On the Number of Anchored Rectangle Packings for a Planar Point Set

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Abstract. We consider packing axis-aligned rectangles $r_1, \ldots, r_n$ in the unit square $[0, 1]^2$ such that a vertex of each rectangle r_i is a given point p_i (i.e., r_i is *anchored* at p_i); and explore the combinatorial structure of all locally maximal configurations. When the given points are lower-left corners of the rectangles, then the number of maximal packings is shown to be at most $2^n C_n$, where C_n is the nth Catalan number. The number of maximal packings remains exponential in n when the points may be arbitrary corners of the rectangles. Our upper bounds are complemented with exponential lower bounds.

1 Introduction

Let P be a finite set of points $p_1, \ldots, p_n$ in the unit square $[0, 1]^2$. An *anchored rectangle packing* for P is a set of axis-aligned empty rectangles $r_1, \ldots, r_n$, that lie in $[0, 1]^2$, are interior-disjoint, and point p_i is one of the four corners of r_i for $i = 1, \ldots, n$. We say that rectangle r_i is *anchored* at p_i. In a *lower-left anchored rectangle packing* (*L-anchored packing*, for short), p_i is the lower-left corner of r_i for all i.

Anchored rectangle packings have applications in map labeling in geographic information systems [15–17] and VLSI design [18]. A fundamental problem is to find the maximum total area $A(P)$ (resp., $A_L(P)$) of the rectangles in an anchored (resp., L-anchored) rectangle packing of P. Allen Freedman conjectured (c.f. [23,24]) that if $(0, 0) \in P$, then P admits an L-anchored rectangle packing of area at least 1/2, that is $A_L(P) \geq 1/2$. The currently known best lower bound in this case is $A_L(P) \geq 0.091$ due to Dumitrescu and Tóth [11].

A rectangle r_i with lower-left anchor $p_i = (a_i, b_i)$, can be parameterized by two variables x_i and y_i such that $r_i = [a_i, x_i] \times [b_i, y_i]$. Consequently, the area of an L-anchored rectangle packing is a continuous multivariable function in $2n$ variables $\sum_{i=1}^{n} \text{area}(r_i) = \sum_{i=1}^{n} (x_i - a_i)(y_i - b_i)$, over a domain determined by the geometric constraints of the packing. We call an L-anchored rectangle packing *maximum* (resp., *maximal*) if it attains the global (resp., a local) maximum of this function. We define *maximum* and *maximal* anchored rectangle packing analogously.

D. Xu et al. (Eds.): COCOON 2015, LNCS 9198, pp. 377–389, 2015.
DOI: 10.1007/978-3-319-21398-9_30

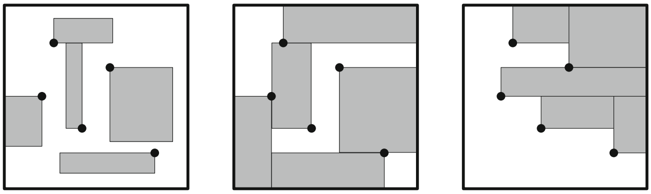

Fig. 1. Left: a set P of 5 points in the unit square $[0,1]^2$ and an anchored rectangle packing for P. Middle: a maximal anchored rectangle packing for P. Right: A maximal L-anchored rectangle packing for P.

For computing the maximum area, $A_L(P)$ or $A(P)$, for a given point set P, it is instrumental to estimate the *number* of maximum packings. It is easily seen that the number of maximum packings is at least exponential in $n = |P|$ if, for example, P contains n points on a diagonal of $[0,1]^2$. The enumeration of locally maximal configurations, which can be computed greedily, combined with reverse search [7] yields a simple strategy for finding the global maximum. In this paper, we control the number of (locally) maximal anchored and L-anchored rectangle packings. For an integer $n \in \mathbb{N}$, let $M(n)$ (resp., $M_L(n)$) denote the largest number of maximal rectangle packings over all sets $P \subset [0,1]^2$ of n noncorectilinear points (two points are *corectilinear* if they have the same x- or y-coordinate).

Results. In this paper, we prove exponential upper and lower bounds for $M_L(n)$ and $M(n)$. Our upper bound for $M_L(n)$ is expressed in terms of the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n} \sim 4^n/(n^{3/2}\sqrt{\pi})$.

Theorem 1. *We have* $\Omega(4^n/\sqrt{n}) \leq M_L(n) \leq C_n 2^n = \Theta(8^n/n^{3/2})$.

Note that both the lower and upper bounds are larger than C_n. The lower bound follows from an explicit construction. The upper bound is the combination of two tight upper bounds. Each L-anchored rectangle packing induces a subdivision of $[0,1]^2$ into "staircases" (*L-subdivisions*, defined in Sec. 3). We show that the number of L-subdivisions for n points is at most C_n, and this bound is attained when the points form an antichain under the product order. We also show that each L-subdivision is induced by at most 2^{n-1} L-anchored rectangle packings, and this bound is attained when the points form a chain under the product order.

