Trends in Mathematics
Research Perspectives CRM Barcelona Vol.3

Maria del Mar González
Paul C. Yang
Nicola Gambino
Joachim Kock
Editors

Extended Abstracts Fall 2013

Geometrical Analysis

Type Theory, Homotopy Theory and Univalent Foundations

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ISSN 2297-0215 ISSN 2297-024X (electronic)
Trends in Mathematics
ISBN 978-3-319-21283-8 ISBN 978-3-319-21284-5 (eBook)
DOI 10.1007/978-3-319-21284-5

Library of Congress Control Number: 2015955730

Mathematics Subject Classification (2010): First part: 53Axx; Second part: 55Pxx

Springer Cham Heidelberg New York Dordrecht London

Printed on acid-free paper

Springer International Publishing AG Switzerland is part of Springer Science+Business Media (www.birkhauser-science.com)

Contents

Part I
Geometrical Analysis

Editors

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Foreword

In this Part of the present volume of the Birkhauser series *Research Perspectives CRM Barcelona* we present 13 Extended Abstracts corresponding to selected talks given by participants in the Geometric Analysis conference that took place at the Centre de Recerca Matemàtica (CRM) from July 1st to 5th, 2013. This conference was a central part of the Intensive Research Programme on Conformal Geometry and Geometric PDE's that took place at the CRM during the summer of 2013. The results presented in this volume constitute a brief overview of current research in the field of Geometric Analysis. This modern field lies at the intersection of many branches of mathematics (Riemannian, Conformal, Complex or Algebraic Geometry, Calculus of Variations, PDE's, etc.) and relates directly to the physical world since many natural phenomena posses an intrinsic geometric content.

Conformal geometry is the study of the set of angle-preserving (conformal) transformations on a space. While in two dimensions this is precisely the geometry of Riemann surfaces, in dimensions three and above this study opens up many new different subjects, leading to the very wide field named conformal geometry.

The first question is to find conformal invariants or, more specifically, conformally covariant operators, that is, operators which satisfy some invariant property under conformal change of metrics on a manifold, and its associated curvature. The model example is the Laplace–Beltrami operator, in relation to the Yamabe problem. The Yamabe equation is a second order, semilinear PDE; we would like to understand higher order or fully non-linear generalizations, such as the Paneitz operators together with Q-curvature, or the σ_k equation. As a consequence, new interesting directions in PDE's have been opened up, where existence or regularity theory is not developed as much. Lately, there has been a lot of interest in the study of non-local, conformally covariant operators of fractional order constructed from Poincaré–Einstein metrics. While they are natural objects in other areas such as probability, their geometrical meaning is not yet well understood.

Particularly, the study of Poincaré–Einstein metrics has been and continues to be a rich source of activity relating conformal and Riemannian geometry. These are complete Einstein metrics which are asymptotically hyperbolic at infinity. Their boundary at infinity invariantly inherits a conformal structure. The asymptotic behavior of the metric encodes a great deal of information about the conformal structure at infinity, and this has led to new constructions and progress in conformal geometry. On the other hand, there are many analytic problems concerning the existence, uniqueness and regularity of Poincaré–Einstein metrics with a given conformal infinity and plenty of open questions. This topic is stimulated by its role in the AdS/CFT correspondence in Physics.

In CR geometry there are formal similarities with conformal geometry. For example, there are conformally covariant operators analogous to the conformal Laplacian and the Paneitz operators. While these operators also come with associated Q-curvature quantities, their geometric/analytic meaning is quite different from conformal geometry. The analysis of these operators is closely connected with the geometry of the pseudoconvex manifolds which they may bound, hence of interest in several complex variables.

Finally, one of the classical topics in Geometric Analysis is the study of variational problems related to the functional area. In this sense, the global theory of minimal and constant mean curvature surfaces in homogeneous three-manifolds, and more generally in Riemannian and sub-Riemannian manifolds, represents today a tremendously active field of new discoveries and challenges. The local models in sub-Riemannian Geometry are the Carnot groups, with a special role played by the Heisenberg group. Applications of minimal surfaces to other subjects include low dimensional topology, general relativity and materials science. Closely related to this topic appears the isoperimetric problem, connecting Geometric Analysis with Geometric Measure Theory.

The editors would like to thank the support of the CRM in the organization of this research programme. We hope it serves as an inspiration for future directions in the field.

Barcelona, Spain — Maria del Mar González
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A Positive Mass Theorem in Three Dimensional Cauchy–Riemann Geometry

Jih-Hsin Cheng, Andrea Malchiodi, and Paul Yang

In this note we summarize the results from [6] on the positive mass problem in 3-dimensional CR (Cauchy–Riemann) geometry.

We consider a compact three dimensional pseudo-Hermitian manifold (M, J, θ) (with no boundary) of *positive Tanaka–Webster class*. This means that the first eigenvalue of the *conformal sublaplacian*

$$L_b := -4\Delta_b + R$$

is strictly positive. Here, Δ_b stands for the sublaplacian of M, and R for the Tanaka–Webster curvature. The conformal sublaplacian rules the change of the Tanaka–Webster curvature under the conformal deformation $\hat{\theta} = u^2\theta$ through the following formula

$$-4\Delta_b u + Ru = \hat{R}u^3,$$

where $\hat{R}$ is the Tanaka–Webster curvature corresponding to the pseudo-Hermitian structure $(J, \hat{\theta})$. The positivity of the Tanaka–Webster class is equivalent to the

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M. del Mar González et al. (eds.), *Extended Abstracts Fall 2013*,
Trends in Mathematics, DOI 10.1007/978-3-319-21284-5_1

condition

$$\mathcal{Y}(J) := \inf_{\hat{\theta}} \frac{\int_M R_{J,\hat{\theta}}\ \hat{\theta} \wedge d\hat{\theta}}{\left(\int_M \hat{\theta} \wedge d\hat{\theta}\right)^{\frac{1}{2}}} > 0, \tag{1}$$

where $\hat{\theta}$ is any contact form. Under the assumption $\mathcal{Y}(J) > 0$ we have that L_b is invertible so, for any $p \in M$, there exists a Green's function G_p for which

$$(-4\Delta_b + R)\, G_p = 16\delta_p.$$

One can show that in CR normal coordinates (z, t), G_p admits the following expansion

$$G_p = \frac{1}{2\pi}\rho^{-2} + A + O(\rho),$$

where A is some real constant and where we have set $\rho^4(z, t) = |z|^4 + t^2$, $z \in \mathbb{C}$, $t \in \mathbb{R}$. Having in mind the Riemannian construction for the blow-up of a compact manifold, we consider the new pseudo-Hermitian manifold with a blow-up of contact form

$$N = \big(M \setminus \{p\},\, J,\, \theta = G_p^2\hat{\theta}\,\big), \tag{2}$$

