On the Complete Width and Edge Clique Cover Problems

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Abstract. A complete graph is the graph in which every two vertices are adjacent. For a graph $G = (V, E)$, the complete width of G is the minimum k such that there exist k independent sets $\mathtt{N}_i \subseteq V$, $1 \leq i \leq k$, such that the graph G' obtained from G by adding some new edges between certain vertices inside the sets $\mathtt{N}_i$, $1 \leq i \leq k$, is a complete graph. The complete width problem is to compute the complete width of a given graph. In this paper we study the complete width problem. We show that the complete width problem is NP-hard on $3K_2$-free bipartite graphs and polynomially solvable on $2K_2$-free bipartite graphs and on $(2K_2, C_4)$-free graphs. As a by-product, we obtain the following new results: the edge clique cover problem is NP-complete on $K_{2,2,2}$-free co-bipartite graphs and polynomially solvable on $K_{2,2}$-free co-bipartite graphs and on $(2K_2, C_4)$-free graphs. We also determine all graphs of small complete width $k \leq 3$.

Keywords: Probe graphs · Split graphs · Bipartite graphs · Complete width · Edge clique cover

1 Introduction

Let $G = (V, E)$ be a simple and undirected graph. A subset $U \subseteq V$ is an *independent set*, respectively, a *clique* if no two, respectively, every two vertices of U are adjacent. The complete graph with n vertices is denoted by K_n. The path and cycle with n vertices of length $n-1$, respectively, of length n, is denoted by P_n, respectively, C_n. For a vertex $v \in V$ we write $N(v)$ for the set of its neighbors in G. A *universal* vertex v is one such that $N(v) \cup \{v\} = V$. For a subset $U \subseteq V$ we write $G[U]$ for the subgraph of G induced by U and $G - U$ for the graph $G[V - U]$; for a vertex v we write $G - v$ rather than $G[V \setminus \{v\}]$.

Given a graph class $\mathcal{C}$, a graph $G = (V, E)$ is called a *probe $\mathcal{C}$ graph* if there exists an independent set $\mathtt{N} \subseteq V$ (of *nonprobes*) and a set of new edges $E' \subseteq \binom{\mathtt{N}}{2}$ between certain nonprobe vertices such that the graph $G' = (V, E \cup E')$ is in the class $\mathcal{C}$, where $\binom{\mathtt{N}}{2}$ stands for the set of all 2-element subsets of $\mathtt{N}$. A graph $G = (V, E)$ with a *given* independent set $\mathtt{N} \subseteq V$ is said to be a *partitioned probe*

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$\mathcal{C}$ *graph* if there exists a set $E' \subseteq \binom{\mathtt{N}}{2}$ such that the graph $G' = (V, E \cup E')$ is in the class $\mathcal{C}$. In both cases, G' is called a $\mathcal{C}$ *embedding* of G. Thus, a graph is a (partitioned) probe $\mathcal{C}$ graph if and only if it admits a $\mathcal{C}$ embedding.

Recently, the concept of probe graphs has been generalized as a width parameter of graph class in [4]. Let $\mathcal{C}$ be a class of graphs. The $\mathcal{C}$*-width* of a graph G is the minimum number k of independent sets $\mathtt{N}_1, \ldots, \mathtt{N}_k$ in G such that there exists an embedding $G' \in \mathcal{C}$ of G such that for every edge xy in G' which is not an edge of G there exists an i with $x, y \in \mathtt{N}_i$. A collection of such k independent sets $\mathtt{N}_i, i = 1, \ldots, k$ is called a $\mathcal{C}$ *witness* (of G'). In the case $k = 1$, G is a *probe* $\mathcal{C}$*-graph*. The $\mathcal{C}$*-width* problem asks for a given graph G and an integer k if the $\mathcal{C}$-width of G is at most k. Graphs of $\mathcal{C}$-width k are also called k-probe $\mathcal{C}$-graph. Note that graphs in $\mathcal{C}$ are, by convenience, 1-probe $\mathcal{C}$-graphs.

In [4], the complete width and block-graph width have been investigated. The authors proved that, for fixed k, graphs of complete width k can be characterized by finitely many forbidden induced graphs, their proof is however not constructive. They also showed, implicitly, that complete width k graphs and block-graph width k graphs can be recognized in cubic time. The case $k = 1$, *e.g.*, probe complete graphs and probe block graphs, has been discussed in depth in [17]. The case $k = 2$ is discussed in [18].

