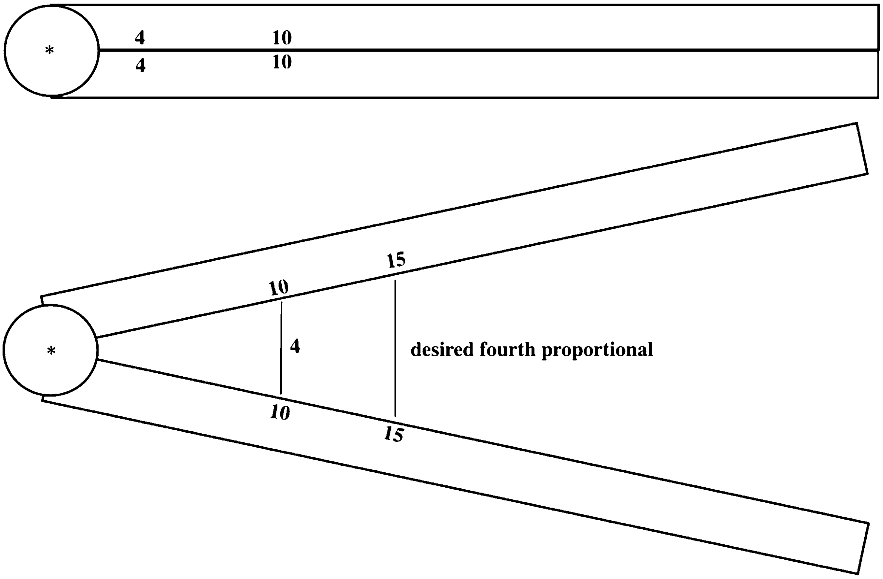

Fig. 2 Given three numbers, to find a fourth in discontinual proportion. Find a value, x, such that $10 : 4 :: 15 : x$ using the Lines of Lines. The dividers are set to the lateral distance from the pivot to the point 4 on one of the lines of lines. The sector is then opened so that the parallel distance between the points marked 10 on the two lines of lines is the same as the distance between the points of the divider. This establishes the transverse or parallel distance of four units between the points marked ten. Without changing the opening angle of the sector, set the dividers so the distance between its points is equal to the transverse or parallel distance between the points marked 15 on the two Lines of Lines, and then, without changing the opening of the dividers, measure the distance between the divider's points on either of the Lines of Lines. One point of the dividers will rest on the pivot point, the other will point to the value engraved on the scale indicating the value of the desired fourth proportional. In our example the value of the desired fourth proportional, x, will equal six

The Lines of Superficies were designed so that the distance from the pivot point to a number engraved on the scale is equal to the area of a square, the length of whose side is the number engraved on the scale at that point. In modern terms the distance to the pivot is the square of the number on the scale, or equivalently, the number inscribed at any point on the scale is the square root of the distance from that point to the pivot of the sector. The purpose of this line is to allow the solution of proportional analogies that relate the ratio area of similar figures to the ratio of the lengths of corresponding sides. This is based on Euclid VI.20 which proves that the area of similar polygons are to each other as the ratio duplicate of that which the corresponding sides have to each other, i.e., in modern terms, the areas of similar polygons varies as the square of the lengths of their corresponding sides. In the case of circles, Euclid XII.2 proves that the areas of circles are to each other as the squares on their diameters. The Lines of Solids provided a similar set of scales which could be used to adjust the linear measure of any regular solid (or a sphere) so

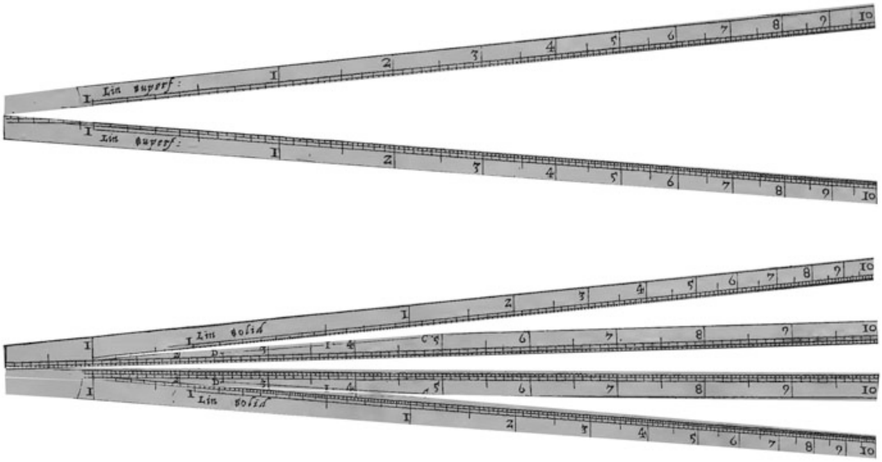

Fig. 3 The legs on the *upper image* (the Lines of Superficies) are labeled 1, 1, 2, 3, . . . , 10. The distance of these labels from the central pivot represent areas of 1, 10, 20, 30, . . . , 100 square units. In other words, the first point labeled 1 represents an area of 1, the second occurrence represents an area of 10, the following labels represent areas of 20, 30, 40, etc. The distance from the pivot point to any such label when measured on the Line of Line (the inner most scales on the *lower image* in this figure) is the linear measure (radius, diameter, or side) of the figure whose area is on the label. The side of the figure with area 100 (or any other value) can be set by changing the angle of the sector opening. The Lines of Solids in the *lower image* is marked 1, 1, 1, 2, 3, . . . , 10, and the distance from the pivot point to these labels represent volumes of 1, 10, 100, 200, 300, . . . , 1000 cubic units, respectively. These distances when measured on the line of lines provide the linear measure of a figure with the associated volume

that the volume was altered according to any given ratio. Conversely, if the volume was changed according to a certain ratio, one could calculate the corresponding ratio by which the side of the solid or the diameter of the sphere would change. See Fig. 3.

For example, given a figure with one side equal to 3 inches representing a 20 acre piece of land, and a similar figure (in the geometric sense of the word) with corresponding side equal to 5 inches, we desire the acreage of the enlarged piece of land. First set the dividers to extend from the pivot to 3 on the Line of Lines. Then open the sector until the transverse distance between 2 and 2 (representing 20 acres) on the Lines of Superficies is equal to the extent of the dividers. Without disturbing the pivot, reset the dividers to extend from the pivot to 5 (inches) on the Lines of Lines. Finally find identical numbers on the Line of Superficies with a transverse or parallel distance of 5 inches. Read the value on the Line of Superficies and that will be your area in acres. In Fig. 4 we see that the plot with a 5 inch side contains 55.5 acres. One can confirm this by noting that the area of the figures will be in the ratio of the square of the sides. Thus $9 : 25 :: 20 : x$ and therefore the requested acreage is 25×20 divided by 9, or 500/9 which is 55 and 5/9 acres.