where $\hat{\theta}$ is chosen so that, near p, it has the behavior described in [6, Proposition 6.5]. With an *inversion of coordinates*, we then obtain a pseudo-Hermitian manifold which has asymptotically the geometry of the Heisenberg group. Starting from this model, we give a definition of asymptotically flat pseudo-Hermitian manifold and we introduce its *pseudo-Hermitian mass* (p-mass) by the formula

$$m(J, \theta) := i \oint_{\infty} \omega_1^1 \wedge \theta := \lim_{\Lambda \to +\infty} i \oint_{S_\Lambda} \omega_1^1 \wedge \theta,$$

$$\left(m(J, \theta) := \lim_{\Lambda \to +\infty} ni \oint_{S_\Lambda} \sum_{\alpha=1}^{n} \omega_\alpha^\alpha \wedge \theta \wedge (d\theta)^{n-1} \ \text{ for } N \text{ of dimension } \ 2n+1 \right)$$

where we have set $S_\Lambda = \{\rho = \Lambda\}$, $\rho^4 = |z|^4 + t^2$, and where ω_1^1 stands for the connection form of the structure. The above quantity is indeed a natural candidate, since it satisfies a property analogous to the Riemannian case:

$$\frac{d}{ds}_{|s=0} \left(-\int_N R(s)\, \theta \wedge d\theta + m(J(s), \theta) \right) = \int_N \big(A_{11}E_{\overline{11}} + A_{\overline{11}}E_{11}\big)\, \theta \wedge d\theta,$$

where R(s) is the Tanaka–Webster curvature corresponding to $(J(s),\ \theta)$, A_{11} is the torsion, and

$$\frac{d}{ds}_{|s=0} J(s) = 2E = 2E_{11}\theta^1 \otimes Z_{\bar{1}} + 2E_{\bar{1}\bar{1}}\theta^{\bar{1}} \otimes Z_1.$$

Moreover, it coincides with the zero-th order term in the expansion of the Green's function for L_b:

$$m(J,\ \theta) = 48\pi^2 A.$$

We prove an integral formula for the p-mass in the spirit of [15]. To state this formula we need to introduce another conformally covariant operator, the CR Paneitz operator

$$P\varphi := 4\left(\varphi_{,\bar{1}}{}^{\bar{1}}{}_1 + iA_{11}\varphi^{,1}\right)^{,1}.$$

The operator P satisfies the covariance property

$$P_{(J,\hat{\theta})} = u^{-4}P_{(J,\theta)}, \qquad \hat{\theta} = u^2\theta, \tag{3}$$

see [10]. We prove then the following integral formula, which holds for an asymptotically flat pseudo-Hermitian manifold N:

$$\frac{2}{3}m(J,\theta) = -\int_N \left|\Box_b\beta\right|^2\theta\wedge d\theta + 2\int_N \left|\beta_{,\bar{1}\bar{1}}\right|^2\theta\wedge d\theta + 2\int_N R\left|\beta_{,\bar{1}}\right|^2\theta\wedge d\theta$$
$$+\frac{1}{2}\int_N \overline{\beta}P\beta\,\theta\wedge d\theta. \tag{4}$$

Here, $\beta\colon N\to\mathbb{C}$ is a function satisfying

$$\beta = \bar{z} + \beta_{-1} + O(\rho^{-2+\varepsilon}) \quad \text{near } \infty, \qquad \Box_b\beta = O(\rho^{-4}),$$

with $\Box_b = -2\beta_{,\bar{1}1}$ and with β_{-1} being a suitable function with homogeneity -1 in ρ.

In the following main theorem we give some general conditions which ensure the non-negativity of the p-mass for blow-ups of compact manifolds, characterizing also the zero case as (CR equivalent to) the standard three dimensional CR sphere.

Theorem 1 (Theorem 1.1 from [6]) *Let M be a smooth, strictly pseudoconvex three dimensional compact CR manifold. Suppose $\mathcal{Y}(J) > 0$, and that the CR Paneitz operator is non-negative. Let $p \in M$ and let θ be a blow-up of contact* *form as in* (2)*. Then,*

(i) $m(J, \theta) \geq 0$*;*
(ii) if $m(J, \theta) = 0$*, M is CR equivalent (or, together with* $\hat{\theta}$*, isomorphic as pseudo-Hermitian manifold) to* S^3 *endowed with its standard CR structure (and its standard contact form).*

The assumptions we give here are conformally invariant, and are needed to ensure the positivity of the right-hand side in (4). By the result in [3], the conditions on $\mathcal{Y}(J)$ and P imply the embeddability of M. We use this property to find a solution of $\Box_b \beta = 0$ with the above asymptotics (and hence, to force the first term in the right-hand side of (4) to vanish): we first find an approximate solution through the expansion of $\Box_b \bar{z}$ at infinity, and then through the analysis of the Szegö projection of this quantity. To obtain the full solvability of $\Box_b \beta = 0$ we then employ a mapping theorem in weighted spaces from [11]. The positivity of the CR Paneitz operator is used instead to control the last term in the right-hand side of (4), showing that it is the sum of a non-negative term and a (negative) multiple of $m(J, \theta)$ which can be reabsorbed into the left-hand side. As a matter of fact, non-negativity of the CR Paneitz operator is preserved under embedded analytic deformations [4].

Theorem 2 (Corollary 1.1 from [6]) *Let M be a smooth, strictly pseudoconvex three dimensional compact CR manifold. Suppose M is an embedded, small enough, analytic deformation of the standard CR three sphere. Let* $p \in M$ *and let* θ *be a blow-up of contact form as in* (2)*. Then, the same conclusions as in the previous theorem hold.*

We construct some examples of structures using the deformation formulas. First, using second variation formulas, we consider perturbations of the spherical structure for which P fails to be non-negative (see also [3]). Then, we derive the first and second variations of the mass near the standard sphere. We also construct examples of manifolds with positive Tanaka–Webster class and negative mass (when the blow-up is done at suitable points). This is in striking contrast with respect to the Riemannian case, where all perturbations of the sphere give rise to blown-up manifolds with positive mass (except for metrics conformally equivalent to the spherical one). We also describe an example of CR structure on $S^2 \times S^1$ with non-negative Paneitz operator and non-vanishing torsion, obtained as a quotient of $\mathbb{H}^1 \setminus \{0\}$.

Our next main goal is to apply Theorem 1 to the study of the CR Yamabe problem, namely finding conformal changes of contact form in order to obtain constant Tanaka–Webster curvature. As for the classical Yamabe problem, the cases $\mathcal{Y}(J) \leq 0$ are more directly treatable (see [7]), while the case $\mathcal{Y}(J) > 0$ is the most difficult one. Calling $\mathcal{Y}_0$ the quotient for the standard CR three sphere, by a result in [12] one always has

$$\mathcal{Y}(J) \leq \mathcal{Y}_0, \tag{5}$$

and, if a strict inequality holds, then the problem is solvable. The strict inequality is needed to ensure compactness of the minimizing sequences in (1). This condition was verified in [13] for (real) dimension greater than or equal to 5, and for non-spherical structures, in the spirit of [1] through some expansions involving the local geometry.

