Unique Covering Problems with Geometric Sets

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Abstract. The Exact Cover problem takes a universe U of n elements, a family $\mathcal{F}$ of m subsets of U and a positive integer k, and decides whether there exists a subfamily(set cover) $\mathcal{F}'$ of size at most k such that each element is covered by exactly one set. The Unique Cover problem also takes the same input and decides whether there is a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ such that at least k of the elements $\mathcal{F}'$ covers are covered uniquely(by exactly one set). Both these problems are known to be NP-complete. In the parameterized setting, when parameterized by k, Exact Cover is W[1]-hard. While Unique Cover is FPT under the same parameter, it is known to not admit a polynomial kernel under standard complexity-theoretic assumptions.

In this paper, we investigate these two problems under the assumption that every set satisfies a given geometric property Π. Specifically, we consider the universe to be a set of n points in a real space $\mathbb{R}^d$, d being a positive integer. When $d = 2$ we consider the problem when Π requires all sets to be unit squares or lines. When $d > 2$, we consider the problem where Π requires all sets to be hyperplanes in $\mathbb{R}^d$. These special versions of the problems are also known to be NP-complete. When parameterizing by k, the Unique Cover problem has a polynomial size kernel for all the above geometric versions. The Exact Cover problem turns out to be W[1]-hard for squares, but FPT for lines and hyperplanes. Further, we also consider the Unique Set Cover problem, which takes the same input and decides whether there is a set cover which covers at least k elements uniquely. To the best of our knowledge, this is a new problem, and we show that it is NP-complete (even for the case of lines). In fact, the problem turns out to be W[1]-hard in the abstract setting, when parameterized by k. However, when we restrict ourselves to the lines and hyperplanes versions, we obtain FPT algorithms.

1 Introduction

The classic Set Cover problem is the following: For a set system $(U, \mathcal{F})$ where U is a finite universe of n elements and $\mathcal{F}$ is a family of subsets of U, is there a

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sub family of at most k sets in $\mathcal{F}$ whose union is U. We say that an element x in U is *covered* by a set S from $\mathcal{F}$ if the set S contains the element x.

For several applications, it turns out that we would like to not only cover elements of U using sets in $\mathcal{F}$, but also cover them uniquely. A common motivation involves problems where covering elements by more than one set leads to noise (for example, wireless networks), so we would like to ensure that an element is covered, but by only one of the sets. This desired refinement manifests itself in the following three natural variations of the Set Cover problem.

– All elements must be covered uniquely by at most k sets. (Exact Cover)
– All elements must be covered, *and* at least k elements must be covered uniquely. (Unique Set Cover)
– At least k elements are covered uniquely. (Unique Cover)

In the first two variants, we are looking for a set cover with additional properties. Note that in the last setting, a valid solution may not be a set cover.

The Exact Cover problem was one of the twenty-one problems shown to be NP-complete by Karp [6]. The Unique Cover problem was introduced by Demaine et al in [1], and it may be considered a natural "maximization" variant of Set Cover, and also a generalization of the Max Cut problem. The Unique Set Cover problem combines elements of both these variants, and is NP-complete as well.

The Geometric Setting. Geometric settings are among the most promising contexts for developing improved algorithms when faced with hardness in a general setting. The geometric nature of the problem opens up several algorithmic possibilities, and this is amply evidenced in the context of approximation algorithms. Many geometric problems are known to admit good approximation algorithms, even PTASes. In particular, the classical set cover and hitting set problems have been very well-explored in the context of geometric objects [5,12]. In this situation, the universe is a point set in d-dimensional Euclidean space, and the sets are defined by intersection of geometric objects with the point set. An object covers a point if it contains it. We study the unique coverage variants for several geometric objects, including lines, hyperplanes, squares, and rectangles.

Our Approach. In this work, we focus on the parameterized complexity of these problems, both in the general abstract setting and carefully considered special cases. In parameterized complexity each problem instance comes with a parameter k and a central notion in parameterized complexity is *fixed parameter tractability (FPT)*. This means, for a given instance (x, k), solvability in time $f(k) \cdot p(|x|)$, where f is an arbitrary function of k and p is a polynomial in the input size. The parameterized problem is said to admit a *polynomial kernel* if there is a polynomial time algorithm (the degree of polynomial is independent of k), called a *kernelization* algorithm, that reduces the input instance down to an instance with size bounded by a polynomial $p(k)$ in k, while preserving the answer.