The machinery developed for the proof of Theorem 1 does not extend to general anchored rectangle packings. Nevertheless, we can prove that the number of maximal (any corner) anchored rectangle packings is exponential

Theorem 2. *There exist constants* $1 < c_1 < c_2$ *such that* $\Omega(c_1^n) \leq M(n) \leq O(c_2^n)$.

We derive an exponential upper bound using the contact graph of the rectangles in a packing. Specifically, we show that the contact graph can be represented by a planar embedding of the contact graph in which the vertices are points in the rectangles, and the edges are represented by polylines with at most one bend per edge. The number of graphs with such an embedding is known to be exponential [12]. We can encode all maximal anchored rectangle packings for P using one such graph and $O(n)$ bits of additional information. This leads to an exponential upper bound.

Remark. In a maximal anchored or L-anchored rectangle packing, we may assume that all vertices of all rectangles lie on one of the $(n+2)^2$ "grid points" induced by the vertical and horizontal lines passing through the n points in P and the corners of $[0,1]^2$ (cf. Sec. 2). This crucial property discretizes the problem, but is insufficient for establishing an exponential upper bound. By choosing the points (x_i, y_i) among the grid points, we obtain only a weak upper bound of $(n-1)^n$ (resp., $(n!)^2$ for L-anchored packings).

Related Work. Packing axis-aligned rectangles in a rectangular container, albeit without anchors, is the unifying theme of several classic optimization problems. The 2D knapsack problem, strip packing, and 2D bin packing involve arranging a set of given rectangles in the most economic fashion [3,8,14]. The maximum weight independent set for rectangles involves selecting a maximal area packing from a set of given rectangles [4]. These optimization problems are NP-hard, and there is a rich literature on the best approximation algorithms. Our problem setup is fundamentally different: the rectangles have variable sizes, but their location is constrained by the anchors. In this sense, it is reminiscent to classic Voronoi diagrams for n points in the plane. However, the Voronoi cells tile the space without gaps. Area maximization problems arise in the context of Voronoi games [5,9], where two players alternately choose points in a bounding box and wish to maximize the total area of the Voronoi cells of their points.

Combinatorial bounds for the number of some other geometric configurations on n points in the plane have been studied extensively. Determining the maximum number of (geometric) triangulations on n points in the plane captivated researchers for decades. The current best upper and lower bounds are $\Omega(8.65^n)$ and $O(30^n)$ [10,20]. Ackerman et. al. [1,2] established an upper bound of $O(18^n/n^4)$ for the number of *rectangulations* of n points in $[0,1]^2$, where a *rectangulation* is a subdivision of $[0,1]^2$ into $n+1$ rectangles by n axis-parallel segments, each containing a given point. This structure is reminiscent of L-subdivisions, defined in Sec. 3, for which we prove a tight upper bound of $C_n \le O(4^n/n^{3/2})$. The number of anchored rectangle packings has not been studied before. It is not known if finding the maximum area of an anchored rectangle packing of n given points is NP-hard.

2 Discretization of Maximal Anchored Rectangle Packings

Let $P \subset [0,1]^2$ be a set of noncorectilinear points $p_1, \ldots, p_n$. The vertical and horizontal lines that pass through the points in P and the edges of the bounding box are called *grid lines*. The *grid points* are the intersections of the grid lines.

It is easy to see that all vertices of a maximal L-anchored rectangle packing must be grid points.

Proposition 1. *If an L-anchored rectangle packing for P has maximal area, then all corners of all rectangles are grid points.*

Proof. Consider an anchored rectangle packing of maximal area. The left and bottom edges are on grid lines. This implies that each rectangle may only expand up and to the right. Because the packing is of maximal area, no rectangle can expand (while other rectangles are fixed). The upper and right edges of each rectangle are necessarily in contact with the bottom and left edges of other rectangles or with the bounding box. This places the upper-right vertex at the intersection of two grid lines and thus on a grid point. We have shown that the lower-left and the upper-right corner of every rectangle is a grid point. From the definition of grid lines, this implies that all corners of all rectangles are grid points. □

The situation is more subtle when the rectangles can be anchored at arbitrary corners. Specifically, a local maximum may be attained at a "plateau" where the configuration can vary continuously while maintaining the same maximal area. A transformation that maintains the total area of the rectangles is called *equiareal.*

Proposition 2. *If an anchored rectangle packing for P has maximal area, then*

- *the local maximum is isolated, and all vertices of all rectangles are grid points, or*
- *there is an equiareal continuous deformation to an anchored rectangle packing in which all vertices of all rectangles are grid points; furthermore, the deformation either creates a contact between two previously disjoint rectangles, or decreases the area of some rectangle to 0.*