The positivity of the mass is instead a more global property, and it enters when G_p has the following expansion near p

$$G_p = c_n \rho^{-2n} + A + O(\rho), \tag{6}$$

where ρ is the Heisenberg distance in CR normal coordinates. It turns out that the term A is a multiple of the mass defined for the blow-up M. We observe that (6) holds for $n = 1$ (dimension 3 case) and for N being spherical of all dimensions.

For such manifolds of dimension greater than or equal to 5 (plus some extra technical condition in dimension 5) with positive CR Yamabe or Tanaka–Webster class, one can prove a positive mass theorem for A (and hence find solutions to the CR Yamabe problem with minimal energy) through another approach [5].

Our next result gives the strict Webster–Sobolev inequality in the three dimensional case (the only one left), if M is not CR equivalent to the standard CR three-sphere, under the same assumptions as in the previous theorem.

Theorem 3 (Theorem 1.2 in [6]) *Suppose we are under the assumptions of Theorem 1. Then, either M is the standard CR three sphere or, if M is not CR equivalent to the standard CR three sphere, one has $\mathcal{Y}(J) < \mathcal{Y}_0$. In both cases, the Tanaka–Webster quotient admits a smooth minimizer.*

The CR Yamabe problem for the case of three-dimensional CR manifolds and for spherical CR manifolds was solved in [8] and [9], respectively (we also refer to [5, 7]). While the proof in these papers relies on topological arguments and may not provide energy extremals, in the spirit of [2], our argument is based on direct minimization and gives an extra variational characterization on the solutions. To prove strict inequality we follow Schoen's argument in [14], finding test functions which resemble a CR bubble at a small scale, and the Green's function G_p at a larger one. More in general, the analysis of the Yamabe problem in the CR case has been so far less precise than the Riemannian case: for example a basic difficulty is the lack of a moving plane method, which is useful in general to derive a priori estimates and to classify entire solutions.

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On the Rigidity of Gradient Ricci Solitons

Manuel Fernández-López and Eduardo García-Río

A complete Riemannian manifold (M, g) is said to be a *gradient Ricci soliton* if there exists a smooth function $f: M \to \mathbb{R}$ such that

$$Rc + H_f = \lambda g, \tag{1}$$

where Rc denotes the Ricci tensor, H_f is the Hessian of the function f, and λ is a real number. The function f is called a *potential function* of the gradient Ricci soliton. For $\lambda > 0$ the Ricci soliton is *shrinking*, for $\lambda = 0$ it is *steady*, and for $\lambda < 0$ it is *expanding*.

In recent years a lot of papers devoted to the study of gradient Ricci solitons have been published. Gradient Ricci solitons are natural extensions of Einstein metrics (if f is constant the Ricci soliton is just Einstein). They are special solutions of the Ricci flow, which are called self-similar solutions. If (M, g_0) (with some potential function f) is a gradient Ricci soliton then the metric evolves along the Ricci flow $\frac{\partial}{\partial t} g(t) = -2Rc(g(t))$, as $g(t) = (1 - 2\lambda t)\varphi_t^*(g_0)$, where φ_t is the 1-family of diffeomorphisms generated by $\nabla f/(1-2\lambda t)$. Moreover, gradient Ricci solitons often arise as singularity models for the Ricci flow. Therefore, it is important to classify gradient Ricci solitons or to understand their geometry. We refer to [3, 8] and the references therein for background on Ricci solitons.

The two most basic examples of gradient Ricci solitons are the Einstein metrics, and the Gaussian soliton when one considers the potential function $f(x) = \lambda|x|^2/2$ on $\mathbb{R}^n$ and the product of an Einstein manifold and a Gaussian soliton with the same

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M. del Mar González et al. (eds.), *Extended Abstracts Fall 2013*,
Trends in Mathematics, DOI 10.1007/978-3-319-21284-5_2

constant. Generalizing these examples, Petersen–Wylie [18] introduced the notion of *rigidity* of gradient Ricci solitons. A Ricci soliton is said to be *rigid* if it is of the form $N \times_\Gamma \mathbb{R}^k$, where N is an Einstein manifold and Γ acts freely on N and by orthogonal transformations on $\mathbb{R}^k$.

1 Classification of Shrinking Gradient Ricci Solitons

Hamilton [11] for $n = 2$, and Ivey [12] for $n = 3$, proved that shrinking solitons have constant sectional curvature. Perelman [17] proved that a non-flat gradient shrinking Ricci soliton with bounded and nonnegative sectional curvature, κ-noncollapsed on all scales for some $\kappa > 0$, must be $\mathbb{S}^3$, $\mathbb{R}\times\mathbb{S}^2$ or a quotient thereof. Ni–Wallach [16] showed that 3-dimensional gradient shrinking Ricci solitons with nonnegative Ricci curvature must be quotients of $\mathbb{R} \times \mathbb{S}^2$ provided that the norm of the curvature tensor has at most exponential growth. Cao–Chen–Zhu [5], obtained the previous result without any assumption on the curvature of the 3-dimensional gradient shrinking Ricci soliton.

The classification of complete locally conformally flat gradient shrinking Ricci solitons has been finally achieved as a result of several works. The compact case was settled by Eminenti–La Nave–Mategazza in [9] who showed that the only possibilities are the standard sphere or one of its quotients. Ni–Wallach [16] showed that a complete locally conformally flat gradient shrinking soliton must be $\mathbb{S}^n$, $\mathbb{R}^n$, $\mathbb{R} \times \mathbb{S}^{n-1}$ or one of their quotients, assuming nonnegative Ricci curvature and at most exponential growth of the norm of the curvature tensor. Cao–Wang–Zhang [6] relaxed the assumption on the Ricci curvature and assumed only that it is bounded from below. Petersen–Wylie [19] got the same result by using a different assumption, indeed $\int_M |Rc|^2 e^{-f} < \infty$, where f is any potential function of the gradient shrinking Ricci soliton. Zhang [20] obtained a classification of all locally conformally flat gradient shrinking Ricci solitons, showing that they have nonnegative curvature operator and the growth of its norm is at most exponential. Munteanu–Sesum [14] gave a different proof, proving that the integral inequality $\int_M |Rc|^2 e^{-f} < \infty$ holds for shrinking gradient Ricci solitons with vanishing Weyl tensor.

Since gradient Ricci solitons are generalizations of Einstein metrics and Einstein metrics have harmonic Weyl tensor, it is natural to study gradient Ricci solitons with harmonic Weyl tensor. We have the following result.