Table 1. A summary of our results

<table>
<tr><td></td><td>Exact Cover</td><td></td><td>Unique Cover</td><td>Unique Set Cover</td></tr>
<tr><td>Parameter</td><td>Size of solution</td><td></td><td colspan="2">Number of elements uniquely covered</td></tr>
<tr><td rowspan="2">Abstract Sets</td><td>W[1]-hard</td><td></td><td>FPT</td><td>W[1]-hard</td></tr>
<tr><td></td><td></td><td>Quadratic Element Kernel</td><td></td></tr>
<tr><td rowspan="2">Lines</td><td>FPT</td><td rowspan="4">VC-Dimension</td><td>FPT</td><td>FPT</td></tr>
<tr><td>Quadratic Kernel</td><td>Quadratic Kernel</td><td>Poly Kernel</td></tr>
<tr><td>Hyperplanes $\mathbb{R}^d$</td><td>$k^{O(d^2)}$
kernel</td><td>$k^{O(d)}$ Instance Kernel
(Quadratic Element Kernel)</td><td>$k^{O(d^2)}$
kernel</td></tr>
<tr><td>Unit Squares</td><td>W[1]-hard</td><td>Poly Kernel</td><td>Open</td></tr>
</table>

Studying the parameterized complexity of geometric problems has interesting implications. On the one hand, a tractability result demonstrates the utility of the geometric structure in contrast with the abstract setting. On the other, a hardness result often has consequences for hardness of approximation; usually it establishes evidence for the non-existence of the EPTAS. This has motivated several studies of geometric problems from a parameterized perspective [8].

Our Results. In this work, we establish the following results, summarized also in Table 1.

Exact Cover. We show that Exact Cover is W[1]-hard even in the restricted setting where all the objects are unit squares (Lemma 1). On the positive side, we show that Exact Cover is FPT for lines (Lemma 2). Further, if the objects are hyperplanes in a d-dimensional Euclidean space, the Exact Cover continues to be FPT parameterized by k and d (Lemma 3).

Unique Cover. For Unique Cover, a simple argument shows that the number of elements in the universe can be bounded by $O(k^2)$ (Lemma 4). This shows that the problem is FPT. It turns out that this also implies a polynomial kernel for various geometric objects (Corollary 2), using the fact that these objects have bounded VC Dimension.

Unique Set Cover. We show that Unique Set Cover is W[1]-hard in the general setting (Lemma 5) and NP-complete when restricted to lines (Lemma 6). On the positive side, we show that the problem is FPT for families of bounded intersection (Lemma 7) and hyperplanes in d dimensions (Lemma 8).

2 Preliminaries

Parameterized Complexity. A parameterized problem Π is a subset of $\Sigma^* \times \mathbb{N}$. Given a parameterized decision problem with input $x \in \Sigma^*$ of size n, and an integer parameter k, the goal in parameterized complexity is to design a deterministic algorithm which decides the membership of the instance (x, k) in Π in time $f(k)n^{\mathcal{O}(1)}$, where f is a function of k alone. Problems which admit such algorithms are said to be fixed parameter tractable (FPT). We call an algorithm, with a running time of $f(k)n^{\mathcal{O}(1)}$, an FPT algorithm, and such a running time, an FPT running time. The theory of parameterized complexity was developed by Downey and Fellows [3]. For recent developments, see the book by Flum and Grohe [4].

Definition 1. *A parameterized problem Π FPT-many-one reduces to another parameterized problem Γ, if there is a polynomial p, computable functions $f, g : N \rightarrow N$, and a Turing machine T such that, given any input instance (x, k), T outputs an instance (x', k') within $f(k)p(|x|)$ time, with $(x, k) \in \Pi$ if and only if $(x', k') \in \Gamma$, and $k' \leq g(k)$.*

There is a hierarchy of problems in parameterized complexity. For the purpose of this paper we will define the class $W[1]$ with respect to a hard problem in this class. The class $W[1]$ is the class of all parameterized problems that FPT-many-one reduce to the k-Clique problem parameterized by k. A parameterized problem is W[1]-hard if there is a FPT-many-one reduction from k-Clique, parameterized by k, to the given problem. It is widely believed that FPT $\subset$ W[1]. For further understanding of the various parameterized classes, refer to Flum and Grohe [4].