Graphs do not contain an induced subgraph isomorphic to a graph H are called H*-free*. More generally, a graph is $(H_1, \ldots, H_t)$*-free* if it does not contain an induced subgraph isomorphic to one of the graphs $H_1, \ldots, H_t$. For two graphs G and H, we write $G + H$ for the disjoint union of G and H, and for an integer $t \geq 2$, tG stands for the disjoint union of t copies of G. The complete k-partite with n_i vertices in color class i is denoted by $K_{n_1,\ldots,n_k}$. For graph classes not defined here see, for example, [2,3,10].

In this paper we study the COMPLETE WIDTH problem (given G and k, is the complete width of G at most k?). We show that

- COMPLETE WIDTH is NP-complete, even on $3K_2$-free bipartite graphs, and
- computing the complete width of a $2K_2$-free bipartite graph (chain graph), and more generally, of a $(2K_2, K_3)$-free graph can be done in polynomial time,
- computing the complete width of a $2K_2$-free chordal graph (split graph), and more generally, of a $(2K_2, C_4)$-free graph can be done in polynomial time.

Moreover, we give structural characterizations for graphs of complete width at most 3.

In the next section we point out a relation between complete width and the most popular notion of edge clique cover of graphs. Then we prove our results in the last three sections. As we will see, it follows from our results on complete width that edge clique cover is NP-complete on $K_{2,2,2}$-free co-bipartite graphs and is polynomially solvable on $K_{2,2}$-free co-bipartite graphs.

2 Complete Width and Edge Clique Cover

An *edge clique cover* of a graph G is a family of cliques (complete subgraphs) such that each edge of G is in at least one member of the family. The minimal cardinality of an edge clique cover is the *edge clique cover number*, denoted by $\theta_e(G)$.

The EDGE CLIQUE COVER problem, the problem of deciding if $\theta_e(G) \leq k$, for a given graph G and an integer k, is NP-complete [13,16,22], even when restricted to graphs with maximum degree at most six [14], or planar graphs [6]. EDGE CLIQUE COVER is polynomially solvable for graphs with maximum degree at most five [14], for line graphs [22,23], for chordal graphs [7,24], and for circular-arc graphs [15].

In [16] it is shown that approximating the clique covering number within a constant factor smaller than two is NP-hard. In [11], it is shown that EDGE CLIQUE COVER is fixed-parameter tractable with respect to parameter k; see also [8] for more recent discussions on the parameterized complexity aspects.

We write $cow(G)$ to denote the complete width of the graph G. As usual, $\overline{G}$ denotes the complement of G. In [4], the authors showed that COMPLETE WIDTH is NP-complete on general graphs, by observing that

Proposition 1 ([4]). *For any graph G, $cow(G) = \theta_e(\overline{G})$*

Proposition 1 and the known results about EDGE CLIQUE COVER imply:

Theorem 1. (1) *Computing the complete width is NP-hard, and remains NP-hard when restricted to graphs of minimum degree at least $n - 7$, and to co-planar graphs.*
(2) *Computing the complete width of graphs of minimum degree at least $n - 6$ and of co-chordal graphs can be done in polynomial time.*

In [5], it is conjectured that EDGE CLIQUE COVER, and thus COMPLETE WIDTH, is NP-complete for P_4-free graphs (also called cographs).

3 Computing Complete Width is Hard for $3K_2$-free Bipartite Graphs

A bipartite graph $G = (V, E)$ is a graph whose vertex set V can be partitioned into two sets X and Y such that for any edge $xy \in E$, $x \in X$ and $y \in Y$. Bipartite graphs without induced cycles of length at least six are called *chordal bipartite*. A *biclique cover* of a graph G is a family of complete bipartite subgraphs of G whose edges cover the edges of G. The *biclique cover number*, also called the *bipartite dimension*, of G is the minimum number of bicliques needed to cover all edges of G.

Given a graph G and a positive integer k, the BICLIQUE COVER problem of G asks whether the edges of G can be biclique covered by at most k bicliques. The following theorem is well known.