On the other hand, if we know the ratio of areas and the side of one of the figures, we can determine the length of the side in the second figure. Suppose we know that

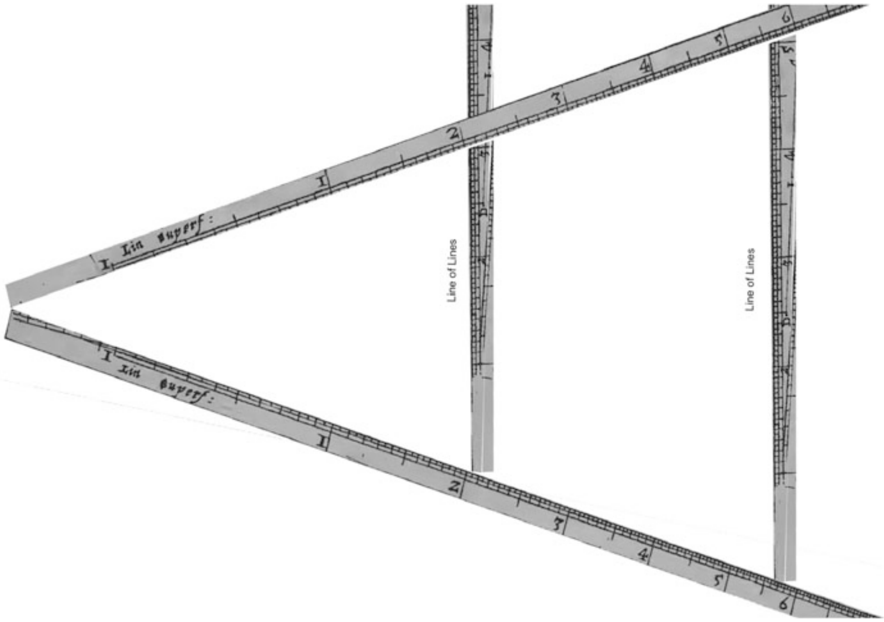

Fig. 4 If a plot with one side measuring 3 inches represents 20 acres of land, what is the area of a similar figure with the corresponding side equal to 5 inches?

the first field contains 2 acres, and the second field contains 5 acres. If the length of a certain side is 4 inches as measured on the first plot, when we draw a similar figure for the 5 acre plot, how long will the line be? Set the dividers to extend from the pivot to 4 on the line of lines and open the sector's pivot until the transverse distance between 2 and 2 on the line of superficies agrees with the dividers.

Without changing the angle at which the sector is open, reset the dividers to extend from 5 to 5 on the line of superficies and measure that extent as a lateral distance on the line of lines to determine the length of the corresponding side on the second plot. This line of lines in combination with the line of superficies could also be used to find squares and square roots of numbers within the accuracy with which the scales could be read.

3.3 Lines of Circular Parts

In his *Introductio in analysin infinitorum* of 1748 Leonhard Euler introduced the modern concept of trigonometric functions as ratios of sides of a triangle (Euler 1748). Prior to that year, trigonometric values were conceived as physical line segments, related to a base circle of arbitrary radius. The lines of circular parts often included lines of chords, sines, tangents, and secants. See Fig. 5.

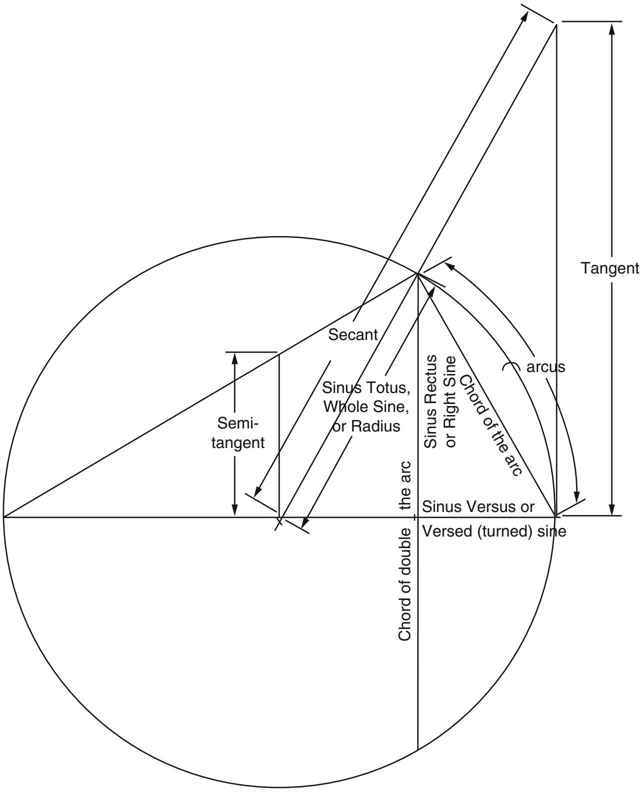

Fig. 5 Prior to the mid eighteenth century, circles of different radii had different trigonometric values and trigonometric tables were based on circles of a particular radius, often referred to as the total sine, or *sinus totus*. Euler’s expanded view of trigonometric functions was not widely adopted by teachers and practitioners of mathematics unit the mid-nineteenth century

3.4 The Particular Lines

If any room remained on the sector after inscribing the Lines of Lines, Sines, Tangents, Chords, Superficies, and Solids, the instrument maker might fill up the unused space with an assortment of lines which would be most useful to the applications of most interest to his client. Some of the more common included extra lines are described below.

3.4.1 The Lines of Quadrature

The purpose of the Lines of Quadrature was to determine the length of the side of a square equal in area to the area of any given circle, or to determine the radius of

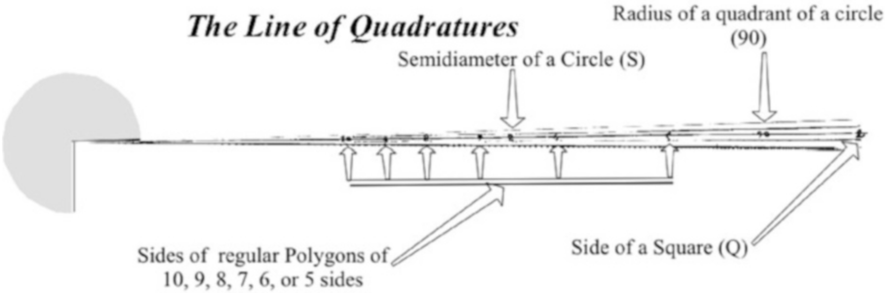

Fig. 6 The Lines of Quadrature are used to determine the length of the side of a square equal in area to a polygon with a side of known length or vice versa

a circle equal in area to any given square, or determine the length of the side of a regular polygon of 5 through 10 sides equal in area to a given circle or a given square.

The Lines of Quadrature were labeled Q, 5, 6, 7, 8, 9, 10, and S. The numerical labels indicate that the transverse distance between like numbers is the length of a regular polygon with that number of sides. The transverse distance between points labeled Q is the side of a square whose area is equal to that polygon. The transverse distance between points labeled S is the length of the semi-diameter (i.e., the radius) of a circle with an area equal to that of the polygon. See Fig. 6. All transverse distances are measured on the Line of Lines.

