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8

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8.1. The Classical Story

One of the loveliest parts of mathematics is the subject of projective curves, developed, notably, by Jacobi, Abel, Riemann, and Poincaré over most of the nineteenth century. "Projective curve" means that you take an irreducible polynomial $p \in \mathbb{C}[x_1, x_2]$, bring each of its terms up to top degree by means of an auxiliary variable x_0, and think of the vanishing of the homogeneous polynomial $p(x_0, x_1, x_2)$ so produced as cutting out a locus or "curve" $\mathbb{X}$ in the projective space $\mathbb{CP}^2$ of triples $(x_0, x_1, x_2) \in \mathbb{C}^3 - 0$ with identification of (complex) lines $c(x_0, x_1, x_2)$: $c \neq 0$ to single points. $\mathbb{X}$ is then a compact, complex manifold of (complex) dimension 1, with possible singularities which may be resolved in a routine way, and conversely: any such manifold arises in this way. Alternatively, the solution x_2 of $p(x_1, x_2) = 0$ is a many-valued "algebraic" function of x_1 and $\mathbb{X}$ is its "Riemann surface".

$\mathbb{X}$ is an (oriented) sphere with $0 \leq g < \infty$ handles attached, this number being its genus. It carries a field $\mathbb{K}$ of functions of "rational character", imitating the common rational functions $\mathbb{K} = \mathbb{C}(x)$ on the sphere ($g = 0$), but more complicated, e.g. for the torus ($g = 1$), $\mathbb{K}$ is an elliptic function field of the form $\mathbb{K} = \mathbb{C}(x)[\sqrt{(x-e_1)(x-e_2)(x-e_3)}]$ with distinct numbers e. Here, it is already a remarkable fact that the algebra carries within it all the geometry: if the $\mathbb{K}$'s are isomorphic *as fields*, then the underlying curves $\mathbb{X}$ have a 1:1 onto "morphism" between them, rationally expressible both forward and back.

$\mathbb{X}$ also carries exactly g independent differentials of the first kind (DFK) of the form $\omega = f(z)dz$ with pole-free f on any little patch with "local parameter" z. These may be organized as follows. Let $a_1, \ldots, a_g$ and $b_1, \ldots, b_g$ be a standard homology basis of $\mathbb{X}$, the a's/b's passing around/through the holes. The standard basis of DFK is specified by requiring $[a_i(\omega_j) : 1 \leq i,\ j \leq g] =$ the identity.

Now form the "Abel sum" as follows: Take (1) a base point $\mathfrak{o} \in \mathbb{X}$, fixed, (2) a variable "divisor" $\mathfrak{P}$ comprising so and so many points $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ of $\mathbb{X}$, (3) paths from $\mathfrak{o}$ to each $\mathfrak{p}$, and write

$$\int_{\mathfrak{O}}^{\mathfrak{P}} \omega \equiv \sum_{i=1}^{n} \int_{\mathfrak{o}}^{\mathfrak{p}_i} (\omega_1, \ldots, \omega_g) = \mathfrak{x} \in \mathbb{C}^g.$$

Here, the paths of integration produce ambiguities, which may be removed by considering $\mathfrak{x}$ modulo "periods", i.e. modulo the lattice $\mathfrak{L} \subset \mathbb{C}^g$ produced by closed paths, from $\mathfrak{o}$ and

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back again, in each summand. In this way, $\mathfrak{x}$ is reduced to an unambiguous point of $\mathbb{C}^g/\mathfrak{L} \equiv$ Jac = the Jacobi variety of $\mathbb{X}$, this being a compact torus because the several periods $a(\omega)$ and $b(\omega)$ span C^g over $\mathbb{R}$. The beautiful theorem of Abel is expressed in this language: $\mathfrak{P}/\mathfrak{Q}$ is the divisor of poles/roots of a function $f \in \mathbb{K}$ if and only if they comprise an equal number of points and have the same image in Jac. Riemann-Roch may also be mentioned here. It states that the class $F \subset \mathbb{K}$ of functions with poles $\mathfrak{P}$ or softer and the class $D \subset$ DFK of differentials of the first kind with these roots or harder are related by $\dim F =$ the number of points in $\mathfrak{P} + 1 - g + \dim D$, showing (in part) how $\mathbb{K}$ and DFK are intertwined.