Theorem 2 ([21,22]). BICLIQUE COVER *is NP-complete on bipartite graphs, and remains NP-complete on chordal bipartite graphs.*

For convenience, a bipartite graph $G = (V, E)$ with a bipartition $V = X \cup Y$ into independent sets X and Y is denoted as $G = (X + Y, E)$. Let $BC(G) = (X + Y, F)$, where $F = \{xy \mid x \in X, y \in Y, \text{ and } xy \notin E\}$. We call $BC(G)$ the *bipartite complement* of $G = (X + Y, E)$. Note that $BC(C_6) = 3K_2$ and $BC(C_8) = C_8$. Hence if G is chordal bipartite, then $BC(G)$ is $(3K_2, C_8)$-free bipartite.

In [4], the authors showed that the complete width problem is NP-complete on general graphs. We now establish our main theorem for sharping that result of [4].

Theorem 3. COMPLETE WIDTH *is NP-complete on bipartite graphs, and remains NP-complete on $(3K_2, C_8)$-free bipartite graphs.*

Proof. We prove this theorem by reducing BICLIQUE COVER to COMPLETE WIDTH.

Let (G, k) be an input instance of the biclique cover problem, where $G = (X + Y, E)$ is a bipartite graph. We construct an input instance (G', k') of the complete width problem as follows.

- G' is the bipartite graph obtained from the bipartite complement $BC(G) = (X + Y, F)$ of G by adding two new vertices x and y and adding all edges between x and vertices in $Y \cup \{y\}$ and between y and vertices in $X \cup \{x\}$. More formally, $G' = (X' + Y', F')$ with $X' = X \cup \{x\}, Y' = Y \cup \{y\}$, and $F' = F \cup \{xu \mid u \in Y \cup \{y\}\} \cup \{yv \mid v \in X \cup \{x\}\}$.
- Set $k' := k + 2$.

We claim that the biclique cover number of G is at most k if and only if the complete width of G' is at most $k' = k + 2$.

First, let $\{B_i \mid 1 \leq i \leq k\}$ be a biclique cover of G, where $B_i = (X_i + Y_i, E_i)$ with $X_i \subseteq X, Y_i \subseteq Y$. Then, as each B_i is a biclique in G, each $N_i = X_i \cup Y_i$ is an independent set in G'. Set $N_{k+1} := X'$ and $N_{k+2} := Y'$. Then it is easy to check that the $k' = k + 2$ independent sets N_i, $1 \leq i \leq k + 2$, form a complete witness of G'. That is, $cow(G') \leq k'$.

Conversely, let $\{N_i \mid 1 \leq i \leq k + 2\}$ be a complete witness of G'. Then we may assume that

$$x, y \notin N_i, 1 \leq i \leq k.$$

(To see this, consider a vertex $u \in X$. As $\{N_i \mid 1 \leq i \leq k + 2\}$ is a complete witness of G', u and x must belong to N_t for some $t \in \{1, \dots, k+2\}$. Therefore, $N_t \subseteq X \cup \{x\} = X'$ because x is adjacent to all vertices in Y'. Clearly, we can replace N_t by X' and, if $x \in N_i$ for some $i \neq t$, replace N_i by $N_i - \{x\}$ to obtain a new witness such that $N_t = X'$ and x is contained only in N_t. Similarly, there is some s such that $N_s = Y'$ and y is contained only in N_s. By re-numbering if necessary, we may assume that $t = k + 1$ and $s = k + 2$.)

Thus, by construction of G', $N_1, \ldots, N_k$ are independent sets in $BC(G)$ and form a complete witness of $BC(G)$. Therefore, $B_i = G[N_i]$, $1 \leq i \leq k$, are bicliques in G forming a biclique cover of G. That is, $\theta_e(G) \leq k$.

Note that if G is chordal bipartite, then $BC(G)$, hence the bipartite graph G' cannot contain $3K_2$ and C_8 as induced subgraphs. □

Theorem 3 and Proposition 1 imply the following corollary.

Corollary 1. EDGE CLIQUE COVER *is NP-complete on $K_{2,2,2}$-free co-bipartite graphs.*

4 Polynomially Solvable Cases

In this section we establish some cases in which COMPLETE WIDTH can be solved in polynomial time. Actually, in each of these cases we will show that the complete width of the graphs under consideration can be computed in polynomial time.

4.1 $2K_2$-free Bipartite Graphs

Bipartite graphs without induced $2K_2$ are known in literature under the name *chain graphs* ([25]) or *difference graphs* ([12]). They can be characterized as follows.

