Homoclinic and Heteroclinic Orbits for a Class of Singular Planar Newtonian Systems

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1 Introduction

The study of existence and multiplicity of solutions of differential equations possessing a variational nature is a problem of great meaning since most of them derives from mechanics and physics. In particular, this relates to Hamiltonian systems including Newtonian ones. During the past 30 years there has been a great deal of progress in the use of variational methods to find periodic, homoclinic and heteroclinic solutions of Hamiltonian systems. Hamiltonian systems with singular potentials, i.e., potentials that become infinite at a point or a larger subset of $\mathbb{R}^n$, are among those of the greatest interest. Let us remark that such potentials arise in celestial mechanics. For example, the Kepler problem with

$$V(q) = -\frac{1}{|q-\xi|}$$

has a point singularity at ξ ($q \in \mathbb{R}^n \setminus \{\xi\}$). In physics, the gradient ∇V of the gravitational potential is called a weak force.

Our presentation is based on [3, 5]. We are interested in conservative dynamical systems involving strong forces. A model potential in a neighbourhood of a singular point ξ is defined by

$$V(q) = -\frac{1}{|q-\xi|^{\alpha}},$$

where $\alpha \geq 2$.

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Consider an autonomous Newtonian system

$$\ddot{q} + \nabla V(q) = 0, \tag{HS}$$

where $q \in \mathbb{R}^2$. Assume that a potential V satisfies the following conditions:

(V_1) $V \in C^1(\mathbb{R}^2 \setminus \{\xi\}, \mathbb{R})$;
(V_2) $V(x) \to -\infty$ as $x \to \xi$;
(V_3) there exist a neighbourhood $\mathcal{N}$ of the point ξ and a function $U \in C^1(\mathcal{N} \setminus \{\xi\}, \mathbb{R})$ such that $|U(x)| \to \infty$ as $x \to \xi$, and $-|\nabla U(x)|^2 \geq V(x)$ for every $x \in \mathcal{N} \setminus \{\xi\}$;
(V_4) $V(x) \leq 0$ and $V(x) = 0$ iff $x \in \{a, b\}$, $a, b \in \mathbb{R}^2 \setminus \{\xi\}$;
(V_5) there is $V_0 < 0$ such that $\limsup_{|x|\to\infty} V(x) \leq V_0$.

The assumption (V_3) due to W.B. Gordon is called a strong force condition, see [2].

Let E denote the Sobolev space

$$\left\{q \in W^{1,2}_{\mathrm{loc}}(\mathbb{R}, \mathbb{R}^2) \colon \int_{-\infty}^{\infty} |\dot{q}(t)|^2 dt < \infty\right\}$$

with the norm

$$\|q\|_E = \left(|q(0)|^2 + \int_{-\infty}^{\infty} |\dot{q}(t)|^2 dt\right)^{\frac{1}{2}}.$$

Set

$$\Lambda = \{q \in E \colon q(t) \neq \xi \text{ for } t \in \mathbb{R}\}.$$

We can define a rotation number map

$$\{q \in \Lambda \colon q(\pm\infty) \in \{a, b\}\} \ni q \longrightarrow \mathrm{rot}_\xi(q) \in \mathbb{Z}$$

as follows. In the polar coordinate system with the pole ξ and the polar axis $l = \{x \in \mathbb{R}^2 \colon x = \xi + s \cdot \vec{\xi a},\ s \geq 0\}$ one has $q(t) = (r(t)\cos\varphi(t), r(t)\sin\varphi(t))$. We can assume that $r(t)$ and $\varphi(t)$ are continuous. (Polar angles are measured counterclockwise from l.) If $q(-\infty) = b$ and $q(\infty) = a$ then

$$\mathrm{rot}_\xi(q) = \left[\frac{\varphi(\infty) - \varphi(-\infty)}{2\pi}\right] + 1.$$

Otherwise,

$$\mathrm{rot}_\xi(q) = \left[\frac{\varphi(\infty) - \varphi(-\infty)}{2\pi}\right],$$

where $[\tau]$ is the integral part of τ. Let us remark that if $q(-\infty) = q(\infty)$ then $\mathrm{rot}_\xi(q)$ is the winding number of the curve q about ξ. Moreover, after a reparametrization, q can be considered to be the continuous image of S^1. Therefore one can associated with q its Brouwer degree with respect to ξ. The Brouwer degree of q equals $\mathrm{rot}_\xi(q)$.

