Chapter 5
Life Insurance: Reserving

5.1 Introduction

The insurer's debt position, which is an obvious implication of the single-premium arrangement, must be realized also when other premium arrangements are adopted. This need clearly emerged in Sect. 4.4.1. We recall that an asset accumulation–decumulation process develops, throughout the policy duration, against the insurer's debt position. A technical tool for assessing the insurer's debt is provided by the so-called mathematical reserve.

The need for assessing the insurer's position with respect to an insurance policy emerges at any time during the policy duration. In particular, we can recognize:

- "ordinary" needs which emerge, for example, in relation to:
  - the balance sheet, which must display the total insurer's debt toward the policyholders;
  - the sharing of profits with the policyholders, which, in particular, can be related to the proportion of assets contributed by each policy;
- "extraordinary" needs, related for example to the interruption of periodic premium payment, and hence the need for assessing the policyholder's credit and then
  - converting the policy into a "paid-up" one, namely a policy for which no further premium payment is required;
  - determining the amount to be paid by the insurer in the case of "surrender".

5.2 General Aspects

We refer to a generic insurance policy, and focus on benefits and net premiums only. That is, we start disregarding expenses and related expense loadings. We assume that the policy term is m, but a generalization to lifelong policies is straightforward, by setting $m = \omega - x$ (where ω denotes as usual the maximum attainable age).

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Let t_1, t_2 denote two integer times (policy anniversaries), with $0 \le t_1 < t_2 \le m$. We define the following notation which proves to be useful when dealing with the definition of the mathematical reserve:

- $Y(t_1, t_2)$ denotes the random present value at time t_1 of the benefits which fall due in the time interval (t_1, t_2);
- $X(t_1, t_2)$ denotes the random present value at time t_1 of the premiums to be cashed in the time interval (t_1, t_2).

Remark 1 The notation just defined generalizes the one we used in Chap. 4 to denote the random present values of benefits and premiums. Indeed, $Y = Y(0, m)$ and $X = X(0, m)$.

Then, we define:

- $\mathrm{Ben}(t_1, t_2) = \mathbb{E}[Y(t_1, t_2)]$, i.e., the expected present value (or actuarial value) at time t_1 of the benefits which fall due in the time interval (t_1, t_2);
- $\mathrm{Prem}(t_1, t_2) = \mathbb{E}[X(t_1, t_2)]$, i.e., the expected present value (or actuarial value) at time t_1 of the premiums to be cashed in the time interval (t_1, t_2).

Remark 2 It is worth commenting in some detail which of the benefits and premiums paid at the extremes of the time interval (t_1, t_2), i.e., at times t_1 and t_2, are included in the quantities $Y(t_1, t_2)$ and $\mathrm{Ben}(t_1, t_2)$ (for benefits), $X(t_1, t_2)$ and $\mathrm{Prem}(t_1, t_2)$ (for premiums).

In general terms, if an amount is paid at a given time t because it is due at the beginning of year $(t, t+1)$, we say that it is paid at time t in advance. Conversely, if it is paid at time t because due at the end of year $(t-1, t)$, then we say that it is paid at time t in arrears. The rule we adopt when defining the flows included in the quantities $Y(t_1, t_2)$, $\mathrm{Ben}(t_1, t_2)$, $X(t_1, t_2)$, and $\mathrm{Prem}(t_1, t_2)$ is the following. Premiums and benefits paid at time t_1 in advance are included, while benefits paid at time t_1 in arrears are excluded. Benefits paid at time t_2 in arrears are included, while premiums and benefits paid at time t_2 in advance are excluded. Actually, the time interval addressed by the quantities $Y(t_1, t_2)$, $\mathrm{Ben}(t_1, t_2)$, $X(t_1, t_2)$, and $\mathrm{Prem}(t_1, t_2)$ runs from the beginning of year (t_1, t_1+1) to the end of year (t_2-1, t_2). Of course, all the flows falling due at a time t, $t_1 < t < t_2$, are included in such quantities. The rule will clearly emerge in Example 5.2.1, as well as in the following sections.

We now assume that the actuarial values rely on the first-order basis, i.e., the pricing basis TB1. The notations Ben' and Prem' reflect this hypothesis.

It is well known that the equivalence principle requires

$$\mathrm{Prem}'(0, m) = \mathrm{Ben}'(0, m) \tag{5.2.1}$$

On the contrary, all the following situations may occur, at least in principle, when intervals shorter than the whole policy duration are referred to:

$$\mathrm{Prem}'(0, t) \lesseqgtr \mathrm{Ben}'(0, t) \tag{5.2.2a}$$

$$\mathrm{Prem}'(t, m) \lesseqgtr \mathrm{Ben}'(t, m) \tag{5.2.2b}$$

(we recall that t is an integer time).

Further, we can find:

$$\text{Prem}'(t, t+1) \lesseqgtr \text{Ben}'(t, t+1) \tag{5.2.3}$$

where the term on the left-hand side denotes, for example, the annual level premium, whereas the term on the right-hand side denotes the natural premium.

Example 5.2.1 Consider a m-year term insurance, providing a unitary benefit (that is, $C = 1$), with single premium Π, or annual level premiums P payable for the whole policy duration. We have:

$$\text{Ben}'(0, m) = {}_m A'_x$$

$$\text{Prem}'(0, m) = \begin{cases} \Pi & \text{in the case of single premium} \\ P\, \ddot{a}'_{x:\overline{m}|} & \text{in the case of annual level premiums} \end{cases}$$

For $t = 1, 2, \ldots, m-1$, we have:

$$\text{Ben}'(t, m) = {}_{m-t} A'_{x+t}$$

$$\text{Prem}'(t, m) = \begin{cases} 0 & \text{in the case of single premium} \\ P\, \ddot{a}'_{x+t:\overline{m-t}|} & \text{in the case of annual level premiums} \end{cases}$$

Further, for $t = 0, 1, \ldots, m-1$, we have:

$$\text{Ben}'(t, t+1) = {}_1 A'_{x+t} = P_t^{[\mathrm{N}]}$$

$$\text{Prem}'(t, t+1) = \begin{cases} \Pi & \text{in the case of single premium, if } t = 0 \\ 0 & \text{in the case of single premium, if } t \geq 1 \\ P & \text{in the case of annual level premiums} \end{cases}$$

❑

5.3 The Policy Reserve

5.3.1 Definition

Refer to the time interval (t, m), with $t = 0, 1, \ldots, m$; let V_t denote the quantity such that:

$$\text{Prem}'(t, m) + V_t = \text{Ben}'(t, m) \tag{5.3.1}$$

Clearly, from Eq. (5.2.1) we obtain

$$V_0 = 0 \tag{5.3.2}$$

Conversely, for $t > 0$, the amount V_t fulfills the equivalence principle given that only "residual" benefits and premiums are referred to.

We note that if $\text{Ben}'(t, m) > \text{Prem}'(t, m)$, then the insurer is in a debt position. Hence, the financing condition can be simply expressed by the inequality $V_t \geq 0$ which means no credit position. From Eq. (5.3.1) we also note that, if $\text{Ben}'(t, m) > \text{Prem}'(t, m)$, the amount V_t together with the future premiums exactly meets the future benefits.

The quantity

$$V_t = \text{Ben}'(t, m) - \text{Prem}'(t, m) \tag{5.3.3}$$

is called the *prospective net reserve*. The adjective "prospective" denotes that the reserve refers to the "future" time interval, namely from time t onwards (the retrospective reserve will be shortly addressed in Sect. 5.3.6), whereas "net" recalls that we are not allowing for expenses and related loadings. Of course, the reserve we have defined is a *policy reserve*, as it refers to an insurance contract (the portfolio reserve will be dealt with in Sect. 6.1). The expression *mathematical reserve* is also used.

As already mentioned, the reserve, defined by (5.3.3), is assessed adopting the pricing basis TB1. Hence, it can be considered a prudential valuation of the insurer's debt. However, as the pricing basis leads to an implicit safety loading, the "degree" of prudence cannot be easily determined. An explicit approach to a safe-side assessment of the reserve will be presented in Sects. 6.1.2 and 6.1.3.

5.3.2 The Policy Reserve for Some Insurance Products

The following examples are straightforward applications of formula (5.3.3), which defines the reserve. If not otherwise stated, we assume unitary benefits. We first consider insurance products financed by annual level premiums. It is understood that, for each product, the premium P must rely on the appropriate formula (see Sect. 4.4.2).

For a whole life insurance, with lifelong premiums, we find:

$$V_t = A'_{x+t} - P\,\ddot{a}'_{x+t} \tag{5.3.4}$$

In the case of s-year temporary premiums, we have:

$$V_t = \begin{cases} A'_{x+t} - P(s)\,\ddot{a}'_{x+t:\overline{s-t}|} & \text{if } t < s \\ A'_{x+t} & \text{if } t \geq s \end{cases} \tag{5.3.5}$$

The reserve of a term insurance, with premiums payable for the whole policy duration, is given by:

$$V_t = {}_{m-t}A'_{x+t} - P\,\ddot{a}'_{x+t:\overline{m-t}|} \tag{5.3.6}$$

For a pure endowment insurance, we have:

$$V_t = {}_{m-t}E'_{x+t} - P\,\ddot{a}'_{x+t:\overline{m-t}|} \tag{5.3.7}$$

and for an endowment insurance:

$$V_t = A'_{x+t,\overline{m-t}|} - P\,\ddot{a}'_{x+t:\overline{m-t}|} \tag{5.3.8}$$

We now address, for $t > 0$, insurance products financed by a single premium. For a pure endowment insurance, we have:

$$V_t = {}_{m-t}E'_{x+t} \tag{5.3.9}$$

whereas for an immediate life annuity in advance, we find:

$$V_t = \ddot{a}'_{x+t} \tag{5.3.10}$$

When a premium arrangement based on single-recurrent premiums is adopted, the reserve can be easily determined via iterated application of the single-premium reserve formula. For example, consider a pure endowment insurance, and assume that, at time t, the amounts $\Delta S_0, \Delta S_1, \ldots \Delta S_{t-1}$ have been financed according to the scheme presented in Sect. 4.4.5 (see relations (4.4.36)). The sum insured cumulated up to time t is then S_t. Hence, the reserve is given by

$$V_t = {}_{m-t}E'_{x+t} \sum_{h=0}^{t-1} \Delta S_h = S_t\,{}_{m-t}E'_{x+t} \tag{5.3.11}$$

In a whole life insurance, the sum assured cumulated up to time t is C_t. Then, we find:

$$V_t = A'_{x+t} \sum_{h=0}^{t-1} \Delta C_h = C_t\,A'_{x+t} \tag{5.3.12}$$

In particular, if $i' = 0$ we have (see (4.4.39)):

$$V_t = \sum_{h=0}^{t-1} \Pi_h \tag{5.3.13}$$

Remark We note that, although the arrangement based on single-recurrent premiums falls in the category of periodic premium arrangements, a reserve formula similar to (5.3.7) (for the pure endowment insurance) or (5.3.5) (for the whole life insurance) cannot be adopted because the amount of premiums payable from time t onwards is, at least in principle, unknown.

5.3.3 The Time Profile of the Policy Reserve

The policy reserve, V_t, is a function of time t. When analyzing its behavior against time, we assume that the insured is alive at time t.

As we have so far assumed that the reserve is calculated by adopting the pricing basis, the reserve itself at the policy issue, namely at time $t = 0$, is equal to zero, whatever the premium arrangement (see (5.2.1) and (5.3.2)). However, in the case of a single premium, Π, it is usual to focus on the reserve immediately after cashing the premium itself, denoted by V_{0+}, hence setting:

$$V_{0+} = V_0 + \Pi = \Pi \tag{5.3.14}$$

As regards the value of the reserve at maturity, i.e., at time m, for a term insurance we clearly have:

$$V_m = 0 \tag{5.3.15}$$

Conversely, for a pure endowment and an endowment insurance with a unitary amount as the benefit in case of survival, we find:

$$V_m = 1 \tag{5.3.16}$$

We now move to the time profile for $t = 1, 2, \ldots$ (thus, restricting the analysis at the policy anniversaries). Since we have chosen numerical life tables (the input of the calculation procedures), although derived from an analytical model (the Heligman–Pollard law), to express mortality assumptions, the time profile of the reserve (the output) can only be analyzed in numerical terms. Notwithstanding, some arguments emerging from the numerical inspection have a wide range of application. A number of examples follow.

Example 5.3.1 The reserve of a single-premium term insurance is plotted in Fig. 5.1, whereas the case of annual level premium is referred to in Fig. 5.2. In both the cases, data are as follows: sum assured $C = 1\,000$, $x = 40$, $m = 10$; the pricing basis is TB1 = (0.02, LT1).

In Fig. 5.3, the reserves corresponding to various ages at entry are plotted. The other data are unchanged. Conversely, Fig. 5.4 displays the reserves related to various policy durations, age $x = 40$. ❑

Fig. 5.1 Term insurance; single premium

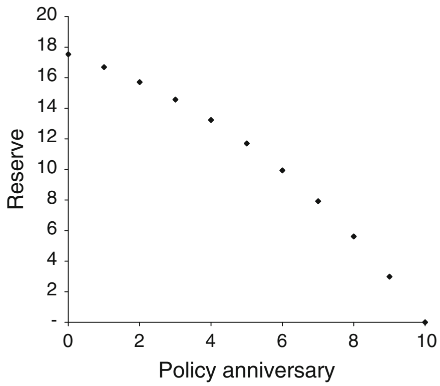


Fig. 5.2 Term insurance; annual level premiums

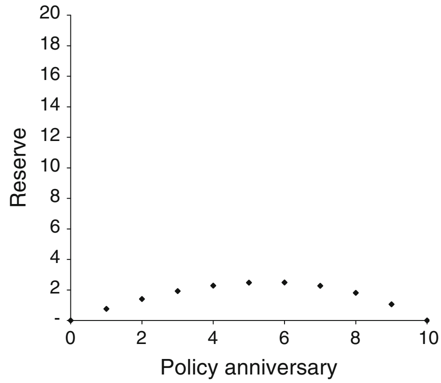
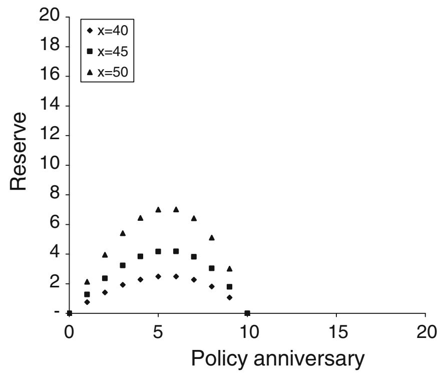


Fig. 5.3 Term insurances, with various ages at entry; annual level premiums



The following features of the reserve of the term insurance should be pointed out.

