The Judicial Analogy for Mathematical Publication

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Abstract Having criticized the analogies between mathematical proofs and narrative fiction in 2000 and between mathematics and playing abstract games in 2008, I want to put forward an analogy of my own for criticism. It is between how the mathematical community accepts a new result put forward by a mathematician and the proceedings of a law court trying a civil suit leading to a verdict. Because it is only an analogy, I do not attempt to draw any philosophical conclusions from it.

1 Judicial Analogy

I am fond of analogies. I find that they improve life much as humour improves life. At CSHPM meetings I spoke on an analogy between mathematics and fiction in 2000 (Thomas 2000) and between mathematics and games in 2008 (Thomas 2008), in both cases pouring cold water—despite my fondness for analogies—on what I viewed as a too enthusiastic espousal of those analogies by others and in the case of fiction even an absurd identification. I have published my reservations about these analogies of others elsewhere (Thomas 2002, 2009). What I want to do here is to offer what I view as an improvement on the fiction analogy with my own undue enthusiasm. You may even think it not particularly closely related to that other analogy, but that is where it started out.

The fiction analogy works—to the extent that it works at all—only for certain documents like explanations and proofs. I once tried to present some mathematics to

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a sympathetic audience to illustrate how narrative-like what I said was. The audience only began sympathetic; their response convinced me that proofs are only a little like stories.[1]

There is, however, one way in which a story is like a proof and also like a paper like this one even when it contains no proofs. A story is an artful linearization of descriptions of situations, props, events, and whatever else is needed simply because it is written down in words. A picture is said to be worth a thousand words, not just a thousand pixels, because it is more effective than its linearization no matter how artful. A fax machine makes of a picture an artless linearization that is not humanly readable. As I have said, this paper is intended to offer a picture but is of course a linearization. Even if it were desirable, it cannot begin in the top left corner and proceed like the cathode rays of an old computer monitor because the picture is only metaphorical. All I can do, so far as I can tell, is to sketch patches of the analogy intended in an order that I find somewhat linked. It is just piling up details like the points of a pointillist painting.

The analogy that I want to point to today is larger in scope than proof/narrative. I want to sketch a comparison between the justice system on the British model, used more or less in all English-speaking countries, and the more public elements of the professional mathematical enterprise.[2] So, on the one hand, we have courts with judges and cases that are tried in terms of laws and the evidence on which cases are decided. And, on the other, we have the knowledge base of the profession into which gatekeepers admit results in terms of postulates, logic, and definitions, where results are put forward by mathematicians in talks and papers. The results to which I refer are mainly theorems, theorems are spelled out in terms, the terms need definitions, and theorems are normally derived from postulates in accordance with some system of logic. Now obviously publishing a theorem isn't much like litigation, but there are some similarities here that may possibly be interesting. Each is publicly making a claim that may be accepted.

One of the philosophical problems to do with mathematics that I think is of some interest is its objectivity. We all agree that mathematical results do not depend on our preferences; they are what they are even if we don't like them. This objectivity is not something that is as true even of the hard sciences; it is an achievement that the ancient Greeks seem to have started us off with. In my opinion (Thomas 2014) it is based on consensus-based definitions. One hears a great deal about axioms and postulates, but as time has passed the importance and complexity of definitions has grown. In the 1970s Michael Spivak (1970–1979) pointed out that recent mathematics has tried to move as much theory as possible from theorems into the definitions influenced by them. One need look no farther than the definite

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[1]The idea does continually spring up. The evening before my writing this footnote, 20th January, 2015, the Oxford Research Centre in the Humanities and the Mathematical Institute at Oxford held an event called "Narrative and Proof" featuring a panel discussion led off with a paper by Marcus du Sautoy entitled "Proof = Narrative" http://new.livestream.com/oxuni/narrativeandproof.

[2]A list of pairs that I see as analogous appears at the end for reference.

integral for an example. His book *Calculus on manifolds* was a popularization of the derivative represented by the Jacobian matrix for an audience that may have heard only of ordinary and partial derivatives. Unlike the proofs of most theorems, definitions are the product of a sometimes slow evolution, much discussion, trial and error, and finally a temporary consensus that can be revised if reason demands. Two hundred years ago mathematicians thought they knew the meaning of convergence of a sequence of functions, and then they found they needed uniform convergence. Then the real numbers, then the integers. New definitions for old concepts—what Carnap called explication. I think that the agreed-upon definitions, which much work is done in terms of, are—as a class—given less attention than they should be.

