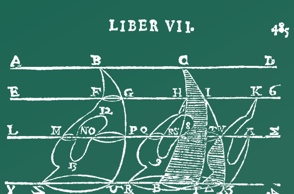
Proceedings of the Canadian Society for History and Philosophy of Mathematics
La Société Canadienne d'Histoire et de Philosophie des Mathématiques

Maria Zack
Elaine Landry
Editors

Research in History and Philosophy of Mathematics

The CSHPM 2014 Annual Meeting in St. Catharines, Ontario



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Research in History and Philosophy of Mathematics

The CSHPM 2014 Annual Meeting in St. Catharines, Ontario

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Introduction

This volume contains 13 papers that were presented at the 2014 Annual Meeting of the Canadian Society for History and Philosophy of Mathematics. The meeting was held on the campus of Brock University in lovely St. Catharines, Ontario, Canada in May 2014. The chapters in the book are arranged in roughly chronological order and contain an interesting variety of modern scholarship in both the history and philosophy of mathematics.

In the chapter "Falconer's Cryptology," Jr. Charles F. Rocca describes the contents of John Falconer's *Cryptomenysis Patefacta* (1685). Falconer's book is one of the very early English language texts written on cryptology and the mathematics underlying Falconer's ciphers is quite interesting. The chapter "Is Mathematics to Be Useful? The Case of de la Hire, Fontenelle, and the Epicycloid" contains Christopher Baltus' discussion of the 1694 work of Philippe de la Hire on the epicycloid. Baltus examines la Hire's mathematics together with some seventeenth century views on the relationship between science and mathematics.

The chapters "The Rise of "the Mathematicals": Placing Maths into the Hands of Practitioners—The Invention and Popularization of Sectors and Scales" and "Early Modern Computation on Sectors" focus on some physical tools used by mathematicians in the seventeenth century. In the chapter "The Rise of "the Mathematicals": Joel S. Silverberg discusses the invention and popularization of mathematical devices called sectors and scales". These carefully crafted instruments facilitated the rapid expansion of sophisticated mathematical problem solving among craftsmen and practitioners in areas as diverse as navigation, surveying, cartography, military engineering, astronomy, and the design of sundials. In the chapter "Early Modern Computation on Sectors," Amy Ackerberg-Hastings uses 27 sectors in the mathematics collection of Smithsonian's National Museum of American History to trace the history of the sector in the seventeenth century Italy, France, and England.

In the chapter "The Eighteenth-Century Origins of the Concept of Mixed-Strategy Equilibrium in Game Theory," Nicolas Fillion examines the circumstances surrounding the first historical appearance of the game-theoretical concept of mixed-strategy equilibrium. What is particularly intriguing is that this technique, commonly associated with twentieth century mathematics, actually originated in the eighteenth century. In the chapter "Reassembling Humpty Dumpty: Putting George Washington's Cyphering Manuscript Back Together Again," Theodore J. Crackel, V. Frederick Rickey, and Joel S. Silverberg discuss the mathematical cyphering books of America's first president George Washington. This paper discusses the provenance of the Washington manuscript and the detective work done by the authors to locate some of the cyphering book's missing pages in other archival collections.

In the chapter "Natures of Curved Lines in the Early Modern Period and the Emergence of the Transcendental," Bruce J. Petrie examines the role of Euler and other mathematicians in the development of algebraic analysis. Euler's *Introductio in analysin infinitorum* (*Introduction to Analysis of the Infinite*, 1748) was part of a body of literature that developed the tools necessary for uncoupling the study of curves from geometry, greatly increasing the number of curves which can be understood and analyzed using functions and functional notation. This paper looks at the development of this uncoupling.

In the chapter "Origins of the Venn Diagram," Deborah Bennett examines the development of what we know today as the Venn diagram. Several mathematicians including Euler and Leibniz used drawings to illustrate logical arguments, and based on this work, the nineteenth century mathematician John Venn ingeniously altered what he called "Euler circles" to become the diagrams that are familiar to us today. In the chapter "Mathematics for the World: Publishing Mathematics and the International Book Trade, Macmillan and Co.," Sylvia M. Nickerson expands our knowledge of the nineteenth century mathematical community by carefully examining the influence that publishers had in developing mathematical pedagogy through the selection and printing of textbooks. This article studies the well-known publisher Macmillan and Company.

The next two chapters look at some interesting aspects of mathematics on the cusp of the twentieth century. In the chapter "The Influence Arthur Cayley and Alfred Kempe on Charles Peirce's Diagrammatic Logic," Francine F. Abeles provides information about the influence that Arthur Cayley and Alfred Kempe had on Charles Peirce's diagrammatic logic. This chapter is a combination of historical information with a carefully annotated bibliography of material found in archival collections. In the chapter "Émile Borel et Henri Lebesgue: HPM," Roger Godard looks at the relationships between Émile Borel's *Les fonctions de variables réelles et les développements en séries de polynômes* (*Functions of Real Variables and Expansions as Polynomial Series*, 1905) and Henri Lebesgue's *Leçons sur les séries trigonométriques* (*Lessons on Trigonometric Series*, 1906) in light of some correspondence between the two mathematicians. Godard says that he wrote this article in French "to reflect the Paris atmosphere at the beginning of the XXth century."

The last two chapters in this volume discuss twentieth century mathematics. In the chapter "The Judicial Analogy for Mathematical Publication," Robert S.D. Thomas examines mathematical analogies using a specific example. Thomas' analogy compares how the mathematical community accepts a new result put forward by a mathematician with the proceedings in a court of law trying a civil suit that leads to a verdict. In the chapter “History and Philosophy of Mathematics at the 1924 International Mathematical Congress in Toronto,” David Orenstein describes the International Mathematical Congress of 1924 held in Toronto, which was organized by J.C. Fields. This paper takes the form of a “narrated slide show” of the event using information from a number of artifacts to give the reader a feel for how the meeting progressed.

This collection of papers contains several gems from the history and philosophy of mathematics, which will be enjoyed by a wide mathematical audience. This collection was a pleasure to assemble and contains something of interest for everyone.

San Diego, CA, USA Maria Zack
Davis, CA, USA Elaine Landry

Editorial Board

The editors wish to thank the following people who served on the editorial board for this volume:

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Falconer's Cryptology

C.F. Rocca Jr.

Abstract *Cryptomenysis Patefacta* by John Falconer is only the second text written in English on the subject of cryptology. We will examine what types of ciphers Falconer addressed, and pay particular attention to some of the math he used. We will also look at what can or can't be said about John Falconer himself.

1 Introduction

In *The Codebreakers* (Kahn 1996, pp. 155–156) David Kahn states that *Cryptomenysis Patefacta; Or, the Art of Secret Information Disclosed Without a Key. Containing, Plain and Demonstrative Rules, for Decyphering*, written by John Falconer and first published in 1685, was only the second text printed in English on the subject of cryptology. Kahn goes on to say that Falconer had made a

> . . . praiseworthy assault on that old bugbear polyalphabetic substitution.

and had given the

> . . . earliest illustration of a keyed columnar transposition cipher . . .

In the next paragraph Khan states that the texts on cryptology from this time period "have a certain air of unreality about them" and that the "authors did not know the real cryptology being practiced." But, he seems willing to exclude John Falconer from this comment. It should be noted, however, that in his text Falconer never actually takes credit for deciphering any particular cipher of any importance.

Falconer's work was significant enough to still be read or at least referenced over the next century and a half, though occasionally with criticism not praise. For example, William Smith referenced Falconer's work in *A Natural History of Nevis, and the Rest of the English Leeward Charibee Islands in America* (Smith 1745 p. 253) published in 1745, stating that he had considered republishing the text as it had become rare and difficult to find. Later in 1772, Philip Thicknese in *A Treatise on the Art of Decyphering, and of Writing in Cypher: With an Harmonic Alphabet*

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(Thicknesse 1772) referenced Falconer in a number of places, later still Falconer and *Cryptomenysis* are listed with other authors and works in the entry on ciphers in the 1819 printing of the reference *Pantologia: a new cabinet cyclopaedia*... (Good et al. 1819). He was even referenced, along with other works, by H.P. Lovecraft in 1928 in "The Dunwich Horror." (Lovecraft 2011, p. 258). Thus, while not perhaps as significant and well known as other works and authors, Falconer and *Cryptomenysis* seem to have been widely enough read and referenced that both Falconer and his work are worthy of closer examination.

2 Dating Falconer's Work

The publication date in Falconer's text is 1685, with a second printing in 1692. Therefore, we know when the text was published, but not necessarily when it was written. In *The Codebreakers* (Kahn 1996, pp. 155–156) Khan states that *Cryptomenysis* came out posthumously in 1685, after Falconer had followed King James II into exile in France where he died. The implication seems to be that Falconer wrote the text before, perhaps long before, it was published.

James Stuart (1633–1701), Duke of York and later King James II of England, went into exile for the first time from about 1648 to 1660. During this time he did serve with the French Army. His second period of exile occurred between 1679 and 1681 after he was accused, due to his Catholicism, of being part of a popish plot to assassinate his brother Charles; a plot that never in fact existed. However, this exile was largely in Brussels and Edinburgh, not France. Finally, in 1688 James II, who had now been king for 3 years, was overthrown in the Glorious Revolution and went into exile in France for good in 1689.

So if Falconer's work was published posthumously and he died in exile in France with James, he must have written his work prior to or during James's earlier exile, 1648–1660. However, there are some issues in accepting this date range for when Falconer could have written *Cryptomenysis* (many of which are pointed out in by Tomokiyo on his website (Tomokiyo 2014) and which we discuss below). In particular the first edition of the text is addressed to Charles Earl of Middleton and secretary of state for King James II; this is an office Charles assumed in 1684. This would seem to imply almost immediately that the work must not have been written long before its publication. However, it could be the case that this preface was not written by Falconer but added by the publisher to appeal to the current ruler, so let us proceed.

2.1 References Within Cryptomenysis

Within his text, John Falconer references a wide variety of other works on cryptology. Two repeatedly referenced works are by John Wilkins and Gaspar Schott. When Falconer refers to Wilkins' writing it is generally to disparage it and to point out ways in which the proposed ciphers could be easily broken. Wilkins' work (Wilkins 1694) was originally published anonymously in 1641, thus references to Wilkin's work do not contradict the possibility that Falconer wrote his work in the 1640's or 1650's. However, when Falconer refers to Wilkins himself it is as Bishop Wilkins, and John Wilkins (1614–1672) did not become Bishop of Chester until 1668. Falconer also refers in many places, and with more respect, to Gaspar Schott's *Magia universalis* (Schott 1657) and his *Schola steganographica* (Schott 1665) published in 1665. Thus these references push the date of writing to 1668 or 1669 at the earliest.

2.2 Motivation for Cryptomenysis

In discussing his motivation for writing his work Falconer expresses his regrets that it is necessary for men to keep secrets, but that it is the case that princes and kings may have need of secrecy. To highlight this point he refers to the recent rebellion by the late Earl of Argyll as described in "An Account of Discoveries Made in Scotland" which was written by George Mackenzie and published in 1685.

Archibald Campbell, the ninth Earl of Argyll, lived from 1629 to 1685. He rebelled against the government/crown of England twice, once in 1681 and again in 1685. The first time he was able to escape to exile, but the second time, when he returned with the intent of overthrowing King James II in favor of the Duke of Monmouth, he was captured and executed. One of the reasons that Falconer refers specifically to this rebellion is because among the Earl's letters were some which were enciphered and had proved difficult to crack.[1] Thus we see the motivation for Falconer's emphasis on analysis as well as encipherment. In fact, there are no less than eight references to the Earl of Argyll and his rebellion against the crown.

Given the frequency with which the late Earl of Argyll is mentioned, it seems unlikely that the text was written prior to 1685. In fact, since Argyll was executed in June of 1685 and Falconer frequently refers to him as the *late* Earl of Argyll it is tempting to place the writing of, or at least completion of, *Cryptomenysis* in the second half of that year. However, in an issue of the London Gazette from May 1685 (Gazette 1685, p. 2) Archibald Campbell is also referred to as the late Earl of Argyll so it is possible that Falconer was referring to the fact that he was no longer Earl, not that he was dead.

While it might have been possible, or even likely, that a later editor or compiler would have changed one or two references, or could have written a new introduction it seems unlikely that they would have rewritten large portions of the text. Based both on the references and the stated motivation in the text it seems likely that it was written not long before it was published. Thus, we can narrow down the writing of

[1] The Earl's letters were decrypted by a Mr. Gray of Crichie, later Lord Gray.

the text to between 1681 (if a new introduction was written) and the end of 1685. But what can we say about Falconer's life and the claim that the work was published posthumously?

2.3 Falconer's Biography

In *The Codebreakers* we find very little information about Falconer and when he seems to have lived (Kahn 1996, pp. 155–156). The source for what little information there is comes from a text in the New York public library labelled *Falconer's Writings* published in 1866 and written by Thomas Falconer. In the appendix to *The Codebreakers* it states that Falconer is not listed in any of standard histories concerning King James including *The Memoirs of James II: His Campaigns as Duke of York*. Neither is Falconer mentioned in either volume of the *Parochial Register of Saint Germain-en-Laye: Jacobite Extracts of Births, Marriages, and Deaths* (Lart 1910) which would seem to indicate that if he did go into exile with James in 1689 then he was not a central member of James's court. Also, in a search of names indexed in Charles Middleton's papers in the British Library and National Archives, Falconer is not listed. So it would seem that John Falconer will be a hard man to get a hold of; let us begin to try by looking at the writings of Thomas Falconer.

Thomas Falconer is apparently a descendent of John Falconer, a great great grandson to be precise. In the mid-nineteenth-century Thomas wrote a number of texts concerning his ancestry, these include (Falconer 1860, 1870) which are largely genealogical, containing general notes, baptismal records, some wills, and records of what is written on monuments. Thomas also compiled a bibliography of his family's writings (Falconer 1866) which is one of the sources listed for many of the Falconer entries in some volumes of the Dictionary of National Biography (Stephen and Lee 1889). In (Falconer 1860, p. 59 in the e-copy), we find the quote:

> This John Falconer was entrusted with the private cypher of James II whom he followed to France where be died

which is the information we had looked at previously. However, Thomas goes on to tell us that John was married to Mary Dalmahoy (1663–1754) on February 14th 1681. Thomas also tells us of Mary and John's children, in particular their son William, Thomas's great grandfather, who became the recorder of Chester and died in 1764 at the age of 65. Curiously, only dates of death and ages are listed for three of Mary and John's four sons. Their first son, born shortly after they married, died young. The other sons, based on the information given, seem to have been all born after 1690 with William, the youngest, born around 1699. Thus, based on the information given to us by Thomas, assuming his facts are accurate, it is impossible for John Falconer to have died prior to the publication of either edition of his text. However, given the lack of birth records for his sons it is possible that they were not born in Scotland or England but in exile with their father in France (though clearly this is not the only possibility).

What is missing from Thomas Falconer's works is any specific reference to John Falconer's parentage. In (Falconer 1860) he lists the baptismal records for several John Falconers, two or three of whom may be the one we are interested in. Based on what we know so far, his marriage date and the dates for his son's lives, the most likely John Falconer that Thomas lists was born to John Falconer and Elizabeth Cant on February 15, 1650. However, if we try to verify that this is the correct John (or Johne as he is sometimes listed) through a database such as Ancestry.com, we are stymied. There are records available for John and Mary's wedding and for the deaths of some of their sons and for Mary (but not John). There are records of a John being born to John Falconer and Elizabeth Cant in 1650, but there is nothing to connect the two. In fact a search for birth and baptismal records for John Falconer in Scotland in the mid to late seventeenth century returns several hundred hits. So, the question becomes what can we know and verify beyond what Thomas wrote down nearly two hundred years later?

For starters the monument inscriptions that Thomas quotes, at least for John and Mary's sons Thomas and William, can still be viewed today. Also, Thomas's comments about John's writings are identical to what was said about him in an obituary for another Thomas Falconer from April 1839 (Pickering 1839, p. 436) and one for a William Falconer in 1825 (Longman et al. 1825, p. 413). It is possible that these were written by Thomas himself, or that he has the same source (likely someone in the family) as the obituaries, but it is still worth noting that they predate his work by over 30 years. Finally, searching for official government records we find Edinburgh poll tax records from 1694–1699, in these is a record of a John Falconer and Mary Dalmahoy living in the same location at that time (Poll Taxes 2014). This last piece of evidence is one of the most interesting since it was not mentioned by Thomas Falconer in (Falconer 1860, 1866, 1870) and places John in Scotland at least 2 years after the second printing of *Cryptomenysis* and 5 years after James II went into exile.

Given the evidence at hand, if it is accurate, we may conclude that *Cryptomenysis* was likely written between 1681 and 1685, more likely in 1685. John Falconer definitely seems to have been alive at the time the work was published. And, it is not immediately clear that he went into exile in France and died there (though there is a general lack of evidence on his life).

3 Contents of *Cryptomenysis*

3.1 Overview

Falciner's text is broken into the five chapters:

- Chapter I: Of Secret Writing and the Resolution Thereof
- Chapter II: Of Secret Information by Signs and Gestures and its Resolution
- Chapter III: Of Cryptology or the Secrecy Consisting in Speech

- Chapter IV: Of Secret Means for Conveying Written Messages
- Chapter V: Of Several Proposals for Secret Informations Mentioned by Trithemius in His Epistle to Arnoldue Bostius, & c.

The first four chapters cover various aspects of cryptology while the fifth is a discussion/defense of the work of Johannes Trithemius, a sixteenth century clergymen and cryptologist who had been accused of dabbling in the occult. Our focus will be on chapter one. We will look at what he covers and how he covers it, with special attention to some of the mathematics he touches on.

Chapter one is divided into six sections and in these Falconer covers all of the following methods of conveying secret messages:

- Basic Monoalphabetic Ciphers
- Polyalphabetic Ciphers
- Keyed Columner Transpositions (by word and by character)
- Hiding the intended message inside an innocuous message
- Transcribing text into a binary or trinary alphabet
- Transcribing text into multiple fonts
- Omnia Per Omnia (where any cipher text may represent any text)
- Shorthand
- Scytale and similar
- Secret inks and such

For each he is careful to, where appropriate, give a reference for where he gathered his information. Also, unlike some earlier authors, he is very careful to give examples of how one might attempt to break the ciphers he is presenting. Further, he takes the time to explain even some of the basic mathematics that he employs with respect to either enciphering or deciphering.

3.2 Specific Examples

Falconer discusses the use of permutations, prime factorizations, and the multiplication principle. Permutations (which he calls combinations) first appear when he calculates the number of possible keys which could be used for a monoalphabetic substitution cipher. He correctly gives this number as 620448401733239439360000 (which is 24!, since he was using to the 24 character Latin alphabet of the time) and references *Schola Steganographia* and *Magia Universalus* by Gaspar Schott ($\approx$1665) as his source for this. However, Falconer then proceeds to give the reader a way to try and understand exactly how large this number is:

> For if one writer in one day write forty pages, every one containing forty combinations, 40 multiplied by 40, gives 1600, the number he completes in one day, which multiplied by 366, the number of days (and more) in a year; a writer in one year shall compass 585600

> distinct rows. Therefore in a thousand million years he could write 585600000000000, which being again multiplied by 1000000000, the number of writers supposed, the product will be 585600000000000000000000, which wants the number of combinations no less than 348484017332394393600000. (Falconer 1685, p. 5)

Later in his text Falconer carefully discusses permutations of letters for a given number of letters and he discusses prime factorizations both of which he then proceeds to use in his exploration of keyed-columnar transposition ciphers.

He begins by giving the number of ways we can arrange any number of letters into different permutations:

Letters		Several ways
1	May be combined	1
2		2
3		6
4		24
5		120
6		720
7		5040
8		40320
9		362880
10		3628800
11		39916800
12		479001600
&c.		

However, he does not stop at just providing the table. Falconer spends three pages carefully walking us through how to generate each new row from the last. For example once we know that there are two combinations of two letters, *AB* and *BA*, he observes that:

> From the Combination of two Letters we find that of 3, for the new Letter added is three times applicable to the former Positions, viz. in the beginning, middle, and end... (Falconer 1685, p. 39)

And, thus we get *CAB*, *ACB*, *ABC*, *CBA*, *BCA*, and *BAC*, which are the six possible combinations of these three letters. He proceeds to describe this process for four and five letters as well.

Once we know all the ways to arrange letters he introduces *"A new Method how to Write Secretly by the Art of Combinations."* First we pick some number of letters, say three, and then some subset of all the possible combinations of those letters. We set these out in a table as follows:

Order of positions		A	B	C
1	CBA			
2	CAB			
3	ACB			
4	BCA			
5	BAC			

so that the combinations of letters at the side indicate the order in which we should fill the columns of each row. To send a message such as *"The quick brown fox jumps over the lazy sleeping dog,"*[2] we fill the table from top to bottom writing the letters one at a time in the order indicated by the combination key for each row. So, in the first row we write down the *"The"* with the *"T"* under the C, the *"h"* under the B, and the *"e"* under the A, because the key for that row is CBA.

Order of positions		A	B	C
1	CBA	E	H	T
2	CAB			
3	ACB			
4	BCA			
5	BAC			

In the next row we put down the *"qui"* with the *"q"* under the C, the *"u"* under the A, and the *"i"* under the B, because CAB is the key for the second row.

Order of positions		A	B	C
1	CBA	E	H	T
2	CAB	U	I	Q
3	ACB			
4	BCA			
5	BAC			

The remaining *"ck"* from *"quick"* and the *"b"* from *"brown"* are placed in the third row in columns A then C then B since the key there is ACB.

[2]This is not Falconer's example but a shorter one which was chosen for demonstration purposes.

Order of positions		A	B	C
1	CBA	E	H	T
2	CAB	U	I	Q
3	ACB	C	B	K
4	BCA			
5	BAC			

We proceed row by row in this manner until all the rows are filled.

Order of positions		A	B	C
1	CBA	E	H	T
2	CAB	U	I	Q
3	ACB	C	B	K
4	BCA	W	R	O
5	BAC	F	N	O

Once each row is filled we return to the top and start the process over with the remaining message. So, the remaining *"x"* from *"fox"* will go in row one column C and the *"ju"* from *"jumps"* goes in columns B and then A.

Order of positions		A	B	C
1	CBA	EU	HJ	TX
2	CAB	U	I	Q
3	ACB	C	B	K
4	BCA	W	R	O
5	BAC	F	N	O

This process of writing down the message letter by letter in the rows according to the order given by the key for each row proceeds until the message is completely copied down.

Order of positions		A	B	C
1	CBA	EUS	HJY	TXZ
2	CAB	UPE	ISE	QML
3	ACB	COP	BEN	KVI
4	BCA	WHO	RRG	OTD
5	BAC	FL	NEG	OA

The enciphered message is then written out from left to right and top to bottom as follows:

△ ĖUS HJẎ TẊZ UPE IṠE QML COP BEN K̇VI WHO RṘĠ OTD FL ṄĖĠ OA

where the triangle is to tell your compatriot how many letters were in your combinations and the dots indicate terminal letters in words. He remarks that these markings aid in decipherment, which he demonstrates, but then he also shows us how we may decrypt such a message without the markings.

If we do not know the number of letters in the key (i.e., the number of columns), but we do know the method of encipherment, we may make an educated guess by examining the divisors of the number of groups of letters. To aid in this Falconer gives a complete process for finding all the divisors of a number. Supposing there are 450 groups of letters, he begins much as we might today by finding the prime divisors, though he uses a table instead of a factor tree. Each time we divide by a prime factor we write the prime underneath the number we are currently factoring and write the result of the division at the top of the next column. In this way factoring 450 gives us this table.

450	225	75	25	5	1
2	3	3	5	5	

He then walks us through a very nice way of delineating all of the possible factors of the number we are interested in, not just the prime factors. He begins by making a new table and writing out all the prime factors with repetition as the headers of the columns. Then he tells us to, starting with the second column, multiply the header for each column by all the numbers to its left by which it has not already been multiplied and write this down in that column. Under the 3 at the top of the second column we place a 6 since $3 \times 2 = 6$, then under the 3 at the top of the third column we place a 9 for $3 \cdot 3$ and an 18 for 3×6 but not a 6 for 3×2 because we already wrote that product down in the previous column. Working from left to right this process will give us all the possible factors:

2	3	3	5	5

→

2	3	3	5	5
	6	9		
		18		

→

2	3	3	5	5
	6	9	10	25
		18	15	50
			30	75
			45	150
			90	225
				450

Since we supposed that there were 450 groups of letters, the number of letters in the key for the cipher must be one of these factors.

In our example there are 15 letter groups so the factors are:

3	5
	15

Assuming it would be pointless to have a 1 letter key and silly to have a 15 letter key, there must have been either 3 or 5 letters in the key.

Curiously, after going through all this trouble to find potential key lengths Falconer uses a completely different method to crack the cipher. He writes each letter group from the enciphered message

△ ĖUS HJẎ TẊZ UPE IṠE QML COP BEN K̇VI WHO RṘĠ OTD FL ṄĖĠ OA

vertically so that we get the following array:

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	E	H	T	U	I	Q	C	B	K	W	R	O	F	N	O
2	U	J	X	P	S	M	O	E	V	H	R	T	L	E	A
3	S	Y	Z	E	E	L	P	N	I	O	G	D		G	

He does this because he recognizes that the first letter of each group (now all in the top row) were the first letters written down. Likewise those in the second row were written down on the second pass through the array and the third row were written down on the third pass through (which explains the blanks). Now we can treat each row as an anagram. When we find a word we can rearrange the columns that the letters of the word are in and we should start to see words under the ones we found. Eventually all these should hopefully give us the message. In our example in the first row we see the *E*, *H*, and *T* that could be the word *THE*. There is also a *Q* that should be with a *U*. Arranging the columns appropriately we get:

	3	2	1	6	4	5	7	8	9	10	11	12	13	14	15
1	T	H	E	Q	U	I	C	B	K	W	R	O	F	N	O
2	X	J	U	M	P	S	O	E	V	H	R	T	L	E	A
3	Z	Y	S	L	E	E	P	N	I	O	G	D		G	

The presence of the word *JUMPS* in the second row and *SLEEP* in the third assures us that we are on the right track. We can also see that the second word in the first row should probably be *QUICK*. Proceeding in this way we complete the message:

	3	2	1	6	4	5	7	9	8	11	12	10	14	13	15
1	T	H	E	Q	U	I	C	K	B	R	O	W	N	F	O
2	X	J	U	M	P	S	O	V	E	R	T	H	E	L	A
3	Z	Y	S	L	E	E	P	I	N	G	D	O	G		

This example is typical of all of Falconer's explanations. He carefully explains and demonstrates not only how to encipher a message but also how to decipher and, if need be, how to decrypt messages. Along the way he always tries to demonstrate to the reader the whys and hows of the work he is doing.

4 Conclusion

John Falconer's *Cryptomenysis* is an interesting study in the history of cryptology for a variety of reasons. Though he was an amateur[3] he still presented a broad view of cryptology including both enciphering and decrypting. His exposition is clear and generally thorough, covering background material, such as the mathematics discussed earlier, when needed. Also, given his stated motivations for undertaking this project and his attitude toward the ninth Earl of Argyll (who he clearly considered a traitor) and Bishop John Wilkins (whose work he did not seem to respect) his text presents some interesting connections to the history of the time.

In this article I focused on only the first chapter of *Cryptomenysis* and on John Falconer's life, therefore there is still plenty of material left to investigate. In particular, there is still little known about John Falconer himself; any further information about him will likely need to come from the papers of those around him; Charles Middleton to whom he addressed his text, his children or grandchildren who seem to have gone on to have successful lives, his very long lived wife, or her family. His numerous reference, both cryptologic and historic, offer various avenues for further study.

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[3]As opposed to individuals like John Wallis, Thomas Phelippes, and the Rossignol family who had been directly employed by royalty in England and France for their skills.

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Is Mathematics to Be Useful? The Case of de la Hire, Fontenelle, and the Epicycloid

Christopher Baltus

Abstract The epicycloid is the path of a point on a circle rolling on another circle. Philippe de la Hire (1640–1718) developed mathematical properties of the epicycloid in a 1694 work. Further, according to Bernard de Fontenelle's *Eloge de M. de la Hire*, where the shape of gear teeth had earlier been "abandoned to the fantasies of workmen," M. de la Hire showed "that these teeth, in order to have all the perfection possible, should be in the form of an arc of the epicycloid." However, despite words praising the utility of mathematics, La Hire's work itself suggests a mathematician with a solution in search of a problem as much as the reverse motivation. La Hire's mathematics is examined, together with the views of Fontenelle and de la Hire on the role of science and mathematics.

1 Introduction

Most of what is known of the life of Philippe de la Hire, born in 1640 at Paris, is from the *Eloge* of Bernard de Fontenelle, written in 1718, after the death of La Hire. Philippe was the son of a painter and seemed destined for the same profession. But even in his teens, geometric aspects of painting, such as perspective, intrigued him. When he travelled to Italy in 1660, both to advance his art and to heal his fragile health, he fell in love with the country and with Greek geometry. He particularly took to the *Conics* of Apollonius. Fontenelle reported, "Geometry began to prevail with him, although dressed in this thorny and frightening form which it took on in the books of the ancients." (Fontenelle 1718, p. 77) Returning to France after 4 years, he worked with engraver Abraham Bosse on some conic section problems that arose in applying principles from Desargues (died 1661) to a treatise on stone cutting. La Hire's first publications were in the 1670's, on conics and projective geometry, but with a short treatise, dated 1676, that began with four pages on

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the cycloid. That treatise included a new derivation of the area under the cycloid, in which thin triangles fill, in the limit, a regions bounded below by the cycloid. La Hire ended the argument saying, "There would be no difficulty in producing the demonstration in the manner of the ancients." (La Hire 1676)

La Hire entered the Academy of Sciences in 1678, and the next year began his long career of public service on a mapping expedition to Brittany.

We see Fontenelle's point of view in his *Eloge*. He described in detail La Hire's career of public works, including the mapping of France, the production of astronomical tables, and the earth leveling for an aqueduct which carried water 25 leagues (about 100 km) to Versailles. La Hire's remarkable work in the projective study of conic sections merited two paragraphs; we see the clarity of La Hire compared favorably to the obscurity of Descartes's geometry. Nearly as much space is devoted to La Hire's *Traité de Gnomonique*, a 1682 work expanded in 1698. Fontenelle's comments on the use of gnomons are indicative of his ideas on mathematics and science. While that science had most often been left to simple workers,

> M. de la Hire clarified *gnomonique* by principles and demonstrations, and reduced it to the most pure and simple operations, and to not change its ancient state he took care to publish demonstrations in a character different from that of the operations, so simple workers could easily discard that which did not accommodate them, for science must manage ignorance. . . .

Before we turn to Fontenelle's report on the epicycloid, we survey the mathematical and mechanical principles involved.

2 Background in Mathematics and Mechanics

An epicycloid is the path of a point on the circumference of a circle—the *generator circle*, which rolls on another, the *base circle.* In Fig. 1 left, the path *GHM* is the epicycloid traced by point *H*.

When the base circle is stretched out to a line, as in Fig. 1 right, the figure generated—path *AEB*—is the cycloid. The name *cycloid*, and the study of its properties, began with Galileo (Whitman 1943, p. 310). Mersenne brought it to the attention of French mathematicians. In 1638, Gilles Persone de Roberval, Descartes, and Fermat each gave a construction of the tangent; Roberval had found the area of the figure around 1634. A printed version of Roberval's work appeared only in 1693 (Roberval 1693).

Roberval gave a simple derivation of the tangent to the cycloid. He observed that the cycloid resulted from two composed movements, that of the center of the generator circle and the rotation of a circle about that center. By the nature of the curve, both movements occur with the same speed: when a point on the circle moves through an arc of length s with respect to the center of the circle, the point of contact of the circle with the base line moves the same distance s, and so the center moves

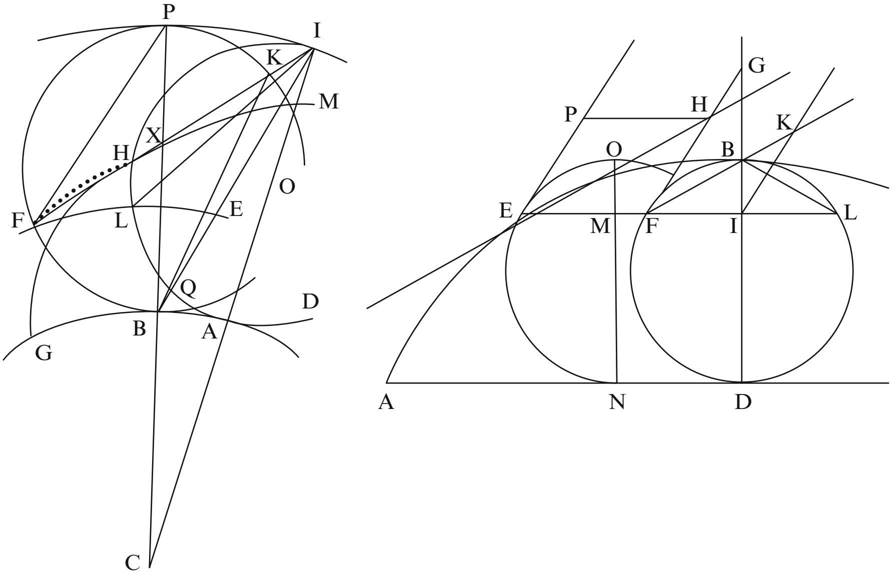

Fig. 1 *Left*: La Hire (1694). Tangent to the epicycloid at H. *Right*: Roberval (1693). Tangent to the cycloid at E

that same distance s. So the tangent at a point E on the cycloid is the diagonal of a parallelogram with equal sides, one horizontal—EF in Fig. 1 right (1693), and the other, FH, tangent to the circle on B, the vertex of the cycloid. (Note that FB is parallel to the tangent.) As might be expected, arguments applied in the case of the cycloid could be extended to the epicycloid. Here, Roberval's argument transfers directly to the epicycloid: in Fig. 1 left, the tangent at H meets the generating circle at I, where I is opposite the point of tangency, A, of the circle on A, H, and I. La Hire's derivation of the tangent, based on Fig. 1 left, 1694, is long and detailed, following the methods of ancient Greek mathematics. It occupies a couple pages in La Hire (1694).

It is worth noting that Roberval's quadrature of the region under the cycloid was extended to the epicycloid by Philippe de la Hire, although several lemmas, with involved proofs, are needed in the transition. Again looking at the 1693 publication, in Fig. 2, of ideas developed much earlier, Roberval found the area under the half-cycloid AKB by introducing a "companion curve" $AOLB$. Point O is found this way: From K on the cycloid draw a parallel to the base, AD, meeting the diameter DB at M and the circle with that diameter at L; then mark O so KO equals LM. We see that the area of the region between the cycloid and the companion curve is equal to that of the half-circle DFB, by Cavalieri's Principle. Further, when the generating circle traces point K, O lies on that diameter of the generating circle which is perpendicular to AD, as the generating circle is just a translation by distance MO of circle DFB.

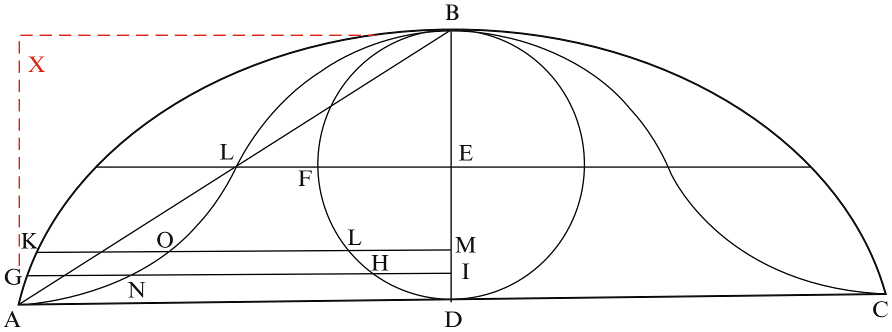

Fig. 2 Area under the half-Cycloid *AKB*, vertex at *B* (Roberval 1693)

Again, as Roberval suggests, if the movement of point *B* from *A* to the vertex of the cycloid is considered as composed of two movements, we see that the companion curve is symmetric about its center-point, *L*.

So the companion curve divides rectangle *AXBD* into two equal areas. When *DB* is $2r$, rectangle *AXBD* has area $2\pi r^2$. It follows that the half-cycloid has area $1.5\pi r^2$.

La Hire adapted Roberval's argument. The companion curve is now *DNOPd*, in Fig. 3. Arc 7*B* equals arc *O* 16. Thus by a Cavalieri's Principle for arcs, half the generating circle equals, in area, the region between the epicycloid and the companion curve. That the companion curve divides region *ADXd* into two equal parts is based on several lemmas which show that the regions formed by radii from *Y* and arcs come in pairs of equal area, such as the shaded regions.

That the epicycloid might serve in the design of gear teeth is less exotic a concept than it might at first seem, given the tangent to the epicycloid and a principle of mechanics. The design appeared at least through 1900 in mechanical engineering texts (Schwamb 1908). The idea did not originate with La Hire. La Hire mentioned that Christiaan Huygens (La Hire 1694, p. ii) independently made the discovery in the 1670s. But La Hire made the first thorough study of the epicycloid, including application of its properties to gear teeth.

Figure 4 accompanies La Hire's mechanical argument that the epicycloid is the correct shape for teeth of two engaged gears. His second argument is based on the principle of the balance. Let the base circles of the gears meet at *D*, while the teeth meet at *E* along an epicycloid *BE* formed with the right circle as the base circle. (The first argument involves forces and makes use of a differential triangle.) We may assume that *HAM* and *BKC* are horizontal. By the property of the tangent to the epicycloid, *NE* is also parallel to *HAM*. Weights *X*, *Z*, and *Y* hang vertically, *Z* on line *KDE*. The moments are provided by weights *X* and *Y*. Now $AM \cdot Z = AH \cdot X$ and $CK \cdot Z = CB \cdot Y$. $AH = AD = AE$. By similar triangles, $\frac{AE}{AM} = \frac{CB}{CK}$. It follows that $X = Y$. To have equal and opposite moments at *E* of weight *Z*, then equal and opposite forces result at *D* (Fig. 4). [See also Schwamb (1908, p. 193), for a clearer argument.]

Figure 5 provides an illustration of La Hire's epicycloid-shaped gear teeth.

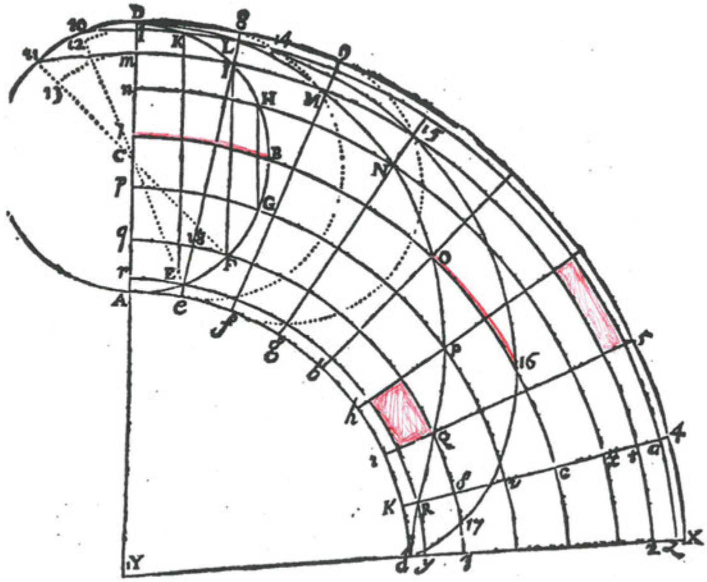

Fig. 3 Area under the half-epicycloid *D*15 16 *d* (La Hire 1694)

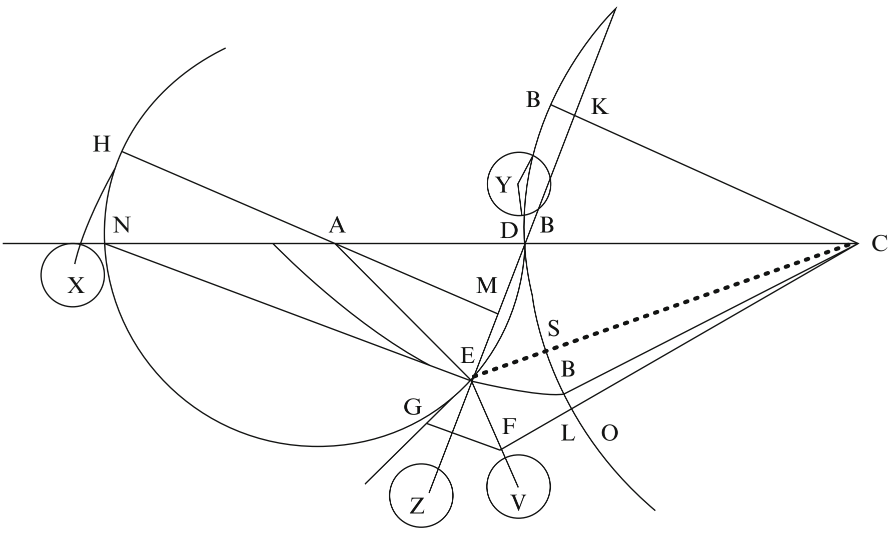

Fig. 4 Mechanical argument for epicycloid (La Hire 1694)

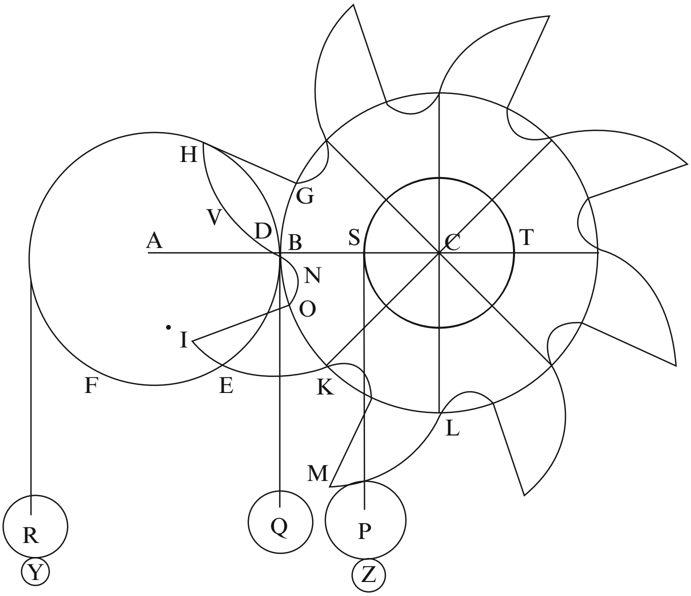

Fig. 5 Epicycloid-shaped gear teeth (La Hire 1694)

3 La Hire and Fontenelle on the Epicycloid

La Hire opened his *Traité des epicycloides, et de leur usage dans les méchaniques* with a statement on applied mathematics:

> There is no part of mathematics which might be of greater use in life than mechanics. But for all the treatises produced until now, having examined only that which is geometric in the science, with no regard for its application, it is not surprising that the greater part of the machines which fill these books are useless and could not be put into use. The figure of gear teeth seems of so little consequence that one had always overlooked it as nothing more than practice, and which should be left entirely to the worker; although, actually, this is something which should have been the most carefully examined. For friction being more or less in proportion to the deviation of the gear teeth, which do all the work of the machines, from the shape that suits them, machines with gear teeth almost never have the effect which was imagined. . . . To overcome this friction one very often needs forces much greater than necessary for what is needed for the movement of the machine itself, and one ordinarily pays no attention to this.

Fontenelle appears to have paraphrased La Hire's introduction in his discussion of the epicycloid.

> Friction, which always destroys a great part of the effect of the machines, is at these points (gear teeth) greater and more harmful than anywhere else. It would have reduced the friction, and, what is more, made all effort equal, by giving the gear teeth a certain shape which would have been determined, necessarily, by geometry. But this is not followed; on the contrary the shape of the teeth is abandoned to the fantasies of workers as something of no consequence, so the machines deceive the hope and calculation of the machinists. M. de

> la Hire found that these teeth, in order to have all the perfection possible, should be in the form of an arc of the epicycloid. His idea was carried out with success at the Chateau de Beaulieu, at 8 leagues from Paris, in a machine to raise water.
>
> It must be said that his idea has only been carried out in that one instance, and, by a certain fatality, among inventions few are useful, and among the useful few are followed. The application of the cycloid to the clock had been much practiced at least in appearance, but one began to recognize little utility.

4 A Sign of the Times

Science in Europe experienced a great change in just the years from Descartes's *Geometry*, 1637, to Newton's *Principia*, 1687. The publications of the Royal Society of London (founded 1660), and of similar societies in other nations, witness the widening curiosity of the times, starting with observation. Just as an example, the 1694 issue of the *Philosophical Transactions* includes an account of the earthquake in Sicily, *De morsu uenenoso canis rabidi* (On the poisonous bite of a rabid dog), observations on a rare kidney stone, observations on an egg found in the fallopian tube of a "lady lately dissected." In comparison, the earliest issues of the *Transactions* were more narrow, concentrating more on antiquities, travel logs, and astronomical observations; even in the lifetime of La Hire, change could be observed.

Fontenelle is emphatic that Newton had it right and Descartes did not, certainly in their physics, and more broadly in their approach to science. In his *Eloge de M. Neuton*, Fontenelle wrote that Descartes "resolved to place himself as master of first principles by certain clear and fundamental ideas of his own, so he would only need to defend the phenomena of Nature as necessary consequences." Newton, on the other hand, did not begin with first principles; he began by observing nature, "gradually advancing to unknown principles, resolved to admit those which could give the chain of consequences" (Fontenelle 1727, p. 160). The results show the superiority of Newton's approach: Descartes brought us vortices while Newton could explain why Saturn and Jupiter deviate from their paths when they are near each other.

His *Eloge* for La Hire exhibits Fontenelle's clear faith that observational and experimental science lead to a better life for humanity, and that mathematics serves science. About La Hire's short work on optics, *Sur les differents Accidents de la Vue*, which appeared in the same 1694 volume, Fontenelle praises it as "physical optics, which supposes the geometric" (Fontenelle 1718, p. 85). The mathematics that serves science is tested by observation. On the astronomical tables that La Hire published in 1702, Fontenelle noted favorably that the tables were from a "long series of diligent observations, and not from a hypothesis on the curves described by the celestial bodies" (Fontenelle 1718, p. 82).

Further, the best science and mathematics aid us not just in material goods but also in a deeper understanding of the world about us. Discussing La Hire's 1695 *Traité de Méchnique*, Fontenelle wrote:

> He (La Hire) was not content with the theory of this science, which he bases on exact demonstrations, but he greatly applied himself to what was most important in the practice of these arts. He rose to the principles of this divine Art, which constructed the universe.

But we should note that Fontenelle is a nuanced thinker. At the same time that he praises observational and experimental science, he is skeptical about certainty, he is skeptical about metaphysics, he is skeptical about the rationality of the human being. We see both his faith and his skepticism in discussing the epicycloid as a model for gear teeth. In his *Eloge*, his doubts appear, as when he noted that the epicycloid-shaped gear teeth and cycloid shapes for the clock have proved to be of little practical value.

We might summarize that Fontenelle felt that mathematics will lead us in the right direction, but use it cautiously and don't expect too much.

La Hire has left us much less by which to read his thought. He is consistent with the thinking of Fontenelle, as we saw in his praise of the useful in mathematics at the opening of his *Traité des epicycloides*. But he is different, too. La Hire is a mathematician, and he works like a mathematician. He is concerned with proper justifications, and the highest standard was the geometric proof in the manner of ancient Greek mathematics. And he concerned himself with the mathematically beautiful, not just the practical. The work that showed the ideal form for gear teeth also included topics quite unrelated to gear teeth, including the quadrature and arc length of the epicycloid and its evolute (another epicycloid). His introduction to La Hire (1694) is centered on applications, but the work itself is devoted as much to what is mathematically beautiful.

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The Rise of "the Mathematicals": Placing Maths into the Hands of Practitioners—The Invention and Popularization of Sectors and Scales

Joel S. Silverberg

Abstract Following John Napier's invention of logarithms in 1614, the remainder of the sixteenth century saw an explosion of interest in the art of mathematics as a practical and worldly activity. Mathematics was no longer the exclusive realm of scholars, mathematicians, astronomers, and occasional gentlemen. Teachers of mathematics, instrument makers, chart makers, printers, booksellers, and authors of pamphlets, manuals, and books developed new audiences for the study of mathematics and changed the public's perception of the status and aims of mathematics itself. The inventions of mathematical instrument makers facilitated the rapid expansion of sophisticated mathematical problem solving among craftsmen and practitioners in areas as diverse as navigation, surveying, cartography, military engineering, astronomy, and the design of sundials.

1 Introduction

During the period between Johann Müller (Regiomontanus) (1436–1476) and Bartolomeo Pitiscus (1561–1613) the entire science of planar and spherical trigonometry was reimagined and systematized. The development of new functions, new tables, new instruments, new applications and new audiences flourished. Increasingly, the geometry of regular shapes was seen to have real-world applications of importance beyond the realm of scholars, and beyond its traditional area of application: astronomy (Finck 1583; Pitiscus 1600; Regiomontanus 1967; Rheticus and Otho 1596; Viete 1579).

Subsequent to this restructuring of trigonometry, a family of instruments that came to be called "sectors" were developed in England and on the continent that greatly broadened the community of practitioners who would come to employ this new trigonometry in areas of more practical concern (Galilei 1606; Gunter 1623;

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Hood 1598). In a very real sense, the mathematics of the heavens was brought down to the realm of earthly concerns and endeavors.

Immediately following this period a second revolution in trigonometric applications was ushered in by the invention of logarithms by Napier (1614, 1618) and the rapid development of tables of the logarithms of both natural numbers and trigonometric values by Napier and Briggs (1619) and others. A second family of mathematical instruments was promptly developed and refined which enabled the easy manipulation of these new logarithmic or artificial values. These instruments were called rules or scales, often referred to as Gunter's Rules or simply Gunters in England (Gunter 1623). Originally engraved on the arms of a cross-staff, they became instruments in their own right, engraved on a two-foot long oblong shaped piece of wood, bone, ivory, or brass, with a multiplicity of scales (most of which were logarithmic scales) which were manipulated with the aid of a pair of dividers.

Due to limitations of time and space, this paper limits its discussion to sectors and scales of English origin. English instruments were not only among the very first to be developed, but it is the English tradition that led to the most highly complex and sophisticated of these instruments, and among those whose use was most widespread and long lasting. The comparison between English instruments and their continental counterparts in Italy, France, and Germany will be reserved for another time and place.

2 The Importance of Proportional Analogies in General Problem Solving

An analogy is a comparison between two things which have some features that are similar and other features which differ. In mathematics, we may wish to compare two lines or two circles or two spheres, but the two similar objects may have different lengths, areas, volumes. This comparison of differences is expressed as a ratio.

A proportional analogy may be understood as an analogy between four things, in which the relationship between A and B is the same as the relationship between C and D, or A:B = C:D, or as in the notation I will use in this paper, A:B :: C:D. In this case A and B are like objects, and C and D are like objects. If the relationship between like objects is expressed as a ratio of some characteristic which differs between the like objects, then a proportional analogy states that, the ratio of the differing characteristic of similar objects A and B is the same as the ratio of the differing characteristic between similar objects C and D, for some characteristic of A and B and some (possibly different) characteristic of C and D.

Finding such a relationship was generally the primary approach to framing and solving many problems for which we today would use arithmetic or algebra. In modern times we generally rely on the algebraic solution of equations or systems of equations. The earlier approach is closely tied to the geometry of similar figures since the ratios of corresponding or homologous sides of two similar figures always have the same ratio. This is perhaps best illustrated by way of an example.

Here is a problem drawn from William Kempe’s English translation (Ramus and Kempe 1592, p. 50) of the arithmetic portion of Pierre de la Ramée’s 1569 book on arithmetic and geometry (Ramus 1569, Book II, Chap. 7, p.29).

> A post goeth from Plimmouth to London in 5 dayes, another commeth from London to Plimmouth more speedily in 4 dayes, admit that they begin their journey both on Monday at [3 of the clock and 20 minutes] in the morning, when and where shall they meete?

A modern student might note that since distance traveled is equal to the product of the rate and duration of travel, it follows that the speed of the slower coach, R_{slower} is equal to the unknown distance between Plymouth and London (let us call it D), divided by 5 days, while the speed of the faster coach R_{faster} would be D divided by 4 days. Rearranging the equation $D = R \times T$ we have $T = \frac{D}{R}$.