- The reserve is, in any case, very small if compared to the sum assured.
- In the case of a single premium, the premium itself is progressively used according to the mutuality mechanism working in the insurer's portfolio, and hence the reserve decreases throughout the policy duration.

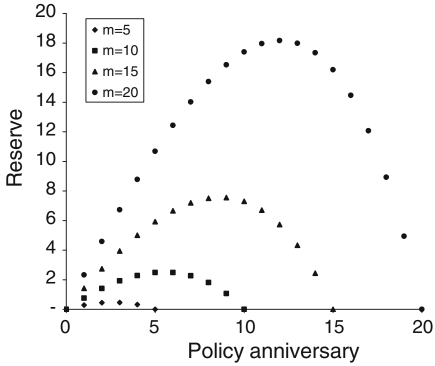


Fig. 5.4 Term insurances, with various durations; annual level premiums

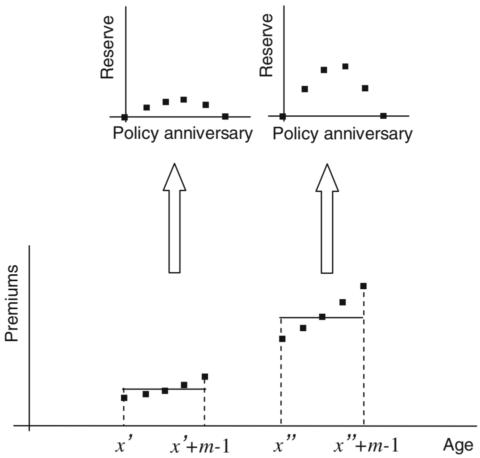


Fig. 5.5 Reserve profile depending on the age at entry (annual level premiums)

- In the case of annual level premiums, the reserve initially grows, since the level premium slightly exceeds the corresponding natural premium (see Sect. 4.4.3, and Example 4.4.3 in particular), then it decreases and is equal to zero at the end, because the insurer has no obligation if the insured is alive at maturity.
- Still in the case of annual level premiums, the reserve profile is higher when the age at entry is higher, for a given policy term; this can be explained in terms of variation of the natural premiums throughout the policy duration (again, see Sect. 4.4.3, and Example 4.4.3; see also Fig. 5.5, in which the solid horizontal lines represent the amount of the level premium for initial ages x' and x'', respectively). A similar argument explains the higher values of the reserve, for a given age at policy issue, when the policy term is greater.

Figure 5.6 explains the variation (either positive or negative) in the reserve value, in the case of annual premiums.

Example 5.3.2 We refer to a decreasing term insurance (see Sect. 4.3.4). Data are as follows: $x = 40$, $m = 10$, $\text{TB1} = (0.02, \text{LT1})$. The sums assured are given by

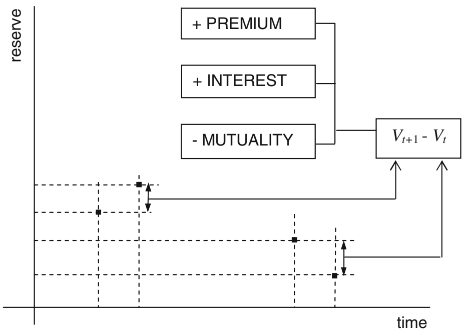


Fig. 5.6 Annual variations in the reserve of a term insurance (annual level premiums)

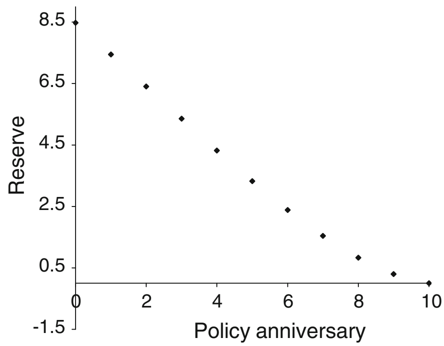


Fig. 5.7 Decreasing term insurance; single premium

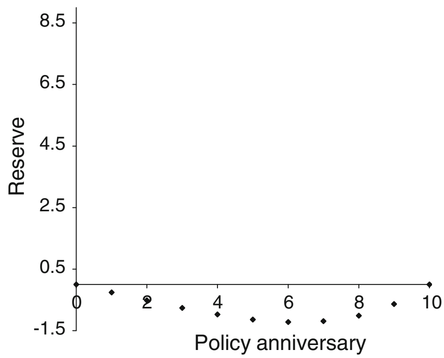


Fig. 5.8 Decreasing term insurance; annual level premiums

$C_{h+1} = \frac{10-h}{10}\,1\,000$, $h = 0, 1, \ldots, 9$. The reserve profile in the case of a single premium is plotted in Fig. 5.7. Conversely, Fig. 5.8 displays the reserve in the case of annual level premiums payable for the whole policy duration. The violation of the financing condition is apparent. Shortening the premium payment period leads to the reserve profiles plotted in Figs. 5.9 and 5.10. In particular, the former shows an insufficient shortening ($s = 8$), whereas the latter displays a feasible arrangement ($s = 7$). ❑

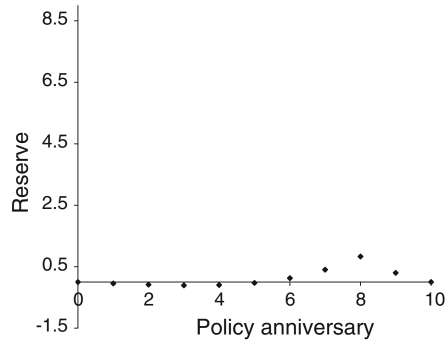


Fig. 5.9 Decreasing term insurance; shortened annual level premiums ($s = 8$)

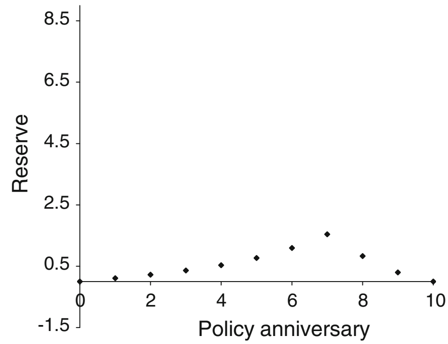


Fig. 5.10 Decreasing term insurance; shortened annual level premiums ($s = 7$)

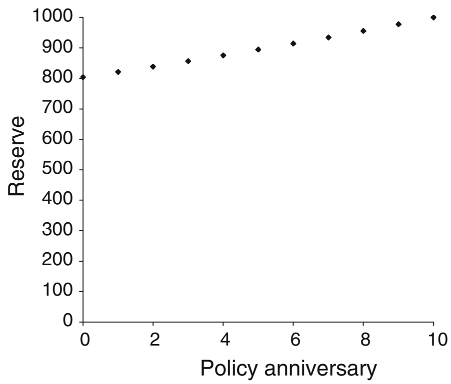


Fig. 5.11 Pure endowment; single premium

Example 5.3.3 The reserve of a single-premium pure endowment is plotted in Fig. 5.11, whereas the case of annual level premium is referred to in Fig. 5.12. In both the cases, data are as follows: sum assured $C = 1\,000$, $x = 40$, $m = 10$, $\text{TB1} = (0.02, \text{LT1})$. ❑

The reserve of a pure endowment is increasing throughout the whole policy duration. Figure 5.13 shows the causes of annual increments in the reserve, in the case

Fig. 5.12 Pure endowment; annual level premiums

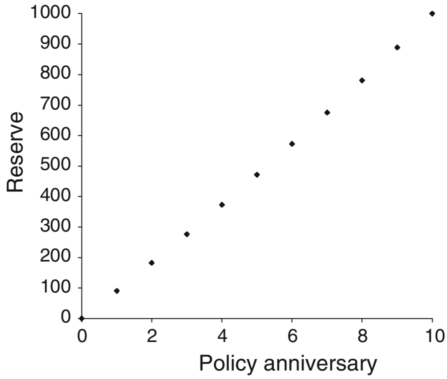
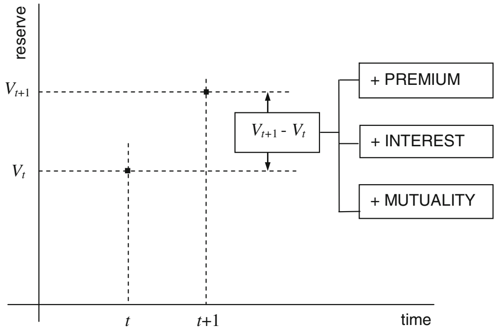


Fig. 5.13 Annual variation in the reserve of a pure endowment (annual level premiums)



of annual premiums. In particular, we recall that each individual reserve is annually credited with a share of reserves released by the insureds who died in that year (see also Case 4a in Sect. 1.7.4, and Fig. 1.24 in particular).

Example 5.3.4 Figures 5.14 and 5.15 refer to an endowment insurance, with single premium and annual level premiums, respectively. Data are as for the pure endowment. ❑

The time profile of the reserve of an endowment insurance almost coincides with that of a pure endowment. In fact, the difference between the two reserves is the reserve of a term insurance (assuming that the same technical basis is adopted in the three insurance products), and hence it is very small, as already noted. It is worth noting, however, that the rationale underlying the annual variations in the reserve of an endowment insurance is quite different. Indeed, the payment of death benefits to insureds who die implies that shares of each individual reserve are annually subtracted from the reserve itself. See Fig. 5.16, in which the mutuality effect works in a negative sense with respect to insureds who are still alive.

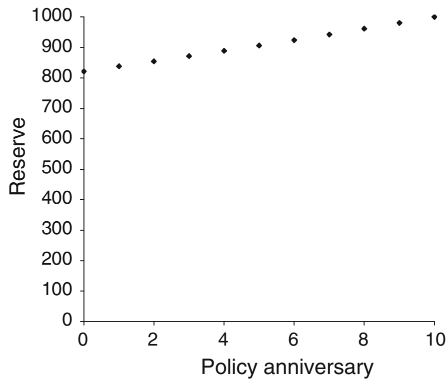


Fig. 5.14 Endowment insurance; single premium

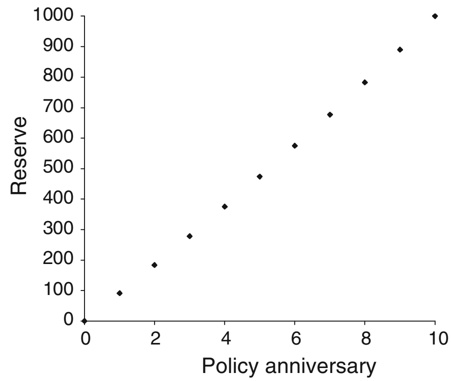


Fig. 5.15 Endowment insurance; annual level premiums

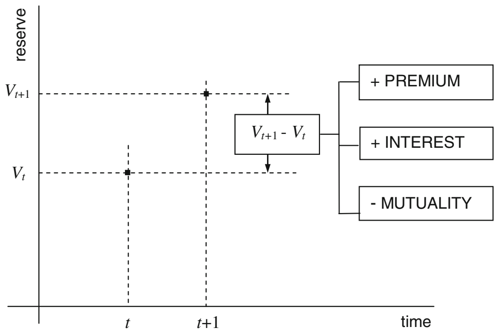


Fig. 5.16 Annual variation in the reserve of an endowment insurance (annual level premiums)

Example 5.3.5 Figures 5.17 and 5.18 refer to a whole life insurance, with single premium and annual level premiums payable for $s = 20$ years, respectively. Data are as follows: $C = 1\,000$, $x = 50$, TB1 = (0.02, LT1). ❑

The time profile of the reserve of a whole life insurance is increasing, in both the case of single premium and annual level premiums, and tends to the sum assured C.

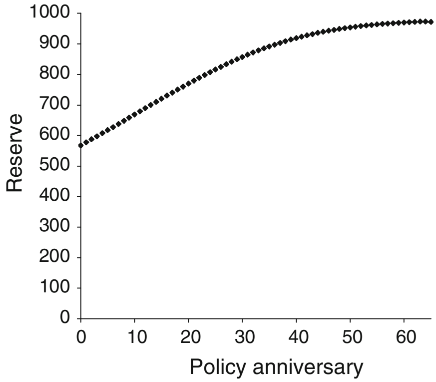


Fig. 5.17 Whole life insurance; single premium

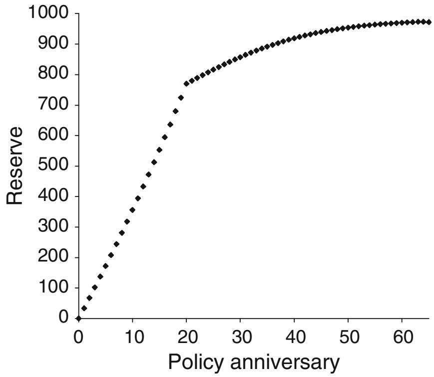


Fig. 5.18 Whole life insurance; temporary annual level premiums

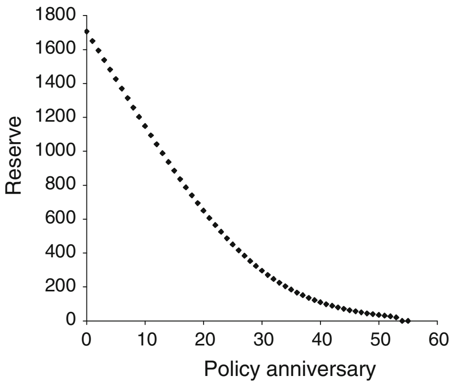


Fig. 5.19 Single-premium life annuity

In the case of annual premiums, we note that, when all the premiums have been paid, the behavior of the reserve coincides with that of the single-premium reserve.

Example 5.3.6 The reserve of a single-premium immediate life annuity (in arrears) is plotted in Fig. 5.19. Data are as follows: $b = 100$, $x = 65$, TB1 $= (0.02, \text{LT4})$. Conversely, Fig. 5.20 shows the time profile of a fund (whose initial amount is equal

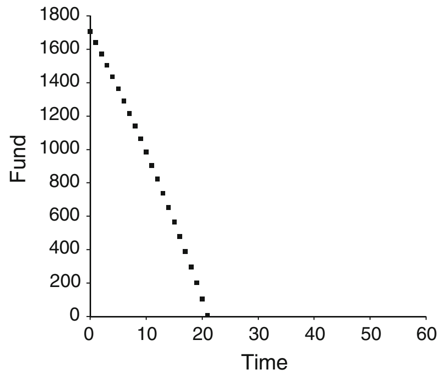


Fig. 5.20 Withdrawal process

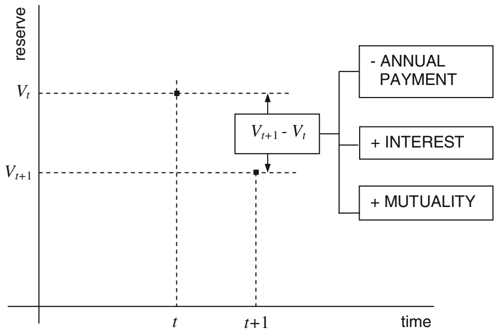


Fig. 5.21 Annual variation in the reserve of an immediate life annuity (single premium)

to the single premium of the immediate life annuity), from which the annual amount $b = 100$ is withdrawn; the interest rate is 0.02. The withdrawal process exhausts the fund in 21 years. On this aspect, see Sect. 1.2.5, Case 4c. ❑

The reserve of an immediate life annuity is decreasing throughout the whole policy duration. Figure 5.21 shows the causes of annual decrements in the reserve. We note, in particular, that the mutuality mechanism works as in the pure endowment. Of course, no mutuality mechanism works in the withdrawal process (see Fig. 1.3). The presence of the mutuality mechanism in the life annuity explains the substantial difference between the time profiles shown in Figs. 5.19 and 5.20, respectively.