Kernelization. A *kernelization* algorithm for a parameterized problem Π is a polynomial time procedure which takes as input an instance (x, k), where k is the parameter, and returns an instance (x', k') such that $(x, k) \in \Pi$ if and only if $(x', k') \in \Pi$ and $|x'| \leq g(k)$ and $k' \leq h(k)$, for some computable functions g, h. The returned instance (x', k') is said to be the kernel for the instance (x, k) of Π.

Problem Definitions. A set system is a pair $(U, \mathcal{F})$, where U is a universe of n elements and $\mathcal{F}$ is a family of m subsets of U. Given a set system $(U, \mathcal{F})$, a set S is said to cover an element $p \in U$ if $p \in S$. An element p is said to be covered *uniquely* by a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ if there is exactly one set in $\mathcal{F}'$ which contains p. The SET COVER problem asks for a smallest collection of subsets whose union covers every element in the universe. We are now ready to define some of the variations of this problem that we consider in our work.

EXACT COVER **Parameter:** k
Input: A set system $(U, \mathcal{F})$ of n elements and m sets, and a positive integer k.
Question: Is there a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ of size at most k that covers every element of U such that each element is contained in exactly one set in $\mathcal{F}'$?

UNIQUE COVER **Parameter:** k
Input: A set system $(U, \mathcal{F})$ of n elements and m sets, and a positive integer k.
Question: Is there a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ such that, among the set of elements covered by $\mathcal{F}'$, there is a subset $S \subset U$, $|S| \geq k$ with each element of S being contained in exactly one set in $\mathcal{F}'$?

Unique Set Cover	**Parameter:** k
Input: A set system $(U, \mathcal{F})$ of n elements and m sets, and a positive integer k.	
Question: Is there a subfamily $\mathcal{F}' \subseteq \mathcal{F}$ that covers every element of U such that there is a subset $S \subseteq U$, $\|S\| \geq k$ where each element of S is contained in exactly one set in $\mathcal{F}'$?	

We consider some notions that will be useful in defining special set systems that we will encounter during our study of these problems. A set system $(U, \mathcal{F})$ is said to have *bounded intersection* when there is a universal constant c such that for any pair of sets $F_1, F_2 \in \mathcal{F}$, $|F_1 \cap F_2| \leq c$. A set system $(U, \mathcal{F})$ is said to be a set system of squares (lines) if U is a subset of n points in $\mathbb{R}^2$ and each set $F \in \mathcal{F}$ is the maximal set of points of U that are contained in a square(line) defined on $\mathbb{R}^2$. Similarly, a set system $(U, \mathcal{F})$ is said to be a set system of hyperplanes if U is a set of n points in $\mathbb{R}^d$, for a fixed positive integer d, and each set $F \in \mathcal{F}$ is the maximal set of points of U that are contained in a hyperplane defined on $\mathbb{R}^d$.

Given a set system $(U, \mathcal{F})$ of n elements and m sets, for every subset $A \subseteq U$ we define the family of sets $\mathcal{F}_A = \{S \cap A | S \in \mathcal{F}\}$.

Definition 2 (VC Dimension). *Let $(U, \mathcal{F})$ represent a set system. A subset $A \subseteq U$ is said to be* shattered *if for every $B \subseteq A$, there exists $F \in \mathcal{F}$ such that $F \cap A = B$. The* Vapnik-Chervonenkis dimension*(or* VC dimension *of $(U, \mathcal{F})$ is the supremum of the sizes of all shattered subsets of U.*

Therefore, in general, the VC dimension of a set system could be infinite. However, set systems of several geometric objects are known to have bounded VC dimension. We refer the reader to [9] for further details on VC Dimension. The following result is known from [13] about set systems of finite VC dimension.