The next item in the classical story is Riemann's theta function: with the prior normalization $A = [a_i(\omega_j)] =$ the identity, it turns out that $B = [b_i(\omega_j)]$ is symmetric, with positive-definite imaginary part, guaranteeing the rapid convergence of the sum

$$\vartheta(\mathfrak{x}) = \sum_{\mathfrak{n} \in \mathbb{Z}^g} e^{2\pi\sqrt{-1}\mathfrak{n}\bullet\mathfrak{x} + \pi\sqrt{-1}\mathfrak{n}\bullet B\mathfrak{n}} = \text{“theta”}$$

for $\mathfrak{x} \in \mathbb{C}^g$. This is almost a function on Jac itself: ϑ is unchanged by addition to $\mathfrak{x}$ of any "real" period $a(\omega)$, while addition of any "imaginary" period $b(\omega)$ multiplies it by a simple exponential factor. It follows that the vanishing of ϑ cuts out a sub-variety Θ of Jac, the so-called "theta divisor". This may be described by Riemann's "vanishing theorem" to the effect that if $\mathfrak{P}$ is comprised of exactly g points $\mathfrak{p}_1, \dots, \mathfrak{p}_g$ in general position and if $\mathfrak{x}$ is its image in Jac, then, with a suitable fixed "Riemann constant" K, $f(\mathfrak{p}) = \vartheta(x - \int_{\mathfrak{o}}^{\mathfrak{p}} + K)$ vanishes simply at $\mathfrak{p} = \mathfrak{p}_1, \dots, \mathfrak{p}_g$ and no place else. This fact can now be used to express the general function $f \in \mathbb{K}$ as a ratio of products of translates of ϑ, comparable to the way an ordinary rational function is written as a ratio of products of translates of x. Here it must be mentioned that ϑ contains all the information about $\mathbb{X}$ in a deeper sense: if the imaginary period matrix $B = [b_i(\omega_j)]$ is known, then so is $\mathbb{X}$, up to a morphism. This is "Torelli's theorem".

These are the bare bones of the classical story.

8.2. Bigger and Better Curves

Think about curves which are 2-sheeted covers of the plane, of which Jacobi's elliptic curve $y^2 = (1-x^2)(1-k^2x^2)$ with $k \neq 0, 1, \infty$ is the genus 1 prototype. But why just a polynomial in x? Why not $y^2 = \Pi_{n\neq 0}(1 - x^2/n^2)$? In short, why not a "transcendental" curve? Now the old (projective) compactness is lost and transcendental points of a new character appear where infinitely many handles pile up, as over $x = \infty$ in the present example.

Historically, a number of attempts were made to come to grips with such curves. Here are a few references: Nevanlinna [26], Myrberg [22, 23, 24], Ahlfors [2], Heins [11], Andreotti [4], and Accola [1]. But I think it is fair to say that these trials, gallant as they may have been, were none of them fully satisfactory, and this for three reasons: (1) It is necessary to deal with the piling up of handles at "transcendental" points of $\mathbb{X}$. (2) It is also pretty obvious that you must bring in transcendental functions to the function field. Already, H.F. Baker [6] understood this point, introducing the analogue of the exponential function which bears his name, together with that of Akhiezer [3], who reinvented it for another reason. (3) To extend the classical story to transcendental curves, it is, if not indispensable, then surely an enormous help to have in mind some concrete problem to which the machinery is to be applied as an aid to finding the best technical conditions: not so wide that the story comes to a stop, not so narrow that no application of any consequence can be accommodated. Baker [7] was almost successful in this respect, too: on p. 48, close inspection will reveal the solution of KdV expressed in the fashion of Its-Matveev [12] reported below. Unfortunately, KdV was unknown to him (Baker), so he did not understand what he had. But this is jumping ahead.

