Chapter 1
An Introduction to Algebraic Quantum Field Theory

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Abstract The algebraic approach to quantum field theory is reviewed, and its aims, successes and limitations are discussed.

1.1 Introduction

The algebraic properties of quantum observables, as discovered by Heisenberg, Born and Jordan and summarized in the canonical commutation relation between position q and momentum p,

$$[q, p] = i\hbar, \tag{1.1}$$

are crucial for the structure of quantum theory. A further crucial structure is the representability of observables as Hilbert space operators, as first discovered by Schrödinger and related to the probability interpretation of quantum mechanics by Born. It was later found that the existence of Hilbert space representations can be guaranteed if the algebra $\mathfrak{A}$ generated by observables is equipped with an involutive structure $A \mapsto A^*$, characterizing the real elements, and a norm which satisfies the C*-property

$$||A^*A|| = ||A||^2. \tag{1.2}$$

This involves the restriction to bounded observables, but this is neither from the point of view of physics (here it amounts to parametrize the possible results of a measurement by numbers in a finite interval) nor from the point of view of mathematics (there one exploits the spectral theorem for selfadjoint operators and considers bounded functions of the operator in question) a loss of generality. It may, however, lead to problems in calculations, since in concrete applications one often has to start from a formula involving unbounded operators. For our general considerations we may ignore these difficulties and consider the algebra of observables as a C*-algebra with unit, i.e. as an involutive unital algebra equipped with a C*-norm, which is

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complete as a normed space. Every such algebra arises as a norm closed algebra of Hilbert space operators, but may have in addition many other inequivalent representations by Hilbert space operators. Then the mathematical question arises which of the possible representations one should choose in order to describe a given physical situation.

In quantum mechanics with finitely many degrees of freedom a partial answer was given by von Neumann. He proved that the algebra generated by unitaries $e^{i\alpha q}$ and $e^{i\beta p}$ with $\alpha, \beta \in \mathbb{R}$ and satisfying Weyl's version of the canonical commutation relation,

$$e^{i\alpha q} e^{i\beta p} = e^{-i\alpha\beta} e^{i\beta p} e^{i\alpha q}, \tag{1.3}$$

has, up to unitary equivalence, only one regular irreducible representation, namely the representation found by Schrödinger. Here regular means that the maps $\alpha \mapsto e^{i\alpha q}$ and $\beta \mapsto e^{i\beta p}$ are strongly continuous.

In quantum field theory with its infinitely many degrees of freedom, the uniqueness result of von Neumann is no longer valid. Nevertheless, it took some time before the physical significance of this fact was realized. The crucial insight was due to Haag who found that theories with translation invariant interactions cannot have a ground state in the Hilbert space describing the vacuum of the free theory. It was then understood that this problem is generic, e.g. thermodynamical equilibrium states cannot be described in terms of density matrices in the vacuum Hilbert space.

Apart from these problems, there is a deeper reason why it is fortunate to separate the construction of observables from the construction of states. This is the apparent conflict between the principle of locality, which in particular governs classical field theory, and the existence of nonclassical correlations (*entanglement*) in quantum systems, often referred to as non-locality of quantum physics. As a matter of fact it turns out that the algebra of observables is completely compatible with the locality principle whereas the states typically exhibit nonlocal correlations. For this reason, *algebraic quantum field theory* is also called *local quantum physics* [63].

The concepts of algebraic quantum field theory were first introduced in a contribution of Haag to the Lille conference 1957 [61]. The main motivation at this time was an explanation why a quantum field theory yields a theory of interacting particles. The Haag-Ruelle scattering theory [62, 85] was a first success of these concepts.

It was then made mathematically precise by using the theory of operator algebras, mainly by Araki [5, 6]. The crucial step of considering the algebras of observables independently of their action on an underlying Hilbert space was performed in a programmatic paper by Haag and Kastler [64]. They formulated the following axioms:

- To each open bounded region $\mathcal{O}$ of Minkowski space $\mathbb{M}$ there is associated a unital C*-algebra $\mathfrak{A}(\mathcal{O})$, interpreted as the algebra of observables which can be measured within the spacetime region $\mathcal{O}$.
- If $\mathcal{O}_1 \subset \mathcal{O}_2$, then there exists an embedding (unital injective *-homomorphism)

$$i_{\mathcal{O}_2\mathcal{O}_1} : \mathfrak{A}(\mathcal{O}_1) \to \mathfrak{A}(\mathcal{O}_2). \tag{1.4}$$

- These embeddings satisfy the compatibility relation

$$i_{\mathcal{O}_3\mathcal{O}_2} \circ i_{\mathcal{O}_2\mathcal{O}_1} = i_{\mathcal{O}_3\mathcal{O}_1} \text{ if } \mathcal{O}_1 \subset \mathcal{O}_2 \subset \mathcal{O}_3. \tag{1.5}$$

- The identity component of of the Poincaré group is represented by automorphisms α_L with the properties

$$\alpha_L \circ \mathfrak{A} = \mathfrak{A} \circ L,$$
$$\alpha_L \circ i_{\mathcal{O}_1\mathcal{O}_2} = i_{L\mathcal{O}_1 L\mathcal{O}_2} \circ \alpha_L,$$
$$\alpha_{L_1} \circ \alpha_{L_2} = \alpha_{L_1 L_2}.$$

- If $\mathcal{O}_1$ is spacelike separated from $\mathcal{O}_2$ and $\mathcal{O} \supset \mathcal{O}_1 \cup \mathcal{O}_2$, then

$$[i_{\mathcal{O}\mathcal{O}_1}(A), i_{\mathcal{O}\mathcal{O}_2}(B)] = 0 \;\forall A \in \mathfrak{A}(\mathcal{O}_1), B \in \mathfrak{A}(\mathcal{O}_2).$$

These axioms do not involve any choice of Hilbert space representations. The system of algebras together with the embeddings and the Poincaré symmetries is called a *local net* (or a *Haag-Kastler net*). It contains the minimal requirements one may pose on observables localized in subregions of Minkowski space. Note, however, that fermionic fields which anticommute at spacelike separation are excluded from the local algebras. It was one of the main goals and finally successes of the algebraic approach that the occurrence of fermionic fields could be derived from the structure of local nets.

The axioms do not specify a dynamical law. The existence of a dynamical law, however, can be implied by a further axiom, namely the *time slice axiom* :

- If $\mathcal{O}_1 \subset \mathcal{O}_2$ contains a Cauchy surface of $\mathcal{O}_2$ (i.e. a hypersurface which is met exactly once by every non-extendible causal curve in $\mathcal{O}_2$), then the embedding $i_{\mathcal{O}_2\mathcal{O}_1}$ is an isomorphism.

The system of local algebras possesses an inductive limit, called the quasilocal algebra $\mathfrak{A}(\mathbb{M})$. It is the unique C*-algebra with embeddings $i_{\mathcal{O}} : \mathfrak{A}(\mathcal{O}) \to \mathfrak{A}(\mathbb{M})$ such that

$$i_{\mathcal{O}_2} \circ i_{\mathcal{O}_2\mathcal{O}_1} = i_{\mathcal{O}_1} \text{ if } \mathcal{O}_1 \subset \mathcal{O}_2,$$

which is generated by the local subalgebras $i_{\mathcal{O}}(\mathfrak{A}(\mathcal{O}))$. Since all embeddings $i_{\mathcal{O}}$ are injective we identify in the following $\mathfrak{A}(\mathcal{O})$ with its image under $i_{\mathcal{O}}$.

One might be worried by the task to specify algebras for all subregions of Minkowski space. But is suffices to associate algebras to so-called diamonds (or double cones). These regions are parametrized by a pair of points (x, y) with x in the chronological future of y and consist of all points z which are in the chronological past of x and the chronological future of y. We denote the set of double cones by $\mathcal{K}$. Algebras of other regions G are then defined as subalgebras of $\mathfrak{A}(\mathbb{M})$ which are generated by the algebras $\mathfrak{A}(\mathcal{O})$ with $G \supset \mathcal{O} \in \mathcal{K}$.

Once the local net is given, the space of states is obtained as a subset of the dual of $\mathfrak{A}(\mathbb{M})$ as a Banach space. Namely, states are just those linear functionals $\omega : \mathfrak{A}(\mathbb{M}) \to \mathbb{C}$ which satisfy the conditions

$$\omega(1) = 1, \tag{1.6}$$

$$\omega(A^*A) \geq 0 \; \forall A \in \mathfrak{A}(\mathbb{M}). \tag{1.7}$$

The values of a state are interpreted as expectation values, and the whole probability distribution $\mu_{\omega,A}$ of measured values $a \in \mathbb{R}$ of an observable $A = A^*$ is obtained from its moments, namely the expectation values of powers of A,

$$\int a^n d\mu_{\omega,A}(a) = \omega(A^n) \, , \; n \in \mathbb{N}. \tag{1.8}$$

Every state induces by the so-called GNS construction a representation π of the quasilocal algebra on some Hilbert space $\mathcal{H}$ together with a distinguished unit vector Ω such that

$$\omega(A) = \langle \Omega, \pi(A)\Omega \rangle,$$

and the set of vectors $\{\pi(A)\Omega | \; A \in \mathfrak{A}(\mathbb{M})\}$ is dense in $\mathcal{H}$.

The density matrices ρ in $\mathcal{H}$ form a family of states,

$$\omega_\rho(A) = \mathrm{Tr}\rho\pi(A),$$

the so-called folium of ω. The crucial mathematical fact is that infinitely dimensional C*-algebras in general have a huge number of disjoint folia. While this is welcome from the physics point of view in order to be able to describe macroscopically different situations, it makes the analysis of the structure of the state space difficult. Hence one of the main objectives in algebraic quantum field theory is to select a suitable subset of the state space which covers the situations of interest and admits a classification.