Theorem 1 (See Conclusion 1.5 in [5]) *Under the assumptions* (V_1)–(V_5)*, the Newtonian system* (HS) *has at least two solutions which wind around* ξ *and join* $\{a, b\}$ *to* $\{a, b\}$*. One of them is a heteroclinic orbit joining the point a to the point b. The second is either heteroclinic with a rotation different from the first, or homoclinic.*

Theorem 1 is a generalization of the result given by Rabinowitz in [6] on the existence of homoclinic solutions in the case $a = b$.

2 Variational Approach

Our approach to the problem of existence and multiplicity of connecting orbits of (HS) is variational. Homoclinic and heteroclinic solutions are global in time. Moreover, they are critical points of an action functional. Therefore it is reasonable to use variational (global) methods to receive them.

For $q \in \Lambda$, set

$$I(q) = \int_{-\infty}^{\infty} \left(\frac{1}{2} |\dot{q}(t)|^2 - V(q(t)) \right) dt.$$

We define the family $\mathcal{F}$ as follows. A subset $Z \subset \Lambda$ is a member of $\mathcal{F}$ iff it has the following properties:

- $I(q) < \infty$ for all $q \in Z$,
- if $p, q \in Z$ then $p(-\infty) = q(-\infty)$ and $p(\infty) = q(\infty)$,
- for each $q \in Z$ and for each $\psi \in C_0^\infty(\mathbb{R}, \mathbb{R}^2)$ there exists $\delta > 0$ such that if $s \in (-\delta, \delta)$ then $q + s\psi \in Z$.

Of course $\mathcal{F}$ is nonempty. We see at once that for example:

$$\Gamma^+ = \{q \in \Lambda : q(-\infty) = a,\ q(\infty) = b \ \wedge \ rot(q) \geq 0\},$$

$$\Gamma^- = \{q \in \Lambda : q(-\infty) = a,\ q(\infty) = b \ \wedge \ rot(q) < 0\},$$

$$\Omega_a^{\pm n} = \{q \in \Lambda : q(\pm\infty) = a \ \wedge \ \pm rot(q) \geq n\},$$

$$\Omega_b^{\pm n} = \{q \in \Lambda : q(\pm\infty) = b \ \wedge \ \pm rot(q) \geq n\},$$

where $n \in \mathbb{N}$, are members of this family. Standard arguments show that if q is a minimizer of I on $Z \in \mathcal{F}$ then q is a classical solution of (HS).

Outline of the Proof of Theorem 1 Set

$$\gamma^{\pm} = \inf\{I(q) : q \in \Gamma^{\pm}\}.$$

Without loss of generality we can assume that $\gamma^- \leq \gamma^+$. Let $\{q_m\}_{m\in\mathbb{N}} \subset \Gamma^-$ and $\{\tilde{q}_m\}_{m\in\mathbb{N}} \subset \Gamma^+$ be minimizing sequences of I on Γ^- and Γ^+, respectively. There exist Q and $\tilde{Q}$ in Λ such that, going to subsequences if necessary, $q_m \rightharpoonup Q$ and $\tilde{q}_m \rightharpoonup \tilde{Q}$ in E. Both Q and $\tilde{Q}$ are connecting orbits of (HS). One can show that $Q \in \Gamma^-$. Whereas $\tilde{Q}$ may belong to Γ^-, Γ^+ or Ω_a^k for a certain $k \in \mathbb{N}$.