Proposition 1. *Let $(U, \mathcal{F})$ be a set system with $|U| = n$ and VC dimension d. Then $|\mathcal{F}| \leq \binom{n}{0} + \binom{n}{1} + \cdots + \binom{n}{d}$*

Hyperplanes An i-flat in $\mathbb{R}^d$ is the affine hull of $i+1$ affinely independent points. The dimension of a (possibly infinite) set of points P, denoted as $dim(P)$, is the minimum i such that the entire set P is contained in an i-flat of $\mathbb{R}^d$ [7].

Observation 1. *[7] For a pair of i-flat H_1 and j-flat H_2 , $1 \leq j, i \leq d-1$, if $H_1 \not\subset H_2$ and $H_1 \not\subset H_2$, then $dim(H_1 \cap H_2) < min\{i, j\}$.*

In this paper we refer to $(d-1)$-flats of $\mathbb{R}^d$ as hyperplanes.

3 Exact Cover

In this section, we consider the Exact Cover problem, parameterized by the number k of sets in a solution family. Since this problem is known to be W[1]-hard, it is natural to introduce properties on the input set family and see whether the added structure makes the problem easier in these special cases. Here, we restrict ourselves to geometric versions, where the universe is a set of points in a real space $\mathbb{R}^d$, for an appropriate integer d, while the set family is such that every set satisfies a particular geometric property.

The W[1]-hardness of Exact Cover was shown in [11]. We give an alternative proof for this W[1]-hardness. This proof, with a little bit of modification, shows that Exact Cover on set systems of unit squares is also W[1]-hard.

Proposition 2 ($\star$). Exact Cover *is* W[1]-*hard.*[1]

Lemma 1 ($\star$). Exact Cover *on set systems of unit squares is* W[1]-*hard.*

Kernels for Exact Cover *with Lines and Hyperplanes.* In contrast with the hardness results that we have seen so far, we now turn to some algorithmic results. First, when we consider our input universe to be a set of n points in $\mathbb{R}^2$ and our sets to be maximal sets of collinear points, we obtain a quadratic kernel using a "high degree" reduction rule. This version of Exact Cover is also NP-complete, and the proof for NP-hardness is very similar to the proof of Lemma 6 given in the Appendix.

Let $(P, \mathcal{L})$ be the input set system. In our discussion, we use the terms sets and lines interchangeably. The input family could contain sets containing single points. Our first reduction rule is taken directly from [7]. We remove all lines (but for one) that pass through exactly one point.

Reduction Rule 1. *For any input point* p*, from the set* $\mathcal{L}_p = \{L | L \in \mathcal{L}$ *and* $L \cap P = \{p\}\}$*, delete all lines but one (say* L_p*).*

Proposition 3. *Reduction Rule 1 is sound.*

Reduction Rule 2. *In an instance*$(P, \mathcal{L}, k)$*, if there is a line* L *that contains at least* $k+1$ *input points then delete* L *and all lines intersecting with points on* L *and decrease* k *by one. All points contained in* L *are deleted from the universe* U*. Formally, the reduced instance is* $(P \setminus L, \mathcal{L} \setminus \{T \mid L \cap T \neq \emptyset\}, k-1)$.

A similar reduction was given in [7] to exhibit a polynomial kernel for Point Line Cover. We show the correctness of this reduction in this case.

Claim. Reduction Rule 2 is sound.

Proof. Suppose there is a solution, $\mathcal{L}'$, for $(P, \mathcal{L}, k)$ which does not contain L. Since $\mathcal{L}'$ is a set cover of size at most k that excludes L, it includes at least $(k+1)$ other lines to cover the points on L, as two lines intersect at at most one point. This contradicts that $\mathcal{L}'$ is a solution of the instance $(P, \mathcal{L}, k)$. Note that this argument shows that *any* valid solution of the instance $(P, \mathcal{L}, k)$ must contain L. Now, consider the subfamily of $\mathcal{L}$ comprising of lines that contain points belonging to L:

$$\mathcal{L}'' := \{T \in \mathcal{L} \mid T \neq L, T \cap L \neq \emptyset\}.$$

[1] The proofs of results marked with a $\star$ are in the Appendix.