The most famous of these selection criteria is the DHR (for Doplicher, Haag and Roberts) criterion which is supposed to select the states of interest for elementary particle physics. These are states which differ from a distinguished state ω_0 only within some bounded region and its causal future and past.

The state ω_0 is interpreted as the vacuum. It is assumed to be a pure state (i.e. it cannot be decomposed into a convex combination of other states) and to be invariant under the Poincaré transformations α_L. Moreover, in its GNS-representation π_0, the associated unitary representation U of the identity component of the Poincaré group given by

$$U(L)\pi_0(A)\Omega = \pi_0(\alpha_L(A))\Omega,$$

which implements the automorphisms α_L,

$$U(L)\pi_0(A)U(L)^{-1} = \pi_0(\alpha_L(A)),$$

is strongly continuous and satisfies the relativistic spectrum condition

$$\mathrm{sp}(U_0) \subset \overline{V_+^*}.$$

Here U_0 is the restriction of U to the translation subgroup, V_+ is the cone of future directed timelike translations, V_+^* the dual cone in momentum space and $\overline{V_+^*}$ its closure. The representation U_0 is generated by mutually commuting selfadjoint operators P_μ, $\mu = 0, \ldots 3$ (the momentum operators)

$$U_0(x) = e^{iPx}\ ,\ \ Px \equiv P_\mu x^\mu, \tag{1.9}$$

whose joint spectrum is by definition the spectrum of U_0.

In the presence of superselected charges there exist other representations π which are translation covariant, i.e. there exists a unitary strongly continuous representation of the translation group, which implements the translation automorphisms and fulfills the spectrum condition.

An important question is whether all observables can be expressed in terms of observables in arbitrarily small regions, as one might expect in models which are defined in terms of observable pointlike localized fields. A precise version of this property is additivity : a representation π satisfies additivity if for every covering of the open region $\mathcal{O}$ by double cones $\mathcal{O}_i$ one has

$$\pi(\mathfrak{A}(\mathcal{O})) \subset (\bigcup_i \pi(\mathfrak{A}(\mathcal{O}_i)))''. \tag{1.10}$$

Here $\mathcal{R}'$ denotes the commutant of a set of operators $\mathcal{R}$, i.e. the set of all bounded operators commuting with $\mathcal{R}$. The bicommutant of a selfadjoint set (a set of operators which is invariant under taking the adjoint) is a von Neumann algebra.[1]

The combination of the axiom of local commutativity with the spectrum condition imposes strong restrictions on the theory. The most remarkable one is the Reeh-Schlieder Theorem [63, 83].

Theorem 1.1.1 *Let (π, U_0) be a translation covariant representation satisfying additivity and the spectrum condition. Let $\Phi \in \mathcal{H}_\pi$ be a cyclic vector for $\mathfrak{A}(\mathbb{M})$, i.e.*

$$\pi(\mathfrak{A}(\mathbb{M}))\Phi \text{ is dense in } \mathcal{H}_\pi,$$

and let $\Psi = U_0(i\beta)\Phi$ with $\beta \in V_+$. Let $\mathcal{O}$ be a nonempty double cone in Minkowski space. Then the set $\pi(\mathfrak{A}(\mathcal{O}))\Psi$ is dense in $\mathcal{H}_\pi$, and Ψ is separating for $\pi(\mathfrak{A}(\mathcal{O}))'$, i.e. $A\Phi = 0$ implies $A = 0$ for $A \in \pi(\mathfrak{A}(\mathcal{O}))'$.

The interpretation of the Reeh-Schlieder Theorem induced an intense discussion in philosophy of science (see e.g. [67]). The theorem does not mean that there is an

[1] A von Neumann algebra is an algebra of Hilbert space operators which is invariant under involution, contains the unit operator and is closed with respect to the weak operator topology.

instantaneous effect of a local operation. A local operation would correspond to the application of a unitary $U \in \mathfrak{A}(\mathcal{O})$, and the set of vectors $\{\pi(U)\Phi | U \in \mathfrak{A}(\mathcal{O})\}$ is total, i.e. its finite linear combinations are dense in $\mathcal{H}_\pi$, but the set itself is not dense in the unit ball of $\mathcal{H}_\pi$. See [29] for a recent discussion.

An important generalization of the Reeh-Schlieder Theorem is the following theorem of Borchers [12]:

Theorem 1.1.2 *Let (π, U_0) as before and assume that the center of $\pi(\mathfrak{A}(\mathbb{M}))$ is trivial. Let $\mathcal{O}, \mathcal{O}_1$ be double cones with $\overline{\mathcal{O}} \subset \mathcal{O}_1$. Then to every nonzero projection $E \in \pi(\mathfrak{A}(\mathcal{O}))''$ there exists an isometry $V \in \pi(\mathfrak{A}(\mathcal{O}_1))''$ such that $VV^* = E$.*

This theorem shows that local algebras contain no nonzero finite dimensional projections. This property of relativistic quantum field theory is quite a surprise if confronted with structures known from nonrelativistic quantum mechanics.

The theorem reminds on the characterization of von Neumann algebras of type III. Namely, von Neumann algebras $\mathcal{N}$ with trivial centre (the so-called factors) can be classified in terms of the equivalence classes of their projections where 2 projections $E, F \in \mathcal{N}$ are called equivalent if there exists some $V \in \mathcal{N}$ such that

$$E = VV^* \text{ and } F = V^*V. \tag{1.11}$$

Type I factors are isomorphic to the algebra of all bounded operators on a Hilbert space $\mathcal{H}$. There the projections are equivalent if the subspaces $E\mathcal{H}$ and $F\mathcal{H}$ have the same dimension. In quantum field theory type I factors occur as the von Neumann algebras generated by $\pi(\mathfrak{A}(\mathbb{M}))$ in an irreducible representation π. Type II factors have projection classes which can be labeled by real numbers. They occur in physics only in extreme situations, for instance for a gas of fermions at infinite temperature. Type III factors are defined by the property that all nonzero projections are equivalent.

Borchers' Theorem above is somewhat weaker than the statement that the von Neumann algebras $\pi(\mathfrak{A}(\mathcal{O}))''$ are factors of type III. Actually, in many cases they are known to be of type III (see, e.g. [47]), and there are general reasons to expect that this is always the case [27].

As mentioned before, one of the early successes of the algebraic approach (in fact, the main motivation for its introduction) is the explanation of the particle structure observed in experiments. Namely, let $(\mathcal{H}, \pi_0, \Omega)$ be the GNS triple associated to the vacuum state ω_0, We assume that the spectrum of the translation group contains an isolated mass shell,

$$\text{sp}(U_0) = \{0\} \cup H_m \cup H_{\geq M} \ , \ 0 < m < M,$$

with $H_m = \{p | p_0 > 0, \, p^2 = m^2\}$ and $H_{\geq M} = \{p | p_0 > 0, \, p^2 \geq M^2\}$. The subspace $\mathcal{H}_1 \subset \mathcal{H}$ corresponding to H_m can be interpreted as 1-particle space. One then can show that the vacuum Hilbert space $\mathcal{H}$ contains also states which can be interpreted as multi-particle states at asymptotic times (Haag-Ruelle scattering theory). It is, however, an open question, whether all states in the vacuum representation admit a particle interpretation (problem of asymptotic completeness) (See [49] for recent progress).

One obvious obstruction to asymptotic completeness is that 1-particle states may exist which are disjoint from the states of the vacuum representation. This is true in particular for fermions, but also holds for bosonic particles which carry a superselected charge, e.g. the W-bosons of the electroweak theory.

In models these states are obtained by enlarging the algebra of observables to a so-called field algebra. In spite of the fact that the elements of the field algebra are not necessarily observable, one postulates that they satisfy local commutation, or, for fermionic fields, anticommutation relations. This allows to extend the construction of multi-particle states to charged particles which then satisfy Bose or Fermi statistics, respectively. A partial justification of this ansatz is the spin statistics theorem which states, under the hypothesis that either Bose or Fermi statistics holds, that only the observed connection between spin and statistics is possible. The ansatz is, however, ad hoc and is in fact not sufficiently general to cover all situations of interest, in particular in less than 4 dimensions when more general statistics are possible.

To incorporate these particles, Doplicher, Haag and Roberts formulated their selection criterion for "states of interest for particle physics". It incorporates the property of charge carrying fields that they satisfy local commutation relations with the observables but does not impose any commutation relations between the fields themselves. In a series of 4 seminal papers [40–43] they analyzed the arising representations and could show that

- The multi-particle structure previously seen in the vacuum sector extends to all DHR states.
- The particles satisfy Bose or Fermi statistics and may have additional degrees of freedom.
- Antiparticles exist and have the same mass and spin as the corresponding particle.

Moreover, Doplicher and Roberts succeeded in [46] to show that the internal degrees of freedom can be understood in terms of representations of a uniquely determined compact group. This led to an improved version of the Tannaka-Krein characterization of the dual of a compact group.

Interestingly, the DHR analysis yields a different structure if applied to quantum field theory in 2 spacetime dimensions [58]. This was used in particular in the operator algebraic approach to conformal field theory as explained in more detail in Rehren's contribution to this book.

The DHR criterion excludes states describing particles with an electric charge because of the associated electric flux which distinguishes these states from the vacuum even at arbitrarily large spacelike separation. This is a generic feature of charged particles in gauge theories [25]. If a particle is massive and if in its representation the energy-momentum spectrum has an upper mass gap, Buchholz and the author could show that the corresponding sector is related to a vacuum state such that the interpolating fields can be localized in an infinitely extended spacelike cone [26]. In 4 dimension this localization allows to apply the DHR analysis, but in 3 dimensions one observes a more general structure similar to the 2 dimensional situation for DHR states.