In the case $\tilde{Q} \in \Gamma^-$ we check that $\gamma^+ = I(\tilde{Q}) + \omega_b^n$, where $\omega_b^n = \inf\{I(q) : q \in \Omega_b^n\}$ and $n = -\mathrm{rot}_\xi(\tilde{Q})$. Moreover, the Hamiltonian system (HS) possesses either a homoclinic solution $p : \mathbb{R} \to \mathbb{R}^2 \setminus \{\xi\}$ such that $p(\pm\infty) = b$ and $\mathrm{rot}_\xi(p) > 0$, or a heteroclinic solution $Q_0 \in \Gamma^-$ such that $\mathrm{rot}_\xi(Q_0) < \mathrm{rot}_\xi(\tilde{Q})$ (see [5, Theorems 1.3, 1.4]). This connecting orbit is obtained as a weak limit of a minimizing sequence of the action functional I on Ω_b^n. □

Fix $Z \in \mathcal{F}$ such that if $q, p \in Z$ then $\mathrm{rot}_\xi(q) = \mathrm{rot}_\xi(p)$. Let

$$z = \inf\{I(q) : q \in Z\}$$

and $\{q_m\}_{m\in\mathbb{N}} \subset Z$ be a sequence such that

$$\lim_{m\to\infty} I(q_m) = z.$$

For each $i \in \mathbb{N}$, set

$$C_i = \overline{\bigcup_{m=i}^{\infty} q_m(\mathbb{R})}.$$

Define

$$SC = \bigcap_{i=1}^{\infty} C_i.$$

Theorem 2 (Shadowing Chain Lemma, See Lemma 3.2 in [3]) *Under the assumptions* (V_1)–(V_5), *there are a finite number of homoclinic and heteroclinic orbits,* $Q_1, Q_2, \ldots, Q_l$, *of the Newtonian system* (HS) *such that*

$$z = I(Q_1) + I(Q_2) + \cdots + I(Q_l)$$

and

$$rot_\xi(q_m) = rot_\xi(Q_1) + rot_\xi(Q_2) + \cdots + rot_\xi(Q_l).$$

Theorem 2 is the starting point to show the existence of infinitely many homoclinic and heteroclinic orbits to the Newtonian system (HS) under certain extra conditions of Bolotin's type (see [1]) on the existence of minimal noncontractible periodic orbits around ξ.

Theorem 3 (M. Izydorek, J. Janczewska) *Let $a = b$. Suppose that $V \in C^{1,1}(\mathbb{R}^2 \setminus \{\xi\}, \mathbb{R})$ satisfies the conditions (V_2)–(V_5) and, moreover,*

(B_1) *there are $T_1 \in (0, \infty)$ and $p_1 \in W^{1,2}([0, T_1], \mathbb{R}^2 \setminus \{a, \xi\})$ such that $p_1(0) = p_1(T_1)$, $rot_a(p_1) = 0$, $rot_\xi(p_1) = 1$ and $\int_0^{T_1} \left(\frac{1}{2}|\dot{p}_1(t)|^2 - V(p_1(t))\right) dt < \lambda_1$;*
(B_2) *there are $T_2 \in (0, \infty)$ and $p_2 \in W^{1,2}([0, T_2], \mathbb{R}^2 \setminus \{a, \xi\})$ such that $p_2(0) = p_2(T_2)$, $rot_a(p_2) = rot_\xi(p_2) = 1$ and $\int_0^{T_2} \left(\frac{1}{2}|\dot{p}_2(t)|^2 - V(p_2(t))\right) dt < \lambda_1$, where*

$$\lambda_1 = \inf\{I(q) \colon q \in \Omega_a^1 \ \wedge \ rot_\xi(q) = 1\},$$

and $rot_a(p_i)$, $rot_\xi(p_i)$ are the winding numbers (Brouwer's degree) of the curve p_i about the point a and ξ, respectively.

Under the above assumptions, there exist infinitely many homotopy classes in $\pi_1(\mathbb{R}^2 \setminus \{\xi\})$ containing at least two geometrically distinct homoclinic (to a) solutions.

The detailed proof of Theorem 3 is contained in the paper [4].

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