5.3.4 Change in the Technical Basis

In some circumstances, the reserve must be calculated by adopting a technical basis (called the *reserving basis*, or *valuation basis*) other than the pricing basis used for determining the premiums. Such a need can arise, for example, because:

- a "realistic" assessment of the insurer's debt is required, in order to single out the safety component included in the reserve;

- an important change in the financial or biometric scenario makes the reserve (assessed according to the pricing basis) either no longer prudential, or conversely too high.

The former issue will be addressed in Sect. 6.1.3; how to allow for the consequences of a change in the scenario is the topic of the present section.

Assume that a significant change in the scenario is accounted for when assessing the reserves. This change can be due, for example, to an important variation observed in the mortality, or to different forecasts about the return on investments. The consequent variation in the reserve (when positive) can constitute a compulsory action, imposed by the supervisory authority.

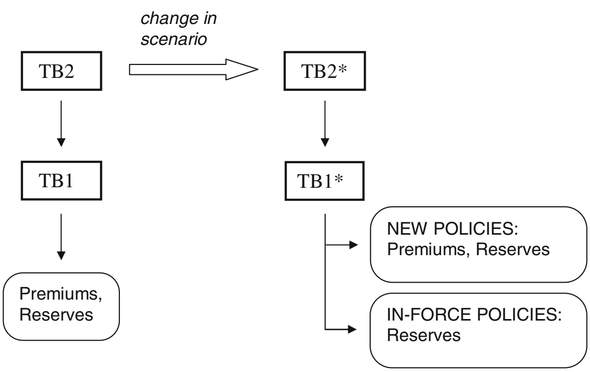
Figure 5.22 sketches the consequences of a change in the scenario. First, the new scenario is expressed by an updated second-order basis, TB2*, which, in its turn, suggests the adoption of a new first-order basis, TB1*. This basis will be used as a pricing basis, and hence adopted in pricing as well as reserving, for policies written after the scenario change. Conversely, premiums of in-force policies cannot be changed, since policy conditions are guaranteed at the policy issue. Thus, for these policies, the basis TB1* is only used to update the reserves.

Several approaches to the reserve updating are available, at least in principle. We focus on some approaches, referring to an endowment insurance with annual level premiums payable for the whole policy duration. As usual, x denotes the insured's age at policy issue, m the duration, C the sum insured in both the cases of death and survival. We assume that the shift in the technical basis occurs at time τ; the updated reserve will be denoted with $V_t^{[u]}$, for $t \geq \tau$. Further, we assume that the shift implies an increase in the reserve; hence, $V_\tau^{[u]} > V_\tau$.

The updated reserve is defined as the amount that, at time τ, together with the actuarial value of the future premiums (whose amount P has been stated at policy issue), meets (according to the equivalence principle) the actuarial value of the future benefits; both the actuarial values rely on the new basis TB1*. In formal terms:

$$V_\tau^{[u]} + P\, \ddot{a}^*_{x+\tau:m-\tau\rceil} = C A^*_{x+\tau,m-\tau\rceil} \tag{5.3.17}$$

Fig. 5.22 Shift to new technical bases because of a change in scenario



In more general terms, the equivalence principle requires that the following condition is fulfilled:

$$(V_\tau + \Delta V_\tau) + (P + \Delta P)\, \ddot{a}^*_{x+\tau:m-\tau\rceil} = CA^*_{x+\tau,m-\tau\rceil} \tag{5.3.18}$$

Condition (5.3.18) is an equation in the two unknowns ΔV_τ and ΔP. Particular solutions of (5.3.18) suggest practicable approaches to the updating problem. It is understood that, whatever is the particular solution chosen, the insurer is charged with both amounts ΔV_τ and ΔP.

1. Set

$$\Delta V_\tau = V_\tau^{[u]} - V_\tau \tag{5.3.19}$$

and hence $\Delta P = 0$; Eq. (5.3.18) reduces to (5.3.17). This approach implies an immediate rise in the reserve (at time τ) and hence turns out to be the most prudential. For all integer t, $t \geq \tau$, we then have:

$$V_t^{[u]} = CA^*_{x+t,m-t\rceil} - P\, \ddot{a}^*_{x+t:m-t\rceil} \tag{5.3.20}$$

2. Less prudential approaches consist in a lower rise, ΔV_τ, in the reserve, that is

$$0 < \Delta V_\tau < V_\tau^{[u]} - V_\tau \tag{5.3.21}$$

followed by premium integrations ("paid" by the insurer), ΔP, which amortize the missing share of the required increment in the reserve, namely the amount $V_\tau^{[u]} - (V_\tau + \Delta V_\tau)$. A particular approach in this category can be of prominent practical interest. Let P^* denote the annual premium according to the pricing basis TB1*, namely the premium such that $P^*\, \ddot{a}^*_{x:m\rceil} = CA^*_{x,m\rceil}$. Then, set

$$\Delta P = P^* - P \tag{5.3.22}$$

From (5.3.18), it follows:

$$\Delta V_\tau = CA^*_{x+\tau,m-\tau\rceil} - P^*\, \ddot{a}^*_{x+\tau:m-\tau\rceil} - V_\tau \tag{5.3.23}$$

It is worth noting that the resulting reserve, $V_\tau + \Delta V_\tau$, coincides with the reserve, $V_\tau^* = CA^*_{x+\tau,m-\tau\rceil} - P^*\, \ddot{a}^*_{x+\tau:m-\tau\rceil}$, which will pertain to new policies issued according to the basis TB1*. Hence, the advantage of this particular approach consists in a reserve accumulation process coinciding with that for the new policies. For all integer t, $t \geq \tau$, we then have:

$$V_t^* = CA^*_{x+t,m-t\rceil} - P^* \ddot{a}^*_{x+t:m-t\rceil} \tag{5.3.24}$$

3. Set $\Delta V_\tau = 0$; hence, from (5.3.18) we obtain:

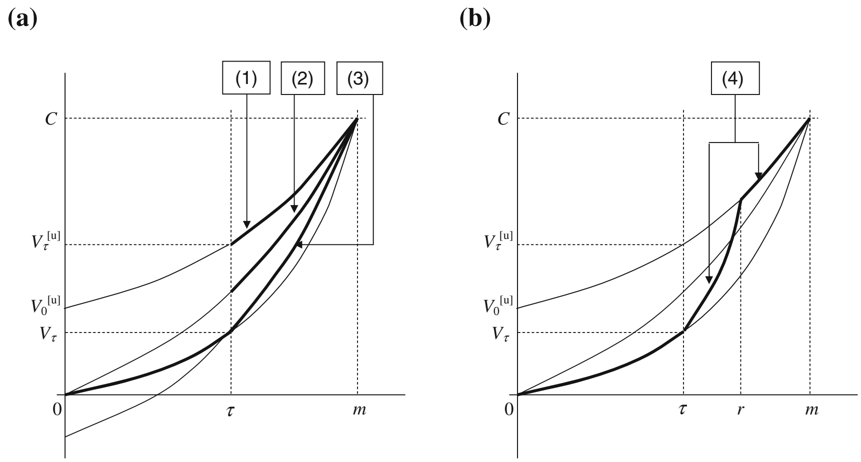


Fig. 5.23 Updating the reserve because of a shift in the technical basis

$$\Delta P = \frac{C A^*_{x+\tau,m-\tau\rceil} - P\,\ddot{a}^*_{x+\tau:m-\tau\rceil} - V_\tau}{\ddot{a}^*_{x+\tau:m-\tau\rceil}} = \frac{V_\tau^{[u]} - V_\tau}{\ddot{a}^*_{x+\tau:m-\tau\rceil}} \tag{5.3.25}$$

Thus, the whole required update in the reserve, that is $V_\tau^{[u]} - V_\tau$, is amortized in $m - \tau$ years. For all integer t, $t \geq \tau$, denoting with $\tilde{V}_t$ the resulting reserve, we then have:

$$\tilde{V}_t = C A^*_{x+t,m-t\rceil} - (P + \Delta P)\,\ddot{a}^*_{x+t:m-t\rceil} \tag{5.3.26}$$

with ΔP given by (5.3.25). Clearly, this approach does not provide a prudential solution.

The solutions we have described lead to the reserve profiles sketched in Fig. 5.23a (for simplicity, the reserve profile is represented by a solid line, i.e., disregarding the jumps corresponding to annual premiums).

Of course, other technical solutions are available, even outside the framework designed by condition (5.3.18). We just mention the following one.

4. Set $\Delta V_\tau = 0$. Let

$$s = \max\{r - \tau, 0\} \tag{5.3.27}$$

with $r < m$. Then, if $s \geq 1$, set:

$$Q = \frac{V_\tau^{[u]} - V_\tau}{\ddot{a}^*_{x+\tau:s\rceil}} \tag{5.3.28}$$

Table 5.1 Updating the reserve because of a shift in the technical basis

t	V_t	$V_t^{[u]}$ (1)	V_t^* (2)	$\tilde{V}_t$ (3)
0	0.00			
1	53.59			
2	108.64			
3	165.21			
4	223.37			
5	283.19			
6	344.75			
7	408.16			
8	473.51	570.03	509.62	473.51
9	540.95	628.54	576.35	545.16
10	610.63	687.83	643.97	617.76
11	682.71	747.99	712.59	691.43
12	757.42	809.14	782.33	766.30
13	835.00	871.41	853.35	842.55
14	915.74	934.97	925.83	920.37
15	1 000.00	1 000.00	1 000.00	1 000.00

Of course, if $s = 0$ we simply have:

$$Q = V_\tau^{[u]} - V_\tau \tag{5.3.29}$$

Hence, the premium integration, Q, amortizes the required increase in the reserve in a period shorter than the residual policy duration (see Fig. 5.23b).

Example 5.3.7 Refer to an endowment insurance with annual level premiums payable for the whole policy duration. Data are as follows: $C = 1\,000$, $x = 50$, $m = 15$, TB1 $= (0.03, \text{LT1})$. The resulting annual premium is $P = 55.13$. At time $\tau = 8$, because of a decrease in interest rates, the technical basis shifts to TB1$^* = (0.01, \text{LT1})$. The resulting annual premium is $P^* = 64.27$. Table 5.1 displays the reserve V_t which relies on the basis TB1, and the reserves $V_t^{[u]}$, V_t^*, and $\tilde{V}_t$, calculated according to formulae (5.3.20), (5.3.24), and (5.3.26), respectively. ❑

5.3.5 *The Reserve at Fractional Durations*

The analysis of the time profile of the reserve has been so far restricted to the policy anniversaries, namely integer durations since the policy issue. The extension to fractional durations is, however, of practical interest. For example, the need for calculating the policy reserve (and the portfolio reserve, as well) at times other than the policy anniversaries arises when assessing the items of the balance sheet.

The calculation of the exact value of the policy reserve at all past durations can be carried out in a time-continuous setting. In such a setting, a mortality law must be available, instead of a numerical life table. In the actuarial practice, however, it is rather common to work in a time-discrete framework (as we are actually doing) and to obtain approximations to the exact value of the reserve via interpolation procedures, in particular by adopting linear interpolation formulae. Here we illustrate the interpolation approach, focussing on some examples.

Consider an insurance policy, for example, a term insurance, with premium arrangement based on natural premiums. The reserve is, of course, equal to zero at all the policy anniversaries, before cashing the premium which falls due at that time; thus $V_t = 0$ for all integer t. Immediately after cashing the premium, the insurer's debt (and the corresponding asset) is clearly equal to the premium itself; hence, denoting with V_{t^+} the reserve after cashing the premium, we have:

$$V_{t^+} = P_t^{[N]}; \quad t = 0, 1, \dots \tag{5.3.30}$$

Then, the premium is used throughout the year according to the mutuality mechanism and, again, we have $V_{t+1} = 0$. At time $t + r$, with $0 < r < 1$, we let:

$$V_{t+r} = (1 - r)\, V_{t^+} = (1 - r)\, P_t^{[N]} \tag{5.3.31}$$

The resulting time profile of the reserve is plotted in Fig. 5.24.

As the second example, we refer to an insurance product (e.g., an endowment insurance) with annual level premiums P. After cashing the premium which falls due at time t, the reserve increases from V_t to

$$V_{t^+} = V_t + P \tag{5.3.32}$$

Then, the linear interpolation yields:

$$V_{t+r} = (1 - r)\, V_{t^+} + r\, V_{t+1} = [(1 - r)\, V_t + r\, V_{t+1}] + (1 - r)\, P \tag{5.3.33}$$

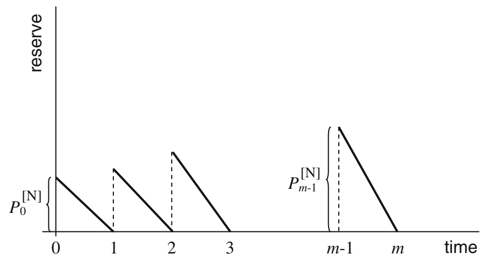


Fig. 5.24 Interpolated reserve profile in the case of natural premiums

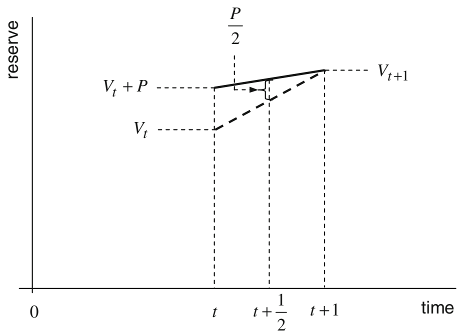


Fig. 5.25 Reserve interpolation in the case of annual level premiums

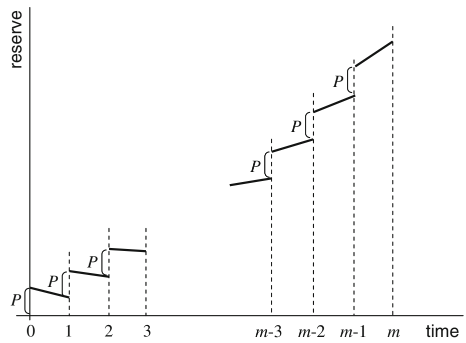


Fig. 5.26 Interpolated reserve profile in the case of annual level premiums: an example

See Fig. 5.25. We note, in particular, the following aspects.