3.4.2 The Lines of Segments

The Lines of Segments were designed to divide a circle of any given diameter into two parts by a chord perpendicular to the diameter in such a way that the areas of the two segments created were in a given ratio, or to find the proportion between the area of the entire circle and that of a given segment thereof.

3.4.3 The Lines of Inscribed Bodies and the Line of Equated Bodies

The Lines of Inscribed Bodies were labeled D, I, C, S, O, T, which signified the length of the sides of a dodecahedron, icosahedron, cube, octahedron, tetrahedron inscribed in a sphere of semidiameter (i.e., radius) S. See Fig. 7.

3.4.4 The Lines of Metals

The scales were calibrated according to the volume of any metal, which would have the same weight as a specified volume of any reference metal. English sectors

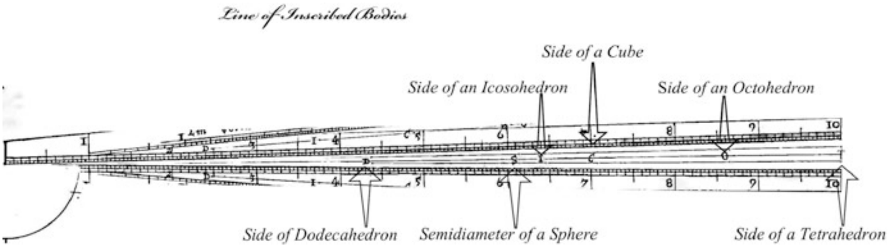

Fig. 7 The Lines of Inscribed Bodies. The transverse distance (from the Line of Lines) was set between any like markings, and the transverse distance measured between any other matching markings (when measured on the Line of Lines) would give the radius of the sphere or the length of the side of a different platonic solid of equal volume

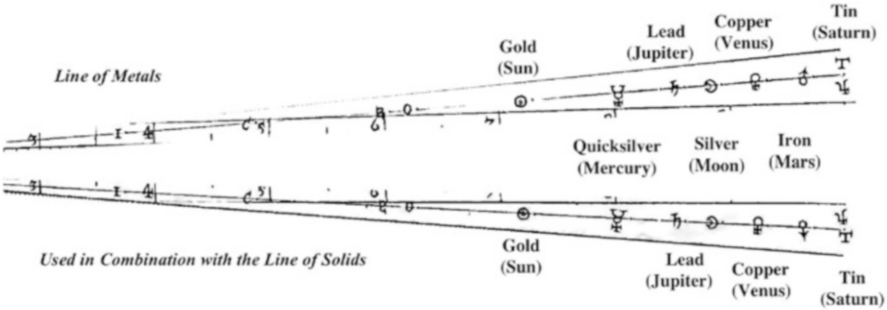

Fig. 8 The lines of metals

provided markings for Gold, Mercury, Lead, Silver, Copper, Iron, and Tin. Each was marked with its alchemical symbol at a distance from the sector pivot related to the volume needed to match the weight of a volume of another metal.

The least dense of the metals (tin) was a distance of 10 from the pivot. The symbol for Gold was that of the sun, for Quicksilver: Mercury, for Lead: Jupiter, for Silver: the moon, for Copper: Venus, for Iron: Mars, and for Tin: Saturn. See Fig. 8. When used in combination with the Line of Solids, the linear measure (most often the radius of a sphere or the side of a cube) which corresponded to the volume determined to have the desired weight. The scale could be used in either direction, i.e., to find the volume of a different metal of equivalent weight, or to find the weight of a different metal of equivalent volume. Such calculations were central to the art of gunnery and artillery.

4 Logarithmic Scales and Rules

Edmund Gunter’s *De sectore et radio* contains a detailed description of the construction and use of his version of Hood’s sector. The English title of this work is *The Description and Use of the Sector, The crosse-staffe and other instruments For* *such as are studious of Mathematical practice.* The work is divided into six "books": the first three books are devoted to the sector and the second three are devoted to the crosse-staffe. The crosse-staffe is composed of the Staffe, the Crosse, and three sights. Gunter's staff was three feet in length, the cross 26 inches long. On the Staffe are inscribed a line of equal parts for measure and protraction, and a line of tangents for measuring of angles, a line conversion between the sea chart and the plain chart, and four lines for working with proportions.

The lines for working with proportions are one of the first appearances of logarithmic scales. Gunter served as the third Gresham Professor of Astronomy (1619–1620). Henry Briggs, his mentor, was the first Gresham Professor of Geometry who left Gresham College in 1619 to become the first Savilian Professor of Geometry at Oxford, and Gunter assumed his position at Gresham upon Briggs's departure for Oxford. It was Briggs who traveled to Edinburgh to visit Napier and worked for 10–15 years to develop and popularize Napier's logarithms. It is not a surprise then to find that the four new lines upon Gunter's Crosse-Staffe were scales of logarithms, log-sines, log-tangents, and log-versed sines. The "bookes of the crosse-staffe" explain how to use these scales together with a set of dividers to solve proportional analogies with a pivoting instrument such as the sector, but with a linear rule. This proved so useful and popular that a separate instrument was developed for purposes of calculation which consisted of a simple oblong shaped rule, inscribed with scales for protraction and measurement. The scales included a line of equal parts, a line of chords, scales for either orthographic or stereographic projection of the sphere onto the plane (a scale of natural sines and a scale of natural semi-tangents), and a set of logarithmic scales (numbers, sines, tangents, secants, and versed sines) for solving proportional analogies and for performing multiplications and divisions. The instrument became known as a Gunter's Rule or a Gunter's Scale and remained in use from the 1620s until the third quarter of the nineteenth century. It survived in a modified form as the slide-rule from the seventeenth century until the invention and commercialization of electronic calculators and computers in the 1970s.

The key to the importance and utility of this new instrument was the ease with which proportional analogies could be resolved with the aid of a variety of logarithmic scales. Since the scales are logarithmic, the logarithm of the ratio of a to b is the distance between the points labeled a and the point labeled b on the line of numbers. This can be used to find a fourth proportional (the Golden Rule or Regula Aurea of problem solving techniques), with two simple movements of a pair of dividers. First open the dividers so that its legs extend from point a to point b on the Line of Numbers. Then move the dividers so that one leg points to the point c. The other leg will point to point d whose value as read on the scale is the desired fourth proportional. See Fig. 9.

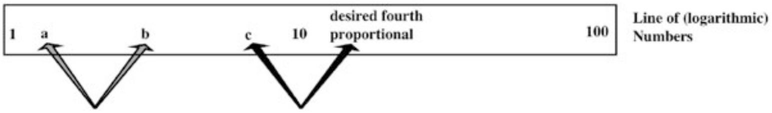

Fig. 9 To solve the proportional analogy A:B :: C: ??, simply set the dividers to extend from A to B in the Line of Numbers, then move the dividers to C and they will extend to the desired fourth proportional. Note that the direction from A to B (whether to the left or to the right) must be the same as the direction from C to the desired proportional

5 Construction and Use of the Scales of the Gunter's Rule

The scales on Gunter's Rule were many, but they can easily be divided into categories. The rule does not have any paired scales as do sectors, since it does not use Euclid's postulate on similar triangles to solve proportions, but instead uses logarithmic scales. There are, however, a small number of non-logarithmic, or natural, scales. There are scales of equal parts which are used for constructing lines of a particular length or for measuring the length of lines. There is a line of chords for measuring angles or laying out angles of a particular size. There are also lines of sines, tangents, semi-tangents, and secants which are not logarithmic. These were not used for solving proportions or performing arithmetic with trigonometric values. Rather, they were used for the geometric construction of both orthographic and stereometric projections of arcs on the surface of a sphere onto the plane. These projections were used in navigation, astronomy, and cartography, and they allowed the practitioner to draw planar projections of spherical triangles, great and lesser circles, etc. and to measure and interpret their properties in the plane.

