Beliefs about the nature of numbers

Lance J. Rips

Abstract Nearly all psychologists think that cardinality is the basis of number knowledge. When they test infants' sensitivity to number, they look for evidence that the infants grasp the cardinality of groups of physical objects. And when they test older children's understanding of the meaning of number words, they look for evidence that the children can, for example, "Give [the experimenter] three pencils" or can "Point to the picture of four balloons." But when people think about the positive integers, do they single them out by means of the numbers' cardinality, by means of the ordinal relations that hold among them, or in some other way? This chapter reviews recent research in cognitive psychology that compares people's judgments about the integers' cardinal and ordinal properties. It also presents new experimental evidence suggesting that, at least for adults, the integers' cardinality is less central than their number-theoretic and arithmetic properties.

1 The Structural Perspective and the Cardinal Perspective on Numbers

Numbers don't lend themselves to psychologists' usual way of explaining how we know about things. The usual explanation is perception. We gain knowledge of many physical objects—such as squirrels and bedroom slippers—by seeing them, and we gain knowledge of many other things by reasoning from perceptual evidence. But numbers and other mathematical objects leave no perceptible traces. Although we might consider mathematical objects as abstractions from things that we can perceive, this explanation faces difficult problems. How could such a psychological process be sufficiently powerful to give us knowledge of *all* the natural numbers (not to mention the numbers from other systems)? We can't possibly abstract them one-by-one. Not all cognitive scientists have given up the abstraction story (see, e.g., [25]), but it's safe to say that it is no longer the dominant view.

L.J. Rips (✉)
Psychology Department, Northwestern University, 2029 Sheridan Road, Swift 314, Evanston, IL 60208, USA
e-mail: rips@northwestern.edu

E. Davis, P.J. Davis (eds.), *Mathematics, Substance and Surmise*,
DOI 10.1007/978-3-319-21473-3_16

Most psychologists who study the development of number knowledge no longer think that we grasp numbers like ninety-five by generalizing from encounters with groups of ninety-five squirrels, ninety-five slippers, and other groups of that size. But these groupings still play a crucial role in current theories. According to these theories, children's ability to count objects is the pivotal step in their acquisition of true number concepts. Learning to count groups of objects—for example, saying "one, two, three, four, five: five squirrels," while pointing to the squirrels one by one—is supposed to transform children's primitive sense of quantities into adult-like concepts of the small positive integers [3]. Moreover, developmental psychologists' standard method for assessing children's knowledge of the meaning of number words like "five" is to ask the children to "Give me five balloons [or other small objects]" from a pile of many, or to ask the children to "Point to the card with five balloons" in the presence of one card with a picture of five balloons and a second card with a picture of four or six [46].

This emphasis on counting (and on picking out the right number of objects) is due to psychologists' belief that the fundamental meanings of number words are cardinalities, in line with earlier theories by Frege [11] and Russell [41]. The meaning of "five," for example, is the set of all sets of five objects (or some similar construction). What children learn when they learn how to count objects is a rule for computing the cardinality of collections for the number words they know. This rule—Gelman and Gallistel's "Cardinal Principle" [13]—is that the meaning of the final word in the count sequence is the cardinality of the collection. So the meaning of "five" is the set size you get when "five" is the final term in a correct application of the counting procedure. Asking a child to give you five balloons from a larger pile is a test of whether the child can use this counting procedure to arrive at the right total.

However, we should consider other possible paths to knowledge of number. Many contemporary philosophers argue that the meaning of a number is given by the position of the number in an appropriate number system [28, 30, 43]. Stewart Shapiro claims, for example, that "there is no more to being the natural number 2 than being the successor of the successor of 0, the predecessor of 3, the first prime, and so on . . . " [43, pp. 5–6], and Michael Resnik puts the point this way:

> The underlying philosophical idea here is that in mathematics the primary subject-matter is not the individual mathematical objects but rather the structures in which they are arranged. The objects of mathematics, that is, the entities which our mathematical constants and quantifiers denote, are themselves atoms, structureless points, or positions in structures. And as such they have no identity or distinguishing features outside a structure. [30, p. 201]

This structural perspective suggests that people's understanding of particular numbers depends on their knowledge of the relations that govern these numbers. In the case of the natural numbers, the key relations would include the facts that every number has a unique immediate successor, that every number except the first has a unique immediate predecessor, and so on. From this point of view, children wouldn't be able to grasp five merely by connecting the word "five" to the set of all five-membered sets, since this connection wouldn't succeed in establishing the critical relations that five bears to the other naturals (e.g., being the successor of four, the successor of the successor of three, and so on).

The distinction between the cardinal perspective and the structural perspective doesn't mean that no connection exists between them. *Frege's Theorem* establishes that the Dedekind-Peano axioms that define the structure of the natural numbers (no number precedes zero, every number has a unique immediate successor, etc.) are derivable in second-order logic from definitions of zero, the ancestral relation, and natural number itself, together with a central fact about cardinality called *Hume's Principle*. This is the idea that the number of things of one kind is the same as the number of things of another kind if and only if there is a one-one relation between these things. For example, the number of squirrels in your backyard is equal to the number of bedroom slippers in your closet if and only if there is a one-one relation between the squirrels and the slippers. (See [17, 47] for expositions of the proof of Frege's Theorem.) You could take this result to mean that cardinality (in the form of Hume's Principle) provides the basis for the structure of the naturals (in the form of the Dedekind-Peano axioms). However, Frege's Theorem by itself does not settle the issue of whether the cardinal perspective or the structural perspective is conceptually prior in people's understanding of the naturals. A version of Hume's Principle is provable from the Dedekind-Peano axioms and the same definitions [16]. So the formal results seem to give us no reason to favor the cardinal perspective over the structural perspective as the conceptually fundamental one (see [23] for a discussion of this issue).

The structural perspective has recently been defended by Øystein Linnebo [21] as a thesis about "our actual arithmetic practice." Linnebo points out that the cardinal perspective seems to predict incorrectly that zero should be easy for children to grasp. Even very young children understand expressions like "allgone cookies" or "no more cookies"; so they understand the cardinality associated with zero cookies [14]. But they take until the end of preschool to understand zero as a number, alongside other integers [45]. Why the age gap if zero is a cardinality [38]? Similarly, older grade-school children understand that the number of natural numbers is infinite [15], but they probably don't gain the concept of the number $\aleph_0$ for this cardinality until much later (if they ever do).

Linnebo has a second argument in favor of the structural perspective, one that's closer to the theme of the present chapter. He maintains [21, p. 228] that reference to a number in terms of its cardinality—for example, to five as the number of squirrels in the yard—doesn't seem "particularly direct or explicit":

> Rather, the only perfectly direct and explicit way of specifying a number seems to be by means of some standard numeral in a system of numerals with which we are familiar. Since the numerals are classified in accordance with their ordinal properties, this suggests that the ordinal conception of the natural numbers is more basic than the cardinal one.

As Linnebo notes, the appeal to directness in thinking about numbers relies on intuition, but he conjectures that results from cognitive experiments might back the claim for the immediacy of the ordinal conception (which we have been calling the "structural perspective").

This chapter asks whether any psychological evidence favors the structural perspective or the cardinal perspective.[1] We can begin by looking at theories and data on how children learn the meaning of the first few positive integers to see whether the evidence supports developmentalists' emphasis on cardinality. I then turn to recent experiments that have compared adults' judgments of cardinality to their judgments of order for further insight on whether our understanding of numbers depends more tightly on one or the other of these two types of information. Finally, the chapter describes some new studies that directly probe the properties that people take as central to number knowledge.

2 The Origins of Number Knowledge

As an example of the role that the cardinal perspective plays in theories of number knowledge, let's consider Susan Carey's influential and detailed proposal about how children learn the meanings of their first few number words [3]. Figure 1 provides a summary of the steps in this process, which Carey calls *Quinian bootstrapping*. In trying to understand the children's progress through this learning regime, let's start by figuring out what the process is supposed to achieve.