- Interpolating between V_t (instead of V_{t^+}) and V_{t+1} would cause an apparent underestimation of the reserve at all times between t and $t+1$ (again, see Fig. 5.25).
- The "use" of the premium P depends on the specific insurance product addressed. For example, if we consider an endowment insurance, the share of the premium used to cover death benefits according to the mutuality mechanism is decreasing throughout the policy duration (as we will see in Sect. 5.4.3); this fact determines a time profile of the reserve like that plotted in Fig. 5.26.

As the third example, we consider an insurance product (for example, a term insurance, or a pure endowment, or an endowment insurance), with a single premium Π. In this case, there is no jump in the reserve profile, but at the payment of the single premium, when the reserve jumps from $V_0 = 0$ to $V_{0^+} = \Pi$. Then, the interpolation procedure is as follows:

$$V_r = (1-r)\, V_{0^+} + r\, V_1 \tag{5.3.34a}$$

$$V_{t+r} = (1-r)\, V_t + r\, V_{t+1} \quad \text{for } t = 1, 2, \dots \tag{5.3.34b}$$

See Fig. 5.27.

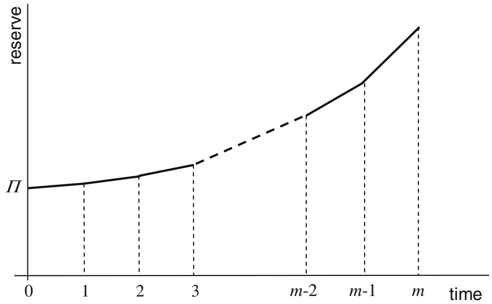


Fig. 5.27 Interpolated reserve profile in the case of single premium: an example

Finally, we refer to single-premium life annuities, providing an annual benefit b. The jumps in the reserve profile correspond to the annual payments of the benefit, as illustrated in Fig. 5.28. For a life annuity in arrears (panel (a)), taking as usual $V_{0^+} = \Pi$, the interpolation is as follows:

$$V_r = (1 - r)\, V_{0^+} + r\,(V_1 + b) \tag{5.3.35a}$$

$$V_{t+r} = (1 - r)\, V_t + r\,(V_{t+1} + b) \ \text{ for } \ t = 1, 2, \ldots \tag{5.3.35b}$$

where $V_t = a'_{x+t}$. For a life annuity in advance (panel (b)), taking again $V_{0^+} = \Pi$, the interpolation is as follows:

$$V_r = (1 - r)\,(V_{0^+} - b) + r\, V_1 \tag{5.3.36a}$$

$$V_{t+r} = (1 - r)\,(V_t - b) + r\, V_{t+1} \ \text{ for } \ t = 1, 2, \ldots \tag{5.3.36b}$$

with $V_t = \ddot{a}'_{x+t}$.

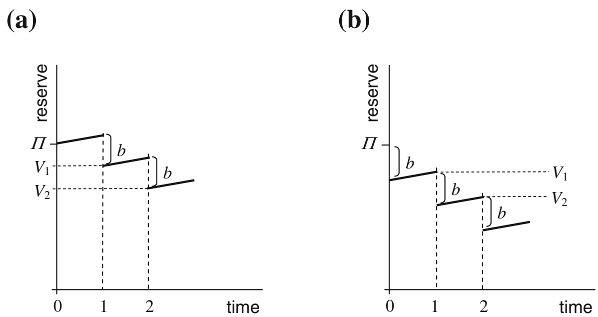


Fig. 5.28 Interpolated reserve profile for life annuities

5.3.6 The Retrospective Reserve

The (prospective) policy reserve has been defined as the balancing term, V_t, which transforms inequality (5.2.2b) into relation (5.3.1). Looking at inequality (5.2.2a), and hence referring to the time interval $(0, t)$, we can define the so-called retrospective reserve.

Let B_t denote the amount such that

$$\text{Prem}'(0, t) = B_t + \text{Ben}'(0, t) \tag{5.3.37}$$

The amount B_t can be interpreted as the actuarial value (at the policy issue) of the benefit that the insurer should pay at time t if the insured decides (at that time) to abandon the contract, stopping the premium payment and renouncing all the benefits which fall due after time t.

Clearly, this interpretation holds if $B_t > 0$, namely $\text{Prem}'(0, t) > \text{Ben}'(0, t)$. Actually, this inequality should be satisfied: indeed, if $B_t > 0$, then the insurer is in a debt position and hence the financing condition is fulfilled.

The benefit, W_t, whose actuarial value at time $t = 0$ is given by B_t, is then defined by the following relation:

$$B_t = W_t \, {}_tE'_x \tag{5.3.38}$$

Hence, we find:

$$W_t = \frac{1}{{}_tE'_x} \left(\text{Prem}'(0, t) - \text{Ben}'(0, t)\right) \tag{5.3.39}$$

The quantity W_t is called the *retrospective reserve*. Note that the term $\frac{1}{{}_tE'_x}$, namely the actuarial accumulation factor (see Sect. 4.2.10), plays the role of referring the valuation at time t.

Remark The interpretation of W_t as the amount to be paid by the insurer in the case the policyholder abandons the contract, although interesting under a theoretical perspective, requires in practice various adjustments. For example, expenses should be accounted for, and penalties could be applied in determining the amount paid by the insurer. We will return on these issues in Sect. 5.7.

The following examples are straightforward applications of formula (5.3.39), which defines the retrospective reserve.

In insurance products which provide a death benefit (term insurance, whole life insurance, and endowment insurance), the insurer's liability is given by the coverage of the death risk over the time interval $(0, t)$. Thus, assuming a unitary benefit, and annual level premiums payable throughout the whole policy duration, we have, for all these products:

$$W_t = \frac{1}{{}_tE'_x} \left(P \, \ddot{a}'_{x:\overline{t}|} - {}_tA'_x\right) \tag{5.3.40}$$

where P denotes the annual premium related to the specific product addressed.

In a pure endowment with annual level premiums, we have:

$$W_t = \frac{1}{{}_tE'_x} P\, \ddot{a}'_{x:\overline{t}|} \tag{5.3.41}$$

as this product does not provide any benefit in the time interval $(0, t)$ (of course, if $t < m$, where m denotes the policy term).

In the case of a single premium (given, according to the equivalence principle, by the actuarial value of the benefits), we have for an endowment insurance:

$$W_t = \frac{1}{{}_tE'_x} (A'_{x,\overline{m}|} - {}_tA'_x) \tag{5.3.42}$$

Replacing $A'_{x,\overline{m}|}$ with A'_x or ${}_mA'_x$, we have the retrospective reserve for the whole life insurance and the term insurance, respectively.

For a single-premium pure endowment, we have:

$$W_t = \frac{1}{{}_tE'_x}\, {}_mE'_x \tag{5.3.43}$$

Remark In spite of the adjective "retrospective," the reserve we are dealing with cannot be interpreted as an ex-post quantification of the "past" liabilities (namely, those preceding time t) of the insurer and the insured. From (5.3.39), it is apparent that the calculation of the retrospective reserve first relies on the valuation at time 0 of the benefits and premiums pertaining to the interval $(0, t)$ (and hence "future" with respect to time 0), and then on a valuation at time t via the actuarial accumulation factor $\frac{1}{{}_tE'_x}$.

Let us go back to the reserve of the single-premium pure endowment (see (5.3.43)). We note that, for this insurance product, the prospective reserve is given by $V_t = {}_{m-t}E'_{x+t}$. Further, we have ${}_mE'_x = {}_tE'_x\; {}_{m-t}E'_{x+t}$ (see (4.2.53)), and hence:

$$W_t = V_t \tag{5.3.44}$$

thus, the prospective and the retrospective reserve coincide. Result (5.3.44) holds under rather general conditions. This topic is beyond the scope of this chapter. So, we will simply provide a further example, and some final remarks as well.

We refer to a whole life insurance, with annual level premium P payable for the whole policy duration. The single premium is, of course, given by A'_x. The following relations hold:

$$A'_x = {}_tA'_x + {}_tE'_x\; A'_{x+t} \tag{5.3.45a}$$

$$\ddot{a}'_x = \ddot{a}'_{x:\overline{t}|} + {}_tE'_x\; \ddot{a}'_{x+t} \tag{5.3.45b}$$

$$P\, \ddot{a}'_x = A'_x \tag{5.3.45c}$$

The prospective reserve for this insurance product is given by (5.3.4). By using relations (5.3.45), we find:

$$V_t = \frac{A'_x - {}_tA'_x}{{}_tE'_x} - P\,\frac{\ddot{a}'_x - \ddot{a}'_{x:\overline{t}|}}{{}_tE'_x} = \frac{1}{{}_tE'_x}\left(P\,a'_{x:\overline{t}|} - {}_tA'_x\right) = W_t \tag{5.3.46}$$

that is, the coincidence between the prospective and the retrospective reserves.

Whenever relations similar to those expressed by formulae (5.3.45) hold, we have the coincidence between the two reserves, provided that the same technical basis is used for both the reserves. However, relations of this type do not hold, for example, in relation to some insurance products which provide benefits depending on the lifetimes of more than one individual. In those products, the reserve at time t depends on which insureds are alive at that time, i.e., on the "status" (either actual or hypothetical) of the insured group, whereas the retrospective reserve, which first requires a valuation at time 0, can only represent a weighted average of the "possible" prospective reserves at time t.

5.3.7 The Actuarial Accumulation Process

To introduce some interesting relations between the reserving process and the premium flows, we will just refer to an example, provided by an m-year term insurance with annual level premiums payable for the whole policy duration. We assume a unitary sum insured.

The natural premiums of the term insurance are expressed by (4.4.27), namely $P_h^{[N]} = {}_1A'_{x+h} = (1+i')^{-1}\,q'_{x+h}$, for $h = 0, 1, \ldots, m-1$. The reserve premiums, $P_h^{[AS]}$, are defined by (4.4.35).

Consider the actuarial value at the policy issue of the reserve premiums pertaining to the first t policy years. This value is given by:

$$\sum_{h=0}^{t-1} P_h^{[AS]}\,{}_hE'_x = P\sum_{h=0}^{t-1} {}_hE'_x - \sum_{h=0}^{t-1} {}_1A'_{x+h}\,{}_hE'_x \tag{5.3.47}$$

From the following relations:

$$\sum_{h=0}^{t-1} {}_hE'_x = \ddot{a}'_{x:\overline{t}|} \tag{5.3.48a}$$

$$\sum_{h=0}^{t-1} {}_1A'_{x+h}\,{}_hE'_x = {}_tA'_x \tag{5.3.48b}$$

we then find that the actuarial value of the reserve premiums can be expressed as follows:

$$P\,\ddot{a}'_{x:t\rceil} - {}_tA'_x = {}_tE'_x\; W_t \tag{5.3.49}$$

(see also (5.3.40)). Finally, we obtain:

$$W_t = \frac{1}{{}_tE'_x} \sum_{h=0}^{t-1} P_h^{[\mathrm{AS}]}\, {}_hE'_x = \sum_{h=0}^{t-1} P_h^{[\mathrm{AS}]} \frac{1}{{}_{t-h}E'_{x+h}} \tag{5.3.50}$$

Thus, the retrospective reserve is the result of the actuarial accumulation of the reserve premiums pertaining to the policy years preceding the time of valuation of the reserve itself. From a more practical perspective, we can say that the retrospective reserve originates thanks to the accumulation of assets exceeding the benefits.

On the one hand, the interpretation relying on the actuarial accumulation of the reserve premiums can be useful in understanding the time profile of the reserve (see Sect. 5.3.3). On the other hand, a different splitting of the annual premium allows us to interpret the policy reserve as the result of a purely financial accumulation process. As we will see in Sect. 5.4, this alternative splitting of the annual premiums is of paramount importance in interpreting the intermediation role of a life insurer.

We just mention that, conversely, the prospective reserve at time t can be expressed as minus the actuarial value (at that time) of the future reserve premiums, namely:

$$V_t = -\sum_{h=0}^{m-t-1} P_{t+h}^{[\mathrm{AS}]}\, {}_hE'_{x+t} \tag{5.3.51}$$

5.4 Risk and Savings

The first topic addressed in this section relates to recursive procedures for the calculation of the policy reserve. Nonetheless, the practical interest of the topic goes well beyond computational aspects. In fact, the topic itself constitutes the starting point for an in depth analysis of the role of a life insurance company. In particular, technical aspects will emerge, concerning the life insurer as a player in both the financial intermediation and the risk pooling process.

5.4.1 A (Rather) General Insurance Product

We refer to an insurance product, with the following characteristics: term m, age at policy issue x, sum insured in the case of death C, sum insured in the case of survival at maturity S, annual level premiums, P, payable for the whole policy duration, and hence given by:

$$P = \frac{C\,{}_{m}A'_{x} + S\,{}_{m}E'_{x}}{\ddot{a}'_{x:\overline{m}|}} \tag{5.4.1}$$

For example,

- setting $S = 0$, $C > 0$, we have the term insurance, with constant sum assured;
- setting $S > 0$, $C = 0$, we find the pure endowment;
- setting $S = C > 0$, we have the (standard) endowment insurance;
- setting $S > C > 0$, we have the endowment insurance with additional survival benefit.

A number of possible generalizations allow us to recognize other insurance products. For example,

- setting $S = 0$, $C > 0$, $m = \omega - x$, we find the whole life insurance;
- setting $S = 0$, and replacing C with a sequence $C_1, C_2, \dots, C_m$, we have the term insurance with varying benefit, and, in particular, the decreasing term insurance;
- replacing P with a sequence $P_0, P_1, \dots, P_{m-1}$, we can represent arrangements based on variable premiums; in particular:
  - with $P_0 = P_1 = \cdots = P_{s-1}$, $P_s = P_{s+1} = \cdots = P_{m-1} = 0$, we have arrangements based on level premiums payable over a shortened period ($s < m$);
  - setting $P_0 > 0$, $P_1 = P_2 = \cdots = P_{m-1} = 0$, we represent the single-premium arrangement;
  - the natural premium arrangement is obviously represented by setting $P_h = P_h^{[N]}$, for $h = 0, 1, \dots, m-1$.

Other generalizations allow us to represent various types of life annuities. Notwithstanding, in what follows we refer to the insurance product defined at the beginning of this section.