The massless situation which is relevant for quantum electrodynamics was recently solved by Buchholz and Roberts [32], based on an older proposal of Buchholz [24].

Algebraic quantum field theory, as may be seen from this overview, was very successful in analyzing the structure of quantum field theory, based on a few plausible axioms. It turned out, however, to be extremely difficult to construct concrete models which are physically interesting and satisfy these axioms. Actually, other approaches to quantum field theory suffer from the same problem, and sometimes these difficulties are taken as an indication for the need to go beyond quantum field theory, or even to give up the requirement of mathematical consistency. It is one aim of the present book to convince the reader that this is premature, and to indicate possible directions for improving the situation.

Let us summarize the known models of algebraic quantum field theory (see also [89] for an overview). There are first the models of free fields describing freely moving particles characterized by irreducible representations of the (covering of) the (connected component of) the Poincaré group. Traditionally these models are constructed in terms of annihilation and creation operators on the Fock space over the corresponding 1-particle space. The construction works for massive particles with spin $s \in \frac{1}{2}\mathbb{N}_0$ and for massless particles with helicities $h \in \frac{1}{2}\mathbb{Z}$. An elegant direct construction was performed in [20]. While physically of limited interest, they have a rich mathematical structure, moreover, they constitute a basis for the interpretation of scattering states, and they serve as a framework for the perturbative formulation of interacting models. Closely related are the generalized free fields, where the single particle space is replaced by a reducible representation of the Poincaré group. In case of a discrete mass spectrum they arise as tensor products of free theories. In the case of a continuous mass spectrum they are usually considered as unphysical, but recently, they were proposed under the name of unparticles [59].

Another class of models are the superrenormalizable models in 2 dimensions, in particular massive scalar fields with a polynomial interaction, and the Yukawa model [60, 87]. These models, unfortunately, seem to have no direct physical application.

A further class consists of conformally invariant theories in 2 dimensions (see the contribution of Rehren to this book). Here a huge class of models has been constructed (see e.g. [75]), and the results from the algebraic approach can be compared to results in other approaches, with mutual fertilization. These models are of high interest for mathematics, but they also have applications to physics, e.g. in critical phenomena of effectively low dimensional systems in condensed matter, and they appear naturally within string theory.

Another class are integrable models in 2 dimensions whose S-matrix satisfies the so-called factorization equations. In case the 2-particle scattering matrix is just multiplication by a pure phase depending on the incoming momenta, the corresponding local nets have been constructed by Lechner [78]. Constructions of more complicated cases have also been performed [1, 2]. In some cases, for example in the sinh-Gordon model with field equation

$$\Box\varphi + \lambda \sinh g\varphi = 0,$$

the algebraic construction can (and should) be compared with a direct construction. It is, however, remarkable that the algebraic construction works also in cases where no corresponding classical dynamics is known. See Chap. 10 for more details.

Motivated by attempts to replace spacetime by a noncommutative space where expected properties of quantum gravity are taken into account (see the contribution of Bahns et al. to this book), one can also construct models by deformation of a given net. This works even in higher dimension, but the models constructed so far satisfy only a restricted version of local commutativity, namely one obtains algebras associated to so-called wedges W, i.e. $W = L\{x \in \mathbb{M}||x^0| < x^1\}$ for some Poincaré transformation L, and observables localized in spacelike separated wedges commute. This allows to construct 2-particle states and to compute the S-matrix for 2-particle scattering, but gives no hint whether a generalization to multi-particle states is possible [31].

A completely new elegant method of construction was recently performed by Barata, Jaekel and Mund (see a forthcoming publication). It starts from the construction of a Haag Kastler net on de Sitter space in terms of Tomita-Takesaki theory. The Minkowski space theory is then obtained by taking the limit of vanishing curvature in the spirit of the scaling limit of Buchholz and Verch [33].

The last class of models I want to mention is obtained by adopting the methods of renormalized perturbation theory to the algebraic framework. There one has to give up the condition that the local algebras are C*-algebras. Instead they are unital *-algebras, where the existence of Hilbert space representations is replaced by the existence of a representation on a space of formal power series $\mathcal{D}[[g]]$ with a dense subspace $\mathcal{D}$ of some Hilbert space where g is the coupling constant. Therefore all numerical predictions of the theory are formal power series in g, and a comparison with experiments requires a truncation of the series. As a matter of fact, in most cases only a few terms of the series can be computed by present techniques, and one finds in many cases an excellent agreement with measurements. Moreover, exploiting the concept of the renormalization group, one obtains different series labeled by a scale μ, and the series at different parameters μ_1, μ_2 are related by replacing the coupling constant by the so-called running coupling constant which typically is a formal power series in g and $\log \mu_2/\mu_1$. Observations at a given scale can then be compared with the truncated series at the same scale, where the running coupling constant is fitted with the data. This improves the agreement with the measured data considerably, and, moreover, confirms the predicted running of the coupling constant.

This weaker version of the Haag-Kastler axioms covers all models of quantum field theory which are relevant for elementary particle physics. Compared to the traditional way of treating perturbation theory, the algebraic approach allows a state independent renormalization and a construction of the local net without any infrared problems. Infrared problems can, in principle, reappear in the construction of states or in the computation of scattering cross sections. But there, the difficulties might have different reasons:

- The corresponding state may not exist, as e.g. the vacuum state for a massless scalar field in 2 dimensions, or a KMS-state of a scalar field with negative temperature.

- The states may belong to different superselection sectors.
- The usual method for construction does not work. This holds e.g. for KMS states of a massive scalar field, where Altherr [4] and Steinmann [88] observed infrared divergences at higher loop order. As was recently shown [57, 79], the problem disappears if the validity of the time slice axiom is exploited so that only the observables localized within a fixed time slice have to be taken into account.

The formalism of AQFT is motivated by the expected interplay between observables in quantum theory and the concept of locality arising from special relativity. In practical applications one often prefers to extend the framework by adding unobservable degrees of freedom. One example is the introduction of anticommuting fields in order to describe particles with Fermi statistics. As mentioned above, this extension of the framework can be intrinsically derived from the Haag-Kastler net of observables by the DHR theory. More generally this holds for all global gauge symmetries, where the compact group whose representations label the superselection sectors is shown to be constructible from the Haag-Kastler net.

A very important question is whether a corresponding statement can also be made for theories with local gauge symmetry, as QED or the standard model of elementary particle physics. In spite of quite a number of works dedicated to this question an answer is presently not known. Several strategies have been probed.

The most direct way is to mimick the traditional construction in terms of auxiliary fields (ghosts etc.). Here one necessarily gives up the framework of C*-algebras, but preserves locality. One obtains the algebra of observables as a resolution of a complex, in the sense of homological algebra [55]. An interesting alternative to this procedure is to enlarge the algebra of local observables by nonlocal quantities but to preserve the representability on a Hilbert space. See [81] for progress in this direction. In case of free abelian gauge theories as e.g. Maxwell's theory without charges, one can give a complete construction of the C*-algebraic Haag-Kastler net.

The most challenging open question is whether gravity can be included into the formalism of AQFT. Since the very concept of a region of spacetime has no intrinsic meaning in a generally covariant theory, a modification of the formalism would be necessary. But gravity is, at presently accessible observations, much weaker than all other known interactions, hence it is a good approximation to treat the spacetime of general relativity as a background which is not influenced by the other quantum fields. This is the motivation to study quantum field theory on generic spacetimes with a metric with Lorentzian signature.

1.2 AQFT on Curved Spacetime

As was first observed by Dimock [36] and Kay [73], AQFT is ideally suited for the treatment of QFT on curved backgrounds. The reason is that, contrary to the situation on Minkowski space, no intrinsic concept of a vacuum state nor of particles exist. The traditional approach to QFT therefore does not work, and attempts to use it lead to inconsistencies. As long as only free fields were discussed, a direct construction can be given on any globally hyperbolic spacetime, and an appropriate version of the Haag-Kastler axioms can be given. Even in free field theory, however, one is interested not only in the field itself, but also in nonlinear expressions in the field as e.g. the energy momentum tensor, in order to estimate the possible back reaction of the quantum fields on the spacetime geometry. It was observed that for free fields there exist states, the so-called Hadamard states, which have similar singularities as the vacuum state on Minkowski space. It was a major breakthrough when Radzikowski [82] observed that the class of Hadamard states can be completely characterized by the wave front set of the 2-point function. This allowed to extend the algebras from the linear fields (or, in order to work in the C*-framework, the exponentials of linear fields (Weyl operators)) to all local Wick polynomials [17]. Moreover, even renormalization could be performed in this extended framework [16].

But there remained an obstruction: namely, due to the absence of nontrivial isometries in generic spacetimes, the covariance axiom of Haag-Kastler is empty, in general. This leads within renormalization theory to the undesirable situation that the renormalization conditions at different points of spacetime cannot be compared with each other. The intuitive feeling is that the removal of divergences should be done in terms of prescriptions depending only on the local geometry. Actually, this idea was already used in Kay's treatment of the Casimir effect [74] and in Wald's discussion of the renormalization of the energy momentum tensor [92]. A precise version of this idea was given in the system of axioms for locally covariant quantum field theory [19] (see also [91] for a preliminary version with application to a spin statistics theorem on curved spacetimes as well as related work of Dimock [37]):

- To each time oriented globally hyperbolic spacetime M there is associated a unital C*-algebra $\mathfrak{A}(M)$.
- To each isometric, time orientation and causality preserving embedding $\chi : M \to N$ there is associated an embedding $\mathfrak{A}\chi : \mathfrak{A}(M) \to \mathfrak{A}(N)$.
- If $\chi : M \to N$ and $\psi : N \to L$ are embeddings of spacetimes as above, then $\mathfrak{A}\psi \circ \chi = \mathfrak{A}\psi \circ \mathfrak{A}\chi$.