At two or three years old, kids are able to recite the numerals in order from "one" to some number like "ten" or "twenty," but they don't yet know how to produce numerals for arbitrary integers in the way you do. They have just a short, finite list, for example, "one, two, three, four, five, six, seven, eight, nine, ten." At this age, they don't understand the cardinal meanings of the words on this list; so if you ask them to give you two balloons from a pile, they can't do it. Then, over an extended period of time—as long as a year or so—they first work out the meaning for the word "one," then for the word "two," then "three," and sometimes "four," again in the sense of being able to give you one, two, three, or four objects in response to a command. At that point, something clicks, and suddenly they're able to give you five things, six things, and so on, up to ten things (or to whatever the last numeral is on their count list). The Quinian bootstrapping theory is supposed to explain this last step, when things finally click: It extends kids' ability to enumerate objects in response to verbal requests from three or four to ten. Post-bootstrap, kids still don't know many of the important properties of the positive integers, but at least they can count out ten things, more or less correctly. In other words, what they've learned is how to count out objects to determine the right cardinality.

[1]This issue might be put by asking whether people think of the first few naturals as cardinal or ordinal *numbers*. However, "cardinal number" and "ordinal number" have special meanings in set theory, and these meanings don't provide the intended contrast. In their usual development (e.g., [8]), the ordinals and cardinals do not differ in the finite range, with which we will be concerned in this chapter. (Both ordinal and cardinal numbers include transfinite numbers—for which they *do* differ—and so extend beyond the natural numbers.)

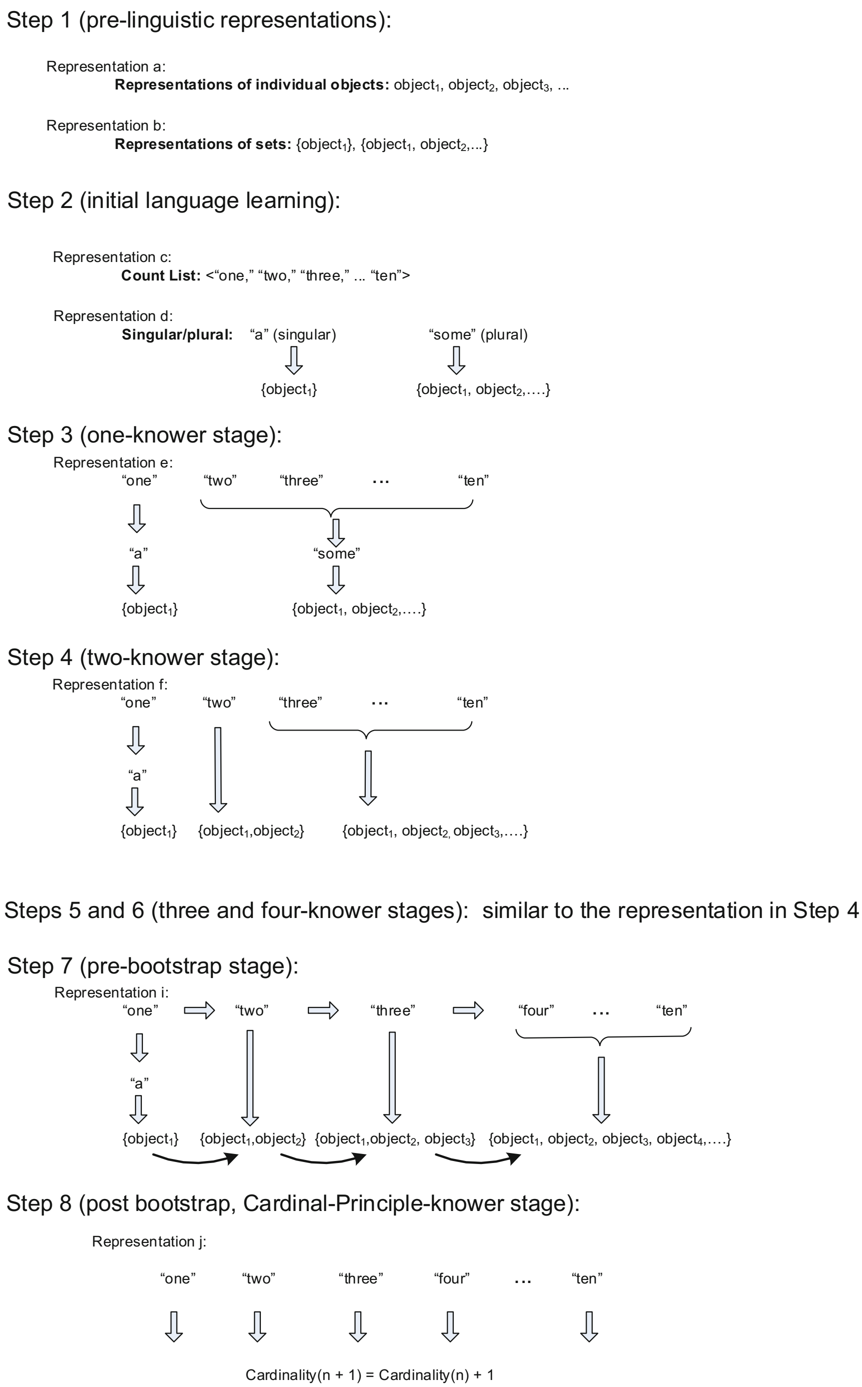


Fig. 1 Steps in the acquisition of the meaning of number words for the first few positive integers, according to Carey's [3] Quinian bootstrapping theory (figure adapted from [32]).

What are the steps that children are supposed to go through in graduating from their pre-bootstrap state of knowledge of number to their post-bootstrap knowledge? Here's a quick tour:

Step 1: At the beginning of this process, children have two relevant innate mental representations. Representations of type (a) in Figure 1 are representations of individual objects—for example, representations of each of three balloons. Representations of type (b) are representations of sets. I'll use $object_i$ to denote the mental representation of a single individual i, $\{object_1, object_2\}$ for the representation of a set containing exactly two individuals, and $\{object_1, object_2, \ldots\}$ to indicate the representation of a set containing more than one individual, but whose total size is unknown. Children at this stage also have representations for approximate cardinality, but they play no role in Carey's bootstrapping story. We'll discuss this approximate number system in Section 3 of this chapter.

Step 2: Language learning at around age two puts two more representations in play. The representation of type (c) in Figure 1 is the memorized list of number words, in order from "one" to some upper limit, which we'll fix for concreteness at "ten." We'll see that this list of numerals doesn't become important until quite late in the process, but it's in place early. The second new representation of type (d) is a mapping between the indefinite determiner "a," as in "a balloon" (or other singular marker in natural language), and the symbol for a singleton $\{object_1\}$.[2] Similarly, there's a mapping between the word "some" (and other plural markers in language) and the symbol for a set of unknown size, $\{object_1, object_2, \ldots\}$. This is the way the children learn the singular/plural distinction in number-marking languages like English, French, or Russian—the difference between "book" and "books."

Steps 3–6: In the next few stages, children learn the meanings of the words "one" through "three" or "four" by connecting them with mental representations of sets containing the appropriate number of things. First, the children think that the word "one" means what "a" means (i.e., $\{object_1\}$) and that the rest of the number words mean what "some" means (i.e., $\{object_1, object_2, \ldots\}$). At this stage, they can give you one balloon if you ask them to, but they're unable to give you two, three, four, or a larger number of balloons. They think all these latter number words mean the same thing, namely, some. But over a period of about a year, they learn to differentiate this second representation. "Two" comes to be connected with $\{object_1, object_2\}$. However, "three," "four," and so on, are still associated with an arbitrary set of more than two elements. Then they learn "three" and occasionally "four" in the same way.

Step 7: Finally, kids are able to notice that a relation exists between the sequence of numerals in the count list and the sequence they can form from their set-based

[2]You might wonder about the use of sets in this construction: Do young children have a notion of set that's comparable to mathematicians' sets? Attributing to children at this age a concept of set in the full-blooded sense would be fatal to Carey's claim that Quinian bootstrapping produces new primitive concepts (e.g., the concept FIVE) that can't in principle be expressed in terms of the child's pre-bootstrap vocabulary. We know how to express FIVE in terms of sets (see, e.g., [8]). So the notion of set implicit in representations like $\{object_1, object_2\}$ is presumably more restrictive than ordinary sets.

representations. They understand this connection because their parents and teachers have leaned on them to count small groups of objects. When children count to "two" while pointing to two squirrels in a picture book, their representation of one thing, $\{object_1\}$, is active, followed by their representation of two things, $\{object_1, object_2\}$. So kids begin to see a relation between the order of the numerals in their count list and the cardinality of the sets that these numerals denote.