5.4.2 Recursive Equations

The policy reserve, at time t, of the insurance product defined above is given by:

$$V_t = \mathrm{Ben}'(t, m) - \mathrm{Prem}'(t, m) = C\,{}_{m-t}A'_{x+t} + S\,{}_{m-t}E'_{x+t} - P\,\ddot{a}'_{x+t:\overline{m-t}|} \tag{5.4.2}$$

We can also write:

$$V_t = C\,{}_{1}A'_{x+t} - P + C\,{}_{1|m-t-1}A'_{x+t} + S\,{}_{m-t}E'_{x+t} - P\,{}_{1|}\ddot{a}'_{x+t:\overline{m-t-1}|} \tag{5.4.3}$$

and, after a little algebra, we get to the following expression:

$$V_t + P = C\,{}_{1}A'_{x+t} + V_{t+1}\,{}_{1}E'_{x+t} \tag{5.4.4}$$

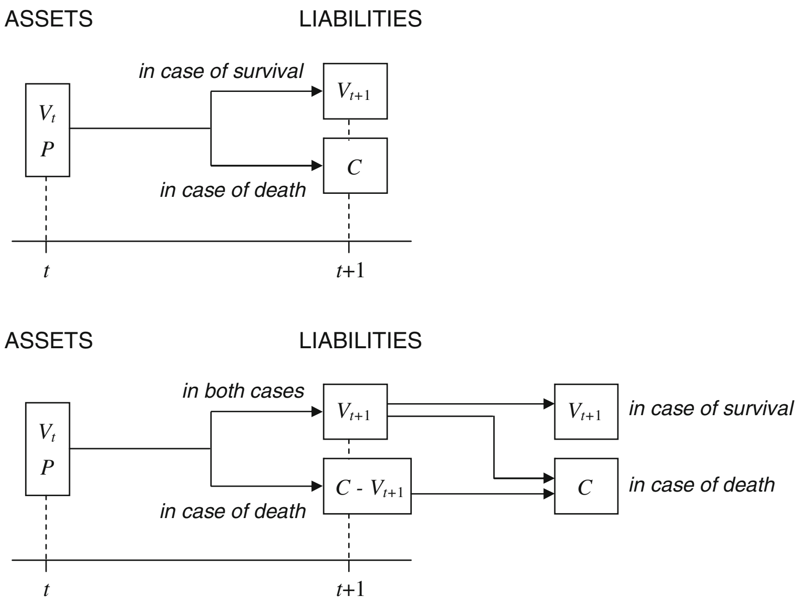


Fig. 5.29 Recursive equations: interpretations

or, in more explicit terms:

$$V_t + P = C\,(1+i')^{-1}\,q'_{x+t} + V_{t+1}\,(1+i')^{-1}\,p'_{x+t} \tag{5.4.5}$$

Recursive Eq. (5.4.5) is called the *Fouret equation* (1891). We note the following features.

- Actuarial values in (5.4.5) are referred at time t, as both the financial and the probabilistic evaluation are referred at that time (that is, the insured is assumed to be alive at time t).
- Equation (5.4.5) describes an "equilibrium" situation in the time interval $(t, t+1)$: the assets available at time t (the reserve V_t and the premium P just cashed) exactly meet the liabilities which fall due at time $t+1$, namely:
  - the sum assured C, in the case of death;
  - the reserve V_{t+1}, which is needed either to continue the policy in the case of survival (if $t+1 < m$), or to be paid as sum S at maturity (if $t+1 = m$)

  (see Fig. 5.29, upper panel).
- The policy reserve can be calculated by an iterative application of (5.4.5): starting from $V_0 = 0$, the equation allows us to calculate $V_1, V_2, \ldots, V_m$ (with the "final" check $V_m = S$); conversely, starting from $V_m = S$, we can calculate $V_{m-1}, V_{m-2}, \ldots, V_0$ (with $V_0 = 0$).

Alternative expressions of (5.4.5) are the following ones:

$$(V_t + P)\,(1 + i') = C q'_{x+t} + V_{t+1}\, p'_{x+t} \tag{5.4.6}$$

$$V_t + P = (C - V_{t+1})\,(1 + i')^{-1}\, q'_{x+t} + V_{t+1}\,(1 + i')^{-1} \tag{5.4.7}$$

$$(V_t + P)\,(1 + i') = (C - V_{t+1})\, q'_{x+t} + V_{t+1} \tag{5.4.8}$$

We note the following aspects.

- In Eqs. (5.4.6) and (5.4.8) (called the *Kanner equation*, 1869), the financial evaluation is referred to time $t + 1$ (whereas the probabilistic evaluation is still referred to time t).
- In Eqs. (5.4.7) and (5.4.8), the reserve V_{t+1} appears as a liability certain at time $t + 1$ (that is, in both the cases of death and survival), whereas the death benefit (if any) is split into two shares,

$$C = (C - V_{t+1}) + V_{t+1} \tag{5.4.9}$$

  namely:

  – the amount $C - V_{t+1}$, which is called the *sum at risk* (or the *net amount at risk*), to stress that it is not yet available but funded (year by year) via the mutuality mechanism;
  – the amount V_{t+1}, which is not "at risk," as the reserve has to be used anyhow (sooner or later)

  (see Fig. 5.29, lower panel).
- In the case of no death benefit ($C = 0$), or a death benefit smaller than the reserve ($C < V_t$), the amount at risk is negative; in these cases, if the insured dies in the year, the sum at risk (the whole reserve, in the case $C = 0$) is released for mutuality and thus contributes to financing the benefits pertaining to the policies still in-force.

Remark 1 It is worth stressing that the term "risk" is used, in this context, according to its traditional actuarial meaning, that is "risk of death." Other risk causes (e.g., investment risks) are not involved.

Remark 2 Recursive Eqs. (5.4.5)–(5.4.8) can be easily interpreted also referring to a portfolio of policies. Let N_t denotes the (given) number of policies in-force at time t, and N_{t+1} the random number of policies in-force at time $t + 1$, namely the number of insureds still alive. Further, let D_t denote the random number of insureds dying in the year; thus, $D_t = N_t - N_{t+1}$. Refer, for example, to Eq. (5.4.6). We can write:

$$(N_t\, V_t + N_t\, P)(1 + i') = N_t\, q'_{x+t}\, C + N_t\, p'_{x+t}\, V_{t+1} \tag{5.4.10}$$

On the left-hand side of Eq. (5.4.10), we find the amount of resources (reserves and premiums) pertaining to policies in-force at time t, cumulated to time $t + 1$. As regards the right-hand side of the equation, we first note that $N_t\, p'_{x+t} = \mathbb{E}[N_{t+1}]$ is the expected number of insureds alive at time $t + 1$, whereas $N_t\, q'_{x+t} = \mathbb{E}[D_t]$ is the expected number of insureds dying in the year. The interpretation of the right-hand side of Eq. (5.4.10) in terms of insurer's expected obligations is then straightforward.

5.4.3 Risk Premium and Savings Premium

From Eq. (5.4.7), we obtain:

$$P = [(C - V_{t+1})\,(1+i')^{-1}\,q'_{x+t}] + [V_{t+1}\,(1+i')^{-1} - V_t] \tag{5.4.11}$$

so that the two following components of the annual premium can be recognized:

$$P_t^{[\mathrm{R}]} = (C - V_{t+1})\,(1+i')^{-1}\,q'_{x+t} \tag{5.4.12a}$$

$$P_t^{[\mathrm{S}]} = V_{t+1}\,(1+i')^{-1} - V_t \tag{5.4.12b}$$

The two components are called the *risk premium* and the *savings premium*, respectively.

The savings premiums maintain the reserving process. In fact, from (5.4.12b) we find:

$$V_{t+1} = (V_t + P_t^{[\mathrm{S}]})\,(1+i') \tag{5.4.13}$$

and then:

$$V_{t+1} = P_0^{[\mathrm{S}]}\,(1+i')^{t+1} + P_1^{[\mathrm{S}]}\,(1+i')^{t} + \cdots + P_t^{[\mathrm{S}]}\,(1+i') \tag{5.4.14}$$

It turns out that the policy reserve is the result of the financial accumulation of the savings premiums. Conversely, the risk premium is the premium of a one-year term insurance to cover the sum at risk.

We note that the two premium components are not necessarily both positive. In particular, if the sum at risk is negative, the risk premium is negative. See the following numerical examples for further details.

Example 5.4.1 Table 5.2 refers to a term insurance, with annual level premiums (denoted by P_t, as in following examples other premium arrangements will be addressed), payable for the whole policy duration. In particular, the decomposition of the annual premium into risk premium and savings premium is displayed. Further, the natural premiums and the time profiles of the reserve and the sum at risk are shown. Data are as follows: $C = 1\,000$, $x = 50$, $m = 10$, $\mathrm{TB1} = (0.02, \mathrm{LT1})$. It is interesting to note that the risk premiums are very close to the natural premiums, as the reserve is very small and hence the sum at risk almost coincides with the sum assured.

Table 5.3 refers to a single-premium term insurance. Clearly, $P_0 = \Pi = C\,{}_mA'_x$. Data are as above. Natural premiums coincide, of course, with those in Table 5.2; in fact, natural premiums only depend on the benefit structure, while they are independent of the specific premium arrangement. All the savings premiums, but the first one, are negative and represent the "use" of the reserve in the mutuality process. ❑

Example 5.4.2 A pure endowment is referred to in Table 5.4. Data are as follows: $S = 1\,000$, $x = 50$, $m = 10$, $\mathrm{TB1} = (0.02, \mathrm{LT1})$. As $C = 0$, the sum at risk is

Table 5.2 Term insurance (annual level premiums)

t	P_t	$P_t^{[N]}$	$P_t^{[R]}$	$P_t^{[S]}$	V_t	$C - V_t$
0	5.40	3.31	3.31	2.09	0.00	–
1	5.40	3.68	3.66	1.74	2.14	997.86
2	5.40	4.08	4.05	1.35	3.95	996.05
3	5.40	4.52	4.49	0.91	5.40	994.60
4	5.40	5.01	4.98	0.42	6.44	993.56
5	5.40	5.56	5.52	−0.12	7.00	993.00
6	5.40	6.17	6.13	−0.73	7.01	992.99
7	5.40	6.84	6.80	−1.40	6.41	993.59
8	5.40	7.58	7.56	−2.16	5.11	994.89
9	5.40	8.41	8.41	−3.01	3.01	996.99
10	–	–	–	–	0	1 000.00

Table 5.3 Term insurance (single premium)

t	P_t	$P_t^{[N]}$	$P_t^{[R]}$	$P_t^{[S]}$	V_t	$C - V_t$
0	48.52	3.31	3.16	45.35	0.00	–
1	0.00	3.68	3.52	−3.52	46.26	953.74
2	0.00	4.08	3.91	−3.91	43.60	956.40
3	0.00	4.52	4.35	−4.35	40.48	959.52
4	0.00	5.01	4.85	−4.85	36.85	963.15
5	0.00	5.56	5.41	−5.41	32.64	967.36
6	0.00	6.17	6.03	−6.03	27.78	972.22
7	0.00	6.84	6.73	−6.73	22.19	977.81
8	0.00	7.58	7.52	−7.52	15.76	984.24
9	0.00	8.41	8.41	−8.41	8.41	991.59
10	–	–	–	–	0.00	1 000.00

negative, and then all the risk premiums are negative; hence we find $P_t^{[S]} > P$ for all t. This means that the premium P is insufficient to maintain the reserving process, which in fact needs for the contributions provided by the reserves of the policies terminating because of the insureds' death. Formally, this feature clearly appears by rewriting Eq. (5.4.8) for the pure endowment; indeed, we find:

$$(V_t + P)\,(1 + i') + V_{t+1}\, q'_{x+t} = V_{t+1} \tag{5.4.15}$$

where the term $V_{t+1}\, q'_{x+t}$ represents the contribution mentioned above. ❑

Example 5.4.3 Table 5.5 refers to a (standard) endowment insurance. Data are as follows: $C = S = 1\,000$, $x = 50$, $m = 10$, TB1 $= (0.02, \text{LT1})$. All the entries in Table 5.5 can be obtained as the sum of the corresponding entries in Tables 5.2

Table 5.4 Pure endowment (annual level premiums)

t	P_t	$P_t^{[N]}$	$P_t^{[R]}$	$P_t^{[S]}$	V_t	$C - V_t$
0	86.30	0.00	−0.29	86.60	0.00	–
1	86.30	0.00	−0.66	86.96	88.33	−88.33
2	86.30	0.00	−1.11	87.41	178.80	−178.80
3	86.30	0.00	−1.66	87.96	271.53	−271.53
4	86.30	0.00	−2.33	88.63	366.68	−366.68
5	86.30	0.00	−3.14	89.45	464.42	−464.42
6	86.30	0.00	−4.12	90.43	564.95	−564.95
7	86.30	0.00	−5.30	91.61	668.48	−668.48
8	86.30	0.00	−6.72	93.02	775.29	−775.29
9	86.30	971.98	−8.41	94.71	885.68	−885.68
10	–	–	–	–	1 000.00	−1 000.00

Table 5.5 Endowment insurance (annual level premiums)

t	P_t	$P_t^{[N]}$	$P_t^{[R]}$	$P_t^{[S]}$	V_t	$C - V_t$
0	91.71	3.31	3.01	88.69	0.00	–
1	91.71	3.68	3.00	88.70	90.46	909.54
2	91.71	4.08	2.95	88.76	182.75	817.25
3	91.71	4.52	2.83	88.87	276.94	723.06
4	91.71	5.01	2.65	89.05	373.12	626.88
5	91.71	5.56	2.38	89.32	471.42	528.58
6	91.71	6.17	2.00	89.70	571.96	428.04
7	91.71	6.84	1.50	90.20	674.90	325.10
8	91.71	7.58	0.84	90.86	780.40	219.60
9	91.71	980.39	0.00	91.71	888.69	111.31
10	–	–	–	–	1 000.00	0.00

and 5.4. We note that all the risk premiums and the savings premiums are positive. This suggests to look at the endowment insurance as the combination of an m-year financial transaction and a sequence of one-year term insurances, as shown in Table 5.6. The interpretation is as follows. An individual, instead of purchasing a m-year endowment insurance with sum insured C, and hence paying the annual premiums P, could in each year:

- invest the amount $P_t^{[S]}$ in a fund, managed by a financial institution, and annually credited with the interest rate i';
- pay the amount $P_t^{[R]}$ to an insurer to buy a one-year term insurance for a sum assured such that the sum itself plus the balance of the fund is equal to C.