Proposition 2 (see [20]). *A bipartite graph $G = (X + Y, E)$ is a chain graph if and only if for all vertices $u, v \in X$, $N(u) \subseteq N(v)$ or $N(v) \subseteq N(u)$.*

Theorem 4. *The complete width of chain graphs can be computed in polynomial time.*

Proof. (Sketch) Observe first that if u, v are two vertices in a graph G such that $N(u) = N(v)$ (in particular, u and v are non-adjacent), then $cow(G) = cow(G-u)$ (if v is not universal in $G-u$) or $cow(G) = cow(G-u)+1$ (otherwise).

Let $G = (X+Y, E)$ be a $2K_2$-free bipartite graph with at least three vertices. As observed above, we may assume that for any pair of vertices u, v of G, $N(u) \neq N(v)$. Thus, $|X| \geq 2, |Y| \geq 2$, and G has at most one non-trivial connected component and at most one trivial component which is then the unique isolated vertex of G. Let us also assume that the isolated vertex (if any) of G belongs to X. By Proposition 2, the vertices of X can be numbered $v_1, v_2, \ldots, v_{|X|}$ such that $N(v_1) \subset N(v_2) \subset \ldots \subset N(v_{|X|}) = Y$. Thus, G is disconnected if and only if v_1 is the isolated vertex of G if and only if $N(v_1) = \emptyset$. Clearly, such a numbering can be computed in polynomial time.

Write $\mathtt{N}_i = \{v_1, \ldots, v_i\} \cup (Y \setminus N(v_i))$, $1 \leq i \leq |X|$. Since $N(v_j) \subset N(v_i)$ for $j < i$, $\mathtt{N}_i$ is an independent set, and since $N(v_{|X|}) = Y$, $\mathtt{N}_{|X|} = X$. In case $N(v_1) \neq \emptyset$, let $\mathtt{N}_{|X|+1} = Y$. (Note that in case $N(v_1) = \emptyset$, *i.e.*, v_1 is the isolated vertex of G, $\mathtt{N}_1 = Y \cup \{v_1\}$.)

We claim that

$$cow(G) = \begin{cases} |X|, & \text{if } N(v_1) = \emptyset \\ |X|+1, & \text{otherwise} \end{cases}$$

Moreover, $\mathtt{N}_1, \ldots, \mathtt{N}_{|X|}$ and $\mathtt{N}_{|X|+1}$ (if $N(v_1) \neq \emptyset$) together form a complete witness of G.

Proof of the Claim: First, to see that the collection of the independent sets $\mathtt{N}_1$, $\ldots$, $\mathtt{N}_{|X|}$ and $\mathtt{N}_{|X|+1}$ (if $N(v_1) \neq \emptyset$) is a complete witness of G, let u, v be two non-adjacent vertices of G. If u and v in X, say $u = v_i$ or $v = v_j$ for some $1 \leq i < j \leq |X|$, then $u, v \in \mathtt{N}_j$. If $u \in X$ and $v \in Y$, say $u = v_i$ for some $1 \leq i \leq |X|$, then $u, v \in \mathtt{N}_i$. So let $u, v \in Y$. In this case, let $i \leq j$ be the smallest integers such that $u \in N(v_i), v \in N(v_j)$. If $i > 1$ then $u, v \notin N(v_1)$, hence $u, v \in \mathtt{N}_1$. Thus, let $u \in N(v_1)$. Then, in particular $N(v_1) \neq \emptyset$ and hence $u, v \in \mathtt{N}_{|X|+1} = Y$.

In particular, $cow(G)$ is at most the right hand side stated in the claim.

Next, observe that the claim is clearly true in case $|X| = 2$. So, let $|X| \geq 3$. Note that in $G - v_1$, $N(v_2)$ is not empty, hence by induction, $cow(G - v_1) = |X \setminus \{v_1\}| + 1 = |X|$ and $\mathtt{N}'_1 = \mathtt{N}_2 \setminus \{v_1\}, \ldots, \mathtt{N}'_{|X|-1} = \mathtt{N}_{|X|} \setminus \{v_1\}$ and $\mathtt{N}'_{|X|} = \mathtt{N}_{|X|+1} = Y$ form a complete witness of $G - v_1$. Now, if $N(v_1) = \emptyset$ then $cow(G) \geq cow(G - v_1) = |X|$, hence $cow(G) = |X|$. So, let $N(v_1) \neq \emptyset$. In this case, for any $u \in N(v_2) \setminus N(v_1)$ and any maximal independent set I of G containing v_1 and u, $\mathtt{N}'_i \not\subseteq I$. Thus, $cow(G) \geq cow(G - v_1) + 1 = |X| + 1$, hence $cow(G) = |X| + 1$. □

Theorem 4 and Proposition 1 imply the following corollary.