Step 8: At long last, then, the children can figure out that advancing by one numeral in the count list is coordinated with adding one object to a set. In other words, what they have learned is how counting, by ticking off the objects with the count list, manages to represent a total. They now have a rule that directly gives them the appropriate number of objects for any of the numerals on their count list, and they no longer need to keep track of the set representations for each numeral. They've learned that:

$$cardinality\,(numeral(n)) = cardinality\,(numeral(n-1)) + 1, \qquad (1)$$

where $numeral(n)$ is the nth numeral in the sequence < "one," "two," . . . , "ten" > and $cardinality(m)$ is the cardinality associated with numeral m.

Recent research puts some kinks in the bootstrap of Figure 1. Many children who seem to have passed through Steps 1–8 by the usual criterion of being able to give the experimenter up to six or eight objects, nevertheless can't say whether a closed box containing ten objects has more items than a box containing six ("The orange box has ten fish in it. The purple box has six fish in it. Which box has more fish?") [20]. Similarly, children in the same position are often unable to say whether a single object added to a box of five results in a box with six objects or with seven objects [6]. The children who fail these tasks know how to recite the count sequence to at least "ten," and they know, for example, that the numeral following "five" is "six" rather than "seven." In terms of (1), they know that $numeral(n-1)$ immediately precedes $numeral(n)$ for the numerals on their count list. So whatever the children have learned about the positive integers at this point doesn't seem equivalent to Rule (1). You might therefore wonder whether a child's ability to give the experimenter six or eight objects in response to a request is enough to show that the child really knows the cardinal meanings of the number words "one" through "ten." Assuming that the child complies with the request to give eight by counting out the objects ("one, two, . . . , eight: eight balloons"), does the child realize that this procedure yields the total number of items? (See [12, 16] for worries about children's initial counting.)

A second pertinent finding from recent research is that children appear to understand the gist of Hume's Principle (the number of F's equals the number of G's iff there's a one-one relation between the F's and the G's) only after they are able to "Give me six" [18, 42]. Before this point, for example, children are able to affirm that a lion puppet with five cupcakes has just as many as a frog puppet with five cupcakes. But then if explicitly told that lion has "five cupcakes," they are often unable to say whether frog has five or six. (The experimenters discouraged the children from counting; so the children had to base their answers on the just-as-many relation.) These results confirm an earlier observation by Gelman and Gallistel [13]: If children have to determine the number of objects in each of two groups of the same cardinality, they sometimes ignore an obvious one-one relation between the groups of objects and instead count both groups separately. This finding would be unexpected if Hume's Principle provided the basis of cardinality for children, in the way it does in the context of Frege's Theorem (see Section 1). But if children's grasp of cardinality at the point when they're able to "Give me six" comes from counting, the results in the preceding paragraph imply that it's only in an anemic sense (at least initially) that has no implications about the relative cardinality of the numerals on the child's count list.

But let's not be too snarky about counting. Let's suppose that counting is one route children could take to learn (eventually) how number words connect to cardinalities. The issue here is: Why assume that children's understanding of the positive integers derives from cardinality (no matter whether they comprehend cardinality through counting, one-one relations, or some other way)? Today, you might dial Gerry on your cell, check the price of a six-pack of Gumball Head at the 7-Eleven, tune your FM to the public radio frequency, check the street address of Pizzeria Due, adjust the toaster to setting 4, note the speed limit on East 55th Street, calculate some averages for a presentation, check the temperature before going out, remind yourself of the date of your trip to Omaha, look online for a shirt with 32" sleeves, note the page of *Fahrenheit 451* that you've managed to reach. None of these encounters with numbers involves cardinalities. So why privilege cardinality? Of course, situations exist where cardinality is crucial. Conducting an inventory, a survey, or an election may be examples. But it is not clear that these contexts provide a reason for thinking that cardinality is the central feature of integer knowledge.[3]

From a structural perspective, what's striking about Steps 1–8 is that they don't teach children much about the positive integers that they didn't already know. At Step 2, children already know the count-list sequence of the number words <"one," "two," . . . , "ten">. They learn in Steps 3–6 how to assign the first few words in this sequence to a representation of a set of appropriate size: $\{object_1\}$, $\{object_1, object_2\}$, $\{object_1, object_2, object_3\}$. Then they finally learn in Steps 7–8 the rule in (1) that pairs the rest of the words in the sequence with a set size. But that's it. They've learned to correlate one 10-item sequence (of the first few number

[3]Many quantitative contexts, including some just mentioned, involve measurement of continuous dimensions (e.g., temperature, time, and length) rather than counting. Current research in developmental psychology suggests that infants are about equally sensitive to the number of objects in a collection (for $n > 3$) and to the continuous extent (e.g., area) of a single object when it is presented alone (see [9] for an overview). This is presumably because the same kind of psychophysical mechanism handles both types of information (see Section 3.1). However, infants' accuracy for the number of objects in a collection (again, for $n > 3$) is better than that for the continuous extent of the objects in the same collection [5]. Of course, children's knowledge of continuous extent, like their knowledge of number, has to undergo further changes before it can support adult uses of measurement (see, e.g., [24]). Some intricate issues exist about children's understanding of continuous quantity that are the topic of current research [39], but because this chapter focuses on knowledge of natural numbers, this brief summary will have to do.

words) with a second, structurally identical 10-item sequence (successive set sizes or cardinalities). Although Rule (1) is perfectly general in applying to any natural number *n*, the domain of *n* is bounded at "one" through "ten," since those are the only numerals that the children know at this point. As a result, the structure of the rest of the positive integers is undetermined [31, 36]. For example, the children don't know on the basis of Steps 1–8 that the integers don't stop at 12 or 73 or that they don't branch at 13 into two independent sequences or that they don't proceed to 100 and then circle back to 20. They can't rule out any of these possibilities because, according to this theory, they don't know the structure of the integers beyond 10.

Carey is clear that Rule (1) is not the only principle that children have to master before they can understand the positive integers [3]. Just after Step 8, they clearly don't yet know the cardinal meanings of "seventy-eight," "seventy-nine," and so on, since these terms are not yet part of their vocabulary. And they still have additional work to do before they understand general facts about the positive integers, such as the fact that every positive integer has a unique successor. But should we even credit them with understanding the meaning of the terms in their count list, "one," "two," ..., "ten," on the basis of the knowledge they gain in Steps 1–8? From the structural perspective, the answer might well be "no" (see [38] for arguments along these lines). From this perspective, children at Step 2 know at least one finite structure of 10 elements. Although Carey treats this count list as a "placeholder structure" and as "numerically meaningless," it is the same structure that they arrive at through Quinian bootstrapping. The only difference is that it is correlated with mental representations of cardinalities. But if you are inclined to say that children's initial <"one," "two," ... , "ten"> is numerically meaningless, shouldn't you say that the same sequence in association with the isomorphic <$\{object_1\}$, $\{object_1, object_2\}$, ..., $\{object_1, object_2, \ldots, object_{10}\}$> is also meaningless? Likewise, for Rule (1), when restricted to the numerals on the children's count list. Of course, this doesn't mean that children have learned *nothing* of importance in proceeding through these steps. They've learned how to calculate the correct number of items in response to the numerals they know. But do they have a better grip on the numbers five or six or ... ten than they had at the start?

As you might expect, Quinian bootstrapping isn't the only proposal on the table about how children learn the cardinal meanings of the small positive integers (see [29] for an alternative and [35] for a critique). However, I won't pursue comparisons to these alternative proposals, since my purpose is to contrast the structural perspective with the cardinal perspective, and the latter perspective is shared by nearly all developmental theories of number learning.[4]

[4]The same is true in education theory, as Sinclair points out in her chapter in this volume. See that chapter for an alternative that is more in line with the structural perspective.

3 Judgments of Order versus Numerosity

Perhaps we can get a better purchase on the cardinal and structural perspectives by looking at the way people make direct judgments about cardinal and structural relations. For example, if people find it easier or more natural to determine the cardinality of sets of five than to determine the relation between five and six (e.g., $5 < 6$), then the cardinal perspective may provide a better fit than the structural perspective to people's apprehension of numbers. But although several studies exist that are relevant to this comparison, some inherent difficulties muddy the implications these studies have for the issue at hand.