Clearly, any solution that contains L does not contain any element of $\mathcal{L}''$, therefore, it is safe to remove these sets from the instance, establishing the correctness of Reduction Rule 2. □

The reduction is very robust, in the sense that a line as described above will belong to any solution of the Exact Cover instance. On exhaustive application of this reduction, a YES instance can have at most k^2 remaining input points, since k lines can only cover k^2 points when each line has at most k points.

Therefore, if our reduced instance has more than k^2 points, we correctly return NO. Otherwise, due to Reduction Rule 1 and by the property of lines, we can also bound the number of lines in the reduced instance to at most k^4. Thus, we have shown the following.

Lemma 2. Exact Cover *on set systems of lines is* FPT*, with a polynomial kernel.*

A natural question is to consider input instances of Exact Cover which are set systems of hyperplanes in $\mathbb{R}^d$. We parameterize the Exact Cover problem in this case by $k+d$. The following Lemma is obtained by extending the reduction rules in [7]. In particular, it is shown in [7] that for any $1 \leq i \leq d-1$, if an i-flat covers more than $k^i + 1$ points, then these points can be replaced with one representative. The crux of the argument is that all of these points are covered "together" by a single hyperplane in any valid solution. In our argument, we further this reduction rule by deleting all hyperplanes that contain a strict subset of these points, because such hyperplanes are automatically forbidden from being a part of any valid solution of Exact Cover. We refer the reader to the Appendix and [7] for further details.

Lemma 3 ($\star$). Exact Cover *on set systems of hyperplanes in* $\mathbb{R}^d$ *is* FPT *parameterized by* $k + d$.

4 Unique Cover

The Unique Cover problem was studied in [11] and was found to be FPT. However, the problem does not have a polynomial kernel unless $NP \subseteq coNP/poly$, as shown in [2]. In this section, we exhibit polynomial sized kernels for several geometric versions, exploiting the geometric property satisfied by each set of the input set system.

Recall that the Unique Cover problem is parameterized by the number of elements that we desire to cover uniquely. To begin with, in the abstract setting, we show that the number of elements in the universe can be bounded by k^2. Note that it is straightforward to bound the sizes of the individual sets in an instance of Unique Cover, with the following observation.

Observation 2. *If there exists* $F_i \in \mathcal{F}$ *such that* $|F_i| \geq k$*, then the given instance is a* YES *instance, with* $\{F_i\}$ *serving as a valid solution.*

We now turn to an argument for bounding the size of the universe in an instance of Unique Cover.

Lemma 4 (⋆). Unique Cover *admits a quadratic element kernel.*

The following result regarding sets of bounded VC Dimension are now implied by Proposition 1 and Lemma 4.

Corollary 1. Unique Cover *on set systems of VC Dimension bounded by a constant d admits a polynomial kernel.*

As an immediate consequence, we get the existence of polynomial kernels in special geometric cases, since these geometric set families have constant VC Dimension.

Corollary 2 (⋆). Unique Cover *admits a polynomial kernel for set systems of lines, hyperplanes, axis-parallel rectangles and disks.*

5 Unique Set Cover

We show that Unique Set Cover is W[1]-hard. However, as with the other problems, assuming geometric properties on the set family provides positive algorithmic results.

Lemma 5 (⋆). Unique Set Cover *is* W[1]*-hard.*

The reduction also shows that Unique Set Cover is NP-hard. In fact, even when we consider the special case when the universe U is a set of n points in $\mathbb{R}^2$ and each set is a line, the Unique Set Cover problem turns out to be NP-hard, as we show in our next lemma.

Lemma 6 (⋆). Unique Set Cover *on set systems of lines is* NP*-complete.*

This NP-hardness reduction is very similar to the NP-hardness reduction for Point Line Cover [10]. Unlike [10], we reduce the problem from 1-IN-3-SAT, instead of 3-SAT.

This implies that Unique Set Cover on set systems with bounded intersection and Unique Set Cover on set systems of hyperplanes are also NP-hard. In the parameterized context, although the problem is W[1]-hard in the general setting, we look at some special cases where this problem can be solved in FPT time. First, we make a few observations.