The judicial analogue is the law. The rule of law is among the few formalities as old as mathematics itself. Laws that are that and not just the whim of the ruler are at least as old as Hammurabi in Mesopotamia, whose law code was on display in the summer of 2013 at the Royal Ontario Museum. That the ruler himself is subject to the laws of the land is one of the main achievements of the Magna Carta in thirteenth-century England. It is certain that some of the features I just recounted for mathematical definitions are also features of laws. They have certainly become progressively more complex, but they do evolve along with the societal consensus that they codify. Sometimes they don't work and have to be replaced. I read recently that the sale of some narcotics was criminalized in the USA only in 1922, and now at least the possession of marijuana is lawful again in some states. Prohibition in North America is another such example. One of the important ways in which laws and definitions are different arises from laws' being the products of nation states and their subordinate units, a difficulty that we are mercifully free of—except for legislatures that define mathematical constants. I wonder whether any but *pi* has ever been noticed by lawmakers.

Analogous to the arguments that are conducted in courtrooms are the arguments with which we try to convince the mathematical world that our results are correct. Fortunately our judges are our colleagues acting as referees for journals and then as readers of journals and listeners to our talks. Cross examination happens at the end of the talk when the one expert present, who has followed the argument, points out by tactfully asking a well-aimed question the error observed or the dubious piece of the argument. Sometimes there are appeals to higher courts. It was a while before the proof of Fermat's Last Theorem was approved of, and I believe that Hales's sphere-packing result is still under study.

Let us look a little more carefully at the process of convincing the mathematical community of the value and truth of our result. Folks are convinced by evidence. What evidence do we present? We either begin with a proof and study the premises and conclusion to find what it proves or we start with premises or conclusion and find a proof that takes us from them to it. Eventually, when we have done these things and are ourselves convinced, we state our result, which of course may be more than just one theorem. This is analogous to the claim of a plaintiff, which needs to be publicly tried before it is recognized. On the basis of our experience of going through the proof, which may easily be like a graph-theory graph in its structure, we write down a linear representation of it that is analogous to testimony at a trial. An example of this is illustrated in a recent paper of Arana (2015) quoting the diagram that Szemerédi supplied in his original proof (Szemerédi 1975) of his eponymous theorem; it was a diagram not of what he was writing about but of the structure of the proof. Frege's Begriffsschrift is so difficult to learn that almost no one has done so because its notation is sensibly (in a way) two-dimensional in its own way. We can also offer non-linear material like diagrams of what is being talked about, which we have to interpret for our audience. Our linear prose is analogous to testimony at the trial. The first thing that we typically do, however, is not to put it publicly on trial but to submit it first to one or more persons privately. Only when their judgement is positive do we take it elsewhere.

Testimony is interesting. Unlike fiction, it is talk that is meant to be believed as said because it is given in evidence. Evidence includes both stuff and testimony, but evidence that is stuff rather than testimony has to be interpreted by testimony to be meaningful as evidential. The murder weapon only matters if someone vouches for its being the murder weapon—if it is associated with both the victim and the murderer, quite possibly by different persons.

2 Testimony

Testimony is something that philosophers have written about. Their main concern is epistemological—whether it should be believed (Adler 2014; Green 2015). They do say things that are relevant to the mathematical analogue. For instance, that testimony transfers what is something else to "the level of things said" (Ricoeur 1972, p. 123); it gets things like the identity of the murder weapon into words where everyone can hear/look at it. In this respect it is a source of objectivity and a basis for judgement. The category of testimony makes literal sense only in the context of influencing a judicial decision. The language is useful by transfer to "situations less codified" (Ricoeur 1972, p. 124), one of which is history, where there is inherent uncertainty. There is a big argument going on in philosophy of history about whether what one might call professional history—the stuff written down—must abjure or embrace testimony. Fortunately in the mathematical context the uncertainty that makes our proof-testimony necessary is meant to be dispelled by it so that at the end there is no uncertainty left and judgement can be rendered easily and firmly. Testimony is meant to be persuasive. Just as justice is served by a correct decision, knowledge is enlarged by the successful proof of a mathematical result. On the other hand, just as there is false testimony there are proofs that are flawed.

When I was writing about the analogy of mathematics to fiction, I pointed out that mathematics is much more like history than like fiction because, however much history it made up in one sense by the historian, to be history it has to operate in recognition of the constraints by what happened in the past, however little may be known about that. If it does not recognize those constraints—and even sometimes when it does to some extent or other—it *is* fiction.

> I should want to stress that as 'fictive' as the historical text may be, its claim is to be a representation of reality. And its way of asserting its claim is to support it by the verificationist procedures proper to history as a science (Ricoeur 1983).

Distinguishing historical fiction from proper history is a problem that historians have.