Corollary 2. *The edge clique cover number of C_4-free co-bipartite graphs can be computed in polynomial time.*

4.2 $(2K_2, K_3)$-free Graphs

We extend Theorem 4 on K_2-free bipartite graphs by showing that COMPLETE WIDTH is polynomially solvable for large class of $2K_2$-free triangle-free graphs.

Theorem 5. *The complete width of $(2K_2, K_3)$-free graphs can be computed in polynomial time.*

Proof. (Sketch) Let G be a $(2K_2, K_3)$-free graph. If G has no induced C_5, then G is $2K_2$-free bipartite, hence we are done by Theorem 4.

So let G contain an induced C_5, say $C = v_1v_2v_3v_4v_5v_1$. As in the proof of Theorem 4, we may assume that $N(u) \neq N(v)$ for any non-adjacent vertices u and v of G. Then it can be shown that C is the non-trivial connected component of G. Thus, $cow(G) = 5$. □

4.3 Split Graphs

A *split graph* is one whose vertex set can be partitioned into a clique Q and an independent set S. For convenience, a split graph is denoted as $G = (Q + S, E)$. It is well known that split graphs can be characterized as follows.

Proposition 3 ([9]). *The following statements are equivalent for any graph G.*

(i) *G a split graph;*
(ii) *G is a $(2K_2, C_4, C_5)$-free graph;*
(iii) *G is a $2K_2$-free chordal graph;*
(iv) *G and $\overline{G}$ are chordal.*

In particular, split graphs are complements of chordal graphs. Hence, by Theorem 1 (2), computing the complete width of split graphs can be done in polynomial time. Below, however, we give here a simple and direct way for doing this. Moreover, our solution will be useful for computing the complete width of pseudo split graphs. The class of pseudo split graphs are not necessarily co-chordal and properly contains all split graphs.

It is not hard to see that a universal vertex is impossible to be a non-probe vertex. Thus in the following, we consider the split graphs $G = (Q + S, E)$ with no universal vertex.

Theorem 6. *For a split graph $G = (Q + S, E)$ with no universal vertex, the complete width of G is either $|Q|$ or $|Q| + 1$.*

Proof. Assume that the complete width of G is k. That is, there is an embedding G' of G such that for every edge xy in G' but not in G there are independent sets $\mathtt{N}_1, \ldots, \mathtt{N}_k$ in G such that $\{x, y\} \subseteq \mathtt{N}_i$ for some i. By the definition, $G[Q]$ is a clique. Thus it is impossible that there are two vertices of Q in the same $\mathtt{N}_i$ for $1 \leq i \leq k$. That is, each $\mathtt{N}_i$ contains at most one vertex in Q. Therefore, the complete width of G is at least $|Q|$.

On the other hand, for each vertex $v \in Q$, let $\mathtt{N}_v = V \setminus N(v)$. Then, each $\mathtt{N}_v$, $v \in Q$, is an independent set. Further, for each $\mathtt{N}_v$, we can fill edges vu, $u \in \mathtt{N}_v - v$. Finally, for the final set S, we make $G[S]$ a clique by filling edge xy for any two vertices $x, y \in S$. The resulting graph is a complete graph. That is, the complete width of G is at most $|Q| + 1$. This completes the proof. □

By Theorem 6, only two cases for determining the complete width of a split graph. For the split graph $G = (Q + S, E)$, let $\mathtt{N}_v = V \setminus N(v)$ for $v \in Q$. We have the following lemma.

Lemma 1. *For a split graph $G = (Q + S, E)$ with no universal vertex, if for any two vertices $x, y \in S$, there is an $\mathtt{N}_v$, $v \in Q$, such that $x, y \in \mathtt{N}_v$, then the complete width of G is $|Q|$; otherwise it is $|Q| + 1$.*

Proof. Assume that for any two vertices $x, y \in S$, there is an $\mathtt{N}_v$, $v \in Q$ such that $x, y \in \mathtt{N}_v$. We show that the complete width of G is $|Q|$. Without loss of generality, we assume all the $\mathtt{N}_v$'s are ordered as the sequence of $\mathtt{N}_1, \mathtt{N}_2, \ldots, \mathtt{N}_{|Q|}$. For completing G into K_n, for each $\mathtt{N}_v$, we fill the edges vu, $u \in (\mathtt{N}_v \cap S)$. Furthermore, assume that $\mathtt{N}_i$ is the last set that contains x and y for any two vertices $x, y \in S$. That is, $\{x, y\} \subseteq \mathtt{N}_i$ but $\{x, y\} \not\subseteq \mathtt{N}_j$ for each $j > i$. Then the edge xy is filled in $\mathtt{N}_i$. By assumption, every edge in $\overline{G}[S]$ can be filled in some $\mathtt{N}_i$. Thus the complete width of G is $|Q|$.