8.3. KdV and All That

A few connections between mechanics and projective curves were known in the nineteenth century. Jacobi in his *Vorlesungen über Dynamik* [13] had used Abel sums to separate variables in the Hamilton-Jacobi equation in connection with the geodesic flow on the surface of a 3-dimensional ellipsoid, etc. Turning things backwards, he even made a mechanical proof of the addition theorem for such integrals in the case of 2-sheeted curves. C. Neumann [25] employed the same trick to integrate a system of initially uncoupled harmonic oscillators, constrained to have their joint displacements move on a fixed sphere. Kovalevskaya's integration of her top by means of theta functions [15] is the most famous instance—this in the face of Picard's advice that theta functions are surely too simple to solve any interesting mechanical problem. These are all examples of "completely integrable" Hamiltonian mechanics, meaning that the system of (say) *2d* degrees of freedom has, in addition to its own Hamiltonian, d-1 more, independent, "commuting" constants of motion, where the adjective signifies that all the Hamiltonian flows produced by these commute, one with another. The number d is maximal in this regard: you cannot have more than d such flows in $2d$ dimensions. But if you *do* have so many, then (in principle) you can change coordinates to convert the original flow into straight-line motion at constant speed and map that back to solve your problem. The only trouble is that there is no effective way to be sure you have enough constants of motion and no effective recipe to find the correct change of coordinates. Jacobi complains of this in his *Vorlesungen*, where he says something like this: "We are supposed to be solving differential equations but of course we don't known how to do it. What we *can* do is look for some geometrically attractive substitution and, having found one that pleases us, seek out mechanical problems that could yield to this particular trick." The moral is that integrability, as defined above, is elusive. I much prefer Hermann Flaschka's practical definition, *to wit*: "You didn't think I could integrate that, but I can!" Take your choice. There are effectively no useful theorems, only a series of beautiful examples; indeed, the whole subject died a sad and early death when Poincaré showed that the 3-body problem is *not* integrable.

But then came startling news in Gardiner, Greene, Kruskal, and Miura [10] that the Korteweg-de Vries equation KdV: $\partial v/\partial t = 3v\partial v/\partial x - (1/2)\partial^3 v/\partial x^3$, descriptive of the leading edge of long waves in shallow water, had an apparently unlimited number of commuting constants of motion and could be integrated explicitly. Now the number of degrees of freedom is $2d = 2\infty$, and it is not enough to exhibit $d = \infty$ constants of motion. Somehow, you must have $d = \infty$ such *exactly*, not one more or less. Be that as it may, KdV was solved in a completely satisfactory way, using (and this is really odd) the quantum-mechanical apparatus of reflection and transmission coefficients ("scattering") for the allied "spectral problem" $-\psi'' + v\psi = \lambda\psi$. I have no space to explain this right now but will come back to it briefly at the end after describing what happens when v is periodic, of period 1, say.

KdV was first expressed in a clever way by P. Lax [16, 17]: with $L\psi = -\psi'' + v\psi$ and $P\psi = (3/2)(v'\psi + 2v\psi') - 2\psi'''$, KdV may be expressed in commutator language as $v^\bullet = L^\bullet = [P, L]$, in which I have obeyed Gelfand's rule that the right-hand side should read P(eter) L(ax). Here L is symmetric and P is skew, which leads at once to the easily verified surmise that the periodic/anti-periodic eigenvalues of L are constants of motion as v moves under KdV; in fact, this a complete list of such constants.

Now $L\psi = \lambda\psi$ is Hill's problem, studied by him in connection with the motion of the moon, and well-understood. There is a simple periodic ground state λ_0 followed by an infinite number of separated, alternately anti-/periodic pairs $\lambda_n^- \le \lambda_n^+$, tending to $+\infty$ like $\pi^2 n^2$. The whole discussion now centers upon a variant of Hill's "discriminant" Δ in the form

$$\Delta^2(\lambda) - 1 = \big(\lambda_0 - \lambda\big)\Pi_{n=1}^{\infty}\big(\lambda_n^+ - \lambda\big)\big(\lambda_n^- - \lambda\big)/\pi^4 n^4.$$