Observation 3. *Let $\mathcal{F}'$ be a solution for* Unique Set Cover*. Then any minimal set cover contained in $\mathcal{F}'$ is also a solution for* Unique Set Cover*.*

Thus, it is enough to find a minimal set cover that covers at least k elements uniquely.

Observation 4. *Any minimal set cover of size at least k is a* Unique Set Cover *solution for a given instance. In particular, if the minimum set cover of the given instance is of size at least k then it is a* YES *instance for the* Unique Set Cover *problem.*

Now, we turn to algorithmic results for special cases of Unique Set Cover.

Sets of Bounded Intersection. First, we consider set systems $(U, \mathcal{F})$ where $\mathcal{F}$ has the property that for any pair of sets $F_1, F_2 \in \mathcal{F}$, $|F_1 \cap F_2| \leq c$.

Lemma 7. Unique Set Cover *for sets of bounded intersection c is* FPT *when parameterized by $c + k$.*

Proof. Construct a minimal set cover $\mathcal{S}$ for the instance. If the size of $\mathcal{S}$ is at least k, then it is a solution for Unique Set Cover, by Observation 4. Similarly, if there is a set $S \in \mathcal{S}$ that contains at least k private elements, then too $\mathcal{S}$ is a solution for Unique Set Cover. Suppose there are at most $k-1$ sets in $\mathcal{S}$ and each set has at most $k-1$ private elements. By pigeonhole principle, there must be a set $S \in \mathcal{S}$ which contains at least $n/(k-1)$ elements of U. The number of elements in S that can belong to other sets of $\mathcal{S}$ is at most $c(k-2)$, because of the bounded intersection property. If $\frac{n}{k-1} - c(k-2) \geq k$ then the given instance is a YES instance. Otherwise, $\frac{n}{k-1} - c(k-2) \leq k-1$, which implies that $n \leq (1+c)(k-1)^2$.

Since the sets have bounded pairwise intersection of at most c, any subset of $c+1$ elements can appear together in at most one set. Therefore, the number of sets are bounded by $n^{c+1} \leq ((1+c)(k-1)^2)^{c+1}$. We can now guess the uniquely covered elements, and the distribution of the uniquely covered elements in a Unique Set Cover solution. Finally, we can check whether there are sets in $\mathcal{F}$ to validate the guess in polynomial time. As the number of guesses is an FPT function, the running time of this algorithm is FPT. $\square$

Hyperplanes in $\mathbb{R}^d$. Next, we consider a geometric set system $(U, \mathcal{F})$ where U is a set of n points in $\mathbb{R}^d$ and sets in $\mathcal{F}$ are defined by hyperplanes in $\mathbb{R}^d$. When $d = 2$, these are lines and this is a special case of sets with bounded intersection. For $d > 2$, hyperplanes do not have this property. Nonetheless, we obtain an FPT algorithm for hyperplanes, by reducing the given instance to an instance of Unique Set Cover for sets with bounded intersection.

Lemma 8. Unique Set Cover *on set systems of hyperplanes in $\mathbb{R}^d$ is* FPT.

Proof. Let U be the universe of n elements and $\mathcal{F}$ be the family of m hyperplanes in $\mathbb{R}^d$. The following Reduction Rule aims at reducing the number of points, while maintaining the Unique Set Cover solution if there exists one.

Reduction Rule 3. *for i from 1 to $d-1$:*
Suppose $P \subseteq U$ is a set of at least k^i points such that $dim(P) = i$, and P is contained in at least one hyperplane of $\mathcal{F}$. Suppose $\mathcal{F}_P \subseteq \mathcal{F}$ is the set of all hyperplanes that contain P. If $\mathcal{F} \setminus \mathcal{F}_P$ is a set cover for U, then we say YES *for our input instance and exit. Otherwise, we delete all but k^i points of P from the universe. If a hyperplane becomes empty, we delete that hyperplane from $\mathcal{F}$.*

