Recent Results and Open Problems on Conformal Metrics on $\mathbb{R}^n$ with Constant Q-Curvature

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1 Constant Q-Curvature Metrics on $\mathbb{R}^{2m}$ and Their Volumes

We consider solutions to the equation

$$(-\Delta)^m u = (2m-1)!e^{2mu} \quad \text{in } \mathbb{R}^{2m}, \tag{1}$$

satisfying

$$V := \int_{\mathbb{R}^{2m}} e^{2mu(x)}dx < +\infty. \tag{2}$$

Geometrically, if u solves (1)–(2), then the conformal metric $g_u := e^{2u}|dx|^2$ has Q-curvature $Q_{g_u} \equiv (2m-1)!$ and volume V (by $|dx|^2$ we denote the Euclidean metric). For the definition of Q-curvature and related remarks, we refer to [2, Chapter 4] or to [6].

Notice that, up to the transformation $\tilde{u} := u + c$, the constant $(2m-1)!$ in (1) can be changed into any positive number, but it is natural to choose $(2m-1)!$ because it is the Q-curvature of the round sphere S^{2m}. This implies that the function $u_1(x) = \log \frac{2}{1+|x|^2}$, which satisfies $e^{2u_1}|dx|^2 = (\pi^{-1})^* g_{S^{2m}}$, is a solution to (1)–(2) with $V = \mathrm{vol}(S^{2m})$ (here, $\pi : S^{2m} \to \mathbb{R}^{2m}$ is the stereographic projection). Translations and dilations of u_1 (i.e., Möbius transformations) then produce a large

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family of solutions to (1)–(2) with $V = \mathrm{vol}(S^{2m})$, namely

$$u_{x_0,\lambda}(x) := u_1(\lambda(x - x_0)) + \log\lambda = \log\frac{2\lambda}{1 + \lambda^2|x - x_0|^2}, \quad x_0 \in \mathbb{R}^{2m}, \quad \lambda > 0. \tag{3}$$

We shall call the functions $u_{x_0,\lambda}$ *spherical* solutions to (1)–(2).

The question whether the family of spherical solutions in (3) exhausts the set of solutions to (1)–(2) has raised a lot of interest and it is by now well understood. For instance, in dimension 2 we have the following result:

Theorem 1 (Chen-Li [5]) *Every solution to* (1)–(2) *with* $m = 1$ *is spherical.*

On the other hand, for every $m > 1$, i.e., in dimension 4 and higher, it was proven by Chang–Chen [3] that the Problem (1)–(2) admits solutions which are non spherical. More precisely:

Theorem 2 (Chang–Chen [3]) *For every* $m > 1$ *and* $V \in (0, \mathrm{vol}(S^{2m}))$ *there exists a solution to* (1)–(2).

Several authors have given analytical and geometric conditions under which a solution to (1)–(2) is spherical (see [4, 14, 16]), and have studied properties of non-spherical solutions, such as asymptotic behavior, volume and symmetry (see [9, 11, 15]). In particular Lin proved:

Theorem 3 (Lin [9]) *Let* u *solve* (1)–(2) *with* $m = 2$. *Then either* u *is spherical (i.e., as in* (3)*) or* $V < \mathrm{vol}(S^4)$.

Spherical solutions are radially symmetric (i.e., of the form $u(|x - x_0|)$ for some $x_0 \in \mathbb{R}^{2m}$) and the solutions given by Theorem 2 might a priori all be spherically symmetric. The fact that this is not the case was proven by Wei–Ye in dimension 4:

Theorem 4 (Wei–Ye [15]) *For every* $V \in (0, \mathrm{vol}(S^4))$ *there exist (several) non-radial solutions to* (1)–(2) *for* $m = 2$.

Remark 5 As recently shown by A. Hyder [7], the proof of Theorem 4 can be extended to higher dimension $2m \geq 4$, yielding several non-symmetric solutions to (1)–(2) for every $V \in (0, \mathrm{vol}(S^{2m}))$, but failing to produce solutions for $V \geq \mathrm{vol}(S^{2m})$. As in the proof of Theorem 2, the condition $V < \mathrm{vol}(S^{2m})$ plays a crucial role.

Theorems 2–4 and Remark 5 strongly suggest that, also in dimension 6 and higher, all non-spherical solutions to (1)–(2) satisfy $V < \mathrm{vol}(S^{2m})$, i.e., (1)–(2) has no solution for $V > \mathrm{vol}(S^{2m})$ and the only solutions with $V = \mathrm{vol}(S^{2m})$ are the spherical ones. Quite surprisingly it turns out that this is not at all the case. In fact, in dimension 6 there are solutions to (1)–(2) with arbitrarily large V:

Theorem 6 (Martinazzi [13]) *For* $m = 3$ *there exists* $V^* > 0$ *such that for every* $V \geq V^*$ *there is a solution* u *to* (1)–(2)*, i.e., there exists a metric on* $\mathbb{R}^6$ *of the form* $g_u = e^{2u}|dx|^2$ *satisfying* $Q_{g_u} \equiv 5!$ *and* $\mathrm{vol}(g_u) = V$.