The logarithmic scales included the Line of Numbers, Line of Sines, Line of Tangents, and Line of Secants. Despite this nomenclature these were scales of common logarithms, log-sines, log-tangent, and log-secants. There was also a scale labeled Versed Sine. That scale was in fact a scale of the logarithm of half of the versed sine of the supplement of the angle marked on the scale. This scale allowed the use of logarithms to solve spherical triangles where either all three sides or all three angles were known. Unlike planar triangles, in spherical trigonometry knowing three angles is sufficient information to determine all three sides. In particular, the determination of local time, and therefore the determination of longitude from celestial observations alone require the solution of such triangles.

The next section will explain how the scales on the sector and on the rule were constructed. With the exception of the logarithmic scales, these scales were not constructed by calculating values and measuring them out on the instruments; neither were they obtained from tables of values and then measured and inscribed or engraved. Instead they were geometrically constructed—often through Euclidean methods, sometimes with an instrumental approach, and occasionally using mechanical or approximate methods for trisecting an angle or squaring a circle.

5.1 Construction of Lines of Equal Parts

The line of lines or line of equal parts requires dividing the length of the sector or scale into 100 equal parts. Although creating a line that is *n* times the length of another is straightforward, given only a straightedge and a pair of dividers, dividing a line into *n* equal parts is not. The details of how this was accomplished are described in Fig. 10.

5.2 Construction of Circular Lines

These scales were generally determined by dividing a quadrant of a circle into 90 divisions, then geometrically constructing the line segments corresponding to various trigonometric values for each division, and using dividers to transfer those lengths onto a linear scale. The radius of the circle was chosen to be equal to the desired length of the scale. Such a diagram was called a Plain Scale. The ingenious diagram in Fig. 11 contains no fewer than nine scales of circular parts.

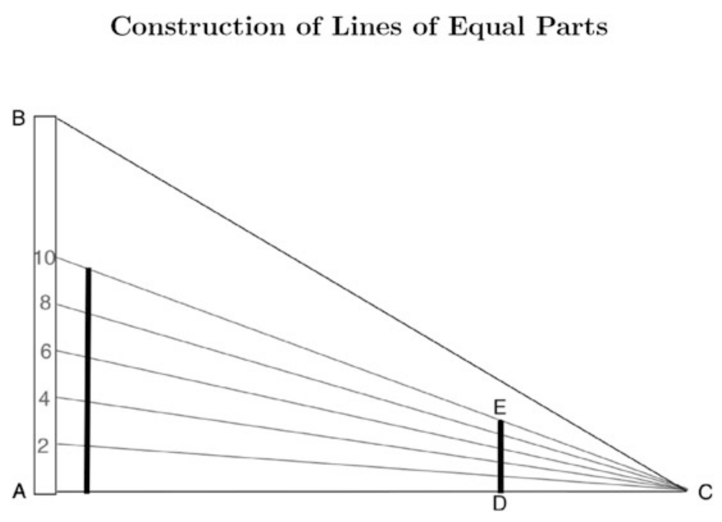

Fig. 10 Lines of Equal Parts. Suppose we wish to divide a line segment DE into five equal parts, labeled 2, 4, 6, 8, 10. Construct a perpendicular to DE at D, extending the line an arbitrary distance to A. Then construct a long line perpendicular to DA at A, extending to B. On line AB mark off 5 equal segment of a convenient length, labeled 2, 4, 6, 8, 10. Draw a line between the point labeled 10 on AB and the point E. Draw a second line from A through D, extending until it meets the previously drawn line at C. Connect the points 2, 4, 6, and 8 on AB to the point C. Line DE is now divided into 5 equal parts which may be labeled as desired

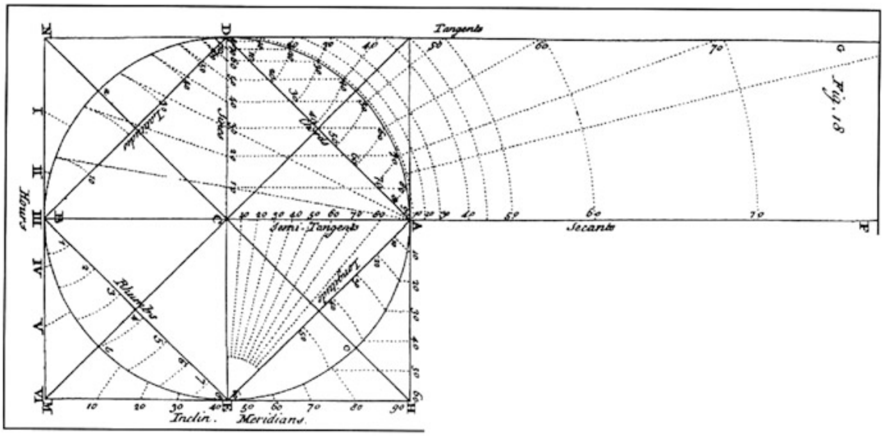

Fig. 11 Construction of the Circular Lines. The arc AD is divided into 9 equal parts, with divisions every 10°. The distance from point D to each of the arc’s divisions is used to construct a line of chords upon the chord AD. The divided arc is used to construct a line of sines on radius CD. Lines of tangents and secants are likewise constructed upon Line DG and AP, respectively. Radius AC is divided into a line of semi-tangents (the tangent of half the angle marked on the scale). Line BE is divided into a line of chords of the 8 points of one quadrant of the compass and is marked Rhumbs. The remaining four scales are specialized scales used for navigation and for the laying out of sundials. All lines are constructed geometrically

5.3 Construction of Lines of Superficies

The Line of Superficies is a line of the square roots of the corresponding value on the Line of Lines, and that of the Line of Solids is a line of cube roots of the corresponding values on the Line of Lines. The reason for this is that the ratio of the length of the sides of similar figures is equal to the ratio of the square roots of their corresponding areas and to the ratio of the cube roots of their corresponding volumes. Since the scale is to be constructed geometrically, the method is to find the mean proportional between the two values, or to find two mean proportionals between two values. Those will be equal to the square root of the product of the two values, or the cube root (and square thereof) of the product of two values. See Fig. 12.

The construction takes unity for one of those values, and the area or volume as the other. The mean proportional will be the side of a square of equal area to the figure. If one determines two mean proportionals between unity and a volume, the first mean proportion will be the side of a cube with the same volume as the original figure.