Since the two coaches started their journey at the same moment, when the coaches meet, they will each have traveled the same amount of time, but will have covered different distances, say x and y, where $x + y = D$. The time that the faster coach traveled is thus $\frac{x}{D/5}$ and must equal the time that the slower coach traveled, $\frac{y}{D/4}$. Simplifying the equation we have $5x = 4y$. Substituting $y = D - x$ in the previous equation yields $5x = 4(D - x)$ or $x = \frac{4}{9}D$ and $y = \frac{5}{9}D$. The time elapsed will be equal to $\frac{5x}{D}$ or $\frac{4y}{D}$, each of which equals $\frac{20}{9}$ of a day, or $2\frac{2}{9}$.

Compare this with the solution given by Kempe.

> Set downe the former propositions thus: the first endeth his journey in 5 dayes, therefore in 1 day he will ende $\frac{1}{5}$ of the journey: the second endeth his journey in 4 dayes, therefore in 1 day, $\frac{1}{4}$: these parts, added together [$\frac{1}{4} + \frac{1}{5}$], are $\frac{9}{20}$ of the journey. Whereupon conclude, seeing [that] $\frac{9}{20}$ of the journey is gone in 1 day, [it follows that], $\frac{20}{20}$[th of the journey], that is [the entirety of the journey, or] 1, is gone in $\frac{20}{9}$, that is $2\frac{2}{9}$ of a day. This is the time of their meeting, to wit, Wednesday at 8 of the clocke and 40 minutes in the fore noone. Then say, the first [post] in 5 dayes goeth 1 [that is, the entire distance from Plymouth to London], therefore in $\frac{20}{9}$ dayes he will go $\frac{4}{9}$ of the journey, which is the place of meeting, and then the second hath gone the rest of the journey, to wit, $\frac{5}{9}$ [of the journey].

Since most problems were framed as proportional analogies, instruments which aided in visualizing and solving them were a useful aid to the mathematical practitioner. Three such instruments are examined in this paper: sectors, plain scales, and Gunter’s scales. There are many seventeenth- and eighteenth-century works devoted to these instruments, but they are frequently incomplete, relying on the reader’s knowledge of conventions that frequently go unexplained, or even unmentioned—conventions that are unknown to the modern reader. Readers of that era had access to physical instruments that are difficult to obtain today, as well as access to tutors, teachers, instrument makers, and shopkeepers anxious to give hands on instruction in their application and use. Such mentors are no longer available to those desirous of understanding these instruments. I have relied primarily upon the writings of Galilei (1606), Hood (1598), Gunter (1623)—the inventors of these instruments—together with manuals written by their contemporaries and close study and experimentation with antique instruments that I have purchased to help me understand how the instruments were used.

3 Description and Use of the Lines of the Sector

The construction of the sector is modeled after the demonstration by Euclid that the corresponding sides of similar triangles are proportional.

> In equiangular triangles the sides about the equal angles are proportional, and those are corresponding sides which subtend the equal angles [Euclid, Book VI, Prop. 4].

In practice, nearly all sectors (excepting those of Thomas Hood) employ pairs of triangles that are not only equiangular (i.e., similar), but also isosceles. The sectors are constructed by joining two legs at a pivot point. The legs are inscribed with pairs of identical scales which originate at the pivot point. A pair of dividers can be used to measure the distance from the pivot of the sector to a number engraved on one of these paired scales on the leg of a sector. We call this a *parallel distance.* The dividers can also be used to measure or set the distance between like numbers engraved on some pair of scales on the two legs of the sector. This we call a *transverse distance*. The pairs of scales are offset in such a way that the transverse distance between paired scales at the same parallel distance from the pivot are the same for each pair of scales. The various pair of scales are designed that certain scales can be used by performing some manipulations on one pair of scales, followed by a manipulation on a second (and different) pair of scales, as we will explain below.

Early sectors contained a pair of scales of equal parts (sometimes called arithmetic lines), a pair of scales of surfaces (geometric lines) and a pair of scales of solids (stereometric lines). There were also polygonal lines, tetragonal lines, and lines of metals. A quadrant was often attached with which to protract or to measure angles (Fig. 1).

3.1 Line of Lines

The purpose of these lines is to solve proportional analogies of the form A:B :: C:D where A, B, and C are known, and D is desired. Euclid VI.4 proves that corresponding parts of similar triangles are proportional. The form of the sector (comprising a pivot point, two lateral or lengthwise lines and two transverse or parallel lines) creates similar triangles. If we open the sector so that the transverse distance between A and A on the lines of lines is equal to C, then the transverse distance measured from B to B will be the desired value of D. See Fig. 2 for a numerical example.

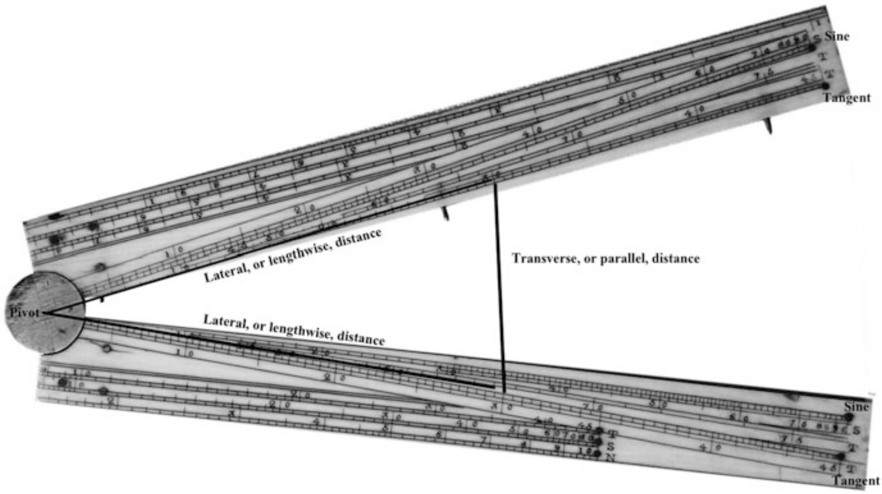

Fig. 1 Each leg of the sector has a scale marked Sine, labeled from 0° to 90°. The lowermost of the scales on each leg are marked Tangent, and are marked from 0° to 45°. The distance from the pivot or center point of the sector to any point on the leg is proportional to the sine or tangent of the angle marked at that point. The right-hand ends of each of these four scales lie on the circumference at a circle centered at the pivot. The length of the chord on this circle connecting the two 90° marks on the Lines of Sines is the same as the length of the chord connecting the two 45° marks on the Line of Tangents. That common value is the radius upon which all trigonometric values are based, and can be set by the user by widening or narrowing the opening of the sector, any transverse or parallel distances changing accordingly

Similarly if two transverse distances and one parallel distance are known the problem may be solved as follows. Open the sector until the transverse distance from A to A is equal to the lateral distance from the pivot point to B. Then open the dividers to a transverse distance equal to C and move the transverse along the legs of the sector until it extends between numerically identical labels on the lateral scales, which will indicate the value of D.

3.2 Line of Superficies

The purpose of these lines is to allow the solution of proportional analogies that relate the ratio of the areas of similar figures to the ratio of the lengths of corresponding sides. The lengths of corresponding sides are measured on the Line of Lines, whereas the areas of the similar figures are measured on the Line of Superficies. These lines may also be used to determine the ratio of areas in similar figures whose ratio of corresponding sides are known, or to determine the ratio by which the side of a given figure must be increased or decreased in order to enlarge or reduce the area according to a desired ratio.

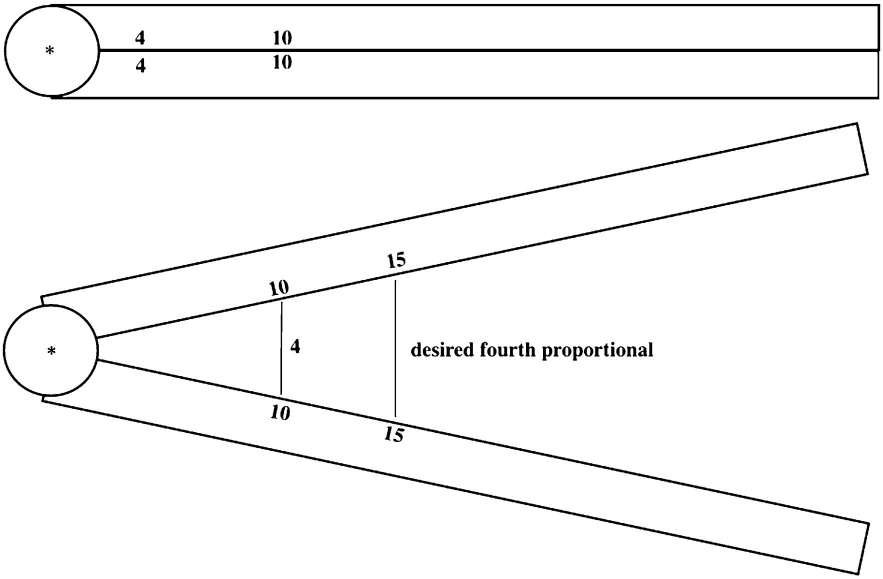

Fig. 2 Given three numbers, to find a fourth in discontinual proportion. Find a value, x, such that $10 : 4 :: 15 : x$ using the Lines of Lines. The dividers are set to the lateral distance from the pivot to the point 4 on one of the lines of lines. The sector is then opened so that the parallel distance between the points marked 10 on the two lines of lines is the same as the distance between the points of the divider. This establishes the transverse or parallel distance of four units between the points marked ten. Without changing the opening angle of the sector, set the dividers so the distance between its points is equal to the transverse or parallel distance between the points marked 15 on the two Lines of Lines, and then, without changing the opening of the dividers, measure the distance between the divider's points on either of the Lines of Lines. One point of the dividers will rest on the pivot point, the other will point to the value engraved on the scale indicating the value of the desired fourth proportional. In our example the value of the desired fourth proportional, x, will equal six

The Lines of Superficies were designed so that the distance from the pivot point to a number engraved on the scale is equal to the area of a square, the length of whose side is the number engraved on the scale at that point. In modern terms the distance to the pivot is the square of the number on the scale, or equivalently, the number inscribed at any point on the scale is the square root of the distance from that point to the pivot of the sector. The purpose of this line is to allow the solution of proportional analogies that relate the ratio area of similar figures to the ratio of the lengths of corresponding sides. This is based on Euclid VI.20 which proves that the area of similar polygons are to each other as the ratio duplicate of that which the corresponding sides have to each other, i.e., in modern terms, the areas of similar polygons varies as the square of the lengths of their corresponding sides. In the case of circles, Euclid XII.2 proves that the areas of circles are to each other as the squares on their diameters. The Lines of Solids provided a similar set of scales which could be used to adjust the linear measure of any regular solid (or a sphere) so

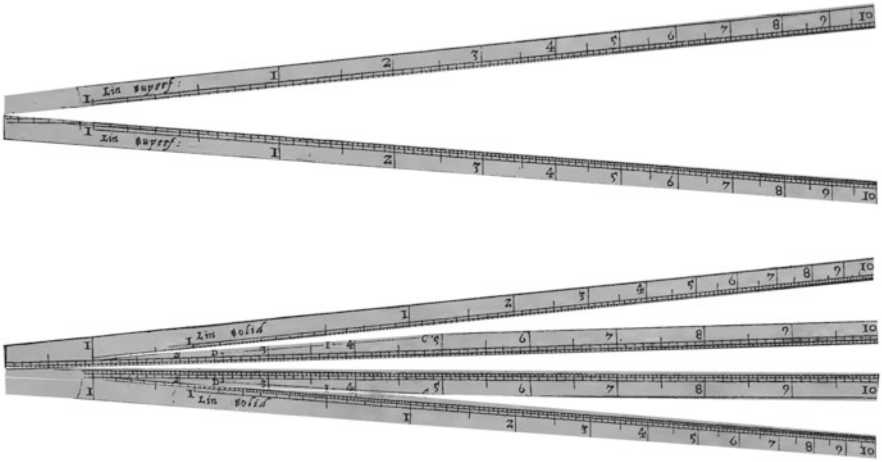

Fig. 3 The legs on the *upper image* (the Lines of Superficies) are labeled 1, 1, 2, 3, . . . , 10. The distance of these labels from the central pivot represent areas of 1, 10, 20, 30, . . . , 100 square units. In other words, the first point labeled 1 represents an area of 1, the second occurrence represents an area of 10, the following labels represent areas of 20, 30, 40, etc. The distance from the pivot point to any such label when measured on the Line of Line (the inner most scales on the *lower image* in this figure) is the linear measure (radius, diameter, or side) of the figure whose area is on the label. The side of the figure with area 100 (or any other value) can be set by changing the angle of the sector opening. The Lines of Solids in the *lower image* is marked 1, 1, 1, 2, 3, . . . , 10, and the distance from the pivot point to these labels represent volumes of 1, 10, 100, 200, 300, . . . , 1000 cubic units, respectively. These distances when measured on the line of lines provide the linear measure of a figure with the associated volume

that the volume was altered according to any given ratio. Conversely, if the volume was changed according to a certain ratio, one could calculate the corresponding ratio by which the side of the solid or the diameter of the sphere would change. See Fig. 3.

For example, given a figure with one side equal to 3 inches representing a 20 acre piece of land, and a similar figure (in the geometric sense of the word) with corresponding side equal to 5 inches, we desire the acreage of the enlarged piece of land. First set the dividers to extend from the pivot to 3 on the Line of Lines. Then open the sector until the transverse distance between 2 and 2 (representing 20 acres) on the Lines of Superficies is equal to the extent of the dividers. Without disturbing the pivot, reset the dividers to extend from the pivot to 5 (inches) on the Lines of Lines. Finally find identical numbers on the Line of Superficies with a transverse or parallel distance of 5 inches. Read the value on the Line of Superficies and that will be your area in acres. In Fig. 4 we see that the plot with a 5 inch side contains 55.5 acres. One can confirm this by noting that the area of the figures will be in the ratio of the square of the sides. Thus $9 : 25 :: 20 : x$ and therefore the requested acreage is 25×20 divided by 9, or 500/9 which is 55 and 5/9 acres.

On the other hand, if we know the ratio of areas and the side of one of the figures, we can determine the length of the side in the second figure. Suppose we know that

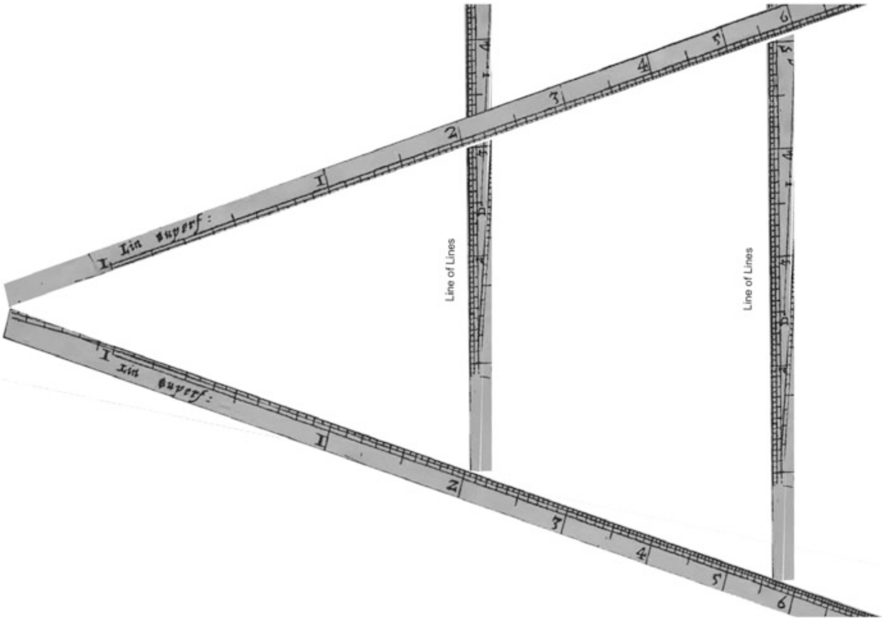

Fig. 4 If a plot with one side measuring 3 inches represents 20 acres of land, what is the area of a similar figure with the corresponding side equal to 5 inches?

the first field contains 2 acres, and the second field contains 5 acres. If the length of a certain side is 4 inches as measured on the first plot, when we draw a similar figure for the 5 acre plot, how long will the line be? Set the dividers to extend from the pivot to 4 on the line of lines and open the sector's pivot until the transverse distance between 2 and 2 on the line of superficies agrees with the dividers.

Without changing the angle at which the sector is open, reset the dividers to extend from 5 to 5 on the line of superficies and measure that extent as a lateral distance on the line of lines to determine the length of the corresponding side on the second plot. This line of lines in combination with the line of superficies could also be used to find squares and square roots of numbers within the accuracy with which the scales could be read.

3.3 Lines of Circular Parts

In his *Introductio in analysin infinitorum* of 1748 Leonhard Euler introduced the modern concept of trigonometric functions as ratios of sides of a triangle (Euler 1748). Prior to that year, trigonometric values were conceived as physical line segments, related to a base circle of arbitrary radius. The lines of circular parts often included lines of chords, sines, tangents, and secants. See Fig. 5.

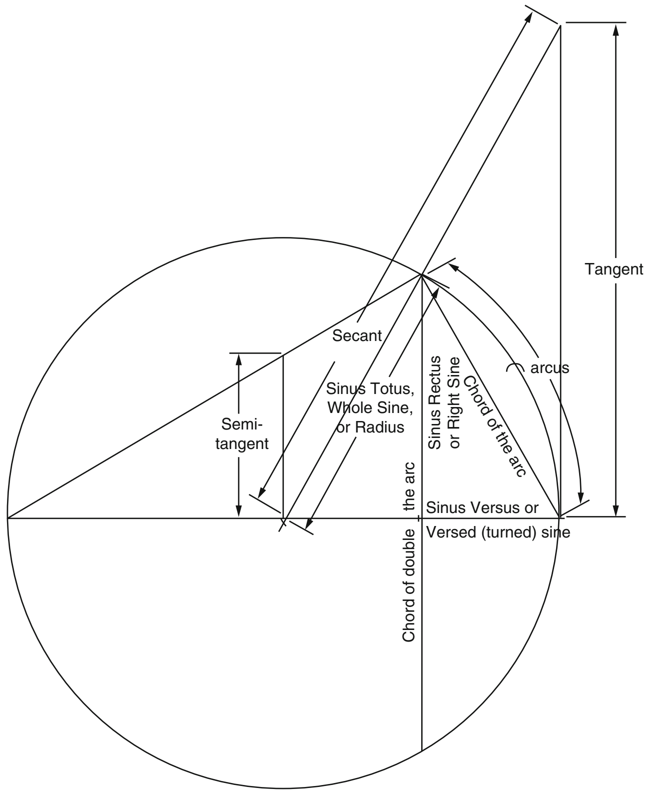

Fig. 5 Prior to the mid eighteenth century, circles of different radii had different trigonometric values and trigonometric tables were based on circles of a particular radius, often referred to as the total sine, or *sinus totus*. Euler’s expanded view of trigonometric functions was not widely adopted by teachers and practitioners of mathematics unit the mid-nineteenth century

3.4 The Particular Lines

If any room remained on the sector after inscribing the Lines of Lines, Sines, Tangents, Chords, Superficies, and Solids, the instrument maker might fill up the unused space with an assortment of lines which would be most useful to the applications of most interest to his client. Some of the more common included extra lines are described below.

3.4.1 The Lines of Quadrature

The purpose of the Lines of Quadrature was to determine the length of the side of a square equal in area to the area of any given circle, or to determine the radius of

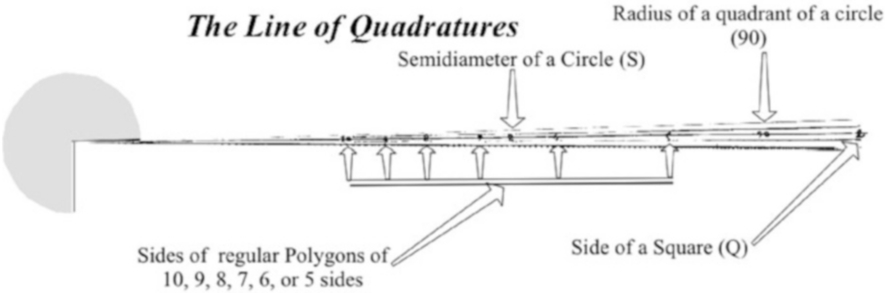

Fig. 6 The Lines of Quadrature are used to determine the length of the side of a square equal in area to a polygon with a side of known length or vice versa

a circle equal in area to any given square, or determine the length of the side of a regular polygon of 5 through 10 sides equal in area to a given circle or a given square.

The Lines of Quadrature were labeled Q, 5, 6, 7, 8, 9, 10, and S. The numerical labels indicate that the transverse distance between like numbers is the length of a regular polygon with that number of sides. The transverse distance between points labeled Q is the side of a square whose area is equal to that polygon. The transverse distance between points labeled S is the length of the semi-diameter (i.e., the radius) of a circle with an area equal to that of the polygon. See Fig. 6. All transverse distances are measured on the Line of Lines.

3.4.2 The Lines of Segments

The Lines of Segments were designed to divide a circle of any given diameter into two parts by a chord perpendicular to the diameter in such a way that the areas of the two segments created were in a given ratio, or to find the proportion between the area of the entire circle and that of a given segment thereof.

3.4.3 The Lines of Inscribed Bodies and the Line of Equated Bodies

The Lines of Inscribed Bodies were labeled D, I, C, S, O, T, which signified the length of the sides of a dodecahedron, icosahedron, cube, octahedron, tetrahedron inscribed in a sphere of semidiameter (i.e., radius) S. See Fig. 7.

3.4.4 The Lines of Metals

The scales were calibrated according to the volume of any metal, which would have the same weight as a specified volume of any reference metal. English sectors

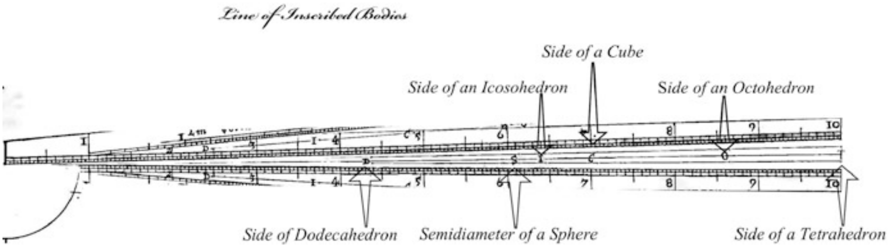

Fig. 7 The Lines of Inscribed Bodies. The transverse distance (from the Line of Lines) was set between any like markings, and the transverse distance measured between any other matching markings (when measured on the Line of Lines) would give the radius of the sphere or the length of the side of a different platonic solid of equal volume

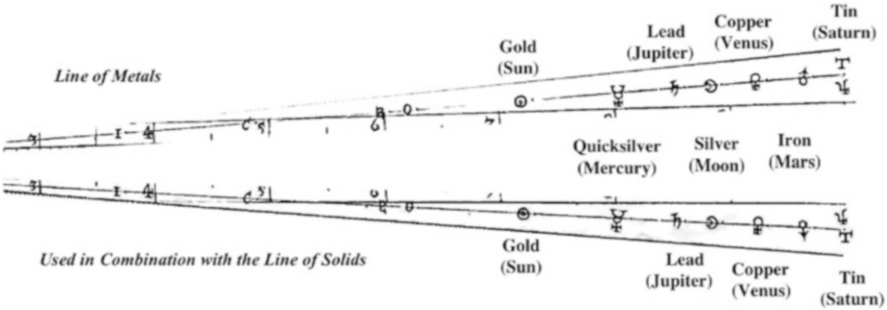

Fig. 8 The lines of metals

provided markings for Gold, Mercury, Lead, Silver, Copper, Iron, and Tin. Each was marked with its alchemical symbol at a distance from the sector pivot related to the volume needed to match the weight of a volume of another metal.

The least dense of the metals (tin) was a distance of 10 from the pivot. The symbol for Gold was that of the sun, for Quicksilver: Mercury, for Lead: Jupiter, for Silver: the moon, for Copper: Venus, for Iron: Mars, and for Tin: Saturn. See Fig. 8. When used in combination with the Line of Solids, the linear measure (most often the radius of a sphere or the side of a cube) which corresponded to the volume determined to have the desired weight. The scale could be used in either direction, i.e., to find the volume of a different metal of equivalent weight, or to find the weight of a different metal of equivalent volume. Such calculations were central to the art of gunnery and artillery.

4 Logarithmic Scales and Rules

Edmund Gunter's *De sectore et radio* contains a detailed description of the construction and use of his version of Hood's sector. The English title of this work is *The Description and Use of the Sector, The crosse-staffe and other instruments For* *such as are studious of Mathematical practice.* The work is divided into six "books": the first three books are devoted to the sector and the second three are devoted to the crosse-staffe. The crosse-staffe is composed of the Staffe, the Crosse, and three sights. Gunter's staff was three feet in length, the cross 26 inches long. On the Staffe are inscribed a line of equal parts for measure and protraction, and a line of tangents for measuring of angles, a line conversion between the sea chart and the plain chart, and four lines for working with proportions.

The lines for working with proportions are one of the first appearances of logarithmic scales. Gunter served as the third Gresham Professor of Astronomy (1619–1620). Henry Briggs, his mentor, was the first Gresham Professor of Geometry who left Gresham College in 1619 to become the first Savilian Professor of Geometry at Oxford, and Gunter assumed his position at Gresham upon Briggs's departure for Oxford. It was Briggs who traveled to Edinburgh to visit Napier and worked for 10–15 years to develop and popularize Napier's logarithms. It is not a surprise then to find that the four new lines upon Gunter's Crosse-Staffe were scales of logarithms, log-sines, log-tangents, and log-versed sines. The "bookes of the crosse-staffe" explain how to use these scales together with a set of dividers to solve proportional analogies with a pivoting instrument such as the sector, but with a linear rule. This proved so useful and popular that a separate instrument was developed for purposes of calculation which consisted of a simple oblong shaped rule, inscribed with scales for protraction and measurement. The scales included a line of equal parts, a line of chords, scales for either orthographic or stereographic projection of the sphere onto the plane (a scale of natural sines and a scale of natural semi-tangents), and a set of logarithmic scales (numbers, sines, tangents, secants, and versed sines) for solving proportional analogies and for performing multiplications and divisions. The instrument became known as a Gunter's Rule or a Gunter's Scale and remained in use from the 1620s until the third quarter of the nineteenth century. It survived in a modified form as the slide-rule from the seventeenth century until the invention and commercialization of electronic calculators and computers in the 1970s.

The key to the importance and utility of this new instrument was the ease with which proportional analogies could be resolved with the aid of a variety of logarithmic scales. Since the scales are logarithmic, the logarithm of the ratio of a to b is the distance between the points labeled a and the point labeled b on the line of numbers. This can be used to find a fourth proportional (the Golden Rule or Regula Aurea of problem solving techniques), with two simple movements of a pair of dividers. First open the dividers so that its legs extend from point a to point b on the Line of Numbers. Then move the dividers so that one leg points to the point c. The other leg will point to point d whose value as read on the scale is the desired fourth proportional. See Fig. 9.

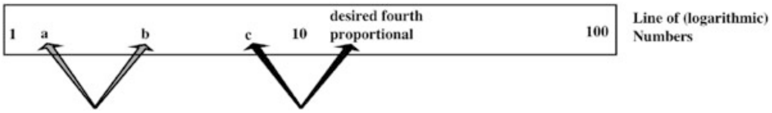

Fig. 9 To solve the proportional analogy A:B :: C: ??, simply set the dividers to extend from A to B in the Line of Numbers, then move the dividers to C and they will extend to the desired fourth proportional. Note that the direction from A to B (whether to the left or to the right) must be the same as the direction from C to the desired proportional

5 Construction and Use of the Scales of the Gunter’s Rule

The scales on Gunter’s Rule were many, but they can easily be divided into categories. The rule does not have any paired scales as do sectors, since it does not use Euclid’s postulate on similar triangles to solve proportions, but instead uses logarithmic scales. There are, however, a small number of non-logarithmic, or natural, scales. There are scales of equal parts which are used for constructing lines of a particular length or for measuring the length of lines. There is a line of chords for measuring angles or laying out angles of a particular size. There are also lines of sines, tangents, semi-tangents, and secants which are not logarithmic. These were not used for solving proportions or performing arithmetic with trigonometric values. Rather, they were used for the geometric construction of both orthographic and stereometric projections of arcs on the surface of a sphere onto the plane. These projections were used in navigation, astronomy, and cartography, and they allowed the practitioner to draw planar projections of spherical triangles, great and lesser circles, etc. and to measure and interpret their properties in the plane.

The logarithmic scales included the Line of Numbers, Line of Sines, Line of Tangents, and Line of Secants. Despite this nomenclature these were scales of common logarithms, log-sines, log-tangent, and log-secants. There was also a scale labeled Versed Sine. That scale was in fact a scale of the logarithm of half of the versed sine of the supplement of the angle marked on the scale. This scale allowed the use of logarithms to solve spherical triangles where either all three sides or all three angles were known. Unlike planar triangles, in spherical trigonometry knowing three angles is sufficient information to determine all three sides. In particular, the determination of local time, and therefore the determination of longitude from celestial observations alone require the solution of such triangles.

The next section will explain how the scales on the sector and on the rule were constructed. With the exception of the logarithmic scales, these scales were not constructed by calculating values and measuring them out on the instruments; neither were they obtained from tables of values and then measured and inscribed or engraved. Instead they were geometrically constructed—often through Euclidean methods, sometimes with an instrumental approach, and occasionally using mechanical or approximate methods for trisecting an angle or squaring a circle.

5.1 Construction of Lines of Equal Parts

The line of lines or line of equal parts requires dividing the length of the sector or scale into 100 equal parts. Although creating a line that is *n* times the length of another is straightforward, given only a straightedge and a pair of dividers, dividing a line into *n* equal parts is not. The details of how this was accomplished are described in Fig. 10.

5.2 Construction of Circular Lines

These scales were generally determined by dividing a quadrant of a circle into 90 divisions, then geometrically constructing the line segments corresponding to various trigonometric values for each division, and using dividers to transfer those lengths onto a linear scale. The radius of the circle was chosen to be equal to the desired length of the scale. Such a diagram was called a Plain Scale. The ingenious diagram in Fig. 11 contains no fewer than nine scales of circular parts.

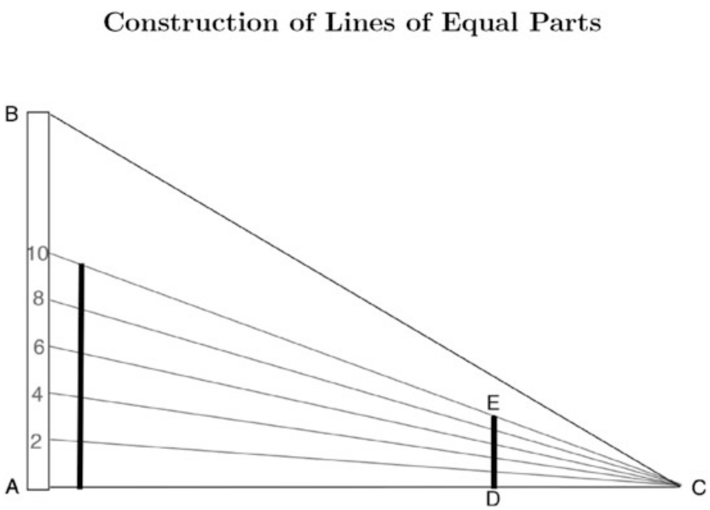

Fig. 10 Lines of Equal Parts. Suppose we wish to divide a line segment DE into five equal parts, labeled 2, 4, 6, 8, 10. Construct a perpendicular to DE at D, extending the line an arbitrary distance to A. Then construct a long line perpendicular to DA at A, extending to B. On line AB mark off 5 equal segment of a convenient length, labeled 2, 4, 6, 8, 10. Draw a line between the point labeled 10 on AB and the point E. Draw a second line from A through D, extending until it meets the previously drawn line at C. Connect the points 2, 4, 6, and 8 on AB to the point C. Line DE is now divided into 5 equal parts which may be labeled as desired

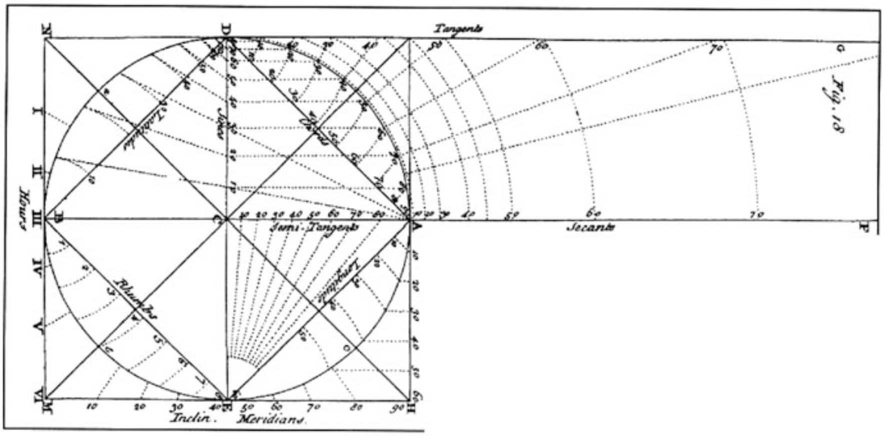

Fig. 11 Construction of the Circular Lines. The arc AD is divided into 9 equal parts, with divisions every 10°. The distance from point D to each of the arc’s divisions is used to construct a line of chords upon the chord AD. The divided arc is used to construct a line of sines on radius CD. Lines of tangents and secants are likewise constructed upon Line DG and AP, respectively. Radius AC is divided into a line of semi-tangents (the tangent of half the angle marked on the scale). Line BE is divided into a line of chords of the 8 points of one quadrant of the compass and is marked Rhumbs. The remaining four scales are specialized scales used for navigation and for the laying out of sundials. All lines are constructed geometrically

5.3 Construction of Lines of Superficies

The Line of Superficies is a line of the square roots of the corresponding value on the Line of Lines, and that of the Line of Solids is a line of cube roots of the corresponding values on the Line of Lines. The reason for this is that the ratio of the length of the sides of similar figures is equal to the ratio of the square roots of their corresponding areas and to the ratio of the cube roots of their corresponding volumes. Since the scale is to be constructed geometrically, the method is to find the mean proportional between the two values, or to find two mean proportionals between two values. Those will be equal to the square root of the product of the two values, or the cube root (and square thereof) of the product of two values. See Fig. 12.

The construction takes unity for one of those values, and the area or volume as the other. The mean proportional will be the side of a square of equal area to the figure. If one determines two mean proportionals between unity and a volume, the first mean proportion will be the side of a cube with the same volume as the original figure.

5.4 Construction of Logarithmic Lines

The logarithmic scales were laid out in a different manner. They were taken from tables of logarithmic values of numbers and logarithms of circular parts, which were then measured out on an accurate scale of equal parts with a set of diagonal scales

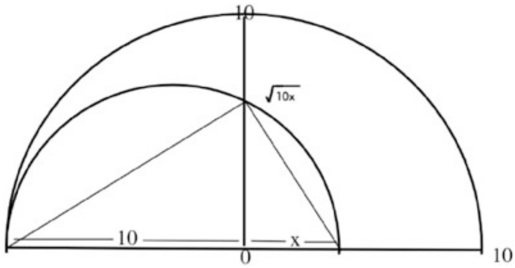

Fig. 12 Construction of the Line of Superficies. The standard construction (Euclid 1956, Book VI, Proposition 13) of the mean proportional between 10 and *x* provides a line segment of length square root of 10*x*. If the segments were 1 and *x*, the altitude of the right triangle formed would be the square root of *x*. The purpose of the larger semicircle in this figure is to emphasize that the value of *x* may range from 0 to 10

Fig. 13 Since the difference of two logarithms is the logarithm of the ratio of their arguments, we may construct the scale from the difference between the position desired and the right-hand end of the scale as the difference of their logarithms

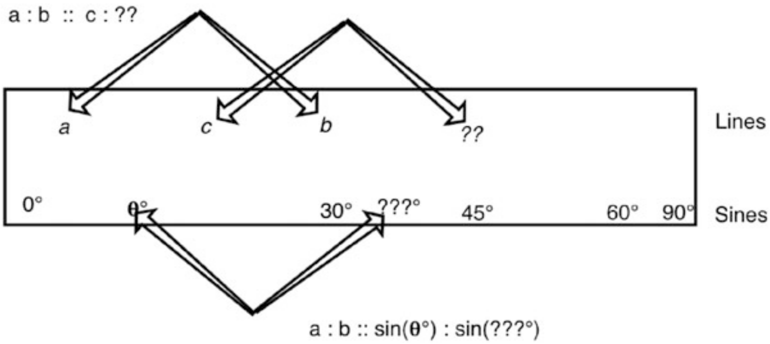

Fig. 14 Application of the artificial (i.e., logarithmic) scales. Using only the Line of (artificial) Numbers, a fourth proportional is determined. Using the Line of Numbers together with the Line of (artificial) sines provides solutions to the planar law of sines. Using only the Line of (artificial) Sines provides solutions to the spherical law of sines

which could measure lengths to three significant figures. These lengths were then transferred to the rule. However, these scales were laid out as measured from the right hand (higher valued) end of the scale, towards the lower valued left end of the scale. These scales have no zero point, but are anchored at the log-sine of a right angle, or at the log-tangent of 45°, etc. See Figs. 13 and 14.

6 Three Applications from Astronomy: Solutions by Sector and Scale Compared

This paper concludes with three practical applications of the sector and scale to solve important problems in spherical trigonometry related to astronomy, navigation, and astrology.

The first problem concerns the determination of the solar declination on any given day of the year. Over the course of a year the position of the sun with respect to the fixed stars moves about a path called the ecliptic. Each day the sun appears to move along a small circle in the heavens, parallel to the celestial equator. During the period from the winter solstice to the summer solstice, that circle gradually moves northward from the tropic of Capricorn to the tropic of Cancer, while the remainder of the year it gradually moves southwards from the tropic of Cancer to the tropic of Capricorn. On each of the equinoxes this circle is coincident with the celestial equator. The angular distance of this small circle, called the day circle, north or south of the celestial equator is known as the solar declination. Knowledge of the declination of the sun on a particular day, together with a measurement of the altitude of the sun above the horizon at its highest point (local apparent noon) allows the navigator to determine his latitude.

The second problem concerns the determination of the azimuth of the sun at sunrise, provided that the solar declination and the observer’s latitude have been determined through calculation and observation. A comparison of the bearing of the rising sun as read from a magnetic compass with the corresponding bearing given by the calculated azimuth provides the amount by which celestial (or true) direction varies from the direction indicated by the magnetic compass. In the seventeenth century it was believed that magnetic variation varied with longitude and that a determination of magnetic variation could be used to determine the longitude of the observer. It was soon discovered that magnetic variation varies with both time and place and could not be used to determine longitude. The measurement of variation could, nonetheless, be used to correct the compass direction to provide true directions, and thus was of considerable value to the navigator.

The third of our sample problems concerns the determination of local solar time from celestial observation. The standardized time zones we use today did not exist until the close of the nineteenth century. Time varied continuously across time zones, and therefore at the very same instant, ships at different longitudes (however small this difference) observed different solar or local times. Since ships are constantly moving, no mechanical clock could be used to determine the local time, thus celestial determination of local time could be used to regulate and correct any timepieces used on ship. From the mid-eighteenth century onward, with the appearance of nautical almanacs recording the positions of the moon with respect to planets, sun, and stars at every hour of the day as seen from some reference meridian, lunar measurements together with determination of local time would provide navigators with knowledge of their longitude. After the development of chronometers, synchronized to time at a reference meridian, the determination of local time when compared to the chronometer time also provided knowledge of longitude. In all of the cases, the ability to determine local apparent time was an important navigational tool.

Having discussed the motivation for each of these problems, let us examine how the use of sector or Gunter's rule was used to approach each problem in turn.

6.1 Problem 1: Given the Distance of the Sun from the Equinoctial Point, to Find Its Declination

During the course of the year, the position of the sun moves along the ecliptic, a great circle at an angle of 23° 30′ to the celestial equator, intersecting the equator at the positions of the vernal and autumnal equinoxes. During the course of a day, the sun (at roughly the same point on the ecliptic) moves in a lesser circle about the celestial north pole. Since the sun moves about 1° along the ecliptic during each day, knowledge of the date determines the distance along the ecliptic from the nearest equinox. That information may be used to determine the distance of the sun (its declination) above or below the celestial equator. A right angled spherical triangle is formed by the celestial equator, the ecliptic, and the meridian of longitude of the sun. See Fig. 15. Using the spherical law of sines, we know that the ratio of

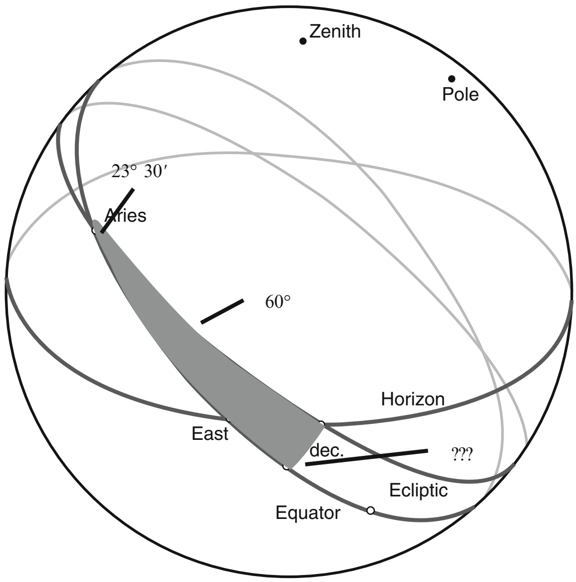

Fig. 15 Given the sun's ecliptic longitude on a particular day (determined by the number of days from since the equinox), to find the solar declination, i.e., the sun's angular distance above or below the equator

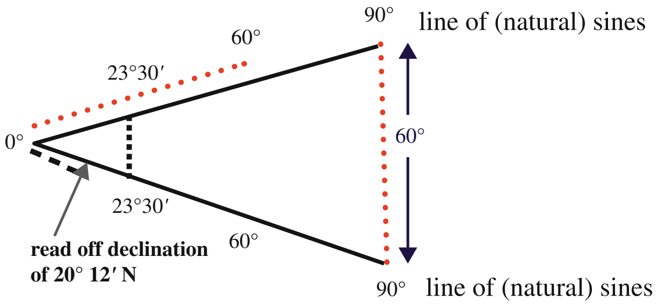

Fig. 16 Use of the sector to solve the spherical law of sines

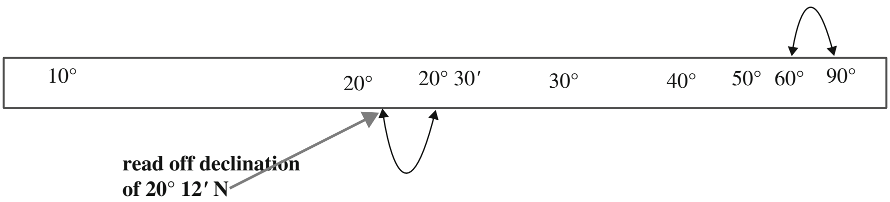

Fig. 17 Use of Gunter’s scale to solve a problem using the spherical law of sines

the sine of an angle to the sine of its opposite side is the same for any angle and its opposite side. Therefore as the sin 90° is to the sin 60° so is sin 23°30′ is to the sine of the declination.

Solution by Means of the Line of (Natural) Sines on the Sector Set the dividers to the distance between 0° and 60° on the Line of Sines. Open the sector until the transverse distance between the 90° marks on each leg of the Line of Sine is equal to the distance previously set on the dividers. Lastly reset the dividers to the distance between the 23°30′ marks on each leg of the Line of Sines, and use those dividers to extend from the pivot point of the sector along either leg of the sector on the Line of Sines, reading the value at that point of the scale as 20°12′, the solar declination. See Fig. 16.

Solution by Means of the Line of Artificial (Logarithmic) Sines on the Gunter’s Scale Performing this calculation is even simpler using the Gunter’s scale. Set the dividers to extend from 90° to 60° on the line of (artificial) sines. Moving in the same direction (right to left), lay off that same distance from 23°30′ on the same line, reading off the value at the other end of the dividers as 20°12′, the desired solar declination. See Fig. 17.

6.2 Problem 2: Given the Latitude of Your Location and the Sun's Declination at Sunrise, to Find Azimuth of the Sun at that Time

Assume that on May 21, the sun's ecliptic longitude is 60° as in Problem 1, above. Then the solar declination will be 20°12′ north of the equator, as previously calculated. Let us further assume that we are observing the sunrise from St. Catharines, ON, Canada, which is at latitude 43°11′ North. What is the azimuth of the sun, that is, how many degrees along the horizon to the East of the North point does the sun rise?

The ortive amplitude or rising amplitude is the angular distance along the horizon between the position of the sun on the horizon at sunrise and the East point of the horizon (Van Brummelen 2013). In the springtime, the sun rises to the north of the East point and sets to the north of the West point. Thus the azimuth is determined by subtracting the ortive amplitude from 90°. See Fig. 18.

As the sine of the complement of the latitude is to the sine of the declination, so is the sine of a right angle (the sinus rectus) to the sine of the desired azimuth. The complement of the latitude of St Catharines is 46°49′.

Thus $\sin 46°49' \ : \ \sin 20°12' \ :: \ \sin 90° \ : \ \sin($ sun's *rising amplitude*). The *azimuth* of the sun is the complement of that angle, i.e., 61°44′. The needle of a magnetic compass does not point to true north, but rather to the magnetic north pole. The amount by which this differs is called magnetic variation. Unfortunately the difference between true and magnetic north varies from year to year and from place to place on the globe. Bearings taken from charts or maps are measured in

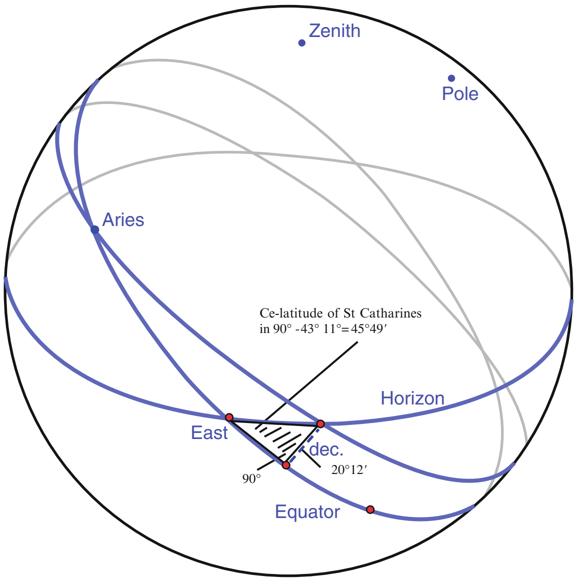

Fig. 18 The right triangle under consideration is the smaller right triangle formed by the solar meridian, the horizon, and the equator. The angle formed by the intersection of the horizon and the equator is the complement of the latitude. The side opposite that angle is the declination previously calculated. The meridian meets the equator, as before at a right angle, and the side opposite the right angle is the desired unknown. Thus the spherical law of sines is again the relationship of interest

the direction from true north. If a navigator took a magnetic bearing of the sun at sunrise and compared it to the azimuth as determined through calculation, the variation could be determined and compass headings could be converted to true directions and vice versa.

Solution by Means of the Sector and the Scale Set the dividers to extend from the pivot of the sector to the point on the Line of (natural) Sines marked 20°12′. Then open the sector so that the dividers extends between the points marked 46°49′ on each leg on the Line of Sines. Then reset the dividers to measure the distance between those points marked 90° and note the extent of the newly reset dividers from the pivot of the sector to a point on the Line of Sines on either leg of the sector. See Fig. 19.

Using the Lines of artificial (logarithmic) Sines, the dividers are set to extend from the known angle to its opposite (known) side. The dividers thus set are used to extend from the second of the known angles to its opposite (but unknown) side. See Fig. 20.

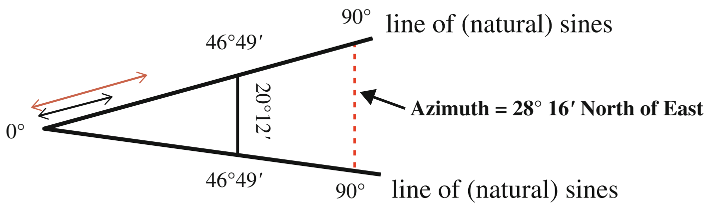

Fig. 19 Solution by sector. The known angle (the complement of the latitude) and its (known) opposite side (the declination) form the leg and the base of the smaller of the two similar isosceles triangles of the sector. The known angle (the right angle) and its (unknown) opposite side (the rising amplitude) form the leg and the base of the larger similar isosceles triangle

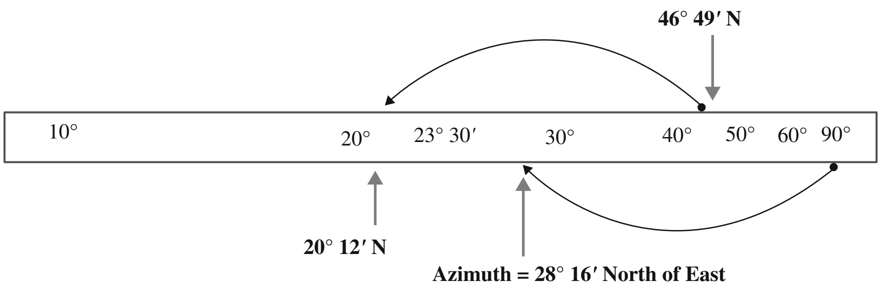

Fig. 20 Solution by Gunter's Scale. The dividers are set to extend from the complement of the latitude to the declination, and moved to extend from 90° to the rising amplitude

6.3 Problem 3: Given the Latitude of Your Location, Declination of the Sun, and the Sun's Altitude, to Find the Local (Solar) Time

This last problem will show a more complicated situation. Here we know three sides of an oblique spherical triangle, but none of the angles. The spherical law of cosines applies in this case:

$$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma,$$

where c is the side opposite angle γ and a and b are the sides adjoining angle γ. Solving for γ we have

$$\cos \gamma = \frac{\cos(c) - \cos(a)\cos(b)}{\sin(a)\sin(b)}.$$

These equations are ill-suited for manipulation with either sector or logarithmic scale, due to the sum or difference of products.

Suppose the sun is observed from St Catharines, ON, to have an altitude of 50° above the horizon on a day when the solar declination is 20° North. The three sides of the "astronomical triangle" formed by the solar position, the pole, and the zenith are the complements of the latitude, the solar declination, and the solar altitude. If we take the latitude of St Catharines to be 43° N, then the sides of the astronomical triangle are 47°, 70°, and 40°—the complements of the latitude, the declination, and the altitude, repectively (Fig. 21).

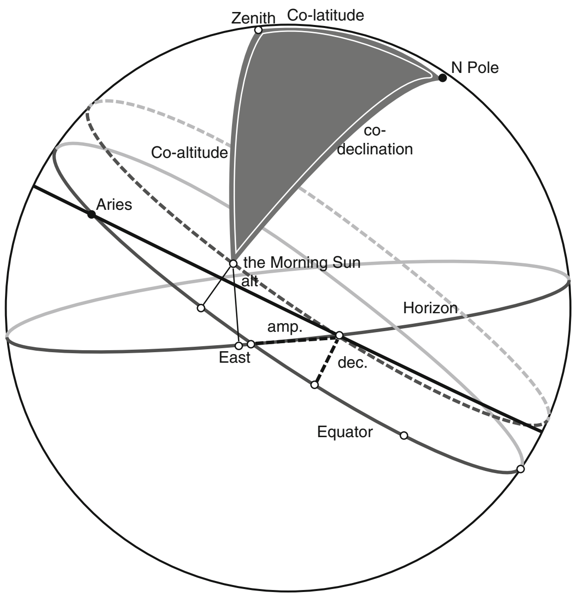

Fig. 21 The complements of the observer's latitude, the solar declination, and the solar altitude form the sides of the *spherical triangle* shaded in the diagram. The angle θ at the celestial north pole, between the meridians of the sun and the observer, measures the local solar time. Angular measure is related to time at the ratio of 15° of arc per hour of time (since a rotation of 360° occurs every 24 h). If the sun is in the east of the South point in the horizon, it is the time before noon, else it is the time since noon

An elegant solution published by Pitiscus between 1595 and 1600 is presented below. The key insight was to replace the law of cosines by a law of versed sines or a law of suversed sines, where the suversed sine of an angle is the versed sine of the supplement of that angle. Algebraically, $\frac{\sin S \sin(S-c)}{\sin a \sin b}$ is equal to half of the suversed sine the angle γ, where S is half of the sum of the three sides of the triangle.