3.1 Distance Effects

To appreciate the difficulties in untangling these perspectives, consider a well-known and well-replicated finding from the earliest days of cognitive psychology [1, 2, 27]. On each trial in this type of experiment, adult participants see two single-digit numerals (e.g., "3" and "8"), one on each side of a screen, and their task is to press a button on the side of the larger number as quickly as possible. (The larger number is randomly positioned at the right or left across trials; so participants can't anticipate the correct position.) One finding from this experiment is that the greater the absolute difference between the numbers, the faster participants' responses. For example, participants are reliably faster to respond that 8 is larger than 3 than that 5 is larger than 3. This may seem surprising, given adults' familiarity with the small positive integers.

The standard explanation for this distance effect is that people automatically map each of the numerals to a mental representation that varies continuously with the size of the number. They then compare the two representations to determine which is larger. In comparing "3" and "5," for example, they mentally represent 3 as an internal quantity of a particular amount, represent 5 as an internal quantity of a larger amount on the same dimension, and then compare the two quantities to determine their relative size. Imagine, for example, that 3 is represented as some degree of neural activation in a particular brain region, and 5 as a larger degree of activation in an adjacent region. Then people find the correct answer to the problem by comparing these degrees of activation. On this account, the distance effect is due to the fact that people find it easier to compare quantities that are farther apart on the underlying dimension. Just as it's easier to determine the heavier of a 3 kg and an 8 kg weight than the heavier of a 3 kg and a 5 kg weight by hefting them, it's easier to determine the larger of 3 and 8 than the larger of 3 and 5. The mental system responsible for this type of comparison goes by a number of aliases (e.g., "mental number line," "analog magnitude system," and "number sense"), but these days most researchers call it the "approximate number system." So will I.

Much evidence suggests that the approximate number system is present in human infants and in a variety of non-human animals (see [7] for a review). Experiments along these lines present two groups of dots, tones, or other non-symbolic items to determine the creatures' sensitivity to differences in the cardinality of these groups. Distance effects appear with these non-symbolic items that echo those found with adults and numerals: The larger the difference in the cardinality between the two groups of objects, the easier they are to discriminate. So you could reasonably suppose that the approximate number system is an innate device specialized for detecting cardinalities (e.g., the number of edible objects in a region), that this device persists in human adults, and that the symbolic distance effect for numerals, described a moment ago, is the result of the same system. What's important in the present context is that what appears to be a primitive system for dealing with cardinality—the approximate number system—may underlie adults' judgments of ordinal relations (e.g., $3 < 8$). If so, the cardinal perspective may be more fundamental than the structural one in human cognition.

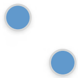
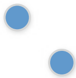
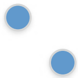
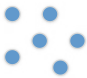
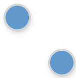
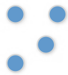
However, these results do not necessarily support the cardinal perspective. Nearly all quantitative physical dimensions—acoustic pressure, luminosity, mass, spatial area, and others—produce distance effects of similar sorts. The perceptual system translates a physical value along these dimensions (e.g., mass) into a perceived value (e.g., felt weight) that can be compared to others of the same type, and comparisons are easier, the greater the absolute value of the difference. The same is true of symbolic stimuli [26]. If participants are asked to decide, for example, which of two animal names (e.g., "horse" or "dog") denotes the larger-sized animal, times are faster the bigger the difference in the animals' physical size. But this effect provides no reason to think that we encode animal sizes as cardinalities. So it is not at all clear that distance effects for numerals depend specifically on representing them in terms of cardinality. Instead, the effects may be due to very general psychophysical properties of the perceptual and cognitive systems.

3.2 Judgments of Order

To make some headway on the cardinal and structural perspectives, we need a more direct comparison of people's abilities to judge cardinal and structural properties. Is it easier for people to assess the size of a set associated with a positive integer than to assess the integer's relation to others? Linnebo's [21] second argument, mentioned in Section 1, seems to predict a negative answer to this question. Several recent studies have attempted a comparison of this sort, but the implications for our purposes are difficult to interpret because of some inherent features of the procedures.

Here's an example: Lyons and Beilock [22] compared a "cardinal task," similar to the standard "Which is larger?" method, described in the preceding subsection, to a novel "ordinal task." Table 1 summarizes the main conditions in the study. In the ordinal task, participants decided as quickly as possible whether triples of single-

Table 1 Summary of Main Conditions from Lyons and Beilock's Comparison of Ordinal and Cardinal Judgments (Adapted from [22, Figure 1])

	Task Type				
Stimulus Items	Ordinal Task (Are the items in either ascending or descending order?)			Cardinal Task (Which item is larger?)	
Numerals					
Close	2	3	4	2	3
Far	2	4	6	2	4
Dots					
Close					
Far					

Entries Provide an Example in which Participants Should Respond "True" in the Ordinal Task and Push the Right-hand Button in the Cardinal Task

digit numerals were correctly ordered (in either ascending or descending sequence) or incorrectly ordered. For instance, participants were to answer "yes" to <2, 3, 4 > or <4, 3, 2 > but "no" to <3, 4, 2>. The elements of the triples could be separated by an absolute difference of one (the "close" triples in Table 1, e.g., <2, 3, 4>) or two (the "far" triples, e.g., <2, 4, 6>). Lyons and Beilock also included a task in which participants made analogous judgments for triples of dots. For example, they decided whether a triple consisting of two dots followed by three dots followed by four dots was correctly ordered. For the cardinal task, participants decided which of two numerals (e.g., 2 or 3) was larger or which of two groups of dots was larger (e.g., a group of two or a group of three dots). As in the ordinal task, the items within a pair could differ by one or by two.

Lyons and Beilock found the usual distance effect in the cardinal task, both for numerals and dots. That is, participants were quicker to respond to the far pairs (e.g., 2 vs. 4) than to the close pairs (e.g., 2 vs. 3). For the ordinal task, however, the results were different for numerals than for the dots. Although the dots showed a distance effect, numerals showed the reverse: The close triples were faster than the far triples. (For related findings, see [10, 44].) The investigators conclude from these findings that the link between mental representations of numerals and cardinalities is less direct than what one might gather from the typical distance effects. Judgments of order for numerals rely on a process distinct from the one governing judgments of cardinality (presumably, the approximate number system). “At the broadest level, the meaning of 6 may thus be determined by both its relation to other symbolic numbers and the computational context in which it rests. This is in keeping with the view that the meaning of symbolic numbers is fundamentally tied to their relations with other symbolic numbers . . . ” [22, p. 17059].

The emphasis on “relations with other symbolic numbers” goes along with what we have been calling the structural perspective. However, the relations in question raise an issue about the type of structure responsible for the data. Lyons and Beilock plausibly suggest that the reversal they observe for ordinal judgments of numerals—faster times for close triples in Table 1—is the result of familiarity with the list of count words. Because college-student participants have rehearsed the count list (“one, two, three, . . . ”) on many thousands of occasions during their lives, they find it easier to recognize numerals as correctly ordered when they appear in adjacent positions on the list (e.g., <2, 3, 4>) than when they are not in adjacent positions (e.g., <2, 4, 6>). fMRI imaging evidence from the same experiment suggests that “one interpretation of these results is thus that ordinality in symbolic numbers is processed via controlled retrieval of sequential visuomotor associations . . . ” [22, p. 17056]. In the case of ordinal judgments for dots, however, participants can’t directly access the count list, but have to compare successively the cardinality of the three groups (e.g., two dots is less than three dots is less than four). These comparisons are similar to those performed in the cardinal task and give rise to similar distance effects.

This interpretation of the ordinal judgments for numerals returns us to the concerns raised at the end of Section 2. According to Lyons and Beilock [22], the structure responsible for the ordinal task with numerals is the rote connections that we form in reciting the count list (“sequential visuomotor associations”). This is the same “placeholder structure” that Carey [3] finds “numerically meaningless.” We can be more charitable than Carey in granting this structure some mathematical significance (and, of course, adults can recite more of the count list than children can). But clearly, this structure isn’t the same as the structure of the positive integers. We can’t individually store the connections between each successive pair of integers since there are infinitely many of them. So the structure responsible for the ordinal judgments of numerals in such tasks must comprise just a short finite segment of the integers if Lyons and Beilock are right in their interpretation. Do people have a deeper structural understanding of the integers?

4 Two Studies of Number Knowledge

The experiments described in the preceding section are timed tasks that call on people’s immediate impression of number, and they sometimes reveal unobvious aspects of those impressions. But if what controls the responses in those experiments is automated cognition, such as rote recitation of the count words, the results may mask deeper features of people’s thinking. We might be able to find out more by quizzing people’s number knowledge directly.