Theorem 1 *Let (M^n, g) be a n-dimensional complete gradient shrinking Ricci soliton whose curvature tensor has at most exponential growth and having Ricci tensor bounded from below. Then (M, g) is rigid if and only if it has harmonic Weyl tensor.*

Note that in the compact case the additional assumptions are not necessary.

It was shown by Munteanu–Sesum [14] that any gradient shrinking Ricci soliton with harmonic Weyl tensor satisfies the integral identity

$$\int_M |\mathrm{div} R|^2 e^{-f} = \int_M |\nabla Ric|^2 e^{-f} < \infty,$$

without assuming any additional assumption (as we did in Theorem 1). Henceforth, [14] shows that the previous result can be stated in the more general situation as: "*a complete gradient shrinking soliton is rigid if and only if the Weyl tensor is harmonic*".

Naber [15] showed that any 4-dimensional complete noncompact shrinking Ricci soliton with bounded nonnegative curvature operator is a finite quotient of either $\mathbb{R}^4$, $\mathbb{S}^3 \times \mathbb{R}$, or $\mathbb{S}^2 \times \mathbb{R}^2$. Kotschwar [13] obtained that a complete rotationally symmetric shriking Ricci soliton is isometric to $\mathbb{S}^n$, $\mathbb{R}^n$, or $\mathbb{S}^{n-1} \times \mathbb{R}$. Chen–Wang [7] obtained that a 4-dimensional gradient shrinking Ricci soliton is a finite quotient of $\mathbb{R}^4$, $\mathbb{S}^4$ or $\mathbb{S}^3 \times \mathbb{R}$, if it is anti-self-dual. This was generalized by Cao–Chen [3] who proved that any 4-dimensional Bach-flat gradient shrinking Ricci soliton is either Einstein, or locally conformally flat and hence, a finite quotient of the Gaussian shrinking soliton $\mathbb{R}^4$ or the round cylinder $\mathbb{S}^3 \times \mathbb{R}$. (Note that any anti-self-dual 4-dimensional manifold is Bach-flat.) For $n > 4$, they obtained that a Bach-flat gradient shrinking Ricci soliton is either Einstein, or a finite quotient of the Gaussian shrinking soliton R^n, or the product $N^{n-1} \times \mathbb{R}$ where N^{n-1} is Einstein.

2 Classification of Steady Gradient Ricci Solitons

In the compact case steady and expanding Ricci solitons are necessarily Einstein. The classification of complete noncompact steady gradient Ricci solitons is also possible in the following cases: assuming that the soliton is locally conformally flat [3]; 4-dimensional and anti-self-dual [7]; Bach-flat with $n \geq 4$, positive Ricci curvature, and scalar curvature attaining its maximum or 3-dimensional with divergence-free Bach tensor [4]; and if it is 3-dimensional, non-flat, and κ-noncollapsed [1] or if it has dimension $n \geq 4$, positive curvature and is assymptotically cilindrical [2]. In all cases one gets that the soliton must be isometric to the Bryant soliton, if non-flat. Bryant showed that, for all $n \geq 3$, there exists a rotationally symmetric steady gradient Ricci soliton on $\mathbb{R}^n$, which is unique up to scaling; it is known as the *Bryant soliton*. It has positive sectional curvature, linear curvature decay, and volume growth of geodesic balls of radius r of the order $r^{(n+1)/2}$.

We only consider the locally conformally flat case. We present a unified and more direct approach towards the classification of complete locally conformally flat gradient Ricci solitons (see [10]). First of all we show the following

Proposition 2 *Let (M^n, g) be a n-dimensional locally conformally flat non-trivial gradient Ricci soliton of dimension n. Then, it is locally (where $\nabla f \neq 0$) isometric to a warped product $(M, g) = ((a, b) \times N, dt^2 + \psi(t)^2 g_N)$, where (N, g_N) is a Riemannian manifold of constant sectional curvature $1, 0$ or -1.*

Our method deals in a unified way the cases $n = 3$ and $n \geq 4$. We assume that the Weyl tensor vanishes (when $n \geq 4$ the manifold is locally conformally flat) and that the Schouten tensor is a Codazzi tensor (when $n = 3$ the manifold is locally conformally flat) simultaneously. Based on the previous result we get that, in the local decomposition as a warped product, N has to be of positive constant curvature, if the soliton is not Ricci-flat. From this, we obtain that the soliton has to be rotationally symmetric and the classification follows from [13] in the shrinking case, and from [3] in the steady case.

Remark 3 We also get that a complete locally conformally flat gradient expanding Ricci soliton with nonnegative curvature operator is also rotationally symmetric. However, the classification of expanding gradient Ricci solitons is much more unclear, because there exists several non-trivial rotationally symmetric expanding gradient Ricci solitons.

3 Other Results

Petersen–Wylie [18] obtained the following characterization of the rigidity of gradient Ricci solitons: a complete gradient Ricci soliton is rigid if and only if it has constant scalar curvature and is radially flat, that is, the sectional curvature of a plane containing ∇f vanishes ($\sec(V, \nabla f) = 0$). Moreover, they obtained a number of weaker conditions guaranteing radial flatness and constant scalar curvature:

(i) the scalar curvature is constant and $\sec(V, \nabla f) \geq 0$, $\forall\, V \in \mathfrak{X}(M)$;
(ii) the scalar curvature is constant, and $0 \leq Rc \leq \lambda$ in the shrinking case or $\lambda \leq Rc \leq 0$ in the expanding case;
(iii) the curvature tensor is harmonic;
(iv) $\sec(V, \nabla f) = 0$, and $Rc \geq 0$ in the shrinking case or $Rc \leq 0$ in the expanding case.

In the same line of the previous result we have the following

Theorem 4 *Let (M, g) be a complete gradient Ricci soliton with constant scalar curvature. Then, the soliton is rigid if one of the following hypothesis holds:*

(i) the principal curvatures of the Ricci tensor are constant;
(ii) (M, g) is curvature homogeneous;
(iii) the rank of the Ricci tensor is constant.

Acknowledgements Supported by projects MTM2009-07756 and INCITE09 207 151 PR (Spain).

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Geometric Structures Modeled on Affine Hypersurfaces and Generalizations of the Einstein–Weyl and Affine Sphere Equations

Daniel J.F. Fox

Affine hypersurface structures (*AH structures*) simultaneously generalize Weyl structures and abstract geometric structures induced on a nondegenerate co-oriented hypersurface in flat affine space. The aim of this note is to define equations for AH structures, called *Einstein* which, for Weyl structures, specialize to the usual Einstein Weyl equations, and, in the case of the AH structure induced on a hypersurface in flat affine space, recover the equations for affine spheres. Additionally, we indicate the simplest constructions of Einstein AH structures that do not arise in either of these manners.