The versed sine of an angle is the difference between the radius or sinus totus and the sine complement of the angle. In modern terms, the versed sine of γ is $1 - \cos\gamma$. and therefore the suversed sine of γ is equal to $1 + \cos\gamma$.

As seen above, the spherical law of cosines, when solved for the unknown angle may be stated as

$$\cos\gamma = \frac{\cos(c) - \cos(a)\cos(b)}{\sin(a)\sin(b)}.$$

Adding $\frac{\sin a \sin b}{\sin a \sin b}$ to both sides, we have

$$1 + \cos\gamma = \frac{\cos c + \sin a \sin b - \cos a \cos b}{\sin a \sin b} = \frac{\cos c - \cos(a+b)}{\sin a \sin b}.$$

Using the trigonometric identity $\cos x - \cos y = 2\sin\frac{x+y}{2}\sin\frac{y-x}{2}$, where $x = c$ and $y = a + b$,

$$\cos c - \cos(a+b) = 2\sin\frac{a+b+c}{2}\sin\frac{a+b-c}{2}$$

Thus, we have

$$\frac{1+\cos\gamma}{2} = \frac{\sin S \sin(S-c)}{\sin a \sin b}.$$

The right-hand side of this equation is computed using a sequence of two proportional analogies. The left-hand side of the equation is one-half of the suversine of angle γ.

The first of the proportional analogies is:

> As the sine of 90° is to $\sin a$, so is the $\sin b$ to "the fourth sine." Using either the sector and a pair of lines of sines or using the Gunter and the scale of log-sines, we find the angle, let us call it x, whose sine is $\sin a \cdot \sin b$.

The second of the proportional analogies is:

> As the value of "the fourth sine" is to $\sin S$, so is $\sin(S-c)$ to the fourth term in this second analogy (which Pitiscus terms the "seventh sine"). The term S is called the half-sum, which is equal to $\frac{a+b+c}{2}$.

As before, using either the sector and a pair of lines of sines or using the Gunter and the scale of log-sines, we find the angle, let us call it y, whose sine is $\frac{\sin S \sin(S-c)}{\sin a \sin b}$, the right-hand side of the equation he seeks to solve.

Thus a value, y, has been found which is a solution to the equation

$$\sin y = \frac{\sin S \sin(S - c)}{\sin a \sin b}.$$

The desired value, γ, is the solution to the equation

$$\frac{1 + \cos\gamma}{2} = \sin y.$$

The half-angle trigonometric identity $\cos^2\frac{\gamma}{2} = \frac{1+\cos\gamma}{2}$ allows us to find a solution, γ to the equation $\sin^2(90° - \frac{\gamma}{2}) = \sin y$. Thus using the scales of superficies on the sector together with a lateral distance equal to the "seventh sine" would allow one to determine $90° - \gamma/2$ and thus the value of γ. On a Gunter's scale, the ability to find square roots by dividing a logarithm by two simplifies the procedure.

However, the solution of this problem was so common, that a special scale, labeled "versed sine" was provided on both sectors, which was in fact a scale of half-suversed sines, on which values of γ were placed at a distance from the pivot of $\frac{1}{2}(1 + \sin(90° - \gamma))$. If the dividers were set to measure the distance between the pivot and y on the scale of sines, then that same distance would extend from the pivot to the point marked γ on the so-called scale of versed sines. A similar logarithmic scale was frequently included on the Gunter's scale. It was labeled the scale of versed sines, but was in fact a scale of logarithms of half-suversed sines.

The solutions to the problem of determining local time using both sector and scale together with a scale of "versed sines" is presented in Fig. 22 and are remarkable in their directness and simplicity.

7 Impact of Sectors and Scales on Mathematics, Science, and Society

We close with a few notes putting these developments into their cultural and historical contexts. The sixteenth and seventeenth centuries were times of tremendous change both in England and on the continent. The effects of the English Reformation, recurring bouts of plague, the English Civil War, Commonwealth, and Restoration (1642–1660), the struggle between Protestant and Catholic sympathizers, the fire of London (1666), exploration and colonization of the New World all had a major impact upon the society in which mathematicians and their students worked and lived. But this period also saw the birth of a panoply of new types of institutions such as Gresham College (1597), the East India Company (1600), the Royal Society (1660), the Christ Hospital's Writing School (1577) and its Royal Mathematical School (1673).

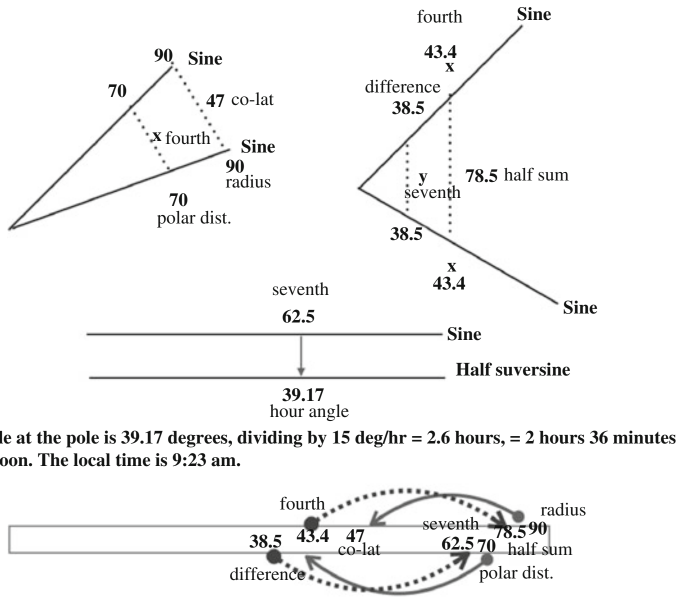

Fig. 22 Determining local time, via sector and via scale

According to Ellis, the first coffee house in Christendom was opened in Oxford in 1650 (Ellis 1956). Within a few years coffee houses were open in London and Cambridge as well, and by the end of that century London could boast of over 2000 coffee houses (O’Connor and Robertson 2006). Not only were the coffee houses meeting places for discussion on topics from science to politics, but lectures were given in them. These were not just impromptu lectures given in the course of discussion, but rather were properly advertised and usually not one-off lectures but rather extended lecture series. Because of this educational function coffee houses were often called the Penny Universities—the name arising since they charged an entrance fee of a penny (Lillywhite 1963; O’Connor and Robertson 2006; Stewart 1999).

Institutions such as Gresham College offered public lectures and demonstrations on topics of scientific and mathematical interest. Works were increasingly written in English (or other vernacular language) rather than scholarly Latin or were promptly translated into English from Latin. Other players in this drama include the recently introduced commercial schools of writing and mathematics and private teachers of “the Mathematicks,” and an increasing number of booksellers, printers, publishers, instrument makers, and sellers of instruments. The fact that Thomas Hood served as “Mathematical Lecturer to the City of London” and dedicated his work on sectors to the auditors of his lectures, rather than to a noble patron, underscores this change in audience and a marked liberalization of accepted ways in which mathematics could be learned.

In the end, the cross cultural milieu changed both the way that mathematicians thought of themselves and the ways in which practitioners viewed the roles of mathematics and science in their trades and professions (Johnston 2005; Taylor 2011, 2013).

As Katie Taylor elegantly summarizes in her article *Vernacular geometry: between senses and reason*,

> In Continental debates about the status of mathematics, the separation of the objects of mathematics from the natural world was a widely cited underpinning for the certainty of mathematics . . . While this debate received relatively little attention in England, it is clear from the instances in which it was touched upon that there was a perceived distinction between the world of reason, geometry's domain, and the world of the senses. In his *Alae seu scalae mathematicae* (1573), Thomas Digges himself set up an opposition between "Queen Reason", responsible for devising the geometrical methods with which stellar observations were to be treated, and the "slave senses", charged with making the observations required to feed into these geometrical methods . . . . Digges went on to stress that these two realms needed to be united to get at truth itself (Taylor 2013).

Meanwhile many practitioners of mechanical and methodological arts embraced the spirit of the newly born scientific revolution and of the applications of mathematical theory to practical concerns and came to view their fields as based upon mathematical and scientific foundations rather than upon craft or artistic traditions, or by appeals to authority (religious or otherwise), tradition, or past practice. The navigator, the designer of buildings and ships, the astrologer, merchant, surveyor, and cartographer came to view themselves as practitioners of the mathematicks and to view the validity of their fields and their practices as supported by a foundation of mathematical theory (whether or not they personally made use of these mathematical underpinnings), laying the foundations for the impressive advances in applied mathematics and the contributions of science and technology to practical matters in the eighteenth and nineteenth centuries.

8 Credits

Figure 1 is a photograph taken by the author of a sector in his private collection. Figures 3, 4, 6, 7, and 8 were derived from images from the 1636 edition of Edmund Gunter's *De sectore et radio*, a work in the public domain, which can be viewed via the Internet Archive at http://www.archive.org/details/descriptionuseof00gunt. Figure 11 was taken from John Robertson's *A treatise of such mathematical instruments, as are usually put into a portable case: containing their various uses in arithmetic, geometry, trigonometry, architecture, surveying*, published in 1775, a work in the public domain and available through ECHO, the European Cultural Heritage Online, which can be viewed at http://echo.mpiwg-berlin.mpg.de/MPIWG:NPREVR4U. All other figures were constructed by the author.

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Early Modern Computation on Sectors

Amy Ackerberg-Hastings

Abstract Before slide rules became widely used in the nineteenth century, European military architects, surveyors, navigators, and other mathematical practitioners performed calculations on sectors. These mathematical instruments have two arms that are joined by a hinge and marked with various proportional, numerical, trigonometric, and logarithmic scales. The user employed a pair of dividers to transfer distances between a sector and a drawing and to measure distances on the scales, effectively creating a series of similar triangles or proportional relationships. Sectors were independently invented on the Italian peninsula and in England at the turn of the seventeenth century; a third version emerged in France by late in the seventeenth century. This paper uses 27 sectors in the Smithsonian's National Museum of American History mathematics collections to trace the history of the instrument.

1 The History of Sectors According to NMAH

While Alexander Matheson (1788–1866) served in the British army in the West Indies and Canada, settled near Perth, Ontario, with other Scottish immigrants after the War of 1812, and helped build the Rideau Canal, he had two sectors in his possession. Sectors are calculating instruments that are usually made of two arms connected with a joint and marked with various proportional, numerical, trigonometric, and logarithmic scales. Distances between two scales or within one scale are measured with a pair of dividers, a drawing compass with points at the ends of both legs. A user often manipulates the dividers to set up problems involving similar triangles. The instruments Matheson owned were made of ivory and manufactured after 1800. Most of their scales prefigured those that characterize slide rules, which made these sectors typical of surviving examples manufactured

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in England but atypical of the larger history of this mathematical calculating instrument. On the other hand, Matheson's background in military surveying and engineering evokes the reasons why mathematical practitioners originally invented sectors.

Matheson's sectors passed to his grandson, Alexander Matheson Richey (1826–1913), who was in the lumber business in the Perth region and then in Chicago. A twentieth-century descendant donated the instruments to the Smithsonian Institution, where they are currently housed in the mathematics collections of the National Museum of American History (NMAH) (Ackerberg-Hastings 2013a). Although the total number of sectors in the collections is relatively small (27 objects, including four found in sets of drawing instruments) and only one dates from the early history of the instrument, this group of objects nevertheless is well suited for telling the story of the sector and how it facilitated early modern computation.

2 Sectors in the Italian Style

The sector is one of several categories of mathematical instruments with multiple nearly-simultaneous inventions. In this case, the earliest version was probably the one introduced by Guidobaldo del Monte (1545–1607) in Italy in the last quarter of the sixteenth century (Drake 1978, p. 12). The instrument had a scale called a line of lines, for solving proportionals and for dividing a line into any number of parts, and a scale called a line of polygons, for constructing regular polygons. Descriptions indicate the device resembled a pair of proportional dividers, with two legs, pointed at both ends, joined in the center by a pivot point that was positioned and tightened by the user so that the distance between the legs represented the desired proportion. Giovanni Paolo Gallucci (1538–ca. 1631) described del Monte's sector in *Della Fabrica et uso di diversi strumenti di Astronomia et Cosmographia* (On the making and use of diverse instruments for astronomy and cosmography, 228 leaves), published in Venice in 1598.

Del Monte was one of Galileo's patrons, which may be how Galileo learned about the instrument. From 1597, he added scales to the sector, which was usually called a "proportional compass" on the Italian peninsula, and modified its design. He moved the hinge away from the center to one end of the device, rendering it more stable. A curved arc between the legs was also marked with scales, and a plumb bob hung from the hinge permitted the device to be used as a level (Van Helden 1995). Galileo eventually settled on a design and arrangement of scales that he published in 1606 in *Le operazioni del compasso geometrico et militare* (Operations of the Geometric and Military Compass), but diagrams were not furnished until the 1640 second edition (Tomash and Williams 2003, pp. 34–35; Istituto 2004). Although only 60 copies were printed initially, this slender volume of 50 pages became widely known, perhaps because it did not take long to read, perhaps because it was by Galileo, and almost certainly at least in part because in

1607 Galileo became embroiled in a priority controversy over Baldassarre Capra's (1580–1626) *Usus et fabrica circini cuiusdam proportionis* (Use and construction of proportional compasses) (1607). Galileo defeated Capra's claims in court. In the next few decades, sectors began to appear from the workshops of instrument makers across Europe.

NMAH owns 3 brass sectors made on the Italian peninsula, one in the early seventeenth century, one in 1683, and one in the eighteenth century. These instruments, along with surveys of online collections such as those at the Museum of the History of Science in Oxford, Harvard University, and the National Maritime Museum, suggest that Italian sectors were typically made of brass and thus represented a significant investment for the mathematical practitioners who purchased one. The objects also suggest that, although military concerns of the Renaissance and early modern periods were clearly influential on designs for the instrument, makers and users did not reach consensus about the form Italian sectors would take.

Thus, the oldest sector in the collections (Fig. 1) bears only two scales, both proportional. One is for dividing a line into fractional parts, and one is for cutting off arcs in a circle. The instrument also has points at the ends of the legs so that the device may be used in a standing position or as dividers, as well as when it was lying on a surface. The other two sectors from the Italian peninsula are marked with 11 and 14 scales, respectively, but only six scales are found on both instruments. The scale labels on all three instruments are mostly in Latin, but the sector made in 1683 also has two scales marked in French that we will see on sectors made in France, *poids des boulets* and *calibre des pieces*. This sector is also the only one marked with the name of its maker, Jacob Lusuerg, who operated a workshop with

Fig. 1 Rectilineal line (proportional scale) on Italian-style sector, early seventeenth century. Catalog number 321678. Negative number 74-1250. Courtesy of the Mathematics Collections, Division of Medicine & Science, National Museum of American History, Smithsonian Institution

his son, Dominicus, in Rome and Modena from 1674 to 1719. No information on the provenance of these sectors is known; the Smithsonian purchased all of them in the 1950s and 1960s from dealers whose sales catalogs did not describe the original owners. As an aside, this group of objects thus also represents changes in museum collecting priorities over time. In this case, when NMAH opened in 1964, curators sought out mathematical treasures to document how science and technology got to the present, befitting the museum's name, Smithsonian Museum of History and Technology. Since 1980, when the building became the National Museum of American History, curators have engaged in the collection, care, and study of objects that reflect the experience of the American people. They have encouraged donations of items with backstories that relate in a concrete way to the story of the United States, broadly defined.

3 Sectors in the French Style

Sectors made in France appear to have emerged in the second half of the seventeenth century. They also were almost all made of brass, and they were engraved with many of the same scales observed on Italian-style sectors. While Italian sectors varied greatly in size, from three to thirteen inches when folded, French sectors were typically around seven inches in the closed position, with a few between three and four inches. The key difference between the instruments, though, is the remarkable uniformity in the selection of scales found on French-style sectors. NMAH owns eleven of these instruments, manufactured between 1677 and 1784. On one side, almost all of them have the following scales: *poid des boulets*, metallic line, line of solids, and line of chords (Fig. 2). On the other side are found scales for *calibre des pieces*, the line of lines, the line of planes, and the tetragonic line (Fig. 3).

The *line of lines* was the base scale for a sector. It was a double scale, which means that the same scale appeared on both legs of the sector. The line of lines was a scale of equal parts, so it was divided into uniform increments. It was used for two purposes, to find the missing value in a proportion and to take measurements that could be transferred to other scales. For example, to solve a problem of the form $10/a = b/x$, the user opened the dividers on one line of lines to the length a. Then, he pivoted the dividers so that the point on the origin moved to the other leg, preserving the length a and widening the sector as needed until the dividers showed that the distance between the sector's legs was a. Next, the user picked up and moved the dividers to the point on one line of lines that represented the length b. The width of the dividers was adjusted so that the other end rested on the point b on the other line of lines scale. This distance between the legs of the sector represented the missing term in the proportion, x. To find the length of x, the user again pivoted and moved the dividers without allowing them to collapse. One end was placed at the origin point of one line of lines; the other rested on the point x. (For instructions for using a sector, see also Heather 1851, pp. 33–42; Sangwin 2003.) In essence, the

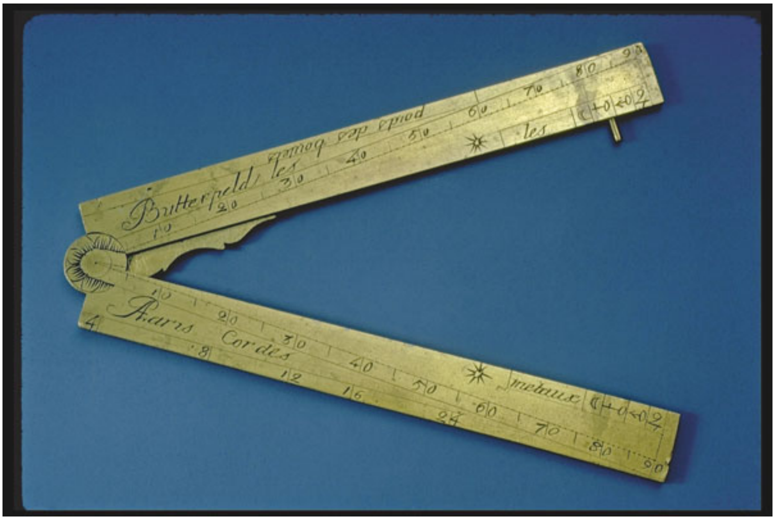

Fig. 2 French-style sector signed by Michael Butterfield of Paris, 1677–1724. Scales shown are a line of chords, metallic line, and *poids des boulets*. Cat. no. 321676. Neg. no. 74-1247. Courtesy of the Mathematics Collections, Division of Medicine & Science, National Museum of American History, Smithsonian Institution

Fig. 3 French-style sector, eighteenth century. Scales shown are *poid des boulets*, line of lines, line of solids, and tetragonic line. Cat. no. 314799. Neg. no. DOR2012-2585. Courtesy of the Mathematics Collections, Division of Medicine & Science, National Museum of American History, Smithsonian Institution

user's motions with the dividers created two overlapping similar triangles. Holding the dividers precisely was vital.

The *line of planes* was another double scale that was used to find ratios between rectangular areas and to calculate square roots. Likewise, the *line of solids* was employed to solve proportions involving volumes and to obtain cube roots. The double scale *line of chords* permitted the construction of angles.

The *tetragonic line* and *metallic line* were both typically found along the fold line of the sector, albeit on opposite faces. The tetragonic line had points representing the sides of regular polygons with the same area, from an equilateral triangle to a 13-sided regular polygon. The user created proportions that provided the areas of regular polygons with other side lengths. The metallic line was marked with alchemical symbols for metals such as gold, lead, silver, copper, iron, and tin, placed proportionately from the sector joint so that balls made from those metals with those radii would weigh the same (Stone 1723, pp. 47–52; Tomash and Williams 2003, pp. 37–39; Istituto 2004).

The line of lines, line of planes, line of solids, and line of chords were multi-purpose scales suitable for a general range of calculations, including those used for military surveying and architecture. The tetragonic line and metallic line were specifically for designing fortifications and for gunnery. The *poid des boulets* and *calibre des pieces* scales were also explicitly for artillery applications. Different types of early modern cannon required different types of shot; each type of shot had a particular volume, weight, and amount of powder. However, the diameters of cannon and thus of shot within a given type varied. The *poid des boulets* scale helped gunners determine the required weight for shot of a particular diameter. Similarly, the *calibre des pieces* scale was used to determine the needed diameter of shot, given the width of the cannon opening and the weight of the shot.

The instruments made in France thus follow Italian-style sectors in their military emphasis, but unlike the Italian instruments, they were standardized in form. They also differ from those made on the Italian peninsula in that nine of the eleven were signed by their makers. These were prominent craftsmen: four are by Michael Butterfield (1635–1724), the expatriate English maker of sundials; two are from the Paris workshop of Nicolas Bion (ca. 1652–1733), best known for his 1709 treatise on the construction of mathematical instruments; one is from Louis Lennel, who advertised himself in 1781 as the official maker for France's king and navy; one is by Pierre Le Maire (1717–1785), who succeeded his father, Jacques Le Maire, in operating a well-known workshop; and one was made by Pierre Martel (1706–1767), a French-speaking engineer, mathematics teacher, maker of mathematical instruments, and geographer who lived in Geneva. Oddly, the *poid des boulets* and *calibre des pieces* scales are depicted in Bion's illustrations but never mentioned in the text (Bion 1709, pp. 29–74). Once again, the provenances for these sectors are unknown, so it is not possible to reconstruct who used them and for what specific purposes.

4 Sectors in the English Style

Sectors were invented almost simultaneously in the Italian lands and in England. In 1598, the same year that Gallucci described del Monte's instrument, Thomas Hood (1556–1620) of London wrote *The Making and Use of the Geometrical Instrument Called a Sector*, in which he indicated that a folding instrument with a hinge at one end, a crossbar or index arm, and sights and mountings for use in surveying was already in common use. Hood also gave the device an English name derived from Euclid's *Elements of Geometry*; the Italians and French favored "compass of proportion." Hood noted that sectors could be purchased from his own instrument maker. These instruments probably had few scales besides a line of lines. Sometime between 1606 and 1623, Edmund Gunter (1581–1626) fiddled with the scales, notably adding logarithmic scales. He also worked out processes for performing navigational calculations and put those scales on the instrument. He described his version in *De sectore et radio*, published in London in 1623 (Fig. 4).

Already, then, there were significant differences between the English and Italian or French versions of the instrument. English-style sectors were not tools specifically for constructing military fortifications and positioning artillery. All of the standard scales on sectors made in England served general purposes and were applicable to a wide variety of real-life problems, including those found on sectors made elsewhere, such as the line of lines and line of chords, and those unique to England, such as logarithmic and trigonometric scales. English-style sectors were found on ships, with military and civilian surveyors, and in the workshops of mechanics and craftsmen. These instruments were also made from additional types

Fig. 4 This detail from the title page of Edmund Gunter's 1623 *De sectore et radio* shows a gentleman making calculations with a sector and dividers. Courtesy of the Smithsonian Libraries, Washington, DC

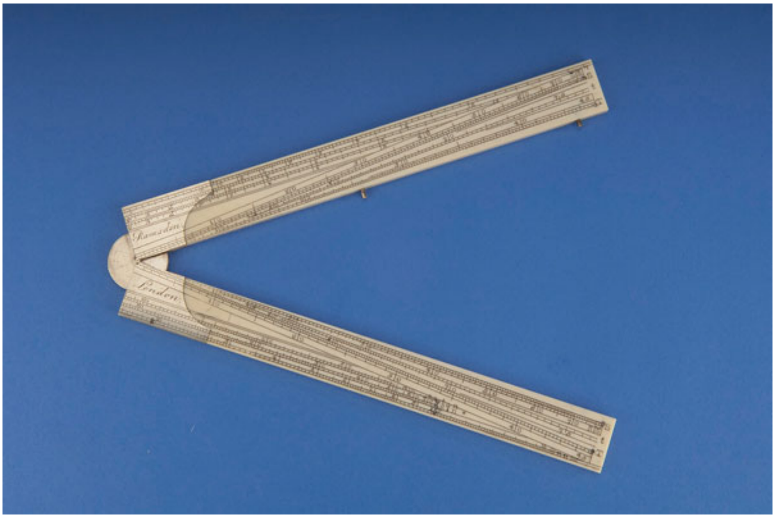

Fig. 5 Ivory English-style sector signed by Jesse Ramsden, London, 1765–1800. Scales shown are logarithmic sines, tangents, and numbers (spanning the hinge); and double scales for sines (10–90°) and tangents (45–75° and 10–45°). Cat. no. 317364. Neg. no. DOR2012-2586. Courtesy of the Mathematics Collections, Division of Medicine & Science, National Museum of American History, Smithsonian Institution

of materials, namely ivory and boxwood, which made them affordable for a larger number of practical mathematicians (Fig. 5). Since these materials were softer than brass, makers often put round brass inserts at the most frequently used points to help the instruments last longer. English sectors were generally a standard size of six inches, so they measured an English foot when fully opened.

NMAH owns 12 English-style sectors, dating between 1712 and the mid-nineteenth century. Although there are no seventeenth-century examples in the collections to document the development, these objects suggest that, in the century after Gunter's treatise appeared, the level of uniformity among English makers became even more pronounced than in France. Additionally, English makers put even more scales on sectors than their counterparts in France. Several of these scales were for horology, which was another unique characteristic of eighteenth-century English-style sectors. The makers of sundials needed scales for *hours*, *chords*, *latitude*, and the *inclined meridian* (Fig. 6). Other scales included *rhumbs* and *longitude* for navigation; *tangents*, *sines*, and *secants* for trigonometry; and *logarithmic numbers*, *sines*, and *tangents*. For single scales (those that appeared on only one leg or across both legs), the user made calculations by measuring distances within the scale. For instance, numbers were multiplied on a logarithmic scale by placing one point of the dividers at the origin of the scale and the other point on the first number to be multiplied, then picking up and moving the first point on the dividers to the second number to be multiplied and reading off the product under the second point of the dividers.

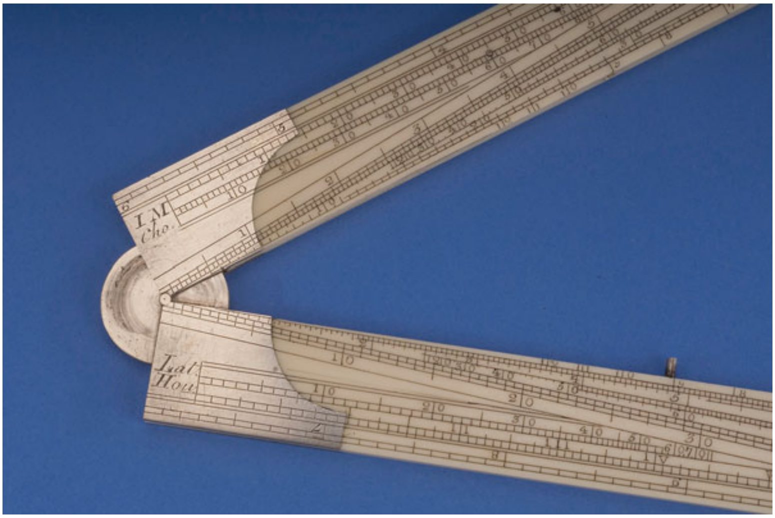

Fig. 6 Close-up of the scales for dialing on the other side of the sector from Ramsden's workshop, 1765–1800: inclined meridian, chords, latitude, and hours. Double scales along the fold line are line of lines, secant line, line of chords, and tetragonal line. A 12-inch ruler divided to 1/10-inch is along the outside edge. Cat. no. 317364. Neg. no. DOR2012-2539. Courtesy of the Mathematics Collections, Division of Medicine & Science, National Museum of American History, Smithsonian Institution

Six of the objects came from prominent London workshops: Richard Glynne, who was active from 1712 to 1729; Jesse Ramsden, open from about 1765 to 1800 and best known for inventing a dividing engine; William Harris, whose shop operated from 1799 to 1839; Thomas Harris and Son (apparently no relation to William), located across the street from the British Museum between 1806 and 1846; Thomas Rubergall, who worked between 1802 and 1854; and James Gilkerson, in business from 1809 to 1825 (Webster Signature Database 2007). The other sectors are unsigned and may have been made later in the nineteenth century than the signed instruments, as they lack the scales for horology. These illustrate yet another characteristic of English-style sectors, which is that they were included not only in finely-made sets of instruments for the wealthy and the most highly skilled practitioners but also in sets marketed to beginning mechanics and students. And, sales of sectors continued until a significantly later period in Great Britain than elsewhere in Europe.

Once again, very little is known of the people who originally owned these instruments. The sector made by Gilkerson and one of the unsigned sectors belonged to Alexander Matheson, mentioned in the introduction. Another of the unsigned sectors was associated with the Drapers, a prominent American scientific family that among other things established an observatory in New York City in 1868. A third unsigned sector was used by another American, Allen A. Jones, while he served in the US Corps of Engineers during World War I and into the 1960s during his career as a civil engineer. He inherited the instrument as part of a set from an uncle or great-uncle; the family believed this relative had worked as a surveyor in the Chicago area before the city was formally founded in 1833 (Ackerberg-Hastings 2013b).

5 Conclusion

This joint tour of the history of the sector and NMAH's holdings with respect to the instrument highlights several themes in early modern computation. For instance, the types of scales and the applications for which they were employed illustrate the diversity of mathematics in the seventeenth and eighteenth centuries as well as the ingenuity of mathematicians, mathematical practitioners, and instrument makers. This is also a story of national differences in the history of science and mathematics, as the purposes and audiences for sectors influenced factors including the types of scales chosen and the materials utilized. Finally, the approach of this paper suggests the possibilities for using museum collections to tell the histories of mathematical instruments. While being on site to handle the objects will always be the best way to learn about them—and the author indeed updated and expanded NMAH catalog descriptions for each of these sectors—increasing digitization of collections will continue to expand access to historic instruments for faculty, students, researchers, and the public.

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The Eighteenth-Century Origins of the Concept of Mixed-Strategy Equilibrium in Game Theory

Nicolas Fillion

Abstract This paper examines the circumstances surrounding the first historical appearance of the game-theoretical concept of mixed-strategy equilibrium. Despite widespread belief that this concept was developed in the first half of the twentieth century, its origins are in fact to be found in the early eighteenth century. After reconstructing the game analysis of Montmort, Waldegrave, and Bernoulli using modern methods and terminology, I argue that their discussion of the concept of solution to a game also anticipated refinements of the concept of equilibrium that we typically associate with the second half of the twentieth century.

1 Introduction

What are the origins of game theory? Answering the question in a way that can enlighten the increasingly broad field of contemporary applications of game theory in economics, biology, psychology, linguistics, philosophy, etc., requires a close examination of both the past and the present state of the discipline. Indeed, asking about the origins of game theory with such purposes in mind is an enterprise that consists in tracing the historical emergence and refinements of game-theoretical methods of analysis and concepts of solution, as well as the improvement of their philosophical, mathematical, and scientific foundations, and in comparing the various stages of development with today's stage.

This paper focuses on the historical emergence of a concept of solution that plays a central role in game-theoretical analyses of strategic games that have no pure strategy equilibria. The concept in question is, of course, that of mixed-strategy equilibrium. What, then, are the origins of the concept of mixed-strategy equilibrium? Perhaps because there is relatively little literature on the history of game theory, almost all academics who are asked this question answer it incorrectly.

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Typically, it is thought that the origins of the concept of mixed-strategy equilibrium and the origins of modern game theory are one and the same, and that the origins of the latter can be portrayed along these lines:

> [. . .] the conventional view of the history of game theory in economics is relatively simple to narrate. It was that von Neumann wrote a paper in the late 1920s on two-person games and minimax. Borel claimed priority but this claim was rejected as mistaken. Then von Neumann and Morgenstern got together in Princeton, wrote their book in 1944, and the word went forth. (Weintraub 1992, p. 7)

Weintraub, however, correctly emphasizes that "this potted history is misleading in all its details," and the collection of essays he edited is meant to rectify the situation. Indeed, insofar as the concept of mixed-strategy equilibrium is concerned, far from finding its origins in the work of von Neumann and Morgenstern (1944), we must instead look back to the beginning of the eighteenth century.

To be clear, the claim is not that a general theory of mixed-strategy equilibria providing general existence proofs had been formulated in the eighteenth century. This general theory is uncontroversially a twentieth century creation. Borel provided the minimax solution of games with a small number of pure strategies (Borel 1921) together with some conjectures about the existence of such solutions in general [see Dimand and Dimand (1992) for a detailed account], and von Neumann (1928) proved the general statement of the minimax theorem for any finite number of pure strategies in any two-person zero-sum game. A few decades later, von Neumann's minimax theorem was generalized by Nash's theorem for mixed-strategy equilibria in arbitrary non-cooperative finite games (Nash 1950).

This being said, concepts and methods are typically older than their foundations, and in this case the first known articulation of the concept of mixed-strategy equilibrium and the first calculation of a mixed-strategy equilibrium are due to an amateur mathematician going by the name Waldegrave. This fact has been noted by many authors [e.g., Todhunter (1865), Fisher (1934), Kuhn (1968), Rives (1975), Hald (1990), Dimand and Dimand (1992), Bellhouse (2007), and Bellhouse and Fillion (2015), to name a few], but the story is to this day not entirely clear.

A first element bringing confusion is that, following Kuhn (1968), the Waldegrave in question is normally identified as James 1$^{\text{st}}$ Earl Waldegrave. See Fig. 1. However, as Bellhouse (2007) pointed out, this is incorrect since Montmort (1713, p. 388) reveals that the Waldegrave in question is a brother of Henry Waldegrave; this leaves Charles, Edward, and Francis as candidates. Bellhouse (2007) argued in favor of Charles, but in a later paper (Bellhouse and Fillion 2015) we have examined calligraphic evidence that shows that Francis Waldegrave deserves the credit for this innovation.

In addition to confusion about who Waldegrave was, many who have written on the topic have harshly judged Waldegrave, Montmort, and Nicolaus Bernoulli, concluding that none of them really had a good grasp of the nature of the problem. For instance, based on a misinterpreted remark, Henny suggests that even though Waldegrave somehow stumbled upon the right solution, he did not have the mathematical skills to demonstrate his result (Henny 1975). Another case in point is the argument by Fisher (1934) that "Montmort's conclusion [that no absolute rule could be given], though obviously correct for the limited aspect in which

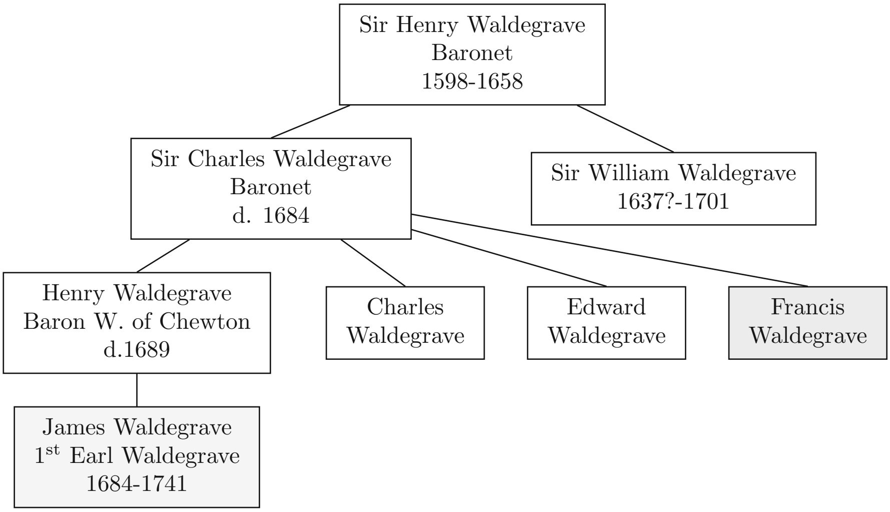

Fig. 1 Some members of the Waldegrave family. Reproduced from Bellhouse (2007). Whereas James is often identified as the one who contributed the mixed-strategy solution, calligraphic evidence suggests that it is Francis (Bellhouse and Fillion 2015)

he viewed the problem, is unsatisfactory to common sense, which suggests that in all circumstances there must be, according to the degree of our knowledge, at least one rule of conduct which shall be not less satisfactory than any other; and this his discussion fails to provide." Once again, Fisher's argument is based on a misinterpretation of Montmort's position; as I will argue in Sect. 3, far from leaving common sense unsatisfied, Montmort was considering a perspective on game theory that only came to the forefront in the second half of twentieth century. In a previous paper (Bellhouse and Fillion 2015), we have translated correspondence involving Montmort, Bernoulli, and Waldegrave that was not published in the second edition of *Essay d'Analyse*; this additional correspondence decisively refutes such claims. Instead, it shows that over the years all three came to a very clear understanding of the situation.

2 Le Her and Its Solution

At the end of *Essay d'Analyze des Jeux de Hazard* (Montmort 1708), Montmort proposed four unsolved problems to his readers. The second one concerns the game Le Her, and its statement is as follows:

> Here is the problem of which we request the solution:
> Three players, Pierre, Paul & Jacques are the only remaining players, and they have only one chip left. Pierre is the dealer, Paul is to his right, & Jacques follows. We request what their odds are with respect to the position they occupy, & in which proportion they should

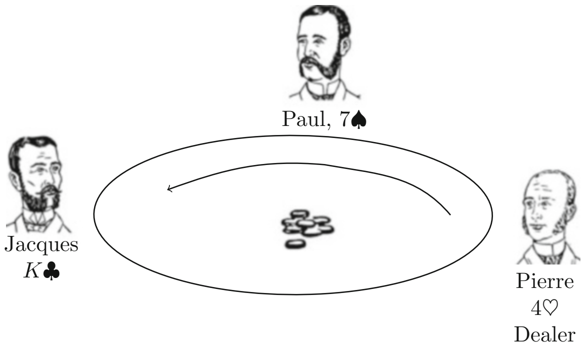

Fig. 2 Setup of players at a card table for Montmort's fourth problem concerning Le Her

> split the pot, if it is, say, 10 coins, if they wanted to share it among themselves without finishing the game. (Montmort 1713, p. 279)

The situation can be represented as in Fig. 2.

The game Le Her that Montmort described in *Essay d'Analyse* is a game of strategy and chance played with a standard deck of 52 playing cards. The dealer Pierre distributes a card face down to Paul, Jacques, and then to himself, and places the remainder of the deck aside. The objective of the game is to end the round with the highest card, where aces are below twos and kings are highest. If there is a tie, the dealer or the player closest to the dealer wins; for example, if all three players end the round with a 10, Pierre wins, whereas if Pierre receives a 9 and Paul and Jacques receive a 10, Jacques wins. The strategic element of the game comes from the fact that the players do not have to stick with the card they are initially dealt. At the beginning, all players look at the card they have received. Then, one after the other, starting with Paul (or whoever is the first player to the right of the dealer), the players have the opportunity to switch their card with that of the player to their right; since the dealer is last, he can switch his card for the card at the top of the deck. However, if a player has a king, he can (and therefore will) refuse to switch his card and simply block the move. In this eventuality the player who attempted the move is stuck with his card.

Consider again Fig. 2. Pierre deals a card to each player, face down, and they look at their cards. Paul goes first, and seeing a 7, he decides to switch. He must switch with Jacques, but Jacques has a king and blocks the switch. It is now Pierre's turn and, being rightly convinced that his 4 will not fare well, he decides to switch his card for the one at the top of the deck. He turns the card and sees a king. His switch is thus automatically blocked,[1] and Jacques wins the hand.

In the second edition of *Essay d'Analyse* (Montmort 1713), Montmort added a fifth part that includes correspondence between Nicolaus Bernoulli, Waldegrave,

[1] Any player would block the switch with a king, and so the rules of the game prescribe the same for the deck. In fact, if Pierre could switch his card for a king, it would give him much better odds of success.

l'Abbé d'Orbais, and himself concerning this game. Despite the fact that Montmort posed the problem for three players, however, their discussion has focused on the two-person case for the sake of simplicity. Their correspondence eventually establishes that there is no pure strategy equilibrium and provides the calculated value for a mixed-strategy equilibrium. Note, however, that none of them actually provide the details of their calculations and instead merely state their results. For the sake of clarity, before examining their correspondence, we will consider a modern approach to analyzing the game and finding its solution.[2]

Let us look at things from Paul's point of view.[3] To get started, observe that if Paul receives a king, it is obviously in his best interest to hold on to it, since switching would guarantee that he would lose. However, if Paul were to hold only to a king and switch with any card of lower value, he would be letting go of strong cards that would very likely win the game, such as jacks and queens. The opportunity cost of this conservative policy would lead to a likely loss (as we will see, around 66 % of the time), so he needs to be more inclusive. At the same time, if Paul were to hold on to an ace, he would lose the round no matter what card Pierre had. Thus, it is easy to see that holding on to an ace and other very low cards is a losing strategy. This outlines the strategic landscape: Paul should hold on to high cards, and switch with low cards. But what exactly should be the threshold value below which it is recommendable for Paul and Pierre to switch? And how should this threshold change as a function of their respective positions? This requires a more extensive analysis.

In general, let (i,j) be the values of the cards dealt to Paul and Pierre, respectively. By "the value" of a card, we mean the ranking $1, 2, \ldots, 13$ of the card notwithstanding its suit, where 1 is an ace, 11 is a jack, 12 is a queen, and 13 is a king. Moreover, we let m be the minimal card value to which Paul holds and n be the minimal card value to which Pierre holds, with the understanding that they switch with any card below this value. Each of the 13 possible holding thresholds constitutes a pure (i.e., non-randomized) strategy, and a choice of a holding threshold for Pierre and Paul is a (pure) *strategy profile*, which we denote $\langle m, n\rangle$. Our first objective is to find the probability $P(\langle m, n\rangle)$ that Paul will win when a certain strategy profile $\langle m, n\rangle$ is employed. We will then compare those probabilities to find which strategy is favorable to Paul, and the extent to which it is favorable.

The probability of Paul winning when the pair of cards (i,j) is dealt and when the strategy profile $\langle m, n\rangle$ is employed, denoted $P_{i,j}(\langle m, n\rangle)$, is simply the product of the probability of dealing (i,j), denoted $P_{i,j}$, and of the probability that Pierre draws a card k that makes Paul win if he decides to draw a card. Let $C_{i,j}(\langle m, n\rangle)$ be the number of cards left in the deck that would make Paul win if Pierre were to draw one of them. Then, we have

[2] See also Hald (1990, pp. 314–322) for a good alternative presentation.

[3] Since it is a zero-sum game, this is an assumption that leads to no loss of generality.

$$P_{i,j}(\langle m,n\rangle) = P_{i,j}\frac{C_{i,j}(\langle m,n\rangle)}{50}\,. \tag{1}$$

If the game is decided without necessitating the drawing of a third card k, then for convenience we will let $C_{i,j}(\langle m,n\rangle)$ be 0 if Paul loses and 50 if he wins. Also, the probability of being dealt (i,j) is simply

$$P_{i,j} = \begin{cases} \frac{4}{52}\cdot\frac{4}{51} & i \neq j \\ \frac{4}{52}\cdot\frac{3}{51} & i = j \end{cases}\,. \tag{2}$$

Moreover, the probability $P_i(\langle m,n\rangle)$ that Paul will win when he is dealt a given card i is a cumulative probability given by

$$P_i(\langle m,n\rangle) = \sum_{j=1}^{13} P_{i,j}(\langle m,n\rangle) = \sum_{j=1}^{13} P_{i,j}\frac{C_{i,j}(\langle m,n\rangle)}{50}\,. \tag{3}$$

Finally, the probability that Paul will win under strategy profile $\langle m,n\rangle$ is given by

$$P(\langle m,n\rangle) = \sum_{i=1}^{13} P_i(\langle m,n\rangle) = \sum_{i=1}^{13}\sum_{j=1}^{13} P_{i,j}\frac{C_{i,j}(\langle m,n\rangle)}{50}\,. \tag{4}$$

For both Paul and Pierre, their decision will consist in comparatively analyzing those values for all combinations of m and n in order to determine how to maximize their respective chances of winning.

As we see, most of the work consists in finding the values of $C_{i,j}(\langle m,n\rangle)$ for various (i,j), and $\langle m,n\rangle$, based on the rules of the game. Instead of writing the function $C_{i,j}(\langle m,n\rangle)$ explicitly as a complicated piecewise function, it might be more instructive to consider a few examples. To begin, suppose the strategy employed is $\langle 8,9\rangle$ and that the dealt cards are $(7,10)$. Paul would then keep any card equal or higher to an 8. But since he has received a 7, he switches with Pierre and gets a 10. Now, Pierre knows that the 7 with which the trade left him is a losing card, and he thus switches it with a card from the deck. By examination of the cases, we find that drawing any ten, jack, or queen would make him win (remember, kings will not work, as they would block his switch), for a total of 11 cards. As any other card would make Paul win, we find that $C_{7,10}(\langle 8,9\rangle) = 50-11 = 39$. Now, suppose that the strategy employed is still $\langle 8,9\rangle$, but now the players are dealt the cards $(9,10)$. In this scenario, Paul does not switch as $9 > 8$, and neither does Pierre as $10 > 9$. As a result, Pierre wins. Following our convention, $C_{9,10}(\langle 8,9\rangle) = 0$ as Pierre didn't draw and won the hand.

For more generality, there is a MATLAB code in Appendix 1 to compute the function $C_{i,j}(\langle m,n\rangle)$ for any admissible values of i,j,m, and n. Using this code, we can easily compute the winning card counts for Paul in any strategy profile. For instance, if we once again consider the strategy profile $\langle 8,9\rangle$, the resulting winning card counts for Paul corresponding to any pair of dealt cards (i,j) are given in Fig. 3.

	Pierre switches								Pierre holds					
Paul\Pierre	1	2	3	4	5	6	7	8	9	10	11	12	13	$P_i(\langle 8,9\rangle)$
Switch														
1	0	7	11	15	19	23	27	31	35	39	43	47	0	0.0358
2	0	0	11	15	19	23	27	31	35	39	43	47	0	0.0350
3	0	0	0	15	19	23	27	31	35	39	43	47	0	0.0337
4	0	0	0	0	19	23	27	31	35	39	43	47	0	0.0319
5	0	0	0	0	0	23	27	31	35	39	43	47	0	0.0296
6	0	0	0	0	0	0	27	31	35	39	43	47	0	0.0268
7	0	0	0	0	0	0	0	31	35	39	43	47	0	0.0235
Hold														
8	31	31	31	31	31	31	31	28	0	0	0	0	0	0.0286
9	35	35	35	35	35	35	35	35	0	0	0	0	0	0.0338
10	39	39	39	39	39	39	39	39	50	0	0	0	0	0.0437
11	43	43	43	43	43	43	43	43	50	50	0	0	0	0.0536
12	47	47	47	47	47	47	47	47	50	50	50	0	0	0.0635
13	50	50	50	50	50	50	50	50	50	50	50	50	0	0.0724
									Total (*i.e.*, $P(\langle 8,9\rangle)$)					0.5118

Fig. 3 Table of winning card counts for Paul for any cards (i, j) when the strategy profile $\langle 8, 9\rangle$ is employed. In the rightmost column are the cumulative probabilities (rounded to four digits) that Paul will win with card i when the strategy profile $\langle 8, 9\rangle$ is employed

With an efficient way of calculating the values of $C_{i,j}(\langle m, n\rangle)$, we can then proceed to finding Paul's advantage under a certain strategy profile using Eqs. (1), (3), and (4). This can also be easily achieved with the MATLAB code given in Appendix 2. As an illustration, the values of $P_i(\langle 8, 9\rangle)$ (i.e., the probabilities of winning with card i under the strategy profile $\langle 8, 9\rangle$) and the probability $P(\langle 8, 9\rangle)$ that Paul will win with this strategy profile are given in Fig. 3.

Since this is a general way of computing the probabilities that Paul will win with a given strategy profile, we can then obtain the probabilities of winning associated with each of the 13 × 13 strategy profiles. The results are displayed in Fig. 4; this information is the basis for our analysis seeking to determine what would be the most advantageous course of action for Paul.

Let us begin our analysis of this information. In what follows, "strategy m" refers to holding any card of value m or higher. Moreover, we say that strategy *m dominates* strategy n if strategy m does at least as well as strategy n against all the strategies that the opponent might employ. First, observe that for Paul strategy 1 is *dominated* by strategy 2, strategy 2 is dominated by strategy 3, strategy 3 is dominated by strategy 4, strategy 4 is dominated by strategy 5, and strategy 5 is dominated by strategy 6. Thus, Paul would be making a clear mistake by holding on to a card lower than a 6. Moreover, observe that strategy 13 is dominated by strategy 12, strategy 12 is dominated by strategy 11, strategy 11 is dominated by strategy 10, strategy 10 is dominated by strategy 9, and strategy 9 is dominated by strategy 8. Thus, only holding on to cards higher than 8 would also be a clear mistake. This captures the idea mentioned earlier that only holding on to very high cards would

Paul \ Pierre	1	2	3	4	5	6	7	8	9	10	11	12	13
1	0.47059	0.44006	0.41508	0.39566	0.38178	0.37345	0.37068	0.37345	0.38178	0.39566	0.41508	0.44006	0.47059
2	0.50643	0.4759	0.45092	0.43149	0.41762	0.40929	0.40652	0.40929	0.41762	0.43149	0.45092	0.4759	0.50643
3	0.53538	0.51005	0.48471	0.4648	0.45044	0.44163	0.43837	0.44066	0.44851	0.4619	0.48084	0.50534	0.53538
4	0.55698	0.53635	0.51572	0.49508	0.47976	0.46998	0.46576	0.46709	0.47397	0.4864	0.504377	0.5279	0.55698
5	0.57074	0.55433	0.53792	0.52151	0.5051	0.49388	0.48821	0.48808	0.49351	0.50449	0.52103	0.54311	0.57074
6	0.57617	0.5635	0.55083	0.53816	0.52549	0.51282	0.50522	0.50317	0.50667	0.51572	0.53032	0.55047	0.57617
7	0.57279	0.56338	0.55397	0.54456	0.53514	0.52573	0.51632	0.51186	0.51294	0.51958	0.53176	0.5495	0.57279
8	0.56012	0.55348	0.54685	0.54021	0.53357	0.52694	0.5203	0.51367	0.51186	0.51567	0.52489	0.53973	0.56012
9	0.53768	0.53333	0.52899	0.52465	0.52037	0.51596	0.51161	0.50727	0.50293	0.50329	0.5092	0.52066	0.53768
10	0.50498	0.50244	0.49991	0.49738	0.49484	0.49231	0.48977	0.48724	0.48471	0.48217	0.48422	0.49183	0.50498
11	0.46154	0.46033	0.45913	0.45792	0.45671	0.45551	0.4543	0.45309	0.45189	0.45068	0.44947	0.45273	0.46154
12	0.40688	0.40652	0.40615	0.40579	0.40543	0.40507	0.40471	0.40434	0.40398	0.40362	0.40326	0.4029	0.40688
13	0.34051	0.34051	0.34051	0.34051	0.34051	0.34051	0.340517	0.34051	0.34051	0.34051	0.34051	0.34051	0.34051

Paul \ Pierre	1	2	3	4	5	6	7	8	9	10	11	12	13
6	0.57617	0.5635	0.55083	0.53816	0.52549	0.51282	0.50522	0.50317	0.50667	0.51572	0.53032	0.55047	0.57617
7	0.57279	0.56338	0.55397	0.54456	0.53514	0.52573	0.51632	0.51186	0.51294	0.51958	0.53176	0.5495	0.57279
8	0.56012	0.55348	0.54685	0.54021	0.53357	0.52694	0.5203	0.51367	0.51186	0.51567	0.52489	0.53973	0.56012

Fig. 4 The matrix on the *right* contains the probabilities $P(\langle m, n\rangle)$ (rounded to the fifth digit) that Paul wins in each of the 13×13 strategy profiles. The matrix on the *left* results from a first round of removal of dominated strategies

be too conservative, while holding on to low cards would be too inclusive, so that the question revolves around which middle value Paul and Pierre should hold on to. By using the method of iterated removal of strictly dominated strategies, this leaves us with the possibilities displayed on the left in Fig. 4, namely holding on to sixes, sevens, or eights.

Now, let us remember that the game is zero-sum, so that Pierre's probabilities of winning are 1 minus those of Paul's. Thus, observing the reduced game on the left in Fig. 4, we see that for Pierre, strategies 1–7 are all dominated by 8. Moreover, the overly conservative strategies are also dominated, i.e., strategies 10–13 are dominated by strategy 9. Thus, assuming that Paul will not play a dominated strategy, the only strategies that are not dominated for Pierre are 8 and 9. However, assuming that Pierre will restrict himself to those two non-dominated strategies, we observe that for Paul 6 is also dominated by 7. Now, it is not possible to further reduce this game by eliminating dominated strategies, so the process of iterated removal of dominated strategies is complete. Whereas we started with 169 possible strategy profiles, we have now identified only four possibilities that are truly viable options for Paul and Pierre. The resulting reduced game is displayed in Fig. 5.