These axioms tell that quantum field theory is a functor $\mathfrak{A}$ from the category of spacetimes, with embeddings as above as morphisms, to the category of unital C*-algebras, with injective unital *-homomorphisms as morphisms. One easily sees that, by restricting the spacetimes to subregions of a fixed spacetime, one obtains an associated Haag-Kastler net. The morphisms contain in particular the inclusions-this yields the isotony of the Haag-Kastler net- and the isometries -this yields the covariance of the Haag-Kastler net.

The remaining Haag-Kastler axioms (local commutativity and time slice axiom) can be easily formulated as additional properties of the functor.

Based on the concept of local covariance, the idea of allowing only operations, which are determined by the local geometry, can be made precise by requiring that they are natural transformations between the functor $\mathfrak{A}$ and other geometrically defined functors on the category of spacetimes. The program of renormalization on generic spacetimes was completed by Hollands and Wald [69, 70]. An important simplification of their construction was recently obtained by Khavkine and Moretti [77].

The axioms above were formulated for a scalar field. For other fields, in particular gauge fields, the relation to topological invariants of the spacetimes may become nontrivial, hence the admissible embeddings in the category of spacetimes have to preserve these structures. See e.g. [9] and references therein. More details on AQFT on curved spacetimes may be found in Chap. 4. See also the recent reviews by Benini et al. [10], by Hollands and Wald [71] as well as another review by Rejzner and the author [56].

One may even extend the given framework to the quantized gravitational field. There one has to split the metric into a background which defines a curved spacetime and a fluctuation which is treated as a quantum field. Since this split is arbitrary one has to prove that the theory does not depend on it. But there remains another difficulty, namely the absence of local observables due to the invariance of the theory under diffeomorphisms. In principle, this can be dealt with by introducing coordinates which are themselves dynamical fields. For a preliminary version of these ideas see [18] (compare also [76] for related ideas in the classical case and the discussion of relative observables within Loop Quantum Gravity [38, 84]).

1.3 Scattering Theory

Why do particles naturally occur in quantum field theory? After all, the theory is formulated without ever using the concept of particles, and as discussed before, no useful particle concept seems to exist for generic curved spacetime. Moreover, classical field theory typically does not admit, in general, an interpretation in terms of particles, only solitons which appear in some special cases remind on particles. Actually, the derivation of the particle structure in quantum field theory on Minkowski space proceeds in two quite different steps. The first is related to the phenomenon of discrete spectra in quantum physics. Therefore, in a translation covariant representation satisfying the spectrum condition, the mass operator

$$M = \sqrt{P_\mu P^\mu} \tag{1.12}$$

may have isolated eigenvalues. It is an old conjecture that the formation of mass eigenstates is related to an almost finite dimensionality of the state space corresponding to a localization within a bounded region with, at the same time, restricted total energy. This scarcity of the state space has been made precise in terms of compactness [65] or nuclearity [30] conditions. While this led to interesting results as split inclusion [39, 44] or the existence of KMS states [28], it was up to now not possible to derive the existence of mass eigenstates from this assumption (for a partial result in this direction see [50]). Therefore, one usually starts from the assumption that the mass operator has an isolated eigenvalue $m > 0$. The corresponding eigenstates are then interpreted as single particle states. We also assume here that the representation contains also a translationally invariant unit vector Ω, unique up to a phase, and call it the vacuum vector. The more general case was treated in [26].

The basic idea is that particle states can be generated from the vacuum vector by applying (almost) local operators, $\Psi = A(t)\Omega$. Since particle states satisfy the Klein Gordon equation

$$(\Box_x + m^2)U_0(x)\Psi = 0, \tag{1.13}$$

they describe freely moving particles, and the operators $A(t)$ generating them may be assumed to be essentially localized near to the actual position of the particle at time t. But then multi-particle states can be generated by products of (almost) local operators

$$\Psi_{1,\dots,n}(t) = A_1(t)\dots A_n(t)\Omega. \tag{1.14}$$

If the particles move with different velocities, the operators will, at large times, be localized in far spacelike separated regions and hence will commute, and thus the vector will depend only on the single particle vectors Ψ_j, $j = 1\dots,n$. This rough idea does not work exactly, but approximately for large times, and one can show that it becomes exact in the limits $t \to \pm\infty$. Hence one obtains two in general different multi-particle states corresponding to each collection of single particle states, the outgoing ($t \to \infty$) and the incoming ($t \to -\infty$) state. The operator mapping the incoming state to the corresponding outgoing state is the scattering matrix, and it can be shown that it is always a partial isometry.

The rigorous argument proceeds at follows: given a single particle state Ψ with compact momentum support and a diamond $\mathcal{O} \in \mathcal{K}$, there is, by the Reeh-Schlieder theorem, for every $\varepsilon > 0$ an operator $A \in \mathfrak{A}(\mathcal{O})$ with $||\Psi - A\Omega|| < \varepsilon$. Let now f be a Schwartz function whose Fourier transform $\tilde{f}$ has a compact support within the forward lightcone, which intersects the energy momentum spectrum only on the mass shell and is equal to 1 on the momentum support of Ψ. Then $A(f) = \int dx \alpha_x(A) f(x)$ is almost local, in the sense that it can be fast approximated by local operators, and

$$||\Psi - A(f)\Omega|| = ||\tilde{f}(P)(\Psi - A\Omega)|| \leq \sup|\tilde{f}|\varepsilon. \tag{1.15}$$

We then consider the family of Schwartz functions $(f_t)_{t\in\mathbb{R}}$ with Fourier transforms

$$\tilde{f}_t(p) = e^{it(\sqrt{p^2}-m)}\tilde{f}(p) \tag{1.16}$$

and observe that the 1-particle vectors

$$\Psi(t) = A(t)\Omega \ \text{ with } A(t) = A(f_t) \tag{1.17}$$

do not depend on t.

The crucial fact is now that the functions f_t are essentially concentrated within the region tV_f, where V_f is the set of 4-velocities $p/\sqrt{p^2}$ with $p \in \operatorname{supp}\tilde{f}$. This follows by applying the argument of the stationary phase to the oscillating integral

$$f_t(x) = (2\pi)^{-4} \int d^4 p \tilde{f}(p) e^{(it(\sqrt{p^2}-m)-ipx)}. \tag{1.18}$$

We now can realize the heuristic idea for the construction of multi-particle states by choosing single particle vectors $\Psi_1, \ldots \Psi_n$ with disjoint momentum support, the corresponding operators $A_i \in \mathfrak{A}(\mathcal{O})$ and the Schwartz functions f_i with Fourier transforms with mutually disjoint velocity supports V_{f_i}. We then find the following theorem:

Proposition 1.3.1 *The limits*

$$\lim_{t\to\pm\infty} A_1(t) \ldots A_n(t)\Omega \tag{1.19}$$

exist.

Proof The statement follows from the fact that the derivative with respect to t is integrable. Namely, using Leibniz' rule and the fact that the single particle vectors $A_i(t)\Omega$ are time independent, we can bound the derivative by

$$||\frac{d}{dt} A_1(t) \ldots A_n(t)\Omega|| \leq \sum_{i<j} ||[\frac{d}{dt} A_i(t), A_j(t)]|| \prod_{k\neq i,j} ||A_k(t)||. \tag{1.20}$$

Since the commutators decay fast, due to the localization of the Schwartz functions $f_{i,t}$, and the norms of the operators $A_i(t)$ are polynomially bounded, the derivative decays fast and hence is integrable.

Clearly, the multi-particle vectors depend only on the corresponding single particle vectors, since the order of factors can be changed arbitrarily. Also their scalar products can be computed in terms of scalar products of single particle vectors:

Proposition 1.3.2 *Let $A_i, f_i, i = 1, \ldots, n$ and $B_j, g_j, j = 1, \ldots k$ be operators and Schwartz functions as above such that* $\mathrm{supp}\tilde{f}_i \cap \mathrm{supp}\tilde{g}_j = \emptyset$ *for $i \neq j$. Moreover, assume that the momentum support of each test function is so small that the set of differences $p - p'$, p, p' elements of the supports of the Fourier transforms of the test functions under consideration, intersects the energy momentum spectrum at most at* $\{0\}$. *Then*

$$\lim_{t\to\pm\infty} \langle A_1(t) \ldots A_n(t)\Omega, B_1(t) \ldots B_k(t)\Omega\rangle = \delta_{nk} \prod_{i=1}^{n} \langle A_i(0)\Omega, B_i(0)\Omega\rangle \tag{1.21}$$

Proof Due to the pairwise disjoint supports of the Fourier transforms of the test functions, the operators $A_i(t)^*$ commute with $B_j(t)$ for $i \neq j$ in the limits $t \to \pm\infty$. Hence the limit of the left hand side is equal to the limit of

$$\langle \prod_{j=k+1}^{n} A_j(t)\Omega, \prod_{i=1}^{k} A_i(t)^* B_i(t)\Omega\rangle \tag{1.22}$$

for $n \geq k$. (For $k \geq n$ we just have to exchange the roles of B's and A's.) But by the assumptions on the smallness of the momentum support of the test functions and on the uniqueness of the vacuum vector Ω, we have

$$A_i(t)^* B_i(t)\Omega = \Omega\langle A_i(0)\Omega, B_i(0)\Omega\rangle. \tag{1.23}$$

Since the momentum support of $\prod_{j=k+1}^{n} A_j(t)\Omega$ is contained in the forward light cone for $n > k$, the scalar product (1.22) vanishes in this case (as also in the case $n < k$), and we arrive at the formula in the proposition. □

The construction of scattering states can now easily be completed. Namely, let $\mathcal{H}_F$ denote the symmetric Fock space over the single particle space. The construction described above yields two densely defined isometries $W_\pm : \mathcal{H}_F \to \mathcal{H}$ with unique extensions to everywhere defined isometries (denoted by the same symbol). The images are the subspaces of incoming and outgoing scattering states, respectively, and the S-matrix is the partial isometry given by $S = W_+ W_-^*$.

