Total Curvature of Complete Surfaces in Hyperbolic Space

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We present a Gauss–Bonnet type formula for complete surfaces in n-dimensional hyperbolic space $\mathbb{H}^n$ under some assumptions on their asymptotic behaviour. As in recent results for Euclidean submanifolds (see Dillen–Kühnel [4] and Dutertre [5]), the formula involves an *ideal defect*, i.e., a term involving the geometry of the set of points *at infinity*.

Let S be a complete surface properly embedded in $\mathbb{H}^n$. Assume further that, when we take the Poincaré half-space model of hyperbolic space $\mathbb{H}^n$,

(i) S extends to a compact smoothly embedded surface with boundary $\overline{S} \subset \mathbb{R}^n$,
(ii) $\overline{S}$ meets the ideal boundary $\partial_\infty \mathbb{H}^n = \mathbb{R}^{n-1}$ orthogonally along a curve C.

The second condition guarantees that S is *asymptotically hyperbolic* in the sense that the intrinsic curvature $K_i(x)$ tends to -1 as $x \to \partial_\infty \mathbb{H}^n$. Note also that if S is minimal and fulfills (i), then condition (ii) is also fulfilled.

We are interested in the total *extrinsic* curvature of S, i.e., the integral on the unit normal bundle $N^1 S$ of the Lipschitz–Killing curvature K. Under the above conditions, this converges and

$$\frac{1}{\omega_{n-2}} \int_{N^1 S} K = \int_S (K_i + 1) = \lim_{\varepsilon \to 0} \int_{S_\varepsilon} (K_i + 1),$$

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where ω_k is the volume of the unit ball in $\mathbb{R}^k$, and $S_\varepsilon = \{x \in S\colon x_n \geq \varepsilon\}$, still in the half-space model. By the Gauss–Bonnet theorem, one easily gets

$$\frac{1}{\omega_{n-2}} \int_{N^1 S} K = 2\pi\chi(S) + \lim_{\varepsilon\to 0}\left(A(S_\varepsilon) - \frac{L(C)}{\varepsilon}\right), \tag{1}$$

where A denotes the hyperbolic area, and L is the Euclidean length in the model. The previous limit is the well-known *renormalized area* of S (cf., [1]).

Our first result is a variation of (1), motivated by the Crofton formula which states that the volume of a submanifold (of $\mathbb{S}^n$, $\mathbb{R}^n$ or $\mathbb{H}^n$) equals the integral of the number of intersection points with all totally geodesic planes of complementary dimension.

Proposition 1 ([6]) *For a surface $S \subset \mathbb{H}^n$ satisfying (i) and (ii),*

$$\frac{1}{\omega_{n-2}} \int_{N^1 S} K = 2\pi\chi(S) + \lim_{\varepsilon\to 0}\left(\int_{\mathcal{L}_\varepsilon} \#(S\cap\ell)d\ell - \frac{2\omega_{n-2}}{\omega_{n-1}}\frac{L(C)}{\varepsilon}\right),$$

where $d\ell$ is a (suitably normalized) invariant measure on the space $\mathcal{L}$ of totally geodesic planes $\ell \subset \mathbb{H}^n$ of codimension 2, and $\mathcal{L}_\varepsilon \subset \mathcal{L}$ contains those planes represented in the model by a half-sphere of radius $r \geq \varepsilon$.

Motivated by Banchoff–Pohl's definition of the *area* enclosed by a space curve (see [2]), we introduce the following functional defined on closed curves $C \subset \mathbb{R}^{n-1} \equiv \partial_\infty \mathbb{H}^n$,

$$I(C) := \lim_{\varepsilon\to 0}\left(\int_{\mathcal{L}_\varepsilon} \lambda^2(C,\ell)d\ell - \frac{2\omega_{n-2}}{\omega_{n-1}}\frac{L(C)}{\varepsilon}\right),$$

where $\lambda(C,\ell)$ denotes the linking number between C and the ideal boundary of ℓ.

Combining this definition with Proposition 1 yields the following Gauss–Bonnet type formula.

Theorem 2 ([6]) *For a surface $S \subset \mathbb{H}^n$ satisfying (i) and (ii),*

$$\frac{1}{\omega_{n-2}} \int_{N^1 S} K = 2\pi\chi(S) + \int_{\mathcal{L}} (\#(S\cap\ell) - \lambda^2(C,\ell))d\ell + I(C),$$

where $d\ell$ is an invariant measure on the space $\mathcal{L}$ of totally geodesic planes $\ell \subset \mathbb{H}^n$ of codimension 2.

The equation above involves no limit as $I(C)$ can be represented by

$$I(C) = \frac{2}{\pi}\int_{C\times C} \cos\tau \sin\theta_p \sin\theta_q \frac{dpdq}{|p-q|^2}, \tag{2}$$

where θ_p (resp. θ_q) is the angle between $p-q$ and C at p (resp. at q), and τ denotes the angle between the two planes through p, q tangent at C in p and q respectively.

Theorem 2 shows in particular that $I(C)$ is invariant under Möbius transformations of C. It is interesting to recall another Möbius invariant for closed space curves: the *writhe* (see [3]). It can be expressed as

$$W(C) = \frac{1}{4\pi} \int_{C\times C} \sin\tau \sin\theta_p \sin\theta_q \frac{dpdq}{|p-q|^2}. \tag{3}$$

This suggests that some connection should exist between I and W. For the moment, this is not known.

It would be nice to have integral representations like (2) where the integrand is Möbius invariant (the same applies to (3)). So far, this is only possible for plane curves.

Theorem 3 ([7]) *For a simple closed curve $C \subset \mathbb{R}^2$,*

$$-I(C) = 2\pi + \frac{1}{\pi} \int_{C\times C} \theta \sin\theta \frac{dpdq}{|p-q|^2} \geq 2\pi,$$

where θ is a continuous determination of the angle between the two circles through p, q that are tangent to C at p and q respectively.

It is not hard to see that the integrand above is invariant under the Möbius group.

References

1. S. Alexakis, R. Mazzeo, Renormalized area and properly embedded minimal surfaces in hyperbolic 3-manifolds. Commun. Math. Phys. **297**(3), 621–651 (2010)
2. T. Banchoff, W. Pohl, A generalization of the isoperimetric inequality. J. Diff. Geom. **6**, 175–192 (1971)
3. T. Banchoff, J.H. White, The behavior of the total twist and self-linking number of a closed space curve under inversions. Math. Scand. **36**, 254–262 (1975)
4. F. Dillen, W. Kühnel, Total curvature of complete submanifolds of Euclidean space. Tohoku Math. J. **57**, 171–200 (2005)
5. N. Dutertre, A Gauss–Bonnet formula for closed semi-algebraic sets. Adv. Geom. **8**, 33–51 (2008)
6. J. O'Hara, G. Solanes, Möbius invariant energies and average linking with circles. Tohoku Math. J. **67**(1), 51–82 (2015). arXiv:1010.3764
7. G. Solanes, Total curvature of complete surfaces in hyperbolic space. Adv. Math. **225**, 805–825 (2010)