For each value of $\lambda \in \mathbb{C}$, $L\psi = \lambda\psi$ has two "multiplicative" solutions ψ_- and ψ_+ with multipliers $m_-/m_+ = \Delta \pm \sqrt{\Delta^2 - 1}$, meaning that $\psi(x+1) = m\psi$. Here, the irrationality $\sqrt{\Delta^2-1}$ comes in, and it is natural to think in terms of the transcendental curve $\mu^2 = \Delta^2 - 1$, lying in two sheets over the complex plane where λ sits. I write $\mathfrak{p} = (\lambda, \sqrt{\Delta^2-1})$ for points of $\mathbb{X}$ and $e(x, \mathfrak{p}) = \psi_\pm(x)$, the signature of the radical $\sqrt{\Delta^2-1}$ in $\mathfrak{p}$ dictating which of the two functions ψ is meant. With the normalization $e(0, \mathfrak{p}) \equiv 1$, this is the Baker-Akhiezer function I spoke of before. Over $\lambda = \infty$, $e(x, \mathfrak{p})$ is of the form $\exp[kx + o(1)]$ with $k = \sqrt{-\lambda}$, $1/k$ serving as local parameter there. This is the only transcendental point of $\mathbb{X}$ where holes pile up, and even there things are pretty nice: the holes are spaced more and more widely and (if v is smooth) of rapidly vanishing diameter, so as to approximate common double points and to be, so to say, self-effacing. The "finite" part of $\mathbb{X}$ imitates a sphere with handles, one such to each open spectral "gap" $\lambda_n^- < \lambda_n^+$, and on the "real oval" a_n covering $[\lambda_n^-, \lambda_n^+]$ is found (a) a simple "immovable" pole of $e(x, \mathfrak{p})$, independent of $0 \le x < 1$, and (b) a simple moving root $\mathfrak{p}_n(x)$ starting at $\mathfrak{p}_n(0) =$ the pole. The projection of this root to the spectral plane is the nth eigenvalue of $Le = \mu e$ subject to $e(x, \mathfrak{p}) = 0$; that is why it moves with x.

Now it is fortunate that

(1) There are plenty of differentials of the first kind.
(2) The moving root divisor of $e(x, \mathfrak{p})$ is in "real position"; i.e. it has just one point in each real oval a.
(3) For such divisors, Abel's sum can be adjusted so as to make perfect sense, mapping the divisor to the (highly compact) real part of a nice Jacobi variety.
(4) The composite map from (a) the class of velocity profiles with common constants of motion, i.e. with fixed Hill's anti-/periodic spectrum, to (b) the root divisor, to (c) the Jacobi variety is 1 : 1 and onto, and better still:
(5) v moves in a complicated way under KdV; likewise the motion of the root divisor is not simple, but (and this is quite a miracle) the corresponding point $\mathfrak{x} \in$ Jac moves in straight lines at constant speed, both under translation of v and under KdV.

It remains to undo this "substitution" to return from the simple motion of $\mathfrak{x}$ to the complicated motion of v. Riemann's theta function, adapted to infinite genus, comes in here. The commuting flows of v, by translation and by KdV, show up as infinitesimal directions $\mathfrak{n}_1$ and $\mathfrak{n}_2$ tangent to Jac, and if now $\mathfrak{x} \in$ Jac corresponds to the initial velocity profile, then the moving profile is

$$v(t,x) = -2\big(\partial^2/\partial x^2\big)\ell n\vartheta\big(\mathfrak{x} + x\mathfrak{n}_1 + t\mathfrak{n}_2\big).$$

This was worked out for $g < \infty$ open gaps, first by S.P. Novikoff [27] and later (independently) by McKean-van Moerbeke [18]. For the general transcendental case ($g = \infty$), see McKean-Trubowitz [19, 20]. The final recipe $v = -2(\ell n\vartheta)''$ is due to Its-Matveev [12]. This is the formula, cited before, that Baker [7] had, not knowing its hydrodynamical interpretation.

8.4. Other Examples

KdV is not the only problem that such machinery can solve. I have said that what we have is a body of examples, lacking much of a general theory, so all I offer here is a selection of these:

(1) Cubic-Schrödinger: $\sqrt{-1}\partial\psi/\partial t = \frac{\partial^2\psi}{\partial x^2} \pm |\psi^2|\psi$, coming from optics, waves in deep water, etc.;
(2) Sine-Gordon: $\partial^2\theta/\partial t^2 = \partial^2\theta/\partial x^2 + \sin\theta$, coming from nineteenth century geometry of surfaces of constant negative curvature, super-conductivity, fiber optics, etc.;
(3) Boussinesq: $\partial^2 v/\partial t^2 = \partial^2 v^2/\partial x^2 - (1/3)\partial^4 v/\partial x^4$, coming from long waves in shallow water once more;

(4) Camassa-Holm: $\partial v/\partial t + v\partial v/\partial x + \partial p/\partial x = 0$ with "pressure" $p(x) = \frac{1}{2}\int e^{-|x-y|}[v^2 + \frac{1}{2}(v')^2]dy$, much simpler than KdV and a lot closer to 3-dimensional Euler: $\partial v/\partial t + (v \bullet \operatorname{grad})v + \operatorname{grad} p = 0$ (Feynman's "dry water").
(5) Kadomtsev-Petviashvilli: $\frac{\partial}{\partial x_1}(\frac{\partial v}{\partial t} - 3v\frac{\partial v}{\partial x_2} - \frac{1}{2}\frac{\partial^3 v}{\partial x_2^3}) + \frac{3}{2}\frac{\partial^2 v}{\partial x_1^2} = 0$, a $(2+1)$-dimensional variant of KdV coming from lasers, etc.