Unfortunately, the construction breaks down in the presence of massless particles. But there, a different method was developed by Buchholz [23] which yields the scattering states of the massless particles. The question, however, about the particle structure for massive particles in the presence of massless particles is not yet completely clarified. Some progress was obtained by Dybalski [48].

1.4 Superselection Sectors

By definition, a superselection sector is an equivalence class of irreducible representations of the quasilocal algebra. We want to characterize representations by their relation to a distinguished representation π_0, the vacuum representation. Let for a double cone $\mathcal{O}$ of Minkowski space $\mathfrak{A}(\mathcal{O}')$ denote the C*-subalgebra of $\mathfrak{A}(\mathbb{M})$ which is generated by the algebras $\mathfrak{A}(\mathcal{O}_1)$ with double cones $\mathcal{O}_1 \subset \mathcal{O}'$ where $\mathcal{O}'$ denotes the spacelike complement of $\mathcal{O}$. The charge carrying fields are then characterized as partial intertwiners F between the vacuum representation π_0 and a representation π which contains charged states,

$$F : \mathcal{H}_{\pi_0} \to \mathcal{H}_\pi,\ F\pi_0(A) = \pi(A)F\ \forall\ A \in \mathfrak{A}(\mathcal{O}'). \tag{1.24}$$

Representations π satisfying the DHR selection criterion are those whose partial intertwiner spaces $\mathcal{F}_\pi(\mathcal{O})$ for every double cone $\mathcal{O}$ generate the representation space, i.e.

$$\mathcal{F}_\pi(\mathcal{O})\mathcal{H}_{\pi_0} \text{ is dense in } \mathcal{H}_\pi,\ \text{ and } \mathcal{F}_\pi(\mathcal{O})^*\mathcal{H}_\pi \text{ is dense in } \mathcal{H}_{\pi_0}. \tag{1.25}$$

This is equivalent to the statement that, after restriction to $\mathfrak{A}(\mathcal{O}')$, the state spaces of π and π_0 coincide (π and π_0 become quasiequivalent after restriction). The condition is usually motivated by the "particle behind the moon"-argument: the charge by which π and π_0 are distinguished might be compensated within an arbitrary region $\mathcal{O}$ which is inaccessible to observations. Note that the condition is satisfied if the charged states can be generated by local fields, but that it is violated if the charge can be measured at spacelike distances, as e.g. the electric charge by Gauss' law in electrodynamics.

The space of partial intertwiners between π_0 and π is a bimodule $\mathcal{F}_\pi(\mathcal{O})$ over $\mathfrak{A}(\mathcal{O})$ with the left and right action

$$A \cdot F \cdot B := \pi(A)F\pi_0(B). \tag{1.26}$$

This fact can be used to define products of fields in terms of tensor products of bimodules $\mathcal{F}_\pi(\mathcal{O})$ and $\mathcal{F}_{\pi'}(\mathcal{O})$. The elements of this tensor product shall then be interpreted as partial intertwiners between π_0 and a new representation $\pi \times \pi'$ corresponding to the composed charges. This product is called fusion in the framework of conformal field theory in 2 dimensions.

As a first step we observe that for $F, G \in \mathcal{F}_{\pi_0}(\mathcal{O})$, the operator F^*G commutes with $\pi_0(\mathfrak{A}(\mathcal{O}'))$. One now adds a crucial maximality condition to the algebra of bounded regions in the vacuum representation called Haag duality,

$$\pi_0(\mathfrak{A}(\mathcal{O}'))' = \pi_0(\mathfrak{A}(\mathcal{O}))''. \tag{1.27}$$

Haag duality implies that the operators F^*G as above are elements of $\pi_0(\mathfrak{A}(\mathcal{O}))''$. In the DHR analysis one further assumes that the partial intertwiner spaces even contain unitary elements, i.e. that the representations restricted to $\mathfrak{A}(\mathcal{O}')$ are even unitarily equivalent, whereas quasiequivalence means unitary equivalence of suitable multiples. We want to go the first steps without this assumption and use instead that, as a consequence of quasiequivalence and Haag duality, the localization regions of the partial intertwiners can be changed by local observables (charge transporter), namely if $\mathcal{O}_1, \mathcal{O}_2 \subset \mathcal{O}$ then $\mathcal{F}_\pi(\mathcal{O}_2) \subset \mathcal{F}_\pi(\mathcal{O}_1)\pi_0(\mathfrak{A}(\mathcal{O}))''$.

In the following we identify for all double cones $\mathcal{O}$ the local algebras $\mathfrak{A}(\mathcal{O})$ with the von Neumann algebra $\pi_0(\mathfrak{A}(\mathcal{O}))''$. Let $\mathfrak{A}_0$ denote the algebra of all local observables, $\mathfrak{A}_0 = \bigcup_{\mathcal{K}} \mathfrak{A}(\mathcal{O})$, and let $\mathcal{F}_\pi = \bigcup_{\mathcal{K}} \mathcal{F}_\pi(\mathcal{O})$ denote the vector space of all partial intertwiners between π_0 and π. Then $\mathcal{F}_\pi$ is a bimodule over $\mathfrak{A}_0$, and $\mathcal{H}_\pi$ is a left module. Moreover, due to Haag duality, $\mathcal{F}_\pi$ has an $\mathfrak{A}_0$ valued scalar product,

$$\langle F, G\rangle := F^*G, \tag{1.28}$$

and $\mathcal{H}_\pi$ is equal to the completion of the tensor product $\mathcal{F}_\pi \otimes_{\mathfrak{A}_0} \mathcal{H}_{\pi_0}$, equipped with the scalar product

$$\langle F \otimes \Phi, G \otimes \Psi\rangle = \langle \Phi, \langle F, G\rangle \Psi\rangle. \tag{1.29}$$

Also the tensor product of two DHR sectors, $\mathcal{F}_{\pi_1} \otimes_{\mathfrak{A}_0} \mathcal{F}_{\pi_2}$ carries an $\mathfrak{A}_0$ valued scalar product

$$\langle F_1 \otimes F_2, G_1 \otimes G_2\rangle := F_2^* \pi_2(F_1^* G_1) G_2, \tag{1.30}$$

and a new Hilbert space representation $\pi_1 \times \pi_2$ is obtained by equipping the left module $\mathcal{F}_{\pi_1} \otimes_{\mathfrak{A}_0} \mathcal{F}_{\pi_2} \otimes_{\mathfrak{A}_0} \mathcal{H}_{\pi_0}$ with the scalar product

$$\langle F_1 \otimes F_2 \otimes \Phi, G_1 \otimes G_2 \otimes \Psi \rangle = \langle \Phi, \langle F_1 \otimes F_2, G_1 \otimes G_2 \rangle \Psi \rangle. \tag{1.31}$$

Intertwiners between DHR representations π and π', i.e. operators $T : \mathcal{H}_\pi \to \mathcal{H}_{\pi'}$ such that

$$T\pi(A) = \pi'(A)T \; \forall A \in \mathfrak{A}_0, \tag{1.32}$$

induce homomorphisms between the corresponding bimodules (denoted by the same symbol)

$$T : \mathcal{F}_\pi \to \mathcal{F}_{\pi'}, \; T : F \mapsto TF. \tag{1.33}$$

The structure we obtain is that of a tensor (or monoidal) C*-category with the DHR representations as objects and the intertwiners as morphisms where the tensor product is constructed in terms of the tensor products of the associated bimodules. This category has the following additional structures:

- It is additive, since direct sums of DHR representations are again DHR representations.
- It has subobjects, since any subrepresentation is again a DHR representation.

It is not a strict category, since the tensor product of bimodules is not strict. As usual we ignore this problem in the treatment of higher tensor products by using Mc Lane's coherence theorem which implies that different choices of brackets in tensor products are uniquely related by a natural isomorphism.