We prove the correctness of this reduction rule by induction on i. When $i = 1$, P is a line with more than k points. We abuse notation and also use P to refer to this collection of points. Suppose $\mathcal{F} \setminus \mathcal{F}_P$ is a set cover for the instance, let $\mathcal{G}$ be a minimal set cover obtained from $\mathcal{F} \setminus \mathcal{F}_P$. Then, by Observation 1, any set in $\mathcal{G}$ can contain at most one point of P. To cover all elements of P, there must be at least $k+1$ sets in $\mathcal{G}$. By Observation 4, we correctly say YES. If no such set cover exists, then we know that any set cover for the input instance must contain at least one hyperplane from $\mathcal{F}_P$. Let $P' \subset P$ be the set of all but k points that are deleted by the Reduction Rule and $P'' = P \setminus P'$. Also, let $\mathcal{F}'$ be the family of hyperplanes that became empty and got deleted from $\mathcal{F}$. Suppose $\mathcal{G} = \{H_1, \ldots, H_l\}$ is a minimal solution for $(U, \mathcal{F}, k)$. Since P is a line with more than k points, there must be at least one H_i that contains all of P. Now, the following cases can occur:

1. Suppose there are two planes $H_i, H_j, 1 \leq i \neq j \leq l$, both of which contain the set P. Then none of the points in P are uniquely covered by this solution. The points which are uniquely covered are not deleted as a result of this Reduction Rule. Also, by definition of minimality, no hyperplane of $\mathcal{G}$ could have become empty after this Reduction Rule was applied. Hence, $\mathcal{G}$ remains a solution for Unique Set Cover in $(U \setminus P', \mathcal{F} \setminus \mathcal{F}', k)$.
2. Suppose $l > k$. Since we have dealt with the case when P is covered be at least 2 hyperplanes of $\mathcal{G}$, we can assume that there is exactly one hyperplane in $\mathcal{G}$ that contains P. There are at least k remaining hyperplanes in $\mathcal{G}$. Since, $\mathcal{G}$ is a minimal solution for Unique Set Cover, each of these remaining hyperplanes cover a point uniquely, and none of these uniquely covered points belong to P. Hence, $\mathcal{G}$ remains a solution for $(U \setminus P', \mathcal{F} \setminus \mathcal{F}', k)$.
3. Finally, suppose $l \leq k$, and as before, let $H \in \mathcal{G}$ be the hyperplane that contains all points in P. Then at most $l-1$ points of P are not uniquely covered. All other points of P must be uniquely covered. In particular, at most $l-1$ points of $P \setminus P'$ are not uniquely covered , which implies that at least $k-l+1$ points in $P \setminus P'$ are uniquely covered by $\mathcal{G}$. By minimality, for each hyperplane H' in $\mathcal{G} \setminus \{H\}$ there is a point $p_{H'}$ that is uniquely covered by H'. Thus, at least $k-l+1$ points from $P \setminus P'$ and $l-1$ points from $\mathcal{G} \setminus \{H\}$ are covered uniquely. Again, by minimality, no hyperplane of $\mathcal{G}$ could have become empty because of application of the Reduction Rule. Hence, $\mathcal{G}$ is a solution in $(U \setminus P', \mathcal{F} \setminus \mathcal{F}', k)$.

On the other hand, let $\mathcal{G}'$ be a minimal solution for $(U \setminus P', \mathcal{F} \setminus \mathcal{F}', k)$. Assume $\mathcal{G}'$ is not a solution for $(U, \mathcal{F}, k)$. Then each point in P'' is covered by a different set in $\mathcal{G}'$. Let this subfamily of $\mathcal{G}'$, with at least k hyperplanes, be $\mathcal{H}'$.Let $G \in \mathcal{F}_P$. Consider $\mathcal{G}' \cup G$, which is clearly a set cover for $(U, \mathcal{F})$. Let $\mathcal{S} \subset \mathcal{G}' \cup G$ be a minimal set cover of $(U, \mathcal{F})$. Suppose $P_1 \subseteq (U \setminus P')$ was a set of k points, such that for each hyperplane H in $\mathcal{H}'$ there is a point in P_1 that it uniquely covers with respect to $\mathcal{G}'$. Let $P_2 \subseteq P_1$ be uniquely covered by $\mathcal{S}$. Each point in $P_1 \setminus P_2$ has exactly one hyperplane in $\mathcal{S}$, other than G, containing it. By the minimality of $\mathcal{S}$, this hyperplane has a point that is uniquely covered by it. Therefore, for all of the points in $P_2 \setminus P_1$, either that point is uniquely covered by G or a hyperplane (other than G) in $\mathcal{S}$ containing it is uniquely covering another point. Therefore $\mathcal{S}$ still covers at least k points uniquely and is a solution for $(U, \mathcal{F}, k)$.