For each of these systems, and more besides, the KdV story, a little modified, is repeated. The flow is expressed by a Lax-type pair $L^\bullet = [P, L]$, permitting the motion to be interpreted as an isospectral deformation of an associated spectral problem $L\psi = \lambda\psi$. The latter produces spectral invariants forming a complete list of commuting constants of motion; a (normally transcendental) multiplier curve comes in, comparable to the Riemann surface of Hill's $\sqrt{\Delta^2 - 1}$; and then the machine rolls: BA functions, root divisors, Abel sums, Jacobi variety and all: The multiplier curves of (1), (2), and (4) lie in two sheets over the plane having 1 or 2 transcendental points; that of (3) in three sheets, with two transcendental points; while (5) leads to effectively arbitrary curves with a limited number of transcendental points.

This sounds simpler than it really is: You have to *find* the Lax pair and its attendant spectral problem, in which connection I must mention the name of V. Zakharov and his indispensable role in this regard. The joke used to be: "You have a problem. Send it to Zakharov. If he sends you the Lax pair (waiting time 2 weeks), then it is integrable; if not, then it is not." This has been the best known way to hit upon Jacobi's "attractive substitution", in accord with Flaschka's "You didn't think I could integrate that, but I can!"

8.5. Aside on Complex Structure

What is it doing here? Nobody really knows, but here's an idle thought. Classical integrability with $2d$ degrees of freedom requires d independent, commuting constants of motion: $H_1, \ldots, H_d$. These functions have vanishing Poisson brackets

$$[H_i, H_j] = \frac{\partial H_i}{\partial P} \bullet \frac{\partial H_j}{\partial Q} - \frac{\partial H_i}{\partial Q} \bullet \frac{\partial H_j}{\partial P} = 0 \quad (i < j),$$

representing "d choose 2" partial differential constraints imposed upon are mere d functions, and if d is 100, say, then "d choose 2" is already 4950 of these. This over-kill reflects the delicacy of the situation (poke it and you break it) and may perhaps imply, what all experience confirms, that this over-kill entails some complex structure in the background.

Besides, it is not only in the periodic case that complex structure is present. I go back to KdV in the "scattering" case when $v(\pm\infty) = 0$. This was solved by GGKM [10] as reported before, put into a proper (integrable) Hamiltonian form by Faddeev-Zakharov [31], and reformulated by Dyson [8] in the following elegant manner, reminiscent of the Its-Matveev formula $v = -2(\ell n\vartheta)''$ of Sect. 8.3. Think of the velocity v as the potential in $\sqrt{-1}\partial\psi/\partial t = -\psi'' + v\psi$ and send in a wave $e^{-\sqrt{-1}kx}$ from $+\infty$: part will be reflected back off v in the form $s_-(k)e^{\sqrt{-1}kx}$ and part will be transmitted in the form $s_+(k)e^{-\sqrt{-1}kx}$; the (complex) numbers $s_\pm(k)$ are, *resp.* the "transmission" and "reflection" coefficients at (real) wave number k. Now turn on KdV: $\partial v/\partial t = 3v/(\partial v/\partial x) - (1/2)\partial^3 v/\partial x^3$. It turns out that the several numbers $s_+(k)$ are (commuting) constants of motion, while $s_-(k)$ is modified by a factor $e^{-\sqrt{-1}4k^3 t}$. Besides, in the absence of bound states, which I ignore for simplicity, the initial reflection coefficient determines the whole KdV flow, as in Dyson's version of the recipe: $v(t,x) = -2(\partial^2/\partial x^2)\ell n\vartheta$, where now ϑ is the Fredholm determinant $\det[I + w(\xi+\eta) : \xi, \eta \geq x]$ with

$$w(\xi + \eta) = \frac{1}{2\pi}\int e^{\sqrt{-1}k(\xi+\eta)} s_-(k) e^{-\sqrt{-1}4k^3 t} dk.$$