Table 5.6 The endowment insurance as a combination of transactions

Year $(t, t+1)$	A m-year financial transaction		A sequence of m one-year term insurances	
	Payment (at time t)	Result (at time $t+1$)	Payment (at time t)	Result (at time $t+1$)
(0, 1)	$P_0^{[S]}$	V_1	$P_0^{[R]}$	$C - V_1$
(1, 2)	$P_1^{[S]}$	V_2	$P_1^{[R]}$	$C - V_2$
(2, 3)	$P_2^{[S]}$	V_3	$P_2^{[R]}$	$C - V_3$
…	…	…	…	…
$(m-2, m-1)$	$P_{m-2}^{[S]}$	V_{m-1}	$P_{m-2}^{[R]}$	$C - V_{m-1}$
$(m-1, m)$	$P_{m-1}^{[S]} = P$	$V_m = C$	$P_{m-1}^{[R]} = 0$	$C - V_m = 0$

It is easy to check that, in both the case of survival and the case of death prior to maturity, the amounts paid by the individual and the benefits obtained by the beneficiaries coincide with the corresponding outflows and inflows of the endowment insurance. It is worth stressing, however, that the "equivalence" between the endowment insurance and the set of transactions described above relies on some important assumptions that, at least to some extent, are rather unrealistic. In particular, the financial transaction should guarantee a constant interest rate i', as a (traditional) endowment insurance does. As regards the one-year term insurances, the life table adopted for calculating the premiums could be changed throughout the m years, thanks to mortality improvements in the population, and hence with an advantage to the insured; on the contrary, if medical examinations are required, the death probabilities could be raised because of worsened health conditions. In conclusion, while the interpretation we have sketched is useful to understand the two-fold role of a life insurance company, it should not be meant as aiming to prove analogies among the results of different transactions. ❑

Example 5.4.4 Table 5.7 refers to a single-premium immediate life annuity (in arrears). Data are as follows: $b = 100$, $x = 65$, TB1 = (0.02, LT4). The technical structure of a life annuity requires a generalization of the recursive equations. First, we set $C = 0$ in Eq. (5.4.6), and then we generalize the equation as follows:

$$(V_t + P_t)(1 + i') = (V_{t+1} + b)\, p'_{x+t} \tag{5.4.16}$$

where $V_0 = 0$, $P_0 = \Pi = a'_x$, and $P_t = 0$, for $t = 1, 2, \ldots$. Equation (5.4.16) allows us to split the annual benefit b in order to single out the resources used to finance the benefit itself. To simplify the notation, we assume $V_0 = \Pi$; hence, we can simply write, for $t = 0, 1, 2, \ldots$:

$$V_t\,(1 + i') = (V_{t+1} + b)\, p'_{x+t} \tag{5.4.17}$$

Table 5.7 Life annuity in arrears (single premium)

t	P_t	$P_t^{[N]}$	$P_t^{[R]}$	$P_t^{[S]}$	V_t	$b = 100$		
						Reserve consumption	Interest	Mortality credit
0	1 706.88	97.48	−9.81	1 716.69	0.00	–	–	–
1	0.00	97.41	−10.72	10.72	1 651.02	55.86	34.14	10.00
2	0.00	97.33	−11.70	11.70	1 594.97	56.04	33.02	10.94
3	0.00	97.23	−12.76	12.76	1 538.81	56.16	31.90	11.94
4	0.00	97.13	−13.89	13.89	1 482.60	56.21	30.78	13.02
5	0.00	97.01	−15.11	15.11	1 426.43	56.18	29.65	14.17
…	…	…	…	…	…	…	…	…
10	0.00	96.16	−22.43	22.43	1 149.01	54.72	24.07	21.21
11	0.00	95.92	−24.16	24.16	1 094.87	54.14	22.98	22.88
12	0.00	95.65	−25.97	25.97	1 041.41	53.46	21.90	24.64
13	0.00	95.35	−27.87	27.87	988.73	52.68	20.83	26.49
14	0.00	95.01	−29.85	29.85	936.93	51.80	19.77	28.43
15	0.00	94.63	−31.90	31.90	886.11	50.82	18.74	30.44
…	…	…	…	…	…	…	…	…
20	0.00	91.93	−43.17	43.17	650.12	44.48	13.89	41.62
21	0.00	91.20	−45.57	45.57	607.16	42.97	13.00	44.03
22	0.00	90.37	−47.99	47.99	565.78	41.38	12.14	46.48
23	0.00	89.46	−50.43	50.43	526.04	39.73	11.32	48.95
24	0.00	88.45	−52.88	52.88	488.01	38.04	10.52	51.44
25	0.00	87.34	−55.32	55.32	451.70	36.30	9.76	53.94
…	…	…	…	…	…	…	…	…

After a little algebra, we find:

$$b = \left[V_t - V_{t+1}\right] + \left[V_t\, i'\right] + \left[V_t\,(1+i')\,\frac{q'_{x+t}}{p'_{x+t}}\right] \tag{5.4.18}$$

The interpretation of (5.4.18) is straightforward:

- the first term on the right-hand side is the amount of benefit financed by the reserve consumption;
- the second term is the amount financed by the interest on the reserve at the beginning of the year (which coincides with the single premium if $t = 0$);

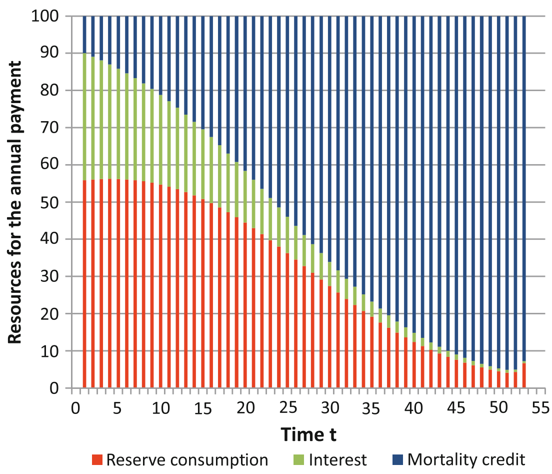


Fig. 5.30 Resources financing the annual benefit

- the third term represents the contribution from the mutuality mechanism, i.e., the *mortality credit*; indeed, it can be easily interpreted rewriting the fraction in terms of the expected number of survivors as follows:

$$\frac{q'_{x+t}}{p'_{x+t}} = \frac{\ell'_{x+t} - \ell'_{x+t+1}}{\ell'_{x+t+1}}$$

where the ℓ''s denote the expected numbers of survivors according to the first-order table.

The splitting of the annual benefit is shown by the last three columns of Table 5.7 and, in graphical terms, by Fig. 5.30. It clearly appears that the mutuality effect becomes more and more important as t increases, because of an increasing mortality among annuitants. To single out the risk and savings components, we generalize Eq. (5.4.7), again setting $C = 0$:

$$V_t + P_t = -V_{t+1}\,(1+i')^{-1}\,q'_{x+t} + V_{t+1}\,(1+i')^{-1} + b\,(1+i')^{-1}\,p'_{x+t} \quad (5.4.19)$$

where (as in (5.4.16)) $V_0 = 0$, $P_0 = \Pi = a'_x$, and $P_t = 0$, for $t = 1, 2, \dots$. From (5.4.19), after a little algebra we obtain:

$$P_t = \left[(V_{t+1} + b)\,(1+i')^{-1} - V_t\right] + \left[(-V_{t+1} - b)\,(1+i')^{-1}\,q'_{x+t}\right] \quad (5.4.20)$$

and then

$$P_t^{[R]} = (-V_{t+1} - b)\,(1+i')^{-1}\,q'_{x+t} \quad (5.4.21a)$$

$$P_t^{[S]} = (V_{t+1} + b)\,(1+i')^{-1} - V_t \quad (5.4.21b)$$

Hence, $P_t^{[R]} < 0$ for all t, and, for $t = 1, 2, \dots,$ as $P_t = 0$ then $P_t^{[S]} = -P_t^{[R]} > 0$. Thus, reserves released by the annuitants dying in the various years maintain the reserves of the surviving annuitants, according to the mutuality mechanism. As regards the natural premiums, we have, for all t:

$$P_t^{[N]} = b\,(1+i')^{-1}\,p'_{x+t} \tag{5.4.22}$$

Table 5.7 shows the numerical results. ❑

5.4.4 Life Insurance Products versus Financial Accumulation

Consider the whole life insurance, financed via single-recurrent premiums, and assume $i' = 0$ (see Sect. 4.4.5). As already noted, according to this arrangement no mortality risk is borne by the insurer. The formal proof is straightforward. In the case of death in year t, the sum paid to the beneficiary is $C_t = \sum_{h=0}^{t-1} \Pi_h$ (see Eq. (4.4.39)); the reserve at time t is $V_t = \sum_{h=0}^{t-1} \Pi_h$ (see Eq. (5.3.13)). Thus, $C_t = V_t$, and hence the sum at risk is equal to zero.

In general, any product in which the death benefit coincides with the policy reserve is just a financial accumulation product. In fact, from

$$C_t = V_t \tag{5.4.23}$$

it follows:

$$P_t^{[R]} = 0 \tag{5.4.24}$$

and hence

$$P_t^{[S]} = P \tag{5.4.25}$$

so that the reserve coincides with the accumulation of the premiums P (or P_t, or Π_t).

A financial accumulation product can be transformed into a "real" insurance product via redefinition of the death benefit C_t, which can be expressed as a function of the reserve V_t, such that the following inequality holds:

$$C_t > V_t \tag{5.4.26}$$

(instead of (5.4.23)). This transformation can be mandatory because of regulation, or can be useful for tax purposes, etc. Some examples follow; in the related figures, the dashed line represents the policy reserve.

1. Choose the amount K, and set:

$$C_t = V_t + K \tag{5.4.27}$$

   Thus, the sum at risk is K; see Fig. 5.31.

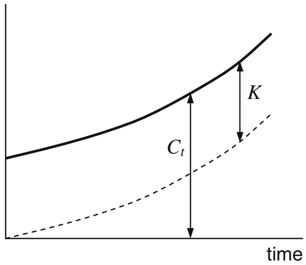


Fig. 5.31 Constant sum at risk

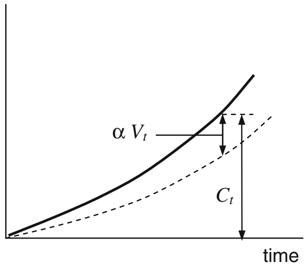


Fig. 5.32 Proportional sum at risk

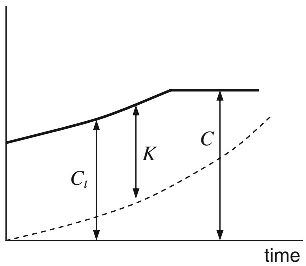


Fig. 5.33 Sum at risk with an *upper* bound

2. Choose the rate α, and set:

$$C_t = (1 + \alpha)\, V_t \tag{5.4.28}$$

   Thus, the sum at risk is $\alpha\, V_t$; see Fig. 5.32.
3. Choose the amounts K and C, and set:

$$C_t = \min\{V_t + K, C\} \tag{5.4.29}$$

   Thus, the sum at risk is $\min\{K, C - V_t\}$; see Fig. 5.33.
4. Choose the amounts K and C, and set:

$$C_t = \max\{V_t + K, C\} \tag{5.4.30}$$

   Thus, the sum at risk is $\max\{K, C - V_t\}$; see Fig. 5.34.

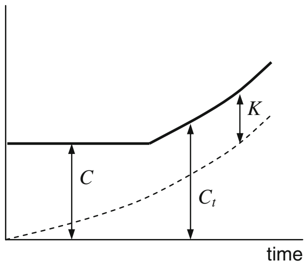


Fig. 5.34 Sum at risk with a *lower* bound

Remark We note that in case 3 above, it may turn out $C_t < V_t$, namely if V_t increases above C. Given the purposes of the policy design, it is not acceptable that $C_t < V_t$. Thus, the death benefit

$$C_t = \max\{\min\{V_t + K, C\}, V_t\} \tag{5.4.31}$$

should rather be considered instead of (5.4.29).

5.5 Expected Profits

The approach to the profit assessment we have described in Sect. 4.3.8 simply relies on a comparison between actuarial values of benefits, namely between the actuarial value calculated by adopting the scenario basis, i.e., TB2, and the actuarial value assumed as the single premium, hence calculated by adopting the prudential basis, i.e., TB1.

A deeper analysis of expected profits requires further steps. In particular:

1. premium arrangements other than that based on a single premium must be allowed for;
2. as life insurance contracts usually have a multi-year duration, it can be useful to attribute a share of the (total) expected profit to each policy year; hence, annual profits are defined, showing the profit emerging throughout the policy duration;
3. further elements, which can constitute sources of profit/loss should be taken into account, and typically
   - expenses and expense loadings;
   - lapses, surrenders, and policy alterations.

Issues 1 and 2 are dealt with in the present section; indeed, the mathematical reserve provides a tool for a "natural" definition of expected annual profits. Conversely, topic 3 will be discussed in Chap. 6, in the framework of a life portfolio analysis.

5.5.1 Expected Annual Profits

We refer to Eq. (5.4.6) which can also be written as follows:

$$(V_t + P)\,(1 + i') - C q'_{x+t} - V_{t+1}\, p'_{x+t} = 0 \tag{5.5.1}$$

Equation (5.5.1) relates to policy year $(t, t+1)$, and expresses a balance between resources (the reserve at the beginning of the year and the premium) and expected obligations (the sum in the case of death and the reserve at the end of the year). The balance relies on the adoption of the same technical basis, namely the first-order basis TB1, in all the elements of Eq. (5.5.1), and this, in its turn, follows from the assumptions adopted in defining the policy reserve (see Sect. 5.3.1).

Conversely, assume that:

- a realistic estimate of the yield from the investment of the amount $V_t + P$ is expressed by the interest rate i'';
- the mortality in the portfolio can be described in realistic terms by probabilities q''_{x+t}.

Thus, the scenario basis TB2 can be introduced into Eq. (5.5.1). The shift to TB2 clearly results in a different meaning of some quantities. Actually, we obtain:

$$(V_t + P)\,(1 + i'') - C q''_{x+t} - V_{t+1}\, p''_{x+t} = \overline{PL}_{t+1} \tag{5.5.2}$$

where $\overline{PL}_{t+1}$ ($\gtreqless 0$) denotes the *expected annual profit/loss* arising from the "distance" between TB1 and TB2. We note that $\overline{PL}_{t+1}$ is referred to time $t+1$, for a policy assumed to be in-force at time t.

Remark Equation (5.5.2) can be easily interpreted also referring to a portfolio of policies. A similar interpretation has been provided for Eq. (5.4.6) (see Remark 2 in Sect. 5.4.2). Let N_t denotes the (given) number of policies in-force at time t, and N_{t+1} the random number of policies in-force at time $t+1$, namely the number of insureds still alive. Then, we can write:

$$N_t\, V_t + N_t\, P + (N_t\, V_t + N_t\, P)\, i'' - N_t\, q''_{x+t}\, C - N_t\, p''_{x+t}\, V_{t+1} = N_t\, \overline{PL}_{t+1} \tag{5.5.3}$$

All quantities can be interpreted as in Eq. (5.4.10). In particular: $N_t\, p''_{x+t} = \mathbb{E}[N_{t+1}]$, $N_t\, q''_{x+t} = \mathbb{E}[D_t]$. Note, however, that the expected numbers are now calculated according to TB2. In Eq. (5.5.3), we can recognize some of the (main) items of the *Profit & Loss Statement* (briefly *P & L*). In general, the P & L Statement refers to a specific period (say, a year) and indicates how the profit/loss originates from income net of expenditure. As we are only addressing one generation of policies, and we are disregarding expenses and related loadings as well as lapses and surrenders, the resulting representation is extremely simplified (see Table 5.8). Further, an obvious adjustment in the benefits is needed when referring to the last year of the policy duration.