Could there be a self-enforcing agreement between Paul and Pierre on a strategy profile, i.e., is there a strategy profile among those four that is such that, if it were played, neither player would gain from changing his strategy? To begin, were Paul to play 7, Pierre would play 8. But if Pierre were to play 8, Paul would play 8 himself. However, if Paul played 8, then Pierre would play 9. Finally, if Paul played 9, Paul would play 7. Thus, as we see, none of the four remaining strategy profiles is a Nash equilibrium. If we were limited to pure strategy equilibria, we would have to conclude that there is no self-enforcing agreement on a pair of strategies, and consequently that it is impossible to uniquely determine the advantage of Paul over Pierre.

However, a standard procedure in such circumstances is to use an extended reasoning that involves chance. Instead of insisting that the game is solved only by identifying a pure strategy profile in which each player is best-responding to the opponent's strategy, we allow players to determine which strategy they will play by using a randomizing device. A solution would then be a probability distribution over the pure strategies that guarantees each player the maximal value of his minimum payoff. This type of solution is now known as a minimax solution in the case of two-person zero-sum games, and more generally as a mixed-strategy equilibrium.

	Pierre	
Paul	8	9
7	0.51186	0.51294
8	0.51367	0.51186

	Pierre	
Paul	8	9
7	2828/5525	2834/5525
8	2838/5525	2828/5525

Fig. 5 Reduction of the game by removing the strictly dominated strategies for both players. On the *left*, values calculated by the MATLAB code provided here; on the *right*, the rational values provided by Montmort (1713, p. 413)

In the game Le Her, each player has the choice between 13 pure strategies. However, the 11 dominated strategies should not be employed, and thus they should be employed with a zero probability. Moreover, Paul will hold on to a seven with a non-zero probability p and he will hold on to an eight with a probability $1 - p$, and Pierre will hold on to an eight with a probability q and he will hold on to a nine with a probability $1 - q$. If we suppose that there is \$1 in the pot, Paul's payoff will vary with respect to p and q as follows:

$$\text{Payoff}(p, q) = p(a_{11}q + a_{12}(1 - q)) + (1 - p)(a_{21}q + a_{22}(1 - q)) \,, \qquad (5)$$

where the a_{ij}s are the entries in the probability matrix of Fig. 5. This function has the characteristic shape of a saddle (see Fig. 6). The so-called *saddle point*, indicated by the dot in the figure, is the probability allocation that constitutes the mixed-strategy equilibrium. To calculate the (p, q) coordinates of the saddle point, we reason that, as it is a zero-sum game, Paul should choose p so as to make Pierre indifferent between holding on to an eight or a nine, and Pierre should choose q so as to make Paul indifferent between holding on to a seven or an eight. This gives us two equations:

$$a_{11}p + a_{21}(1 - p) = a_{12}p + a_{22}(1 - p) \qquad (6)$$

$$a_{11}q + a_{12}(1 - q) = a_{21}q + a_{22}(1 - q). \qquad (7)$$

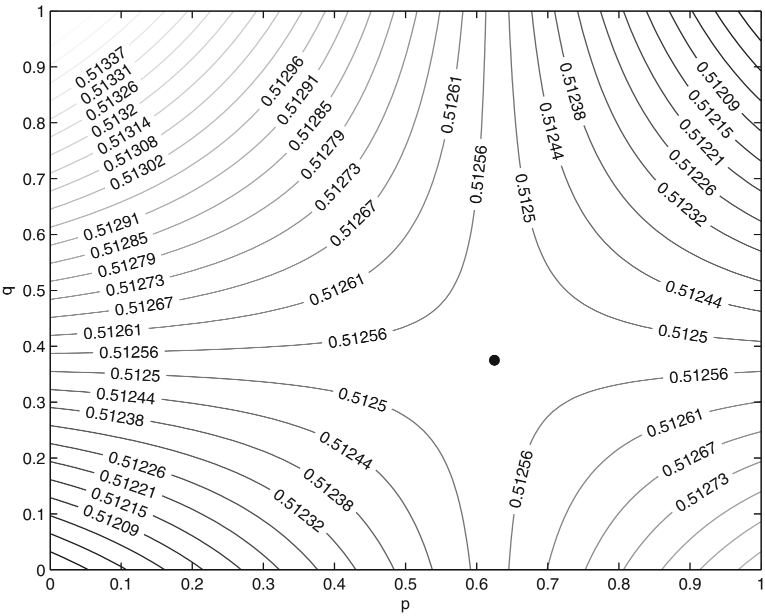

Fig. 6 Contour plot for Paul's payoff as a function of p and q. The saddle point is indicated by the *dot*

Solving the two equations gives us

$$p = \frac{3}{8} \qquad (1-p) = \frac{5}{8} \qquad q = \frac{3}{8} \qquad (1-q) = \frac{5}{8}\,. \tag{8}$$

Thus, Paul should hold on to the 8 slightly more often than to a seven, and Pierre should hold on to a nine slightly more often than to an eight, just as Waldegrave had found.

3 Jouer au Plus Fin

After the publication of his *Essay d'Analyse* (1708), Montmort sent copies of his book to Johann Bernoulli, and this sparked a correspondence on Le Her involving Montmort, Nicolaus Bernoulli, Waldegrave, and l'Abbé d'Orbais. The correspondence began in 1710 and went on until 1715. The earlier part of this correspondence—slightly corrected and edited by Montmort—was published as part V of the second edition of *Essay d'Analyse* (1713). Despite initial confusion, they reached a surprisingly refined understanding of the various ways in which Montmort's problem could be claimed to be solved.

The main elements of the published correspondence are the ones explained earlier. Firstly, they discuss their respective calculations of the probability that Paul and Pierre will win under different strategy profiles and establish that they are in agreement in this respect. Furthermore, after some hesitation they come to an agreement concerning the fact that there is no fixed point on which both players can settle so that, in modern terminology, there is no pure strategy equilibrium. Moreover, the idea of using a randomizing device to determine which strategy is to be employed is introduced, most likely by Waldegrave. The randomizing device in question is a bag in which we put a number of black and white counters. In addition, Waldegrave advances the idea that the ratio of counters that will be optimal for Paul is 5 to 3, whereas it is 3 to 5 for Pierre. We have examined this discussion in detail elsewhere (Bellhouse and Fillion 2015). Here, I will examine their later debate on whether this mixed-strategy profile is solving Montmort's problem in a completely satisfactory way.

Montmort thought that the calculation of this optimal mixed strategy was not fully answering the question, and that a full answer was in fact impossible to obtain on purely mathematical grounds. This claim was vehemently opposed by Bernoulli and, two centuries later, by Fisher. Despite having a clear understanding of the lack of pure equilibrium and of the existence of the optimal mixed strategy $\langle[5/8, 3/8], [3/8, 5/8]\rangle$, Montmort makes the following claim:

> But how much more often must he switch rather than hold, and in particular what he must do (hic & nunc) is the principal question: the calculation does not teach us anything about that, and I take this decision to be impossible. (p. 405)

The problem, according to him, is that the calculation of the ratios 5:3 and 3:5 does not tell us how probable, in fact, it is that the players will play each strategy, and as a result he claims that a solution is impossible.

The way in which he explains his reasoning is interesting. He believes that it is impossible to prescribe anything that guarantees the best payoff, because the players might always try, and indeed good players *will* try, to deceive the other players into thinking that they will play something they are not playing, thus trying to outsmart each other (Montmort uses the French "jouer au plus fin"). Waldegrave also emphasizes this point by considering the probability that a player will not play optimally:

> What means are there to discover the ratio of the probability ratio that Pierre will play correctly to the probability that he will not? This appears to me to be absolutely impossible [...]. (p. 411)

Thus, both Montmort and Waldegrave claim that the solution of the game is impossible, but Bernoulli does not.

Their disagreement concerns what it means to "solve" the game Le Her. Bernoulli claims that the solution is the strategy that guarantees the best minimal gain—what we would call a minimax solution—and that as such there is a solution. However, despite understanding this "solution concept," Montmort and Waldegrave refuse to affirm that it "solves" the game, since there are situations in which it might not be the best rule to follow, namely, if a player is weak and can be taken advantage of.

Bernoulli disagreed with their views on the relation between "establishing a maxim" and solving the problem of Le Her. As he explains in two letters from 1714,

> [o]ne can establish a maxim and propose a rule to conduct one's game, without following it all the time. We sometimes play badly on purpose, to deceive the opponent, and that is what cannot be decided in such questions, when one should make a mistake on purpose. (p. 144)

The reason for which he considers the mixed strategy profile $\langle[5/8, 3/8], [3/8, 5/8]\rangle$ the solution is that playing $[5/8, 3/8]$ constitutes the best advice that could be given to Paul:

> If, admitting the way of counters, the option of 3 to 5 for Paul to switch with a seven is the best you know, why do you want to give Paul another advice in article 6? It suffices for Paul to follow the best maxim that he could know. (p. 191b)

And he continues: "It is not impossible at this game to determine the lot of Paul."

Montmort will finally reformulate his position and reply to Bernoulli's insistence that the game has been solved in a letter dated 22 March 1715. In this letter, he also stresses the relation between solving a game and advising the players. However, he distinguishes between the advice that he would put *in print*, or give to Paul *publicly*, and the advice he would give to Paul *privately*. In his view, the public advice would unquestionably be the mixed strategy with $a = 3$ and $b = 5$, since it is the one that demonstrably brings about the lesser prejudice. However, in the course of an actual game in which Paul would play an ordinary player who is "not a geometer," he would mutter a different advice that could allow Paul to take advantage of his opponent's weakness. In his view, the objective of an analysis of a game such as Le Her is not only to provide a rule of conduct to otherwise ignorant players, but also to warn them about the potential advantages of using *finesse*.

It is clear that the disagreement is not based on confusion, but instead on the fact that they are using different concepts of solution. Bernoulli's concept is in essence the concept of minimax. However, the concept of solution Montmort and Waldegrave have in mind further depends on the probability of imperfect play (i.e., on the skill level of the players). Thus, in addition to the probability of gain with a pure strategy and the probability allocation required to form mixed strategies, their perspective on the analysis of strategic games also requires that we know the probability that a player will play an inferior strategy. However, this probability distribution is not possible to establish on purely mathematical grounds, and it is in this sense that there is no possible solution to this problem. Ultimately, the view they defended is that one should not decide what to do in a strategic game based on minimax payoff, but instead based on expected payoff.

The probability distribution over the set of mixed strategies that Montmort considers has an epistemic and subjective character that echoes the Bayesian tradition. Moreover, both Montmort and Waldegrave suggests that this probability distribution can be interpreted as capturing one's expectations of the type of player against whom one is playing, which is another key methodological component of Bayesian game theory. Finally, as it utilizes this probability distribution to capture the possibility that players are weak, Montmort's perspective on solutions is sensitive to the sort of concerns that were addressed by early works on bounded rationality and on games of incomplete information. Thus, far from being "contrary to common sense," Montmort's perspective on the solution of games is in many respects similar to the perspectives that led to key developments in game theory in the second half of the twentieth century.

Appendix 1: Calculating Paul's Winning Card Counts

Computing the value of $C_{i,j}(\langle m,n\rangle)$ for various (i,j), and $\langle m,n\rangle$ is conceptually simple but can be very long and tedious by hand. The following MATLAB code is a series of nested loops examining all eventualities for any given strategy profile $\langle m,n\rangle$ and any pair of cards (i,j)[4]:

```
function Cijmn = LeHerCfct(i,j,m,n)
% Calculates the number of cards that makes Paul win when he plays
% against the dealer Pierre. i is Paul's card, j is Pierre's card, m
% is the minimal value Paul holds on to, and n is the minimal value
```

[4]The code was written for explanatory rather than efficiency purposes, which in any case has little practical importance for a problem of that size.

```
% Pierre holds on to.
if i>=m                  %Paul doesn't switch
    if j>=n              %Pierre doesn't switch
        if i<=j          %Paul loses
            Cijmn = 0;
        elseif i>j       %Paul wins
            Cijmn = 50;
        end;
    elseif j<n           %Pierre switches
        if i==13
            Cijmn = 50; %Paul wins with a king.
        elseif i>j       %Pierre has lower card, so drawing 13 loses
            Cijmn = i*4-1;
        elseif i<j       %Pierre has winning card, so drawing 13 wins
            Cijmn = (i-1)*4;
        elseif i==j
            Cijmn = (i-1)*4;
        end;
    end;
elseif i<m               %Pierre switches
    if j==13
        Cijmn = 0;       %Pierre blocks and wins
    elseif j~=13         %Pierre can't block
        if j<=i
            Cijmn = 0;   %Paul switches and loses
        elseif i<j
            Cijmn = 4*j-1; %Paul switches and Pierre draws
        end;
    end;
end;
end
```

Using this code, we can compute the winning card counts for Paul in any strategy profile. To use the example from the text, if we consider the strategy profile $\langle 8, 9\rangle$, the winning card counts for Paul corresponding to any pair of dealt cards (i, j) could be computed with a simple function such as this:

```
function C = Cmatrix(m,n)
% Calculates the winning card counts for Paul for all pairs of cards
% (i,j) that could be dealt.
for i=1:13
    for j=1:13
        C(i,j) = LeHerCfct(i,j,m,n);
    end;
end;
end
```

The result of this computation is thus obtained in a straightforward way. As an illustration, the winning card counts for any pair of cards (i, j) when the strategy profile $\langle 8, 9\rangle$ is employed is given in Fig. 3.

Appendix 2: Calculating Paul's Advantage

Even with an efficient way of calculating the values of $C_{i,j}(\langle m, n \rangle)$, calculating Paul's advantage under a certain strategy profile using Eqs. (1), (3), and (4) can be long and tedious by hand. This can be done with the following MATLAB code:

```
1  function [P,PiCmn]=ProbPaulWinning(m,n)
2  % Generates a matrix of P_{i,j} for all pairs of cards (i,j).
3  Pij = (4/52)*(4/51)*ones(13,13);
4  Pij = Pij-diag((4/52)*(1/51)*ones(13,1));
5  % Obtain the winning card counts:
6  C=Cmatrix(m,n);
7  % Generates the probabilities P_i(<m,n>) of winning with an i given a
8  % strategy profile <m,n>:
9  for i=1:13
10 PiCmn(i) = sum(Pij(i,:).*(1/50).*C(i,:));
11 end
12 % Finally, the probability of winning with a given profile <m,n>:
13 P = sum(PiCmn);
14 end
```

Lines 3–4 generate a probability matrix for Eq. (2), line 6 calls the function defined above to compute the $C_{i,j}(\langle m, n \rangle)$, lines 9–11 compute the probabilities $P_i(\langle m, n \rangle)$ of Eq. (3), and line 13 computes the probability $P(\langle m, n \rangle)$ that Paul will win with the strategy profile $\langle m, n \rangle$.

The last step involved in the analysis of this game consists in finding the probabilities of winning associated with each of the 13×13 strategy profiles. This can be done by simply executing the following code:

```
function PM=ProbMatrix
for m=1:13
    for n=1:13
        PM(m,n)=ProbPaulWinning(m,n);
    end
end
end
```

The results are displayed in Fig. 4, and the significance of each entry is discussed in the text.

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Reassembling Humpty Dumpty: Putting George Washington's Cyphering Manuscript Back Together Again

Theodore J. Crackel, V. Frederick Rickey, and Joel S. Silverberg

Abstract Soon after we began the study of George Washington's cyphering manuscript we realized that some of the pages were missing. To understand how this happened, we shall first discuss the provenance of the cyphering books. Then we present some "missing" pages that we have located, provide evidence that there are still more missing pages, and describe the detective work involved in situating these pages in the manuscript.

1 Introduction

While a teenager, George Washington compiled two cyphering books, recording what he was learning about decimal arithmetic, geometry, trigonometry, logarithms, and surveying. We prefer to use the standard eighteenth-century term "cyphering book" which indicates that the contents deal with arithmetic, rather than the more general "copy book" which allows for the inclusion of other subjects such as penmanship and collections of vocabulary and proverbs. In doing this, we follow the usage of Ellerton and Clements (2012 and 2014) from whom we have learned a great deal about cyphering books. Only a few of the thousand cyphering books that they have examined contain any material on decimal arithmetic. Washington is unusual, but certainly not unique, in this regard, for he has a section on decimal arithmetic. In a previous paper we explained that Washington, beginning at age ten,

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learned the arithmetic of whole numbers from a book which he owned, *The Young Man's Companion* (1727) by William Mather. Thus there was no need for him to transcribe this material into his cyphering books (Crackel et al. 2014).

Most of the original pages of Washington's cyphering books are at the Library of Congress and are available online at

http://memory.loc.gov/ammem/gwhtml/gwseries1.html

The library designates them "School Copy Book: Volume 1" and "School Copy Book: Volume 2." They consist of 111 and 68 images, respectively. We shall refer, e.g., to the 22nd image in the first of these books as I.22, and similarly for other images. In this paper we discuss several pages of manuscript that are not at the Library of Congress which we are able to determine precisely where they go in the cyphering books. For example, Fig. 4 pictures page I.21$\frac{1}{2}$v. The $\frac{1}{2}$ indicates that it was originally between images I.21 and I.22. The "v" indicates that it is a verso.[1] The current location of these leaves is given in the text below.

Our long-term goal is to produce a volume that contains images, transcriptions, sources, and annotations of each page of the cyphering books so that this material will be accessible to a wide audience with diverse interests. Our goal in this paper is to discuss a few pages of the manuscript that are not at the Library of Congress and to place them in their proper place in the manuscript. We shall also argue that there are pages of the manuscript that are still missing. To set the stage for this we first discuss the history of the manuscript.

2 Provenance of the Manuscript

The young George Washington, like many contemporaries, realized that his cyphering books could be a useful reference. Possibly their teachers had suggested this. Today these cyphering books provide important evidence of what and how he learned about mathematics. Jared Sparks (1789–1866), who prepared the first comprehensive edition of Washington's *Writings*, commented upon the "remarkable…care with which" the young man's cyphering books "were kept, the neatness and uniformity of the handwriting, the beauty of the diagrams, and a pre-

[1]Terminological explanation: The word "image" in this paper refers to a single web page of the "George Washington Papers at the Library of Congress, 1741–1799: Series 1a" whose URL was given above. A "leaf" is a single sheet of manuscript, consisting of two "sides," the "recto" or front (which is to be read first) and the "verso" or back. We are able to distinguish recto from verso as the leaves were originally in "books" and so while content usually determines the order, this is supplemented by physical evidence such as rough edges that were originally in the gutter of the cyphering books. When Washington was a youth, textbooks were expensive so students created their own. If they could afford it, and Washington probably could, students purchased bound blank books. If not, they folded several sheets of paper into a little booklet and held them together by stitching in the gutter. We do not know which approach Washington took, for his cyphering books have been disbound and trimmed, so evidence of sewing holes or bindings has mostly disappeared.

cise method and arrangement in copying out tables and columns of figures" (Sparks 1834–1837, vol. 2, p. 411).[2]

After Washington's death on December 14, 1799, his voluminous papers—including the cyphering books that illustrate his course of study in mathematics beyond simple arithmetic—were bequeathed to a nephew, Bushrod Washington (1762–1829), an associate justice of the US Supreme Court. At Martha Washington's request, however, most of the papers were first sent to John Marshall (1755–1835), who had agreed to write a biography of the General (Marshall 1974–1995, vol. 4, p. 34). By the middle of 1802 all but two trunks of Washington's papers were in Marshall's hands, but the cyphering books appear to have remained with Justice Washington (Founders Online, Bushrod Washington to Alexander Hamilton, Nov. 21, 1801).

Marshall, who served as Chief Justice of the US Supreme Court from 1801 to 1835, completed his five volume biography of Washington in 1807, but held on to the documents for many years—pleading the desire to publish an abridged edition of the work. It was not until 1820, when Bushrod Washington learned that Marshall had allowed some of the papers to be "mutilated by rats and injured by damp," that he asked that the papers be returned [Founders Online, Bushrod Washington to James Madison, Sep. 14, 1819]. They were all back in Washington's hands by 1823, but only briefly (Marshall 1794–1995, vol. 9, pp. 58–59, 334).

In 1827, Jared Sparks convinced Justice Washington to allow him to publish a select edition of George Washington's papers. He took the bulk of the papers, including the school papers, to Boston and kept many of them until 1837, when his final volume was published.

A detailed provenance of these papers, beyond what has already been said, is not important to our work here (See Washington 1983, pp. xiv–xx, for further detail). Suffice it to say that in the early 1830s the Washington family offered to sell the papers to the Federal Government. This was ultimately done in two lots. The first lot, the public papers (the largest portion), were sold to the Federal government in 1834. Then, in 1849, the government purchased the balance of the documents—George Washington's private papers, including the cyphering books and other school papers. Both parcels were then lodged at the State Department where they stayed throughout the balance of the century. Then, in 1903, they were transferred to the Library of Congress where they reside to this day.

What is of interest to us, in the context of this paper, is that while varying portions of these papers were in the hands of the Washingtons (from 1800 to 1849) and Jared Sparks (from 1827 to 1837) large numbers, including some elements of the cyphering books, were given away with an utter disregard for

[2] Sparks illustrated volume one of his *Writings of George Washington* with facsimiles from the cyphering books and surveys (Sparks 1834–1837, vol. 1, opposite p. 8), but we have not been able to place these in their proper place in the manuscript.

the integrity of the collection.[3] Bushrod Washington, even while the great bulk of the documents were in John Marshall's hands, was able to respond generously to requests for Washington memorabilia—regularly dispensing letters and documents in the President's own hand. Such gifts, wrote William Buell Sprague in 1816, were "much sought for, and considered by every one as perfectly invaluable" (Adams 1893, vol. 1, p. 389). Sprague, himself, a tutor to some of the children in the extended Washington family, was given access to the papers and allowed to take what he wanted, providing only he should leave copies in their stead. In all he took some 1500 letters, the bulk of which were letters addressed to Washington, but carrying his endorsement. The total number of documents that Bushrod Washington gave away in his lifetime is unknown, but appears to have been substantial. In the early 1820s, with all the papers now back under his control, Bushrod Washington dispersed them with renewed vigor. We know of some of these gifts—among the grander were four of Washington's diaries, some of which have found their way back into the collections of research libraries; others remain unaccounted for. In late 1829, Bushrod Washington died and title to the manuscripts passed to Congressman George Corbin Washington. He proved at least as generous in disposing of Washington materials as his predecessor. Their largess, we now know, extended to gifting pages of Washington's cyphering books.

Jared Sparks, who had control of most of the documents for almost a decade before they were transferred to the government, was famous for giving them away. In some cases he seemed to believe that the documents would be better preserved in hands other than the Washingtons. He barely had the papers 6 months when he wrote James Madison: "I have collected several of Genl Washington's autograph letters, which I intend to distribute in different parts of Europe, in public libraries and other institutions, where they will be preserved with great care, and to much better purpose than in the hands of individuals, among whose private papers they will be subject to repeated accidents and eventual loss" (Founders Online, Jared Sparks to James Madison, Dec. 29, 1827).

Even after the private papers were transferred to the federal government in 1849, an unknown number, including pages of the cyphering books, were withheld and continued to be given away by both Sparks and the Washington family. For years after the last sale they would continue to share documents with friends, acquaintances, and others who might ask. In 1838, after supposedly having returned the documents (of which, the official papers had already been sold to the government) Sparks, quite revealingly, complained that the "Washington papers are all returned, and I am nearly drained of autographs, but I will send you two or three" (Adams 1893, vol. 2, p. 325). Sparks may have overstated his difficulties in 1838, but by the late 1850s both Sparks and the Washington family were all running out of documents to hand out. In 1857, George Washington Parke Custis, the General's adopted grandson, was forced to send a friend a "Relic" taken from the accounts

[3]Marshall, so far as we can ascertain, did not succumb to the temptation to give away any of the documents he held.

that the General had kept of the Custis estates. In his covering letter he wrote: "I am now cutting up fragments from old letters & accounts ... to supply the call for any thing that bears the impress of his venerated hand. One of my correspondents says 'send me only the dot of an i or the cross of a t, made by his hand, & I will be content.' " (George Washington Parke Custis to John Pickett, Apr. 17, 1857, F. W. Smith Library).

In 1863 Sparks was also apologizing. He had already begun to hand out leaves of the cyphering book. A leaf now at Cornell and a leaf at the Historical Society of Pennsylvania, which you will see below, were removed from the cyphering books by Sparks. Now, however, he was reduced to sending "a fragment of Washington's handwriting"—in this case the top two inches of a leaf from Washington's cyphering book that dealt with elements of surveying. "The autograph collectors have so far exhausted my stock, that I have now none to spare, which would be of any service to you," he wrote in apology for sending a mere fragment. (American Philosophical Society, Jared Sparks to Miss Whitwell, Nov. 5 1863, Feinstone Collection, no. 2125). But, matters only got worse. Shortly thereafter he was reduced to sending smaller and smaller fragments of the cyphering book leaves. In April of 1865—less than a year before his death—Sparks was giving away fragments of the cyphering books little more than two by three inches and possibly some smaller. One of these, given to E. Q. Hodges, is at the Morristown NHP , but we have not yet located where it comes from in the cyphering books.

John C. Fitzpatrick, editor of the first twentieth century publication of the Washington Papers, believed that "the greatest loss came as a result of" putting the documents into the hands of Jared Sparks (Fitzpatrick 1931–1944, vol. 1, p. xlix). Still, as we have seen, it is clear that the Washington family, including the Custis in-laws, handed out a very large number as well. The combined loss, thought uncounted, was large indeed. With the death, first, of George Washington Parke Custis in 1857 and then of Jared Sparks in 1866, the loss of Washington documents (and fragments) appears to have slowed to a stop. Still, the number of George Washington letters and papers in private hands is very large and the number sold (or resold) each year is likewise remarkable.

Precisely how many pages of the cyphering books are missing and how many were dispensed as fragments is yet unknown. Continued study of the surviving elements of his cyphering books may yet yield a more precise estimate.

3 A Page That the Library of Congress Did Not Digitize

After several months of studying and transcribing the Washington cyphering books, we decided that it would be helpful to see the originals, for we were aware that pages were missing and out of order. We were fortunate that Dr. Julie Miller, early American history specialist, Manuscript Division, Library of Congress, permitted Crackel and Rickey to see the originals in 2012.

While examining the two volumes and comparing them to the copy of the manuscript that we downloaded from the Internet, we discovered that one page had not been microfilmed and hence was not on line (perhaps because one page had been digitized twice, II.43 and II.44). The missing page is the recto of image II.44 and appears in print here as Fig. 1 for the first time.

At the bottom right of Fig. 1, which we shall designate as II.44r (r for recto), a small 94 has been added, probably in the 1880s, when the cyphering books were

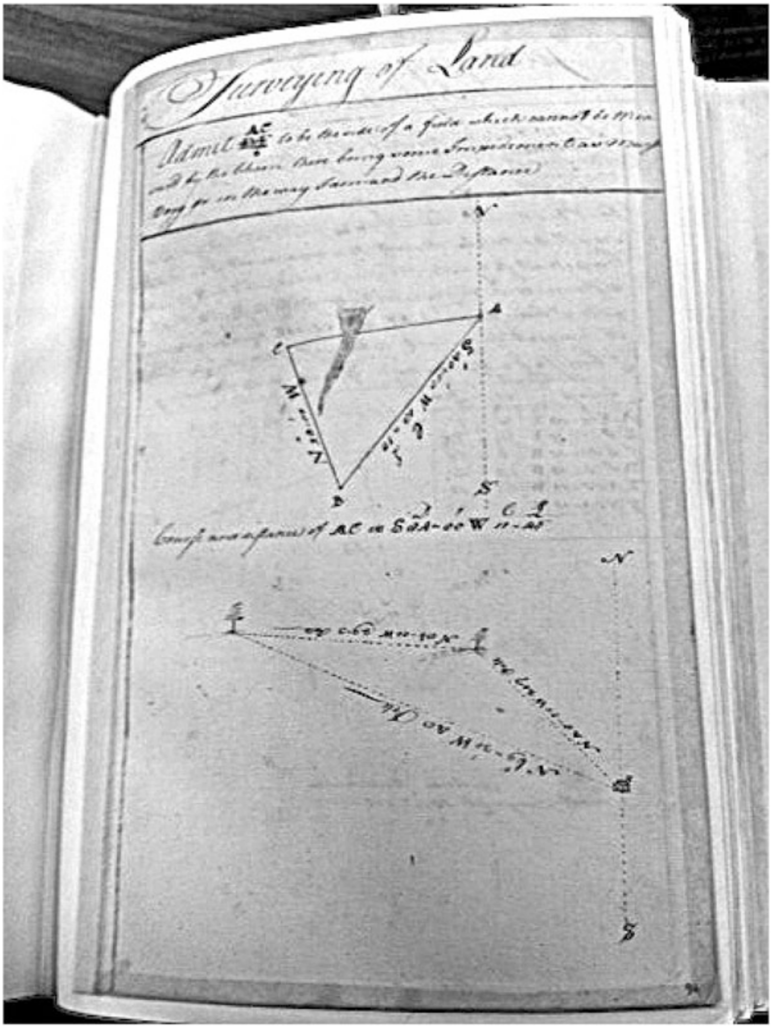

Fig. 1 Leaf II.44r, which is the recto of leaf II.44, was not digitized

(re)arranged by someone at the State Department. These numbers only appear on the rectos of leaves.[4]

When one examines II.44r and II.44, Figs. 1 and 10, one notes that there are faint marks where the ink has bled through from one side of the paper to the other. Bleed-through is caused when the paper is thin or the ink is applied heavily. The paper that Washington used is not particularly thin, but his quill pen often applied the ink heavily.

Since this page is hard to read, we transcribe the text at the top:

> Surveying of land
>
> Admit ~~*DF*~~ *AC* to be the side of a field which cannot be Measur'd by the Chain there being some Impediment as Marsh Bogg &c in the way I demand the distance

Between the two diagrams in Fig. 1, Washington states that the length of *AC* in the top diagram is $\overset{\mathcal{C}}{11}$—$\overset{\mathcal{L}}{45}$, i.e., 11 chains and 45 links.[5] However, he gives no clue as to how he obtained this.

Using information on image II.42, which we do not reproduce here, and from *The Compleat Surveyor* (Leybourn 1675 edition, pp. 199–201 and 220–222), we can explain the process that Washington used to find the distance *AC*. The points *A*, *B*, and *C* are determined at the outset by a tree, a pile of stones, or in some other way. From *A* one can see *B* and *C* and from *B* one can see *C* and *A*. Washington placed his surveyor's compass at *A* and read off the bearing of *AB* as $S\ \overset{\circ}{40}\ ..\ \overset{\prime}{00}\ W$, i.e., 40 degrees west of south. Then he read off the bearing of *AC* as $S\ \overset{\circ}{84}\ ..\ \overset{\prime}{30}\ W$, which he writes below the diagram. This makes it appear that it is part of the answer, but it is information needed to draw the plat.[6] Then he set his chainmen off towards *B*, staying behind to observe that they walked in a straight line. They determined the distance from *A* and *B* to be $\overset{\mathcal{C}}{13}$—$\overset{\mathcal{L}}{10}$

Then the compass was set up at *B* and the bearing to *C* was determined to be $N\ \overset{\circ}{20}\ ..\ \overset{\prime}{00}\ W$. This was done in the field.

[4] image I.22, Fig. 4, has a small 22 at the lower right, indicating that it too is a recto. Images II.44 and II.45, Figs. 10 and 11, do not have small numbers as they are versos.

The recto of the Cornell leaf, Fig. 3, which is the bottom half of a leaf, does not have a small number on it as it was never in the possession of the State Department or the Library of Congress.

[5] The standard Gunter's chain was 4 poles or 66 feet in length and was divided into 100 links. But in Virginia, a 33 foot chain, Fig. 4, was used because of the heavily wooded land. The Virginia chain was divided into 50 links. So the links on the two chains have the same length.

[6] A "plat" is a diagram, drawn to scale, of a "plot" of land. The plat contains information about the plot of land, such as the lengths of the sides of the plot, the bearings of the sides (angles the sides make with north), and distinguishing features such as a tree at a corner of the plot or a stream cutting one of its sides. We will use this terminology consistently, even though Washington did not.

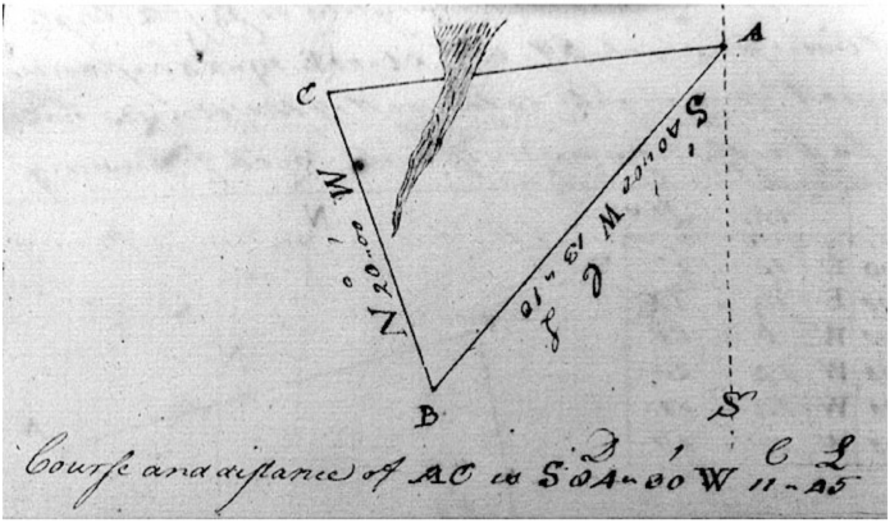

Fig. 2 The upper diagram on II.44r. Note the long “s” in “Course” and “distance”

Returning to his home base, Washington platted the data, obtaining the scale model given in Fig. 2. Then he took the map distance *AC* in his dividers and carried it to his scale where he read off the distance as $\overset{C}{11}$—$\overset{L}{45}$.[7]

4 The Cornell Leaf

The first of Washington’s cyphering books begins with a 19-page section on geometry, images I.2–I.20 (I.1 is a modern title page), the first page bears his signature and the date “August 13th 1745.” There is no mathematical reason why the subject of surveying should immediately follow the sequence of geometrical problems, yet it does. Image I.20, the last page of the geometry problems, is a recto, and I.21, the corresponding verso, contains the first page of the sequence of pages on surveying. There is additional evidence that these two pages comprise a single leaf of the manuscript, since it can be seen that the circle on I.20 has bled through to I.21 and flourishes after the word “Surveying” on I.21 has bled through to I.20. There is a distinct change in Washington’s handwriting from the geometry section to the surveying section. This suggests that he completed the geometry section and then put this cyphering booklet aside. Later, when he was compiling the work on surveying he realized that he had these blank pages and so used them.

[7] Washington used this same process to solve the problem on image I.46.

There are two sections of Washington's cyphering books that deal with surveying, the first of which begins on image I.21 as follows:

> Surveying
> Is the Art of Measuring Land and it consists of 3 Parts
>
> 1st The going round and Measuring a Piece of Wood Land and
> 2d Platting the Same and
> 3d To find the Contents thereof and first how to Measure a Piece of Land

The first two parts of surveying are described on this page by Washington, but, the third part, finding the area of the region, is not included in the manuscript at the Library of Congress. This argues for a missing page. Philander D. Chase,[8] observed that

> The upper parts of the second and third pages of the surveying section [i.e., the leaf following I.22], which include the third step in drawing the boundaries of a tract and a sketch of a surveyor's chain, drawing compass, and plotting scale, are at Cornell University, Ithaca, N.Y. (Chase 1998, pp. 187–188).

Since the image in Fig. 3 is faint, we transcribe the text here:

> Thirdly To find the content thereof 1st find it in Square Poles and divide by 160 the Square Poles in an Acre gives the Content in Acres & if any thing remain Multiply it by 4 & divide by 160 the Quotient in Roods[9]
> NB. If your figure is not a Square or a Triangle it must be reduced to Squares or Triangles and the content found of every Square or Triangle Severally & added together for ye content of ye whole Plot.

This text fits nicely right after I.21 which ends "So have you done the Second Operation." We believe the Cornell leaf[10] is the second leaf of the section on surveying, for it is unlikely that there should be an additional leaf explaining the first two parts of surveying.

This raises the question of what was on the top half of the recto of the Cornell leaf (we will see below that it is the bottom half of a leaf). We conjecture that it contained an example that would elucidate the first two steps, say a plat of a region whose area is to be found. Moreover, we believe that the top half of the verso (above the chain and dividers) has the computation of the content or area of that plot. If we find a source for the introduction of this surveying section it likely would provide an answer to this question. Of course, it would be wonderful to locate the top of this leaf.

[8] After earning a Ph.D. at Duke, Chase came to Charlottesville in 1973 to work on the *Papers of George Washington* as a documentary editor, and as editor-in-chief from 1998 to 2004, and stayed until his retirement in 2008.

[9] A rood is an English unit of area, equal to one quarter of an acre. An acre is 10 square poles.

[10] This half-page is reproduced thanks to Division of Rare and Manuscript Collections, Cornell University Library. The material is in the Letters of Washington, Franklin, and Lafayette, 1744–1830, 4600 Bd. Ms. 548++. It is in Folder 1, which is titled "Washington." This volume, which was disbound by the Cornell librarians, was prepared by Jared Sparks (1789–1866) and purchased by Cornell University in January 1872 (Sparks 1871, pp. 211–212).

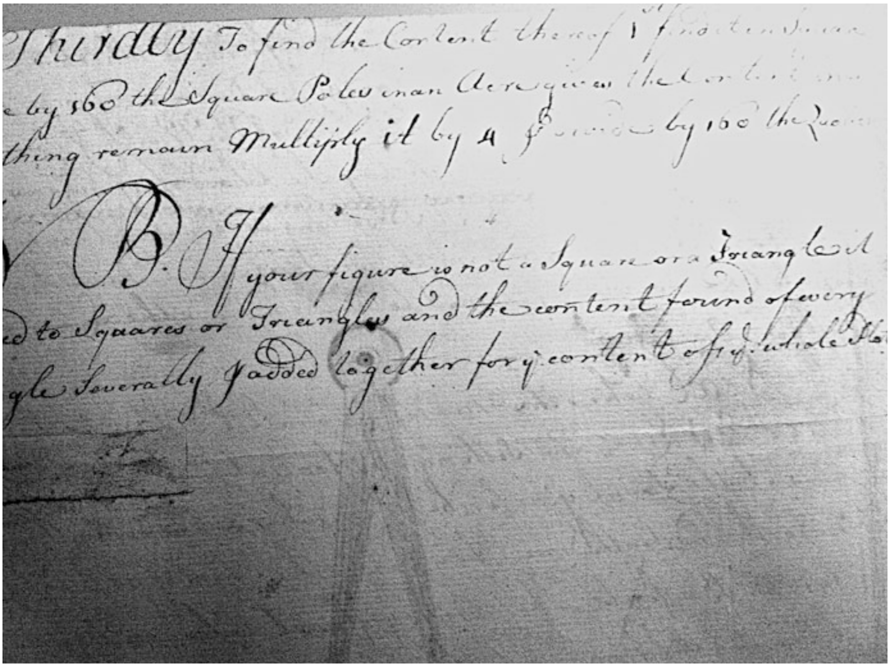

Fig. 3 Image I.21$\frac{1}{2}$r, the recto of the Cornell half-leaf. Division of Rare Books and Manuscript Collections, Cornell University Library

To see what follows the Cornell leaf it is necessary to look carefully at the dividers in Fig. 4. About half way down and mostly to the left of the dividers there is a faint image of a tilted rectangle that has transferred from the bottom of image I.22 (Fig. 5).[11]

Also, just above the chain there are two faint $\vee$ s, one through the words "The Chain" and the other around the word "Feet." The bottoms of these $\vee$ s are just above the chain. These two faint images were caused by ink that has transferred from the bottom of image I.22, Fig. 5, where they are the bottoms of the two squares (one cannot see the tops of the squares because we have trimmed the page).

Also note that the drawing of the dividers in Fig. 4 has transferred to I.22, Fig. 5, providing further evidence that I.21$\frac{1}{2}$v and I.22 were once together. Thus there is ample evidence that the Cornell leaf once was between I.21 and I.22. Consequently, we have designated the recto of this leaf as I.21$\frac{1}{2}$r (Fig. 3) and the verso as I.21$\frac{1}{2}$v (Fig. 4).

There are several interesting things about this image that bear on surveying. But first note that the date at the top, 1746, is not in Washington's hand, but was written

[11]Thanks to Elaine McConnell, Rare Book Curator at West Point, for suggesting the cataloger's word, "transfer."

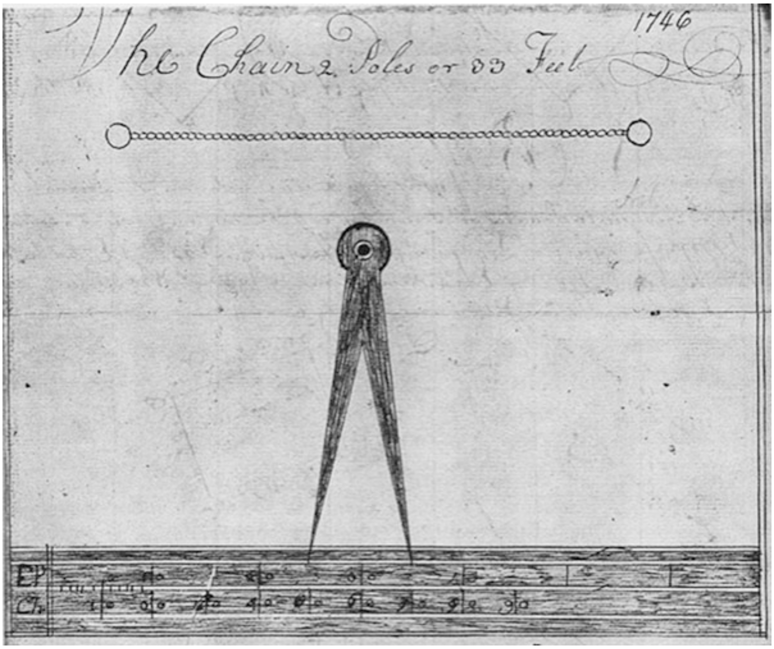

Fig. 4 Page I.21$\frac{1}{2}$v, the verso of the Cornell half-leaf. It is followed by image I.22, Fig. 5. Division of Rare Books and Manuscript Collections, Cornell University Library

by Jared Sparks. Examination of the original reveals that "Æt. 14" is written at the lower right, again by Sparks (but not visible in Fig. 4). Washington was born February 11, 1731/32, and the first cyphering book was dated August 13, 1745. Thus Sparks was incorrect that Washington was 14 years old when he began this cyphering book; his age was 13 years, 6 months, and 2 days.

The verso of this leaf, Fig. 4, which is just half a leaf, and contrary to Chase, is the bottom half of that same leaf, is often reproduced as it contains a nice image of a surveyor's chain, dividers, and two scales.

Rather than drawing the standard Gunter Chain—the 4 pole 66 foot chain that was commonly used in surveying—Washington has drawn "The Chain 2 Poles or 33 Feet" and explains its choice later in the manuscript:

> Because the two Pole Chain is most in use among Surveyors Measuring Lines in Virginia & other American parts I shall chiefly insist on that Measure it being y^{e} best for Wood Land. [II.30; quoted in Chase 1998, p. 163].

As was standard at this time, the handles on the chain are round; later chains had triangular handles.

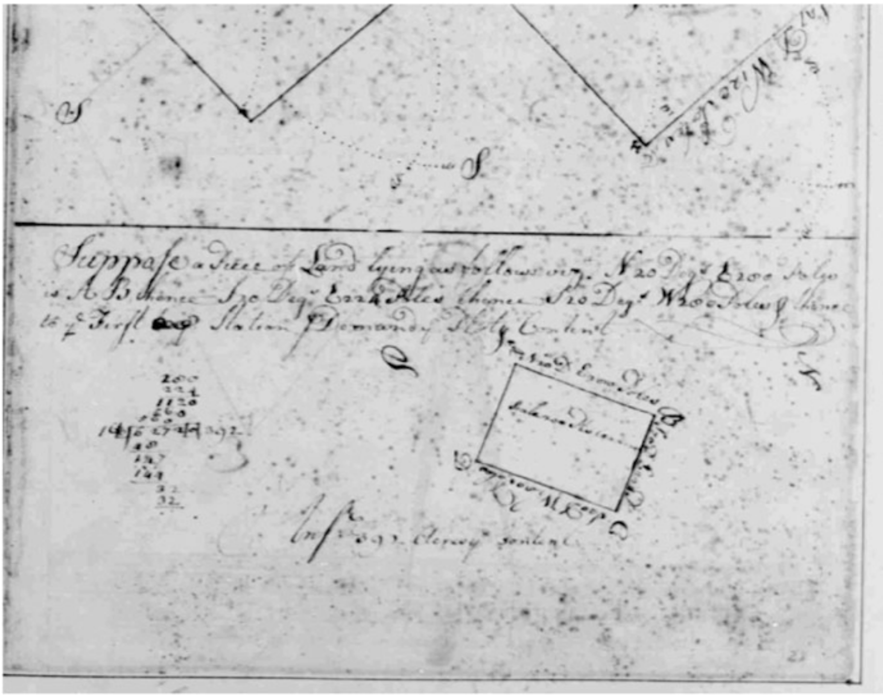

Fig. 5 The bottom of image I.22

Contrary to what the text in Fig. 4 says, the object is a pair of dividers, or simply dividers, not a compass. A compass, or sometimes a pair of compasses, is a device for drawing circles. The word “pair” is used because a compass has two arms, one for holding the pencil or pen that scribes the circle, the other is pointed and determines the center of the circle. Dividers look similar, but they have two sharp points. The sharp points are crucial for taking the distance between two points on a plat. The two points are placed on the two points of interest, and then the dividers are carried over, without changing the opening, to a scale of Equal Parts (EP) or, perhaps, a diagonal scale. The distance is read off the scale. The second scale at the bottom of Fig. 4 is a Scale of Chords (Ch). When platting, this scale was used to lay down angles with precision; see Crackel et al. (2014) for an explanation of how to use a scale of chords to plot angles.

5 The Dartmouth Leaf

This leaf was discovered serendipitously. Rickey was at Dartmouth College in 2004 attending a conference on “Scientific Instrument Collections in the University” and doing archival work. The archivist mentioned that they had a manuscript

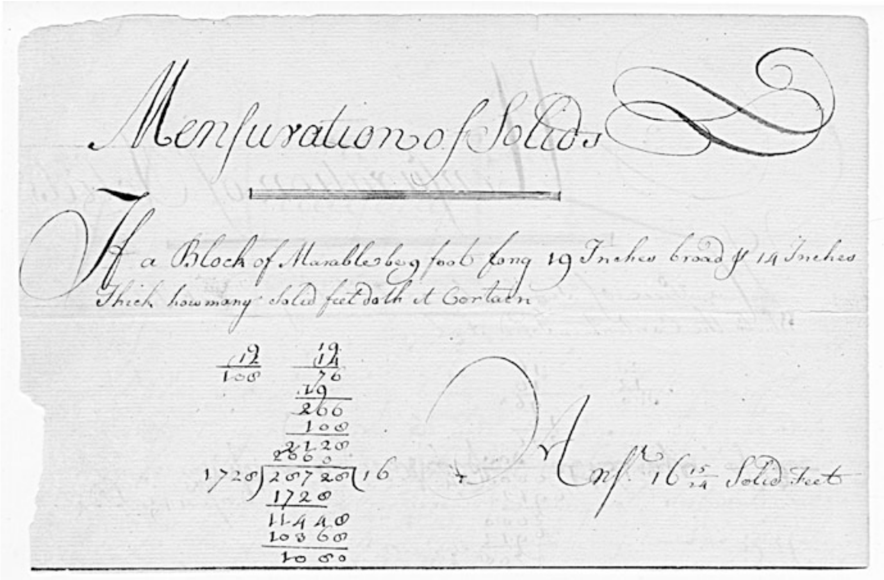

Fig. 6 Recto of the Dartmouth leaf (I.30$\frac{1}{2}$r-top-Dartmouth)

in Washington's hand that contained mathematics. It was a none-too-interesting problem on computing volume. When we were working on the cyphering books he remembered this leaf and so wrote and asked the Dartmouth archivists for a scan of the leaf.[12]

The two slanted, but roughly vertical, faint lines through the word "Solids" in Fig. 6 were caused by the transfer of ink from the "M" in "Mensuration" on image I.30, Fig. 7. Similarly the faint horizontal line running just above the words "Block of Marble" comes from the dark horizontal line on image I.30.

The faint tan colored lines near the top of the verso of the Dartmouth leaf, Fig. 8, have transferred from the dark lines around the word "Gauging" in Fig. 9. Consequently the Dartmouth leaf belongs between I.30 and I.31.

Here is a transcription of the text in Fig. 6:

> Mensuration of Solids
> If a Block of Marble be 9 foot long 19 Inches broad & 14 Inches Thick how many Solid feet doth it Contain

On the first page of the section on "Solid Measure" Washington gave a rule for doing problems like this:

[12]The original of this half-leaf is at the Rauner Special Collections Library Archive at Dartmouth College, Hanover, NH 03755. It bears the title: "Mensuration of solids" attributed to a 12-year old Washington," MS-1033. We thank the archivists for supplying scans of both sides.

Fig. 7 The top portion of I.30, which precedes the Dartmouth leaf and which has transferred to the recto of that leaf

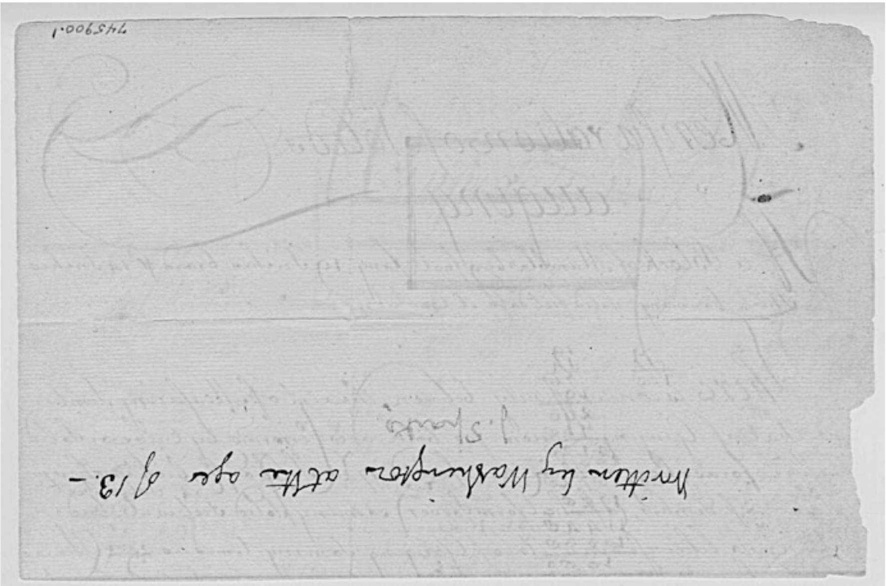

Fig. 8 Verso of the Dartmouth leaf (I.30$\frac{1}{2}$v-top-Dartmouth)

> To Multiply the Length & Breadth together & that Product by the Depth or thickness & the Last Product will be Contents in Cubic Inches which if Timber or Stone divide by 1728 (the Cubic Inches in a foot Solid) & the Quotient gives the Contents in Solid Feet

This rule appears with the exact same wording in *The Instructor; or, the Young Man's Best Companion* by George Fisher (Fifth edition, 1740), p. 224.