5.4 Construction of Logarithmic Lines

The logarithmic scales were laid out in a different manner. They were taken from tables of logarithmic values of numbers and logarithms of circular parts, which were then measured out on an accurate scale of equal parts with a set of diagonal scales

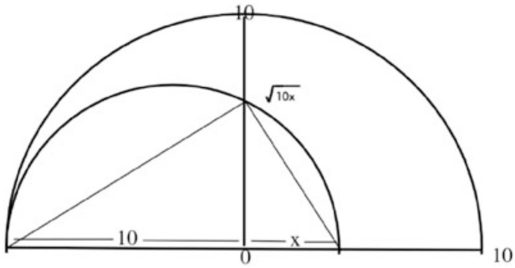

Fig. 12 Construction of the Line of Superficies. The standard construction (Euclid 1956, Book VI, Proposition 13) of the mean proportional between 10 and *x* provides a line segment of length square root of 10*x*. If the segments were 1 and *x*, the altitude of the right triangle formed would be the square root of *x*. The purpose of the larger semicircle in this figure is to emphasize that the value of *x* may range from 0 to 10

Fig. 13 Since the difference of two logarithms is the logarithm of the ratio of their arguments, we may construct the scale from the difference between the position desired and the right-hand end of the scale as the difference of their logarithms

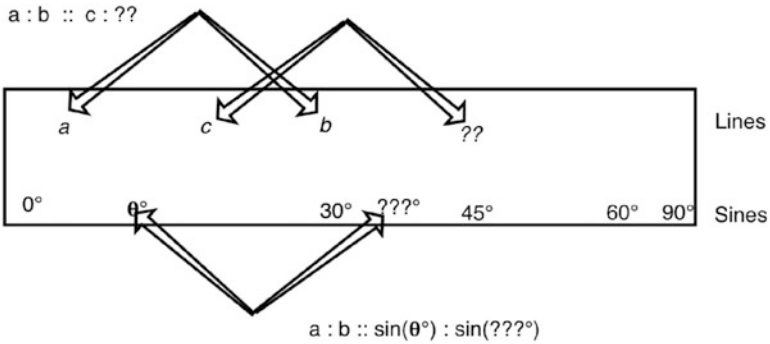

Fig. 14 Application of the artificial (i.e., logarithmic) scales. Using only the Line of (artificial) Numbers, a fourth proportional is determined. Using the Line of Numbers together with the Line of (artificial) sines provides solutions to the planar law of sines. Using only the Line of (artificial) Sines provides solutions to the spherical law of sines

which could measure lengths to three significant figures. These lengths were then transferred to the rule. However, these scales were laid out as measured from the right hand (higher valued) end of the scale, towards the lower valued left end of the scale. These scales have no zero point, but are anchored at the log-sine of a right angle, or at the log-tangent of 45°, etc. See Figs. 13 and 14.

6 Three Applications from Astronomy: Solutions by Sector and Scale Compared

This paper concludes with three practical applications of the sector and scale to solve important problems in spherical trigonometry related to astronomy, navigation, and astrology.

The first problem concerns the determination of the solar declination on any given day of the year. Over the course of a year the position of the sun with respect to the fixed stars moves about a path called the ecliptic. Each day the sun appears to move along a small circle in the heavens, parallel to the celestial equator. During the period from the winter solstice to the summer solstice, that circle gradually moves northward from the tropic of Capricorn to the tropic of Cancer, while the remainder of the year it gradually moves southwards from the tropic of Cancer to the tropic of Capricorn. On each of the equinoxes this circle is coincident with the celestial equator. The angular distance of this small circle, called the day circle, north or south of the celestial equator is known as the solar declination. Knowledge of the declination of the sun on a particular day, together with a measurement of the altitude of the sun above the horizon at its highest point (local apparent noon) allows the navigator to determine his latitude.

The second problem concerns the determination of the azimuth of the sun at sunrise, provided that the solar declination and the observer’s latitude have been determined through calculation and observation. A comparison of the bearing of the rising sun as read from a magnetic compass with the corresponding bearing given by the calculated azimuth provides the amount by which celestial (or true) direction varies from the direction indicated by the magnetic compass. In the seventeenth century it was believed that magnetic variation varied with longitude and that a determination of magnetic variation could be used to determine the longitude of the observer. It was soon discovered that magnetic variation varies with both time and place and could not be used to determine longitude. The measurement of variation could, nonetheless, be used to correct the compass direction to provide true directions, and thus was of considerable value to the navigator.

The third of our sample problems concerns the determination of local solar time from celestial observation. The standardized time zones we use today did not exist until the close of the nineteenth century. Time varied continuously across time zones, and therefore at the very same instant, ships at different longitudes (however small this difference) observed different solar or local times. Since ships are constantly moving, no mechanical clock could be used to determine the local time, thus celestial determination of local time could be used to regulate and correct any timepieces used on ship. From the mid-eighteenth century onward, with the appearance of nautical almanacs recording the positions of the moon with respect to planets, sun, and stars at every hour of the day as seen from some reference meridian, lunar measurements together with determination of local time would provide navigators with knowledge of their longitude. After the development of chronometers, synchronized to time at a reference meridian, the determination of local time when compared to the chronometer time also provided knowledge of longitude. In all of the cases, the ability to determine local apparent time was an important navigational tool.

Having discussed the motivation for each of these problems, let us examine how the use of sector or Gunter's rule was used to approach each problem in turn.

6.1 *Problem 1: Given the Distance of the Sun from the Equinoctial Point, to Find Its Declination*

During the course of the year, the position of the sun moves along the ecliptic, a great circle at an angle of 23° 30′ to the celestial equator, intersecting the equator at the positions of the vernal and autumnal equinoxes. During the course of a day, the sun (at roughly the same point on the ecliptic) moves in a lesser circle about the celestial north pole. Since the sun moves about 1° along the ecliptic during each day, knowledge of the date determines the distance along the ecliptic from the nearest equinox. That information may be used to determine the distance of the sun (its declination) above or below the celestial equator. A right angled spherical triangle is formed by the celestial equator, the ecliptic, and the meridian of longitude of the sun. See Fig. 15. Using the spherical law of sines, we know that the ratio of

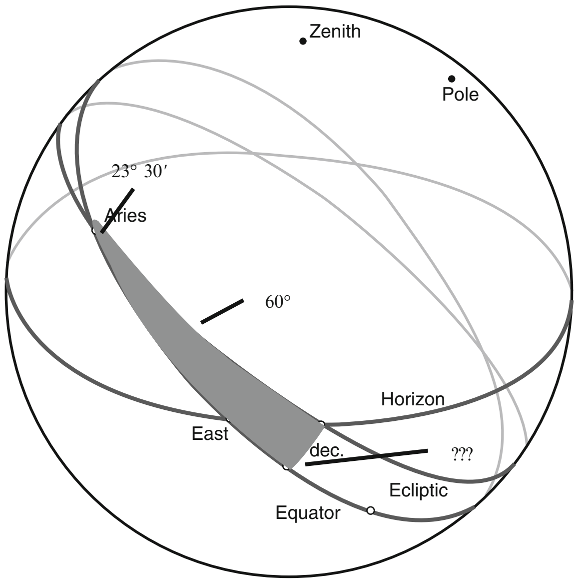

Fig. 15 Given the sun's ecliptic longitude on a particular day (determined by the number of days from since the equinox), to find the solar declination, i.e., the sun's angular distance above or below the equator