Table 5.8 Actuarial values as items of a P & L statement

P & L statement	
Income	
Premiums	$N_t\,P$
Income from investments	$(N_t\,V_t + N_t\,P)\,i''$
Expenditure	
Benefits paid	$\mathbb{E}[D_t]\,C$
Change in liabilities	$\mathbb{E}[N_{t+1}]\,V_{t+1} - N_t\,V_t$
Profit	$N_t\,\overline{PL}_{t+1}$

5.5.2 Splitting the Annual Profit

We now refer to Eq. (5.4.8), written as follows:

$$(V_t + P)\,(1 + i') - (C - V_{t+1})\,q'_{x+t} - V_{t+1} = 0 \tag{5.5.4}$$

Adopting the scenario basis TB2, as in Eq. (5.5.2), we have:

$$(V_t + P)\,(1 + i'') - (C - V_{t+1})q''_{x+t} - V_{t+1} = \overline{PL}_{t+1} \tag{5.5.5}$$

Then, by subtracting (5.5.4) from (5.5.5), we obtain the so-called *contribution formula* (proposed by S. Homans, 1863):

$$(V_t + P)\,(i'' - i') + (C - V_{t+1})\,(q'_{x+t} - q''_{x+t}) = \overline{PL}_{t+1} \tag{5.5.6}$$

which suggests the splitting of the expected annual profit into two terms:

$$\overline{PL}^{[\text{fin}]}_{t+1} = (V_t + P)\,(i'' - i') \tag{5.5.7a}$$

$$\overline{PL}^{[\text{m/l}]}_{t+1} = (C - V_{t+1})\,(q'_{x+t} - q''_{x+t}) \tag{5.5.7b}$$

The quantity $\overline{PL}^{[\text{fin}]}_{t+1}$ is the *financial margin*, namely the component of the expected annual profit originated by the spread between the interest rates, $i'' - i'$. Clearly, as $V_t + P > 0$, the financial margin is positive if and only if $i'' > i'$.

The component $\overline{PL}^{[\text{m/l}]}_{t+1}$ is the *mortality/longevity margin*, which arises from the difference between the mortality rates at the various ages. We note that:

- if $C - V_{t+1} > 0$, the mortality/longevity margin is positive if and only if $q'_{x+t} > q''_{x+t}$;
- if $C - V_{t+1} < 0$, the mortality/longevity margin is positive if and only if $q'_{x+t} < q''_{x+t}$.

Table 5.9 Term insurance: expected profits

t	V_t	$\overline{PL}_t$	$\overline{PL}_t^{[\text{fin}]}$	$\overline{PL}_t^{[\text{m/l}]}$
0	0.00	–	–	–
1	0.76	0.14	0.02	0.12
2	1.40	0.16	0.03	0.13
3	1.92	0.18	0.03	0.15
4	2.29	0.20	0.04	0.16
5	2.48	0.22	0.04	0.18
6	2.49	0.24	0.04	0.20
7	2.27	0.27	0.04	0.22
8	1.81	0.29	0.04	0.25
9	1.06	0.31	0.04	0.27
10	0.00	0.33	0.03	0.30

Thus, the sign of the sum at risk is the driving factor in the choice of the life table to be adopted in the first-order basis, TB1, in order to obtain implicit safety loadings, and hence positive expected profits. For pricing insurance products with a positive sum at risk (for example: the term insurance, the whole life insurance, and the endowment insurance), a life table with a mortality higher than that actually expected in the portfolio should be chosen. On the contrary, products with a negative sum at risk (the pure endowment and the life annuities) require a mortality assumption lower than the mortality actually expected.

Example 5.5.1 Table 5.9 refers to a term insurance. Policy data are as follows: $C = 1\,000$, $x = 40$, $m = 10$; annual level premiums, P, are payable throughout the whole policy duration. The pricing basis is $\text{TB1} = (0.02, \text{LT1})$; we then find: $P = 1.93$. Expected profits are calculated by adopting the second-order basis $\text{TB2} = (0.03, \text{LT2})$. We note that the poor financial content of the term insurance implies very low financial profits, whereas more important contributions to the expected profits come from the mortality assumptions.

Table 5.10 refers to an endowment insurance. Policy data are as follows: $C = 1\,000$, $x = 50$, $m = 15$. Annual level premiums, P, are payable throughout the whole policy duration. The pricing basis is $\text{TB1} = (0.02, \text{LT1})$; we then find: $P = 59.54$. Expected profits are calculated by adopting the second-order basis $\text{TB2} = (0.03, \text{LT2})$. Unlike the term insurance, the endowment insurance has important financial contents, so that the spread between interest rates originates significant contributions to the expected profits. On the contrary, mortality profits are low, and definitely decreasing as the sum at risk decreases. As we will see in Sect. 7.3, the financial profit is shared with policyholders, through an adjustment of benefits. ❑

Table 5.10 Endowment insurance: expected profits

t	V_t	$\overline{PL}_t$	$\overline{PL}_t^{[\mathrm{fin}]}$	$\overline{PL}_t^{[\mathrm{m/l}]}$
0	0.00	–	–	–
1	57.54	0.91	0.60	0.32
2	116.11	1.50	1.17	0.33
3	175.74	2.10	1.76	0.34
4	236.46	2.70	2.35	0.35
5	298.33	3.32	2.96	0.36
6	361.40	3.94	3.58	0.36
7	425.75	4.57	4.21	0.36
8	491.45	5.20	4.85	0.35
9	558.59	5.85	5.51	0.34
10	627.30	6.50	6.18	0.32
11	697.70	7.15	6.87	0.28
12	769.96	7.81	7.57	0.24
13	844.26	8.47	8.29	0.18
14	920.85	9.14	9.04	0.10
15	1 000.00	9.80	9.80	0.00

5.5.3 The Expected Total Profit

The sequence of expected profits/losses $\overline{PL}_1, \overline{PL}_2, \dots, \overline{PL}_m$, which are originated yearly by the policy, can be interpreted as a temporary life annuity. The expected present value of this annuity, $\overline{PL}$, according to the scenario basis TB2, is given by:

$$\overline{PL} = \sum_{t=0}^{m-1} \overline{PL}_{t+1}\,(1+i'')^{-(t+1)}\,{}_t p''_x \tag{5.5.8}$$

which can be interpreted as the expected value of the total profit/loss, expressed as a present value at time 0.

It is possible to check that, assuming $V_0 = 0$ and $V_m = S$, and plugging Eq. (5.5.2) into (5.5.8), we find the following expression:

$$\overline{PL} = \sum_{t=0}^{m-1} P\,(1+i'')^{-t}\,{}_t p''_x - \sum_{t=0}^{m-1} C\,(1+i'')^{-(t+1)}\,{}_{t|1}q''_x - S\,(1+i'')^{-m}\,{}_m p''_x \tag{5.5.9}$$

in which the policy reserve does not appear. We note that the result expressed by (5.5.9) holds thanks to the use of the TB2 for discounting the expected annual profits.

Equation (5.5.9) can also be written as follows:

$$\overline{PL} = \sum_{t=0}^{m-2} {}_t p''_x \, (1+i'')^{-(t+1)} \left[P\,(1+i'') - C q''_{x+t} \right] + {}_{m-1} p''_x \, (1+i'')^{-m} \left[P\,(1+i'') - C q''_{x+m-1} - S\, p''_{x+m-1} \right] \quad (5.5.10)$$

The quantities in brackets, namely

$$\overline{CF}_{t+1} = P\,(1+i'') - C q''_{x+t}; \quad t = 0, 1, \ldots, m-2 \quad (5.5.11a)$$

$$\overline{CF}_m = P\,(1+i'') - C q''_{x+m-1} - S\, p''_{x+m-1} \quad (5.5.11b)$$

represent the *expected annual cash flows*, referred to a policy in-force at time t or $m-1$, respectively, each cash flow being cumulated at the end of the relevant year.

Thus, the expected total profit is the expected present value of the life annuity which consists of the expected annual cash flows. In formal terms:

$$\overline{PL} = \sum_{t=0}^{m-1} \overline{CF}_{t+1} \, (1+i'')^{-(t+1)} \, {}_t p''_x \quad (5.5.12)$$

Hence, the reserve profile affects the expected annual profits and then the emergence of profit throughout time, i.e., the *timing of the profit*, while it does not affect the total amount of the expected profit.

Example 5.5.2 We refer to the insurance products addressed in Example 5.5.1. We find

- for the term insurance: $\overline{PL} = 1.93$;
- for the endowment insurance: $\overline{PL} = 55.90$.

❑

The following example can help in understanding the effect of the reserve on the emerging of expected profits.

Example 5.5.3 Refer to an endowment insurance with annual premiums payable for the whole policy duration. Data are as follows: $C = 1\,000$, $x = 50$, $m = 15$; TB1 $= (0.02, \text{LT1})$, TB2 $= (0.03, \text{LT2})$. Figure 5.35 displays the policy reserves calculated with the interest rates 0, 0.02 (namely i'), and 0.04; possible negative values have been replaced by 0.

Table 5.11 shows the annual profits, $\overline{PL}_t^{(0.00)}$, $\overline{PL}_t^{(0.02)}$, and $\overline{PL}_t^{(0.04)}$, corresponding to the three reserve profiles. It clearly emerges that high reserve values (compared to those obtained using the interest rate $i' = 0.02$) imply a heavy expected loss in the first year, which is recovered by positive expected profits in the following

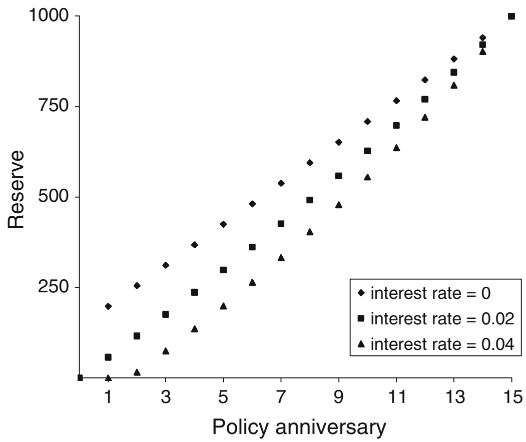


Fig. 5.35 Endowment insurance (annual level premiums)

Table 5.11 Endowment insurance (annual level premiums)

t	$V_t^{(0.00)}$	$\overline{PL}_t^{(0.00)}$	$V_t^{(0.02)}$	$\overline{PL}_t^{(0.02)}$	$V_t^{(0.04)}$	$\overline{PL}_t^{(0.04)}$
0	0.00	–	0.00	–	0.00	–
1	198.08	−139.20	57.54	0.91	0.00	58.28
2	254.82	8.01	116.11	1.50	16.10	41.90
3	311.50	9.72	175.74	2.10	74.82	−0.37
4	368.13	11.42	236.46	2.70	135.75	−0.95
5	424.72	13.12	298.33	3.32	199.00	−1.55
6	481.32	14.82	361.40	3.94	264.71	−2.17
7	537.95	16.51	425.75	4.57	333.02	−2.83
8	594.66	18.21	491.45	5.20	404.11	−3.51
9	651.51	19.89	558.59	5.85	478.16	−4.24
10	708.55	21.58	627.30	6.50	555.39	−5.00
11	765.86	23.26	697.70	7.15	636.07	−5.81
12	823.54	24.95	769.96	7.81	720.48	−6.66
13	881.69	26.63	844.26	8.47	808.99	−7.58
14	940.46	28.31	920.85	9.14	902.00	−8.56
15	1 000.00	30.00	1 000.00	9.80	1 000.00	−9.62
	$\overline{PL}^{(0.00)} = 55.90$		$\overline{PL}^{(0.02)} = 55.90$		$\overline{PL}^{(0.04)} = 55.90$	

years. Conversely, low reserve values lead to an accelerated emerging of expected profits, compensated by expected losses in the following years. Of course, we find $\overline{PL}^{(0.00)} = \overline{PL}^{(0.02)} = \overline{PL}^{(0.04)} = 55.90$ (see Example 5.5.2). ❑

Some results, which emerge from Example 5.5.3, can be generalized. In particular, it can be proved that:

- a lower interest rate adopted in the reserve calculation implies higher reserve values, and hence a "delay" in profit emerging;
- a higher interest rate adopted in the reserve calculation implies lower reserve values, and hence an "acceleration" in profit emerging.

5.5.4 Cash Flows, Profits, and Premium Margins

By comparing Eq. (5.5.11) to Eq. (5.5.2), we find (as $V_m = S$) the following relations:

$$\overline{PL}_{t+1} = \overline{CF}_{t+1} + V_t\,(1+i'') - V_{t+1}\,p''_{x+t}; \quad t = 0, 1, \dots, m-2 \qquad (5.5.13a)$$

$$\overline{PL}_m = \overline{CF}_m + V_{m-1}\,(1+i'') \qquad (5.5.13b)$$

In respect of the annual profit/loss, the role of the policy reserve, and its change in particular, then consists in attributing shares of premiums to policy years, so shifting from "cash-based" valuations (the $\overline{CF}$'s) to "pertinence-based" valuations (the $\overline{PL}$'s).

Example 5.5.4 Profit profile and cash flow profile are compared in Tables 5.12 and 5.13, which refer to a term insurance and an endowment insurance, respectively. Policy data and technical bases TB1 and TB2 are as in Example 5.5.1. ❑

Of course, different time profiles of the reserve lead to different premium attributions and hence, as shown in Example 5.5.3, to different profit profiles. A very particular reserve profile and the related profit profile will be presented in Sect. 5.5.5.

Moreover, specific profit profiles can be generated by adopting a different approach to profit assessment. An interesting approach is described in what follows.