We now want to investigate the commutation relations between spacelike separated partial interwiners. Let $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$ with $\mathcal{O}_1 \subset \mathcal{O}_2'$, and let π_1, π_2 be DHR representations. We define a right module homomorphism

$$\varepsilon(\pi_1, \pi_2) : \mathcal{F}_{\pi_1} \otimes_{\mathfrak{A}_0} \mathcal{F}_{\pi_2} \to \mathcal{F}_{\pi_2} \otimes_{\mathfrak{A}_0} \mathcal{F}_{\pi_1} \tag{1.34}$$

by

$$\varepsilon(\pi_1, \pi_2) F_1 \otimes F_2 = F_2 \otimes F_1 \tag{1.35}$$

with $F_i \in \mathcal{F}_{\pi_i}(\mathcal{O}_i)$, where we used that $\mathcal{F}_\pi = \mathcal{F}_\pi(\mathcal{O})\mathfrak{A}_0$, $\mathcal{O} \in \mathcal{K}$ for all DHR representations π. ε is called the statistics operator. We first check that it is well defined. Namely,

$$\sum_i F_{1,i} \otimes F_{2,i} \cdot A_i = 0 \text{ iff } \sum_{ij} A_i^* F_{2,i}^* \pi_2(F_{1,i}^* F_{1,j}) F_{2,j} A_j = 0, \tag{1.36}$$

with $F_{k,i} \in \mathcal{F}_{\pi_k}(\mathcal{O}_k)$ and $A_i \in \mathfrak{A}_0$. But $F_{k,i}^* F_{k,j} \in \mathfrak{A}(\mathcal{O}_k)$, hence

$$F_{2,i}^* \pi_2(F_{1,i}^* F_{1,j}) F_{2,j} = F_{2,i}^* F_{2,j} F_{1,i}^* F_{1,j} = F_{1,i}^* F_{1,j} F_{2,i}^* F_{2,j} = F_{1,i}^* \pi_1(F_{2,i}^* F_{2,j}) F_{1,j}. \tag{1.37}$$

We conclude that $\sum_i F_{1,i} \otimes F_{2,i} \cdot A_i = 0$ implies $\sum_i F_{2,i} \otimes F_{1,i} \cdot A_i = 0$. Moreover, we see that ε is norm preserving and that the induced operator on $\mathcal{H}_{\pi_1 \times \pi_2}$ is unitary.

The definition of ε depends on the choice of the double cones $\mathcal{O}_1$ and $\mathcal{O}_2$. But clearly ε does not change if we replace the double cones by smaller ones $\tilde{\mathcal{O}}_i \subset \mathcal{O}_i$, $i = 1, 2$. We now may deform the pair of spacelike separated double cones in the following way: Let $(\mathcal{O}_1^n, \mathcal{O}_2^n)_{n=1,\ldots,2N+1}$ be a sequence of pairs of spacelike separated double cones such that $\mathcal{O}_i^{2n\pm 1} \subset \mathcal{O}_i^{2n}$, $n = 1, \ldots N$, $i = 1, 2$. Then ε defined for the pair $(\mathcal{O}_1^1, \mathcal{O}_2^1)$ coincides with ε defined for the pair $(\mathcal{O}_1^{2N+1}, \mathcal{O}_2^{2N+1})$. We conclude that in d dimensional Minkowski space with $d > 2$ the statistics operator is unique, whereas in 2 dimensions there are two choices. The crucial properties of the statistics operator are summarized in the following theorem:

Theorem 1.4.1 *1. ε is a bimodule homomorphism.*

2. In $d > 2$ dimensions we have

$$\varepsilon(\pi_1, \pi_2)\varepsilon(\pi_2, \pi_1) = 1. \tag{1.38}$$

3. Let π_1, π_2, π_3 be DHR representations. Then we have the identity

$$(\varepsilon(\pi_2, \pi_3) \otimes 1)(1 \otimes \varepsilon(\pi_1, \pi_3))(\varepsilon(\pi_1, \pi_2) \otimes 1) = (1 \otimes \varepsilon(\pi_1, \pi_2))(\varepsilon(\pi_1, \pi_3) \otimes 1)(1 \otimes \varepsilon(\pi_2, \pi_3)). \tag{1.39}$$

Proof 1. Let $A \in \mathfrak{A}(\mathcal{O})$ for some $\mathcal{O} \in \mathcal{K}$. Choose $\mathcal{O}_1, \mathcal{O}_2 \in \mathcal{K}$ such that the three double cones are pairwise spacelike separated. Let $F_i \in \mathcal{F}_{\pi_i}(\mathcal{O}_i)$, $i = 1, 2$. Then

$$A \cdot F_1 \otimes F_2 = F_1 \otimes F_2 \cdot A \text{ and } A \cdot F_2 \otimes F_1 = F_2 \otimes F_1 \cdot A, \tag{1.40}$$

hence

$$\varepsilon(\pi_1, \pi_2)(A \cdot F_1 \otimes F_2) = \varepsilon(\pi_1, \pi_2)(F_1 \otimes F_2 \cdot A) =$$
$$F_2 \otimes F_1 \cdot A = A \cdot F_2 \otimes F_1 = A \cdot \varepsilon(\pi_1, \pi_2)(F_1 \otimes F_2).$$

2. This follows from the uniqueness of ε.
3. This relation can be easily checked by application to $F_1 \otimes F_2 \otimes F_3$ with $F_i \in \mathcal{F}_{\pi_i}(\mathcal{O}_i)$ with pairwise spacelike separated double cones $\mathcal{O}_i$, $i = 1, 2, 3$. □

The statistics operators with the properties listed above equip the tensor category with a braiding or even a symmetry in $d > 2$. Physically it means that we can derive the commutation relations of fields and don't have to rely on an a priori choice.

An important question is whether the category is rigid, i.e. whether every π has a conjugate $\overline{\pi}$, unique up to equivalence. A DHR representation $\overline{\pi}$ is conjugate to π if there is an isometric intertwiner R such that

$$\overline{\pi} \times \pi(A)R = R\pi_0(A) \; \forall A \in \mathfrak{A}_0. \tag{1.41}$$

If π describes states of a particle we expect that its conjugate describes states of the corresponding antiparticle. Actually, in case the mass spectrum in π has an isolated eigenvalue, the existence of $\overline{\pi}$ could be proven [26, 52].

Theorem 1.4.2 *Let π be a translation covariant irreducible DHR representation of $\mathfrak{A}_0$ which satisfies the spectrum condition and whose mass spectrum contains an isolated eigenvalue. Then there exists a conjugate representation, which is irreducible and translation covariant and has the same mass spectrum as π.*

Actually, the theorem derived in [26, 52] is much stronger. In particular one does not need to know in advance that the representation π satisfies the DHR criterion. Instead one derives the existence of a vacuum representation π_0 such that π satisfies a weakened form of the DHR criterion where double cones have to be replaced by so-called spacelike cones. These are sets of the form

$$S = x + \bigcup_{\lambda>0} \lambda\mathcal{O} \tag{1.42}$$

with a double cone $\mathcal{O}$ whose closure is spacelike to the origin. If π_0 satisfies Haag duality for spacelike cones, one can redo the DHR analysis and one can directly construct the conjugate representation. There is, however, an important difference in the analysis of the statistics. Namely, due to the different localization one finds symmetry only in dimension $d > 3$. This allows the existence of particles with braid group statistics (plektons, anyons) in 3 dimensions.

We now restrict ourselves to DHR representations π with unitary partial intertwiners $V \in \mathcal{F}_\pi(\mathcal{O})$ for all $\mathcal{O} \in \mathcal{K}$, and we use the fact that the vacuum representation satisfies the conditions of Borchers theorem. Hence products, finite direct sums and subrepresentations also have unitary partial intertwiners. It is an interesting question how far the analysis of superselection sectors can be carried through for the general case. For progress in this direction see e.g. [86].

Let $V \in \mathcal{F}_\pi(\mathcal{O})$ be unitary. Then one can replace π by a unitarily equivalent representation ρ on the vacuum Hilbert space $\mathcal{H}_{\pi_0}$ by

$$\rho(A) = V^*\pi(A)V. \tag{1.43}$$

ρ is actually an endomorphism of $\mathfrak{A}_0$ by Haag duality, and it is localized in $\mathcal{O}$ in the sense that it acts trivially on all algebras $\mathfrak{A}(\mathcal{O}_1)$ with $\mathcal{K} \ni \mathcal{O}_1 \subset \mathcal{O}'$. Moreover, by the assumed existence of unitary elements in the partial intertwiner spaces $\mathcal{F}_\pi(\mathcal{O}_1)$ for all $\mathcal{O}_1 \in \mathcal{K}$ one finds endomorphisms ρ_1 localized in $\mathcal{O}_1$ which are equivalent as representations. But, again by Haag duality, the unitary operator in $\mathcal{H}_{\pi_0}$ which implements the equivalence is an element of $\mathfrak{A}(\mathcal{O}_2)$ for $\mathcal{K} \ni \mathcal{O}_2 \supset \mathcal{O} \cup \mathcal{O}_1$, hence ρ and ρ_1 are conjugate by inner automorphisms of $\mathfrak{A}_0$. Products of representations π_i now correspond to compositions of endomorphisms ρ_i in opposite order, $\pi_1 \times \pi_2 \simeq \rho_2 \circ \rho_1$.

The partial intertwiners $F \in \mathcal{F}_\rho(\mathcal{O})$ are identified with elements (ρ, F) of the so-called field bundle where $F \in \mathcal{B}(\mathcal{H}_{\pi_0}))$ with $\rho(A)F = FA$ for $A \in \mathfrak{A}(\mathcal{O}')$. If

$U \in \mathfrak{A}_0$ such that Ad$U \circ \rho$ is localized in $\mathcal{O}$, then $\rho(A)U^* = U^* A$ for $A \in \mathfrak{A}(\mathcal{O}')$, hence $UF \in \mathfrak{A}(\mathcal{O}')' = \mathfrak{A}(\mathcal{O})$ and $F \in \mathfrak{A}_0$. The tensor product of the bimodules $\mathcal{F}_{\rho_i}$ is the field bundle product of [42],

$$(\rho_1, F_1) \otimes (\rho_2, F_2) = (\rho_2\rho_1, \rho_2(F_1)F_2). \tag{1.44}$$

Homomorphisms of bimodules are now given by operators $T \in \mathcal{B}(\mathcal{H}_{\pi_0})$ which satisfy the intertwiner relation

$$\sigma(A)T = T\rho(A), \ A \in \mathfrak{A}_0, \tag{1.45}$$

and where $T \in \mathfrak{A}(\mathcal{O})$ if σ and ρ are localized in $\mathcal{O}$. The statistics operator, in particular, can now be expressed in terms of endomorphisms and unitaries which move the localization regions of endomorphisms into spacelike separated regions.