On the other hand, if there is no $\mathtt{N}_i$ contains x and y for some $x, y \in S$, then there is no way to fill x, y in $\mathtt{N}_1, \mathtt{N}_2, \ldots, \mathtt{N}_{|Q|}$. Therefore the complete width of G is $|Q| + 1$. □

By Lemma 1, for any two vertices $x, y \in S$, we can check whether there is a vertex $v \in Q$ such that both xv and yv are not in E. By using adjacency matrix of G, all the work can be done in $O(n^3)$ time. Thus, we have the following theorem.

Theorem 7. *The complete width of split graphs can be computed in polynomial time.*

4.4 Pseudo-split Graphs

Graphs without induced $2K_2$ and no induced C_4 are called *pseudo-split graphs*. Thus, by Proposition 3, the class of pseudo-split graphs properly contains the class of split graphs. Note that a pseudo-split graph may contain an induced C_5, hence it need not be co-chordal. Pseudo-split graphs can be characterized as follows.

Theorem 8 ([1,19]). *A graph is pseudo-split if and only if its vertex set can be partitioned into three sets Q, S, C such that Q is a clique, S is an independent set, C induces a C_5 or is empty, there are all possible edges between Q and C, and there are no edges between S and C.*

Note that it can be recognized in linear time if a graph is a pseudo split graph, and if so, a partition stated in Theorem 8 can be found in linear time [19].

Theorem 9. *The complete width of pseudo-split graphs can be computed in polynomial time.*

Proof. Let $G = (V, E)$ be a pseudo-split graph without universal vertices. Let $V = Q + S + C$ be a partition as in Theorem 8. We may assume that $C \neq \emptyset$ otherwise we are done by Theorem 7.

So let C be the induced $C_5 = v_1v_2v_3v_4v_5v_1$. Then, clearly, the $|Q| + 5$ independent sets $V - N(v)$, $v \in Q$, and $S \cup \{v_i, v_{i+2}\}$ (indices are taken modulo 5), $1 \leq i \leq 5$, can be used for completing G. Thus, by Theorem 6, and by noting that $cow(C_5) = 5$, we have $cow(G) = |Q| + 5$. □

Theorem 4 and Proposition 1 imply the following corollary (note that the complement of a pseudo-split is also a pseudo-split graph).

Corollary 3. *The edge clique cover number of pseudo-split graphs can be computed in polynomial time.*

5 Graphs of Small Complete Width

We describe in this section graphs of small complete width $k \leq 3$. These are particularly $2K_2$-free and our descriptions are good in the sense that they imply polynomial time recognition for these graph classes.

Complete Width-1 and Complete Width-2 Graphs

A *complete split graph* is a split graph $G = (Q + S, E)$ such that every vertex in the clique Q is adjacent to every vertex in the independent set S. Such a partition is also called a *complete split partition* of a split graph. Note that if the complete split graph $G = (Q + S, E)$ is not a clique, then G has exactly one complete split partition $V = Q \cup S$. Furthermore, each vertex in Q is a universal vertex.

Graphs of complete width one can be characterized as follows.

Theorem 10 ([17]). *The following statements are equivalent.*

(i) *G is a probe complete graph;*
(ii) *G is a $\{K_2 + K_1, C_4\}$-free graph;*
(iii) *G is a complete split graph.*

The *join* $G \star H$ is obtained from $G + H$ by adding all possible edges xy between any vertex x in G and any vertex y in H. Graphs of complete width at most two can be characterized as follows.

Theorem 11 ([18]). *A graph G is a 2-probe complete graph if and only if G is $\{2K_2, P_4, K_3 + K_1, (K_2 + K_1) \star 2K_1, C_4 \star 2K_1\}$-free.*

Complete Width-3 Graphs

Substituting a vertex v in a graph G by a graph H results in the graph obtained from $(G - v) \cup H$ by adding all edges between vertices in $N_G(v)$ and vertices in H. The *Net* consists of six vertices a, b, c, a', b' and c' and six edges $aa', bb', cc', a'b', b'c'$ and $a'c'$.