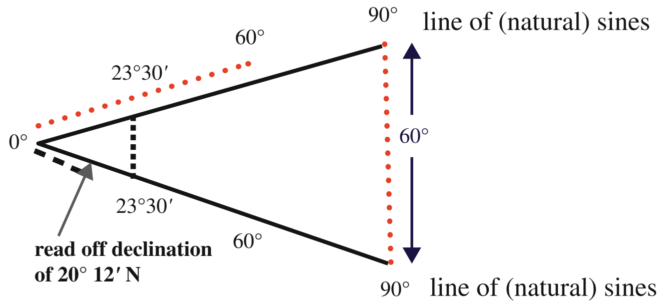

Fig. 16 Use of the sector to solve the spherical law of sines

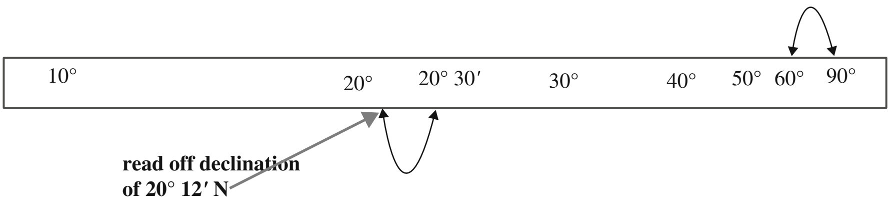

Fig. 17 Use of Gunter's scale to solve a problem using the spherical law of sines

the sine of an angle to the sine of its opposite side is the same for any angle and its opposite side. Therefore as the sin 90° is to the sin 60° so is sin 23°30′ is to the sine of the declination.

Solution by Means of the Line of (Natural) Sines on the Sector Set the dividers to the distance between 0° and 60° on the Line of Sines. Open the sector until the transverse distance between the 90° marks on each leg of the Line of Sine is equal to the distance previously set on the dividers. Lastly reset the dividers to the distance between the 23°30′ marks on each leg of the Line of Sines, and use those dividers to extend from the pivot point of the sector along either leg of the sector on the Line of Sines, reading the value at that point of the scale as 20°12′, the solar declination. See Fig. 16.

Solution by Means of the Line of Artificial (Logarithmic) Sines on the Gunter's Scale Performing this calculation is even simpler using the Gunter's scale. Set the dividers to extend from 90° to 60° on the line of (artificial) sines. Moving in the same direction (right to left), lay off that same distance from 23°30′ on the same line, reading off the value at the other end of the dividers as 20°12′, the desired solar declination. See Fig. 17.

6.2 Problem 2: Given the Latitude of Your Location and the Sun's Declination at Sunrise, to Find Azimuth of the Sun at that Time

Assume that on May 21, the sun's ecliptic longitude is 60° as in Problem 1, above. Then the solar declination will be 20°12′ north of the equator, as previously calculated. Let us further assume that we are observing the sunrise from St. Catharines, ON, Canada, which is at latitude 43°11′ North. What is the azimuth of the sun, that is, how many degrees along the horizon to the East of the North point does the sun rise?

The ortive amplitude or rising amplitude is the angular distance along the horizon between the position of the sun on the horizon at sunrise and the East point of the horizon (Van Brummelen 2013). In the springtime, the sun rises to the north of the East point and sets to the north of the West point. Thus the azimuth is determined by subtracting the ortive amplitude from 90°. See Fig. 18.

As the sine of the complement of the latitude is to the sine of the declination, so is the sine of a right angle (the sinus rectus) to the sine of the desired azimuth. The complement of the latitude of St Catharines is 46°49′.

Thus $\sin 46°49' : \sin 20°12' :: \sin 90° : \sin($ sun's *rising amplitude*). The *azimuth* of the sun is the complement of that angle, i.e., 61°44′. The needle of a magnetic compass does not point to true north, but rather to the magnetic north pole. The amount by which this differs is called magnetic variation. Unfortunately the difference between true and magnetic north varies from year to year and from place to place on the globe. Bearings taken from charts or maps are measured in

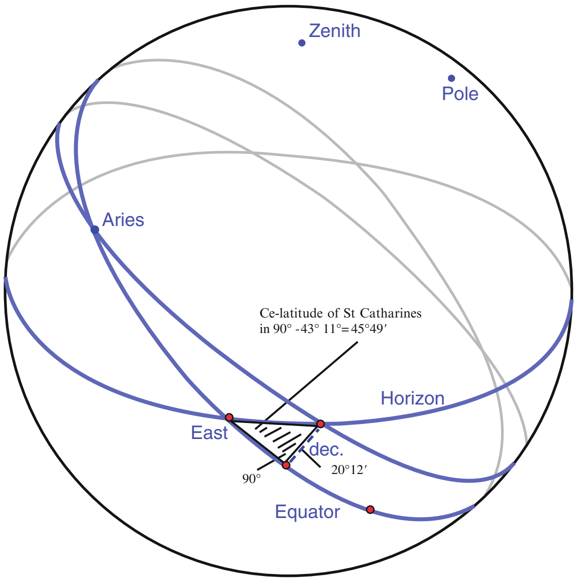

Fig. 18 The right triangle under consideration is the smaller right triangle formed by the solar meridian, the horizon, and the equator. The angle formed by the intersection of the horizon and the equator is the complement of the latitude. The side opposite that angle is the declination previously calculated. The meridian meets the equator, as before at a right angle, and the side opposite the right angle is the desired unknown. Thus the spherical law of sines is again the relationship of interest

the direction from true north. If a navigator took a magnetic bearing of the sun at sunrise and compared it to the azimuth as determined through calculation, the variation could be determined and compass headings could be converted to true directions and vice versa.

Solution by Means of the Sector and the Scale Set the dividers to extend from the pivot of the sector to the point on the Line of (natural) Sines marked 20°12′. Then open the sector so that the dividers extends between the points marked 46°49′ on each leg on the Line of Sines. Then reset the dividers to measure the distance between those points marked 90° and note the extent of the newly reset dividers from the pivot of the sector to a point on the Line of Sines on either leg of the sector. See Fig. 19.

Using the Lines of artificial (logarithmic) Sines, the dividers are set to extend from the known angle to its opposite (known) side. The dividers thus set are used to extend from the second of the known angles to its opposite (but unknown) side. See Fig. 20.