Proof. Consider a maximal anchored rectangle packing for P. Suppose that at least one rectangle has a vertical or horizontal edge not on a grid line. Assume first that a vertical edge of a rectangle is not on a grid line. Let ℓ be the vertical line through the leftmost such edge. Denote by L the set of rectangles whose right edges intersect ℓ, and R the set of rectangles whose left edges intersect ℓ. We can deform the rectangles in L and R simultaneously by translating ℓ. The sum of heights of rectangles in L equals the sum of heights of rectangles in R, otherwise translating ℓ in one of the two possible directions increases the total area before ℓ becomes a grid line. When ℓ shifts to the left, the rectangles in L shrink and may potentially reach 0 area; while the rectangles in R expand and may potentially reach another rectangle. However, because of the choice of ℓ, all edges of such a rectangle lie on grid lines. Translate ℓ until the area of a rectangle in L drops to 0, or the left edge of a rectangle in R reaches the boundary of a new rectangle or the bounding box. Repeat this operation for the next leftmost line ℓ until all vertical edges are on grid lines.

Note that horizontal edges were not affected by the above transformations. We can now deform the horizontal edges of the rectangles (independent of the vertical edges) by repeating the argument starting with the topmost horizontal line. Necessarily, all vertices of all rectangles become grid points. □

In the remainder of the paper, we consider maximal anchored rectangle packings in which the vertices of all rectangles are grid points.

3 Lower-Left Anchored Rectangle Packings

The key tool for the proof of Theorem 1 is a subdivision of the unit square $[0,1]^2$ into staircase polygons, defined below. Let $P = \{p_1, \ldots, p_n\}$ be a set of noncorectilinear points in $[0,1]^2$. We may assume that $(0,0) \notin P$ (by scaling P, if necessary, since maximality is an affine invariant). Let $q = (0,0)$ denote the lower-left corner of $[0,1]^2$.

An *L-shape* for a point p_i $(i = 1, \ldots, n)$ is the union of a horizontal and a vertical segment whose left and bottom endpoint, respectively, is p_i. Refer to Fig. 2(a). An *L-subdivision* for P is formed by n L-shapes for p_i $(i = 1, \ldots, n)$ such that the top and right endpoint of each L-shape lies in another L-shape or the boundary of $[0,1]^2$. The L-shapes subdivide $[0,1]^2$ into $n+1$ simple polygons, called *staircases*. By construction, the lower-left corner of each staircase is either $q = (0,0)$ or a point in P. The upper-right vertices of a staircase are called *steps* of the staircase.

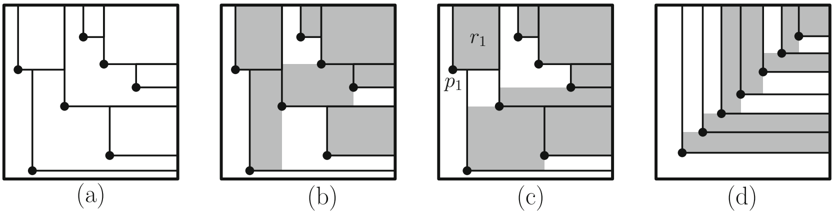

Fig. 2. (a) An L-subdivision for P. (b) An L-subdivion induced by a maximal L-anchored rectangle packing. (c) Maximal anchored rectangles in the staircases that do not form a maximal L-anchored rectangle packing: rectangle r_1 could expand. (d) For n points on the line $y = x$, $M_L(P) = 2^{n-1}$.

Proposition 3. *In every L-subdivision for P, the n staircases anchored at the points in P jointly have at most* $2n - 1$ *steps.*

Proof. Each step of a staircase is either the upper-right corner of $[0,1]^2$, or a top or right endpoint of an L-shape. Every such point is the step of a unique staircase. The n L-shapes yield $2n$ steps, and the upper-right corner of $[0,1]^2$ yields one step. The staircase anchored at $q = (0,0)$ has at least two steps, hence the remaining staircases jointly have at most $2n + 1 - 2 = 2n - 1$ steps. □

Maximal L-anchored packings versus L-subdivisions

Proposition 4. *For every maximal L-anchored rectangle packing of P, there is an L-subdivision such that rectangle r_i lies in the staircase anchored at p_i for $i = 1, \ldots, n$.*