Now, assume $i > 1$ and the Induction Hypothesis is true for all $j < i$. By arguments similar to the base case, the reduction rule is sound for i. The full proof is in the Appendix. $\square$

We exhaustively apply this Reduction Rule. At the end, any hyperplane contains at most k^{d-1} points. Let $\mathcal{G}$ be a minimal set cover for the instance. If there are at least $k+1$ hyperplanes in $\mathcal{G}$, then due to Observation 4, we correctly say YES. Otherwise, there are at most k hyperplanes in the set cover $\mathcal{G}$, which implies that $|U| \leq k^{(d-1)} \cdot k$. The number of hyperplanes that can contain these points is at most $(k^d)^d$. Thus, we have a kernel for the problem. For the algorithm, we guess k points $P \subseteq U$ that are uniquely covered by a solution and the family $\mathcal{G}$ of at most k hyperplanes that are responsible for this unique coverage. Let $\mathcal{F}^P = \{H | H \in \mathcal{F} \setminus \mathcal{G}, \exists p \in P \text{ s.t } p \in H\}$. We check whether the family $\mathcal{F} \setminus \mathcal{F}^P$ is a set cover or not. There are at most $\mathcal{O}(k^{kd^2})$ possible pairs $(P, \mathcal{G})$. Thus the problem is FPT. $\square$

References

1. Demaine, E.D., Feige, U., Hajiaghayi, M., Salavatipour, M.R.: Combination Can Be Hard: Approximability of the Unique Coverage Problem. SIAM J. Comput. () **38**(4), 1464–1483 (2008)
2. Dom, M., Lokshtanov, D., Saurabh, S.: Kernelization Lower Bounds Through Colors and IDs. ACM Trans. Algorithms **11**(2), 13:1–13:20 (2014). doi:10.1145/2650261
3. Downey, R.G., Fellows, M.R.: Parameterized Complexity, p. 530. Springer-Verlag (1999)
4. Flum, J., Grohe, M.: Parameterized Complexity Theory (Texts in Theoretical Computer Science. An EATCS Series). Springer-Verlag New York Inc (2006)
5. Hochbaum, D.S., Maass, W.: Fast approximation algorithms for a nonconvex covering problem. Journal of algorithms **8**(3), 305–323 (1987)
6. Karp, R.M.: Reducibility Among Combinatorial Problems. Complexity of Computer Computations, pp. 85–103 (1972)
7. Langerman, S., Morin, P.: Covering things with things. Discrete & Computational Geometry **33**(4), 717–729 (2005)
8. Marx, Dániel: Efficient approximation schemes for geometric problems? In: Brodal, Gerth Stølting, Leonardi, Stefano (eds.) ESA 2005. LNCS, vol. 3669, pp. 448–459. Springer, Heidelberg (2005)
9. Matoušek, J.: Lectures on discrete geometry, vol. 108. Springer, New York (2002)
10. Megiddo, N., Tamir, A.: On the Complexity of Locating Linear Facilities in the Plane. Oper. Res. Lett. **1**(5), 194–197 (1982). doi:10.1016/0167-6377(82)90039-6
11. Misra, N., Moser, H., Raman, V., Saurabh, S., Sikdar, S.: The Parameterized Complexity of Unique Coverage and Its Variants. Algorithmica, 517–544 (2013)
12. Mustafa, N., Ray, S.: PTAS for geometric hitting set problems via local search. In: Proceedings of the 25th annual symposium on Computational geometry, pp. 17–22. ACM (2009)
13. Vapnik, V.N., Chervonenkis, A.Y.: On the uniform convergence of relative frequencies of events to their probabilities. Theory of Probability & Its Applications **16**(2), 264–280 (1971)