Graphs of complete width at most 3 can be characterized as follows.

Theorem 12. *Let Q be the set (possibly empty) of universal vertices of the graph G. G is a 3-probe complete graph if and only if $G - Q$ has at most one non-trivial connected component which is obtained from the Net by substituting the vertices by (possibly empty) independent sets.*

Proof. (Sketch) First, assume that G is a 3-probe complete graph, and let $\mathtt{N}_1, \mathtt{N}_2, \mathtt{N}_3$ be a complete witness of G. Then G is $2K_2$-free and $Q = V(G) \setminus \mathtt{N}_1 \cap \mathtt{N}_2 \cap \mathtt{N}_3$ is the set of all universal vertices of G. Moreover, as G is $2K_2$-free, $G - Q$ has at most one non-trivial connected component and $I = \mathtt{N}_1 \cap N_2 \cap \mathtt{N}_3$ is the set of all isolated vertices of $G - Q$.

The non-trivial connected component N of $G - Q$ is partitioned into the following six independent sets. $I_{12} = (\mathtt{N}_1 \cap \mathtt{N}_2) \setminus \mathtt{N}_3$, $I_{13} = (\mathtt{N}_1 \cap \mathtt{N}_3) \setminus \mathtt{N}_2$, $I_{23} = (\mathtt{N}_2 \cap \mathtt{N}_3) \setminus \mathtt{N}_1$, $I_1 = \mathtt{N}_1 \setminus (\mathtt{N}_2 \cup \mathtt{N}_3)$, $I_2 = \mathtt{N}_2 \setminus (\mathtt{N}_1 \cup \mathtt{N}_3)$, and $I_3 = \mathtt{N}_3 \setminus (\mathtt{N}_1 \cup \mathtt{N}_2)$. Then there are all possible edges between I_1 and I_2, I_3, I_{23}, between I_2 and I_1, I_3, I_{13}, between I_3 and I_1, I_2, I_{12}, and there are no other edges between these independent sets. Thus, N is obtained from the Net by substituting the vertices by these six independent sets.

Next, assume that $G - Q$ has at most one non-trivial connected component which is obtained from the Net by substituting the vertices by (possibly empty) independent sets. Let I be the set of trivial connected components and let N be the non-trivial connected component of $G - Q$. Let N be obtained from the Net by substituting its vertices a, b, c, a', b', c' by independent set $I_a, I_b, I_c, I_{a'}, I_{b'}, I_{c'}$, respectively. Then $\mathtt{N}_1 = I \cup I_a \cup I_{b'} \cup I_{c'}$, $\mathtt{N}_2 = I \cup I_b \cup I_{a'} \cup I_{c'}$, and $\mathtt{N}_3 = I \cup I_c \cup I_{a'} \cup I_{b'}$ from a complete witness of G. □

We note that, using modular decomposition, one can recognize graphs obtained from the Net by substituting vertices by independent sets in linear time. Hence Theorem 12 gives a linear time recognition for 3-probe complete graphs. We also remark that there is a characterization for 3-probe complete graphs by 14 forbidden induced subgraphs.

6 Conclusion

In this paper we have shown that COMPLETE WIDTH is NP-complete on $3K_2$-free bipartite graphs (equivalently, EDGE CLIQUE COVER is NP-complete on $K_{2,2,2}$-free co-bipartite graphs). So, an obvious open question is: What is the computational complexity of COMPLETE WIDTH on $2K_2$-free graphs? Equivalently, what is the computational complexity of EDGE CLIQUE COVER on C_4-free graphs? We have given partial results in this direction by showing that COMPLETE WIDTH is polynomially solvable on $(2K_2, K_3)$-free graphs and on $(2K_2, C_4)$-free graphs. (Equivalently, EDGE CLIQUE COVER is polynomially solvable on $(C_4, 3K_1)$-free graphs and on $(C_4, 2K_2)$-free graphs.)

Another interesting question is the following. The time complexities of many problems coincide on split graphs and bipartite graphs, *e.g.*, the dominating set problem. However, for the complete width problem, they are different, one is in P and the other is in NP-complete. Trees are a special class of bipartite graphs. Many problems become easy on trees. However, we do not know the hardness of the complete width problem on trees.