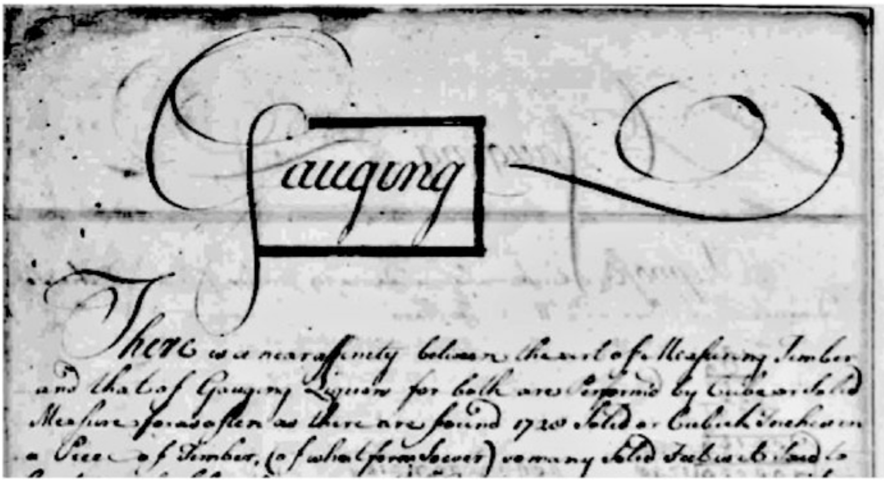

Fig. 9 The top of I.31 which follows the Dartmouth leaf and which has transferred to the verso of that leaf

Washington converted the measurements to inches (a step that he neglected to copy into his cyphering book) and divided by 1728, the number of cubic inches in a cubic foot. The answer he gives is $16\frac{15}{24}$ Solid Feet. There is no work showing the reduction of the fraction $\frac{1080}{1728}$. Probably he worked this out on his slate but did not copy it into his cyphering book. We have no explanation as to why Washington did not further reduce the fraction to $\frac{5}{8}$, especially since one-eighth of a cubic foot is easy to visualize.

Although the writing in Fig. 8 is upside down, the image is topside up. Jared Sparks has noted "Written by Washington at the age of 13 —", but his name was added by someone else.

The thin dark line at the bottom of Fig. 6 indicates that there was another example below, for Washington never drew such a line unless there was more below. We were astonished to discover that the lower half of this leaf is at the Historical Society of Pennsylvania.

6 A Leaf at the Historical Society of Pennsylvania

In the Dreer Collection at the Historical Society of Pennsylvania there is a half-leaf of manuscript that is most interesting. It is not the mathematics that is interesting, but the leaf itself. At the top is written:

> Having the Breadth & Depth of a Piece of Timber or Stone to know how much in Length of it will make Solid Foot
> Rule Multiply one by the other and let be a Divisor to 1728

Two problems follow. He does not state them in words, but simply does the computations. For the first he multiplies "27 Inches Broad" by "12 inches." This computation reveals that Washington knows his twelves multiplication tables, for he immediately writes down the answer with no intermediate work (there are numerous instances of this in the manuscript). When he multiplied 27 by 12, his mental process must have been something like this: 12 times 7 is 84; write down the 4 and carry 8; 12 time 2 is 24 plus the carried 8, which makes 32. The product is 324.

When he divides 1728, the number of cubic inches in a cubic foot, by 324, he obtains 5 with a remainder of 108. Then he reduces the fraction and gives the result:

Ans^r $5\frac{1}{3}$ in Length makes a Solid Foot

The second problem requires the multiplication of 19 by 17. The product 323 is very close to the previous problem but he just gives the answer as 5 feet in length to make a solid foot. He ignores the remainder, 113, probably because the fraction 113/323 is irreducible.

The notation "G. W. / 1745 / age 13" at the bottom right of this half-leaf is not in Washington's handwriting. The verso of the leaf identifies the writer as Jared Sparks:

> The within is the writing and cyphering of General Washington when about 13 years of age. It appears to have formed part of his cyphering book. It was given me on the 22^d Feb^y 1832 the Centenary of his birth, by Mr. Jared Sparks, the Editor of his correspondence.
>
> Robert Gilmor[13]
> 1832

There is no doubt that this leaf, I.30$\frac{1}{2}$r, is the bottom half of the Dartmouth leaf, Fig. 6, for just to the right of the second computation (where he multiplies 19 by 17) there is transfer of ink from I.30. Of course, the mathematical topics are related on the two leaves.

The verso of leaf I.30$\frac{1}{2}$ was originally blank. This is unusual and happens only a few times in the cyphering books. However, the section on "Solid Measure" consists of only two leaves, I.29–I.30 and I.30$\frac{1}{2}$, and those one and a half pages are all the paper Washington needed for this topic. The next leaf, I.31, which is a recto, begins a section on "Gauging" (Fig. 9).

7 A Missing Leaf

In trying to determine how the leaves were originally ordered in Washington's cyphering books, we have paid careful attention to the context—both mathematical and linguistic—from one image to the next. Here is an example that indicates a missing leaf.

[13]Robert Gilmore (1774–1848) was a wealthy Philadelphia merchant, patron of the arts, philanthropist, and an avid collector of manuscripts and autographs.

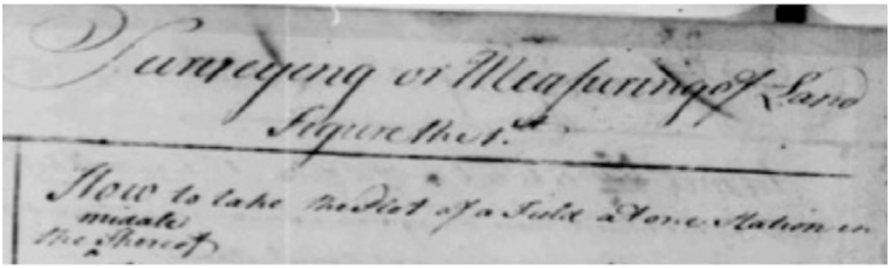

Fig. 10 Detail of the top of image II.44

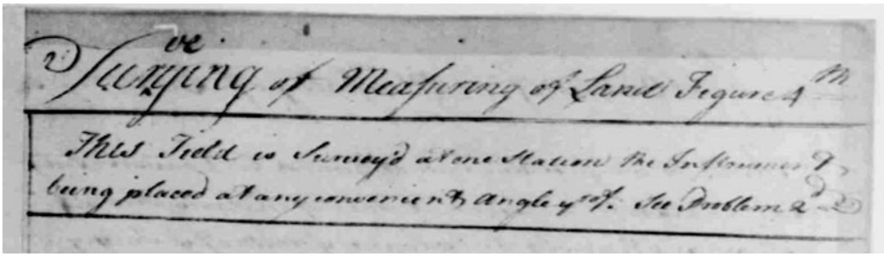

Fig. 11 Detail of the top of image II.45

At the top of image II.44, Fig. 10, which is a verso, we find:

Surveying or the Measuring of Land
Figure the 1$^{\text{st}}$

and at the top of image II.45, Fig. 11, which is a recto, we find:

Sur$^{\text{ve}}$ying or the Measuring of Land Figure 4$^{\text{th}}$

So where are the second and third figures? Clearly a leaf of manuscript is missing. But where is it?

8 Conclusion

Every individual or organization that has ever had control of Washington's cyphering books has done something to them. Bushrod Washington and Jared Sparks gave papers to friends and autograph collectors, and we are certain that we have not found all of the pages of the cyphering books that they removed. The State Department disbound the cyphering books—or perhaps they came to them that way—and they used current archival methods to mount and rebind them. Finally, the Library of Congress—again using the best archival techniques of the day—removed the State Department bindings and arranged the pages the way they thought they should be into two volumes. This is how they remain today.

We have located several of the dispersed leaves and placed them in their proper order but exactly how many are missing is still to be determined. Our work has allowed us to determine that the pages of Washington's cyphering books, as they are ordered today, are not in their original order. We have determined how several of the leaves should be reordered, but additional work needs to be done to ascertain the order of the extant leaves. A report on that work will be left to another time.

> I had no problems at the beginning, and now I have nothing but problems!
>
> *Proofs and Refutations* (1976)
> Imre Lakatos

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Natures of Curved Lines in the Early Modern Period and the Emergence of the Transcendental

Bruce J. Petrie

Abstract The transition from the geometric study to the algebraic study of curved lines changed how early modern mathematicians understood the origins of these mathematical objects. To know a mathematical object was to know its nature. To know its nature was to know its origin. The emergence of the transcendental classification was a consequence of a shift in how mathematicians understood the natures of mathematical objects because they were classified according to their natures. This nature was useful to determine which objects were appropriate for geometrical study, especially when applied to curves. The development of calculus provided the tools necessary for algebraic analysis to uncouple the study of curves and geometry, greatly increasing the number of curves which could be made known. The geometrical motivation for classifying curves was rendered obsolete and was replaced by focusing on functional relationships between variables. The nature of mathematical objects inherited the nature of the algebraic expression used to represent them.

1 Introduction

To know a mathematical object was to know its nature. To know its nature was to know its origin. The history of early modern mathematics reveals that the nature of mathematical objects change. When the nature of a mathematical object changed, what was changing was how mathematicians understood the origins of that object. René Descartes and Leonhard Euler used different classification rules to categorize curves. Investigating these rules exposes how their perceptions on the origins of mathematical objects differ. The transition from the geometric study (as performed by Descartes) to the algebraic study of curved lines (as performed by Euler) changed how early modern mathematicians understood the origins of these mathematical objects. The emergence of the transcendental class was a consequence of this transition because objects were classified according to their natures.

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In Henk Bos's *Redefining Geometrical Exactness*, he details how Descartes enlarged the class of *bona fide* geometrical curves. Using new and more complicated construction techniques, Descartes argued that a number of curves which did not admit straight-edge and compass construction could still be known exactly (Bos 2001, pp. 251–253). This also affected how he categorized curves although Descartes was not consistent in his approach (Bos 1981, p. 298). For instance, Descartes would categorize curves based on construction methods but also upon their roles in geometry (Bos 1981, pp. 295–298). Bos also informs us that there were a variety of ways in which curves could be made known in the early modern period with varying degrees of satisfaction (Bos 1981, p. 296).

By focusing on one such method, geometric construction, the relationship between knowing an object and knowing its origins becomes clear. The emergence of the transcendental class occurs when this relationship is imported into algebra. While others, such as Leibniz, used "transcendental" to describe objects which "transcend algebra," the classification is articulated clearly in the mathematics of Leonhard Euler. He distinguished between the origins of algebraic and transcendental curves and did so using his concept of function.

2 Known and Nature in Early Modern Mathematics

The representation of curves was like an introduction. It began the process of making that curve known. There were various ways to represent a curve in the early modern period. For instance, curves could be represented by a name, by an equation, or visually through an illustration. Visual representation provided an intuitive means to communicate and perform geometry, but this became contested in the early modern period with the rise of algebraic analysis.

The process of interacting with geometric objects (or even modest visual representations of them) was the core component of the practice of geometry in Greek antiquity and through which the nature of those objects could be understood. To understand the nature of a mathematical object was to know its origins or source. When Apollonius discussed the conic sections, it was within the historical context of Greek geometry. Michael N. Fried and Sabetai Unguru caution us that "to understand Apollonius's *Conica* historically presupposes understanding its discourse as a geometrical discourse in which showing, pointing, and drawing play crucial roles" and they continue "together, these... show not only that Apollonuis's elements of conic sections are geometrical in nature but also in what sense they are" (Fried and Unguru 2001, pp. 107–108). Texts such as Apollonius's *Conics* and Euclid's *Elements* instructed the reader to either physically (through drawing) or mentally manipulate objects (or our visual representations of them) that were spatial and tangible. In this manner, geometric objects, such as curves, were constructed and drawn. Geometric construction was also how mathematical objects became known. Aristotle explicitly articulated what it meant to know something (such as a curve): "For we do not think we know a thing until we are acquainted with its primary conditions or first principles, and have carried our analysis as far as its simplest elements" (Aristotle 2000). Construction provided the connection between the mathematical object and the first principles of geometry. Therefore construction revealed the nature and origins of mathematical objects. The conic sections originated as the intersections of planes with cones. This was how they were geometric in nature.

Curves were classified according to their natures. This can be seen in the classification rules of Pappus (discussed below) and then the later changes to those rules made by Descartes and Euler. In all three scenarios, when the origins of a curved line were not well understood—if at all—the curve's status as a knowable mathematical object was in question because knowing an object was coupled with knowing the nature of that object. When the nature of a mathematical object changed, what was changing was how mathematicians understood the origins of that object.

3 Pappus: Plane, Solid, and Line-Like Curved Lines

A Latin translation of Pappus's *Collection* was printed in 1588 and it strongly influenced early modern geometry. Its publication served as an exemplar of geometric practice in early modern Europe (Bos 2001, p. 3). In it, Pappus classified problems, such as those resulting in the construction of a curve, based on what was necessary to perform the construction.

> The ancients stated that there are three kinds of geometrical problems, and that some of them are called plane, others solid, and others line-like; and those that can be solved by straight lines and the circumference of a circle are rightly called plane because the lines by means of which these problems are solved have their origin in the plane. But such problems that must be solved by assuming one or more conic sections in the construction, are called solid because for their construction it is necessary to use the surfaces of solid figures, namely cones. There remains a third kind that is called line-like. For in their construction other lines than the ones just mentioned are assumed, having an inconstant and changeable origin, such as spirals, and the curves that the Greeks call <tetragonizousas>, and which we can call "quadrantes," and conchoids, and cissoids, which have many amazing properties (Bos 2001, p. 38).

Pappus presented three categories of curves (plane, solid, and line-like) and each category was based on the object's nature which was inherited from what was necessary to construct it. The third class, line-like, had "inconstant and changeable" origins. While Pappus did study these curves, they could not be rigorously constructed and therefore their origins were in question. The inability to connect these objects to the first principles of geometry meant that they could not be known in the same sense as planar curves. Consider the following three curved lines: the arc, parabola, and spiral. The circular arc can be constructed using straight lines and circles (in this case just a circle) so it was classified as a planar curve because the tools required to construct it were planar. These planar curves were called geometrical because they satisfied the rigorous geometrical requirements of Euclidean construction. Even though the parabola itself lies within a single plane, its construction required using more than planar objects. It involved using a cone which was a solid by nature and therefore the parabola was classified to be a solid curve. The nature of the parabola was inherited from the solid used to construct it. The spiral and other line-like curves, however, could not be constructed—at all—using the accepted geometric techniques of the time. Therefore it was not possible to connect them to first-principles and their status as known mathematical objects could not be established. As such, both solid and line-like curves were considered mechanical not geometrical.

4 Descartes: Geometrical and Mechanical Curved Lines

René Descartes believed that Pappus erroneously excluded acceptable curves from geometry. He discussed this at length in Book II of his *Geometry* which is appropriately titled "On the Nature of Curved Lines." He argued that even though solid and line-like curves could not be constructed using lines and circles many could still be known with the same exactness as planar curves. For example, consider the following:

> It is true that conic sections were never freely received into ancient geometry… if we think of geometry as the science which furnishes a general knowledge of the measurement of all bodies, then we have no more right to exclude the more complex curves than the simpler ones, provided they can be conceived of as described by a continuous motion or by several successive motions, each motion being completely determined by those which precede; for in this way an exact knowledge of the magnitude of each is always obtainable (Descartes 1925, p. 43).

Descartes argued that constructing some solid or line-like curves was possible but doing so required extending the geometer's toolset. This resulted in the modification of the geometric-mechanical classification of curves. Geometrical curves were curves which could be constructed using this broader geometric toolbox and mechanical curves were those which could not be. His dichotomy split curves into two categories: those which were appropriate for geometrical study and those which were not. In *Redefining Geometrical Exactness*, Henk Bos reveals the struggle and variation of this process. Descartes' revisions were inconsistent and conflicting. In order to classify curves as either geometrical or mechanical, Descartes appealed to motion, equations, and how a curve was used but his application was not uniform. Surprisingly, Descartes may have appealed to equations but he did not believe equations were an appropriate representation of a curve and only a method of analysis (Bos 1981, p. 297). By appealing to motion, geometry was no longer restricted to straight-edge and compass construction but it could also involve other tracing devices such as a set-square or ruler-and-slide. Descartes also employed previously constructed curves as tracing devices-in-motion which resulted in a hierarchy of drawing tools. Curves traced by these new (and much more complicated devices) could be constructed but not known in the traditional sense.

Even though Descartes reclassified some mechanical curves as geometrical, there were many mechanical curves which he believed did not yield to geometrical analysis.

> Probably the real explanation of the refusal of ancient geometers to accept curves more complex than the conic sections lies in the fact that the first curves to which their attention was attracted happened to be the spiral, the quadratix, and similar curves, which really do belong only to mechanics, and are not among those curves that I think should be included here, since they must be conceived of as described by two separate movements whose relation does not admit of exact determination (Descartes 1925, p. 44).

Descartes again appealed to motion (or separate motions in this case) and the incommensurability between certain motions which inhibits exactness. In the case of the spiral, tracing it involved a translation and a rotation and that relationship could not be known exactly so the spiral remained a mechanical curve.

Descartes' changes to Pappus's classification of curves reflected his dissatisfaction with how geometers from antiquity understood the nature of certain curved lines which he considered geometrical but fell into Pappus's solid or line-like categories. These curves were *bona fide* geometrical objects whose origins were not suspicious and could be known exactly but required more complicated constructions. With a new and larger geometer's toolbox, the nature and origins of geometric curves changed.

5 Euler: Algebraic and Transcendental Curved Lines

The development of algebra and invention of calculus had a profound effect on how mathematicians understood the nature of curved lines. New curves which could not be constructed in either the classical sense or the Cartesian sense were becoming objects of popular study. Two geometrical problems, the curve of quickest descent and the shape of a hanging chain, had solutions which were mechanical curves, the brachistochrone and the catenary. Both curves were knowable using calculus but were unknowable in Cartesian geometry. Other curves also appeared as the solutions to indefinite integrals and yet, again, were not constructible in the traditional senses. Yet these were all legitimate solutions to genuine geometrical problems. One such curve, the natural logarithm, was the solution to $f(x) = \int \frac{1}{x}dx$. Bos reminds us that the large number of new curves being studied in the early modern period necessitated rethinking how curves were introduced, defined, or even described: how they were made known (Bos 1981, p. 296). It was simply not practical to know all these new curves by name or how they were drawn.

Leonhard Euler articulated a new framework for the study of curves and algebra proved to be invaluable for knowing and representing the large number of new curves being studied. Euler introduced functions.

> A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities (Euler 1990, p.3).

Euler's definition described the relationship between variables using a single analytic expression (i.e., an algebraic equation). The structure of the functions was used by Euler to classify the objects they represented, such as curves.

> *The principal distinction between functions, as to the method of combining the variable quantity and the constant quantities is here set down.* Indeed, it depends on the operations by which the quantities can be arranged and mixed together. These operations are addition, subtraction, multiplication, division, raising to a power, and extraction of roots.... Beside these operations, which are usually called algebraic, there are many others which are transcendental, such as exponentials, logarithms, and others which integral calculus supplies in abundance (Euler 1990, p. 4, emphasis in original).

Euler distinguished between algebraic and transcendental functions and this depended upon the operations involved. Pay special attention to Euler's statement: "and others which integral calculus supplies in abundance." Many new curves which appeared as solutions to indefinite integrals could not be known or expressed using the algebraic operations Euler listed and were only expressible in integral form. For instance, consider $f(x) = \int \frac{1}{x} dx$, which we call the natural logarithm; the integral of this algebraic expression could not be expressed using a single finite analytic expression composed with only algebraic operations. These transcendental curves did not originate from construction, tracing, or other geometric means. The nature and origins of these curves were found through algebra and calculus. These curves inherited their algebraic and transcendental natures from the functional relationships between variables.

Curves which could not be known visually or through construction could be known algebraically. The primary means to represent a curve in Euler's mathematics was no longer visual, but used functions. These legitimate mathematical objects could originate from algebraic processes and be mathematically manipulated without ever knowing what they looked like visually. Their natures were no longer geometric or mechanical but algebraic or transcendental. Curves which could be expressed using finite analytic expressions with only algebraic operations were algebraic and those which could not be were transcendental.

6 Conclusion

In order to understand the emergence of the transcendental class, it is necessary to also understand that the nature of objects points to their origins because curves were classified according to their natures. In other words, transcendental curves and geometric curves had different origins. More to the point, there were curves (which, *prima facie*, should be intuitive geometric objects) that were not considered geometric because they did not have geometric origins.

In Greek antiquity, Aristotle argued that objects could not be known until they were connected to first principles. This was accomplished for mathematical objects using geometric construction. Pappus classified curves according to their natures and the methods of construction were the basis for those natures. Line-like curves did not have clear origins and thus they could not be established as known mathematical objects. Many curves which fell into Pappus's line-like category would be called transcendental by today's mathematicians. The transcendental classification was the consequence of the changing nature (geometric to algebraic) of curved lines and how mathematicians understood the origins of those curves.

Descartes was responsible for many developments in Western algebra but this did not carry over to his modifications of classical Greek geometry. He revised Pappus's classification rules because he felt they unnecessarily excluded acceptable and knowable geometric curves. His classification rules, like Pappus's, appealed to the geometric origins and construction of curves but he distinguished between the motions used to construct a curve and the combination of some motions meant that those curves could not be known exactly. He maintained the geometrical-mechanical distinction but enlarged the geometrical category by including curves which could be drawn using more complicated tracing devices. Descartes, too, was classifying curves based on their origins, or nature, and curves with unclear origins (such as mechanical curves with inexact construction) were not knowable mathematical objects. In other words, curves were either geometrical or mechanical in nature and Descartes' classification rules clarified which curves were appropriate for geometrical study. We can see that as Descartes changed the classification rules, what was changing was the understanding of the origins, or nature, of such objects. However, the origins were still based on geometric construction. The influx of new curves with the invention of calculus (especially integral calculus) made a geometrical basis for knowing curves impractical.

Euler was the architect of a new paradigm of mathematics called algebraic analysis and it rested upon his concept of function. Euler used functions to represent curves. They expressed a relationship between variables and provided a means to know a curve and manipulate it mathematically without ever needing to see it sketched out visually—if that was even possible. No longer did every curve have to be traced or named to become known; knowing the function was sufficient. This development resulted in a profound shift regarding the natures of curves. Curves were no longer classified using their geometric origins but inherited the nature of the functions used to represent them. That is, curves were no longer classified according to whether or not they were *bona fide* geometrical objects but by the functional relationship between variables which could be algebraic or transcendental. This is pertinent because not all curves could be known using the standard algebraic operations associated with geometric construction. Since some curves had their origins in calculus they could only be made known using functions and so the geometrical-mechanical distinction did not apply to them. Therefore the geometric-mechanical distinction of curves was replaced by something much less intuitive, an algebraic-transcendental distinction. It was in this manner that transcendental curves became knowable mathematical objects. The transcendental classification was a consequence of curves originating outside of traditional geometry. The classification rule changes were the consequence of a shift in how early modern mathematicians understood the nature and origins of curved lines.

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Origins of the Venn Diagram

Deborah Bennett

Abstract Venn diagrams have turned out to be visual tools that are enormously popular, but diagrams to help visualize relationships between classes or concepts in logic had existed prior to those of John Venn. The use of diagrams to demonstrate valid logical arguments has been found in the works of a few early Aristotelian scholars and appeared in the works of the famed mathematicians Gottfried Wilhelm Leibniz and Leonhard Euler. In a 1686 fragment (which remained unpublished for over 200 years), the universal genius Leibniz illustrated all of Aristotle's valid syllogisms through circle drawings. In 1761, the much-admired master mathematician Euler used almost identical diagrams to explain the same logical syllogisms. One hundred and twenty years later, John Venn ingeniously altered what he called "Euler circles" to become the familiar diagrams attached to Venn's name. This paper explores the history of the Venn diagram and its predecessors.

1 Introduction

Nearly everyone has seen the familiar overlapping circles created by John Venn. Advertisers use the diagrams to instruct their market; journalists use the diagrams to exhibit political and social interactions; and one pundit has said that *USA Today* could not exist without Venn diagrams. Venn diagrams have been a standard part of the curriculum of introductory logic, serving as a visual tool to represent relations of inclusion and exclusion between classes, or sets. When logic and sets entered the "new math" curriculum in the 1960s, the Venn diagram joined the mathematics curriculum as well, sometimes as early as elementary school where students first encountered sorting and classifying.

But Venn's diagrams did not simply appear on the mathematical horizon fully formed; they evolved from diagrams predating Venn. Long before their use for analyzing set relationships, Venn's diagrams and diagrams similar to Venn's were used to illustrate valid or invalid arguments in logic—in particular, arguments in the form of 3-line Aristotelian syllogisms. In his 1881 book *Symbolic Logic*, Venn

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acknowledged that he had been anticipated in these ideas and devoted a chapter of historical notes on the evolution of the diagrams for analyzing propositions. With attribution to earlier influences, he stated that the "practical employment" of these diagrams dated to Leonhard Euler in 1761 (Venn 1881, p. 422). But prior to Euler, the foreshadowing of instructional diagrams of this sort has been credited to Raymond Llull (1232–1316?), Juan Luis Vives (1493–1540), Giulio Pace (1550–1635), and Gottfried Leibniz (1646–1716).

2 Early Influences

The thirteenth-century Majorcan monk and Aristotelian logician, Raymond Llull, utilized a variety of diagrams in his treatises. He wrote on topics as varied as the sciences, medicine, law, psychology, military tactics, grammar and rhetoric, mathematics, chivalry, ethics, and politics; he also wrote poems and erotic allegories. Llull was variously referred to as Lull, Lul, Lullius, or Lully, and because he experienced mystical visions of Christ, Llull also became known as *Doctor Illuminatus*. At the age of 83, when he refused to stop trying to convert Muslims to Christianity based on logic and rational debate, he died after being stoned by an angry mob. Llull's master project, which he deemed his "art," was an attempt to relate all forms of knowledge by mechanically manipulating symbols and combinatorial diagrams. Within his prolific works can be found numerous systems of organizing and classifying information using pictorial methods such as trees, ladders, and wheels (Gardner 1958).

Hints of Venn's familiar overlapping circles were also to be found within a compilation of Llull's work. In his works was found a diagram of four overlapping circles, each with a different label: *esse* (existence or being), *verum* (truth), *bonum* (goodness), and *unum* (unity) (Llull 1609, p. 109). Was Llull trying to demonstrate the intersection of being, truth, beauty, and unity (God)? The use of two disjoint circles to indicate qualities with nothing in common, such as truth and falsity, was found frequently in other parts of his work. Llull's "art" was but a step towards his ambition to use logic as a semi-mechanical method of demonstration translating across linguistic frontiers (Sales 2011; Dalton 1925). Llull was certainly controversial, but ultimately, very influential, his work studied for centuries after his death (Gardner 1958).

Another early influence in the use of diagrams to visualize the validity of an Aristotelian syllogism came from the Valecian scholar, Juan Luis Vives. Sometimes considered the "father" of modern psychology, Vives wrote on early medicine, emotions, memory, functions of the soul, the education of women, and relief of the poor. Of interest here is his 1555 work, *De Censura Veri (On the Assessment of Truth)*, a treatise discussing the Aristotelian proposition and the forms of argumentation. Several histories have mentioned the triangles employed by Vives to demonstrate an Aristotelian syllogism (Sales 2011; Nubiola 1993; Venn 1881). The three triangles (they really look like **V**'s) and their positioning with one inside

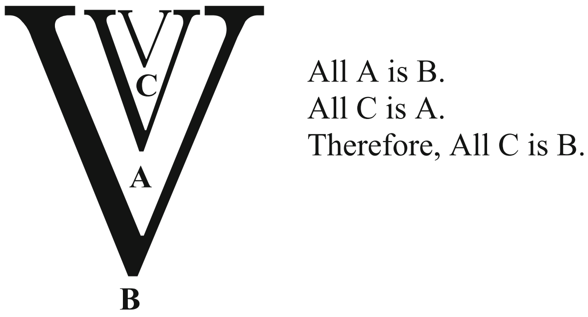


Fig. 1 Juan Luis Vives 1555 De Censura Veri

the other very much suggested the three circles, one inside another, that were later seen when Leibniz and Euler diagrammed this same syllogism. The Vives's diagram is shown in Fig. 1. Next to the diagram, Vives wrote, "If some part of the *first* holds the whole of the *second*, and some part of the *second* holds the whole of the *third*, the whole of the *third* is held by the *first*: that is, if three triangles are drawn, of which one, *B*, is the greatest and holds another (triangle) *A*, the third being the smallest contained within *A*, which is *C*, and we say if all of the *second* is the *first* and all of the *third* is the *second*, all of the *third* is the *first*" (translation by Walt Jacob). Without the diagram, Vives's argument would be very difficult to follow, but this is reported to be the only diagram of its kind in Vive's work. *De Censura Veri* went through hundreds of editions and translations and was widely read during the century after publication, so the diagram may have been noticed by others.

Although there is no evidence that Aristotle employed diagrams in this way, some historians have suggested that the Aristotelian scholar, Giulio Pace (Latin name Julius Pacius a Beriga), may have used such diagrams in his translations of Aristotle. An Italian jurist and scholar, Pace was quite well known. In fact his edition of Aristotle's *Organon*, complete with commentary, became a standard, yielding 11 editions between 1584 and 1623. Pace incorporated extensive use of symbolism and diagrams to demonstrate Aristotle's logic in his 1584 translation of the *Organon*. However, a thorough examination of a 1619 edition of Pace's translation and commentary revealed no Venn-like diagrams. Pace's commentaries are filled with figures of all types—circles, semi-circles, trees, and triangles—but none were used to enlighten the reader regarding the relationships among the terms of the propositions in the Aristotelian syllogisms (Aristotle 1619).

3 Leibniz

Unnoticed in John Venn's 1881 historical notes, circle diagrams to illustrate all of the valid Aristotelian syllogisms had appeared in the 1686 papers of Gottfried Wilhelm Leibniz. Having taught himself Latin when he was about 8 years old, Leibniz soon gained access to his father's library (his father was a professor of philosophy at the University of Leipzig) where he studied logic in the Aristotelian tradition. Leibniz claimed that at age 13 or 14, he was “filling sheets of paper with wonderful meditations about logic” (Leibniz 1966, p. x). Having entered the University of Leipzig at age 14, Leibniz gained his first Bachelor’s degree at age 16; by age 21 he had completed a second Bachelor’s degree, a Master’s degree, and a doctorate in law.

As a courtier in the service of the Dukes of Hanover in Germany, Leibniz was able to travel on a variety of scientific, political, and diplomatic projects where he sought out the great intellects of his time. Leibniz was a frequent visitor at Académie Royale des Sciences in Paris and traveled to London where he was elected to the Royal Society. Leibniz exchanged letters with most of the eminent scientists and scholars; libraries that house Leibniz’s correspondence have estimated that the documents include about 15,000 letters from and to about 1100 correspondents.

In a fragment entitled *De Formae Logicae Comprobatione per Linearum Ductus* (*On the proof of logical forms by the drawing of lines*) Leibniz recorded a catalog of circle (or ellipse) diagrams for the entirety of the valid Aristotelian syllogisms (Leibniz 1903). Leibniz scholar and translator, G. H. R. Parkinson, judged that this undated 18-page fragment was written around the same time as the 1686 document *Generales Inquisitiones* (Leibniz 1966, p. xxxviii). *De Formae Logicae* was not published until 1903 when it appeared in *Opuscules et fragments inédits de Leibniz* (*Work and unedited fragments of Leibniz*). Figure 2 illustrates one such diagram for the proposition “All B is C.”

The circles, however, never seem to be the main point of Leibniz’s article—after all, its title emphasized a method of drawing lines, not circles. The opening sentence of the document read “I have recently been reflecting on the proof of Logical Form by the drawing of lines” (translation by Walt Jacob). Each of Leibniz’s circle diagrams was accompanied by his line diagram method using parallel lines segments of different lengths; Leibniz did not discuss or explain the circles but seemed to be more intent on exhibiting his line diagrams. In several other fragments, he provided extensive explanations of the line notation to illustrate logical arguments. However, another individual is credited with originating the logic line diagrams.

According to the Scottish philosopher Sir William Sterling Hamilton and John Venn (and others to this day), the Swiss mathematician, Johann Heinrich Lambert, originated the line-segment diagram method of displaying relationships between concepts in propositions (Venn 1881, p. 430; Hamilton 1874, p. 256; Lambert 1764).

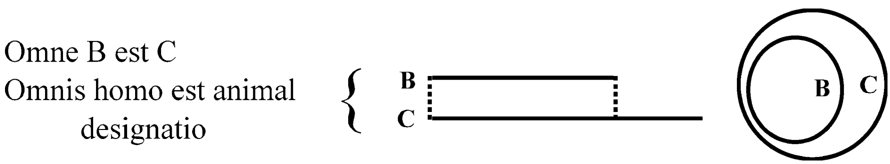


Fig. 2 Leibniz’s line diagrams alongside his circle diagrams circa 1686

Lambert first published his linear methods in his 1764 *Neues Organon*, presumably named after Aristotle's *Organon*. Lambert's 1764 line diagrams and Leibniz's 1686 line diagrams were strikingly similar.

According to some historians, the use of circles to discover the validity of a syllogism first entered the literature in the work of Johann Christoph Sturm, published in 1661 (preceding Leibniz's papers). In *Universalia Euclidea*, Sturm used circles, not to prove, but to highlight evidence in Euclid's propositions on proportions as he reproved them (Sholz 1961). Leibniz and Sturm were familiar with each other's work in philosophy and had the same professor, Erhard Weigel, at Jena University in Germany where Leibniz had studied briefly one summer in 1663 (Bullynck 2013).

Another individual mentioned as the possible "first" logician to use diagrams for the demonstration of the whole of the Aristotelian syllogistics was Christian Weise (1642–1708). In 1691, dramatist and Rector Christian Weise (1642–1708) published a booklet on Aristotelian syllogisms called *Nucleus Logicae* (Hamilton 1874; Venn 1881; Sholz 1961). In 1712 after Weise was dead, the document was revised and republished as *Nucleus logicae Weisianae* under the supervision of Johann Christian Lange, Professor of Philosophy at Giessen. Historians report having seen only the Weise/Lange edition. Sir William Hamilton (1874) related that circles and squares were used to represent propositions in a syllogism. Historian Sholz (1961) confirmed having seen these diagrams and commented that Lange had turned Weise's insignificant 72-page booklet into an 850-page opus, hinting that Lange may have added the drawings to the 1712 edition. Lange dedicated the 1712 *Nucleus logicae Weisianae* to the Berlin Academy, and historian Sholz suggested that this was a tribute to Leibniz, the Academy's founder and first president (Sholz 1961, p. 119).

Leibniz's 1686 circle diagrams and line diagrams went unpublished (and possibly unnoticed) for over 200 years. Although Leibniz amassed an impressive quantity of papers and letters, little was published during his lifetime, and the publications of his mathematics and philosophical work after his death were often unorganized and undated—leaving "a daunting impression of chaotic profusion" (Leibniz 1966, p. ix). Sir William Hamilton, in his 1874 *Lectures on Logic* stated,

> That the doctrines of Leibnitz [sic], on this and other cardinal points of psychology, should have remained apparently unknown to every philosopher of this country, is a matter not less of wonder than of regret, and is only to be excused by the mode in which Leibnitz gave his writings to the world. His most valuable thoughts on the most important subjects were generally thrown out in short treatises or letters, and these, for a long time, were to be found only in partial collections, and sometimes to be laboriously sought out, dispersed as they were in the various scientific Journals and Transactions of every country of Europe; and even when his works were at length collected, the attempt of his editor to arrange his papers according to their subjects (and what subject did Leibnitz not discuss?) was baffled by the multifarious nature of their contents (Hamilton 1874, p. 180).

However, the world did take notice when, in 1761, Leonhard Euler published almost identical circle diagrams to explain the valid Aristotelian syllogisms (Euler 1770). Euler did not claim originality; in fact, the diagrams were contained in study materials intended to represent the state of current knowledge.

4 Euler

Leonhard Euler's diagrams were originally a part of his correspondence with a student and as such were meant for instructional purposes. While Euler was at the Berlin Academy in Prussia, he penned the now famous *Letters to a German Princess, on Different Subjects in Physics and Philosophy* (*Lettres à une Princesse D'Allemagne*), written to Princess Charlotte Ludovica Luisa of Anhalt-Dessau (or Friederike Charlotte of Brandenburg-Schwedt), second cousin to Frederick the Great, King of Prussia. Euler had been asked to tutor the 15-year-old princess and her younger sister, and in 234 letters, written from 1760 to 1762, Euler taught lessons in physics, philosophy, mechanics, astronomy, optics, and acoustics. In 1768, the letters were published as a three-volume book where they enjoyed tremendous popularity. They were published in most European languages and the French edition went through 12 printings. The *Letters* were considered to be popular science of the day; they explained new discoveries of the time in a way that lay people could understand and enjoy.

When the first English translation of the letters appeared in 1795, its translator, Henry Hunter, reported that he embarked on the translation project because he felt that a work such as Euler's *Letters to a German Princess*, which was so well known and so esteemed over the entire European continent, should become known to British young people through their own language (Euler 1802, pp. xiii–xiv). Hunter also marveled at how unusual it was that a young woman of Euler's time had wished to be educated in the sciences and philosophy when most young women of the late eighteenth century were encouraged to learn little more than the likes of cross-stitch (Euler 1802, p. xix).

Euler's circle diagrams are contained in the letters Euler wrote instructing the Princess in Aristotelian and Stoic logic; they were written within a 3-week period and comprised about 50 pages in the 3-volume publication of letters. Although Euler's explanation of the valid Aristotelian syllogisms was much more detailed than that of Leibniz in *De Formae Logicae*, the circle diagrams were identical to those that Leibniz had used.

Euler, a mathematician of the highest order, has often been praised for his ability to explain complex ideas simply. In a 1787 Paris edition of the *Letters*, the Marquis de Condorcet noted that the *Letters* had acquired a celebrity through the reputation of the author, the choice and importance of the subjects, and the clarity of elucidation of those subjects. Condorcet considered the *Letters* to be a treasury of science (Euler 1802, p. xxvii). It was no wonder that Euler's name became attached to the syllogistic circle diagrams. To this day, many references continue to describe them as "Euler Circles."

Both Euler and Leibniz set out their diagrammatic systems so that each circle represented a term within a two-term statement or proposition. The circles were drawn one inside the other, overlapping, or non-intersecting, depending on the relationship between the two terms. Both men displayed how each of the four types of Aristotelian propositions would be represented using circles. Figure 3 reveals how similar they were.

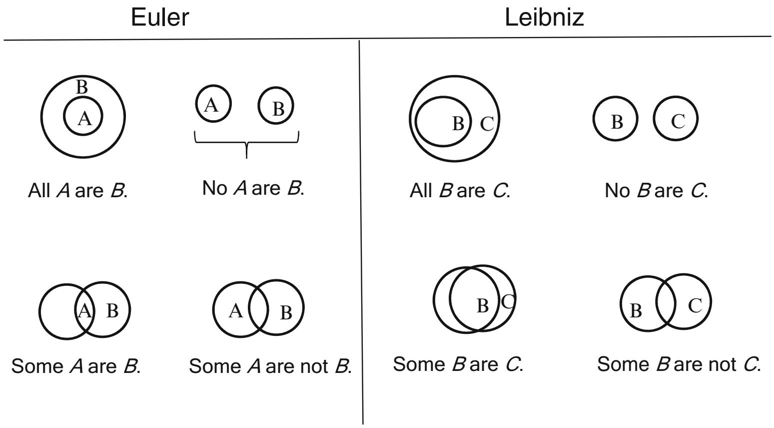


Fig. 3 The Euler/Leibniz circles for the Aristotelian propositions

Euler chose the symbols **A** and **B** to label the terms (called the subject and the attribute). Leibniz chose the labels **B** and **C** (but appeared to have originally used **A** and **B** and then changed his mind). Both men wrote their term labels *inside* the circle representing the term. For the "Some . . . are . . . " proposition, the *choice* and *location inside the circles' intersection* of the term label (**A** or **B** or **C**) indicated which term was the subject and which the attribute. The fact that the subject label was located inside the overlapping region of the circles affirmed rather than denied inclusion (for Euler, some part of **A** was indeed included in **B**). That is entirely unnecessary since for this type of proposition, it is immaterial which term is the subject and which the attribute: if *Some S are P* is true, then it is also true that *Some P are S*. Whereas in the "Some . . . are not . . . ." proposition the location of the label was outside the common region, indicating some part (for Euler) of **A** was definitely not any part of **B**. This turns out to be an extremely awkward notation for this type of proposition since *Some S are not P* and *Some P are not S* are not equivalent statements so their diagrams ought not look the same. "Some dogs are not poodles" is true, while "Some poodles are not dogs" is not. However, both Euler and Leibniz adopted this convention, and then later both applied it inconsistently.

Names for the four types of Aristotelian proposition were invented as a mnemonic device to aid students studying Aristotle's logic and trying to commit the rules to memory. Named after the vowels, "All are" was called an **A** proposition; "None are" was called an **E** proposition; "Some are" was called an **I** proposition; and "Some are not" was called an **O** proposition. Historians say that the letters come from *AffIrmo* (**A** and **I** propositions affirmed something) and *nEgO* (**E** and **O** propositions denied something). The simplest form of each of these types of propositions included two terms, a subject (S) and an attribute or predicate (P). The propositions were: All S are P. (**A**); No S are P. (**E**); Some S are P. (**I**); Some S are not P. (**O**). Three-line syllogisms were formed with three propositions, two serving as premises and the third a concluding proposition. Aristotle showed that some 3-line combinations of the statements lead to a valid argument and some do not.

The Aristotelian syllogisms can be discussed without any reference to the **A**, **E**, **I**, and **O** notation, and that is what Euler did in his first few letters on logic. Leibniz chose to include the notation with his diagrams and that seems to be the reason why he decided against using the term labels **A** and **B** and used **B** and **C** instead. Using the label **A** could cause confusion with the **A**-type proposition. In fact, in the 1903 publication of Leibniz's fragments, editor Louis Couturat indicated in footnotes that Leibniz had, several times, slipped up by using the label **A** when he meant to use the label **C**; Leibniz appeared to have changed his mind about which labels to use for the terms in the diagrams (Leibniz 1903, p. 292).

Neither Leibniz nor Euler claimed credit for the circle diagrams (Leibniz did claim invention of the line diagrams). And although Euler and Leibniz were not contemporaries, the two men were connected through other mathematicians and correspondents. Two of Leibniz's most enthusiastic followers were Jakob and Johann Bernoulli of Switzerland who disseminated his work throughout Europe after his death in 1716 (Dunham 1990). Euler studied mathematics under Johann Bernoulli and was a close friend of Bernoulli's son, Daniel. Leibniz and Euler shared correspondents in Johann Bernoulli and his nephew Nicolaus Bernoulli. Euler may have seen Leibniz's circles through their common colleagues; or both men may have seen the diagrams in the works of another. It is curious that neither of them treated the circle diagrams as if they were a new idea, yet the diagrams have not appeared in other scholarship of that period.

5 Venn

In the 1880s the English mathematical community was buzzing about the revolutionary symbolic logic methods put forward by George Boole in *An Investigation of the Laws of Thought: On Which Are Founded the Mathematical Theories of Logic and Probabilities* in 1854. In July of 1880, John Venn wrote an article entitled, "On the Diagrammatic and Mechanical Representation of Propositions and Reasonings," that was published in *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science*. In his article, Venn proposed a new kind of logic diagram with definite advantages over the previous diagrams in analyzing logical statements. It was Venn's goal that his diagrams would meet the demands of the new Boolean algebra.

John Venn's lectures in logic at Cambridge University formed the basis of his 1881 book, *Symbolic Logic*, where he more fully described his new diagrammatic method. On the prevalence of contemporary diagrammatic methods Venn commented that of 60 logical treatises published during the last century that he had (rather haphazardly) consulted, 34 of them had appealed to the use of diagrams, nearly all making use of the Eulerian scheme (Venn 1881). John Venn was, of course, referring to diagrams that had become known as Euler circles.

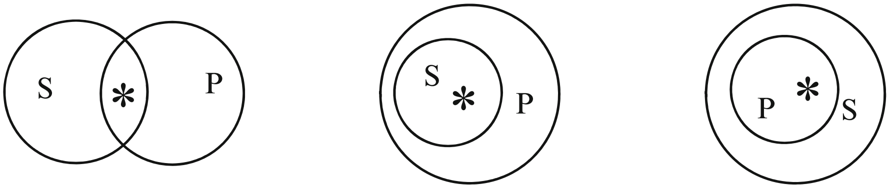


Fig. 4 Alternate possibilities for "Some S are P"

Venn enumerated several shortcomings of Eulerian circle diagrams as he introduced a new way of displaying the circles which he considered to be an improvement over the existing diagrammatic methods. Venn acknowledged that the use of *circles* is entirely arbitrary. Whatever the closed figure used, the purpose of the diagrams was always the same—an attempt to arrange the two or more closed figures to illustrate the mutual relation of inclusion or exclusion of the classes denoted by the terms employed in the syllogisms (Venn 1881, p. 52). One of Venn's objections to the Euler-type diagrams was that certain fairly simple propositions led to more than one possible diagram. For example, if the proposition "Some S are P" was true, with imperfect knowledge it was possible that "All S are P" or All P are S" was also true. So to represent all possibilities, three diagrams ought to be drawn as in Fig. 4. Of the three possible diagrams only one represented the proposition, but without further information it was uncertain which diagram should be used. Three (or more) analyses might be required.

A second objection raised by Venn was that he wanted the diagrams *to aid* in the task of working out a conclusion from premises, and he claimed that the Euler circles could only be drawn *after* the problem had been solved. Furthermore, the analysis of syllogisms had evolved to encompass far more complicated syllogisms than the 3-term, 2-premise syllogisms of Aristotle. The Eulerian system was not equipped to deal with disjunctive statements like, "All X is either Y and Z, or not-Y" and "If any XY is Z, then it is W" (Venn 1880, p. 13). Venn mentioned this deficiency, but he indicated that the older system ought not be criticized for its failure to negotiate statements more complicated than the ones for which the system was invented when he said, "it should be understood that the failure of the older method is simply due to its attempted application to a somewhat more complicated set of data than those for which it was designed" (Venn 1880, p. 14).

In the system of Leibniz and Euler (depending on the type of proposition being made), each new set of premises required a completely different kind of drawing. Venn declared that this was an essential defect of these systems—that each new proposition required a new diagram from the beginning. On the other hand, every one of Venn's diagrams began with the same drawing. Each of Venn's diagrams began with a number of circles equal to the numbers of terms (classes) to be analyzed in a syllogism. The circles, representing the classes, overlapped in such a way as to create compartments and each compartment represented a unique subclass. The underpinnings of Boole's logic rested upon consideration of all combinations of the terms involved—combinations that Venn called subclasses. For two terms, say X and Y, there were four subclasses—things that were both X and Y, things that were X but not Y, things that were Y but not X, and things that were neither X nor Y. For three terms, there were eight subclasses.

So, without needing to know the import of the proposition, every Venn diagram of two classes began with the exact same drawing of two overlapping circles, creating compartments for each of the four distinct subclasses. Since every diagram began in exactly the same way, Venn's compartmentalized circles served as "graph paper" from which the analysis of the syllogism could begin. Venn even suggested that a stamp could be created so that the "graph paper" for the diagrams was ready-made (Venn 1880, p. 16).

When Venn introduced his two-circle diagram to represent two classes, he emphasized that the diagram did not as yet represent a proposition or a relationship between **X** and **Y**, but represented a "framework into which propositions can be fitted" (Venn 1880, p. 6). All points inside the circle labeled **X** were regarded as members of **X**, and all points outside the same circle were regarded as **not-X**. The same applies to the circle labeled **Y**. In Venn's case, the location of the term labels (**X** and **Y** in this case) was irrelevant and had no significance. They could be located anywhere that was convenient. The four subclasses were represented by the compartments—inside both **X** and **Y**, inside **X** but not **Y**, inside **Y** but not **X**, and inside neither **X** nor **Y** nor both (outside the space of both circles). See Fig. 5.

To represent the relationships between the terms of Aristotelian propositions, Venn added shading or markings onto the same one diagram. Shading a compartment was an indication that the subclass was empty, while a small cross or asterisk in a compartment indicated that something existed in that subclass (in other words, it was not empty). The shaded compartments and the crosses in compartments tell something definitive about the relationships between the terms, while compartments devoid of shading or a cross were an indication of the lack of knowledge. Venn commented, "How widely different this plan is from that of the old-fashioned Eulerian diagrams will be readily seen. One great advantage consists in the ready way in which it lends itself to the representation of successive increments of knowledge as one proposition after another is taken into account, instead of demanding that we should endeavor to represent the net result of them all at a stroke" (Venn 1881, p. 113). The four types of Aristotelian propositions using Venn's method are shown in Fig. 6.

Every Venn diagram involving three classes began with the exact same drawing of three circles, overlapping to create eight compartments representing the subclasses. Figure 7 illustrates Venn's 3-term diagram that would be used for analysis of all syllogisms involving three terms. Venn had originated the diagram that has become so familiar today.

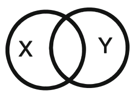


Fig. 5 Venn's template for all two-term propositions

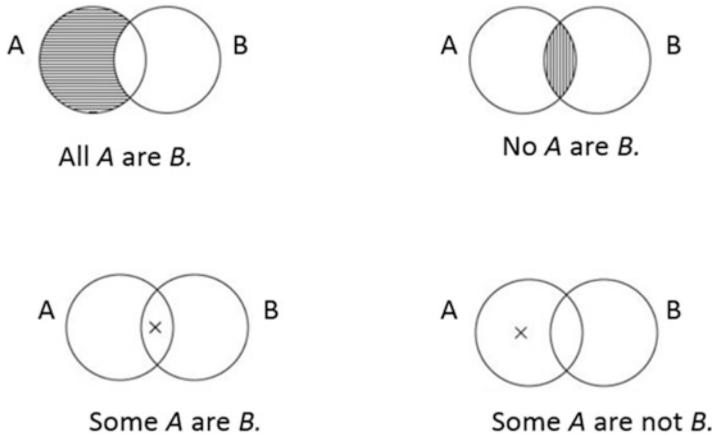
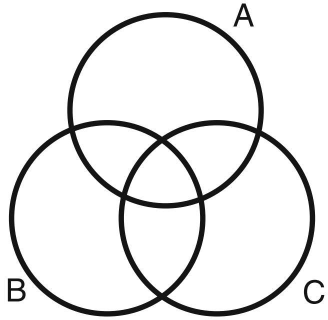


Fig. 6 Venn's diagrams for the four Aristotelian propositions

Fig. 7 Venn's template for all 3-term syllogisms



Venn realized that for four terms, it was impossible to arrange four circles in such a way as to produce 16 compartments. He suggested that the figure could be drawn with some shape other than a circle, "any closed figure will do as well as a circle, since all that we demand of it, in order that it shall adequately represent the contents of a class, is that it shall have an inside and an outside, so as to indicate what does and what does not belong to the class" (Venn 1880, p. 6). Venn's solution for four terms was four overlapping ellipses. When drawn as in Fig. 8, there were 15 compartments plus the region outside of all of the ellipses for a total of 16 compartments. For example, the region marked with the cross symbol **x** was the subclass of things which had the attribute of **X**, **Y**, and **Z** (the symbol was inside those ellipses) and did not have the attribute of **W** (the **x** symbol was outside that ellipse).

For five terms, Venn was unable to find a satisfactory arrangement of ellipses (although modern mathematicians have been able to create symmetrical 5-set diagrams using ellipses); Venn proposed the diagram that can be seen in Fig. 9. This diagram has the unfortunate feature that region **Z** is a donut-shaped region, or annulus. The ellipse in the center of **Z** was actually a hole, so that compartment was outside **Z**.

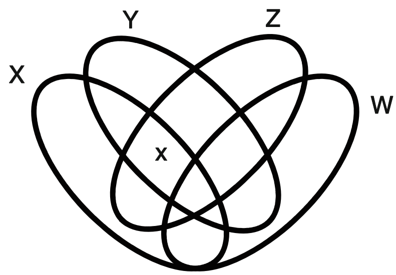


Fig. 8 Venn's suggestion for analysis of 4-term syllogisms

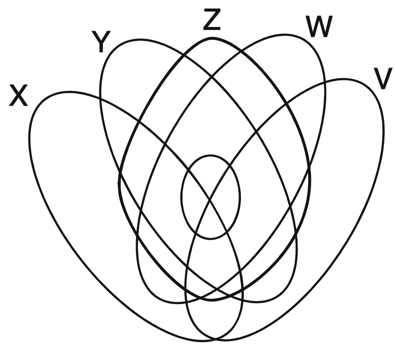


Fig. 9 A Venn diagram for 5 terms

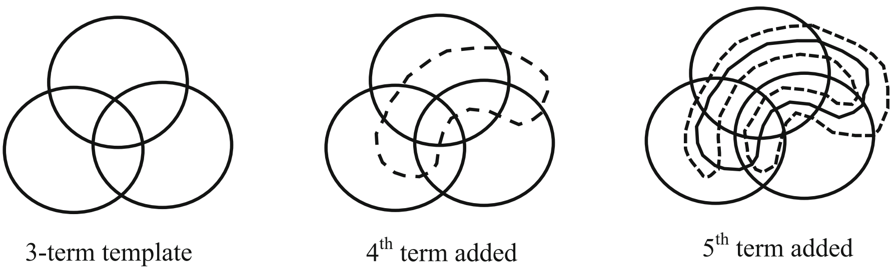


Fig. 10 Venn's method for creating larger diagrams

Venn suggested another interesting diagrammatic format for 4 or more terms as shown in Fig. 10. Working with the 3-term template, Venn added a horseshoe-shaped figure so that its outline divided each compartment it passed through into exactly two compartments. A (3-term) diagram having 8 compartments became a (4-term) diagram having 16 compartments. Venn thought that this technique could be repeated indefinitely.