A Weyl structure on a n-manifold M comprises a conformal structure $[h]$ and a torsion-free connection ∇ such that $\nabla_i H_{jk} = 0$, where $[h]$ is identified with the weighted tensor $H_{ij} = |\det h|^{-1/n} h_{ij}$. Equivalently, for every $h \in [h]$, there is a one-form $\gamma_i \in \Gamma(T^*M)$ such that $\nabla_i h_{jk} = 2\gamma_i h_{jk}$. When $n > 2$, the Einstein equations for a Weyl structure demand that the symmetric trace-free Ricci tensor $R_{(ij)} - \frac{1}{n} R h_{ij}$ vanishes, where $R = h^{ij} R_{ij}$ and h^{ij} is the inverse of h_{ij}. Note that the Ricci tensor need not be symmetric. Its skew symmetric part is given by $2R_{[ij]} = -nF_{ij} = nd\gamma_{ij}$ and does not depend on the choice of $h \in [h]$. When $n = 2$ the Einstein–Weyl equations, like the usual metric Einstein equations, are vacuous. For the usual Einstein equations the traced differential Bianchi identity implies that the scalar curvature is constant, and constant scalar curvature can be regarded as the 2-dimensional analogue of the Einstein condition. Likewise, for the Einstein–Weyl equations, it follows from the Bianchi identities that $\nabla_i R + n\nabla^p F_{ip} = 0$, where indices are raised and lowered using H_{ij} and its inverse H^{ij}, and where $R = H^{ij} R_{ij}$. Calderbank [1, 2] proposed this equation as the definition of Einstein–Weyl structures in two dimensions, and constructed solutions (see also [4]). This

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M. del Mar González et al. (eds.), *Extended Abstracts Fall 2013*,
Trends in Mathematics, DOI 10.1007/978-3-319-21284-5_3

point of view was important for identifying an appropriate notion of Einstein AH structure.

Formally, AH structures can be defined by relaxing the compatibility condition defining Weyl structures. First, however, it is convenient to change the perspective slightly. Let the pair $([\nabla], [h])$ comprise a projective structure, $[\nabla]$, meaning an equivalence class of torsion-free connections having the same unparameterized geodesics, and a conformal structure $[h]$. There is a unique *aligned* representative $\nabla \in [\nabla]$ distinguished by the requirement that $\nabla_i H_{jk}$ be completely trace-free. The pair $([\nabla], [h])$ is an AH structure if $\nabla_i H_{jk}$ is completely symmetric, that is $\nabla_{[i} H_{j]k} = 0$. Equivalently, for every $h \in [h]$, there is $\gamma_i \in \Gamma(T^*M)$ such that $\nabla_{[i} h_{j]k} = \gamma_{[i} h_{j]k}$. The *cubic torsion* of the AH structure is $\mathcal{L}_{ij}{}^k = H^{kp}\nabla_i H_{jp}$. The connection $\bar{\nabla} = \nabla + \mathcal{L}_{ij}{}^k$ is the aligned representative of the *conjugate* AH structure $([\bar{\nabla}], [h])$. Its cubic torsion is $-\mathcal{L}_{ij}{}^k$. Conjugacy is an involution on the space of AH structures and its fixed points are exactly Weyl structures.

The second fundamental form Π of an immersed hypersurface M in an $(n+1)$-manifold N with connection $\mathbb{D}$ is the symmetric normal-bundle valued tensor defined by taking $\Pi(X, Y)$ to be the projection onto the normal bundle of $\mathbb{D}_X Y$, where X and Y are tangent to M. Since the difference tensor of projectively equivalent connections has the form $2\sigma_{(i}\delta_{j)}{}^k$ for some $\sigma_i \in \Gamma(T^*M)$, Π depends only on the projective equivalence class $[\mathbb{D}]$, and not on $\mathbb{D}$ itself. The immersion is *nondegenerate* if Π is. When the target $(N, \mathbb{D})$ is a flat affine space, an equivalent condition is for the Gauss map to the projectivization of the dual vector space, assigning to $p \in M$ the annihilator of the tangent space T_pM, to be an immersion. In this case the pullback via the Gauss map of the flat projective structure on projective space yields a flat projective structure $[\bar{\nabla}]$ on M. Together with a co-orientation of M, meaning an orientation of its normal bundle, the second fundamental form determines a conformal structure on M. A vector field W transverse to M determines an induced connection ∇, a metric h_{ij} representing Π, a shape operator $S_i{}^j$, and a one-form τ_i by the usual formulas, $\mathbb{D}_X Y = \nabla_X Y + h(X, Y)W$ and $\mathbb{D}_X W = -S(X) + \tau(X)W$. Here, X and Y are tangential to M and $\nabla_X Y$ and $-S(X)$ are the tangential parts of $\mathbb{D}_X Y$ and $\mathbb{D}_X W$, respectively. With $\tilde{W} + f(W + Z)$ in place of W, the induced $\tilde{\nabla}$ and $\tilde{h}$ are given by $\tilde{\nabla}_X Y = \nabla_X Y - h(X, Y)Z$, and $\tilde{h}_{ij} = f^{-1}h_{ij}$. In particular, h generates the conformal structure induced by Π and the given co-orientation. Allowing projective changes of $\mathbb{D}$ and arbitrary changes of W, the equivalence class $\{\nabla\} = \{\nabla + 2\alpha_{(i}\delta_{j)}{}^k - \beta^k h_{ij}\}$ generated by any induced connection depends neither on the choice of transversal X nor on the choice of $\mathbb{D}$ within its projective equivalence class. This class $\{\nabla\}$ is the *conformal projective* equivalence class of ∇. Observe that $2\nabla_{[i} h_{j]k} = -2\tau_{[i} h_{j]k}$, so that any induced connection ∇ generates with $[h]$ an AH structure. The pair $(\{\nabla\}, [h])$ is called a *Codazzi projective structure*, and should be viewed as a generalization of the notion of conformal structure. In this analogy, the AH structures generating a Codazzi projective structure correspond to the individual metrics representing a conformal structure; in particular, the difference tensor of the aligned representatives of two AH structures generating the same conformal projective structure has the form $2\alpha_{(i}\delta_{j)}{}^k - h_{ij}h^{kp}\alpha_p$ of the difference tensor of the Levi–Civita connections of conformal metrics. The induced ∇ is in general not the aligned representative of the AH structure $([\nabla], [h])$ it generates. In fact, there is a unique choice of transverse direction such that this is the case, and this choice is the *affine normal direction*. A distinguished affine normal vector field can be selected by requiring that the volume density induced by the corresponding metric h coincides with that induced from some volume form on the ambient space parallel with respect to a particular ambient connection $\mathbb{D}$. The AH structure conjugate to the AH structure $([\nabla], [h])$ induced via the affine normal is $([\bar{\nabla}], [h])$, where $[\bar{\nabla}]$ is the flat projective structure induced via the conormal Gauss map. In summary, a nondegenerate hypersurface in a projectively flat space carries a conformal projective structure which admits a distinguished subordinate AH structure, that determined by the affine normal, for which the conjugate AH structure is the projectively flat one induced via the conormal Gauss map.