Proof. Let $r_1, \ldots, r_n$ be an L-anchored rectangle packing for $p_1, \ldots, p_n \in [0,1]^2$. For each $i = 1, \ldots, n-1$, successively draw an L-shape as follows (refer to Fig. 2(b)). First extend the bottom edge of r_i to the right until it hits the bounding box, the left edge of another rectangle, or a previously drawn L-shape. Similarly, extend the left edge of r_i up until it hits the bounding box, the bottom edge of another rectangle, or a previously drawn L-shape. The n L-shapes form an L-subdivision. By construction, the L-shapes are disjoint from the interior of the rectangles $r_1, \ldots, r_n$, hence each rectangle lies in a staircase. Since the lower-left corner of each staircase is $q = (0,0)$ or a point in P, each staircase with lower-left corner $p_i \in P$ contains the rectangle anchored at p_i. □

In the L-subdivision described in Proposition 4, each rectangle r_i ($i = 1, \ldots, n$) is a maximal rectangle within a staircase polygon. However, the converse is not necessarily true. Choose maximal rectangles, in all staircases, with lower-left corners in P. This need not produce a maximal L-anchored rectangle packing for P. See an example in Fig. 2(c). Nevertheless, we can derive an upper bound for $M_L(P)$.

Proposition 5. *In every L-subdivision for P, $|P| = n$, there are at most 2^{n-1} possible ways to choose a maximal rectangle in each staircase whose lower-left corner is in P. This bound is the best possible.*

Proof. If the staircases anchored at the n points in P have $t_1, \ldots, t_n$ steps, then there are precisely $\prod_{i=1}^n t_i$ different ways to choose a maximal anchored rectangle in each. By Proposition 3 and the arithmetic-geometric mean inequality yields

$$\prod_{i=1}^n t_i \le \left(\frac{1}{n}\sum_{i=1}^n t_i\right)^n = \left(2 - \frac{1}{n}\right)^n < 2^n. \tag{1}$$

The maximum of $\prod_{i=1}^n t_i$ subject to $\sum_{i=1}^n t_i = 2n-1$ and $t_1, \ldots, t_n \in \mathbb{N}$ is attained when the $t_i' s$ are distributed as evenly as possible, say, $t_1 = \ldots = t_{n-1} = 2$ and $t_n = 1$. Consequently, $\prod_{i=1}^n t_i \le 2^{n-1}$. This upper bound is attained when the points in P form a chain in the product order (e.g., points on the line $y = x$), then $n-1$ staircases have 2 steps, and the staircase incident to $(1,1)$ has only 1 step (Fig. 2(d)). □

Let $S(P)$ be the number of all L-subdivisions for a noncorectilinear point set P; and let $S(n) = \max_{|P|=n} S(P)$. By Proposition 5, we have $M_L(P) \le S(P) 2^n$ and $M_L(n) \le S(n) 2^n$.

The Number of L-subdivisions. We prove a tight upper bound for $S(n)$, the maximum number of L-subdivisions for a set of n points in the unit square. Our upper bound is expressed in terms of the nth Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n} \sim 4^n/\sqrt{\pi n^3}$.

Lemma 1. *For every $n \in \mathbb{N}$, we have $S(n) = C_n$.*

Proof. Lower bound. Let P be a set of n points that form an antichain under the product order (e.g., points on the line $y = 1 - x$). In this case, each staircase anchored at a point in P is a rectangle, and it is well known [21,22] that the number of rectangular subdivisions is the Catalan number C_n. Hence $S(P) = C_n$ in this case.

Upper Bound. Let P be an arbitrary noncorectilinear set of n points in $[0, 1]^2$. We may assume that the points $p_1, \ldots p_n$ are sorted by their x-coordinates, that is, $x_1 < \ldots < x_n$. If the points form an antichain under the product order, then their y-coordinates are monotone decreasing, and $S(P) = C_n$. Otherwise, we incrementally modify the y-coordinates of the points to become monotone decreasing such that the number of L-subdivisions increases. In each incremental step, we modify the y-coordinate of one point.

Suppose that the points in P do not form an antichain under the product order; and i is the smallest index such that the points with larger indices, $\{p_j \in P : j > i\}$, form an antichain and are incomparable to all other points (refer to Fig. 3). Let Z_0 be the minimum axis-aligned rectangle incident to $(0, 1)$ that contains the points $p_1, \ldots, p_{i-1}$; and let Z_1 be the minimum axis-aligned rectangle incident to $(1, 0)$ that contains the points $p_{i+1}, \ldots, p_n$. By the choice of i, the boxes Z_0 and Z_1 are on opposite sides of the vertical line $x = x_i$, as well as a horizontal line below $y = \min_{1 \le k \le i} y_k$. Let p'_i be the intersection of these two lines.