As mentioned earlier, Venn thought that stamps could be created for three-, four-, and five-term figures so that the figures would not have to be drawn each time an analysis was made. He also suggested creating a figure in cardboard and cutting out the compartments while leaving the boundary lines so that the compartments would be like the pieces of a child's puzzle. Beginning with all the compartments in their original places and, instead of shading the empty compartments, compartments could simply be removed as they got eliminated. In this way, one could put all the puzzle pieces back when starting on a new problem (no paper wasted). Venn

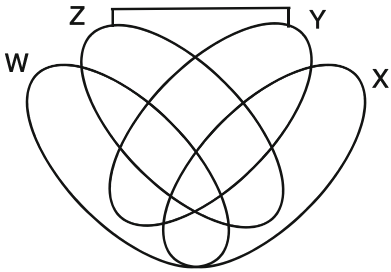
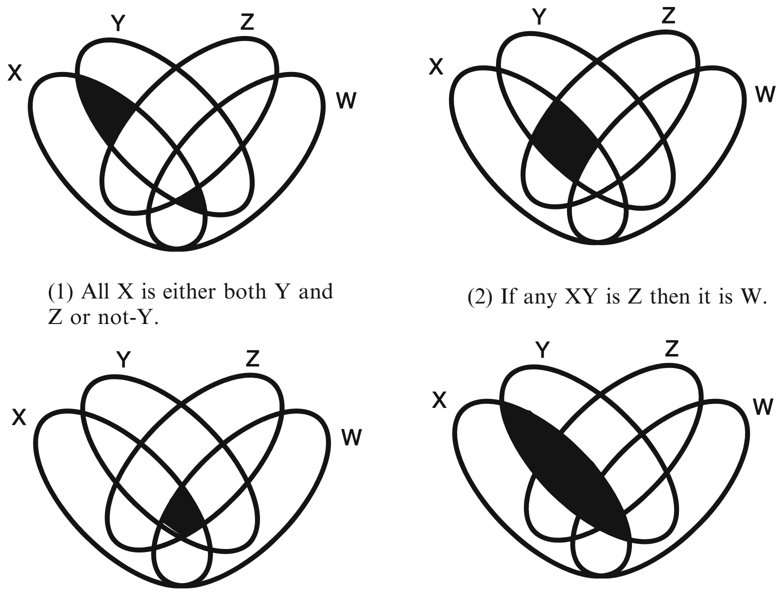


Fig. 11 Venn's plan for a "logic machine"



(1) All X is either both Y and Z or not-Y.

(2) If any XY is Z then it is W.

(3) No WX is YZ.

Conclusion: No X is Y.

Fig. 12 Venn's demonstration of his method

developed plans for a logic machine, based on his diagram method. The "logic machine" was really just a three-dimensional version of the suggested puzzle where the pieces dropped through a hole instead of being removed like the puzzle pieces. In Fig. 11, Venn's logic machine plans revealed a new development: an extra compartment at the top of the ellipses. He indicated that this compartment represented the region outside all of the ellipses.

An illustration of Venn's method was provided as Venn demonstrated using the following complex group of premises: (1) All X is either both Y and Z or not-Y; (2) If any XY is Z then it is W; (3) No WX is YZ. As each premise was added, information was acquired about combinations that could not exist, and additional compartments were eliminated as indicated by the shading in Fig. 12. Finally, after the shadings were completed Venn observed that the diagram made obvious what the conclusion ought to be: X and Y are mutually exclusive or "No X is Y" (Venn 1880, p. 13).

John Venn had modified the earlier logic circle diagrams so that his diagrammatic method would parallel Boole's system and enable a visual representation of it. Today, the diagrams have evolved even further, modified through the use of color and size (where color or size has additional meaning in the diagram). Venn (and those before him) would probably be astounded that a small visual tool like the diagrams would have proliferated into so many spheres of society. A Google search on "Venn Diagram" produces 1,470,000 hits, and a search through You Tube produces 16,500 videos on the Venn diagram. Searching an academic library database for "Venn diagram" produces applications well beyond the syllogism in areas as diverse as bioinformatics, mental health, and ethical reasoning. There is no doubt about the impact of the diagrams; they have become pervasive in popular culture.

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Mathematics for the World: Publishing Mathematics and the International Book Trade, Macmillan and Co.

Sylvia Marie Nickerson

Abstract Several historians including Andrew Warwick, Joan Richards, and Tony Crilly have offered explanations for why a stale culture of mathematics existed in nineteenth-century England. Nineteenth-century British culture did not generally regard mathematics as capable of failure, growth, or change. This paper argues that a significant contributing influence to this climate was book publishers, and the publisher Macmillan and Company in particular. From 1850 to 1900 Macmillan published hundreds of thousands of mathematical textbooks through industrialized book production. Macmillan distributed these pedagogical materials throughout the UK, Canada, the USA, Australia, India, and elsewhere. Motivated by profits from sales, and abetted by the efforts of their collaborator on mathematical subjects, Isaac Todhunter, Macmillan perpetuated a stale, Cambridge-centric image of mathematics among subsequent generations of mathematical learners in educational contexts around the world. How some of these books may have shaped the pedagogical experience of Canadian mathematician J. C. Fields during his high school and undergraduate education in mathematics is considered.

1 Introduction

An underexplored facet within the history of mathematics is how the book trade, and in particular book publishers, shaped knowledge formation in mathematics. Recently historians have recognized the study of book history and printed culture as a fertile ground from which relevant questions may be raised about texts, their origin, and influence within culture (Simon 2012, p. 340). Previously historians have often presumed authors to be the sole producers of textual ideas, while ignoring the mediating role that publishers and the allied trades of printers, distributors, advertisers, booksellers, illustrators, and libraries, played in the molding and communication of information and ideas through printed texts (Topham 2000, p. 560). The present work aims to question assumptions of authorial primacy within the

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history of mathematics and suggests that book publishers in addition to authors, played a role in shaping mathematical culture.

The focus of this study is the mathematical publications of the British book publisher Macmillan and Company in the second half of the nineteenth century. Macmillan's production and distribution of mathematical books influenced the development of mathematical culture in the contexts in which their mathematical books were read and used. The Canadian mathematician John Charles Fields used several Macmillan textbooks during his high school and university education in Hamilton and Toronto, Canada (Riehm and Hoffman 2011, pp. 15–17, 21). Through Fields' life one can evaluate the influence of book culture on the mathematician. Fields was both a reader of many mathematical textbooks, and later in life he authored one research monograph about complex functions (Fields 1906). Isaac Todhunter, by comparison, authored many mathematical class books and collaborated with the Macmillan family in the development of their mathematical publication program, formidable in its multiplicity of publications (see Appendices 1 and 2). This paper examines how this author–publisher partnership evolved within the Cambridge University community, how the Cambridge context influenced their publications, and suggests that these pedagogical materials formed first impressions about mathematics on several generations of mathematical students. Through these numerous publications, Macmillan and Todhunter wielded influence upon mathematical pedagogy within the Anglophone context during the late nineteenth-century period.

2 John Charles Fields' Mathematical Studentship

The Canadian mathematician John Charles Fields (1863–1932) was born and raised in Hamilton, Ontario. At the time of Field's youth in Hamilton, new models of public education had been enacted in Ontario since the mid-century (Parvin 1965, pp. 6–26). Fields arrived on the scene at just the right moment to attend the original model school for elementary public education, Hamilton Central School, and the second public high school in Ontario, Hamilton Collegiate Institute (Smith 1910, p. 83). Fields had excellent teachers at these schools. One of his science teachers at Hamilton Collegiate Institute, J. W. Spencer, had received his PhD in geology from Göttingen, Germany. George Dickson, a chemist, was principal. Beyond their teaching roles, both Dickson and Spencer were active in their scientific fields. W. H. Ballard taught mathematics, and under his tutelage, many students were successful in the matriculation examinations in mathematics at the University of Toronto. Two students from Fields' class at Hamilton Collegiate Institute went on to achieve PhDs in mathematics: J. C. Fields and Milton Haight (Riehm and Hoffman 2011, pp. 12–14).

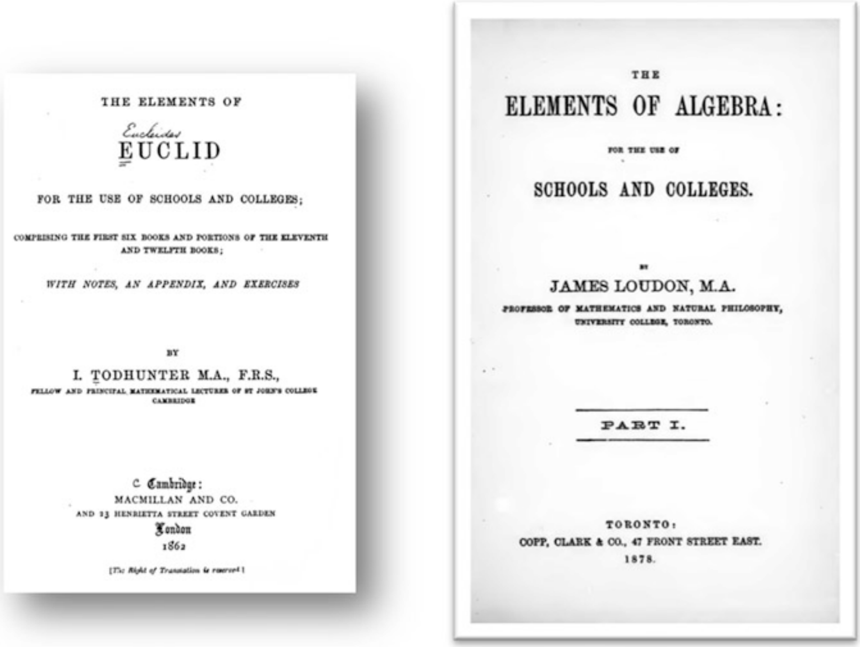
Records from Hamilton Collegiate Institute indicate what books were used in Fields' course of mathematical study. During his time as a student there in the 1870s, he read the Canadian edition of *Elementary Algebra* (1877) by J. Hamblin Smith (of Gonville & Caius College Cambridge), which contained an appendix by Alfred Baker, a mathematical tutor at the University of Toronto. *The Elements of Algebra; for the Use of Schools and Colleges*, Part I (Toronto: Adam, Stevenson and Co., 1873) by James Loudon was on the curriculum. James Loudon was also a tutor, and later professor of mathematics at the University of Toronto. Another book by Hamblin Smith, *Geometry, the Elements of Geometry containing Book I to VI and portions of books XI and XII of Euclid with a Selection of Examinations Papers by Thomas Kirkland, M.A.* (1877) was used at Hamilton Collegiate.

Smith's 1877 *Geometry* and Smith's 1877 *Elementary Algebra* were "Canadian" editions of what had been originally British-published books. Typically, a Canadian edition of a British textbook was mostly identical to the British original with the addition of a preface, exam questions, and the name of a prominent Canadian teacher on the book's frontispiece. Several excerpts from positive testimonials or reviews from Ontario newspapers can also frequently be found in Canadianized editions. The Canadian edition of Hamblin Smith's *Geometry*, for example, fit these features, as it included exam papers by Thomas Kirkland of Toronto's Normal School. Fields was also exposed to a Toronto edition of Isaac Todhunter's *The Elements of Euclid for the Use of Schools and Colleges, comprising the first six books and portions of the eleventh and twelfth books* (Toronto, new edition, 1876). In the frontispiece of the book Todhunter was identified as Fellow and Principal Mathematical Lecturer at St. John's College, Cambridge (Riehm and Hoffman 2011, pp. 15–17) (Fig. 1).

In 1880 after high school, Fields entered the University of Toronto to study mathematics. University records contain the courses in mathematics that were taught at the university at this time. Elaine McKinnon Riehm and Frances Hoffman uncovered the subjects and books presented to Fields as a student at the University of Toronto in the 1880s (see Fig. 2). Several of these books were published in London by the publisher Macmillan, including Todhunter's *Spherical Trigonometry*, Tait and Steele's *Dynamics of a Particle*, and Boole's *Differential Equations* (Fig. 3).

After Fields graduated from the University of Toronto's math department, he went on to study mathematics at Johns Hopkins University. After completing a PhD degree at Johns Hopkins in 1887, he taught for a few years. Then Fields undertook an extended period of further study in Europe, from 1892 to 1900. He attended classes at the Collège de France and the Sorbonne in Paris, after which he moved to Germany where he attended more lectures at the University of Berlin. In France he came to know the work of Henri Poincaré, Émile Picard, Paul Painlevé, and Paul-Émile Appell, and in Germany that of Karl Weierstrass and Georg Riemann (Riehm and Hoffman 2011, pp. 32, 37–38).

In 1900 Fields returned to Toronto, and spent the rest of his life as a mathematics teacher at the University of Toronto. In a 1908 sketch he wrote about his mathematical life, Fields made this reflection about his education in Canada: "Fifteen years after I had received my grounding in the calculus I discovered, I am ashamed to say, in a German University, the University of Berlin, that it had been taught to me falsely, irremediably and fundamentally falsely" (Riehm and Hoffman 2011, pp. 21–22).


THE ELEMENTS OF
EUCLID
FOR THE USE OF SCHOOLS AND COLLEGES;
COMPRISING THE FIRST SIX BOOKS AND PORTIONS OF THE ELEVENTH AND TWELFTH BOOKS;
WITH NOTES, AN APPENDIX, AND EXERCISES
BY
I. TODHUNTER M.A., F.R.S.,
FELLOW AND PRINCIPAL MATHEMATICAL LECTURER OF ST JOHN'S COLLEGE CAMBRIDGE
Cambridge:
MACMILLAN AND CO.
AND 23 HENRIETTA STREET COVENT GARDEN
London
1862
[The Right of Translation is reserved]

THE
ELEMENTS OF ALGEBRA:
FOR THE USE OF
SCHOOLS AND COLLEGES.
BY
JAMES LOUDON, M.A.
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PART I.
TORONTO:
COPP, CLARK & CO., 47 FRONT STREET EAST.
1878.

Fig. 1 Mathematical books used in J. C. Fields' education in Hamilton, Ontario (ca. 1876). *Right*: Isaac Todhunter, *The Elements of Euclid*, London: Macmillan and Co., 1862. (British edition). *Left*: James Loudon, *The Elements of Algebra* ... Toronto: Copp Clark & Co. 1878 (first published 1873)

First Year: Algebra; Euclid; Plane Geometry; Salmon's *Analytical Conic Sections*

Second Year: Elements of statics and Dynamics; Newton's Principia, Sec. I (Main's ed.); *Differential Calculus* (Williamson's); Solid Geometry; *Spherical Trigonometry* (Todhunter's); *Theory of Equations* (Todhunter's)

Third Year: Elements of Hydrostatics and Optics; *Analytical Statics* (Minchin's); *Dynamics of a Particle* (Tait and Steele's); Newton's Principia, Secs. II and III; Geometrical Optics (Jamin); *Hydrostatics* (Besant's); Rigid Dynamics; Eulerian Integrals

Fourth Year: Elements of Astronomy, Acoustics, Heat. Mathematics: 1. Modern Geometry; Salmon's *Conic Sections*, Chaps. 4, 9, 14, 15; Salmon's *Higher Plane Curves* Chaps. 1-4. 2. Modern Algebra; Salmon's *Higher Algebra*, Chaps. 1-9. 3. *Differential Equations*; Boole, Chaps 1-12. 4. Theory of Probability. 5. *Plane Astronomy*; Godfray. 6. Quaternions.

Fig. 2 Mathematical books used in J. C. Fields' education at the University of Toronto, ca. 1880–1884. List derived from the University of Toronto Calendar, 1883/84, reproduced in Riehm and Hoffman (2011, p. 21)

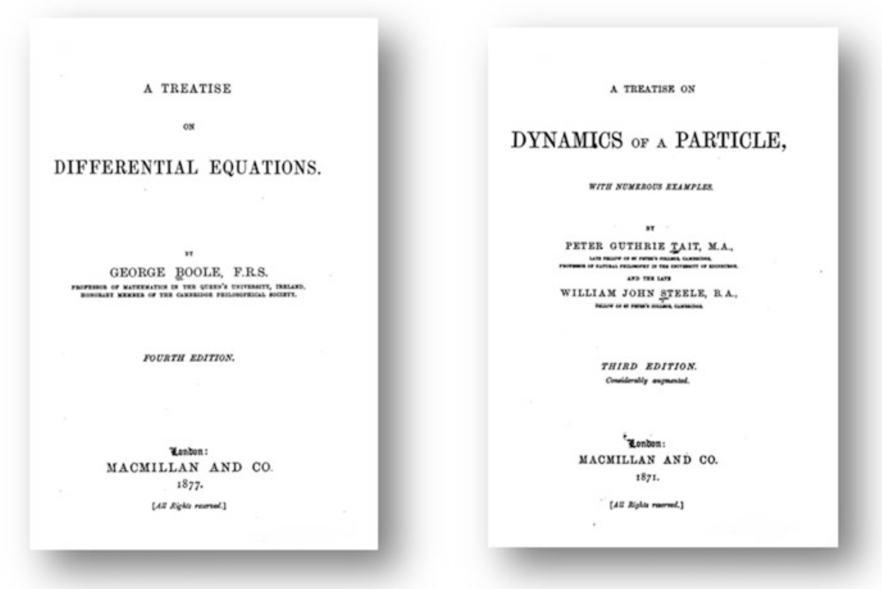


Fig. 3 Frontispieces from mathematical books used in J. C. Fields' education in Toronto (ca. 1880–1884). *Right*: George Boole, *A Treatise on Differential Equations*, London: Macmillan and Co. (4th edn.) 1877. *Left*: P. G. Tait and W. J. Steele, *A Treatise on Dynamics of a Particle, with numerous examples*, London: Macmillan and Co. (3rd edn.) 1871

3 Isaac Todhunter's Mathematical Textbooks

Isaac Todhunter is known to history as having been the British author of many nineteenth-century mathematical textbooks (Barrow-Green 2001, p. 189). Indeed, Todhunter was the author of some of the mathematical textbooks assigned to Fields during his Canadian studentship in mathematics. Todhunter published over forty mathematical books with the publishing company Macmillan between the years 1843 and his death in 1884 (see Appendix 1). Many of Todhunter's books reached unprecedented levels of circulation in educational contexts inside and outside Britain in the latter half of the nineteenth century (Eliot 2002, p. 23; Eliot 1994, p. 13) (Fig. 4).[1]

Todhunter's books were used as course texts by the Universities of Manchester, Leeds, Liverpool, Edinburgh, and Bristol, as well as becoming set texts for schools

[1] Simon Eliot has defined any book whose total print-run exceeded 10,000 copies as unambiguous publishing successes for Macmillan (Eliot 2002, p. 20). Many of Todhunter's textbooks were printed in numbers greater than 10,000, see Appendix 1.

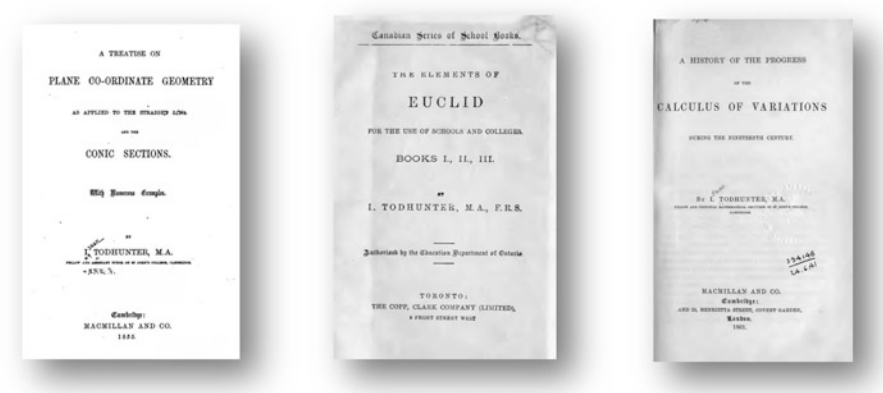


Fig. 4 From *right*: The frontispieces from some of Todhunter's textbooks: I. Todhunter, *A Treatise on Plane Co-ordinate Geometry*... Cambridge: Macmillan and Co., 1855. I. Todhunter, *The Elements of Euclid for the use of Schools and Colleges*, Books I, II, III. Toronto: Copp, Clark Company, 1876. I. Todhunter, *A History of the Progress of the Calculus of Variations During the Nineteenth Century*. Cambridge and London: Macmillan and Co., 1861

in the USA, Canada, and Australia (Barrow-Green 2001, pp. 187–189). In 1878 Todhunter wrote to his wife, "there is a library of mathematical books provided by the Civil Service Commission [of India] for the use of the Examiners. It consists of fourteen volumes, ten of which are by myself" (quoted in Barrow-Green 2001, p. 184). A decade after he died, twelve of Todhunter's textbooks were still recommended as useful reading material for the Cambridge undergraduate (Besant 1893, pp. 33–38).

One well-known student of mathematics who attended Cambridge in the 1890s, and thus who likely made use of Todhunter's textbooks, was the philosopher and logician Bertrand Russell (Crilly 1999, p. 131). Russell studied mathematics at Cambridge from 1890 to 1893. Like Fields, Russell also found his mathematical education lacking, and he later reflected that his mathematical studentship involved practices and materials that were outmoded and out of date. He reflected in his memoir, *My Philosophical Development*, that as a Cambridge undergraduate he had never studied nor heard of most contemporary German or French mathematicians, including Karl Weierstrass, Richard Dedekind, Georg Cantor, Gottlob Frege, or Giuseppe Peano (Russell 1959, p. 39). He wrote that "the 'proofs' that were offered of mathematical theorems [at Cambridge] were an insult to the logical intelligence" and "my teachers offered me proofs which I felt to be fallacious and which, as I learned later, had been recognized as fallacious" (Russell 1956, p. 20; Russell 1959, pp. 37–38).

4 Publishing and the History of Mathematics

Fields and Russell went on to become scholars who made original contributions to mathematics. Yet both men were critical of their mathematical educations. In light of their comments about their mathematical education in Canada and England, we might ask, why did these contexts present mathematics as a tool, but not introduce it as a craft? Why had the creative aspect of mathematics been deadened in these educational contexts?

Several historians have offered perspective on this question. In her 1988 book *Mathematical Visions*, Joan Richards presents the explanation that mathematics was not developed creatively within nineteenth-century English culture because the English viewed mathematics as a tool in which to discipline the mind, rather than as a subject capable of development. Richards argues that the English viewed mathematics as an empirically flawless body of knowledge and did not, in general, regard mathematics, in particular geometry, as capable of growth, failure, or change. As such, mathematics held special cultural value in English culture and society, for its ability to provide an exemplar, or norm, for truth (Richards 1980, pp. 346, 363). Richards argues that this image of mathematics was expressed in England's educational structures, and shaped the teaching of mathematics at the University of Cambridge (Richards 1988, chapter five).

Andrew Warwick's book, *Masters of Theory*, goes some way towards explaining the state of late nineteenth-century English mathematical culture by way of training and pedagogy, contending that Cambridge University's culture of mathematics at that time was largely focused on examinations in which one performed mixed or applied mathematics using synthetic methods (Warwick 2003, p. 434). The emphasis on utility in mathematics and its applications, Warrick suggests, may have deadened sensitivity to questions internal to the subject, or interest in new methods and theories being developed simultaneously in continental Europe (Warwick 2003, pp. 505–506).

In their paper about the history of mathematics in Canada prior to 1945, Tom Archibald and Louis Charbonneau claim that in order to develop a mathematical culture, three key aspects of a society must play a role. Firstly, a population or governing regime must place value on the acquisition of mathematical skill. Secondly, an educational infrastructure, including teachers, teaching materials, the curriculum, and its objectives, is critical in shaping and developing a local practice of mathematics. Thirdly, they note, people who are interested in mathematics rely on the activities of the book trade: publishers and printers make available basic mathematical knowledge (Archibald and Charbonneau 1995, p. 1).

Richard's work, mentioned above, has commented on the first of these requirements for a mathematical culture. She has described how the cultural value imbued in mathematics affected its practice in Victorian England. The second aspect—that of educational structures and teaching—has been examined by Warwick, while Karen Hungar Parshall and Tony Crilly have explored Arthur Cayley and James Joseph Sylvester as mentors and teachers at Cambridge who did not develop close mentoring relationships with students (Crilly 1999, pp. 146, 151–152; Parshall 2006). According to Archibald and Charbonneau's criteria, the as-yet unexamined factor influencing the culture and practice of mathematics in Victorian England is the book trade.

How might this last element, the book trade element of Archibald and Charbonneau's criteria for mathematical culture, relate to the practice of mathematics in the nineteenth-century Anglophone context? In the case of Fields in Canada and Russell at Cambridge, British book publishers helped perpetuate a stale and dated image of mathematics through the pedagogical materials they produced. In the 1860s and 1870s, British publishers increasingly applied steam power and new industrialized methods of binding and printing to book production (Twyman 1998, p. 70). The application of these new technologies combined with the lessening of paper tax and the cost of raw materials helped bring about a revolution in the numbers of produced and distributed books (Eliot 1994, p. 107). These conditions increased the circulation of mathematical books as well, now that this genre, traditionally difficult and costly to produce, could be reproduced more effectively, cheaply, and numerously than ever before (Rider 1993, pp. 111–113). In turn, publishers who employed these techniques were part of an expanding colonial economy, which spread these numerous and relatively cheap textbooks into educational contexts inside and outside Britain (Feather 2006, pp. 115–116). As a result, the stale approach to mathematics contained in these books, many of which originated within the Cambridge context, helped perpetuate this image of the subject in Britain at large and in British colonial places where foreign trade took place. Macmillan Company is one example of a British publisher whose textbooks played this role within the development of nineteenth-century mathematics.

5 Macmillan and Company as a Mathematical Publisher

From 1843 to 1850, through leverage and hard work, the Macmillan brothers Daniel and Alexander came to run the bookshop at number one Trinity Street, Cambridge. This was one of the most prominent bookselling locations in Cambridge, located across from the university's Senate House and beside the University Church of St. Mary's, and within sight of the gates of Kings College. In the climate of teaching and learning at Cambridge in the 1840s and 1850s, many professors had little reason to be in direct contact with their students (Crilly 1999, pp. 151–152; Barrow-Green and Gray 2006, p. 328). The Macmillan's bookshop, however, served as a common ground, a place where both professors and students frequented (Morgan 1943, p. 30). The brothers' lodging above the shop became a sort of little college in itself, where Cambridge men stopped in for "a pipe and a chat", to discuss books, God and social reform, before attending the Sunday sermon (Morgan 1943, p. 34).

During their Cambridge bookselling days, the brothers Daniel and Alexander Macmillan began investing heavily in the publication of mathematics, reflecting the central role this subject held within the Cambridge University curriculum (see Foster 1891, pp. 2–14, listing Macmillan's publications for the years 1845–1850). Initially they bought copyrights to produce and sell established mathematical textbooks, including the twelfth edition to Thomas Lund's *A Companion to Wood's Algebra* (1848), the second edition of J. C. Snowball's *The Elements of Mechanics* (1846), the fifth edition of Snowball and Lund's *Cambridge Course of Elementary Natural Philosophy* (1845), and the third edition to J. Hymers *A Treatise on Plane and Spherical Trigonometry* (1847). Buying these copyrights was expensive for Macmillan. For example, Thomas Lund commanded approximately 80 % of the profit on his 1851 book, *A Short and Easy Course of Algebra*, and 84 % of the profit for his twelfth edition of Wood's *Algebra*.[2] Soon the Macmillans began developing new relationships that produced textbook material that was more cheaply acquired. With Snowball, and eventually with Isaac Todhunter, the Macmillans negotiated to split profits equally between author and publisher (commonly known as the "half-profits" agreement, during the period).[3]

The most important influence shaping Macmillan's mathematical publications in the 1840s and 1850s was Isaac Todhunter, a student at Cambridge from 1844 to 1848, and a person the brothers met through the bookshop and its associated social gatherings (Morgan 1943, pp. 30, 37). Over 30 years Todhunter published over forty mathematical books with Macmillan, ranging from school and college texts to specialized monographs in mathematics and the history of mathematics (see Appendix 1). Correspondence and business records from the company suggest that Todhunter provided advice to the Macmillans about their publishing projects in education and mathematics, by reviewing manuscripts, looking over proofs, recommending new authors, and suggesting the format and presentation of educational works (Nickerson 2012, p. 37). Todhunter connected new authors, such as George Boole and Barnard Smith, with the Macmillan family. Books written by these authors displaced the stock-and-trade textbooks that the two brothers had first published when they began investing in mathematical books.

In 1858 Alexander Macmillan opened an office for their business in London. This brought the Macmillan Company into contact with new spheres of intellectuals. Alexander Macmillan, the surviving brother, began hosting soirees in which artists, writers, historians, theologians, and men of science gathered and discussed, quote, "Darwin and conundrums with general jollity pleasantly intermixed" (Morgan 1943, p. 52). These gatherings, at the Macmillan London office, were informally known as the Tobacco Parliaments. Just as their Cambridge bookshop and its associated social gatherings had facilitated the development of mathematical authors for the company, social evenings in London began to shift Macmillan's focus from mathematical textbooks to developing more broadly as publishers of books on science (Foster 1891, pp. 63–201, listing Macmillan's publications for 1860–1870). During the

[2]Printed catalogue from June 1, 1864, BL Ad. MS 54791, Publications Catalogues with Manuscript Additions, Macmillan Archive, British Library, London, UK.

[3]Printed Catalogue from June 21, 1851, BL Ad. MS 54790, Publications Catalogues with Manuscript Additions, Macmillan Archive, British Library, London, UK.

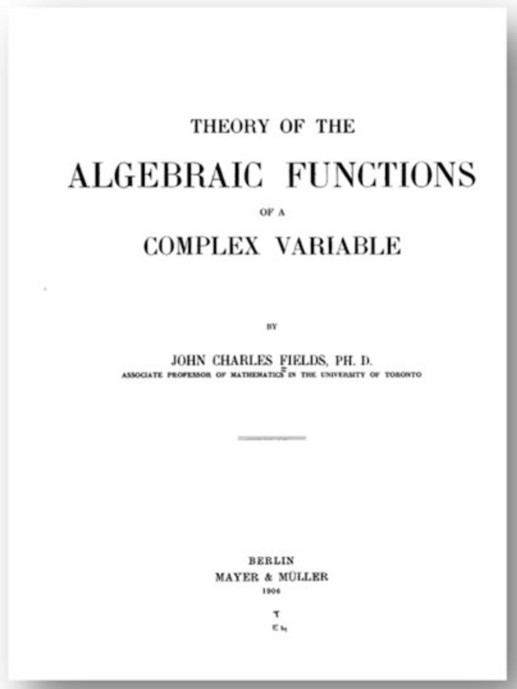
period from 1860 to 1875, as the business became ever more successful, Macmillan continued to reproduce editions of their mathematical textbooks in ever greater quantities, alongside their development of new books series in science and science education (see Appendix 2).

Mathematical textbooks, especially on the scale in which Macmillan produced them, had greater reach than most other printed sources in mathematics, and formed first impressions upon many students. The values expressed by these textbooks and the image of mathematics presented by them were closely tied to the cultural values about mathematics that were embodied at Cambridge University in the nineteenth century. Mathematical materials embodying the approach to mathematics embedded in the Cambridge University curriculum proceeded to publication in textbook form at Macmillan. This image of mathematics was then exported around the world, forming the first impression of mathematics on many students who used these books as course texts.

6 Book Trade Influence: Research Monographs Versus Textbooks

After J. C. Fields completed his Hamilton high school education and study in mathematics at the University of Toronto, he had the benefit, somewhat unusual at that time, of obtaining further education. He completed his PhD at Johns Hopkins followed by several more years of post-doctoral study in Europe. His exposure to mathematics continued in these contexts. At Johns Hopkins, he was introduced to books on differential equations by Charles Auguste Briot, Jean Claude Bouquet, Gaston Floquet, and Lazarus Fuchs, treatises on the theory of functions by Charles Hermite, Broit, Bouquet and Fuchs, and sources on elliptic and Abelian functions by Cayley, Alfred Clebsch and Paul Gordan (Barnes 2007, pp. 9–11). Fields kept detailed notebooks that recorded the lectures he attended during his postdoctoral study in France and Germany. These record lectures Fields' attended by Georg Frobenius, Fuchs, Kurt Hensel, Hermann A. Schwarz, Georg Hettner, Johannes Knoblauch, Ernst Steinitz, and Max Plank. These notes contain topics in number theory, analytic geometry, synthetic projective geometry, algebraic equations, hyperelliptic functions, differential equations, Abelian integrals, theory of functions of a complex variable, Fourier series, Cantor's theory of transfinite cardinals, and theoretical physics (see Barnes 2007, pp. 13–18).

Fields returned to Canada in the spring of 1900 at the age of 37. After many years of his mathematical education and various perambulations around the world, Fields mathematical ambitions culminated in his single-author monograph, *Theory of the Algebraic Functions of a Complex Variable*, published in 1906 by Mayer and Müller and Acta Mathematica. Faced with a dearth of printers capable of handling his special monograph, Fields's acquaintance Gösta Mittag-Leffler in Sweden offered to assist in making arrangements for the publication (Riehm and Hoffman 2011, p. 62) (Fig. 5).


THEORY OF THE

ALGEBRAIC FUNCTIONS

OF A

COMPLEX VARIABLE

BY

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BERLIN

MAYER & MÜLLER

1906

Fig. 5 Frontispiece to J. C. Fields, Theory of the Algebraic Functions of a Complex Variable, Berlin: Mayer & Müller, 1906

What relationship Fields' 1906 book may have had to mathematical research in its own day is still largely undetermined. Until recently, Fields' life and work had remained relatively unexplored. Historians who have offered an opinion seem to agree that Fields' *Algebraic Functions* is difficult to understand from a contemporary point of view.[4] It develops its own theory of algebraic curves, without reference to the dominant methods employed in Fields' own time (Barnes 2007, p. 3). The work presented is therefore alien to the modern reader, and, as such, difficult to enter into and evaluate. Apart from the undetermined status of its intellectual content, however, it can be seen to have significant symbolic content. It was one of the most sophisticated and ambitious contributions to research mathematics made by someone who grew up and made his career in a country that was, in 1906, still a remote place in which to be create pure mathematics.

Fields received notoriety for this symbolic achievement. In the popular press, the Toronto *Globe* noted the book's publication, giving its title and a brief description, however admitting that the book "convey[ed] little to the lay mind" (Riehm and Hoffman 2011, p. 63). A notice in the New York *Nation* commented

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[4]This was Craig Fraser's interpretation in his talk, "J. C. Fields and the Utility of Mathematics" at Mills Library, McMaster University, Hamilton, Ontario, 24 November 2014, 1:30–2:30 p.m.

"From the eminence here attained one is permitted to behold readily a variety of classic propositions that have hitherto been found only by difficult and circuitous paths . . . " (Riehm and Hoffman 2011, p. 63). A professional review of the book by J. I. Hutchison appeared in the *Bulletin of the American Mathematical Society*, describing the work as "not intended as a treatise or a textbook on the theory of algebraic functions along any of the well-established lines of treatment. It is, on the contrary, a new and distinctive mode of approach to this class of functions" (quoted in Riehm and Hoffman 2011, p. 64).

While Fields' book had been a significant intellectual accomplishment, apart from notoriety for having published it, Fields' book seems to have left little intellectual impact on the development of mathematics in Canada or abroad. Marcus Barnes, in his 2007 MA thesis about Fields life and mathematics, described the reception of Fields monograph as lukewarm (Barnes 2007, p. 70). Riehm and Hoffman also point out in their biography of Fields that his influence as a mathematician was somewhat limited in that he, like Sylvester and Cayley, attracted few protégés.

By contrast, consider the influence that Macmillan as a book publisher, may have had on the English culture of mathematics. Macmillan's success as a publisher enabled the small and local context of Cambridge, out of which they developed their mathematical materials, to become amplified and exported, through the mass production of books, to foreign places around the world. Macmillan textbooks did a lot of work in the contexts in which they were used as pedagogical materials. Macmillan books brought to places like Canada the identities of the authors, the ideas, and the values about mathematics expressed by these publications. In Britain, the Macmillan publishing company had been a node around which a community of intellectuals and culture makers organized. During each decade in which the Macmillan brothers developed their publications list, the values of the social context of Cambridge, and its mathematical education system, in which the Macmillans and their closest advisors resided, became manifest in the physical object of the books they produced. Through the success and mass production of these books, the Macmillans amplified the influence of Cambridge University and its mathematical community outward into many far-flung places around the world.

Note on Macmillan and Company Sources

Records for Macmillan and Company are held in several collections. There are documents from Macmillan held at several locations in the UK: at the Palgrave–Macmillan head offices in Basingstoke, in the Special Collections department at Reading University, and in the Department of Manuscripts at the British Library. This paper makes use of the last of these collections. The Macmillan Archive (British Library Add MS 54786–56035) comprises nineteenth- and twentieth-century correspondence and business records from the publishing firm.

Included in Macmillan's many extensive records of their publishing activity are a series of production ledgers, the Editions Books, which list the number of books ordered, date of publication, name of printer, type and date of paper ordered, etc., for each published title. The first Editions Book covering Macmillan's publications to the year 1892 is held at the Palgrave–Macmillan head office in Basingstoke. Subsequent volumes are found in the British Library. For convenience sake, the British Library holds a CD-ROM copy of the first Editions Book as a complement to their Macmillan collections, and it was the British Library's CD-ROM copy consulted by this author. The tables in Appendices 1 and 2 were compiled from this source. However it should be noted that this source is not officially a part of the British Library's manuscript collections, and so the CD-ROM is not listed in their catalogue or in the records of manuscripts. By bringing attention to this I hope to alleviate any confusion for the reader wishing to locate the source.

Acknowledgements The author would like to thank Craig Fraser, June Barrow-Green, and Nicholas Griffin for their helpful comments on past iterations of the present work. Research conducted for this article was undertaken while funded by a Michael Smith Foreign Study Supplement and a Canada Graduate Scholarship (SSHRC). Errors are the author's own.

A.1 Appendix 1: Isaac Todhunter's Publications with Macmillan, 1843–1889

Year of first appearance	Title	Price[a]	Total copies printed[b]
1852	A Treatise on the Differential Calculus and the Elements of the Integral Calculus	10s. 6d.	24, 250
1853	A Treatise on Analytical Statics	10s. 6d.	9000
1855	A Treatise on Plane Co-ordinate Geometry	10s. 6d.	27, 700
1857	A Treatise on the Integral Calculus and its Applications	10s. 6d.	17, 500
1858	Algebra for the use of Colleges and Schools	7s. 6d.	138, 500
1858	Answer to Mr. Lund's Attack on Mr. Todhunter	Not known	2000
1858	Examples of Analytical Geometry of Three Dimensions	4s.	4000
1859	Plane Trigonometry	5s.	86, 500
1859	Spherical Trigonometry for the use of Colleges and Schools	4s. 6d.	32, 530
1861	A History of the Progress of the Calculus of Variations During the Nineteenth Century	12s.	500
1861	An Elementary Treatise on the Theory of Equations	7s. 6d.	13, 500
1862	The Elements of Euclid	3s. 6d.	525, 000
1863	Algebra for Beginners	2s. 6d.	693, 000
1865	A History of the Mathematical Theory of Probability from the Time of Pascal to That of Laplace	18s.	1000

Year of first appearance	Title	Price[a]	Total copies printed[b]
1866	Trigonometry for Beginners	2s. 6d.	108, 500
1867	Mechanics for Beginners	4s. 6d.	56, 000
1868	Key to Algebra for Beginners	6s. 6d.	21, 000
1869	Mensuration for Beginners	2s. 6d.	215, 000
1870	Key to Algebra for the use of Colleges and Schools	10s. 6d.	14, 000
1871	Researches in the Calculus of Variations	6s.	500
1873	The Conflict of Studies	10s. 6d.	1000
1873	A History of the Mathematical Theories of Attraction and the Figure of the Earth	24s.	500
1873	Key to Trigonometry for Beginners	8s. 6d.	6000
1874	Key to Plane Trigonometry	10s. 6d.	7000
1875	An Elementary Treatise on Laplace's Functions, Lamé's Functions, and Bessel's Functions	10s. 6d.	1000
1876	An Abridged Mensuration with Numerous Examples for Indian Students	1s.	5000
1876	Macmillan's Series of Text-Books for Indian Schools: Algebra for Indian Students	2s. 6d.	10, 000
1876	Macmillan's Series of Text-Books for Indian Schools: The Elements of Euclid for the use of Indian Students	2s.	27, 000
1876	Macmillan's Series of Text-Books for Indian Schools: Mensuration and Surveying for Beginners	2s.	42, 000
1877	Natural Philosophy for Beginners, Part I	3s. 6d.	11, 000
1877	Natural Philosophy for Beginners, Part II	3s. 6d.	6000
1878	Key to Mechanics for Beginners	6s. 6d.	4000
1880	Key to Exercises in Euclid	6s. 6d.	9500
1886	Key to Todhunter's Mensuration for Beginners (by L. McCarthy)	7s. 6d.	2750
1887	Solutions to Problems Contained in Plane Coordinate Geometry (ed. C. W. Bourne)	10s. 6d.	1000
1888	Key to Todhunter's Differential Calculus (by H. St. J. Hunter)	10s. 6d.	2750
1889	Key to Todhunter's Integral Calculus (by H. St. J. Hunter)	10s. 6d.	2250

Isaac Todhunter's publications with Macmillan 1843–1889 (Source: Macmillan's First Editions Book, British Library)

Calculations of lifetime print-run given in this chart were calculated from Macmillan's first editions book only

[a]Price, given in shillings (s.) and pence (d.) refers to either the most stable price or the price on first printing.

[b]Some titles may have been reprinted beyond the last year given here

A.2 Appendix 2: Macmillan's Mathematical Books with Print Runs Greater Than 100,000, 1843–1889

Total copies printed	Year of first printing	Last year printed[a]	Author	Title	Price[b]
693,000	1863	1917	I. Todhunter	Algebra for Beginners	2s. 6d.
608,000	1885	1937	H.S. Hall and S.R. Knight	Elementary Algebra	3s. 6d.
597,500	1854	1920	B. Smith	School Arithmetic	4s. 6d.
525,000	1862	1903	I. Todhunter	Euclid	3s. 6d.
430,000	1865	1906	B. Smith	Shilling Book of Arithmetic, Part I	1s.
362,000	1872	1925	B. Stewart	Science Primers: Physics	1s.
295,000	1889	1931	H.S. Hall and S.R. Knight	A Textbook of Euclid's Elements, Parts I & II, Books I–IV	3s.
270,000	1887	1930	H.S. Hall and S.R. Knight	A Textbook of Euclid's Elements, Part I, Book I & II	2s.
253,000	1888	1932	H.S. Hall and S.R. Knight	A Textbook of Euclid's Elements, Book I–IV & XI	4s. 6d.
215,000	1869	1931	I. Todhunter	Mensuration for Beginners with Numerous Examples	2s. 6d.
211,000	1879	1929	J. Thornton	First Lessons in Bookkeeping	2d. 6d.
210,500	1889	1922	H.S. Hall and F.H. Stevens	A Textbook of Euclid's Elements for the Use of Schools, Book I	1s.
206,500	1886	1929	J.B. Lock	Arithmetic for Schools	4s. 6d.
176,100	1887	1938	H.S. Hall and S.R. Knight	Higher Algebra	7s. 6d.
173,240	1881	1929	S.P. Thompson	Elementary Lessons in Electricity and Magnetism	4s. 6d.
167,000	1874	1920	J.N. Lockyer	Science Primers: Astronomy	1s.
153,500	1866	1928	B. Smith	Shilling Book of Arithmetic with Answers	1s. 6d.
138,500	1858	1911	I. Todhunter	Algebra for Colleges and Schools	7s. 6d.
138,380	1887	1936	J.T. Bottomley	Four Figure Mathematical Tables	2s. 6d.
133,500	1870	1934	W.S. Jevons	Elementary Lessons on Logic	3s. 6d.

Total copies printed	Year of first printing	Last year printed[a]	Author	Title	Price[b]
121,000	1886	1932	C. Smith	Elementary Algebra	4s. 6d.
108,500	1866	1921	I. Todhunter	Trigonometry for Beginners	2s. 6d.
104,500	1866	1901	B. Smith	Key to Shilling Book of Arithmetic	6d.
104,000	1882	1937	J.B. Lock	Treatise on Elementary Trigonometry	4s. 6d.
102,500	1870	1911	B. Stewart	Lessons in Elementary Physics	4s. 6d.
102,000	1872	1910	J. Brook-Smith	Arithmetic in Theory and Practice	3s. 6d.

Macmillan's mathematical books with print runs greater than 100,000, 1843–1889 (Source: Macmillan's First Editions Book, British Library)
Calculations of lifetime print-run given in this chart were calculated from Macmillan's first editions book only
[a]Some titles were reprinted beyond the last year given here
[b]Price, given in shillings (s.) and pence (d.) refers to either the most stable price or the price on first printing.

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The Influence of Arthur Cayley and Alfred Kempe on Charles Peirce's Diagrammatic Logic

Francine F. Abeles

Abstract This paper is dedicated to the memory of Irving H. Anellis and represents joint work on the historical sources of Charles Sanders Peirce's (1839–1914) diagrammatic logic. Arthur Cayley (1821–1895) and Alfred Bray Kempe (1849–1922) contributed to the logic of relations and its applications to geometry and foundations of geometry. This paper gives an overview of sources related to analytical trees and diagrams which were inspirational for Peirce's development of his existential graphs. Much of the material upon which this paper draws consists of unpublished manuscripts from the Peirce Edition Project at the University of Indianapolis where for many years my collaborator Irving Anellis was a member of the research staff.

1 Introduction

Despite his facility with, and original contributions to algebraic logic, Charles Sanders Peirce's (1839–1914) was primarily and essentially a visual thinker. His earliest extant writings, beginning at least in 1859, utilized diagrams. Peirce's childhood study of chemistry, his post-master's degree, a Sc. B. in chemistry *summa cum laude* from Harvard College's new Lawrence School of Science, and a lifelong interest in chemistry also contributed to his interest in the use of diagrams in chemistry to investigate the construction of molecules in organic chemistry, based upon the carbon ring which became the basis for chemical structure theory. Peirce initially saw logic as a classificatory science like chemistry, so it is not remarkable that one of the earliest of his publications in logic was "On a Natural Classification of Arguments," from 1868 (Peirce 1868).

Peirce's first publication on graphs, his entitative graphs for propositional logic, appeared in 1897 in *The Monist*. We know that he began working on the topic in 1886. Peirce was influenced in this work by several mathematicians, particularly

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Arthur Cayley (1821–1895), Arthur Bray Kempe (1849–1922), William Kingdon Clifford (1845–1879), and James Joseph Sylvester (1814–1897) (Houser 1997, pp. 1–22). Peirce's early work on his diagrammatic logic directly benefited from his reading of Kempe and Cayley's papers and from his subsequent correspondence with Kempe. Cayley employed chemical diagrams to represent algebraic invariants, and Kempe developed a theory of logical forms.

2 Peirce's Entitative Graphs

Peirce spent much of his professional endeavors working on graphs and diagrams and in search of improvements over those, such as Euler's, which were already available. Peirce's existential graphs (diagrams for logical expressions) arose in part from his work on truth-functional logic and in the main out of his experimentation with graphical or diagrammatic methods for analyzing logical propositions and proofs. Unlike the diagrammatic systems of Alexander MacFarlane (MacFarlane 1879), Alan Marquand (Marquand 1881), John Venn (Venn 1880), and Charles Dodgson (Dodgson 1887), all of which were intended to handle syllogisms, Peirce used his graphs for his logic of relations which he developed in the period 1870–1882 (Peirce 1870, 1880; Peirce n.d.(g); Macfarlane 1881–1883). He considered a relation as either a set of n-tuples or as one of three relative concepts: monads, dyads, and triads. Relations are necessary for analyzing logical propositions. By the early 1890s, Peirce thought that all relations could be represented as dyadic and triadic relations, and that all polyadic relations of tetradic or greater order could be expressed as relative products of triadic relations. In Fig. 1, h is the monad "__is a man," and d is the monad "___is mortal." In Fig. 2, l is the dyad "___loves___." (Hartshorne and Weiss 1933a, p. 301).

The first extant published appearance of his entitative graphs is found in his paper "The Logic of Relatives" (Peirce 1897b) in *The Monist*, where he employed them to elucidate the exposition of the algebra of relatives in as non-technical a manner as possible for philosophical readers. There Peirce explicitly tells us that his system for graphically representing relational propositions was inspired by his study of chemistry, and refers to Kempe and Clifford. In explicitly explaining logical relations in terms of chemical bonding, he cites the chemical bonding theory

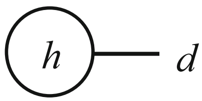

Fig. 1 Monad h (is man) and monad d (is mortal). Reprinted by permission of the publisher from Collected Papers of Charles Sanders Peirce, Volumes I–VI, edited by Charles Hartshorne and Paul Weiss, p. 301, Cambridge, MA: The Belknap Press of Harvard University Press, Copyright ©1931, 1932, 1933, 1934, 1935, 1958, 1959, 1960, 1961, 1963, 1986 by the President and Fellows of Harvard College

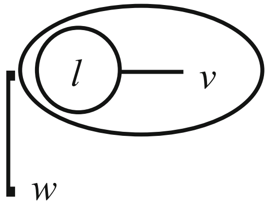

Fig. 2 Dyad l (loves) joining monads w and v. Reprinted by permission of the publisher from Collected Papers of Charles Sanders Peirce, Volumes I–VI, edited by Charles Hartshorne and Paul Weiss, p. 301, Cambridge, MA: The Belknap Press of Harvard University Press, Copyright ©1931, 1932, 1933, 1934, 1935, 1958, 1959, 1960, 1961, 1963, 1986 by the President and Fellows of Harvard College

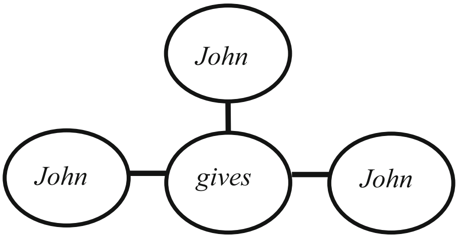

Fig. 3 John gives John to John. Reprinted by permission of the publisher from Collected Papers of Charles Sanders Peirce, Volumes I–VI, edited by Charles Hartshorne and Paul Weiss, p. 301, Cambridge, MA: The Belknap Press of Harvard University Press, Copyright ©1931, 1932, 1933, 1934, 1935, 1958, 1959, 1960, 1961, 1963, 1986 by the President and Fellows of Harvard College

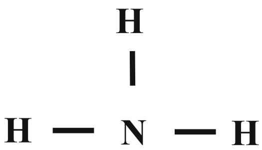

Fig. 4 Diagram of ammonia. Reprinted by permission of the publisher from Collected Papers of Charles Sanders Peirce, Volumes I–VI, edited by Charles Hartshorne and Paul Weiss, p. 301, Cambridge, MA: The Belknap Press of Harvard University Press, Copyright ©1931, 1932, 1933, 1934, 1935, 1958, 1959, 1960, 1961, 1963, 1986 by the President and Fellows of Harvard College

of the German chemist Julius Lothar von Meyer (1830–1895). Peirce does not give a reference for Meyer's work whose single major conceptual advance over his immediate predecessors was seeing *valence*, the number that represents the combining power of an atom of a particular element, as the link among members of each family of elements and as the pattern for the order in which the families were themselves organized (Von Meyer 1864). In the two figures above, Peirce provides an example of how logical relations can be explained in terms of chemical bonding. In Fig. 3, the proposition, "John gives John to John" structurally corresponds to ammonia which is shown in Fig. 4 (Hartshorne and Weiss 1933a, p. 296).