4.1 What Types of Properties Do People Connect to Numbers?

As one way to find out whether people understand numbers in terms of cardinal or structural features, I asked 20 undergraduates from an introductory psychology class to list properties for each of seven numbers. The numbers were: "zero," "minus eight hundred forty-nine," "square root of 2," "three," "seventy-one hundred ninety-three," "twenty-nine billion and ninety-one," and "eighty-three septendecillion and seventy-six." These were spelled out, as in the preceding list. I picked the last four of these items to coincide with those from an earlier experiment [33]; the rest were chosen to contrast with the positive integers. Results from the three biggest numbers did not differ greatly in this experiment, and of these, I'll report only those for 7193 (with a few exceptions, noted later).

Participants saw these numbers one-at-a-time on a computer screen, in a new random order for each participant. The participants were told that the experiment concerned their knowledge of number properties. They were asked to think of ten properties for each number and to type them into spaces provided on the screen. The instructions cautioned them that they should "try not to just free associate—for example, if a number happens to remind you of your father, do NOT write down 'father.'" After the participants had finished listing properties for all the numbers, a new set of instructions asked them to rate the importance of each of these properties. A participant saw a series of screens, each containing one of the original numbers and a property that the participant had listed earlier for that number. For example, if the participant had listed "is an odd number" for three in the first part of the experiment, then he or she saw in the second part a request to rate the importance of "is an odd number" for three. A 0–9 rating scale appeared on the same screen, with "0" labeled "not at all important" and "9" labeled "extremely important."

We can get some idea of the nature of the properties that the participants produced by classifying each property token into one of the following categories. Examples of actual properties from the participants' lists appear in parentheses after the category name:

Cardinality (e.g., "is nothing" for zero)

Magnitude (overall size, e.g., "is a big number," "small," "very large")

Number line (e.g., "three integers away from zero on the number line," "exact middle of the number line," "is between 1 and 2 on the number line")

Number system membership (e.g., "integer," "is a rational number," "is not a rational number")

Arithmetic comparison (e.g., "smaller than 10," "is bigger than one million" "between 2 and 4")

Arithmetic operations (e.g., "multiplied by itself will give 2," "any number added to zero is the same number," "zero divi[d]ed by 1 is zero")

Number-theoretic properties (in a loose sense in which, e.g., "is an odd number," "is negative," "is not a prime," "has an imaginary square root," "is a factor of 21" are number-theoretic)

Numeral properties (properties of the written shape or spelling of the number, e.g., "contains one digit," "has a comma in it," "7 in the thousands place," "spelled with four letters")

Non-numeric properties (e.g., "is significant," "is important to mathematics," "common")

Other (e.g., "no idea," "I would like to have this much candy," "I don't like square roots")

Figure 2 (solid lines and circles) shows the frequency of tokens in these ten categories for each number. The first thing to notice is the very small number of tokens for cardinality. Zero produced a few such mentions—variations on "nothing"—but no one listed "describes 7193 objects" (or anything of the kind) for 7193. In fact, none of the property categories that might be linked to the approximate number system—the cardinality, magnitude, and number line categories—received more than a few mentions, as you can see from the first three points of each of the Figure 2 curves. Of course, no one would expect participants to mention cardinality for -849 or $\sqrt{2}$, but it's notable that 0, 3, and 7193 also received small frequencies.

By contrast, mention of number systems (e.g., "is an integer," "is not rational") was relatively frequent, especially for $\sqrt{2}$, the sole non-integer and non-rational in this group. Similarly, number-theoretic properties (e.g., "is prime," "is divisible by 3") were popular responses for the integers.

Although participants did not often produce properties having to do with arithmetic comparison or operations, zero did yield a fairly large number of operations properties, no doubt because of its special role in addition and multiplication. For example, one participant mentioned, "the product of zero with any other number is zero itself," and another wrote, "anything plus zero is itself." Properties of numerals (either for number words or symbols) were not especially common, except for 7193 ("four digit number," "no repeated digits") and, to a lesser extent, -849 ("has one 8," "looks like this '-849'").

You might argue that the very low frequency for cardinality is due to the fact that each natural number has just one cardinality, whereas it has many properties of other types (e.g., many number-theoretic properties). So perhaps the frequency of mention just reflects the actual number of available tokens per type. But although a natural number has just one cardinality, it nevertheless denotes the size of an infinite number of sets. Participants could have said that three is the number of bears in the fairy tale, the number of stooges in the film comedies, the number of instrumentalists or vocalists in a trio, the number of vertices or angles or sides in a triangle, the number of people that's a crowd, the number of deities in the trinity, the number of events in a triathlon, the number of rings in a circus, the number of races in the triple crown, the number of children in a set of triplets, the number of outs in a turn at bat, the number of isotopes of carbon, the number of novels in a trilogy, the number of wheels on a tricycle, the number of leaders in a triumvirate, the number of panels in a triptych. But no one mentioned any of these or any other three-membered groups in response to three. Moreover, for the numbers 3, 7193, 29,000,000,091, and $(8.3 \times 10^{55}) + 76$, none of the participants mentioned cardinality even once. Of course, we asked participants to list number "properties," and perhaps this way of phrasing the question militated against their writing down items related to cardinality. For example, the number of things in a

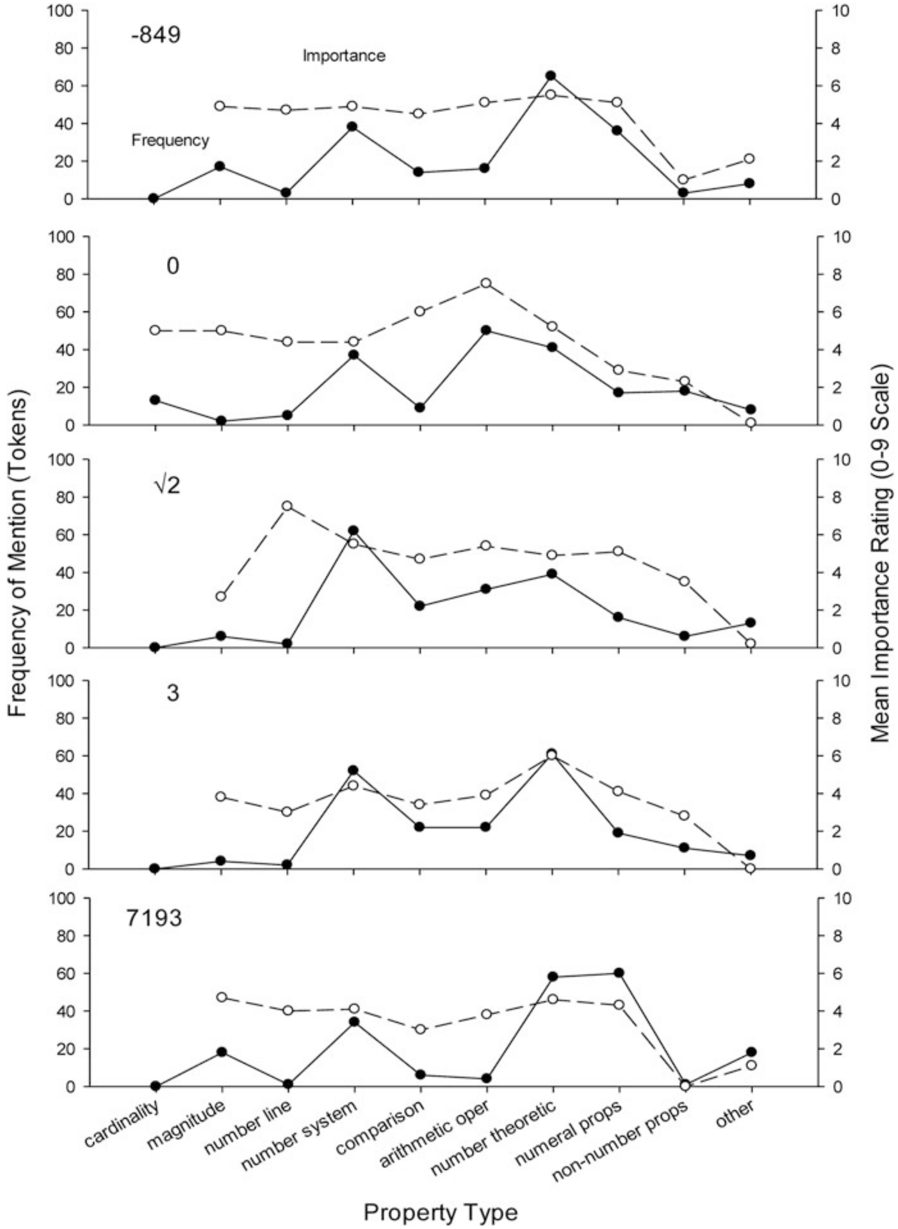


Fig. 2 Frequency of mention of properties of different types (solid lines and symbols) and mean rated importance of the same properties (dashed lines and open symbols) from Study 1. Points on the importance curves are missing if participants listed no properties of that type.

triple or of vertices in a triangle might have seemed too extrinsic to qualify as a property of three. But the question at issue in this study is, in fact, what participants believe to be intrinsic to their number concepts. What kind of information about a number is central to people's beliefs about the number's nature? If participants find "being an integer" and "being divisible by 3" to be number properties but not "being the number of items in a triple," then this suggests that cardinality may not be what organizes their conception of numbers.