Namely, let $(\rho_i, F_i) \in \mathcal{F}_{\rho_i}(\mathcal{O}_i), i = 1, 2$ with spacelike separated double cone $\mathcal{O}_1, \mathcal{O}_2$. Choose unitaries $U_i \in \mathfrak{A}_0$ such that $\rho_i' := \mathrm{Ad} \circ \rho_i$ is localized in $\mathcal{O}_i$. Then $F_i' := U_i F_i \in \mathfrak{A}(\mathcal{O}_i)$ and in (1.44) we obtain for the second entry on the right hand side

$$\rho_2(F_1)F_2 = \rho_2(U_1^* F_1')U_2^* F_2' = \rho_2(U_1^*)U_2^* \rho_2'(F_1')F_2' = \rho_2(U_1^*)U_2^* F_1' F_2' \,. \tag{1.46}$$

Inserting this into the formula for the definition of the statistics operator we obtain

$$\varepsilon(\rho_1, \rho_2) = \rho_1(U_2^*)U_1^* U_2 \rho(U_1). \tag{1.47}$$

By passing from the representations to the endomorphisms we get a new category where the objects are the DHR endomorphisms and the arrows are intertwiners between these endomorphisms. This category is equivalent as a braided (and in $d > 2$ symmetric) tensor C*-category to the previous one (after adding the requirement of the existence of unitary partial intertwiners). In contrast to the previous one it is small (the objects form a set) and strict (the tensor product is now the composition of endomorphisms and hence associative).

Let us return to the question of rigidity (the existence and uniqueness of conjugates). This question is closely related to the classification of statistics. In order to characterize conjugates $\overline{\rho}$ of an DHR endomorphism ρ up to equivalence one requires in addition to the existence of an isometric intertwiner R from the vacuum id to the representation $\overline{\rho}\rho$ the existence of an isometry $\overline{R}$ from id to $\rho\overline{\rho}$ (this is automatically fulfilled by setting $\overline{R} = \varepsilon(\overline{\rho}, \rho)R$) and the equations

$$\overline{R}^* \rho(R) = R^* \overline{\rho}(\overline{R}) = \lambda 1, \ \lambda \neq 0. \tag{1.48}$$

(In category theory, it is common to normalize the intertwiners such that $\lambda = 1$. This, however, is in general not compatible with the required isometry.) We now compute λ in terms of the statistics operators. We find

$$\overline{R}^* \rho(R) = R^* \varepsilon(\overline{\rho}, \rho)^* \rho(R) \varepsilon(\mathrm{id}, \rho) = R^* \varepsilon(\overline{\rho}, \rho)^* \varepsilon(\overline{\rho}\rho, \rho) R = R^* \overline{\rho}(\varepsilon(\rho, \rho)) R, \tag{1.49}$$

where we used $\varepsilon(\mathrm{id}, \rho) = 1$, $\rho(R)\varepsilon(\mathrm{id}, \rho) = \varepsilon(\overline{\rho}\rho, \rho)R$ and $\varepsilon(\overline{\rho}\rho, \rho) = \varepsilon(\overline{\rho}, \rho)$ $\overline{\rho}(\varepsilon(\rho, \rho))$. The completely positive map

$$\phi : \mathfrak{A}_0 \to \mathfrak{A}_0,\ \phi(A) = R^*\overline{\rho}(A)R$$

is a left inverse of ρ. In the case of a symmetry, it can be used for analysing the representation of the permutation group S_n generated by the operators $\rho^k(\varepsilon(\rho, \rho))$, $k = 0, \ldots, n-1$ in terms of the *statistics parameter* $\lambda = \phi(\varepsilon(\rho, \rho))$. One finds that, by positivity, the allowed values of λ are $\lambda \in \{\pm\frac{1}{d} | d \in \mathbb{N}\}$. The natural number d is called the statistical dimension within the DHR theory. As discovered by Longo [80], it coincides with the square root of the Jones index [72] of the inclusion $\rho(\mathfrak{A}(\mathcal{O})) \subset \rho(\mathfrak{A}(\mathcal{O}'))'$ which characterizes the degree of violation of Haag duality in the representation ρ (independently of the choice of $\mathcal{O} \in \mathcal{K}$). This relation remains valid also in the case where the symmetry has to be replaced by braiding. The statistical dimension coincides with the general notion of dimension of objects in symmetric tensor categories.

In case of a braiding, as it is the generic situation in 2 dimensions, one can approach a similar classification of the representation of the braid group B_n. This was successful under additional conditions on the spectrum of the statistics operators $\varepsilon(\rho, \rho)$. If e.g. it contains only two points as in the symmetric case, the representations can be classified in analogy to the representations of the symmetric group [58].

In the symmetric case, the DHR endomorphisms with finite statistical dimension form a rigid symmetric tensor C*-category. It was proven by Doplicher and Roberts that such a category is equivalent to a category of representations of a compact group G with a distinguished element k of order 2, where the representation spaces are graded by the eigenvalues of k and where the symmetry σ on the tensor product of representations is chosen such that for eigenvectors v, w

$$\sigma\ v \otimes w = \pm w \otimes v. \tag{1.50}$$

Here the minus sign appears only when both vectors have eigenvalue -1. Both, G and k are uniquely determined. The authors then exploit this equivalence and construct a net of von Neumann algebras $\mathfrak{F}$ on some Hilbert space $\mathcal{H}$ with the following properties:

- $\mathfrak{F}$ satisfies the condition of isotony.
- The group G (the gauge group) acts by automorphisms $g \mapsto \alpha_g$ such that $\alpha_g(\mathfrak{F}(\mathcal{O})) = \mathfrak{F}(\mathcal{O})$ for all $\mathcal{O}$.
- If $\mathcal{O}_1$ is spacelike to $\mathcal{O}_2$ and $F_i \in \mathfrak{F}(\mathcal{O}_i)$ with $\alpha_k(F_i) = +F_i$ (bosonic) or $-F_i$ (fermionic), then $F_1F_2 = \pm F_2F_1$, where the minus sign holds if both factors are fermionic.
- There is an action of the covering of the connected component of the Poincaré group $L \mapsto \alpha_L$ by automorphisms of $\mathfrak{F}$, such that

$$\alpha_L(\mathfrak{F}(\mathcal{O})) = \mathfrak{F}(L\mathcal{O}) \text{ and } \alpha_L\alpha_g = \alpha_g\alpha_L\ \forall g \in G. \tag{1.51}$$

- The subalgebras of fixed points under the gauge group,

$$\mathfrak{A}(\mathcal{O}) = \{A \in \mathfrak{F}(\mathcal{O}) | \alpha_g(A) = A \ \forall g \in G\}, \tag{1.52}$$

  form a net, which is equivalent to the original Haag-Kastler net and where the Poincaré symmetry of $\mathfrak{A}$ derives from the action $L \mapsto \alpha_L$ on $\mathfrak{F}$.
- Each irreducible DHR representation with finite statistics is equivalent to a subrepresentation of $\mathfrak{A}$ on $\mathcal{H}$.
- The net $\mathfrak{F}$ acts irreducibly on $\mathcal{H}$ in the following strong sense: Let $\mathcal{O}$ be a double cone. Then the only bounded operators which commute with $\mathfrak{F}(\mathcal{O})$ and $\mathfrak{A}(\mathbb{M})$ are the multiples of the identity.

For more details see the original papers [45, 46] and for a nice review [68].

The analysis described above concerns the sector structure on Minkowski space. In conformally invariant theories it is often useful to replace Minkowski space by its conformal completion. In such a spacetime the partially ordered set of double cones is no longer directed which leads to a modification of the DHR theory. More generally, in globally hyperbolic spacetimes, new phenomena might occur due to topological obstructions. See [21] and references therein for more information.

1.5 Structure of Local Algebras

The Haag-Kastler axioms, together with the existence of a vacuum representation, yield already quite a number of informations on the structure of local algebras. It was first observed on the example of the free scalar field that the local algebras are factors of type III [5, 6]. At that time, type III algebras were not well understood and were considered to be pathological. It was observed that the type III property is due to the sharp localization within a spacetime region, and Borchers conjectured that there exist intermediate type I factors between local algebras associated to regions $\mathcal{O}_1, \mathcal{O}_2$ with $\overline{\mathcal{O}_1} \subset \mathcal{O}_2$, a property which later was named the split property. Actually, Buchholz was able to prove this conjecture for the free scalar field [22].

A crucial progress in the structural analysis came with the advent of the Tomita-Takesaki theory. Namely, given a von Neumann algebra N with a cyclic and separating vector ξ, a situation present for local algebras due to the Reeh-Schlieder theorem, one can define an antilinear operator S with domain $N\xi$ by

$$SA\xi = A^*\xi, \ \ A \in N. \tag{1.53}$$

S is in general an unbounded operator. It is closable, and we denote the closure with the same symbol. The big surprise is that its polar decomposition

$$S = J\Delta^{1/2}, \ \ \Delta = S^*S \text{ (the } \textit{modular operator}) \tag{1.54}$$

has the following remarkable properties:

- J is an antiunitary involution (the *modular involution*) , and $J\Delta = \Delta^{-1}J$.
- $JNJ = N'$.
- The unitaries Δ^{it} implement automorphisms of N (the *modular automorphisms*).
- The state induced by ξ satisfies the KMS condition with respect to the time evolution given by the modular automorphisms, with inverse temperature -1.

An immediate question is whether one can determine the modular structure for local algebras and their cyclic and separating vectors, in particular the vacuum, and whether these operations have a physical interpretation.