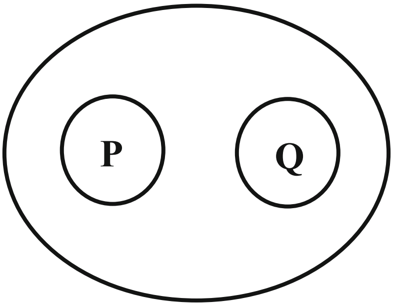

Fig. 5 *P and Q* as an entitative graph

P Q

Fig. 6 *P and Q* as an existential graph

In the fully developed system of entitative graphs, the surface of the graph was a sheet which represented a truth-theoretic plane, and the letters representing the terms of the calculus were connected by lines representing the relations between these terms. A "cut" in the sheet, depicted by a circle around a letter representing a term in the universe of discourse, indicated a hole in the sheet, and thus represented the negation, or falsity, of the encircled term. In his existential graphs, the next phase of his work, Peirce used a similar graphical technique to deal with quantified propositions. In the entitative graphs, *P* together with *Q*, i.e., their concatenation, means *P or Q* while in the existential graphs it would mean *P and Q*. Figure 5 depicts *P and Q* in the entitative graphs and Fig. 6 depicts *P and Q* in the existential graphs.

3 The Development of Existential Graphs

Peirce became dissatisfied with entitative graphs even before the issue of *The Monist* (1897) in which they appeared was printed, that is, while he was still checking the galley proofs. He wrote to the journal's editor, Paul Calvin Carus (1852–1919) to describe his new system, the existential graphs, hoping to delay the publication of his paper so as to publish it with existential graphs.

The earliest dated record we have of existential graphs describes what is called the *sheet of assertion*, a blank sheet which represents everything that is true in the universe of discourse, appears in 1903 (Peirce 1903c). This document came from the final decade of Peirce's life when he was preparing for his Lowell Institute lectures of 1903–1904. Peirce held that improvement in reasoning requires, first of all, a study of deduction, and that for this task, an unambiguous and simple system of expression is needed (Peirce 1903a, Ms. #450). The system in which reasoning is broken up into its smallest fragments by means of diagrams is the system of existential graphs which Peirce continued to develop in terms of fourteen conventions. In a very closely related set of notes for the Lowell lectures, he explained that existential graphs provide a system for expressing any assertion with precision and that they are not intended to facilitate but to analyze necessary reasoning (i.e., deduction) (Peirce 1903b, Ms. #454). The system is introduced by means of four basic conventions, called "principles," and four rules or "rights" of transformation. In 1905, Peirce presented his entitative graphs and existential graphs, and explicitly compared the existential graphs to chemical graphs (Peirce 1905b).

Peirce held his existential graphs to be his "chef d'ouevre" and in the manuscript on "The Basics of Pragmaticism" (Peirce 1905a), he presents an elementary discussion of existential graphs, which he termed "quite the luckiest find that has been gained in exact logic since Boole." Existential graphs have three component graphs: Alpha, Beta, and Gamma which correspond to propositional, predicate, and modal logic, respectively. Descriptions of the more complex Beta, and Gamma graphs can be found in the *Collected Papers of Charles Sanders Peirce*, vol. IV, *The Simplest Mathematics* (Hartshorne and Weiss 1933b).

It is interesting to note that in the undated notebook "On Existential Graphs as an Instrument of Logical Research," Peirce wrote that he discovered existential graphs late in 1896, but that he was practically there some 14 years before (Peirce n.d.(o), Ms. #498). This suggests that at least some of the undated manuscripts that deal with graphs might date from the early 1880s, while Peirce was still writing and publishing on his algebraic development of logic.

4 Arthur Cayley's Influence on Pierce

Arthur Cayley's early work includes using trees to represent algebraic relations. Many of his papers were brought to Peirce's attention by the chemist Allan Douglas Risteen (1866–1932) while they were working together on the *Century Dictionary* (1883–1909). Nathan Houser notes (Houser 2010, pp. xi–xcvii) that Peirce listed many of Cayley's papers when he wrote to Risteen in a letter of June 10, 1891 (Risteen 1891). This list included:

On the Theory of Analytical Form Called Trees (Cayley 1857)

On the Theory of Analytical Form Called Trees, Second Part (Cayley 1859)

The Theory of Groups: Graphical Representation (Cayley 1878)

On the Theory of Analytical Form Called Trees (Cayley 1881).

Risteen, who had served as Peirce's assistant at the US Coastal Survey and later earned a doctorate from Yale University as a student of Josiah Willard Gibbs (1839–1903), is remembered chiefly for his *Molecules and the Molecular Theory of Matter* (Risteen 1895). In addition, Peirce wrote about Cayley's 1870 "Memoir on Abstract Geometry from the point of view of the Logic of Relatives" (Peirce n.d.(g)). And Cayley's paper, "On the Theory of Analytical Form Called Trees" (Cayley 1881)

Fig. 7 A tree for $n = 4$

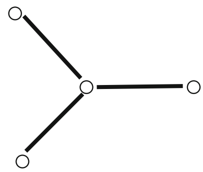

Fig. 8 A tree for $n = 4$

appeared in the 1881 volume of the *American Journal of Mathematics*, where Peirce, whose article "On the Logic of Number" also appeared, was certain to have seen it (Peirce 1881).

In a paper of 1874, "On the Mathematical Theory of Isomers," Arthur Cayley reported that removing the hydrogen atoms from the diagrams corresponding to the alkanes (paraffins) produces a tree in which each vertex has degree 1, 2, 3, or 4 . . . So enumerating these isomers is equivalent to counting trees having this property. Figures 7 and 8 show two different configurations of trees for $n = 4$.

Houser notes in particular the role of Cayley's tree diagrams, writing that, upon studying Cayley's tree diagrams and their application to chemistry "[i]t occurred to Peirce that Arthur Cayley's diagrammatic method of using branching trees to represent and analyze certain kinds of networks based on heritable or recurrent relations would be useful for his work on the algebra of the copula and his investigation of the permutations of propositional forms by the rearrangement of parentheses." (Houser 2010, p. xlviii). It was around this time, in the Spring of 1891, that Peirce wrote "On the Number of Dichotomous Divisions: A Problem of Permutations" (Peirce 1891a) in which he used binary trees, inspired by Cayley's work on analytic trees, to compute the number of propositional forms containing any number of copulas (see also Peirce 1891b, Ms. #73).

5 Alfred Bray Kempe

Among those having an influence upon Peirce's thought in developing diagrammatic tools, it was Kempe who had the most direct tangible influence that left a detectable residue in the *Nachlaß*. In the 1880s and 1890s, there was substantial interaction between Peirce and Kempe, and a correspondence ensued (Grattan-Guinness 2002, p. 327). The earliest contact for which documentation is available is the receipt by Peirce of Kempe's 1886 "A Memoir on the Theory of Mathematical Form" (Kempe 1886). Their correspondence includes pages from Kempe's 1890 article "The Subject Matter of Exact Thought," with a note directing Peirce's attention to that article (Kempe 1890).

Peirce was influenced by Kempe's diagrammatic method which used dots and lines and was based upon chemical diagrams. Grattan-Guinness followed Peirce's own dating to assign January 15, 1889 as the point of departure for his conceiving, on the basis of his study of Kempe's work, the idea for developing his entitative graphs and existential graphs (Grattan-Guinness 2000, p. 140). Peirce wrote extensively about Kempe's diagrams in the period 1886–1897.

In an unfinished and fragmented article intended for *The Monist*, "The Bed-Rock beneath Pragmaticism," Peirce (Peirce 1905b) cited Kempe's 1886 "A Memoir on the Theory of Mathematical Form" as an "invaluable, very profound, and marvellously strong contribution to the science of Logic" (Peirce 1905b). And his copy of it is heavily annotated (Peirce 1889).

Beginning with his paper "A Memoir on the Theory of Mathematical Form" (1886), Kempe worked out his theory of linear triads, based to a large extent on his interaction with Peirce. Kempe's next paper "Note to A Memoir on the Theory of Mathematical Form" (Kempe 1887) is very much a reply to Peirce's comments and criticisms, and in particular to Peirce's letter to Kempe of January 17, 1887. Likewise, Kempe's "The Theory of Mathematical Form: A Correction and Clarification" (Kempe 1897) is a reply to the same letter and to similar comments made by Peirce in his paper "The Logic of Relatives" (Peirce 1897a, pp. 168ff.). Peirce's manuscript "Reply to Mr. Kempe (K)" (Peirce 1897b, pp. 5–6, 11, 15–19), is a reply to Kempe's 1897 reply to Peirce's discussion of entitative graphs in "The Logic of Relatives." And it is devoted to demonstrating that, despite differences in detail with Kempe's graphs and his own, and despite Kempe's claim to the contrary, tetrads can be rewritten as triads, but not as monads or dyads. Peirce remarked that when we pursue the idea of Mr. Kempe's system, we arrive at a result that each graph consists of monads, dyads, or triads. Starting from the tetrad Peirce reduces the tetrad to two connected triads (Peirce 1897b, pp. 6, 11). (See also Peirce n.d.(f), where Peirce compares Kempe's graphs with his own existential graphs.)

In three undated manuscripts, Peirce's general reaction to Kempe's work focused on technical issues concerning the problem of the symmetry and asymmetry of combinations of relations. Here Peirce expressed particular concern for what he considered Kempe's careless use of the difference between *distinguished* and *undistinguished* terms in his representation of n-ads (Peirce n.d.(a)–n.d.(c)). In his paper "On the Relation between the Logical Theory of Classes and the Geometrical Theory of Points" (Kempe 1889–1890, p. 149), Kempe employed the notion of a base system of linear triads determined by a set of five algebraic laws. To help Kempe clarify the difference, Peirce (in Peirce n.d.(a), pp. 3–4) used as an example the claim that supposing $ab \cdot c$ and $ba \cdot c$ are indistinguishable only asserts that a and b have exactly the same relationship to the system, and not that $a = b$. In another two-page undated manuscript on Kempe, Peirce praised Kempe's mathematical powers and native instinct for doing logic, but was critical of "his sad want of training" in logic, and offered specific criticisms (Peirce n.d.(e)). (See also Peirce n.d.(d).)

6 Conclusion

Peirce's lifelong interest in chemistry, and 1863 graduate degree in chemistry from Harvard College contributed significantly to his interest in the use of diagrams to investigate the construction of molecules in organic chemistry. Later on, Cayley's work on counting trees (analytical trees), and Kempe's graphing method for geometry based on the relations between points, formed a large part of Peirce's inspiration for creating his diagrammatic systems for logic. After the end of his short professorial career from 1879 to 1884, at The Johns Hopkins University, Peirce definitively turned away from algebraic logic to the development of graphical logic, inventing first his entitative graphs and then his more powerful existential graphs.

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Émile Borel et Henri Lebesgue: HPM

Roger Godard

Abstract We discuss some chapters of two books, 1) *Les fonctions de variables réelles et les développements en séries de polynômes*, written by Émile Borel in 1905, and 2) *Leçons sur les séries trigonométriques* by Henri Lebesgue in 1906. In 2) Lebesgue utilized an historical approach for the presentation of Fourier series. Both books were published by Gauthier-Villars in Paris who was the scientific editor at that time. Both books showed the state of mathematical knowledge and they represented an important pedagogical effort. They belonged to a collection directed by Émile Borel. We also comment upon some letters written by Henri Lebesgue to Borel around 1903–1906.

Résumé

On discute de quelques chapitres de deux livres, 1) *Les fonctions de variables réelles et les développements en séries de polynômes*, écrit par Émile Borel en 1905, et 2) *Leçons sur les séries trigonométriques* par Henri Lebesgue en 1906. Dans 2) Lebesgue utilisa une approche historique pour l'introduction aux séries de Fourier. Les deux livres furent publiés par Gauthier-Villars à Paris qui était l'éditeur scientifique à cette époque. Les deux livres montrèrent l'état des connaissances et représentèrent un réel effort pédagogique. Ils ont appartenu à une collection dirigée par Émile Borel. On commente aussi certaines lettres écrites by Lebesgue à Borel entre 1903 et 1906.

1 Introduction: mathématiques et pédagogie au début du XX^{e} siècle

On a beaucoup parlé des réformes de l'enseignement secondaire et du futur des mathématiques à la fin du XIX^{e} siècle et au début du XX^{e} siècle (Coray et al. 2003). On connaît la boutade d'Émile Borel en 1907 (Furinghetti 2011): « [. . .] *une des raisons pour lesquelles l'enseignement secondaire se perfectionne lentement,*

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c'est que l'enseignement que l'on donne ne peut différer beaucoup de celui qu'on a reçu ». Mais que se passa-t-il pour l'enseignement supérieur. Dans cet article, on discute de quelques chapitres de deux livres, 1) *Les fonctions de variables réelles et les développements en séries de polynômes*, écrit par Émile Borel en 1905, et 2) *Leçons sur les séries trigonométriques* par Henri Lebesgue en 1906. Dans 2) Lebesgue utilisa une approche historique pour l'introduction aux séries de Fourier. Les deux livres furent publiés par Gauthier-Villars à Paris qui était l'éditeur scientifique à cette époque. Les deux livres montrèrent l'état des connaissances et représentèrent un effort pédagogique.

2 Émile Borel : homme d'action

L'œuvre d'Émile Borel est bien connue. Elle fut notamment décrite par Louis de Broglie (de Broglie), Maurice Fréchet (Fréchet 1965, 1967) et sa propre femme Marguerite (1957) (Marbo 1968). Brièvement, il naquit à Saint Affrique dans le Midi de la France en 1871. Reçu premier à la fois à l'École polytechnique, et à l'École normale supérieure (Ens), il choisit l'Ens et les sciences mathématiques. Ayant fait sa thèse de doctorat sous la direction de Gaston Darboux en 1893, il supervisera celle d'Henri Lebesgue sur l'intégration des fonctions. Sa femme Marguerite était la fille du mathématicien Paul Appell. Borel devint l'ami de Paul Painlevé, un futur Président du conseil (premier ministre) qui contribua plus tard à faire entrer Borel dans la politique. Leurs amis intimes étaient Jean Perrin, Paul Langevin, Charles Maurin. Les Lebesgue furent aussi plusieurs fois invités. L'influence qu'il avait acquise permit à Borel d'aider à la création de l'Institut Henri Poincaré à Paris en 1928 et du Centre national de la recherche scientifique (Cnrs) en 1936. Émile Borel mourut en 1956. Les publications de Borel furent plus de 300 dont trente-cinq sont des livres.

3 Émile Borel comme professeur : commentaires sur *Les fonctions de variables réelles et les développements en séries de polynômes*

À vingt-six ans, Borel est nommé maître de conférences (professeur agrégé) à L'Ens dont il devint le directeur des études scientifiques. Il publie son premier livre « *Leçons sur la théorie des fonctions* » en 1898 chez l'éditeur Gauthier-Villars. Cinquante petits livres de 130 pages environ sur des sujets très précis avec les travaux les plus récents, furent ainsi publiés par Gauthier-Villars dont dix par Borel lui-même. Parmi les contributeurs, on note les noms de Borel (Borel 1900, 1905, 1972), Baire (Baire 1905), Lebesgue (Lebesgue 1904, 1906), de la Vallée Poussin (de la Vallée Poussin 1919), Volterra, Bernstein (Bernstein 1926), Montel, Lévy, Riesz, etc. Ils correspondaient à des cours de l'École normale, la Sorbonne ou du Collège de France. Citons Fréchet sur un commentaire de Collingwood (Fréchet 1967, p. 17):

> *« Borel a rendu un important service aux mathématiques en présentant, grâce à cette collection, une synthèse des plus récents travaux sur l'application de la théorie des ensembles à la théorie des fonctions, à une époque où ces idées n'étaient pas encore très répandues. Cette collection reste un des principaux monuments mathématiques de cette époque.* »

Borel a décrit lui-même ses objectifs dans ses préfaces. Par exemple, en 1900, il écrit que dans cette série, les petits livres sont en principe, complètement indépendants, sur un sujet bien délimité, pour aller assez vite et d'arriver en peu de leçons à s'approcher des limites actuelles de la science En 1921, alors que beaucoup de livres ont déjà été publiés, il précise qu'il est indispensable de simplifier et systématiser les résultats acquis dans une discipline, en vue d'en permettre l'acquisition plus aisée à ceux qui cultivent des disciplines différentes. Donc les monographies spécialisées sont autant importantes que les travaux originaux. Borel insiste sur le caractère vivant des monographies et l'excellent accueil de l'éditeur Gauthier-Villars. Nous n'avons pas eu accès aux échanges de lettres entre Borel et Gauthier-Villars ou celles des différents auteurs, mais la collection Gauthier-Villars fut rachetée par Dunod et se trouve à Caen[1] en Normandie.

Nous avons pris comme exemple pour étudier et illustrer l'approche de Borel, son livre *« Leçons sur les fonctions de variables réelles et les développements en séries de polynômes* » à partir de notes qui furent rédigées par Maurice Fréchet sur un cours à l'Ens pendant l'hiver 1903–1904. Le livre fut publié en 1905, soit un an après. Le livre contient cinq chapitres avec des notes supplémentaires de M. P. Painlevé, H. Lebesgue, É. Borel, soit 159 pages au total. Borel commence par faire une révision sur les ensembles qui occupent une place prépondérante dans la collection, et au chapitre 4, il étudie la représentation des fonctions continues par des séries de polynômes et il commence par le théorème fondamental de Weierstrass. Il cite le théorème de Weierstrass de la façon suivante, qui celle qui est la plus connue:

> « *Étant donné une fonction $f(x)$ continue dans un intervalle (a,b), incluant les extrémités, on peut trouver un polynôme $P(x)$ tel qu'on ait à l'intérieur de cet intervalle:*
>
> $$|f(x) - P(x)| < \varepsilon \quad (1)$$
>
> [2]*où ε est un nombre positif, donné à l'avance, et aussi petit qu'on veut* », tandis qu'en 1919, C. de la Vallée Poussin énoncera les deux théorèmes de Weierstrass comme :

[1] Voir note en fin d'article.

[2] Lebesgue, dans la lettre CVII indique un article très élégant de Meray datant de 1896 donnant la première preuve que la formule d'interpolation de Lagrange était instable.

1. *Toute fonction continue dans un intervalle (a, b) peut être développée en série uniformément convergente de polynômes dans cet intervalle.*
2. *Toute fonction continue de période 2π peut être développée en série uniformément convergente d'expressions trigonométriques.*

Parmi les différentes preuves, on trouve celle de Weierstrass (1885), celle de Runge (1885), de Picard, de Lerch (1892), de Volterra (1897), de Lebesgue (1898), de Mittag-Leffler (1900), de Bernstein en 1912. et Borel commentera et prouvera toutes les preuves, puis il discutera de l'extension aux fonctions de plusieurs variables. Fréchet dira « *que Borel a toujours donné une préférence exclusive aux méthodes constructives par rapport aux méthodes descriptives* »... Enfin dans le même chapitre, il présentera les méthodes d'interpolation. Ce chapitre est un trésor d'idées et montre l'état des sciences ! Remarquons que l'énoncé de Borel n'indique pas quel serait le polynôme d'approximation, mais en 1901, Runge (Runge 1901) montrera qu'il ne sera pas un polynôme d'interpolation de Lagrange avec des données équidistantes. Ce résultat de Runge avait aussi été donné par Borel indépendamment. Enfin, en 1913, S. Bernstein (Bernstein 1913; Davis 1963) prouvera que ses polynômes correspondaient aux critères de Weierstrass. Ce petit livre de Borel montre son enthousiasme mais aussi la fécondité de son approche car il sera suivi par Charles de la Vallée Poussin en 1908 (de la Vallée Poussin 1908) sur l'interpolation, et les livres de la Vallée Poussin en 1919 (de la Vallée Poussin 1919) et celui de Bernstein en 1926 (Bernstein 1970) dans la même collection. Notons que dans le même chapitre 4, Borel étudie la méthode d'approximation de Tchebicheff (Tchebicheff 1961) et la thèse de M. Paul Kircherberger parue en 1902 (Borel 1905, p. 82). Cette thèse reprend d'une façon plus rigoureuse la méthode de Tchebicheff. Ceci est la preuve des efforts de Borel pour stimuler la rapidité de la communication des résultats.

On ne commentera ici que brièvement la preuve élémentaire de Lebesgue en 1898 (Lebesgue 1898) et qui constitue son premier article, car cette preuve est typique de l'approche de Lebesgue en mathématiques, voir « simplement des choses simples » (Lebesgue 1898, p. 138). Aussi, cet article va intéresser Borel et sera l'objet d'échanges d'idées entre Lebesgue et lui. On a retrouvé dans le bureau de Borel la correspondance de Lebesgue à Borel et les discussions sur le théorème de Weierstrass des années 1900 (Bru and Dugac 1991, lettres XVII, XIX, XXXVIII, XL, CVIII) mais malheureusement, Lebesgue n'avait pas gardé les lettres de Borel. Ces lettres sont un témoignage de cette époque mathématique.

Dans sa preuve de la démonstration de Lebesgue, de la Vallée Poussin (de la Vallée Poussin 1919) observe qu'on peut approcher autant qu'on veut une courbe continue par une ligne polygonale faite de segments de droites continus entre eux. Si, les couples $(x_1, y_1), (x_2, y_2), \ldots, (x_n, y_n)$ sont les sommets de cette ligne polygonale, on peut représenter l'approximation affine par des fonctions :

$$\varphi_k(x) = |x - x_k| + (x - x_k) \tag{2}$$

Prenons comme fonction d'interpolation une combinaison linéaire de ces fonctions:

$$F(x) = a_0 + \sum_{k=1}^{n-1} a_k \varphi_k(x) \tag{3}$$

où les constantes $a_0, a_1, \ldots, a_n$ sont déterminées par substitution d'un système triangulaire d'équations linéaires en forçant $F(x_k) = y_k$. D'après l'équation (3), on voit que de la Vallée Poussin a construit une fonction globale de polynômes connus localement. Cette interpolation correspond à une interpolation spline affine. Lebesgue remarqua que pour la preuve du théorème de Weierstrass, tout revenait à poser :

$$\begin{aligned} x^2 &= 1 - u^2, \\ |x| &= \sqrt{1 - u^2} \end{aligned} \tag{4}$$

Et de développer $\sqrt{1 - u^2}$ par la formule du binôme qui converge uniformément dans son intervalle de convergence. En 1919, Charles de la Vallée Poussin soulignera que la démonstration de Lebesgue était simple « *mais ne fournit qu'une approximation médiocre*. »

4 Henri Lebesgue : l'homme

La carrière d'Henri Lebesgue fut très différente de celle d'Émile Borel, ainsi que son approche pédagogique. Il publiera deux livres importants dans la collection Borel chez Gauthier-Villars sur l'intégration de fonctions et les séries trigonométriques.[3] Il se décrit lui-même comme un homme timide. Kenneth O. May écrira (May 1966):

> *«Bien que les idées de Lebesgue ont continué à dominer, plus persuasives en analyse, l'homme Lebesgue n'eut pas l'influence qu'on pouvait espérer. Il n'était pas un homme politique ayant des intérêts ni dans la politique partisane en dehors de la communauté académique ni dans les jeux d'homme requis pour la puissance académique.... Il resta sceptique au sujet de la valeur de son travail.*
>
> *Au temps de son élection à l'Académie, Lebesgue avait publié quatre-vingt-dix livres et papiers sur la théorie des ensembles, la théorie des mesures, les séries trigonométriques, l'approximation polynomiale, le calcul des variations, les probabilités géométriques, la topologie, et la géométrie algébrique. »*

Lebesgue naquit à Beauvais en 1875. En 1894, il entre à l'École normale supérieure, il passe sa thèse de doctorat en 1902 sur l'intégration de fonctions. En

[3]Dans sa lettre XIX à Borel (fin 1903-janvier 1904), il annonce son intention d'écrire un livre sur les séries trigonométriques, mais qu'il « est en humeur de flemmard ».

1902 il est nommé à la Faculté de Rennes. Il donne le cours Peccot[4] au Collège de France sur les intégrales de Riemann et de Lebesgue en 1902–1903 (Lebesgue 1904) et sur les séries trigonométriques (Lebesgue 1906) en 1904–1905. Il est nommé à la Sorbonne en 1910 et élu à l'Académie de sciences en 1922 dans la section géométrie. Lebesgue créa 19 cours pour le Collège de France et il écrivit plus de 165 articles, papiers et communications ainsi que des livres (Perrin 1971; Félix 1974; Hochkirchen 2003). Il mourut en 1941. Ses Œuvres scientifiques contiennent cinq volumes. Elles furent publiées par l'Enseignement Mathématique à Genève en 1972 (Lebesgue 1972). Elles contiennent aussi tous les hommages de ses amis et collaborateurs (Lebesgue 1972, tome 1, pp. 31–88).

Kenneth O. May dira qu'après son élection à L'Académie des sciences, et pendant les vingt prochaines années, Henri Lebesgue continuera à écrire sur les sujets qui l'intéressaient précédemment, mais plus d'un point de vue philosophique, pédagogique et historique. Il écrira notamment 13 articles de son vivant pour *l'Enseignement Mathématique*, et huit autres seront publiés après sa mort. Il écrivit aussi 10 autres articles pour *L'enseignement scientifique*, deux articles pour la *Revue de l'enseignement des science*s et deux autres articles dans la *Revue Enseignement secondaire pour jeunes filles*. Enfin, en 1933, il publia un article pour les sévriennes[5] dans leur bulletin. Mais il n'eut pas l'influence de Borel pour des réformes de l'enseignement.

5 Henri Lebesgue comme professeur

Dans la collection Borel chez Gauthier-Villars, Henri Lebesgue se distingue des autres contributeurs par son désir d'utiliser l'histoire des mathématiques dans son enseignement. Il s'explique qu'il ne peut pas le faire dans ses *Leçons sur l'intégration et la recherche de fonctions primitives* (Lebesgue 1904) en 1903:

> « *les vingt leçons que comprend cet ouvrage ont été consacrées à l'étude du développement de la notion d'intégrale. Un historique complet n'aurait pas pu tenir en vingt leçons.* »

Il expliquera son approche dans la préface de la deuxième édition du livre sur l'intégration en 1926 :

> *«Bien que cette première édition avait paru, à certains, audacieusement et volontairement remplie de nouveautés un peu scandaleuses, elle était l'œuvre d'un timide qui, sur les sept Chapitres qu'il avait écrit, en avait consacré six à l'exposé des recherches antérieures avant d'aborder les travaux que l'on considérait comme révolutionnaires . . . Il croyait en effet, et*

[4] La fondation Peccot au Collège de France octroie depuis 1900 des bourses à de jeunes chercheurs de moins de 30 ans et elle est encore active. Et dans la lettre XVI, il dit que s'il est nommé au Cours Peccot, il donnera un cours sur les séries trigonométriques. Enfin dans la lettre XLVIII datée du 17 novembre 1904, il annonce que le Collège de France lui octroiera 4000 francs au lieu de 3000 pour le cours Peccot.

[5] Henri Lebesgue était professeur à l'École normale pour jeunes filles à Sèvres.

il croit encore, que pour faire œuvre utile il faut marcher dans l'une des voies ouvertes par les travaux antérieurs; qu'on risquerait trop, en agissant autrement, de créer une science sans rapport avec le reste des mathématiques. »

Dans la prochaine section, on analysera la présentation de Lebesgue pour les séries trigonométriques, mais son approche est purement didactique avec des objectifs différents de ce que disait par exemple, H. Poincaré en 1908 au quatrième congrès des mathématiciens à Rome (Poincaré 1913, p. 369) :

« *Pour prévoir le futur des mathématiques, la vraie méthode est d'étudier son histoire et son état présent.* »

Poincaré suggère une méthode «d'extrapolation » un peu présomptueuse, mais si on veut étudier l'histoire des mathématiques, jusqu'où devons-nous remonter? Doit-on parler d'un futur immédiat ou d'un futur lointain. Lebesgue, plus proche de Descartes, nous propose une « longue chaîne de raisonnements », liée à l'évolution des idées et des techniques de preuves et de la rigueur. Il veut aussi rendre hommage aux scientifiques qui l'ont précédé et l'état des connaissances récentes. Il écrira en 1930 que ce qui l'intéresse est l'histoire de l'acquisition d'un fait mathématique (Leconte 1956, pp. 225–226). Aussi le sujet qu'il avait choisi sur les séries trigonométriques, bien qu'il fût parfaitement adapté à la théorie des fonctions et au deuxième théorème de Weierstrass sur l'approximation de fonctions (voir section 3), était un sujet étroit. Il aurait manqué de matériel s'il n'avait pas adopté une approche historique. Par exemple, il n'avait pas suivi cette méthode dans sa thèse. Lebesgue ne sera jamais un historien des mathématiques, mais il cultivera un goût pour l'histoire des mathématiques (Lebesgue 1958). Dans son livre sur les séries trigonométriques, il s'inspirera d'un article écrit par Sachse en 1880 : « *Essai historique sur la représentation d'une fonction arbitraire d'une seule variable par une série trigonométrique* » (Sachse 1880), et de la thèse de Riemman de 1854 (Riemann 1854). Mais aussi il aura connaissance plus tard de l'article de Burkhardt et Esclangon[6] pour *l'Encyclopédie mathématique* (Burkhardt 1993).

6 Commentaires sur l'histoire des mathématiques dans les *Leçons sur les séries trigonométriques*

« *En m'occupant des séries trigonométriques, j'ai eu surtout pour but de montrer l'utilité que pouvait avoir, dans l'étude des fonctions discontinues de variable réelle, la notion d'intégrale que j'ai introduite dans ma thèse.* » dit Lebesgue en 1903 dans son article aux *Annales de l'É.N.S.* sur les séries trigonométriques (Lebesgue 1903). Il écrira en tout six articles sur les séries trigonométriques dont un très important en 1905 pour *Mathematische Annalen* (Lebesgue 1905).

[6]Dans les lettres à Borel CIII, CIV, CV, CVI, il critique Molk, et l'encyclopédie mathématique [20].

Le livre sur les séries trigonométriques réunit les leçons faites au Collège de France pendant l'année 1904–1905 au cours de la fondation Claude-Antoine Peccot. Il comprend une introduction, un chapitre sur la détermination des coefficients des séries trigonométriques, un chapitre sur la théorie élémentaire de Fourier, un autre sur les séries de Fourier convergentes, celui sur les séries de Fourier quelconques, enfin un chapitre sur les séries trigonométriques quelconques.

Dans le chapitre 1, il présente un historique classique et succinct des séries trigonométriques sur l'obtention des coefficients d'Euler-Fourier avec les travaux d'Euler, Daniel Bernoulli, Lagrange puis Fourier et le problème des fonctions arbitraires au XVIII^{e} siècle et du temps de Fourier. Il donnera en plus des commentaires très utiles en bas de page notamment un mémoire de Riemann sur l'historique des séries de Fourier (Riemann 1854). Lebesgue montre qu'il s'intéresse aux applications en exposant les méthodes d'interpolation de Clairaut et de Lagrange, mais il ne réalisera pas que ces formules d'interpolation possèdent la propriété fondamentale de convergence en tout point, donc qu'elles peuvent s'appliquer à des fonctions ayant des discontinuités ce que ne permet pas l'interpolation lagrangienne. Ce que fera de la Vallée Poussin en 1908.

Dans le chapitre II, il aborde la sommation de séries trigonométriques et le processus de convergence. Là aussi, Lebesgue fait un bref historique du problème en présentant les méthodes de sommation d'Euler, de Lagrange, de Fourier et d'Abel. Concernant le théorème fondamental sur la convergence en tout point, voyons ce que disait Lejeune Dirichlet en 1829:

> « *Si la fonction $f(x)$, dont toutes les valeurs sont supposées finies et déterminées, ne présente qu'un nombre fini de solutions de continuité entre les limites $-\pi$ et $+\pi$, et si en outre elle n'a qu'un nombre déterminé de maxima et de minima entre ces mêmes limites, la série* (*de Fourier trigonométrique*)*, dont les coefficients sont les intégrales dépendant de la fonction $f(x)$, est convergente et a une valeur généralement exprimée par :*
>
> $$\frac{1}{2}\left[f(x+\varepsilon)+f(x-\varepsilon)\right]$$
>
> *où ε désigne un nombre infiniment petit.* » (Lejeune Dirichlet 1829)

Sachse dira que les conditions de Dirichlet étaient des conditions suffisantes, mais non nécessaires. Il fallait examiner ensuite les cas d'une fonction où 1) elle devenait infinie en un ou plusieurs points, 2) qui avait un nombre infini de discontinuités, 3) qui avait un nombre infini de maxima et de minima. Lebesgue l'énoncera de la façon suivante :

> « *Si l'intervalle* $(0, 2\pi)$ *peut être partagé en un nombre fini d'intervalles partiels dans chacun desquels la fonction f admet une dérivée à variation bornée,*[7] *la série de Fourier de f est partout convergente. Elle converge uniformément vers f dans tout intervalle ne contenant aucun point de discontinuité de f, en un point de discontinuité la série tend vers la moyenne arithmétique des valeurs vers lesquelles f tend quand la variable s'approche du point de discontinuité.* »

[7] Cette notation de fonctions à variation bornée est due à Jordan. Une fonction peut avoir une infinité dénombrable de points de discontinuité réguliers.

Alors bien sûr, le résultat précédent nous ramène au théorème de Weierstrass sur la représentation approchée de fonctions par des polynômes ou des séries trigonométriques (Borel 1905, ch. IV; 12, pp. 48–49. On échantillonne une fonction continue $f(x)$ dans un intervalle (α, β). On peut toujours la supposer périodique de période T et telle que $\psi(x)$ soit une ligne polygonale continue qui coïncide avec $f(x)$ aux points d'échantillonnage. Alors on peut approximer $\psi(x)$ par une suite finie de séries trigonométriques. Cette suite de Fourier peut être elle-même développée en série de Taylor uniformément convergente. On la transforme en polynôme en conservant suffisamment de termes dans la série de Taylor. Lebesgue ramène le théorème II de Weierstrass[8] au théorème I de la section 3.

Puis Lebesgue abandonne l'approche chronologique, mais il prendra soin chaque fois de préciser l'apport de ses prédécesseurs[9] : Riemann, Fatou, Lipschitz, Dini, Jordan, du Bois–Reymond, Schwarz, Fejér, Hurwitz, Cantor, Heine, etc. Dans le tome 1 de ses Œuvres scientifiques, Lebesgue expliquera tous ses travaux sur les séries de Fourier dans un résumé de 5 pages (Lebesgue 1972, pp. 131–136; 36, pp. 1046–1048; 37, pp. 806–812).

Finalement, rappelons que l'approche historique peut avoir ses limitations et qu'elle n'est pas toujours la meilleure car elle peut engendrer des méandres ou avoir des omissions importantes. Ainsi, la convergence quadratique, omise par Lebesgue, montre l'importance des moindres-carrés et les liens qui assemblent ensemble les différentes parties de l'analyse des séries et l'approximation de fonctions. En 1857, Bertrand présenta à l'Académie des sciences un compte-rendu de deux pages et demi écrit par Plarr (Plarr 1857) dont le titre était :

> "*Note sur une propriété commune aux séries dont le terme général dépend des fonctions X_n de Legendre, ou des cosinus ou sinus des multiples de la variable.*"

Plarr montra que si on veut minimiser l'erreur quadratique globale entre une fonction bornée, continue par morceaux et une série tronquée consistant de polynômes de Legendre ou de séries trigonométriques, les coefficients associés à ces polynômes étaient automatiquement les coefficients de Fourier-Legendre ou d'Euler-Fourier. En 1907, Riesz et Fisher généralisèrent la méthode de Plarr.

Pendant l'année 1927–28, Lebesgue ré-enseigna les séries trigonométriques au Collège de France et les recherches récentes en insistant sur les travaux de Fatou[10] sur la formule de Poisson. Nous ne croyons pas qu'il reprît l'approche historique dans ce cours.

Pour illustrer la contribution de Lebesgue aux séries de Fourier, on a choisi, quelques théorèmes importants dans un livre récent sur les intégrales de Lebesgue et les séries de Fourier. Les lecteurs pourront comparer avec les énoncés du théorème de Dirichlet par lui-même et par Lebesgue :

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[8] Voir aussi les travaux de Lerch et Volterra.

[9] L'ordre de ces noms correspond à leur citation dans le livre de Lebesgue.

[10] voir, Acta Mathematica, 30, pp. 335–400, 1906.

Théorème 1: Soit $f(x)$ appartenant à $C^2[-\pi, +\pi]$. Alors la série de Fourier converge uniformément vers f sur $[-\pi, +\pi]$

Lemme de Riemann-Lebesgue: Si $f \in L[-\pi, +\pi]$ et $\{a_k\}_{k=1}^{+\infty}$ et $\{b_k\}_{k=1}^{+\infty}$ sont les coefficients de Fourier de f, alors $\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = 0$

Théorème 2: Soit $f \in L[-\pi, +\pi]$ et les deux f'_R et f'_L existent, alors la série de Fourier converges vers $\frac{f(x_+)+f(x_-)}{2}$ en x.

Note: L'éditeur Dunod à Paris a transféré les livres de Gauthier-Villars à l'Institut Mémoire de l'édition contemporaine (IMEC), qui est dans la ville de Caen en Normandie. Vous pouvez contacter IMEC via les sites internet:

http://www.imec-archives.com/fonds_archives_fiche.php?i=GTV

courriel: chercheurs-ardenne@imec-archives.com

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Références

The Judicial Analogy for Mathematical Publication

R.S.D. Thomas

Abstract Having criticized the analogies between mathematical proofs and narrative fiction in 2000 and between mathematics and playing abstract games in 2008, I want to put forward an analogy of my own for criticism. It is between how the mathematical community accepts a new result put forward by a mathematician and the proceedings of a law court trying a civil suit leading to a verdict. Because it is only an analogy, I do not attempt to draw any philosophical conclusions from it.

1 Judicial Analogy

I am fond of analogies. I find that they improve life much as humour improves life. At CSHPM meetings I spoke on an analogy between mathematics and fiction in 2000 (Thomas 2000) and between mathematics and games in 2008 (Thomas 2008), in both cases pouring cold water—despite my fondness for analogies—on what I viewed as a too enthusiastic espousal of those analogies by others and in the case of fiction even an absurd identification. I have published my reservations about these analogies of others elsewhere (Thomas 2002, 2009). What I want to do here is to offer what I view as an improvement on the fiction analogy with my own undue enthusiasm. You may even think it not particularly closely related to that other analogy, but that is where it started out.

The fiction analogy works—to the extent that it works at all—only for certain documents like explanations and proofs. I once tried to present some mathematics to

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a sympathetic audience to illustrate how narrative-like what I said was. The audience only began sympathetic; their response convinced me that proofs are only a little like stories.[1]

There is, however, one way in which a story is like a proof and also like a paper like this one even when it contains no proofs. A story is an artful linearization of descriptions of situations, props, events, and whatever else is needed simply because it is written down in words. A picture is said to be worth a thousand words, not just a thousand pixels, because it is more effective than its linearization no matter how artful. A fax machine makes of a picture an artless linearization that is not humanly readable. As I have said, this paper is intended to offer a picture but is of course a linearization. Even if it were desirable, it cannot begin in the top left corner and proceed like the cathode rays of an old computer monitor because the picture is only metaphorical. All I can do, so far as I can tell, is to sketch patches of the analogy intended in an order that I find somewhat linked. It is just piling up details like the points of a pointillist painting.

The analogy that I want to point to today is larger in scope than proof/narrative. I want to sketch a comparison between the justice system on the British model, used more or less in all English-speaking countries, and the more public elements of the professional mathematical enterprise.[2] So, on the one hand, we have courts with judges and cases that are tried in terms of laws and the evidence on which cases are decided. And, on the other, we have the knowledge base of the profession into which gatekeepers admit results in terms of postulates, logic, and definitions, where results are put forward by mathematicians in talks and papers. The results to which I refer are mainly theorems, theorems are spelled out in terms, the terms need definitions, and theorems are normally derived from postulates in accordance with some system of logic. Now obviously publishing a theorem isn't much like litigation, but there are some similarities here that may possibly be interesting. Each is publicly making a claim that may be accepted.

One of the philosophical problems to do with mathematics that I think is of some interest is its objectivity. We all agree that mathematical results do not depend on our preferences; they are what they are even if we don't like them. This objectivity is not something that is as true even of the hard sciences; it is an achievement that the ancient Greeks seem to have started us off with. In my opinion (Thomas 2014) it is based on consensus-based definitions. One hears a great deal about axioms and postulates, but as time has passed the importance and complexity of definitions has grown. In the 1970s Michael Spivak (1970–1979) pointed out that recent mathematics has tried to move as much theory as possible from theorems into the definitions influenced by them. One need look no farther than the definite

[1]The idea does continually spring up. The evening before my writing this footnote, 20th January, 2015, the Oxford Research Centre in the Humanities and the Mathematical Institute at Oxford held an event called "Narrative and Proof" featuring a panel discussion led off with a paper by Marcus du Sautoy entitled "Proof = Narrative" http://new.livestream.com/oxuni/narrativeandproof.

[2]A list of pairs that I see as analogous appears at the end for reference.

integral for an example. His book *Calculus on manifolds* was a popularization of the derivative represented by the Jacobian matrix for an audience that may have heard only of ordinary and partial derivatives. Unlike the proofs of most theorems, definitions are the product of a sometimes slow evolution, much discussion, trial and error, and finally a temporary consensus that can be revised if reason demands. Two hundred years ago mathematicians thought they knew the meaning of convergence of a sequence of functions, and then they found they needed uniform convergence. Then the real numbers, then the integers. New definitions for old concepts—what Carnap called explication. I think that the agreed-upon definitions, which much work is done in terms of, are—as a class—given less attention than they should be.

The judicial analogue is the law. The rule of law is among the few formalities as old as mathematics itself. Laws that are that and not just the whim of the ruler are at least as old as Hammurabi in Mesopotamia, whose law code was on display in the summer of 2013 at the Royal Ontario Museum. That the ruler himself is subject to the laws of the land is one of the main achievements of the Magna Carta in thirteenth-century England. It is certain that some of the features I just recounted for mathematical definitions are also features of laws. They have certainly become progressively more complex, but they do evolve along with the societal consensus that they codify. Sometimes they don't work and have to be replaced. I read recently that the sale of some narcotics was criminalized in the USA only in 1922, and now at least the possession of marijuana is lawful again in some states. Prohibition in North America is another such example. One of the important ways in which laws and definitions are different arises from laws' being the products of nation states and their subordinate units, a difficulty that we are mercifully free of—except for legislatures that define mathematical constants. I wonder whether any but *pi* has ever been noticed by lawmakers.

Analogous to the arguments that are conducted in courtrooms are the arguments with which we try to convince the mathematical world that our results are correct. Fortunately our judges are our colleagues acting as referees for journals and then as readers of journals and listeners to our talks. Cross examination happens at the end of the talk when the one expert present, who has followed the argument, points out by tactfully asking a well-aimed question the error observed or the dubious piece of the argument. Sometimes there are appeals to higher courts. It was a while before the proof of Fermat's Last Theorem was approved of, and I believe that Hales's sphere-packing result is still under study.

Let us look a little more carefully at the process of convincing the mathematical community of the value and truth of our result. Folks are convinced by evidence. What evidence do we present? We either begin with a proof and study the premises and conclusion to find what it proves or we start with premises or conclusion and find a proof that takes us from them to it. Eventually, when we have done these things and are ourselves convinced, we state our result, which of course may be more than just one theorem. This is analogous to the claim of a plaintiff, which needs to be publicly tried before it is recognized. On the basis of our experience of going through the proof, which may easily be like a graph-theory graph in its structure, we write down a linear representation of it that is analogous to testimony at a trial. An example of this is illustrated in a recent paper of Arana (2015) quoting the diagram that Szemerédi supplied in his original proof (Szemerédi 1975) of his eponymous theorem; it was a diagram not of what he was writing about but of the structure of the proof. Frege's Begriffsschrift is so difficult to learn that almost no one has done so because its notation is sensibly (in a way) two-dimensional in its own way. We can also offer non-linear material like diagrams of what is being talked about, which we have to interpret for our audience. Our linear prose is analogous to testimony at the trial. The first thing that we typically do, however, is not to put it publicly on trial but to submit it first to one or more persons privately. Only when their judgement is positive do we take it elsewhere.

Testimony is interesting. Unlike fiction, it is talk that is meant to be believed as said because it is given in evidence. Evidence includes both stuff and testimony, but evidence that is stuff rather than testimony has to be interpreted by testimony to be meaningful as evidential. The murder weapon only matters if someone vouches for its being the murder weapon—if it is associated with both the victim and the murderer, quite possibly by different persons.

2 Testimony

Testimony is something that philosophers have written about. Their main concern is epistemological—whether it should be believed (Adler 2014; Green 2015). They do say things that are relevant to the mathematical analogue. For instance, that testimony transfers what is something else to "the level of things said" (Ricoeur 1972, p. 123); it gets things like the identity of the murder weapon into words where everyone can hear/look at it. In this respect it is a source of objectivity and a basis for judgement. The category of testimony makes literal sense only in the context of influencing a judicial decision. The language is useful by transfer to "situations less codified" (Ricoeur 1972, p. 124), one of which is history, where there is inherent uncertainty. There is a big argument going on in philosophy of history about whether what one might call professional history—the stuff written down—must abjure or embrace testimony. Fortunately in the mathematical context the uncertainty that makes our proof-testimony necessary is meant to be dispelled by it so that at the end there is no uncertainty left and judgement can be rendered easily and firmly. Testimony is meant to be persuasive. Just as justice is served by a correct decision, knowledge is enlarged by the successful proof of a mathematical result. On the other hand, just as there is false testimony there are proofs that are flawed.

When I was writing about the analogy of mathematics to fiction, I pointed out that mathematics is much more like history than like fiction because, however much history it made up in one sense by the historian, to be history it has to operate in recognition of the constraints by what happened in the past, however little may be known about that. If it does not recognize those constraints—and even sometimes when it does to some extent or other—it *is* fiction.

> I should want to stress that as 'fictive' as the historical text may be, its claim is to be a representation of reality. And its way of asserting its claim is to support it by the verificationist procedures proper to history as a science (Ricoeur 1983).

Distinguishing historical fiction from proper history is a problem that historians have.

Recently testimony has attracted attention from philosophers of information (Floridi 2014) and of mathematics (Geist et al. 2010).[3] There is no doubt much to be said about the place of testimony in mathematics. This paper's narrow focus is on testimony as a way of looking at the important category, published proofs. Geist et al. (2010) are mainly concerned with testimony as a way of looking at referee reports. The testimony of a referee, either for or against publication, is important but somewhat orthogonal to the acceptance of published work. Negative referee testimony, ranging from matters of taste to counterexamples, has the effect, if any, of preventing publication without significant improvement. Positive referee testimony simply disappears once its job is done. Geist et al. are concerned, not with the reception of mathematical work, but with the stage of being published. Referee testimonies or opinions are personal, fallible, and not as important as the community reception of work once published. They are also based on wildly varying amounts of study. The correspondent in the judicial analogy is, it seems to me, the advice of the plaintiff's legal team. Referees act as surrogates for the mathematical community in the decision to publish, just as one of the pre-trial functions of the lawyers is to help with the decision to go to court. Like the reviewing of manuscripts, this job can be done well or badly and is done on the basis of differing levels of expertise and study. Referees are often thought of as gatekeepers, a function that they do serve, but I think of them also as acting partly on behalf of the author in advising how work can be improved based on how it will strike the intended audience and whether the reputation of the author will be harmed or helped by publication. It would be a case of perverse incentives for a young scholar to publish something that in the short run helped to get a first job or tenure but affected long-term reputation adversely.

I said that the main interest of philosophers in testimony is in whether it should be believed, whether it produces knowledge in its hearer. Plato's strictures on knowledge make it particularly difficult to attain what philosophers are prepared to call "knowledge". This is obviously a much bigger problem to a judge of testimony in a courtroom or to a historian. Steps are taken in context to improve (I cannot say "ensure") veracity. In courts typically testimony is sworn or when the witness can't do that because of age or dim-wittedness, the importance of truth-telling is impressed on the witness. And the vogue for the so-called oral history is a way of singling out recollections from ordinary history, which is distinct in at least being cobbled together from as many recollections as are worth collecting. On the other hand, in mathematics the testimony of someone competent that has proved

[3] I am grateful to an anonymous referee for reminding me of this paper, of which I was nominally aware, having reviewed (in a weak sense, "made note of") the book in which it appears (Thomas 2012).

something says how it was done. Any similarly competent hearer ought to be able to reconstruct much the same experience from reading the testimony. In this it is distinct—in a thoroughly positive way—from eyewitness recollection, however sworn or amalgamated. Testimony in its normal meaning asks to be believed on the say-so of the witness, but testimony of mathematical experience invites the hearer to join in the experience, its intersubjectivity being the chief indicator of its truth and objectivity. To see its truth, it should not matter who you are. In his ground-breaking book *Testimony*, the Australian philosopher Coady (1992) quoted Russell (1927, p. 150) in this connection, "I mean here by 'objective' not anything metaphysical but merely 'agreeing with the testimony of others' ".

What some recent philosophers have said about testimony is both to observe that an enormous amount of what we know is dependent upon testimony and to attempt to justify this obvious fact despite Plato's discouragement. One way in which the world has changed since Plato's day is relevant. Twenty-five hundred years ago, one was personally dependent for a lot of what one learned first from one's elders and then from one's contemporaries, but that was as far as the dependency went. Now we have whole areas of life that are based on intellectual work over many years accessible only by testimony. Scientific research in particular, including mathematics, is just not feasible without all of the background knowledge built up over time, much of which one has learned from testimony. It is only in principle that one can replicate old results. Undergraduate experiments are a replication of only a few high spots in the history of science, and the same is true to a lesser extent of what one learns in mathematics. Poincaré and Hilbert were perhaps the last mathematical know-it-alls. We are lucky that mathematical knowledge among scientific knowledge is uniquely learnable that way.

One of the things that I think is interesting about this analogy is the different way that authority works in the two contexts. In litigation up to the point of reaching a verdict, the judge needs to be competent in law because he has to keep under control the advocates of the parties to the dispute, who need to be learned in the law because it is the framework within which the dispute is being settled—one of the reasons it is safer to settle disputes out of court. My barrister brother-in-law has said that a civil suit that goes to trial has at least one party that is making a mistake. In addition to those competent in the law, there are sometimes expert witnesses. Their competence pertains to the interpretation of evidence. The authorities cited by the lawyers will be law and precedents; authorities cited by the experts will be published scientific facts. None of this pertains to making the judgement on the case; it is all peripheral. If the trial is before a jury an important point is that the members of the jury do not need to be experts on anything. Compare this with the analogue. What authorities do we cite in our mathematical arguments? Postulates, definitions and previous mathematical results, nothing else being relevant. It is the jury that needs to be competent. Where personal authority comes into the mathematical scene is in the orthogonal judgements of the importance or depth or beauty of the result if true. Such judgements really are orthogonal, since one can judge a failed theorem to be important enough to continue pursuing a proof; the four-colour theorem had that status from Heawood's disproof of Kempe's attempt to the successful proof of Appel and Haken, a period of nearly a century.

One needs, in exploring such a comparison, to keep in mind that one is comparing two things that are different to see ways in which they are similar. Coady has been studying testimony for 40 years and appears to have put the topic on the philosophical map. As he pointed out, other philosophers have for some time studied the different sorts of thing that we do when we say or write words. "Asserting, testifying, objecting, and arguing all have the same or similar illocutionary points—roughly to inform an audience that something is the case ... (Coady 1992, p. 43, referring to works of John Searle 1979; Searle and Vanderveken 1985)." But there are distinctions. Testimony, which is not proof but is believed, is believed on the say-so of the witness.

> When we believe testimony we believe what is said because we trust the witness. This attitude of trust is very fundamental, but it is not blind. As (Eighteenth-century Scottish philosopher Thomas) Reid noted, the child begins with an attitude of complete trust in what it is told, and develops more critical attitudes as it matures. None the less, even for adults, the critical attitude is itself founded upon a general stance of trust, just as the adult awareness of the way memory plays us false rests upon a broader confidence in recollective powers. (Coady 1992, p. 46), citing (Reid 1764, VI, xxiv)

While our proofs are testimony to our having gone through the proof process, and the proof may be believed up to a point on our say-so, our testimony is also a challenge to every reader to go through the proof and be convinced for oneself. It is not normal for one to publish several persons' versions of one's proof for corroboration as several witnesses may be called in court to corroborate the testimony of the first of them. We do construct new proofs of old results but that is only occasionally to guarantee their truth. It is the reader that is called upon to corroborate a printed proof by first-hand experience. In Pollard's (2014) review of Hersh's (2014) *Experiencing Mathematics*, he quotes Hersh putting it this way.

> When Hardy (for example) makes a discovery, he explains how other mathematicians can verify his claim, by following a certain sequence of steps, to arrive at 'seeing it'. And those directions are 'the proof'!