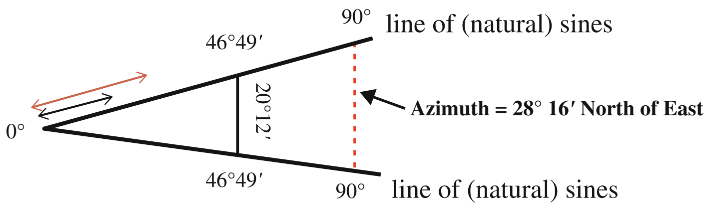

Fig. 19 Solution by sector. The known angle (the complement of the latitude) and its (known) opposite side (the declination) form the leg and the base of the smaller of the two similar isosceles triangles of the sector. The known angle (the right angle) and its (unknown) opposite side (the rising amplitude) form the leg and the base of the larger similar isosceles triangle

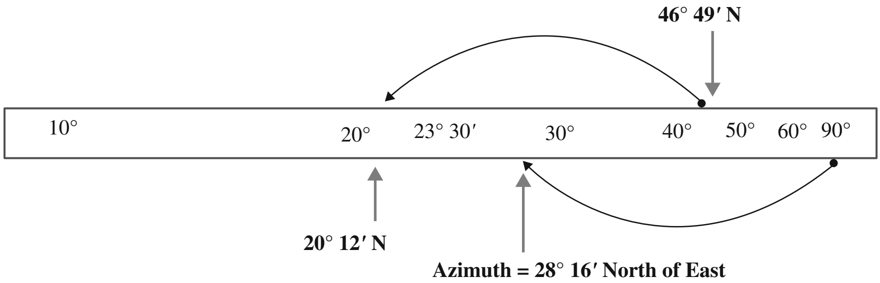

Fig. 20 Solution by Gunter’s Scale. The dividers are set to extend from the complement of the latitude to the declination, and moved to extend from 90° to the rising amplitude

6.3 Problem 3: Given the Latitude of Your Location, Declination of the Sun, and the Sun's Altitude, to Find the Local (Solar) Time

This last problem will show a more complicated situation. Here we know three sides of an oblique spherical triangle, but none of the angles. The spherical law of cosines applies in this case:

$$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma,$$

where c is the side opposite angle γ and a and b are the sides adjoining angle γ. Solving for γ we have

$$\cos \gamma = \frac{\cos(c) - \cos(a)\cos(b)}{\sin(a)\sin(b)}.$$

These equations are ill-suited for manipulation with either sector or logarithmic scale, due to the sum or difference of products.

Suppose the sun is observed from St Catharines, ON, to have an altitude of 50° above the horizon on a day when the solar declination is 20° North. The three sides of the "astronomical triangle" formed by the solar position, the pole, and the zenith are the complements of the latitude, the solar declination, and the solar altitude. If we take the latitude of St Catharines to be 43° N, then the sides of the astronomical triangle are 47°, 70°, and 40°—the complements of the latitude, the declination, and the altitude, repectively (Fig. 21).

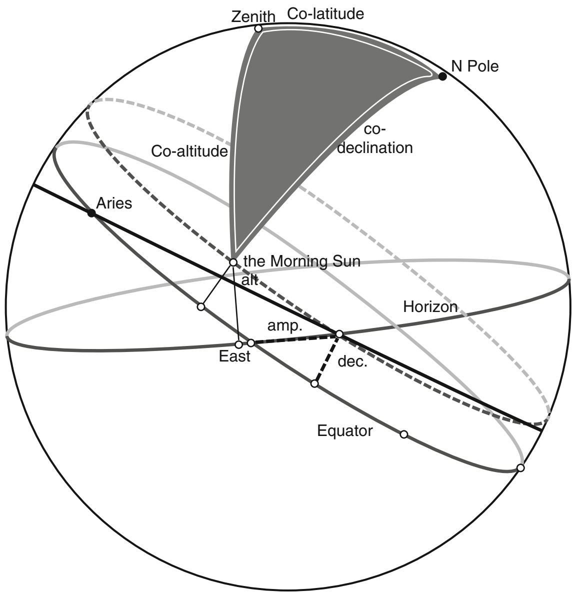

Fig. 21 The complements of the observer's latitude, the solar declination, and the solar altitude form the sides of the *spherical triangle* shaded in the diagram. The angle θ at the celestial north pole, between the meridians of the sun and the observer, measures the local solar time. Angular measure is related to time at the ratio of 15° of arc per hour of time (since a rotation of 360° occurs every 24 h). If the sun is in the east of the South point in the horizon, it is the time before noon, else it is the time since noon

An elegant solution published by Pitiscus between 1595 and 1600 is presented below. The key insight was to replace the law of cosines by a law of versed sines or a law of suversed sines, where the suversed sine of an angle is the versed sine of the supplement of that angle. Algebraically, $\frac{\sin S \sin(S-c)}{\sin a \sin b}$ is equal to half of the suversed sine the angle γ, where S is half of the sum of the three sides of the triangle.

The versed sine of an angle is the difference between the radius or sinus totus and the sine complement of the angle. In modern terms, the versed sine of γ is $1-\cos\gamma$. and therefore the suversed sine of γ is equal to $1+\cos\gamma$.

As seen above, the spherical law of cosines, when solved for the unknown angle may be stated as

$$\cos\gamma = \frac{\cos(c) - \cos(a)\cos(b)}{\sin(a)\sin(b)}.$$

Adding $\frac{\sin a \sin b}{\sin a \sin b}$ to both sides, we have

$$1+\cos\gamma = \frac{\cos c + \sin a \sin b - \cos a \cos b}{\sin a \sin b} = \frac{\cos c - \cos(a+b)}{\sin a \sin b}.$$

Using the trigonometric identity $\cos x - \cos y = 2\sin\frac{x+y}{2}\sin\frac{y-x}{2}$, where $x = c$ and $y = a + b$,

$$\cos c - \cos(a+b) = 2\sin\frac{a+b+c}{2}\sin\frac{a+b-c}{2}$$

Thus, we have

$$\frac{1+\cos\gamma}{2} = \frac{\sin S \sin(S-c)}{\sin a \sin b}.$$

The right-hand side of this equation is computed using a sequence of two proportional analogies. The left-hand side of the equation is one-half of the suversine of angle γ.

The first of the proportional analogies is:

> As the sine of 90° is to $\sin a$, so is the $\sin b$ to “the fourth sine.” Using either the sector and a pair of lines of sines or using the Gunter and the scale of log-sines, we find the angle, let us call it x, whose sine is $\sin a \cdot \sin b$.

The second of the proportional analogies is:

> As the value of “the fourth sine” is to $\sin S$, so is $\sin(S-c)$ to the fourth term in this second analogy (which Pitiscus terms the “seventh sine”). The term S is called the half-sum, which is equal to $\frac{a+b+c}{2}$.

As before, using either the sector and a pair of lines of sines or using the Gunter and the scale of log-sines, we find the angle, let us call it y, whose sine is $\frac{\sin S \sin(S-c)}{\sin a \sin b}$, the right-hand side of the equation he seeks to solve.