Recently testimony has attracted attention from philosophers of information (Floridi 2014) and of mathematics (Geist et al. 2010).[3] There is no doubt much to be said about the place of testimony in mathematics. This paper's narrow focus is on testimony as a way of looking at the important category, published proofs. Geist et al. (2010) are mainly concerned with testimony as a way of looking at referee reports. The testimony of a referee, either for or against publication, is important but somewhat orthogonal to the acceptance of published work. Negative referee testimony, ranging from matters of taste to counterexamples, has the effect, if any, of preventing publication without significant improvement. Positive referee testimony simply disappears once its job is done. Geist et al. are concerned, not with the reception of mathematical work, but with the stage of being published. Referee testimonies or opinions are personal, fallible, and not as important as the community reception of work once published. They are also based on wildly varying amounts of study. The correspondent in the judicial analogy is, it seems to me, the advice of the plaintiff's legal team. Referees act as surrogates for the mathematical community in the decision to publish, just as one of the pre-trial functions of the lawyers is to help with the decision to go to court. Like the reviewing of manuscripts, this job can be done well or badly and is done on the basis of differing levels of expertise and study. Referees are often thought of as gatekeepers, a function that they do serve, but I think of them also as acting partly on behalf of the author in advising how work can be improved based on how it will strike the intended audience and whether the reputation of the author will be harmed or helped by publication. It would be a case of perverse incentives for a young scholar to publish something that in the short run helped to get a first job or tenure but affected long-term reputation adversely.

I said that the main interest of philosophers in testimony is in whether it should be believed, whether it produces knowledge in its hearer. Plato's strictures on knowledge make it particularly difficult to attain what philosophers are prepared to call "knowledge". This is obviously a much bigger problem to a judge of testimony in a courtroom or to a historian. Steps are taken in context to improve (I cannot say "ensure") veracity. In courts typically testimony is sworn or when the witness can't do that because of age or dim-wittedness, the importance of truth-telling is impressed on the witness. And the vogue for the so-called oral history is a way of singling out recollections from ordinary history, which is distinct in at least being cobbled together from as many recollections as are worth collecting. On the other hand, in mathematics the testimony of someone competent that has proved

[3] I am grateful to an anonymous referee for reminding me of this paper, of which I was nominally aware, having reviewed (in a weak sense, "made note of") the book in which it appears (Thomas 2012).

something says how it was done. Any similarly competent hearer ought to be able to reconstruct much the same experience from reading the testimony. In this it is distinct—in a thoroughly positive way—from eyewitness recollection, however sworn or amalgamated. Testimony in its normal meaning asks to be believed on the say-so of the witness, but testimony of mathematical experience invites the hearer to join in the experience, its intersubjectivity being the chief indicator of its truth and objectivity. To see its truth, it should not matter who you are. In his ground-breaking book *Testimony*, the Australian philosopher Coady (1992) quoted Russell (1927, p. 150) in this connection, "I mean here by 'objective' not anything metaphysical but merely 'agreeing with the testimony of others' ".

What some recent philosophers have said about testimony is both to observe that an enormous amount of what we know is dependent upon testimony and to attempt to justify this obvious fact despite Plato's discouragement. One way in which the world has changed since Plato's day is relevant. Twenty-five hundred years ago, one was personally dependent for a lot of what one learned first from one's elders and then from one's contemporaries, but that was as far as the dependency went. Now we have whole areas of life that are based on intellectual work over many years accessible only by testimony. Scientific research in particular, including mathematics, is just not feasible without all of the background knowledge built up over time, much of which one has learned from testimony. It is only in principle that one can replicate old results. Undergraduate experiments are a replication of only a few high spots in the history of science, and the same is true to a lesser extent of what one learns in mathematics. Poincaré and Hilbert were perhaps the last mathematical know-it-alls. We are lucky that mathematical knowledge among scientific knowledge is uniquely learnable that way.

One of the things that I think is interesting about this analogy is the different way that authority works in the two contexts. In litigation up to the point of reaching a verdict, the judge needs to be competent in law because he has to keep under control the advocates of the parties to the dispute, who need to be learned in the law because it is the framework within which the dispute is being settled—one of the reasons it is safer to settle disputes out of court. My barrister brother-in-law has said that a civil suit that goes to trial has at least one party that is making a mistake. In addition to those competent in the law, there are sometimes expert witnesses. Their competence pertains to the interpretation of evidence. The authorities cited by the lawyers will be law and precedents; authorities cited by the experts will be published scientific facts. None of this pertains to making the judgement on the case; it is all peripheral. If the trial is before a jury an important point is that the members of the jury do not need to be experts on anything. Compare this with the analogue. What authorities do we cite in our mathematical arguments? Postulates, definitions and previous mathematical results, nothing else being relevant. It is the jury that needs to be competent. Where personal authority comes into the mathematical scene is in the orthogonal judgements of the importance or depth or beauty of the result if true. Such judgements really are orthogonal, since one can judge a failed theorem to be important enough to continue pursuing a proof; the four-colour theorem had that status from Heawood's disproof of Kempe's attempt to the successful proof of Appel and Haken, a period of nearly a century.