The first breakthrough was obtained by Bisognano and Wichmann [11]. They showed that for a generic Haag-Kastler net which is generated by Wightman fields in a specific way the problem could be solved for the algebra associated to the wedge $W = \{x \in \mathbb{M} | |x^0| < x^1\}$ and the vacuum. They proved that the modular automorphisms coincide with Lorentz boosts in the x^1-direction (hence the vacuum looks like a thermal state for a uniformly accelerated observer, an observation which was made independently at about the same time by Unruh [90] as an analog to Hawking radiation of black holes, see [34] for a discussion of the physical interpretation), and they showed that the modular involution coincides with the P_1CT-transformation where parity P is replaced by reflection P_1 of the x^1-coordinate only. As a consequence, Haag duality holds for wedges, since $P_1TW = W'$, a property called essential Haag duality, a concept introduced by Roberts.

There were also a few other cases where the modular structures could be uncovered, but for the generic case not much is known. There is, however, a very remarkable theorem of Borchers [13, 51] which can be seen as an abstract version of the Bisognano-Wichmann-Theorem. Namely, let M be a von Neumann algebras with a cyclic and separating vector Ω together with a strongly continuous 1-parameter unitary group $t \mapsto U(t)$ with the properties:

- The generator of U is positive.
- $U(t)\Omega = \Omega$.
- For $t > 0$, $\mathrm{Ad}U(t)$ maps M into itself.

Then, for the modular operator Δ and the modular involution J associated to the pair (N, Ω), the following relations hold:

$$\Delta^{it}U(s)\Delta^{-it} = U(e^{-2\pi t}s), \tag{1.55}$$

$$JU(s)J = U(-s). \tag{1.56}$$

These are the relations found by Bisognano and Wichmann for the wedge algebra on the vacuum where U describes the future directed lightlike translations within the wedge.

There is also a partial converse of Borchers' Theorem found by Wiesbrock [7, 93]. Namely, let N, M be von Neumann algebras acting on the same Hilbert space with a joint cyclic and separating vector Ω such that $N \subset M$ and that the modular automorphisms associated to M map (for $t > 0$) the algebra N into itself,

$$\mathrm{Ad}\Delta_M^{it}(N) \subset N, t > 0. \tag{1.57}$$

(This was termed *half sided modular inclusion*.) Then there exists a uniquely determined unitary 1-parameter group $a \mapsto U(a)$ with a positive generator which satisfies the relations above, such that

$$\mathrm{Ad}U(1)(N) = M. \tag{1.58}$$

It was shown by Borchers, Wiesbrock et al. that by using these theorems one can construct covariant Haag-Kastler nets from a finite family of half sided modular inclusions where the necessary number depends on the dimension of spacetime. Unfortunately, up to now, explicit examples of half sided modular inclusions have been found only within Haag-Kastler nets, so this method has not yet led to new examples of AQFT. See [14] for a very detailed overview which also contains many more references. See also [35] for an attempt to interpret the modular automorphisms as a state dependent intrinsic time evolution with applications to quantum gravity.

In spite of the fact that an explicit determination of the action of modular automorphisms on the local algebras was possible only in special cases, some general features could be established which finally led to the determination of the local von Neumann algebras up to isomorphy under plausible conditions. To explain this we first review the classification of type III algebras by Connes. Namely, Connes showed that the intersection of the spectra of all modular operators of a factor of type III with respect to its cyclic and separating vectors is one of the following possibilities:

- $\{0, 1\}$ (type III_0),
- $\{0, \lambda^n, n \in \mathbb{Z}\}$ for some $0 < \lambda < 1$ (type III_λ),
- $\mathbb{R}_+$ (type III_1).

One finds that in all known examples the algebras $\mathfrak{A}(\mathcal{O})$ are factors of type III_1. Moreover, under plausible assumptions, the type III_1 property is generic for theories generated by Wightman fields with a well behaved short distance scaling limit [53]. Again, this originally looked, from the mathematics perspective, as the least understood possibility. It soon turned out, however, that these factors have very nice properties. First of all, Haagerup [66] could show that there is up to isomorphy only one hyperfinite factor of type III_1, where hyperfinite means that the algebra is generated by an increasing sequence of full matrix algebras. It is exactly the factor already found for local algebras of free field theories. Hyperfiniteness of the local algebras is implied by the split property mentioned in the introduction. The split property, finally, can be derived from the so-called nuclearity condition [27, 30]. This condition may be understood as the condition that the partition function in a finite spatial volume is finite. Technically, it says that for each double cone $\mathcal{O}$ and each finite $\beta > 0$ the map

$$T_{\mathcal{O},\beta} : \mathfrak{A}(\mathcal{O}) \to \mathcal{H},\ A \mapsto e^{-\beta H} A\Omega \tag{1.59}$$

is nuclear. Here the algebra is represented on the vacuum Hilbert space $\mathcal{H}$, H is the Hamiltonian (the generator of time translations in a given Lorentz frame) and Ω is the unit vector inducing the vacuum state. A linear map f between Banach spaces X, Y is nuclear if it can be written in the form

$$f(x) = \sum y_n \langle l_n, x \rangle \tag{1.60}$$

with $y_n \in Y$, $l_n \in X^*$, the dual of X, and $\sum_n ||y_n|| \cdot ||l_n|| < \infty$. The infimum $\nu(f)$ over these sums for all such representations of f is called the nuclearity index. If $\nu(T_{\mathcal{O},\beta})$ behaves in the way expected from a partition function,

$$\nu(T_{\mathcal{O},\beta}) < e^{(\beta_0/\beta)^n}, \tag{1.61}$$

for some $\beta_0 > 0$ (depending on $\mathcal{O}$) and some $n \in \mathbb{N}$ (this was shown to be true for the free scalar field), then one finds that the split property holds and that the theory has KMS states for all $\beta > 0$ [28]. On the other hand, there are generalized free fields which violate the nuclearity condition as well as the split property.

Besides implying hyperfiniteness, the split property has many other nice consequences, which may be summarized in the universal localizing map [44]. Namely, let

$$\Lambda = (\mathfrak{A}(\mathcal{O}_1) \subset \mathfrak{A}(\mathcal{O}_2)) \tag{1.62}$$

be a split inclusion. Then there is a unit vector $\Omega_\Lambda \in \mathcal{H}$ which induces the product state

$$\omega_\Lambda(AB') = \omega_0(A)\omega_0(B'), \ A \in \mathfrak{A}(\mathcal{O}_1), B' \in \mathfrak{A}(\mathcal{O}_2'). \tag{1.63}$$

One then defines a unitary $U : \mathcal{H} \to \mathcal{H} \otimes \mathcal{H}$ by

$$UAB'\Omega_\Lambda = A\Omega \otimes B'\Omega. \tag{1.64}$$

The universal localizing map Φ_Λ now maps $\mathcal{B}(\mathcal{H})$ into the algebra $\mathfrak{A}(\mathcal{O}_2)$,

$$\Phi_\Lambda(A) = U^*(A \otimes 1)U, \tag{1.65}$$

in such a way that it acts trivially on $\mathfrak{A}(\mathcal{O}_1)$. We conclude that all operators on the Hilbert space are mapped by this procedure into the larger of the two local algebras. In particular, global charges and momentum operators can be localized in such a way that not only their action on the smaller algebra is preserved but also their spectrum. This is quite a surprise, since an integral over a charge density with a suitable test function would typically have a different spectrum than the global charge operator.

The split property allows to decouple the observables of the smaller region completely from the observables in the spacelike complement of the larger region, in the sense that normal states[2] exist (the product states) for which all correlations between

[2]states induced by density matrices.

these observables vanish. On the contrary, the fact that the local algebras are of type III implies that normal states always are correlated. They are even necessarily entangled, i.e. they cannot be uniformly approximated by convex combinations of product states. This mathematical fact is e.g. responsible for the Hawking effect; in any Hadamard state the algebra of observables outside the horizon is of type III and hence the state must be entangled with observables behind the horizon. (See [54] for a field theoretical derivation of the Hawking effect and, as an alternative, the recently postulated fire walls at the horizon which should deform the algebra into the type I case [3].)

Actually, far from being pathological, the type III_1-property of local algebras has many nice aspects. In particular, one can move by the adjoint action of unitary elements through the state space and finds almost transitivity, in the sense that each orbit is norm dense in the space of normal states. This is exploited in the Buchholz-Roberts analysis of superselection sectors for QED [32] and also in the discussion of transition probabilities in [29]. See also [94] and references therein for further information.

1.6 Conclusions

Algebraic Quantum Field Theory is an approach to quantum field theory which is in its aims essentially equivalent to other approaches, as e.g. the path integral approach or an approach based on canonical quantization of classical field theory. It offers some conceptual advantages compared with other approaches, in particular the separate discussion of observables and states which allows to incorporate the locality principle into the theory. Moreover, it is fully rigorous. In its formulation with C*- and von Neumann algebras the rich mathematical structure can be exploited and leads to an understanding of particle statistics and global gauge symmetries. Moreover, apparent contradictions between nonrelativistic quantum mechanics and relativistic quantum field theory find their natural explanation in the different structures of the occuring algebras of observables. The operator algebraic formulation is, on the other hand, rather rigid, which makes it difficult to deform a given model. Nevertheless, first examples have been obtained which satisfy a weakened form of the axioms. It is, however, also possible to relax the conditions on the operator algebras in order to make contact with the way QFT is treated in other approaches. The concepts from AQFT have turned out to be especially fruitful for the perturbative construction of interacting quantum field theories on curved spacetimes, a problem which could not been solved in other approaches.

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