Mean importance ratings for the same properties also appear in Figure 2 (dashed lines and open circles). The means for the cardinality and number-line properties are based on only a small number of data points, as the frequency curves show. Omitting these latter categories, a statistical analysis indicates that, across all the numbers, participants rated the number-theory, arithmetic-operations, and number-system properties as more important than the non-numeric properties. They also rated "other" properties lowest in importance, as you would hope. No further reliable differences in importance appeared among the property categories (adjusting for the number of comparisons). However, arithmetic operations are especially important for zero, probably for the reasons mentioned earlier, and number-theory properties (e.g., "are prime") are important for three. (7193 is also prime, but participants probably didn't recognize it as such.) These peaks contribute to a statistically reliable difference in the shape of the importance curves in the figure. Keep in mind, though, that the properties contributing to these importance ratings are the ones that the same participants produced in the first part of the experiment. It might be useful to look at an independent measure of the importance of number properties. The study in the next section provides a measure of this sort.

4.2 The Centrality of Number Properties

In a second study of number properties, a new group of participants decided whether a given property of an integer "was responsible for" a second property of the same integer. The properties included:

Cardinality (phrased as "being able to represent a certain number of objects")
Number system membership ("being an integer")
Arithmetic ("being equal to the immediately preceding integer plus one")
Position in the integer sequence ("being between the immediately preceding and the immediately following integers"), and
Numeric symbol ("being represented by a particular written symbol").

On one trial, for example, participants were asked to consider whether "an integer's ability to represent a certain number of objects is responsible for its being equal to the immediately preceding integer plus one." The instructions told participants, "By 'responsible' we mean that the first property is the basis or reason for the second property." The participants answered each of the responsibility questions by clicking a "yes" or a "no" button on the screen.

In a preliminary version of this experiment, I asked participants about the properties of specific integers from Study 1 (0, 3, 7193, and 29,000,000,091). For example, participants had to decide whether three's property of representing three objects was responsible for its being equal to $2 + 1$. The results from this pilot study, however, suggested that participants were interpreting these questions in a way that depended on the specificity of the properties. Asked whether being an integer was responsible for three's being able to represent three objects, for example, many participants answered "no," apparently because merely being an integer was consistent with being able to represent any number of objects. Some participants explicitly mentioned (in their written comments after the experiment) a sufficiency criterion of this sort: "If the second part had to be true because of the first part, then I selected that the first property was responsible for the second." To avoid this problem, the study reported here rephrased the properties to be about integers in general. For example, participants were asked "whether being an integer was responsible for the integer's ability to represent a certain number of objects."

Participants saw all possible pairs of the five properties listed above (in both orders). For example, the participants decided both whether "an integer's ability to represent a certain number of objects is responsible for its being equal to the immediately preceding integer plus one" and whether "an integer's being equal to the immediately preceding integer plus one is responsible for its ability to represent a certain number of objects." Thus, there were 20 key questions about integer properties. In addition, the experiment included five catch trials with questions that were intended to be obviously true or false (e.g., "Is an integer's ability to represent a certain number of items responsible for its color?"). The presentation order of the full set of 25 questions was random. Fifty participants were recruited from Amazon Mechanical Turk for this study, but three were excluded for making errors on three or more of the catch trials.

We can get some idea of the centrality of a numeric property by looking at participants' willingness to say that the property was responsible for the others. For example, if cardinality is a critical property, then participants should be willing to say that "the ability to represent a certain number of objects" is responsible for other properties. Of course, a participant's overall likelihood of endorsing a property in this way will depend on the other properties on our list and on the particular phrasing of these properties. Still, these scores provide a hint of the relative importance of the property types.

Figure 3 illustrates the results in a way that may help bring out the comparison of the properties' importance. The circles at the pentagon's vertices represent the properties listed earlier (cardinality, arithmetic, and so on). The arrows between the circles correspond to the participants' judgments of whether the property p_1 at the arrow's tail is responsible for the property p_2 at the arrow's head. The numbers on these arrows are the proportion of participants who agreed that p_1 was responsible for p_2. For example, the arrow from arithmetic to cardinality is labeled .62 and indicates that 62 % of the participants thought that the arithmetic property ("being equal to the immediately preceding integer plus one") was responsible for cardinality ("being able to represent a certain number of objects"). One measure

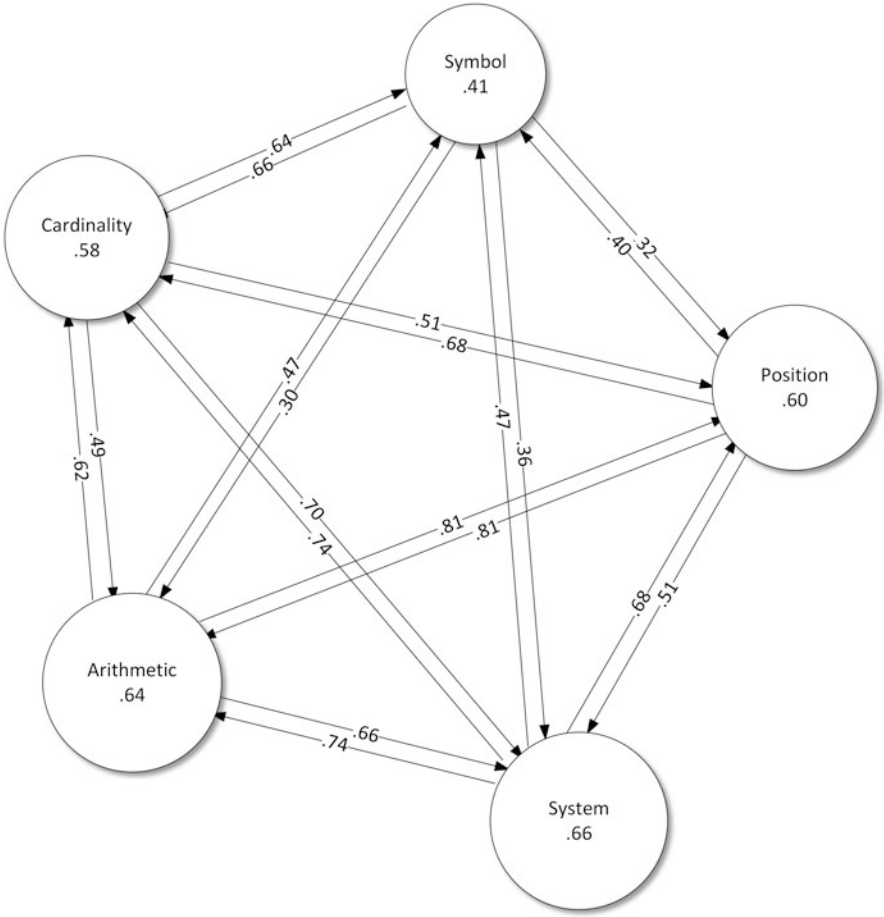


Fig. 3 Judgments of "responsibility" for number properties. Each arrow in the diagram represents participants' decisions as to whether the property at the base of the arrow is responsible for the property at the head of the arrow. Numbers on the arrows are the proportion of participants who said "yes." Numbers in the circles and their areas represent the means of the outgoing arrows.

of a property's centrality is its average responsibility for the other properties—the average of the proportions on the outgoing arrows from that property. These averages appear within the circles, and the areas of the circles are approximately proportional to these averages. The bigger the circle, the more central the property.