Now comparison with Its-Matveev is instructive: (1) $|s_-|^2 + |s+|^2 \equiv 1$, so it is only the phase of s_- that moves (in "straight lines" at constant speed), by addition of $2kx$ in response to translation by x, and by $-4k^3t$ in response to KdV, which I take to mean that the phase of s_- lives in some "Jacobi variety". (2) ϑ as a function of that phase must be Riemann's function. (3) There must be some kind of BA function and an Abel sum mapping its root divisor to phase s_-, and so forth. This speculation could be confirmed by a simple experiment: Take your favorite profile v, decaying rapidly at $\pm\infty$, periodize it as in $\overline{v}(x) = \sum_{\mathbb{Z}} v(x+np)$, apply Its-Matveev to that, and hope that Riemann's theta will pass over into Dyson's determinant as the period $p \uparrow \infty$. This was done most elegantly by Venakides [30] and in a more synthetic way by Ercolani-McKean [9]. In fact, Dyson's ϑ, properly complexified, imitates Riemann's ϑ in nearly every aspect, a nice instance being a nearly perfect analogue of Riemann's description of his theta divisor alluded to in Sect. 8.1, for which see Kempf [14]. But enough of such details.

What I want to suggest is that the language and the technology of projective curves, like everything stemming from complex structure, is extraordinarily robust: Pushed in favorable directions, it will enable you to recognize old projective friends going about their business in new ∞-dimensional settings—such friends as, formerly, you might have thought to have no algebraic meaning at all.

8.6. The Present Book

I come back to my theme: that to get a hold of transcendental curves, you need a specific problem you want to solve. It will tell you what to do: what technical conditions are favorable and, too, by what machinery to compute. Speaking for a moment classically, the introduction of moduli permits the most *efficient* description of a projective curve. But efficiency is in the eye of the beholder. It depends what you want to do. If you want to *compute* anything, efficient moduli are hopeless, since it is next to impossible ($g \leq 1$ excepted) to decode them into useful information. How about *inefficient* moduli? For example, a Hill's curve is determined by a single profile v with the right constants of motion. Horribly redundant to be sure, but what an advantage here with the whole machinery of Hill's equation behind you for help in computation.

The present book arises out of evidence obtained in this way, as to the "true" idea of a transcendental curve, at once effectively computable and flexible enough to cover all applications that have come to light so far. The class of curves described here is very wide. They are "small" deformations of a sphere with widely spaced double points (nodes), pasted together out of patches and handles under strict (but not too strict) rules of behavior, so that everything is true that should be true. I cannot enter now into the details, which are scrupulously explained, nothing too much or too little. It will be enough to say that, some few mysteries aside, $\mathbb{K}$, DGK, BA=Baker-Akhiezer, Abel's sum, Jacobi variety, Riemann's theta and its vanishing theorem, Riemann-Roch, and Torelli's theorem, too, all survive in a robust form. Here are included (1) very general 2-sheeted curves, as for KdV, but more; (2) "heat curves" connected to Kadomtsev-Petviashvilli [(5) of Sect. 8.4], imitating most any classical curve you could want; (3) "Fermi curves" which enter into a separate, very elaborate story about superconductivity, not explained here. These are the chief examples spelled out in detail.

The few complaints I have may be quickly told. (1) A little more chat would have been welcome, both mathematical and historical. The exposition is determinedly technical, and while it is lucid and done with much care, it could be discouraging to the immature reader. These may find Schmidt [28] helpful. (2) I would have liked to see more about Riemann-Roch, but for this there is the splendid article of Merkl [21]. (3) And last: the omission of an index makes it nearly impossible to browse.

Be that as it may. This is a big piece of work, brought to a very successful conclusion after some 10 laborious years, for which the authors have my warmest congratulations.

But isn't it all remarkable, this whole story? I mean, who would have guessed that projective curves could help us to understand long waves in shallow water, not to mention fiber optics, self-transparency, and so on and on. And who would ever have thought that physical problems of this type could be any guide to algebraic geometry. I will take just a moment to explain this last allusion. Riemann's theta function involves the imaginary periods $B = [b_i(\omega_j)]$. These Riemann matrices are special: They are symmetric with positive imaginary part, and something more, something hidden, owing to their origination from a curve. What is it? Well, there is an Its-Matveev-type formula expressing the solution of Kadomtsev-Petviashvilli in terms of Riemann's theta function, and this works *only if* the B appearing in the theta sum of Sect. 8.1 is a bona fide Riemann matrix. S.P. Novikoff conjectured this; the proof is due to Arbarello and de Concini [5] and to Shiota [29]. You never can tell.

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