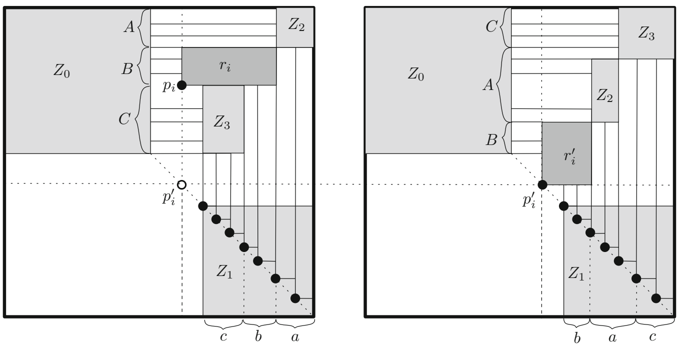

Fig. 3. Left: A schematic image of an L-subdivision D for P. Right: The corresponding L-subdivision D' for the modified point set P'.

We move point p_i to p'_i. Denote by P' the modified point set. In order to show $S(P) \le S(P')$, we construct an injective map $f : S(P) \to S(P')$. For every L-subdivision D of the point set P, we construct a unique L-subdivision $D' = f(D)$ of the modified point set P'. Let D be an L-subdivision of P (Fig. 3, left). Since no other point dominates p_i, the staircase anchored at p_i is a rectangle, that we denote by r_i. We introduce some notation. Some horizontal segments of L-shapes of points in Z_0 cross the right edge of Z_0: Let A, B, and C, respectively, denote the number of L-shapes whose horizontal segments pass above r_i, hit the left edge of r_i, and pass below r_i. Similarly, some vertical segments of L-shapes of points in Z_1 cross the top edge of Z_1: Let a, b, and c, respectively, denote the number of L-shapes whose vertical segments pass right of r_i, hit the bottom edge of r_i, and end strictly below the bottom edge of r_i. Let Z_2 be the axis-aligned rectangle that contains all intersections between the A horizontal segments passing above r_i and the a vertical segments right of r_i. Similarly, let Z_3 contain the intersections between the C horizontal segments passing below r_i and the c vertical segments that end strictly below r_i.

We can now define the L-subdivision D' for the modified point set P'. The arrangement of L-shapes restricted to the boxes Z_0 and Z_1 remains the same. Consequently, $A + B + C$ horizontal segments exit the right edge of Z_0, and $a + b + c$ vertical segments exit the top edge of Z_1. Draw an L-shape for the point p'_i such that it blocks the B lowest horizontal segments that exit Z_0 and the b leftmost vertical segments that exit Z_1. Group the remaining horizontal (resp., vertical) segments into bundles of size A and C (resp., a and c). Let the groups of size A and a intersect in the same pattern as in Z_2, and the groups of size C and c as in Z_3. This completes the description of the L-subdivision D'. By construction $D' = f(D)$ is a unique L-subdivision, and the function f is injective. □

We are now ready to prove Theorem 1.

Theorem 1. *We have* $\Omega(4^n/\sqrt{n}) \leq M_L(n) \leq C_n 2^n = \Theta(8^n/n^{3/2})$.

Proof. Let P be a set of n noncorectilinear points in the unit square. By Proposition 4, every maximal L-anchored rectangle packing for P can be constructed by considering an L-subdivision for P, and then choosing a maximal rectangle from each staircase anchored at a point in P. By Lemma 1, we have $S(P) \leq S(n) = C_n$ L-subdivisions for P. By Proposition 5, there are at most 2^{n-1} ways to choose maximal rectangles in the staircases. Consequently, we have $M_L(P) \leq S(P)2^{n-1} \leq C_n 2^{n-1}$.

Even though both Proposition 5 and Lemma 1 are tight, their combination is not tight, since they are attained on different point configurations: n points that form a chain or an antichain under the product order. Our lower bound

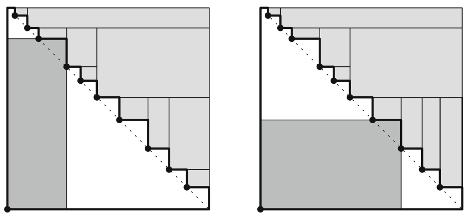

Fig. 4. One point at the origin and $n - 1$ points on the line $y = 1 - x$

is based on the following construction (refer to Fig. 4). Place one point at the origin and $n-1$ points on the line $y = 1-x$. The L-shape of the first point is contained in the boundary of $[0,1]^2$, and the last $n-1$ points admit $C_{n-1} = \frac{1}{n}\binom{2n-2}{n-1} \sim 4^{n-1}/\sqrt{(n-1)^3\pi}$ L-subdivisions. The first point has a staircase with n steps ($t_1 = n$), all other staircases are rectangles ($t_i = 1$, for $i = 2, \ldots, n$). Consequently, $M_L(P) = S(P)\prod_{i=1}^{n} t_i = C_{n-1} \cdot n = \Theta(4^n/\sqrt{n})$, as required. □

4 General Anchored Rectangle Packings

In this section, we prove Theorem 2. We show that a maximal anchored rectangle packing for a point set P can be reconstructed from the contact graph of the rectangles, and from $O(n)$ bits of additional information. Since a maximal rectangle packing may contain rectangles of 0 area (cf. Proposition 2), we need to be careful defining contact graphs.