One needs, in exploring such a comparison, to keep in mind that one is comparing two things that are different to see ways in which they are similar. Coady has been studying testimony for 40 years and appears to have put the topic on the philosophical map. As he pointed out, other philosophers have for some time studied the different sorts of thing that we do when we say or write words. "Asserting, testifying, objecting, and arguing all have the same or similar illocutionary points—roughly to inform an audience that something is the case ... (Coady 1992, p. 43, referring to works of John Searle 1979; Searle and Vanderveken 1985)." But there are distinctions. Testimony, which is not proof but is believed, is believed on the say-so of the witness.

> When we believe testimony we believe what is said because we trust the witness. This attitude of trust is very fundamental, but it is not blind. As (Eighteenth-century Scottish philosopher Thomas) Reid noted, the child begins with an attitude of complete trust in what it is told, and develops more critical attitudes as it matures. None the less, even for adults, the critical attitude is itself founded upon a general stance of trust, just as the adult awareness of the way memory plays us false rests upon a broader confidence in recollective powers. (Coady 1992, p. 46), citing (Reid 1764, VI, xxiv)

While our proofs are testimony to our having gone through the proof process, and the proof may be believed up to a point on our say-so, our testimony is also a challenge to every reader to go through the proof and be convinced for oneself. It is not normal for one to publish several persons' versions of one's proof for corroboration as several witnesses may be called in court to corroborate the testimony of the first of them. We do construct new proofs of old results but that is only occasionally to guarantee their truth. It is the reader that is called upon to corroborate a printed proof by first-hand experience. In Pollard's (2014) review of Hersh's (2014) *Experiencing Mathematics*, he quotes Hersh putting it this way.

> When Hardy (for example) makes a discovery, he explains how other mathematicians can verify his claim, by following a certain sequence of steps, to arrive at 'seeing it'. And those directions are 'the proof'!

Why is it that personal testimony is relevant when everyone agrees that mathematical proof is the prototype of objectivity? This question was suggested to me by a paper (Montaño 2012) on mathematical aesthetics by Ulianov Montaño. In order to discuss the beauty of a mathematical proof more satisfactorily than usual (McAllister 2005; Rota 1997), he draws a distinction between the proof as a *mathematical object*, of which a fully formalized version is the best expression, and the proof as an *intentional object*, the utterly subjective content of one's mind as one rehearses or contemplates the proof, almost always informal. Just as a picture is beautiful or not as *seen* and a piece of music is beautiful or not as *heard*, a proof is beautiful or not in one's mind, not as written down. It would be a category mistake

Fig. 1 Analogical pairs

Justice system	Mathematical publishing
Courts	Journals
Plaintiff	Author
Case	Paper
Suit	Theorem(s)
Law	Background mathematical knowledge
Non-verbal evidence	Non-verbal presentation
Testimony, argument	proof(s)
Judge/jury	Mathematical community
Legal team	Referees
Cross examination	Critical questions
Expert witnesses	—
Winning/losing	Acceptance/rejection
Appeal to higher court	Lengthy controversy
Justice	New public knowledge

to attribute beauty to a written formal proof as to a written musical score.[4] This distinction seems to me useful to describe how it is that personal testimony of the subjective experience of becoming convinced of something has probative value. That testimony, if expertly done for a proof that is actually valid, is a recipe, as Pollard and Hersh suggest, for sharing in the experience and the conviction.[5]

Testimony is talk that is taken as evidence. What of evidence that is not testimony? Is anything to be learned from that comparison? What is the mathematical-proof analogue of exhibits at a trial? It seems to me that it is whatever is not self-interpreting, mainly diagrams but also anything that is not prose, that cannot be read out in words. All ordinary speech is human communication and so *is* an interpretation, but a diagram or a formal proof does not interpret itself. Someone must tell us what a diagram or formal proof is *of* and show us what in it corresponds to what we are talking about. It must be interpreted to us or be embedded in a situation of which we know a standard interpretation. There has been some movement since Frege's invention of his Begriffsschrift in 1879 toward formality in proving—as distinct from rigor, which he did not invent. That has been instructive, but it is obviously a process that can go only so far because such formalities require interpretation to have meaning.

[4]At the end of my presentation it was pointed out to me by Michael Williams that there are those that find pleasure in reading computer code, an activity that seems to me comparable to reading musical scores without hearing the music—actually or virtually.

[5]The other comment made to me at the end of my presentation, by someone whose name I did not record, was that a feminist view of testimony explicitly considers the standpoint of the speaker. While in most circumstances this is an important feature of testimony, it does not seem to matter in the mathematical context because what matters is so much the recipe for having the appropriate experience oneself rather than the anything at all to do with the witness.

I am far from claiming that this is the only way to look at this matter, and I am newly enough come to it that I am not even sure that it is a good way, but perhaps it merits consideration. A summary of the analogy can be seen in Fig. 1. I apologize for the lack of philosophical conclusions.

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