The curvature $R_{ijkl} = R_{ijk}{}^{p}H_{pl}$ of an AH structure means the curvature of its aligned representative ∇. There are three principles useful in understanding the curvature. It can be decomposed by symmetries, it can be decomposed into its self-conjugate and anti-self-conjugate parts, and its pieces can be isolated depending only on the underlying conformal projective equivalence class. These three points of view lead to essentially the same tensors, which are now briefly summarized. There are two possible rank two traces of R_{ijkl}, namely the ordinary Ricci trace $R_{ij} = R_{pij}{}^{p}$, and the trace $R_{ip}{}^{p}{}_{l}$. All further traces lead to a multiple of the weighted scalar curvature $R = H^{ij}R_{ij}$, and the trace-free symmetric Ricci tensor and the trace-free symmetric conjugate Ricci tensor span the space of trace-free rank two traces. The Weyl curvature is the completely trace-free part of R_{ijkl}. It decomposes as $W_{ijkl} = A_{ijkl} + E_{ijkl}$, where the self-conjugate Weyl tensor A_{ijkl} has the symmetries of a metric curvature tensor and the anti-self-conjugate Weyl tensor E_{ijkl} has the symmetries of a symplectic curvature tensor. These two tensors are invariant under conformal projective equivalence. There are corresponding self-conjugate and anti-self-conjugate Cotton tensors, which are invariant, respectively, when A_{ijkl} or E_{ijkl} vanishes. In four dimensions, when the anti-self-conjugate Weyl and Cotton tensors vanish, there is a Bach tensor that directly generalizes the Bach tensor of a Weyl structure. A key role in understanding appropriate generalizations of the Einstein condition is played by the conformal projectively invariant one-form A_i defined by

$$\mathcal{L}^{abc}E_{iabc} = 2(2-n)A_i = (n-2)\big(\nabla^{p}F_{ip} + \tfrac{1}{n}\nabla_i R + 2\nabla^{p}\{A\}_{ip} - \mathcal{L}_i{}^{pq}\{W\}_{pq}\big), \tag{1}$$

where the brackets $\{\cdot\}$ indicate the trace-free symmetric part, and A_{ij} and W_{ij} are the self-conjugate and full Schouten tensors, respectively, which are certain linear combinations of the symmetric Ricci and conjugate Ricci tensors and of RH_{ij}. An AH structure is *conservative* if $A_i = 0$. The *naive Einstein* equations require the vanishing of the trace-free symmetric parts of the Ricci and conjugate Ricci tensors. However, these conditions are inadequate to generate (via the Bianchi identities) anything like the constancy of the scalar curvature, and appear too flabby to give rise to a good theory. In the presence of the naive Einstein equation, (1) gives

$\mathcal{L}^{abc}E_{iabc} = 2(2-n)A_i = (n-2)(\nabla^p F_{ip} + \frac{1}{n}\nabla_i R)$. The vanishing of this expression is a consequence of the Einstein–Weyl equations that can be regarded as generalizing constancy of the scalar curvature, and was shown by Calderbank to give a good notion of Einstein–Weyl equations in two dimensions. Coupled with the conformal projective invariance of A_i, this suggests defining an AH structure to be *Einstein* if it is naive Einstein and conservative. By (1), an AH structure with vanishing anti-self-conjugate Weyl tensor is conservative. Coupled with the possibility of constructing a good generalization of the Bach tensor when the anti-self-conjugate Weyl and Cotton tensors vanish, this suggests considering the stronger conditions of the naive Einstein equations plus the vanishing of the anti-self-conjugate Weyl tensor, and possibly also of the anti-self-conjugate Cotton tensor. While it is particularly interesting to construct Einstein AH structures satisfying these stronger conditions, it is not yet clear to what extent they should be regarded as part of the Einstein condition.

By definition, an AH structure is Einstein if and only if its conjugate is Einstein. An affine hypersurface is an *affine sphere* if its affine normals meet in a point or are all parallel. Equivalently, its shape operator is a multiple of the identity. The AH structures induced on a nondegenerate affine hypersurface are Einstein if and only if the hypersurface is an affine sphere. Since the anti-self-conjugate Weyl and Cotton tensors vanish for a conjugate projectively flat AH structure such as that induced via the affine normal, these AH structures automatically satisfy the stronger conditions discussed in the previous paragraph. By a theorem of Cheng and Yau, the interior of a sharp convex cone is foliated in a unique way by hyperbolic affine spheres asymptotic to the boundary of the cone. When this theorem is applied to the cone over the universal cover of a convex flat real projective manifold M, the equiaffine metrics of the affine spheres yield a canonical homothety class of metrics which, together with the given flat projective structure, generates an Einstein AH structure on M. Since convex flat real projective manifolds abound (see [5]), this provides many Einstein AH structures. If these were the only examples of Einstein AH structures there would be no point in introducing the formalism described here. The simplest example of an Einstein AH structure that is neither Weyl nor projectively nor conjugate projectively flat is the following. Let $G = SU(n)$ and define on the Lie algebra $\mathfrak{su}(n)$, regarded as skew-Hermitian matrices, the one-parameter family of commutative, nonassociative multiplications

$$X \circ Y = t\mathrm{i}(XY + YX - \tfrac{2}{n}\,\mathrm{tr}(XY)I).$$

Let h be the bi-invariant Riemannian metric determined by the negative of the Killing form, let D be its Levi–Civita connection, and define a bi-invariant torsion-free connection ∇ by $\nabla_X Y = D_X Y + \frac{1}{2}X \circ Y$. Then $([\nabla], [h])$ is an Einstein AH structure with self-conjugate Weyl and Cotton tensors. Many exact Einstein AH structures $([\nabla], [h])$, for which a representative metric $h \in [h]$ having vanishing γ is flat, can be constructed as follows. Let D be the Levi–Civita connection of the Euclidean metric h on $\mathbb{R}^n$, and let P be a harmonic homogeneous cubic polynomial such that the square of the norm of the Hessian of P is a nonzero constant multiple of the quadratic form corresponding to the Euclidean metric. Then $\nabla = D - P_{ijp}P^{kp}$, where P^{ij} is the bivector dual to the Hessian P_{ij} of P, generates with $[h]$ an exact Einstein AH structure with self-conjugate curvature. In all dimensions $n > 3$ there are P for which the resulting Einstein AH structure is neither projectively flat nor conjugate projectively flat. The simplest examples are the following. A Steiner triple system is a collection $\mathscr{B}$ of 3 element subsets of $\bar{n} = \{1, \dots, n\}$ such that every two element subset of $\bar{n}$ is contained in exactly one $B \in \mathscr{B}$. For $I = abc \in \mathscr{B}$ let $x_I = x_a x_b x_c$. For any choice of $\epsilon_I \in \{\pm 1\}$ the polynomial $P_{\mathscr{B},\epsilon}(x) = \sum_{I \in \mathscr{B}} \epsilon_I x_I$ has the desired properties. The simplest nontrivial example is the seven element Fano projective plane $\mathscr{B} = \{123, 145, 167, 246, 257, 347, 356\}$, yielding

$$P(x) = x_1x_2x_3 + x_1x_4x_5 + x_1x_6x_7 + x_2x_4x_6 + x_2x_5x_7 + x_3x_5x_6 + x_3x_4x_7. \qquad (2)$$