The *contact graph* of a rectangle packing is a graph $G = (V, E)$, where V corresponds to the set of vertices, E to the set of edges, and two vertices are connected by an edge iff the corresponding rectangles have positive area and intersect in a nontrivial line segment; or one rectangle has 0 area and lies on the boundary of the other rectangle. It is easy to see that the contact graph of a rectangle packing is planar. However, the number of n-vertex planar graphs is super-exponential [13]. The number of graphs reduces to exponential with suitable geometric conditions. For a set P of n points in the plane, for example, the number of straight-line graphs with vertex set P is only exponential. An $\exp(O(n))$ bound was first shown by Ajtai et al. [6] using the crossing number method. The current best upper bound is $O(187.53^n)$, due to Sharir and Sheffer [20]. The contact graphs of any anchored rectangle packings for P can be embedded in the plane such that the vertex set is P, but these graphs cannot always be realized by straight-line edges. It turns out that a weaker condition will suffice: a *1-bend* embedding of a planar graph $G = (V, E)$ is an embedding in which the vertices are distinct points in the plane, and the edges are polylines with one bend per edge (that is, each edge is the union of two incident line segments). Frankeke and Tóth [12] proved recently that for every n-element point set, the number of such graphs is at most $\exp(O(n))$.

Lemma 2. *Let $P = \{p_1, \ldots, p_n\}$ be a noncorectilinar set in $[0,1]^2$. The contact graph of every maximal anchored rectangle packing for P has a 1-bend embedding in which the vertex representing rectangle r_i is point p_i for $i = 1, \ldots, n$.*

The proof would be straightforward if the anchors were in the interior of the rectangles. In that case, we could simply choose a *bend point* on the common boundary between two rectangles in contact, and then draw a 1-bend edge between their anchors via the bend point. When the anchors are at corners of the rectangles, we need to be more careful to prevent any overlap between adjacent edges.

Proof. Let $r_1, \ldots, r_n$ be a maximal anchored rectangle packing for P. For every $i = 1, \ldots, n$, point p_i is a corner of the rectangle r_i. For every two rectangles

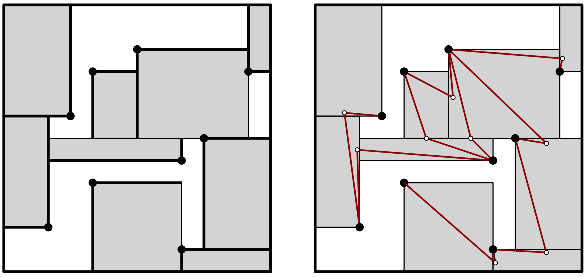

Fig. 5. Left: A maximal anchored rectangle packing for P. Thick lines indicate the L-shapes incident to the points in P. Right: A 1-bend embedding of the contact graph of the rectangles.

in contact, r_i and r_j, choose an arbitrary *preliminary* bend point $q_0(i,j)$ on the common boundary of r_i and r_j.

Define the *L-shape* anchored at p_i as the union of the two edges of r_i incident to p_i. Note that the relative interiors of the n L-shapes are pairwise disjoint, since the points $p_1, \ldots, p_n$ are noncorectilinear. Let the bend point $q(i,j) = q_0(i,j)$ if the preliminary bend point is not on an L-shape or if one of the rectangles has 0 area. Otherwise, assume $q_0(i,j)$ is on the L-shape of p_i. Then choose a bend point $q(i,j)$ in the interior of r_i in a sufficiently small neighborhood of the preliminary point $q_0(i,j)$. Now, for any two rectangles in contact, r_i and r_j, draw a 1-bend edge between p_i and p_j via $q(i,j)$. No two edges cross or overlap, and hence we obtain a 1-bend embedding of the contact graph of the rectangles. □

For a fixed point set P, by Lemma 2, the contact graph of every maximal anchored rectangle packing admits a 1-bend embedding on the vertex set P. However, several maximal anchored rectangle packings may yield the same contact graph (as an abstract graph). We show that all maximal anchored rectangle packings for P can be encoded by their contact graphs and $O(n)$ bits of additional information. By Proposition 2, we may assume that all vertices of a maximal rectangle packing are grid points. Furthermore, we may also assume that there is no equiareal continuous deformation that creates a new contact or reduces the area of a rectangle to 0.

Fix a noncorectilinear point set $P = \{p_1, \ldots, p_n\}$. Every maximal anchored rectangle packing $r_1, \ldots, r_n$ is encoded by the following information:

(1) The contact graph G of the rectangles $r_1, \ldots, r_n$;
(2) for $i = 1, \ldots n$, an indicator variable σ_i such that $\sigma_i = 0$ iff area$(r_i) = 0$;
(3) for $i = 1, \ldots n$, the position of the anchor p_i in r_i (lower-left, lower-right, etc.);
(4) for each edge (i,j) of G, the orientation of the line segment $r_i \cap r_j$.