Why is it that personal testimony is relevant when everyone agrees that mathematical proof is the prototype of objectivity? This question was suggested to me by a paper (Montaño 2012) on mathematical aesthetics by Ulianov Montaño. In order to discuss the beauty of a mathematical proof more satisfactorily than usual (McAllister 2005; Rota 1997), he draws a distinction between the proof as a *mathematical object*, of which a fully formalized version is the best expression, and the proof as an *intentional object*, the utterly subjective content of one's mind as one rehearses or contemplates the proof, almost always informal. Just as a picture is beautiful or not as *seen* and a piece of music is beautiful or not as *heard*, a proof is beautiful or not in one's mind, not as written down. It would be a category mistake

Fig. 1 Analogical pairs

Justice system	Mathematical publishing
Courts	Journals
Plaintiff	Author
Case	Paper
Suit	Theorem(s)
Law	Background mathematical knowledge
Non-verbal evidence	Non-verbal presentation
Testimony, argument	proof(s)
Judge/jury	Mathematical community
Legal team	Referees
Cross examination	Critical questions
Expert witnesses	—
Winning/losing	Acceptance/rejection
Appeal to higher court	Lengthy controversy
Justice	New public knowledge

to attribute beauty to a written formal proof as to a written musical score.[4] This distinction seems to me useful to describe how it is that personal testimony of the subjective experience of becoming convinced of something has probative value. That testimony, if expertly done for a proof that is actually valid, is a recipe, as Pollard and Hersh suggest, for sharing in the experience and the conviction.[5]

Testimony is talk that is taken as evidence. What of evidence that is not testimony? Is anything to be learned from that comparison? What is the mathematical-proof analogue of exhibits at a trial? It seems to me that it is whatever is not self-interpreting, mainly diagrams but also anything that is not prose, that cannot be read out in words. All ordinary speech is human communication and so *is* an interpretation, but a diagram or a formal proof does not interpret itself. Someone must tell us what a diagram or formal proof is *of* and show us what in it corresponds to what we are talking about. It must be interpreted to us or be embedded in a situation of which we know a standard interpretation. There has been some movement since Frege's invention of his Begriffsschrift in 1879 toward formality in proving—as distinct from rigor, which he did not invent. That has been instructive, but it is obviously a process that can go only so far because such formalities require interpretation to have meaning.

[4]At the end of my presentation it was pointed out to me by Michael Williams that there are those that find pleasure in reading computer code, an activity that seems to me comparable to reading musical scores without hearing the music—actually or virtually.

[5]The other comment made to me at the end of my presentation, by someone whose name I did not record, was that a feminist view of testimony explicitly considers the standpoint of the speaker. While in most circumstances this is an important feature of testimony, it does not seem to matter in the mathematical context because what matters is so much the recipe for having the appropriate experience oneself rather than the anything at all to do with the witness.

I am far from claiming that this is the only way to look at this matter, and I am newly enough come to it that I am not even sure that it is a good way, but perhaps it merits consideration. A summary of the analogy can be seen in Fig. 1. I apologize for the lack of philosophical conclusions.

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History and Philosophy of Mathematics at the 1924 International Mathematical Congress in Toronto

David Orenstein

Abstract When the University of Toronto hosted the International Mathematical Congress (IMC) in August 1924, the prime organizer, University of Toronto mathematician John Charles Fields (1863–1932) insisted the papers cover a wide range of mathematical topics: algebra, analysis, astronomy, engineering, statistics, and history and philosophy of mathematics. Section VI of the Congress covered History, Philosophy and Didactics of Mathematics. There were in total 13 papers in the published proceedings: seven full Communications and six Abstracts. Five were historical, six philosophical and only two pedagogical. In Section VI the American algebraist G. A. Miller looked at "The History of Several Mathematical Concepts" including "the unknown" and "permutations", going back to the ancient Egyptians and Greeks. Miller also presented in Toronto on algebra, looking at commutativity in Abelian subgroups. The great Italian logician Giuseppe Peano, who had also presented in Zurich in 1897 at the IMC and then in Cambridge in 1912, spoke in simplified Latin "De Aequalitate", On Equality. The Swiss educator Henri Fehr contributed to the pedagogical programmes at four other IMCs (1904, 1908, 1912 and 1932), focusing in Toronto on the university's preparation of high school mathematics teachers. Florian Cajori, the great American historian of mathematics, discussed mathematical notation in two different papers: its history in geometry and a programme for its improvement. This paper examines the role of both History and Philosophy of Mathematics at the Toronto IMC.

1 Introduction

A study in the History of Mathematics can have many possible foci. It can be about a person, a book, an equation, or an institution. It can also be about an event: an occurrence at a certain place and at a certain time. This research project was inspired by the discovery that in the summer of 1924 Georges Lemaître (1894–1966), of Big Bang fame, accompanied his graduate supervisor at the University of Cambridge,

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Arthur Eddington (1882–1944), to Toronto. They were there for the International Mathematical Congress (IMC) held at the University of Toronto (Orenstein 2012).

The first “International Mathematical Congress [was] held in connection with the World’s Columbian Exposition Chicago 1893”. This World’s fair at Chicago included among its many activities a World Congress of Mathematics and Astronomy. On Monday, August 21, they met jointly with G. H. Hough in the chair and Felix Klein spoke on “The Present State of Mathematics”. Then the astronomers and the mathematicians decided to divide into separate meetings. The mathematicians met for another 5 days of papers (Moore et al. 1896).

This Chicago Congress is usually considered to be the precursor to the formal series of International Mathematical Congresses (IMC) or International Congresses of Mathematicians (ICM).

2 J. C. Fields and the Toronto IMC

The 1924 Congress had been slated for the USA, but when the American Mathematical Society (AMS) refused to maintain the post-World War I exclusion of mathematicians from the Central Powers, University of Toronto mathematics professor John Charles (JC) Fields (1863–1932) offered to host it at his university. The offer was accepted with great relief. On short notice Fields successfully organized the Congress, with the strong backing of his colleagues and his institution (Riehm and Hoffman 2011).

When he made the offer to host the Congress, Fields was serving a 2-year term (1923–1925) on the American Mathematical Society (AMS) Council where he had previously served in 1910–1912. L. E. Dickson and L. P. Eisenhart as delegates to the 1920 Strasbourg Congress had “tendered an invitation for the Congress of 1924 to be held in the United States without having consulted the AMS. By 1922 it was clear that financial backing was unobtainable in the United States, with the restrictions imposed by the IMU [International Mathematical Union]. Hence it was fortunate that the Dominion of Canada, which (under the inspiration of J. C. Fields enthusiasm) had done so much to promote research, should offer to arrange for an International Congress of Mathematicians in 1924”. (Archibald 1938)

Fields certainly had the organizational experience to lead such an endeavour. When the University of Toronto had hosted the American Association for the Advancement of Science (AAAS) in December 27–31, 1921, he was Chairman of “The Local Committee for the Second Toronto Meeting In charge of all local arrangements” and also the local representative for Section A: Mathematics. Both the AMS and the Mathematical Association of America (MAA) held Toronto sessions under the auspices of Section A. (AAAS 1921)

When Fields saved the AMS from international embarrassment he was also working with his friend and colleague Physics Professor J. C. McLennan and also Robert Falconer, the university’s president, on organising the fourth Canadian

Meeting of the British Association for the Advancement of Science (BAAS). Previous meetings had been held in Montreal (1884), Toronto (1897), and Winnipeg (1909) (Riehm and Hoffman 2011).

With the acquisition of the IMC, Fields switched his focus, but still served the BAAS as one of two Local Secretaries, an official representative of the Royal Canadian Institute, and on the Local Sectional Committee of Section A: Mathematics and Physics (BAAS, 1924). Furthermore, "The Association was welcomed to Toronto ... by Prof. J. C. Fields, F. R.S., President of the Royal Canadian Institute on behalf of that body" (BAAS 1925).

When Fields' colleague, Samuel Beatty, reported on "The Progress of Mathematics in Canada", at the 1938 special symposium on the history of science in Canada at the 1938 Summer Meeting of the AAAS in Ottawa, he began by recalling "Fields had presented a report to Section III of the Royal Society of Canada, dealing with the development of the idea of research in mathematics ... " (Beatty 1939).

For Beatty, "Fields by his insistence on the value of research, as well as by his published papers has ... done most ... to advance the cause of mathematics in Canada ... . [H]is gift of being able to see a complicated situation as a whole" naturally led him "to algebraic functions and later ... algebraic numbers ... [H]is book [*Theory of the Algebraic Functions of a Complex Variable*] came out in 1906, after he had joined the staff in Toronto ... . His last great paper [see Fig. 1] was presented to the International Mathematical Congress at its Toronto meeting, 1924, and appeared in its *Proceedings*", using his theory "to furnish an analogous theory of algebraic numbers. Failing health prevented" complete success with the paper (Beatty 1939).

U. of T. Mathematics Professor and Dean of the Faculty of Arts, Alfred Tennyson DeLury, joined Fields on the Organizing Committee. He also served as Introducer for Section VI: History, Philosophy, and Didactics (Fields 1928b).

In 1921 DeLury had spoken on history of mathematics to University of Toronto *students* at their Mathematical and Physical Society (MPS). "After a brief musical entertainment Professor De Lury [spoke] on the life of Evariste Galois ... [to make] the study of mathematics more interesting. [I]n 1830 [Galois] wrote three important

A FOUNDATION FOR THE THEORY OF IDEALS

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The object of the paper is to lay a foundation for the theory of the ideals in particular and for the theory of the algebraic numbers in general on lines parallel to those on which the writer has developed the theory of the algebraic functions of one variable. With this object in view, it will be convenient to make use of Hensel's conception of the rational p-adic numbers. Any rational number is said to be divisible or not divisible by a prime p according as the numerator of the number in its reduced form is or is not divisible by p.

Fig. 1 Transcription of the Start of John Charles Fields 1924 IMC Paper (Fields 1928a) memoirs . . . and became attached to a club of revolutionaries . . . . [Because] of an obscure duel he met his death at age twenty" (*Varsity* 1921).

Another of Fields' Mathematics colleagues, Professor J. L. Synge, worked intensely on the organizational details as IMC Secretary. At the beginning of the 1924–1925 academic year it was his turn to speak to the MPS. "Excellent refreshments were served to a large crowd including . . . several professors and several of their wives. Mrs. J. L. Synge poured tea . . . . Prof. J. L. Synge spoke on the International Mathematical Congress, outlining it's history . . . . He commended the efforts of Dr. J. C. Fields in securing funds and stimulating interest" (*Varsity* 1924).

3 The Schedule of the 1924 IMC

At 2:30 in the afternoon, at University College (see Fig. 2), on Monday, August 11, 1924, Dean A. T. Delury convened "Section VI: History, Philosophy, Didactics" at the International Mathematical Congress hosted by the University of Toronto. The section was chaired by Professor F. Cajori, with Professor L. C. Karpinski serving as Secretary.

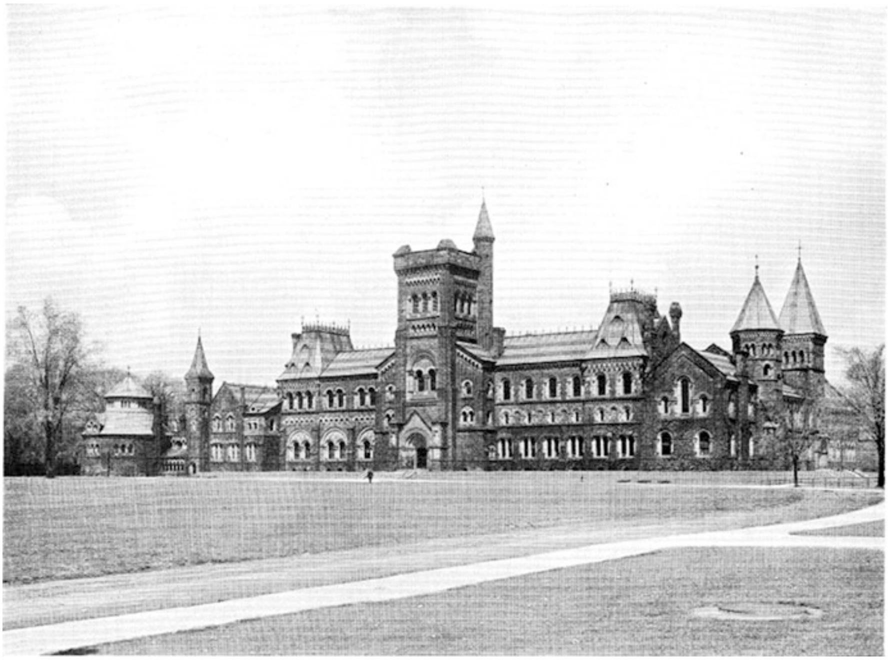


Fig. 2 University College. Image from (Fields 1928b) and reprinted with permission of the University of Toronto Press

Fig. 3 Convocation Hall. Image from Fields (1928b) and reprinted with permission of the University of Toronto Press

The first session had to be fairly brief because they were expected to be at the York Club for a Garden Party hosted by Physics Professor and Mrs. J. C. McLennan at 4:30. Their morning had been spent at the Opening Session for the Congress held at the University's Convocation Hall (see Fig. 3), a large domed structure reminiscent of the Roman Pantheon. Following that they joined "a group photograph of the Members of the Congress ... in front of the Physics Building" (Fields 1928b).

Section VI reconvened (see Fig. 4) on Tuesday, August 12, at 9 a.m. for a longer session this time. The next general scheduled event wasn't until after lunch. At 2:30 p.m., Professor Francesco Severi delivered the plenary lecture on "*géométrie algébraique*" (Algebraic Geometry).

It was then off to another Garden Party, this time at Government House and under the auspices of "His Honour Henry Cockshutt, Lieutenant-Governor of Ontario and Mrs. Cockshutt". That evening they "were entertained at a Conversazione at Hart House (see Fig. 5) by the University of Toronto and the Royal Canadian Institute", the co-sponsor of the Congress (Fields 1928b).

On the programme (see Fig. 6) "A Stringed Quartette in the Music Room .... The Band of the Royal Grenadiers ... in the Quadrangle" and Mr. Merrill Denison's Canadian drama "Brothers-in-Arms" in the Hart House Theatre. On the athletic side, fencing directed by the Canadian Champion, Mr. Charles Walters and an indoor

GENERAL SESSION AND SECTIONAL MEETINGS

GENERAL SESSION

Following the Opening Session a General Session of the Congress was held for the election of Officers. On the nomination of Professor de la Vallée Poussin, Professor J. C. Fields was elected President of the Congress, and the following Vice-Presidents were elected: Professors B. Bydžovský, F. M. Da Costa Lobo, L. E. Dickson, Senator F. Faure, Professors H. Fehr, L. E. Phragmén, S. Pincherle, E. Schou, C. Servias, C. Stormer, W. van der Woude, W. H. Young, and S. Zaremba.

Professors J. L. Synge and L. V. King were elected General Secretaries of the Congress.

Following the General Session a group photograph of the members of the Congress was taken in front of the Physics Building.

2:30 p.m.	Sections I, II, III(a), III(b), IV(a), IV(b), V, and VI, having been separately installed by the Introducers, papers were read and discussed.
4:30 p.m.	The members of the Congress were entertained at a Garden Party at the York Club by Professor and Mrs. J. C. McLennan.
8:30 p.m.	Professor Carl Stormer delivered his lecture on "Modern Norwegian Researches on the Aurora Borealis."

TUESDAY, AUGUST 12

9:00 a.m.	Sections I, II, III(a), III(b), IV(a), IV(b), V, and VI met separately. Papers were read and discussed.
2:30 p.m.	Professor F. Severi delivered his lecture on "géométrie algébraique".
4:30 p.m.	The members of the Congress were entertained at a Garden Party at Government House by His Honour Henry Cockshutt, Lieutenant-Governor of Ontario and Mrs. Cockshutt.
8:30 p.m.	The members of Congress were entertained at a Conversazione in Hart House by the University of Toronto and the Royal Canadian Institute.

WEDNESDAY, AUGUST 13

9:00 a.m.	Sections I, II, III(a), IV(a), and V met separately. Papers were read and discussed.
11:30 a.m.	Professor É. Cartan delivered his lecture on "La théorie des groupes at les recherches récentes de géométrie différentielle".
3:00 p.m.	The honorary degree of D.Sc. was conferred by the University of Toronto on the following delegates to, and members of, the Congress: Sir William Bragg, Professor Charles de la Vallée Poussin, Professor G. Koenigs, The Honourable Sir Charles A. Parsons, Professor F. Severi, Professor W. Stekloff. Following the conferment, the members of the Congress were entertained at a Garden Party given by the University of Toronto.
8:30 p.m.	Professor W. H. Young delivered his lecture on "Some characteristic features of Twentieth Century pure mathematical research".

Fig. 4 Transcription of Schedule for the 1924 ICM (Fields 1928b)

baseball game in the large gymnasium. Canadian paintings were hung throughout the building and "[A]n interesting collection of prints and photographs of old ... Toronto" were displayed in the Sketch Room. After satisfying their cultural hunger, physical hunger was assuaged by the refreshments served in the Great Hall (see Fig. 7) after 10 p.m. (Hart House 1924).

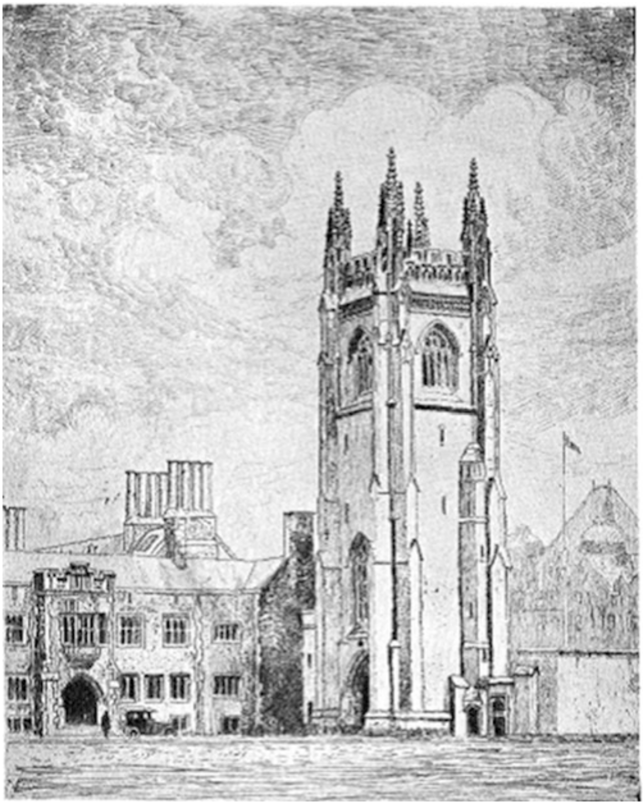


Fig. 5 Hart House. Image from Fields (1928b) and reprinted with permission of the University of Toronto Press

Tuesday morning was the last session of History, Philosophy, and Didactics, so they were free to attend other sessions to pursue other mathematical interests, attend further plenary lectures or the special University of Toronto Convocation awarding the honorary Doctor of Science to Francesco Severi among others. Or they might pursue the many offers of hospitality in Toronto.

On Thursday, August 14, the members of Congress crossed to Niagara, where they inspected the generating stations (see Fig. 8) at Queenston and at Niagara Falls and had lunch at the Clifton Inn. The participants in the ICM were guests of the Power Commission. After viewing the Falls (see Fig. 9) and travelling the Gorge Route, the group returned by boat to Toronto (Fields 1928b).

The Congress continued for two more days of sectional meetings, plenary lectures and entertainments.

PROGRAMME

Two presentations of the Canadian drama "Brothers-in-Arms" under the direction of the author, Mr. Merrill Denison, will be given in the theatre of Hart House by the courtesy of the Hart House syndics. The first performance commences at 8:30 p.m. and the second at 9:30 p.m.

*****

Guests are requested to enter the theatre by the outside entrance and to be in their seats at the times stated above.

*****

Three lecturettes will be given concurrently, commencing at 9:15 p.m.

Sir Richard Padget ………………Lecture Room

Professor A. Coleman…………………..Library

Dr. F. A. E. Cres………………..Reading Room

*****

A stringed Quartette will play in the Music Room from 8:45 p.m. to 10:45 p.m.

*****

The Band of the Royal Grenadiers will play in the Quadrangle.

An exhibition of fencing under the direction of the Canadian Fencing Champion, Mr. Charles Walters, will be given in the Fencing Room, commencing at 9:00 p.m.

*****

An indoor baseball game will take place in the large Gymnasium at 8:30 p.m.

*****

The attention of the guests is drawn to the paintings throughout the building which, with the exception of a few belonging to the House, have been loaned by Canadian artists for exhibition on this occasion.

*****

An interesting collection of prints and photographs of old scenes in and about Toronto may be seen in the Sketch Room.

*****

Refreshments will be served in the Great Hall after 10:00 p.m.

*****

Ushers placed throughout the House are for the purpose of directing guests.

Fig. 6 Transcription of the Program for Conversazione, August 12, 1924 (Hart House 1924)

4 History, Philosophy, Didactics Section at the 1924 IMC

There were 16 papers delivered by 14 different speakers in the two sessions of Section VI: History, Philosophy and Didactics. Only 13 were published (7 complete papers and 6 abstracts) in the two volume *Proceedings* (see Fig. 10), edited by Fields with the help of an editorial committee (see Fig. 11) that included Professors Ettore Bertolotti and L.C. Karpinski, members of Section VI (Fields 1928b).

5 Four Leaders: Fehr, Cajori, Miller, Peano

Four of our early twentieth century colleagues: a leading mathematics educator, Henri Fehr; an accomplished historian of mathematics, Florian Cajori; the great foundationist and philosopher of mathematics, Giuseppe Peano; and the path-breaking algebraist and avocational historian, George Abram Miller were all at the ICM in Toronto in 1924.

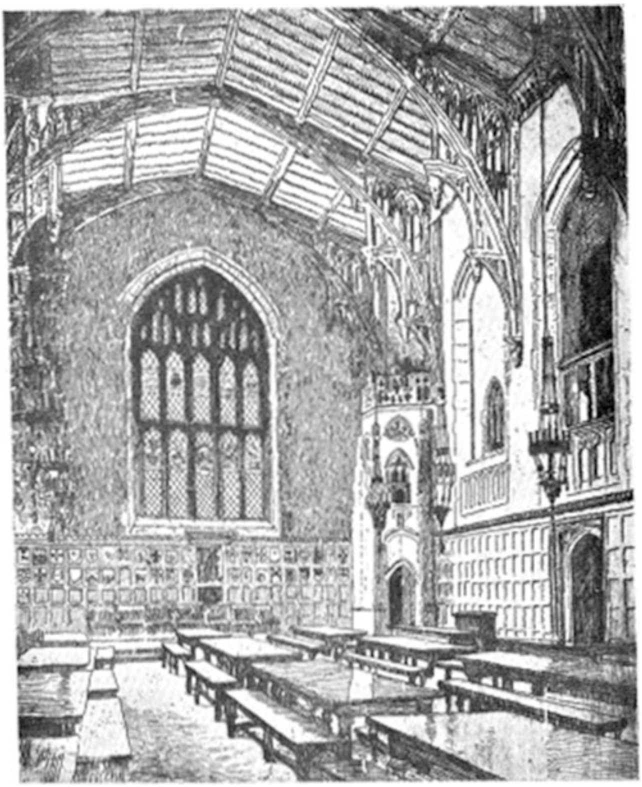


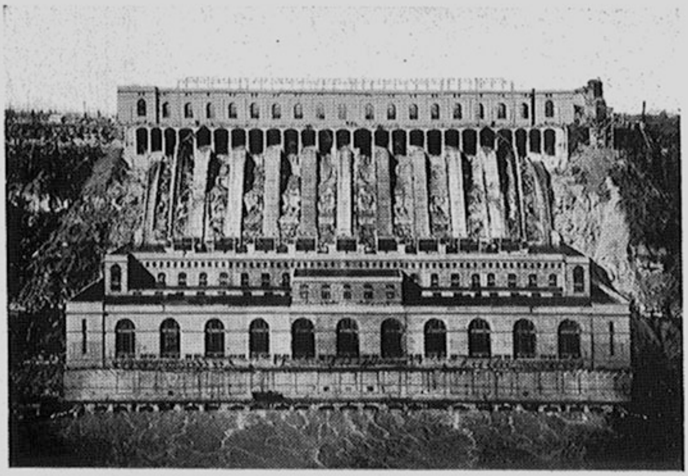
Fig. 7 Great Hall. Image from Fields (1928b) and reprinted with permission of the University of Toronto Press

5.1 Henri Fehr (1870–1954)

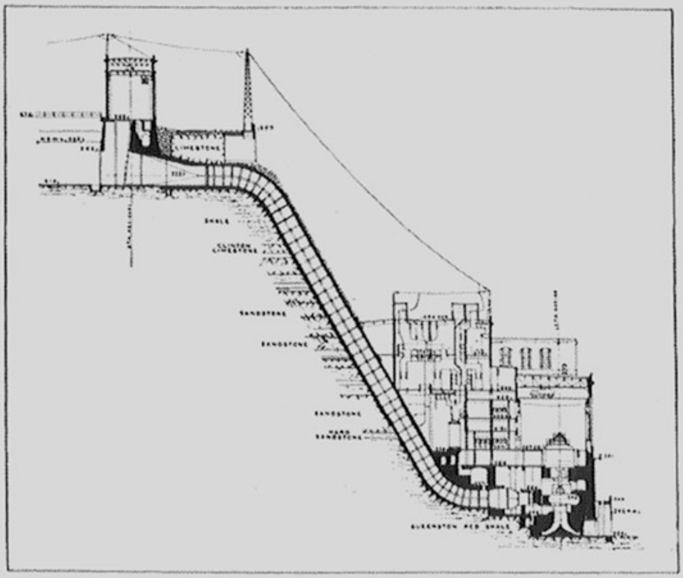
Henri Fehr was a professor at Switzerland's University of Geneva and a co-founder, in 1899, and a co-editor of the journal *L'Enseignement mathématique: Méthodologie et organisation de l'enseignement, philosophie et histoire des mathématiques* (Mathematics Teaching: Teaching Methodology and Organisation, Philosophy and History of Mathematics).

Fehr offered strong support for the Toronto IMC in his journal. "Des pourparlers sont engagés en vue de l'organisation d'un congrès international de mathématiques qui aura lieu à Toronto au début du septembre 1924, comme suite à la réunion que la British Association tiendra au Canada l'an prochain. Nous ne manquerons pas de renseigner nos lecteurs sur la programme de ce congrès" (Fehr 1923).

In a later issue of the same volume of *L'Enseignement mathématique*, he would be able to specify the Congress would run "from Monday August 11 to Saturday August 16" and to announce (in English) the subjects of the six Sections.



Queenston-Chippawa power house with nine units installed. It is located in the gorge of the Niagara river, the screen house standing on the top of the cliff and concrete covered steel pipes leading the water down the face of the cliff to the turbines. The power house stands on the edge of the river, is about 560 feet long and reaches half-way up the cliff.



Cross section through screen house and power house, showing pipe line down face of cliff with transmission line above.

Fig. 8 Images of Niagara Falls Power Station. Image from Fields (1928b) and reprinted with permission of the University of Toronto Press

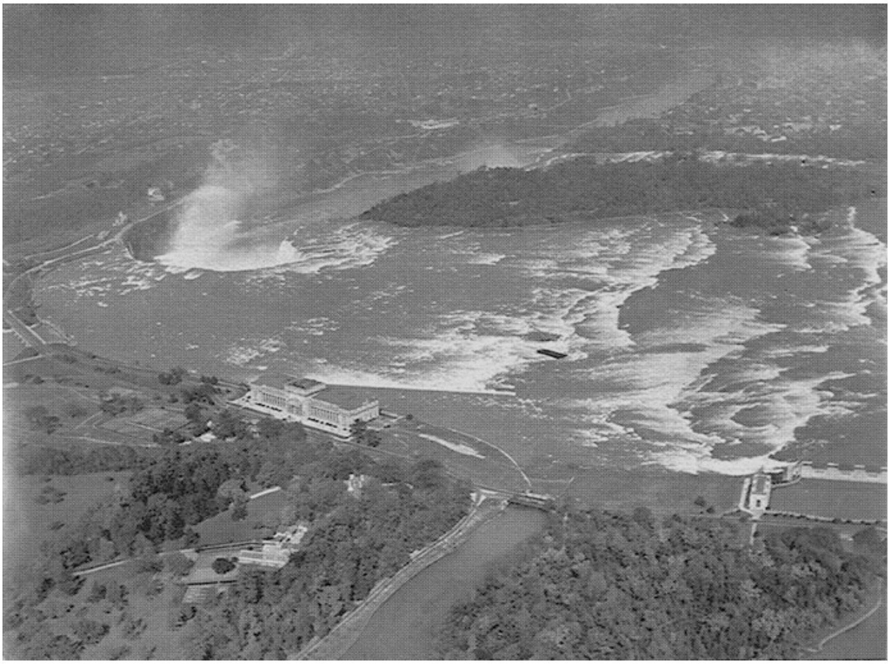


Fig. 9 Image of Niagara Falls. Image from Fields (1928b) and reprinted with permission of the University of Toronto Press

He also noted, importantly, that "Le Congrès sera organisé conformément aux dispositions prévues par les statuts du Conseil international des recherches". That is to say, mathematicians from the erstwhile Central Powers (Germany, Austria-Hungary) would be barred from attendance. Fehr also held out the promise of a variety of scientific excursions after the Cogress, especially one to Vancouver (Fehr 1924a).

In the last issue before the IMC, Fehr noted that this would be the first time that an international mathematical congress would be held on the American continent, clearly discounting Chicago 1893. Fehr is also grateful that the organizing committee led by J.C. Fields has provided generous support for representatives of universities and learned societies to attend (Fehr 1924b).

In the March 1925 issue, he briefly summarized, as previously promised, "Le Congrès international de mathématiques de Toronto" (The Toronto International Congress of Mathematics).

> Le Congrès international de mathématiques qui vient d'avoir lieu à Toronto, sous les auspices de l'Université de Toronto et de L'Institut Royal Canadien, a réuni plus de quatre cents mathématiciens. Grâce au généreux appui du Comité canadien, un grand nombre de sociétés savantes et de hautes Ecoles ont pu se faire présenter au Congrès.
>
> The International Congress of Mathematics that has just taken place in Toronto, under the auspices of the University of Toronto and the Royal Canadian Institute, brought together

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Fig. 10 Transcription of the Table of Contents from Proceedings (Fields 1928b)

> more than four hundred mathematicians. Thanks to the generous support of the Canadian Committee a large number of learned societies and higher institutions were represented at the Congress (Fehr 1925a).

From Fehr's overview of Congress business we note that Fields, as chairman of the Organizing Committee, spoke to "la séance solenelle d'ouverture" (the formal opening session) and was elected President of the acclamation and, at the General Assembly of the International Mathematical Union, he was elected its Honorary President. Bortollotti was placed on the Bibliographical Commission and Fehr himself became a vice-president of the Union.

Because the City of Toronto hosted the Annual Meeting of the British Association at the same time, members of both congresses had the opportunity to meet in many sessions and to mingle at the fine receptions and congress excursions.

Fehr fondly remembers "la brilliante soirée arranged by the University organisée par l'Université dans les belles salles de Hart House." (the brilliant soirée in the beautiful rooms of Hart House) and of the Transcontinental Voyage he says in English "It was a most wonderful Trip" (Fehr 1925a).

EDITORIAL COMMITTEE	
Professor J. C. Fields, *Chairman*	University of Toronto
Professor R. C. Archibald	Brown University
Professor G. D. Birkhoff	Harvard University
Professor Ettore Bortolotti	University of Bologna
Professor J. Chapelon	Ecole Polytechnique and University of Toronto
Professor A. B. Coble	Johns Hopkins University
Professor D. R. Curtiss	Northwestern University
Professor L. P. Eisenhart	Princeton University
Professor E. R. Hedrick	University of California (Southern Branch)
Professor L. C. Karpinski	University of Michigan
Professor A. J. Kempner	University of Colorado
Professor A. E. Kennelly	Harvard University
Professor F. R. Moulton	University of Chicago
Mr. J. Patterson	Dominion Meteorological Observatory
Professor T. M. Putnam	University of California
Professor H. L. Rietz	University of Iowa
Professor J. L. Synge	University of Toronto

Fig. 11 List of Editorial Committee Members (Fields 1928b)

> Fort bien organisée dans les moindres détails, ce voyage laissera un souvenir inoubliable à tous ceux qui ont eu le privilège d'y prendre part. Que M. le Prof. Fields ..., le père du Congrès et leur chef pendant pendant le voyage transcontinental, reçoive ici, l'expression de toute notre gratitude.
>
> Incredibly well-organised down to the smallest details, this trip will leave an unforgettable memory for all of those who had the privilege of joining it. And may Prof. Fields, the father of the Congress and our leader during the Transcontinental Voyage, receive here the expression of all our thanks (Fehr 1925a).

Fehr concludes with a listing of all the plenary lectures and sessional papers (Fehr 1925a).

The copy of Fehr's book that I consulted is an offprint to be found among the Old Classification in the subbasement of the University of Toronto's Gerstein science library. It's inscribed in English "with kindest remembrances/H. Fehr" and stamped "Library/University of Toronto/August 24, 1925" (Fehr 1925b).

Fehr's contribution to the session (see Fig. 12), "L'Université et la préparation des professeurs de mathématiques" (The University and the Preparation of Mathematics Teachers), deals with the education of secondary school mathematics teachers. Fehr believed that they should study in depth the fundamental principles of mathematics as well as mathematical methodology and pedagogical principles. With the candidates taking an active role, their sessions should consist of the study of basic concepts, the role of definitions and the examination of classical mathematics treatises (Fehr 1928). Unfortunately, the *Proceedings* only provides an abstract.

Fehr had previously given papers at the 1904, 1908 and 1912 International Mathematics Congresses and would again in 1932.

L'UNIVERSITÉ ET LA PRÉPARATION DES PROFESSEURS DE MATHÉMATIQUES

PAR M. HENRI FEHR
Professeur à l'Université de Genève, Genève, Suisse.

L'auteur examine le role qui doit jouer L'Université dans la preparation des professeurs de mathématiques de l'enseignement secondaire. Bien que la recherche scientifique doive rester au premier plan du but de l'enseignement supérieur, l'Université ne doit pas perdre de vue sa mission vis a vis de l'enseignement secondaire auquel elle doit fournir de bons maîtres.

Lautre s'attache plus particulièrement a la partie scientifique de la préparation professionelle des maitres. Elle doit comprendre not arment une étude approfondie des principes fondamentaux des mathématiques, ainsi que de la méthodologie et de la didactique mathématique. Cet enseignement ne doit pas être doetrne sous la forme d'un cours ayant un caractère dogmatique, mals plueňe sous la forme de conférences auxquelles les candidate eux-mêmes sont appelés à prendre une part active. C'est ici qu'il convient d'appliquer la devise américaine *learning by doing* (apprendre un agissant). Ces conférences, faites sous la direction d'un professeur, suivant un plan bien cordonné, comprendront, par exemple, l'étude des concepts fondamentaux, le role des définitions en mathématiques, l'esamen de traites classiques en usage dans des principaux paya, etc.

Il y a aussi lieu de signaler l'avare de la Commission internationale de l'enseignement mathématique et de faire connaitre les documents relatifs aux pays environnants.

Fig. 12 Henri Fehr's 1924 IMC Abstract Transcribed (Fehr 1928)

Fehr was interested in the preparation of teachers but he was a mathematician in his own right. His mathematical accomplishments can be represented by two books available in the University of Toronto's main science collection. The first is *Application de la méthode vectorielle de Grassman à la géométrie infinitésimale* (The Application of Grassman's Vector Methods to Infinitesimal Geometry) which was Fehr's *Docteur ès sciences* thesis at the University of Geneva. Published in Paris in 1899, it entered the University of Toronto Library September 26, 1903. Fehr says:

> [Dans l]a méthode de Grassman ... la multiplication extérieure et la multiplication intérieure jouent un role très important. Elle conduit aisément à la résolution d'une foule de problèmes... en Géométire, mais encore dans toutes les branches qui se rattachent à la science de l'étendue.
>
> In Grassman's method the outer and inner products play an important role. It leads readily to the resolution of a host of problems ... not only in Geometry but also in other studies that depend on extension in space (Fehr 1899).

It's a very *algebraic* geometry, with only seven sketchy diagrams on its 91 pages and Fehr cites Peano's 1888 *Calculo geometrico*. After an introduction to the basics (vector operations, determinants, equations of curves and surfaces) Fehr describes the differential geometry of space curves and surfaces, especially various curvatures. It's very reminiscent of my *Schaum's Outline of Vector Analysis*.

Fehr's *Enquête de L'Enseignement Mathématique sur la méthode de travail des mathématiciens* (*Enseignement Mathématique*'s Investigation Into the Working Methods of Mathematicians) which was published in 1908, asked 30 questions, followed by the answers of various European and North American mathematicians, both signed and anonymous, accompanied by quotations of deceased mathematicians (Fehr 1908).

In 1932, Fehr memorialized both Fields and Peano on the same page of *L'Enseignment mathématique*. After reminding readers that Fields had been President "in 1924, of the 7th International Mathematics Congress", Fehr declared Fields to have been a "Loyal friend of Europe" and a devoted supporter of international scholarly cooperation. "In difficult circumstances ... he had organized the Toronto Congress and had carried through to publication the two magnificent volumes" of *Proceedings* (Fehr 1932).

When Fields saw that the Congress had had a financial surplus, he had decided, with the necessary approvals, that "this balance would establish a fund to award, every four years, two prizes in Mathematics in the form of gold medals." He had hoped to present his idea to the 1932 Zurich ICM. Fehr says "Thus it is with deep sorrow, when on arrival at the Zurich Congress, we learned of the premature death of J. C. Fields. Everyone, who had the privilege of travelling to Toronto eight years ago, will remember him strongly and gratefully" (Fehr 1932).

Of Peano, Fehr declared: "His contributions to the principles of analysis have become classics and are justly admired for their clarity and brilliant simplicity". Peano was also a dedicated popularizer of both differential geometry and Interlingua. Strangely no mention is made of Peano's fundamental work in philosophy of mathematics (Fehr 1932).

5.2 Florian Cajori (1859–1930)

Florian Cajori (University of California, Berkeley) published two complete papers in the *Proceedings*, both dealing with mathematical notation: "Uniformity of Mathematical Notations – Retrospect and Prospect" and "Past Struggles Between Symbolists and Rhetoricians in Mathematical Publications".

In "Symbolists and Rhetoricians" (see Fig. 13) Cajori focuses on the struggle between English geometers: from William Oughtred's 1648 "translation of the tenth book of Euclid into language largely ideographic, using about forty new symbols", to Robert Simson's 1756 *Elements*, "presenting Euclid unmodified, he avoided all mathematical signs". By the late nineteenth century and presently English geometries contain a moderate amount of symbolism, a victory of the golden mean.

Though Peano's *Formulaire de mathématiques* "practically disposes with ordinary language and expresses all propositions of mathematics by means of a small number of signs, the efforts of the previous forty years to express all mathematics in ideographic form was not being supported by mathematicians in general (Cajori 1928b).

PAST STRUGGLES BETWEEN SYMBOLISTS AND RHETORICIANS IN MATHEMATICAL PUBLICATIONS

BY PROFESSOR FLORIAN CAJORI
University of California, Berkeley, California, U. S. A.

For many centuries there has been a conflict between individual judgements, on the use of mathematical symbols. On the one side there are those who, in geometry, for instance, would employ hardly any mathematical symbols, on the other side are those who insist on the use of ideographs and pictographs almost to the exclusion of ordinary writing. The real merits or defects of the two extreme views cannot be ascertained by *a priori* argument; they rest upon experience and must therefore be sought in the study of the history of our science.

Fig. 13 Transcription of the Start of Florian Cajori's Second 1924 IMC Paper (Cajori 1928b)

UNIFORMITY OF MATHEMATICAL NOTATIONS
RETROSPECT AND PROSPECT

BY PROFESSOR FLORIAN CAJORI
University of California, Berkeley, California, U. S. A.

In mathematical notations mathematicians are not profiting by the teachings of history. As one surveys mathematical writings of the last five centuries, certain facts arrest the attention. It is noticeable, for example, that no one individual can invent an extended system of symbols which all mathematicians will adopt. W. Oughtred used one hundred and fifty symbols, many of his own design. Of the latter only one, the St. Andrew's cross for multiplication is still in general use. The long lists of symbols framed by P. Hérigone in the seventeenth century, and by C. F. Hindenburg in the eighteenth century have passed away. These and more recent experiences indicate that mathematical symbols, being for community use, must be adopted by the community; they cannot be forced on it.

Fig. 14 Transcription of the Start of Florian Cajori's First 1924 IMC Paper (Cajori 1928a)

"Retrospect and Prospect" (see Fig. 14) is more programmatic. Its examples are more algebraic and analytical, such as the various forms of the letter "D" in and around calculus, with partial derivatives alone having 35 varieties of notation.

Arthur Cayley's committee reported in 1875 to the British Association that "'uniformity in notation tends toward a common language and would assist the dissemination of mathematical knowledge.' Though our mathematical sign language is heterogeneous and contradictory, this lack is supplied by the spirit of the mathematician. Much might have been achieved with greater symbolic uniformity. Mathematicians must break with their extreme individualism on a matter intrinsically communistic, and organize strong international committees to adopt new and reject outgrown symbols, with publications acting accordingly" (Cajori 1928a).

COMMUTATIVE CONJUGATE CYCLES IN SUBGROUPS OF THE HOLOMORPH OF AN ABELIAN GROUP

BY PROFESSOR G. A. MILLER
University of Illinois, Urbana, Illinois, U.S.A.

If K represents any regular substitution group the holomorph H of may be defined as the substitution group composed of all the substitutions on the letters of K which transform K into itself, and the group of isomorphisms of K may be defined as the subgroup formed by all the substitutions of H which omit a given letter. In the present article it will be assumed that K is Abelian, and all the subgroups H of which involve K and have the property that all their conjugate cycles are commutative will be determined. For the sake of clearness we shall first consider the case when K is cyclic and has an order of the form p^m, p being a prime number.

Fig. 15 Transcription of the Start of George Abram Miller's IMC Algebra Paper (Miller 1928a)

HISTORY OF SEVERAL FUNDAMENTAL MATHEMATICAL CONCEPTS

BY PROFESSOR G. A. MILLER
University of Illinois, Urbana, Illinois, U.S.A.

One of the most fundamental practices in mathematics is the utilization of a symbol for an unknown number, both as an operand and also as an operator. In the work of Ahmes there appear various examples in which an unknown is used as an operand, giving rise to equations of the first degree in which we find practically a special symbol for the unknown with the suggestive meaning of heap. When the unknown is squared, or two unknowns are multiplied together, the unknown is evidently used as an operator as well as an operand. In these forms it is found in Egyptian papyri which may be as old as the work of Ahmes itself.

Fig. 16 Transcription of the Start of George Abram Miller's IMC History Paper (Miller 1928b)

5.3 George Abram Miller (1863–1951)

Miller's Section VI paper is on the related topic: "History of Several Fundamental Mathematical Topics" His other paper (see Fig. 15) was delivered to Section I Algebra, Theory of Numbers, Analysis: "Commutative conjugate cycles in subgroups of the holomorph of an Abelian group", another contribution to the basic development of Group Theory (Miller 1928a).

Miller's history paper (see Fig. 16) displays a broader historical range than Cajori's papers. Starting with Ahmes', ancient Egyptian use of "heap" for the concept of the unknown, he also notes Diophantus' use of "number" and Aryabhata's "small sphere". Miller also examines the concepts of number, system of postulates, function and (of course) group.

Although the concept of "group" is older than "unknown" a special name appears only with Ruffini's late eighteenth century "permutation". Cauchy used "system of conjugate substitutions." The "group" and "unknown" concepts are in close contact from the closure property of any group, first defined in 1912 in Weber's *Algebra*. A group centres attention on totalities and hence it tends to larger views.

Miller objects to stating that Greeks solved the quadratic equation or that the geometric solution to the cubic is a mediaeval Arab discovery. Greeks and Arabs had not reached the stage when complex roots could be considered.

For Ahmes, numbers were a group with respect to multiplication. The extension to a group under addition was achieved with a satisfactory theory of negative numbers at the beginning of the nineteenth century. With omitting the identity of addition comes the domain of rationality.

Each of the five fundamental concepts is related to both elementary and advanced mathematics, exhibiting the continual enrichment of the elementary by the higher parts (Miller 1928b).

5.4 Giuseppe Peano (1858–1932)

Giuseppe Peano notes in his short paper (see Fig. 17) or long abstract "De Aequalitate" (On Equaltiy):

> "L'articolo qui pubblicato è scritto in <Latino sine flexione> nella quale lingua tutte le parole sono Latine sotto forma del tema (ablativo o imperative); non c'è grammaticà." (The article published here is written in 'latin without inflection' in which language all the words are Latin in a set form (ablative for nouns or imperative for verbs); it's not grammatical).

Peano's two previous ICM papers "Logica matematica" (Mathematical Logic) presented in 1897, and "Delle Propisizioni esistenzali" (On Existence Proposition), presented in 1912, were delivered in Italian, though at the latter ICM he tried unsuccessfully to convince the powers that be to allow "Latine sine flexione".

DE AEQUALITATE

DEL PROFESSORE G. PEANO
Univesrsità di Torino, Torino, Italia

Aequalitate es indicato per ae, initiale de "aequatur" deformato in ∞ ab Vieta (mortuo in 1603) ad Leibniz (m. 1716). Recorde, a. 1557, introduce signo = , adoptato ab Newton (m. 1727), et nunc de usu universale.

Relatione = habe tres proprietate sequente:

1. x=x
2. si x=y, tunc y=x
3. si x=y, et y=z, tunc x=z.

Fig. 17 Transcription of the Start of Peano's 1924 IMC Abstract (Peano 1928)

In 1912, Peano wrote (see Fig. 18) to Bertrand Russell, who was chairing Peano's IMC session using the "THE FIFTH INTERNATIONAL CONGRESS OF MATHEMATICIANS / CAMBRIDGE, 1912" letterhead (Peano 1912):

> Dear Sir
>
> My *latino sine flexione* is *Italian*, for it is intelligible at first glance by any Italian. It is more Italian than the *Second Circular*. I could have it attested that it is Italian by the Italians present at the Congress. But its advantage over Italian is that it is also intelligible to non-Italians. I will be able to deliver a declaration that it agrees with the rules, and is a communication. The latter is very short and I believe it will be a worthwhile experiment.
>
> Yours devotedly
>
> "G. Peano"

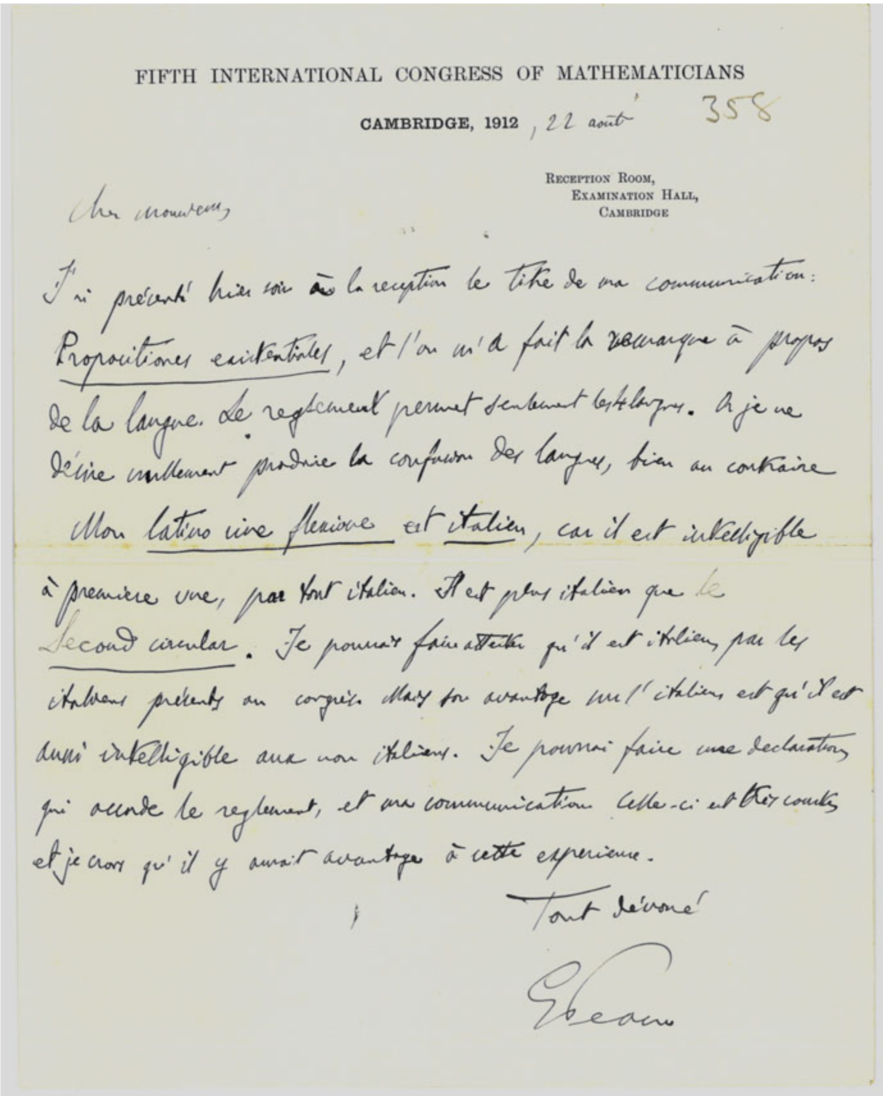
In his 1924 paper (Fig. 17) Peano began: "Aequalitate es indicate per ae, intiale de <aequatur>" (Equality is indicated by "ae", the beginning of "aequator"). He goes on to say that Recorde introduced parallel lines for equality in 1557, to was used by Newton from there became universally accepted.

Peano defines the equality relation and rewrites it in logico-mathematical symbols and describes and sources the transitivity, symmetric and reflexive properties. These three properties are independent, as Peano demonstrates, by showing there are relations that have *two* properties but not all three, using square arrays of plus and minus signs (Peano 1928).

Greater detail and insight can be found in the University of Toronto Mathematics Library's copy of Peano's *Notation de logique mathématique: Introduction au Formulaire de mathématique* (1894). This copy that has a bookplate from Stillman Drake, the great Galileo scholar and shows a price of $4.00. The *Formulaire*, says Peano, was conceived as an answer to Leibniz's "projet de créer une écriture universelle, dans laquelle toutes les idées composées fussent exprimées au moyen des idées simples, selon des règles fixes" (project to create a universal system of writing in which all composite ideas would be expressed by means of accepted signs for simple ideas, flowing fixed rules) (Peano 1894).

Peano provides clear definitions of his symbols starting with Classes of numbers and running through their Relations and Operations, Logical Operators, Propositions, Functions and Inverse Functions and finally Definitions. For proofs, "Les règles de la logique, pour transformer un ensemble d'hypothèses dans la thèse a prouver, sont analogues aux lois de l'Algèbre pour transformer un ensemble d'équations ...." (The rules of logic, for transforming a set of hypotheses into the result to be proved, are analogous to the laws of Algebra for transforming a set of equations ....).

Peano concluded that Leinbnz's problem is thus solved, for reduced to symbols the propositions would take up less space than any bibliography on the given topic, though for historic reasons the author's name could be attached to each result (Peano 1894). As we saw earlier, Cajori didn't share his assurance.


FIFTH INTERNATIONAL CONGRESS OF MATHEMATICIANS

CAMBRIDGE, 1912, 22 août 358

RECEPTION ROOM,
EXAMINATION HALL,
CAMBRIDGE

Cher monsieur,

J'ai présenté hier soir à la reception le titre de ma communication: Propositiones existentiales, et l'on m'a fait la remarque à propos de la langue. Le reglement permet seulement les 4 langues. Or je ne désire nullement produire la confusion des langues, bien au contraire. Mon latino sine flexione est italien, car il est intelligible à première vue, par tout italien. Il est plus italien que le Second circular. Je pourrais faire attester qu'il est italien, par les italiens présents au congrès. Mais son avantage sur l'italien est qu'il est aussi intelligible aux non italiens. Je pourrai faire une declaration qui accorde le reglement, et ma communication. Celle-ci est très courte, et je crois qu'il y aurait avantage à cette experience.

Tout dévoué

G Peano

Fig. 18 Peano's Letter to Russell. Reprinted by permission of the William Ready Division of Archives and Research Collections, McMaster University Library, Hamilton, Canada

6 Conclusion

In this paper you've encountered some of the many connections of the History and Philosophy of Mathematics sessions, at the 1924 International Mathematical Congress in Toronto and there are many more areas in the *History* of the History and Philosophy of Mathematics that can be pursued.

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