Other polynomials with the desired properties include the Cartan cubic isoparametric polynomials, the cubic forms defining the multiplications on the Nahm algebras of compact simple Lie algebras (see [6]), and the cubic form of the Griess algebra preserved by the monster finite simple group. While much more can be said, including general structural statements involving conditions on curvatures, it is out of the scope of the present note to do more than introducing the admittedly complicated formalism. The preceding examples show that the formalism admits solutions more general than the Einstein–Weyl structures and affine spheres that motivated it, and the reader is referred to [3] for a preliminary exposition of further developments.

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Submanifold Conformal Invariants and a Boundary Yamabe Problem

A. Rod Gover and Andrew Waldron

While much is known about the invariants of conformal manifolds, the same cannot be said for the invariants of submanifolds in conformal geometries. Codimension one embedded submanifolds (or *hypersurfaces*) are important for applications to geometric analysis and physics. An extremely interesting example is the Willmore equation

$$\bar{\Delta} H + 2H(H^2 - K) = 0, \tag{1}$$

for an embedded surface Σ in Euclidean 3-space $\mathbb{E}^3$. Here, H and K are, respectively, the mean and Gauß curvatures, while $\bar{\Delta}$ is the Laplacian induced on Σ. We shall term the left hand side of this equation the *Willmore invariant*; as given, this quantity is invariant under Möbius transformations of the ambient $\mathbb{E}^3$. A key feature is the linearity of its highest order term, $\bar{\Delta} H$. This linearity is important for PDE problems, but also means that the Willmore invariant should be viewed as a fundamental curvature quantity.

In the 1992 article [1], Andersson, Chrusciel and Friedrich, building on the works [2, 3, 8], identified a conformal surface invariant that obstructs smooth boundary asymptotics for a Yamabe solution on a conformally compact 3-manifold (and gave some information on the obstructions in dimension $d > 3$). It is straightforward to show that this invariant is the same as that arising from the variation of the Willmore energy; in particular its specialisation to surfaces in $\mathbb{E}^3$

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M. del Mar González et al. (eds.), *Extended Abstracts Fall 2013*,
Trends in Mathematics, DOI 10.1007/978-3-319-21284-5_4

agrees with (1). We show how tools from conformal geometry can be used to describe and compute the asymptotics of the Yamabe problem on a conformally compact manifold. This reveals higher order hypersurface conformal invariants that generalise the curvature obstruction found by Andersson, Chrusciel and Friedrich. In particular, for hypersurfaces of arbitrary even dimension this yields higher order conformally invariant analogues of the usual Willmore equation on surfaces in 3-space. The construction also leads to a general theory for constructing and treating conformal hypersurface invariants along the lines of holography and the Fefferman–Graham programme for constructing invariants of a conformal structure via their Poincaré–Einstein and "ambient" metrics [4].

1 The Problem

Given a Riemannian d-manifold (M, g) with boundary $\Sigma := \partial M$, one may ask whether there is a smooth real-valued function u on M satisfying the following two conditions:

(1) u is a defining function for Σ (i.e., Σ is the zero set of u, and $\boldsymbol{d}u_x \neq 0 \; \forall x \in \Sigma$);
(2) $\bar{g} := u^{-2}g$ has scalar curvature $\mathrm{Sc}^{\bar{g}} = -d(d-1)$.

Here $\boldsymbol{d}$ is the exterior derivative. We assume $d \geq 3$ and all structures are C^∞.

Assuming $u > 0$ and setting $u = \rho^{-2/(d-2)}$, part (2) of this problem gives the Yamabe equation. The problem fits nicely into the framework of conformal geometry. Recall that a conformal structure c on a manifold is an equivalence class of metrics where the equivalence relation $\hat{g} \sim g$ means that $\hat{g} = \Omega^2 g$ for some positive function Ω. The line bundle $(\Lambda^d TM)^2$ is oriented and for $w \in \mathbb{R}$ the bundle of *conformal densities* of weight w, denoted $\mathcal{E}[w]$, is defined to be the oriented $(w/2d)$-root of this (we use the same notation for bundles as for their smooth section spaces). Locally, each $g \in c$ determines a volume form and, squaring this, globally a section of $(\Lambda^d T^*M)^2$. So, on a conformal manifold (M, c) there is a canonical section $\boldsymbol{g}$ of $S^2T^*M \otimes \mathcal{E}[2]$ called the conformal metric. Thus each metric $g \in c$ is naturally in one-to-one correspondence with a (strictly) positive section τ of $\mathcal{E}[1]$ via $g = \tau^{-2}\boldsymbol{g}$. Also, the Levi–Civita connection ∇ of g preserves τ, and hence $\boldsymbol{g}$. Thus we are led to the conformally invariant equation on a weight 1 density $\sigma \in \mathcal{E}[1]$,

$$S(\sigma) := \big(\nabla\sigma\big)^2 - \frac{2}{d}\sigma\Big(\Delta + \frac{\mathrm{Sc}}{2(d-1)}\Big)\sigma = 1, \tag{2}$$

where $\boldsymbol{g}$ and its inverse are used to raise and lower indices, $\Delta = \boldsymbol{g}^{ab}\nabla_a\nabla_b$ and Sc means $\boldsymbol{g}^{bd}R_{ab}{}^a{}_d$, with R the Riemann tensor. Choosing $c \ni g = \tau^{-2}\boldsymbol{g}$, Eq. (2) becomes exactly the PDE obeyed by the smooth function $u = \sigma/\tau$ solving part (2) of the problem above. Since u is a defining function this means σ is a *defining density* for Σ, meaning that it is a section of $\mathcal{E}[1]$, its zero locus $\mathcal{Z}(\sigma) = \Sigma$, and

$\nabla\sigma_x \neq 0\ \forall x \in \Sigma$. For our purpose we only need to treat the problem formally (so it applies to any hypersurface):

Problem 1 Let Σ be an embedded hypersurface in a conformal manifold (M, c) with $d \geq 3$. Given a defining density σ for Σ, find a new, smooth, defining density $\bar{\sigma}$ such that

$$S(\bar{\sigma}) = 1 + \bar{\sigma}^{\ell} A_{\ell}\,, \tag{3}$$

for some $A_\ell \in \mathcal{E}[-\ell]$, where $\ell \in \mathbb{N} \cup \infty$ is as high as possible.