We now show that we can uniquely reconstruct a maximal anchored rectangle packing from this information.

Lemma 3. *For every noncorectilinear point set P, every code described above determines at most one maximal anchored rectangle packing for P, which can be (re)constructed in polynomial time.*

Proof. We are given the points $p_1, \ldots, p_n$, and for every $i = 1, \ldots, n$, we know which corner of the rectangle r_i is p_i. To reconstruct the rectangles r_i ($i = 1, \ldots, n$), it is enough to find the corner of r_i opposite to p_i, which we denote by (x_i, y_i). That is $r_i = [\min(a_i, x_i), \max(a_i, x_i)] \times [\min(b_i, y_i), \max(b_i, y_i)]$. We determine the parameters x_i (resp., y_i) with the following strategy.

> Consider a rectangle r_i, and assume without loss of generality that p_i is the lower-left corner of r_i. If r_i is not in contact with any rectangle r_j such that $r_i \cap r_j$ is vertical and $a_i < a_j$, then $x_i = 1$ (that is, r_i extends to the right edge of the bounding box $[0, 1]^2$). If r_i is in contact with a rectangle r_j such that the segment $r_i \cap r_j$ is vertical, $a_i < a_j$, and the anchor p_j is the lower-left or upper-left corner of r_j, then we have $x_i = a_j$. Analogous conditions determine y_i in some cases.

We now show that our assumptions from Proposition 2 ensure that the above strategy determines x_i and y_i for all $i = 1, \ldots, n$. If the above strategy fails to find x_i, then r_i is in contact with a rectangle r_j such that the segment $r_i \cap r_j$ is vertical, $a_i < a_j$, but p_j is the lower-right or upper-right corner of r_j. In this case, we call (r_i, r_j) a *horizontal pair*. Analogously, if the strategy does not find y_i, then r_i is part of some *vertical pair* (r_i, r_j). The horizontal (resp., vertical) pairs define a subgraph of the contact graph, that we denote by G_H (resp., G_V). Each connected component C of the graph G_H (resp., G_V) corresponds to rectangles whose left or right edge lies on some common vertical (resp., horizontal) line ℓ.

Consider a component C of G_H (the argument is analogous for G_V). The line ℓ must be right of all lower-left and upper-left anchors of rectangles in C, and left of all lower-right and upper-right anchors. Suppose that there exists a maximal anchored rectangle packing that satisfies these constraints. Denote by $L \subset C$ (resp., $R \subset C$) the set of rectangles whose right (resp., left) edges lie on ℓ. Similarly to the proof of Proposition 2, we deform the rectangles in L and R simultaneously by translating ℓ. If the sum of heights of rectangles in L and R differ, then translating ℓ in one of the two possible directions increases the total area, contradicting maximality. If the sum of heights of rectangles in L and R are equal, then translating ℓ in any direction is an equiareal deformation. We can now translate ℓ left until the area of a rectangle in L drops to 0 or a rectangle in R is in contact with a new rectangle on the left of ℓ. This contradicts our assumption that equiareal deformations create neither new contacts nor new rectangles of 0 area. Consequently, G_H (resp., G_V) is the empty graph, there are neither horizontal nor vertical pairs, and the above strategy uniquely determines x_i and y_i for all $i = 1, \ldots, n$. □

Theorem 2. *There exist constants $1 < c_1 < c_2$ such that $\Omega(c_1^n) \leq M(n) \leq O(c_2^n)$.*

Proof. The combination of Lemmas 2 and 3 yields the upper bound. Theorem 1 gives the lower bound.

5 Conclusions

We have considered two variants of anchored rectangle packings: the anchors p_i were required to be either the lower-left or arbitrary corners of the rectangles r_i. We could consider a variant that we call *relaxed anchored rectangle packing*, where the anchors p_i are contained in the rectangles r_i. In this case, the maximum area of a rectangle packing is always 1, since the bounding box can be subdivided into n parallel strips, each containing a point in P. Note that a rectangle $r_i = [x_i, x_i'] \times [y_i, y_i']$ is now described by 4 variables. In a relaxed anchored rectangle packing, however, a local maximum need not attain the global maximum. Nevertheless, the technique of Section 4 extends to this variant: each maximal rectangle packing can be reconstructed from the contact graphs of the rectangles (which has an embedding using polylines with at most one bend per edge), and $O(1)$ bits of additional information per rectangle. Consequently, the number of locally maximal packings for an n-element point set is bounded by $\exp(O(n))$.