The mean responsibility score for the system property ("being an integer") is the largest of these items. This may be due in part to the fact that two of the other properties—the arithmetic and position properties—mentioned the word "integer." For example, participants were asked whether "being an integer" was responsible for "being between the immediately preceding and immediately following integers." However, most participants (74 %) also thought that being an integer was responsible for cardinality ("being able to represent a certain number of objects"), whose phrasing did not include the word "integer." This suggests that participants saw membership in the number system itself as an important factor.

The arithmetic and position properties also had relatively large responsibility scores in these data. The arithmetic property was just the +1 relation (i.e., "being equal to the immediately preceding integer plus one"), and the importance that participants attached to it may suggest that they view a successor function as central to the integers' identity. The same may be true of the positional property, which was specified in terms of being between the preceding and the following integer.

The numeric symbol property (e.g., "being represented by a particular written symbol") received lowest responsibility scores. Many participants probably answered "no" to questions about the symbols' responsibility because of the arbitrary nature of the symbol. In the pilot study mentioned earlier, one participant wrote in his or her debriefing comments, "I did not think that the symbol for any number could be responsible for anything, since the designated symbols for numbers are arbitrary." According to another, "Does a number's numerical symbolization really have to do with anything? After all, it could simply be an arbitrary symbol." Section 5 of this chapter discusses the implication of this result for the structural perspective.

Responsibility scores for cardinality ("being able to represent a certain number of objects") were about midway between those for the system property and symbol property. Cardinality does somewhat better here than we might have predicted from its very infrequent mention in the preceding study, where the participants had to produce their own properties for the numbers (see Figure 2, panels 2, 4, and 5). The results from Study 1 could be put down to the obviousness of the connection between an integer and its cardinality. Maybe it literally goes without saying that "three" denotes three entities. But the same is probably true of several of the other properties on our list. Another more likely possibility is that because the cardinality property fixes its referent in a nonarbitrary way, some participants in the present study may regard cardinality as sufficient for the other properties. For example, if a number has cardinality 3, then it *is* 3, and hence, is the successor of 2, is between 2 and 4, is an integer, and is symbolized by "3." Other participants, however, may regard cardinality as too "incidental" to an integer to allow it to be responsible for its purely mathematical properties, and this same feeling may explain why cardinality went unmentioned in Study 1. The final section tries to sketch what "incidental" could amount to in the mathematical domain.

5 Conclusions

What implications do the two studies of Section 4 have for the cardinal and structural perspectives? On the one hand, Study 1 suggests that cardinality isn't something that often occurs to people when they are asked to consider the properties of an integer. They don't immediately take a cardinality perspective under these circumstances.

Instead, the properties of the integer that come readily to mind are ones that might be called "number internal," such as being odd or positive or prime. In the case of special numbers, such as 0, these number-internal properties also include arithmetic or algebraic features, such as being an additive identity. On the other hand, Study 2 shows that some people acknowledge the importance of cardinality in relation to an integer's other properties. Once we know that an integer has a particular cardinality, say 7193, we can predict all its other properties, for example, that it is the successor of 7192 and the predecessor of 7194. Cardinality can, in this way, single out an integer uniquely, yet people may not typically use it in thinking about the integer as such.

Perhaps something of the same is true of numbers in general. In applied math, numbers denote specific times, spatial coordinates, velocities, temperatures, money, IQ, and what have you. But we don't think of a number as intimately connected with these instances (e.g., a particular position, time, or value). A shift in coordinates will assign a different number to the same spatial point. A change in currency will associate a different number with the same value for a good. Numbers provide models for time, space, utility, and other dimensions, as measurement theory makes clear (see, e.g., [19, 40]), but only up to certain well-defined transformations. The numbers themselves have an identity that's independent of their role in the model. In the same way, you could view the use of integers to denote cardinalities as another application of a number-based model to the size of sets.

Proponents of the cardinality perspective still have some cards to play in defending the idea that cardinality is conceptually fundamental. For example, they could take the view that cardinality provides children's first entry point to the natural numbers, even if children outgrow this perspective as they gain further knowledge of number properties in school. Or they could claim that cardinality is central to adults' thinking about the naturals, even though adults don't often mention it when explicitly asked about number properties. But although defenses of this kind may turn out to be right, not much evidence exists to back them. We noticed in Section 2 that it's hard to see how cardinality can by itself advance children's knowledge of the naturals. Mere apprehension of numerosity through the approximate number system can't provide the right properties for the naturals (e.g., the perceived difference in numerosity between 14 items and 15 items is smaller than the perceived difference between 4 and 5 in this system). Perhaps the more sophisticated use of cardinality in children's initial counting creates the link to the naturals, but we found in Section 2 that it's difficult to make this case without begging the question against the structural perspective (see [31, 37] for more on this theme). Similarly, the results we reviewed in Section 3 show that, although numerosity may influence adults' judgments about the integers in tasks like numeral comparison, there's a catch: The mental process most likely responsible for such effects is again the approximate number system, which delivers distorted information about cardinality. Maybe true cardinality underlies adult intuitions about the naturals, but what reasons support such an assumption?

The structural perspective may be better able to cope with the results. But we should enter a couple of qualifications about the support that these studies lend to the structural view. First, we've seen that participants didn't often mention properties connected to numerals (except for the larger numbers) in Study 1, and they didn't judge them as especially important in producing a number's other properties in Study 2. This seems reasonable in view of the arbitrary nature of the symbols. But if "the only perfectly direct and explicit way of specifying a number seems to be by means of some standard numeral in a system of numerals" [21], then you might have expected numerals to play a more important role in the results. Notice, though, that in these experiments no variations occurred in the manner in which the numbers were specified. In Study 1, a numeral was given to participants explicitly (as an English phrase), and they responded based on this numeral. In Study 2, no numerals appeared. So the results are still compatible with the possibility that numerals within a standard system provide an easier way to denote numbers than other possibilities, such as the set of sets containing that number of elements.

Second, in Study 1, the frequency of positional properties was fairly low. We classified these properties as arithmetic comparisons in that study (see Figure 2), and they included items such as "is greater than -1" (in the case of 0), "is smaller than four" (in the case of 3), and "is less than 10000" (in the case of 7193). Properties of this sort comprised only 7.3 % of tokens across the five numbers in Figure 2 and produced a mean importance rating of 4.5, which is the midpoint of the 0–9 scale. Why did participants fail to produce these positional properties and fail to rate them as important? The reason may be similar to the one that makes cardinality properties unpopular. People may feel that relations like being between 7192 and 7194 in the integer sequence are external to the number 7193 itself.

What come to mind more readily for the numbers in Study 1 are properties like being even or positive—properties we classified as "number-theoretic" in the loose sense of the first study. These properties were either the most often mentioned (for -849 and 3) or the second most often mentioned (for 0, $\sqrt{2}$, and 7193), as Figure 2 indicates. We can unpack properties like these in relational terms—being positive is being greater than 0, and being even is being divisible by 2—but participants may see them (at least at first thought) as something intrinsic to the numbers themselves. My guess is that much the same is true for most other individual concepts (e.g., UNCLE FRED, WALDEN COLLEGE): The properties of these concepts that are easiest to access are ones that we represent as monadic.

It may be an important fact about individual concepts that we mentally organize them around central properties of this sort and take other, more peripheral properties to be the products of the central ones. Uncle Fred may have traits like being genteel or pig-headed that we see as non-relational and intrinsic, but that are responsible for many other aspects of his personality and behavior. In planning our interactions with him, or in predicting what he will do at the party or believe about big government or want for his birthday, we consult and extrapolate from these central properties. The suggestion here is that this type of thinking may carry over to individuals in abstract domains, such as numbers. We don't personify numbers in the way Lewis Carroll did ("'Look out now, Five! Don't go splashing paint over me like that!' 'I couldn't help it,' said Five, in a sulky tone; 'Seven jogged my elbow.'" [4, chap. 8]). But we might think about five as centrally positive, odd, and prime because these properties are handy in making inferences about this number.