References

1. Blázsik, Z., Hujter, M., Pluhár, A., Tuza, Z.: Graphs with no induced C_4 and $2K_2$. Discrete Mathematics **115**, 51–55 (1993)
2. Brandstädt, A., Le, V.B., Spinrad, J.P.: Graph Classes: A Survey. SIAM Monographs on Discrete Mathematics and Applications, Philadelphia (1999)

3. Chandler, D.B., Chang, M.-S., Kloks, T., Peng, S.-L.: Probe Graphs, (2009). http://www.cs.ccu.edu.tw/~hunglc/ProbeGraphs.pdf
4. Chang, M.-S., Hung, L.-J., Kloks, T., Peng, S.-L.: Block-graph width. Theoretical Computer Science **412**, 2496–2502 (2011)
5. Chang, M.-S., Kloks, T., Liu, C.-H.: Edge-clique graphs of cocktail parties have unbounded rankwidth, (2012). arXiv:1205.2483 [cs.DM]
6. Chang, M.-S., Müller, H.: On the tree-degree of graphs. In: Brandstädt, A., Le, V.B. (eds.) WG 2001. LNCS, vol. 2204, pp. 44–54. Springer, Heidelberg (2001)
7. Ma, S., Wallis, W.D., Wu, J.: Clique covering of chordal graphs. Utilitas Mathematica **36**, 151–152 (1989)
8. Cygan, M., Pilipczuky, M., Pilipczuk, M.: Known algorithms for EDGE CLIQUE COVER are probably optimal. In: Proc. SODA, 1044–1053 (2013)
9. Foldes, S., Hammer, P.L.: Split graphs. Congressus Numerantium, No. XIX, 311–315 (1977)
10. Golumbic, M.C.: Algorithmic Graph Theory and Perfect Graphs. Annals of Discrete Math., vol. 57, 2nd edn. Elsevier, Amsterdam (2004)
11. Gramm, J., Guo, J., Hüffner, F., Niedermeier, R.: Data reduction and exact algorithms for clique cover. ACM Journal of Experimental Algorithmics **13** (2008). Article 2.2
12. Hammer, P.L., Peled, U.N., Sun, X.: Difference graphs. Discrete Applied Mathematics **28**, 35–44 (1990)
13. Holyer, I.: The NP-completeness of some edge-partition problems. SIAM Journal on Computing **4**, 713–717 (1981)
14. Hoover, D.N.: Complexity of graph covering problems for graphs of low degree. Journal of Combinatorial Mathematics and Combinatorial Computing **11**, 187–208 (1992)
15. Hsu, W.-L., Tsai, K.-H.: Linear time algorithms on circular-arc graphs. Inf. Process. Lett. **40**, 123–129 (1991)
16. Kou, L.T., Stockmeyer, L.J., Wong, C.K.: Covering edges by cliques with regard to keyword conflicts and intersection graphs. Comm. ACM **21**, 135–139 (1978)
17. Le, V.B., Peng, S.-L.: Characterizing and recognizing probe block graphs. Theoretical Computer Science **568**, 97–102 (2015)
18. Le, V.B., Peng, S.-L.: Good characterizations and linear time recognition for 2-probe block graphs. In: Proceedings of the International Computer Symposium, Taichung, Taiwan, December 12–14, 2014, pp. 22–31. IOS Press (2015). doi:10.3233/978-1-61499-484-8-22
19. Maffray, F., Preissmann, M.: Linear recognition of pseudo-split graphs. Discrete Applied Mathematics **52**, 307–312 (1994)
20. Mahadev, N.V.R., Peled, U.N.: Threshold Graphs and Related Topics. Annals of discrete mathematics, vol. 56. Elsevier, Amsterdam (1995)
21. Müller, H.: On edge perfectness and classes of bipartite graphs. Discrete Math. **149**, 159–187 (1996)
22. Orlin, J.: Contentment in graph theory: covering graphs with cliques. Indagationes Mathematicae **80**, 406–424 (1977)
23. Pullman, N.J.: Clique covering of graphs IV. Algorithms. SIAM Journal on Computing **13**, 57–75 (1984)
24. Raychaudhuri, A.: Intersection number and edge clique graphs of chordal and strongly chordal graphs. Congressus Numer. **67**, 197–204 (1988)
25. Yannakakis, M.: Node-delection problems on bipartite graphs. SIAM Journal on Computing **10**, 310–327 (1981)