Table 5.12 Term insurance (annual level premiums)

t	$\overline{PL}_t$	$\overline{CF}_t$	$(P - P'')\,(1+i'')$
1	0.14	0.90	0.23
2	0.16	0.78	0.23
3	0.18	0.65	0.23
4	0.20	0.51	0.23
5	0.22	0.35	0.23
6	0.24	0.17	0.23
7	0.27	−0.03	0.23
8	0.29	−0.25	0.23
9	0.31	−0.49	0.23
10	0.33	−0.76	0.23

Table 5.13 Endowment insurance (annual level premiums)

t	$\overline{PL}_t$	$\overline{CF}_t$	$(P - P'')\,(1 + i'')$
1	0.91	58.28	4.84
2	1.50	57.95	4.84
3	2.10	57.58	4.84
4	2.70	57.17	4.84
5	3.32	56.72	4.84
6	3.94	56.22	4.84
7	4.57	55.66	4.84
8	5.20	55.04	4.84
9	5.85	54.36	4.84
10	6.50	53.60	4.84
11	7.15	52.76	4.84
12	7.81	51.82	4.84
13	8.47	50.79	4.84
14	9.14	49.65	4.84
15	9.80	−938.67	4.84

Refer to Eq. (5.5.9), and set:

$$\ddot{a}''_{x:\overline{m}|} = \sum_{t=0}^{m-1} (1 + i'')^{-t}\; {}_t p''_x \tag{5.5.14}$$

$$ {}_m A''_x = \sum_{t=0}^{m-1} (1 + i'')^{-(t+1)}\; {}_{t|1} q''_x \tag{5.5.15}$$

$$ {}_m E''_x = (1 + i'')^{-m}\; {}_m p''_x \tag{5.5.16}$$

The expected total profit can be expressed as follows:

$$\overline{PL} = P\, \ddot{a}''_{x:\overline{m}|} - C\, {}_m A''_x - S\, {}_m E''_x \tag{5.5.17}$$

Let P'' denote the "second-order premium," namely the annual level premium calculated by adopting the scenario basis TB2, such that:

$$P''\, \ddot{a}''_{x:\overline{m}|} = C\, {}_m A''_x + S\, {}_m E''_x \tag{5.5.18}$$

The expected total profit/loss can then be expressed as the actuarial value of the temporary life annuity whose items are the annual *premium margins*:

$$\overline{PL} = (P - P'')\, \ddot{a}''_{x:\overline{m}|} \tag{5.5.19}$$

The following aspects should be stressed.

- The result expressed by Eq. (5.5.19) is extremely intuitive: indeed, the expected total profit/loss is due to the difference between the premium charged to the policyholder (P) and the premium fulfilling the equivalence principle under realistic assumptions (P''), which clearly leads to a zero expected profit.
- Equation (5.5.19) generalizes to the case of annual premiums the result expressed by Eq. (4.3.27) for the single-premium arrangement.
- According to Eq. (5.5.19), we could assume as the expected annual profit the amount

$$\overline{PL}_t = (P - P'')\,(1 + i''); \quad t = 1, 2, \ldots, m \tag{5.5.20}$$

so originating a flat profit profile. Note, however, that this can lead to a significant acceleration in the emerging of profits (see Example 5.5.5).

Example 5.5.5 From Tables 5.12 and 5.13, which refer to a term insurance and an endowment insurance, respectively, it clearly appears that, in both the insurance products, the assumption (5.5.20) leads to a significant acceleration in the profit profile. ❑

5.5.5 Expected Profits According to Best-Estimate Reserving

Consider the expected present value of future benefits net of future premiums, according to the scenario basis TB2, that is, in formal terms:

$$V_t^{[\mathrm{BE}]} = C\,{}_{m-t}A''_{x+t} + S\,{}_{m-t}E''_{x+t} - P\,\ddot{a}''_{x+t:\overline{m-t}|} \tag{5.5.21}$$

The quantity $V_t^{[\mathrm{BE}]}$ is usually called the *best-estimate reserve*.

In particular, we have:

$$V_0^{[\mathrm{BE}]} = C\,{}_{m}A''_{x} + S\,{}_{m}E''_{x} - P\,\ddot{a}''_{x:\overline{m}|} \tag{5.5.22}$$

and hence (see Eqs. (5.5.18) and (5.5.19)):

$$V_0^{[\mathrm{BE}]} = (P'' - P)\,\ddot{a}''_{x:\overline{m}|} = -\overline{PL} \tag{5.5.23}$$

Thus, the quantity $-V_0^{[\mathrm{BE}]} = \overline{PL}$ represents the "value" of the policy (at the time of its issue), meant as the expected present value of profits/losses originated by the policy itself throughout its duration.

Assume now, for the policy reserve V_t, the following values:

$$V_0 = 0; \quad V_t = V_t^{[\mathrm{BE}]},\ \text{for } t = 1, 2, \ldots, m-1; \quad V_m = S \tag{5.5.24}$$

By using Eq. (5.5.2), with the reserves as defined by (5.5.24), after a little algebra we obtain the following results:

$$\overline{PL}_1 = -V_0^{[\mathrm{BE}]}\,(1+i'') \tag{5.5.25a}$$

$$\overline{PL}_t = 0; \quad t = 2, \ldots, m \tag{5.5.25b}$$

Thus, the expected total profit/loss completely emerges in the first policy year.

Remark The particular profit profile originated by the best-estimate reserve witnesses the existence of two basic approaches to profit emerging. The *Deferral & Matching* approach is a traditional feature of actuarial models. The basic idea underlying this approach is that the total profit arises progressively throughout time. The profit assessment procedure basically consists of two steps:

- assessment of annual results (typically: cash flows and profits);
- calculation of the total profit as the expected present value of annual results.

The *Assets & Liabilities* approach is a feature of financial models. The profit assessment procedure basically consists of two steps:

- the total profit is given by the difference between the value of assets (e.g., the single premium, or the credit for future periodic premiums) and the value of liabilities (the insurer's obligations);
- possible annual profits are only given by changes in the values of assets and liabilities.

5.6 Reserving for Expenses

Equation (5.3.3) defines the "net reserve," in which benefits and net premiums are only involved. We can extend the definition, and thus define the "total reserve," in which expenses and loading for expenses are also included:

$$V_t^{[\mathrm{tot}]} = \mathrm{Ben}'(t,m) + \mathrm{Exp}'(t,m) - \mathrm{Prem}'(t,m) - \mathrm{Load}'(t,m) \tag{5.6.1}$$

where $\mathrm{Exp}'(t,m)$ and $\mathrm{Load}'(t,m)$ represent the actuarial values at time t of future expenses and expense loadings, respectively, calculated according to the first-order basis. It turns out that $V_t^{[\mathrm{tot}]}$ can be determined including the future expenses in the insurer's liabilities and directly accounting for the expense-loaded premiums instead of the net premiums.

Of course, we also have

$$V_t^{[\mathrm{tot}]} = V_t + V_t^{[\mathrm{E}]} \tag{5.6.2}$$

where $V_t^{[\mathrm{E}]} = \mathrm{Exp}'(t,m) - \mathrm{Load}'(t,m)$ just allows for expenses and expense loadings.

Notwithstanding, it is much more useful to deal separately with the various expense components and the related loadings. First, we note that the need for reserving arises because of a time-mismatching between the insurer's inflow and outflow streams. So, as regards expenses and related loadings, we can exclude premium collection expenses, as these are supposed to occur at the same time the relevant loading is cashed.

Acquisition costs can also be excluded from further analysis in the case of a single premium. Conversely, in the case of periodic premiums payable for s years, we can define the (negative) *acquisition cost reserve*, which in fact represents the insurer's credit for the related loadings to be cashed in future years:

$$V_t^{[A]} = \begin{cases} -\Lambda^{[A]}\, \ddot{a}'_{x+t:s-t\rceil} & \text{for } t \le s-1 \\ 0 & \text{for } t \ge s \end{cases} \tag{5.6.3}$$

The quantity

$$V_t^{[Z]} = V_t + V_t^{[A]} \tag{5.6.4}$$

is called the *Zillmer reserve*. In general, we have $V_t^{[Z]} \le V_t$, and in particular, in the first policy years, we may find $V_t^{[Z]} < 0$. We note that the Zillmer reserve implies a "clearing" between insurer's credit and debt, and, (also) for this reason, in many countries zillmerization is not allowed when assessing the balance sheet portfolio reserve.

General administration expenses do not originate any reserve if the premiums are payable for the whole policy duration, that is if $s = m$. On the contrary, if $s < m$ the *reserve for general administration expenses* is defined as follows:

$$V_t^{[G]} = \begin{cases} \gamma\, C\, \ddot{a}'_{x+t:m-t\rceil} - \Lambda^{[G]}\, \ddot{a}'_{x+t:s-t\rceil} & \text{for } t \le s-1 \\ \gamma\, C\, \ddot{a}'_{x+t:m-t\rceil} & \text{for } t \ge s \end{cases} \tag{5.6.5}$$

In the case of a single premium, we have:

$$V_t^{[G]} = \gamma\, C\, \ddot{a}'_{x+t:m-t\rceil} \tag{5.6.6}$$

In some countries (in particular in Continental Europe), it is usual to define the following reserve:

$$V_t^{[I]} = V_t + V_t^{[G]} \tag{5.6.7}$$

which is called in Germany the *Inventardeckungscapital*.

It is easy to prove that the reserve, $V_t^{[tot]}$, allowing for all the expenses and the related loadings, as well as for benefits and net premiums, can be expressed as follows:

$$V_t^{[tot]} = V_t + V_t^{[A]} + V_t^{[G]} \tag{5.6.8}$$

Example 5.6.1 We refer to the insurance products and the related data considered in Example 4.5.1. Tables 5.14 and 5.15 display the various reserves allowing for expenses and related loadings. ❑

Table 5.14 Whole life insurance (level premiums; $s = 15$)

t	V_t	$V_t^{[A]}$	$V_t^{[G]}$	$V_t^{[Z]}$	$V_t^{[I]}$	$V_t^{[tot]}$
0	0.00	0.00	0.00	0.00	0.00	0.00
1	42.57	−18.85	0.76	23.72	43.33	24.48
2	85.79	−17.68	1.55	68.11	87.34	69.66
3	129.69	−16.49	2.35	113.20	132.04	115.55
4	174.28	−15.27	3.17	159.01	177.45	162.18
5	219.57	−14.03	4.02	205.54	223.59	209.56
…	…	…	…	…	…	…
12	559.31	−4.60	10.74	554.71	570.05	565.45
13	611.76	−3.11	11.86	608.64	623.62	620.50
14	665.46	−1.58	13.02	663.88	678.49	676.90
15	720.56	0.00	14.25	720.56	734.81	734.81
16	730.68	0.00	13.74	730.68	744.42	744.42
…	…	…	…	…	…	…
24	807.47	0.00	9.82	807.47	817.29	817.29
25	816.33	0.00	9.37	816.33	825.70	825.70
…	…	…	…	…	…	…

Table 5.15 Endowment insurance (level premiums; $s = m = 15$)

t	V_t	$V_t^{[A]}$	$V_t^{[G]}$	$V_t^{[Z]}$	$V_t^{[I]}$	$V_t^{[tot]}$
0	0.00	0.00	0	0.00	0.00	0.00
1	57.54	−34.52	0	23.02	57.54	23.02
2	116.11	−32.38	0	83.73	116.11	83.73
3	175.74	−30.19	0	145.54	175.74	145.54
4	236.46	−27.97	0	208.49	236.46	208.49
5	298.33	−25.70	0	272.63	298.33	272.63
…	…	…	…	…	…	…
12	769.96	−8.43	0	761.53	769.96	761.53
13	844.26	−5.70	0	838.56	844.26	838.56
14	920.85	−2.90	0	917.95	920.85	917.95
15	1 000.00	0.00	0	1 000.00	1 000.00	1 000.00

5.7 Surrender Values and Paid-Up Values

As mentioned in Sect. 4.1.2, the calculation of surrender values and paid-up values should account for the policyholder's credit at the time of the contract alteration. The net policyholder's credit (that is, the amount which allows for benefits, expenses and expense-loaded premiums) is given by the reserve $V_t^{[tot]}$, defined by (5.6.8). As this reserve coincides in many cases with the Zillmer reserve $V_t^{[Z]}$ (see, for instance, Table 5.15 in Example 5.6.1) we just focus on the Zillmer reserve.

The *surrender value*, denoted as R_t, can be determined as follows:

$$R_t = \varphi(t)\, V_t^{[Z]} \tag{5.7.1}$$

Note that the function $\varphi(t)$ $(0 \leq \varphi(t) \leq 1$, and usually equal to 0 for $t = 1, 2$ only), aims at penalizing the surrendering policyholders. Commonly, the penalty decreases as t increases, and to this purpose the function should be increasing. The penalty can be justified as follows:

- from a legal point of view, the policyholder breaks the contract;
- from an economic point of view, the insurer can recover, via the penalty, future profits expected from the contract.

Other formulae are also commonly adopted in insurance practice. For endowment insurance products, with maturity at time m and annual level premiums payable for m years, the so-called *proportional rule* is frequently adopted. If C denotes the sum insured, we have:

$$R_t = \frac{t}{m}\, C\, (1 + i^*)^{-(m-t)} \tag{5.7.2}$$

Thus, a share of the sum insured, proportional to the number of annual premiums already paid, is discounted at a rate i^*, higher than the interest rate in the technical basis. Formula (5.7.2) can be justified looking at the time profile of the policy reserve in an endowment insurance, which is very close to a linear profile (see, for example, Fig. 5.15). The discounting rate i^* can be used as a parameter to allow for both zillmerisation and penalty.

To illustrate the *reduction* of the sum insured when converting an insurance contract into a paid-up one, we refer to a m-year pure endowment, with sum insured S and annual level premiums payable for the whole policy duration.

Assume that the policyholder asks for the reduction at time t, namely after paying the annual premiums at times $0, 1, \ldots, t-1$. A share of the Zillmer reserve, $V_t^{[Z]}$, is then used (as a "single" premium) to finance the paid-up contract, namely the reduced benefit at maturity, $S^{[red]}$, and the general administration expenses (quantified as results from (4.5.6a)) for the residual duration.

In formal terms, $S^{[red]}$ is the solution of the following equation:

$$\bar{\varphi}(t)\, V_t^{[Z]} = S^{[red]} \left({}_{m-t}E'_{x+t} + \gamma\, \ddot{a}'_{x+t:\overline{m-t}|} \right) \tag{5.7.3}$$

The function $\bar{\varphi}(t)$ $(0 < \bar{\varphi}(t) \leq 1)$ determines a penalty charged to the policyholder when shifting to the paid-up contract, and can be justified similarly to the surrender penalty (see above). However, as the contract goes on, we usually have $\bar{\varphi}(t) \geq \varphi(t)$.

Formula (5.7.3) relies on the equivalence principle, and hence leads to a result consistent with this actuarial calculation principle. Nonetheless, other (approximate) formulae are often adopted in insurance practice. For example, according to the *proportional rule*, in an endowment insurance with maturity at time m and annual level premiums payable for m years, the amount $S^{[\mathrm{red}]}$ can be determined as follows:

$$S^{[\mathrm{red}]} = \frac{t}{m} S \tag{5.7.4}$$

5.8 References and Suggestions for Further Reading

All the actuarial textbooks on life insurance mathematics and technique deal with the calculation of reserves. Hence, the reader can refer to Bowers et al. (1997), Dickson et al. (2013), Gerber (1995), Gupta and Varga (2002), Koller (2012), Norberg (2002), Promislow (2006), and Rotar (2007).

The traditional approach to the profit assessment at the policy level is proposed by Promislow (2006), whereas Gupta and Varga (2002) place special emphasis on mortality profits.