This isn't quite the structural perspective that we get in the philosophy of mathematics, but it has its own structural aspects. Structuralists in philosophy see numbers as atoms with no internal structure, as the quotations from Resnik [30] and Shapiro [43] in Section 1 make clear. Their content is exhausted by the position they have in a relevant number system, as given by an appropriate set of axioms. In one way, this seems right for psychological concepts of numbers, as well. Our mental representations of a natural number, for example, had better conform to the usual Dedekind-Peano axioms, since otherwise it's difficult to see in what sense they could represent that number [38]. In another way, however, mental representations about a natural number include a richer domain of facts that we use in dealing with typical mathematical tasks, including calculation and proof (see [34] for the distinction between "representations of" and "representations about" objects and categories). What the studies reported here suggest is that information of the latter sort, at least among college students, may organize itself around, not cardinality, but instead properties like primality that may be more helpful in mathematical contexts.

Acknowledgement Thanks to Ernest Davis, Jacob Dink, and Nicolas Leonard for comments on an earlier version of this chapter and to John Glines, Jane Ko, and Gabrielle McCarthy for their help with the experiments described in Section 4.

References

1. W.P. Banks, M. Fujii and F. Kayra-Stuart, "Semantic congruity effects in comparative judgments of magnitudes of digits," *Journal of Experimental Psychology: Human Perception and Performance*, vol. 2 (1976) 435–447.
2. P.B. Buckley and C.B. Gillman, "Comparisons of digits and dot patterns," *Journal of Experimental Psychology*, vol. 103 (1974) 1131–1136.
3. S. Carey, *The origin of concepts*, Oxford University Press, New York, NY, 2009.
4. L. Carroll, *Alice's adventures in wonderland*, Macmillan and Co., London, 1866.
5. S. Cordes and E.M. Brannon, "Attending to one of many: When infants are surprisingly poor at discriminating an item's size," *Frontiers in Psychology*, vol. 2 (2011) doi: 10.3389/fpsyg.2011.00065.
6. K. Davidson, K. Eng and D. Barner, "Does learning to count involve a semantic induction?," *Cognition*, vol. 123 (2012) 162–173.
7. S. Dehaene, *The number sense: how mathematical knowledge is embedded in our brains*, Oxford University Press, New York, 1997.
8. K. Devlin, *The joy of sets*, 2nd ed., Springer, Berlin, 1993.
9. L. Feigenson, "The equality of quantity," *Trends in Cognitive Sciences*, vol. 11 (2007) 185–187.
10. M.S. Franklin, J. Jonides and E.E. Smith, "Processing of order information for numbers and months," *Memory & Cognition*, vol. 37 (2009) 644–654.
11. G. Frege, *The foundations of arithmetic: A logico-mathematical enquiry into the concept of number*, 2nd rev. ed., Northwestern University Press, Evanston, IL, 1980.
12. K.C. Fuson, *Children's counting and concepts of number*, Springer-Verlag, New York, 1988.

13. R. Gelman and C.R. Gallistel, *The child's understanding of number*, Harvard University Press, Cambridge, Mass., 1978.
14. C. Hanlon, "The emergence of set-relational quantifiers in early childhood", in: *The development of language and language researchers: Essays in honor of Roger Brown*, F.S. Kessel, ed., Lawrence Erlbaum Associates Inc, Hillsdale, NJ, 1988, pp. 65–78.
15. P.M. Hartnett, "The development of mathematical insight: From one, two, three to infinity," *Dissertation Abstracts International*, vol. 52 (1992) 3921.
16. R.G. Heck Jr, "Cardinality, counting, and equinumerosity," *Notre Dame Journal of Formal Logic*, vol. 41 (2000) 187–209.
17. R.G. Heck Jr, "Frege's theorem: An overview", in: *Frege's theorem*, Oxford University Press, Oxford, UK, 2011, pp. 1–39.
18. V. Izard, A. Streri and E.S. Spelke, "Toward exact number: Young children use one-to-one correspondence to measure set identity but not numerical equality," *Cognitive Psychology*, vol. 72 (2014) 27–53.
19. D.H. Krantz, R.D. Luce, P. Suppes and A. Tversky, *Foundations of measurement*, Academic Press, New York, 1971.
20. M. Le Corre, "Children acquire the later-greater principle after the cardinal principle," *British Journal of Developmental Psychology*, vol. 32 (2014) 163–177.
21. Ø. Linnebo, "The individuation of the natural numbers", in: *New waves in philosophy of mathematics*, O. Bueno, Ø. Linnebo, eds., Palgrave, Houndmills, UK, 2009, pp. 220–238.
22. I.M. Lyons and S.L. Beilock, "Ordinality and the nature of symbolic numbers," *The Journal of Neuroscience*, vol. 33 (2013) 17052–17061.
23. J. MacFarlane, "Double vision: two questions about the neo-Fregean program," *Synthese*, vol. 170 (2009) 443–456.
24. K.F. Miller, "Measurement as a tool for thought: The role of measuring procedures in children's understanding of quantitative invariance," *Developmental Psychology*, vol. 25 (1989) 589–600.
25. K.S. Mix, J. Huttenlocher and S.C. Levine, *Quantitative development in infancy and early childhood*, Oxford University Press, Oxford, UK, 2002.
26. R.S. Moyer and S.T. Dumais, "Mental comparison," *Psychology of Learning and Motivation*, vol. 12 (1978) 117–155.
27. R.S. Moyer and T.K. Landauer, "Time required for judgements of numerical inequality," *Nature*, vol. 215 (1967) 1519–1520.
28. C. Parsons, *Mathematical thought and its objects*, Cambridge University Press, New York, 2008.
29. S.T. Piantadosi, J.B. Tenenbaum and N.D. Goodman, "Bootstrapping in a language of thought: A formal model of numerical concept learning," *Cognition*, vol. 123 (2012) 199–217.
30. M.D. Resnik, *Mathematics as a science of patterns*, Oxford University Press, Oxford, UK, 1997.
31. G. Rey, "Innate and learned: Carey, Mad Dog nativism, and the poverty of stimuli and analogies (yet again)," *Mind & Language*, vol. 29 (2014) 109–132.
32. L.J. Rips, "Bootstrapping: How not to learn", in: *Encyclopedia of the sciences of learning*, N.M. Seel, ed., Springer, Berlin, 2012, pp. 473–477.
33. L.J. Rips, "How many is a zillion? Sources of number distortion," *Journal of Experimental Psychology: Learning, Memory, and Cognition*, vol. 39 (2013) 1257–1264.
34. L.J. Rips, *Lines of thought: Central concepts in cognitive psychology*, Oxford University Press, Oxford, UK, 2011.
35. L.J. Rips, J. Asmuth and A. Bloomfield, "Can statistical learning bootstrap the integers?" *Cognition*, vol. 128 (2013) 320–330.
36. L.J. Rips, J. Asmuth and A. Bloomfield, "Do children learn the integers by induction?" *Cognition*, vol. 106 (2008) 940–951.
37. L.J. Rips, J. Asmuth and A. Bloomfield, "Giving the boot to the bootstrap: How not to learn the natural numbers," *Cognition*, vol. 101 (2006) B51-B60.

38. L.J. Rips, A. Bloomfield and J. Asmuth, “From numerical concepts to concepts of number,” *Behavioral and Brain Sciences*, vol. 31 (2008) 623–642.
39. L.J. Rips and S.J. Hespos, “Divisions of the physical world: Concepts of objects and substances,” *Psychological Bulletin*, vol. 141 (2015).
40. F.S. Roberts, *Measurement theory with applications to decisionmaking, utility, and the social sciences*, Addison-Wesley, Reading, Mass., 1979.
41. B. Russell, *Introduction to mathematical philosophy*, Allen & Unwin, London, 1919.
42. B.W. Sarnecka and C.E. Wright, “The idea of an exact number: Children’s understanding of cardinality and equinumerosity,” *Cognitive Science*, vol. 37 (2013) 1493–1506.
43. S. Shapiro, *Philosophy of mathematics: Structure and ontology*, Oxford University Press, New York, 1997.
44. E. Turconi, J.I.D. Campbell and X. Seron, “Numerical order and quantity processing in number comparison,” *Cognition*, vol. 98 (2006) 273–285.
45. H.M. Wellman and K.F. Miller, “Thinking about nothing: Development of concepts of zero,” *British Journal of Developmental Psychology*, vol. 4 (1986) 31–42.
46. K. Wynn, “Children’s acquisition of the number words and the counting system,” *Cognitive Psychology*, vol. 24 (1992) 220–251.
47. E.N. Zalta, “Frege’s theorem and foundations for arithmetic”, in: *Stanford encyclopedia of philosophy*, E.N. Zalta, ed., 2014.