The proof of Theorem 6 is based on a ODE argument: one considers radial solutions to (1)–(2), so that (1) reduces to an ODE. Precisely, given $a \in \mathbb{R}$ let $u = u_a(r)$ be the solution of

$$\begin{cases} \Delta^3 u = -120 e^{6u} \quad \text{in } \mathbb{R}^6 \\ u(0) = u'(0) = u'''(0) = u'''''(0) = 0, \quad u''(0) = -a, \quad u''''(0) = 1. \end{cases} \tag{4}$$

Then one shows that

$$\int_{\mathbb{R}^6} e^{6u_a} dx < \infty \text{ for } a \text{ large}, \quad \lim_{a\to\infty} \int_{\mathbb{R}^6} e^{6u_a} dx = \infty.$$

In particular the conformal metric $g_{u_a} = e^{2u_a}|dx|^2$ of constant Q-curvature $Q_{g_{u_a}} \equiv 5!$ satisfies

$$\mathrm{vol}(g_{u_a}) < \infty \text{ for } a \text{ large}, \quad \lim_{a\to\infty} \mathrm{vol}(g_{u_a}) = \infty. \tag{5}$$

Theorem 6 then follows from (5) and the remark that the quantity $\mathrm{vol}(g_{u_a})$ is a continuous function of a when a is sufficiently large (this seems to be false in general if $a > 0$ is not large enough).

The proof of Theorem 2, which is variational and based on the sharpness of Beckner's inequality [1], does not extend to the case $V > \mathrm{vol}(S^{2m})$. On the other hand with the previous ODE approach one can prove that, at least when $m \geq 3$ is odd, Theorem 2 extends as follows.

Theorem 7 (Martinazzi [13]) *Set* $V_m := \frac{(2m)!}{4(m!)^2}\mathrm{vol}(S^{2m}) > \mathrm{vol}(S^{2m})$. *Then, for* $m \geq 3$ *odd and for every* $V \in (0, V_m]$*, there is a non-spherical (but radially symmetric) solution* u *to* (1)–(2)*, i.e., there exists a metric on* $\mathbb{R}^{2m}$ *of the form* $g_u = e^{2u}|dx|^2$ *satisfying* $Q_{g_u} \equiv (2m-1)!$ *and* $\mathrm{vol}(g_u) = V$.

The condition $m \geq 3$ odd is (at least in part) necessary in view of Theorems 1 and 3, but the case $m \geq 4$ even is open. Notice also that, when $m = 3$, Theorems 6 and 7 guarantee the existence of solutions to (1)–(2) for

$$V \in (0, V_m] \cup [V^*, \infty),$$

but do not rule out that $V_m < V^*$ and the existence of solutions to (1)–(2) is unknown for $V \in (V_m, V^*)$. Could there be a gap phenomenon?

We remark that the case m even is more difficult to treat since the ODE corresponding to (1), in analogy with (4), becomes

$$\Delta^m u(r) = (2m-1)! e^{2mu(r)}, \quad r > 0,$$

whose solutions can blow up in finite time (i.e., for finite r) if the initial data are not chosen carefully (contrary to what happens when m is odd).

2 Negative Curvature and Odd Dimension

It is natural to investigate how large the volume of a metric $g_u = e^{2u}|dx|^2$ on $\mathbb{R}^{2m}$ can be, also with constant and negative Q-curvature $Q_{g_u} < 0$. Again with no loss of generality we assume $Q_{g_u} \equiv -1$. In other words, consider the problem

$$(-\Delta)^m u = -e^{2mu} \quad \text{on } \mathbb{R}^{2m}, \tag{6}$$

subject to condition (2). Although for $m = 1$ it is easy to see that Problem (6)–(2) admits no solution for any $V > 0$, when $m \geq 2$ we have

Theorem 8 (Martinazzi [10]) *For any $m \geq 2$ Problem (6)–(2) has solutions for some $V > 0$.*

Using the fixed point argument from [15] and a compactness result from [12], Hyder–Martinazzi recently proved:

Theorem 9 (Hyder–Martinazzi [7]) *For any $m \geq 2$ and any $V > 0$ Problem (6)–(2) has solutions.*

Also the odd-dimensional case is interesting, but more delicate since (1) becomes a non-local equation for $m = (k+1)/2$, $k \in \mathbb{N}$. Building upon previous results from [3, 9, 16], we recently proved the following existence result:

Theorem 10 (Jin–Maalaoui–Martinazzi–Xiong [8]) *Fix $m = 3/2$. For every $V \in (0, 2\pi^2]$, Problem (1)–(2) has a solution (where $(-\Delta)^{\frac{3}{2}}$ needs to be suitably defined). Moreover, if u is a non-spherical solution to (1)–(2), then $V < 2\pi^2 =$ $\mathrm{vol}(S^3)$.*

It is interesting to compare the volume restrictions of Theorems 3 and 10 for dimension 3 and 4, with the results of Theorems 6 and 7 for dimension 6 and higher. It is then natural to ask what does the situation look like in dimension 5, i.e., when $m = 5/2$.

Conjecture 11 Problem (1)–(2) for $m = 5/2$ admits solutions for some values of $V > \mathrm{vol}(S^5)$.

In other words, we conjecture that dimension 5 is similar to dimension 6 more than to dimension 4. The intuition behind this is that the kernel of $(-\Delta)^{5/2}$ contains polynomials of degree 4, just as the kernel of $(-\Delta)^3$, while the kernels of $(-\Delta)^{3/2}$ and $(-\Delta)^2$ contain polynomials of degree 2 but not of degree 4, which is crucial in the proofs of Theorems 3 and 10. On the other hand, we remark that there seems to be no chance to extend the proofs of Theorem 6 to dimension 5, since ODE techniques do not fit well in a non-local framework.

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