Analogous problems arise for anchored packings with other simple geometric shapes, such as circular disks or positive homothets of some convex body. For packings with object of bounded description complexity, the configuration space can be parameterized with $O(n)$ variables, and some of the techniques developed here do generalize. However, several crucial steps in our work have relied on properties of axis-aligned rectangles. Determining the maximum area covered by a packing remains open for both anchored and L-anchored rectangle packings. For other geometric shapes (e.g., circular disks), finding the maximum area covered by relaxed anchored variants is already a challenging problem.

References

1. Ackerman, E.: Counting problems for geometric structures: rectangulations, floorplans, and quasi-planar graphs, PhD thesis, Technion (2016)
2. Ackerman, E., Barequet, G., Pinter, R.: On the number of rectangulations of a planar point set. J. Combin. Theory, Ser. A **113**(6), 1072–1091 (2006)
3. Adamaszek, A., Wiese, A.: Approximation schemes for maximum weight independent set of rectangles. In: Proc. 54th FOCS. IEEE (2013)
4. Adamaszek, A., Wiese, A.: A quasi-PTAS for the two-dimensional geometric knapsack problem. In: Proc. 26th SODA. SIAM (2015)
5. Ahn, H.-K., Cheng, S.-W., Cheong, O., Golin, M., van Oostrum, R.: Competitive facility location: the Voronoi game. Theoret. Comput. Sci. **310**, 457–467 (2004)
6. Ajtai, M., Chvátal, V., Newborn, M., Szemerédi, E.: Crossing-free subgraphs. Annals Discrete Math. **12**, 9–12 (1982)
7. Avis, D., Fukuda, K.: Reverse search for enumeration Discrete Appl. Math. **65**, 21–46 (1996)
8. Bansal, N., Khan, A.: Improved approximation algorithm for two-dimensional bin packing. In: Proc. 25th SODA, pp. 13–25. SIAM (2014)
9. Cheong, O., Har-Peled, S., Linial, N., Matoušek, J.: The one-round Voronoi game. Discrete Comput. Geom. **31**, 125–138 (2004)

10. Dumitrescu, A., Schulz, A., Sheffer, A., Tóth, C.D.: Bounds on the maximum multiplicity of some common geometric graphs. SIAM J. Discrete Math. **27**(2), 802–826 (2013)
11. Dumitrescu, A., Tóth, C.D.: Packing anchored rectangles. In: Proc. 23rd SODA, pp. 294–305. SIAM (2012); and Combinatorica **35**(1), 39–61 (2015)
12. Francke, A., Tóth, C.D.: A census of plane graphs with polyline edges. In: Proc. 30th SoCG, pp. 242–250. ACM Press (2014)
13. Giménez, O., Noy, M.: Asymptotic enumeration and limit laws of planar graphs. J. AMS **22**, 309–329 (2009)
14. Harren, R., Jansen, K., Prädel, L., van Stee, R.: A $(5/3 + \varepsilon)$-approximation for strip packing. Comput. Geom. **47**(2), 248–267 (2014)
15. Kakoulis, K.G., Tollis, I.G.: Labeling algorithms, chap. 28. In: Tamassia, R. (ed.) Handbook of Graph Drawing and Visualization. CRC Press (2013)
16. Knuth, D., Raghunathan, A.: The problem of compatible representatives. SIAM J. Discete Math. **5**, 36–47 (1992)
17. van Kreveld, M., Strijk, T., Wolff, A.: Point labeling with sliding labels. Comput. Geom. **13**, 21–47 (1999)
18. Murata, H., Fujiyoshi, K., Nakatake, S., Kajitani, Y.: VLSI module placement based on rectangle-packing by the sequence-pair. IEEE Trans. CAD Integrated Circuits and Systems **15**(12) (1996)
19. Santos, F., Seidel, R.: A better upper bound on the number of triangulations of a planar point set. J. Combin. Theory, Ser. A **102**, 186–193 (2003)
20. Sharir, M., Sheffer, A.: Counting plane graphs: cross-graph charging schemes. Combinat. Probab. Comput. **22**, 935–954 (2013)
21. Stanley, R.: Problem k^8 in Catalan addendum to Enumerative Combinatorics, vol. 2, May 25, 2013. http://www-math.mit.edu/~rstan/ec/catadd.pdf
22. Thomas, H.: New combinatorial descriptions of the triangulations of cyclic polytopes and the second higher Stasheff-Tamari posets. Order **19**(4), 327–342 (2002)
23. Tutte, W.: Recent Progress in Combinatorics: Proceedings of the 3rd Waterloo Conference on Combinatorics, May 1968. Academic Press, New York (1969)
24. Winkler, P.: Packing rectangles. In: Mathematical Mind-Benders, pp. 133–134, A.K. Peters Ltd., Wellesley (2007)