Thus a value, y, has been found which is a solution to the equation

$$\sin y = \frac{\sin S \sin(S-c)}{\sin a \sin b}.$$

The desired value, γ, is the solution to the equation

$$\frac{1+\cos\gamma}{2} = \sin y.$$

The half-angle trigonometric identity $\cos^2\frac{\gamma}{2} = \frac{1+\cos\gamma}{2}$ allows us to find a solution, γ to the equation $\sin^2(90° - \frac{\gamma}{2}) = \sin y$. Thus using the scales of superficies on the sector together with a lateral distance equal to the "seventh sine" would allow one to determine $90° - \gamma/2$ and thus the value of γ. On a Gunter's scale, the ability to find square roots by dividing a logarithm by two simplifies the procedure.

However, the solution of this problem was so common, that a special scale, labeled "versed sine" was provided on both sectors, which was in fact a scale of half-suversed sines, on which values of γ were placed at a distance from the pivot of $\frac{1}{2}(1 + \sin(90° - \gamma))$. If the dividers were set to measure the distance between the pivot and y on the scale of sines, then that same distance would extend from the pivot to the point marked γ on the so-called scale of versed sines. A similar logarithmic scale was frequently included on the Gunter's scale. It was labeled the scale of versed sines, but was in fact a scale of logarithms of half-suversed sines.

The solutions to the problem of determining local time using both sector and scale together with a scale of "versed sines" is presented in Fig. 22 and are remarkable in their directness and simplicity.

7 Impact of Sectors and Scales on Mathematics, Science, and Society

We close with a few notes putting these developments into their cultural and historical contexts. The sixteenth and seventeenth centuries were times of tremendous change both in England and on the continent. The effects of the English Reformation, recurring bouts of plague, the English Civil War, Commonwealth, and Restoration (1642–1660), the struggle between Protestant and Catholic sympathizers, the fire of London (1666), exploration and colonization of the New World all had a major impact upon the society in which mathematicians and their students worked and lived. But this period also saw the birth of a panoply of new types of institutions such as Gresham College (1597), the East India Company (1600), the Royal Society (1660), the Christ Hospital's Writing School (1577) and its Royal Mathematical School (1673).

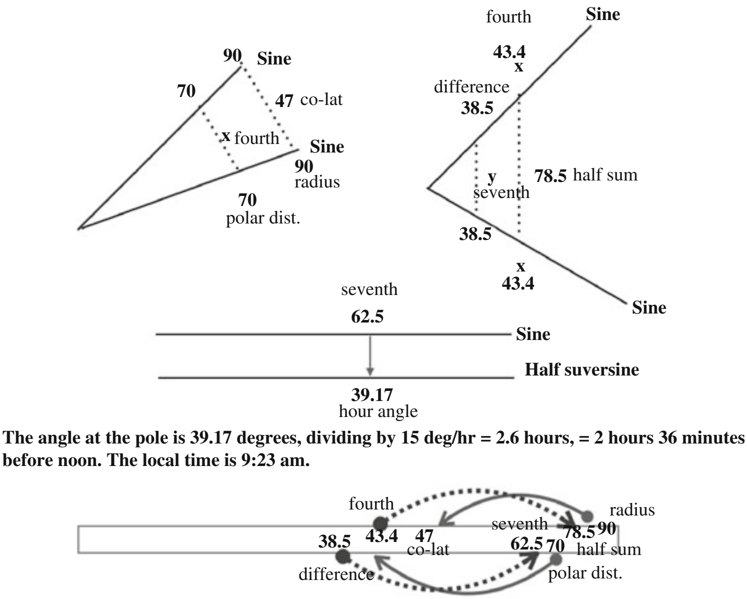

Fig. 22 Determining local time, via sector and via scale

According to Ellis, the first coffee house in Christendom was opened in Oxford in 1650 (Ellis 1956). Within a few years coffee houses were open in London and Cambridge as well, and by the end of that century London could boast of over 2000 coffee houses (O’Connor and Robertson 2006). Not only were the coffee houses meeting places for discussion on topics from science to politics, but lectures were given in them. These were not just impromptu lectures given in the course of discussion, but rather were properly advertised and usually not one-off lectures but rather extended lecture series. Because of this educational function coffee houses were often called the Penny Universities—the name arising since they charged an entrance fee of a penny (Lillywhite 1963; O’Connor and Robertson 2006; Stewart 1999).

Institutions such as Gresham College offered public lectures and demonstrations on topics of scientific and mathematical interest. Works were increasingly written in English (or other vernacular language) rather than scholarly Latin or were promptly translated into English from Latin. Other players in this drama include the recently introduced commercial schools of writing and mathematics and private teachers of “the Mathematicks,” and an increasing number of booksellers, printers, publishers, instrument makers, and sellers of instruments. The fact that Thomas Hood served as “Mathematical Lecturer to the City of London” and dedicated his work on sectors to the auditors of his lectures, rather than to a noble patron, underscores this change in audience and a marked liberalization of accepted ways in which mathematics could be learned.

In the end, the cross cultural milieu changed both the way that mathematicians thought of themselves and the ways in which practitioners viewed the roles of mathematics and science in their trades and professions (Johnston 2005; Taylor 2011, 2013).

As Katie Taylor elegantly summarizes in her article *Vernacular geometry: between senses and reason*,

> In Continental debates about the status of mathematics, the separation of the objects of mathematics from the natural world was a widely cited underpinning for the certainty of mathematics ... While this debate received relatively little attention in England, it is clear from the instances in which it was touched upon that there was a perceived distinction between the world of reason, geometry's domain, and the world of the senses. In his *Alae seu scalae mathematicae* (1573), Thomas Digges himself set up an opposition between "Queen Reason", responsible for devising the geometrical methods with which stellar observations were to be treated, and the "slave senses", charged with making the observations required to feed into these geometrical methods .... Digges went on to stress that these two realms needed to be united to get at truth itself (Taylor 2013).

Meanwhile many practitioners of mechanical and methodological arts embraced the spirit of the newly born scientific revolution and of the applications of mathematical theory to practical concerns and came to view their fields as based upon mathematical and scientific foundations rather than upon craft or artistic traditions, or by appeals to authority (religious or otherwise), tradition, or past practice. The navigator, the designer of buildings and ships, the astrologer, merchant, surveyor, and cartographer came to view themselves as practitioners of the mathematicks and to view the validity of their fields and their practices as supported by a foundation of mathematical theory (whether or not they personally made use of these mathematical underpinnings), laying the foundations for the impressive advances in applied mathematics and the contributions of science and technology to practical matters in the eighteenth and nineteenth centuries.

8 Credits

Figure 1 is a photograph taken by the author of a sector in his private collection. Figures 3, 4, 6, 7, and 8 were derived from images from the 1636 edition of Edmund Gunter's *De sectore et radio*, a work in the public domain, which can be viewed via the Internet Archive at http://www.archive.org/details/descriptionuseof00gunt. Figure 11 was taken from John Robertson's *A treatise of such mathematical instruments, as are usually put into a portable case: containing their various uses in arithmetic, geometry, trigonometry, architecture, surveying*, published in 1775, a work in the public domain and available through ECHO, the European Cultural Heritage Online, which can be viewed at http://echo.mpiwg-berlin.mpg.de/MPIWG:NPREVR4U. All other figures were constructed by the author.

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