2 The Main Results

Here we use the notation $\mathcal{O}(\sigma^{\ell})$ to mean plus $\sigma^{\ell} A$ for some smooth $A \in \mathcal{E}[-\ell]$.

Theorem 2 *Let Σ be an oriented embedded hypersurface in (M, c), where $d \geq 3$. Then,*

(i) *there is a distinguished defining density $\bar{\sigma} \in \mathcal{E}[1]$ for Σ, unique up to $\mathcal{O}(\bar{\sigma}^{d+1})$, such that*

$$S(\bar{\sigma}) = 1 + \bar{\sigma}^{d} B_{\bar{\sigma}}\,, \tag{4}$$

where $B_{\bar{\sigma}} \in \mathcal{E}[-d]$ is smooth on M. Given any defining density σ, then $\bar{\sigma}$ depends smoothly on (M, c, σ) via a canonical formula $\bar{\sigma}(\sigma)$.

(ii) *$\mathcal{B} := B_{\bar{\sigma}(\sigma)}\big|_{\Sigma}$ is independent from σ, and it is a natural invariant determined by (M, c, Σ).*

For any defining density $\bar{\sigma}$ satisfying (4), it is straightforward to calculate $\mathcal{B}$ (although a bit tedious). For $d = 3$ we obtain

$$\mathcal{B} = 2\big(\bar{\nabla}_{(i}\bar{\nabla}_{j)\circ} + H\,\mathring{\mathrm{II}}_{ij} + R^{\top}_{(ij)\circ}\big)\mathring{\mathrm{II}}^{ij}\,, \tag{5}$$

where $\mathring{\mathrm{II}}_{ij}$ is the trace-free part of the second fundamental form II_{ij}, $R^{\top}_{(ij)\circ}$ is the trace-free part of the projection of the ambient Ricci tensor along Σ, and $\bar{\nabla}$ is the Levi–Civita for the metric on Σ induced by g. Equation (5) agrees with [1, Theorem 1.3] and [6] and, by using the Gauß–Codazzi equations, agrees with (1) for Σ in $\mathbb{E}^3$. (We note that (4) is consistent with [1, Lemma 2.1].)

For $d \geq 5$ odd, the *obstruction density* $\mathcal{B}$ of Theorem 2 has a linear highest order term, namely $\bar{\Delta}^{(d-1)/2}H$ (up to multiplication by a non-zero constant). So, $\mathcal{B}$ is an analogue of the Willmore invariant; it can be viewed as a fundamental conformal curvature invariant for hypersurfaces; as an obstruction it is an analogue of the Fefferman–Graham obstruction tensor [4]. We see this as follows.

From the algorithm for calculating $\mathcal{B}$ one easily concludes that it is a natural invariant (in terms of a background metric); indeed, it is given by a formula polynomially involving the second fundamental form and its tangential (to Σ) covariant derivatives, as well as the curvature of the ambient manifold M and its covariant derivatives. To calculate the leading term we linearise this formula by computing the infinitesimal variation of $\mathcal{B}$. It suffices to consider an $\mathbb{R}$-parametrised family of embeddings of $\mathbb{R}^{d-1}$ in $\mathbb{E}^d$, with corresponding defining densities σ_t and such that the zero locus $\mathcal{Z}(\sigma_0)$ is the $x^d = 0$ hyperplane (where x^i are the standard coordinates on $\mathbb{E}^d = \mathbb{R}^d$) so that $\mathcal{B}|_{t=0} = 0$. Then applying $\frac{\partial}{\partial t}|_{t=0}$ (denoted by a dot) we obtain the following:

Proposition 3 *The variation of the obstruction density is given by*

$$\dot{\mathcal{B}} = \begin{cases} a \cdot \bar{\Delta}^{(d+1)/2}\dot{\sigma} + \text{lower order terms}\,, & d-1 \textit{ even, with } a \neq 0 \textit{ a constant,} \\ \text{non-linear terms,} & d-1 \textit{ odd.} \end{cases}$$

This establishes the result, as in this setting the highest order term in the variation of mean curvature is $\bar{\Delta}\dot{\sigma}/(d-1)$. It also shows that when n is odd the general formula for $\mathcal{B}$ may be expressed so that it has no linear term.

2.1 A Holographic Approach to Submanifold Invariants

Given a conformal manifold (M, c) and a section σ of $\mathcal{E}[1]$, one may construct density-valued conformal invariants that couple the data of the jets of the conformal structure with the jets of the section σ. In the setting of Theorem 2, consider such an invariant U, say, which uses the section $\bar{\sigma}$ of the theorem. Suppose that at every point, U involves $\bar{\sigma}$ non-trivially, but uses no more than its d-jet of $\bar{\sigma}$. Then it follows from the first part of Theorem 2 that $U|_\Sigma$ is determined by (M, c, Σ) and so it is a conformal invariant of Σ. On the interior, the formula for U as calculated in the scale $\bar{\sigma}$ (so using $\bar{\sigma}$ to trivialize the density bundles) is then a regular Riemannian invariant of $(M, \bar{g})$ (where $\bar{g} = \bar{\sigma}^{-2}\boldsymbol{g}$) which corresponds holographically to the submanifold invariant $U|_\Sigma$.

3 The Ideas Behind the Proofs

On a conformal manifold (M, c), although there is no canonical connection on TM, there is a canonical linear connection $\nabla^{\mathcal{T}}$ on a rank $d+2$ vector bundle known as the tractor bundle and denoted $\mathcal{E}^A$ in an abstract index notation. A choice of metric

$g \in c$ determines an isomorphism

$$\mathcal{E}^A \stackrel{g}{\cong} \mathcal{E}[1] \oplus T^*M[1] \oplus \mathcal{E}[-1].$$

This connection preserves a metric h_{AB} on $\mathcal{E}^A$ that we may therefore use to raise and lower tractor indices. For $V^A = (\sigma, \mu^a, \rho)$ and $W^A = (\tau, \nu^a, \kappa)$ this is given by

$$h(V, W) = h_{AB}V^A W^B = \sigma\kappa + \boldsymbol{g}_{ab}\mu^a\nu^b + \rho\tau =: V.W.$$

Closely linked to $\nabla^{\mathcal{T}}$ is an important, second order conformally invariant operator $D^A\colon \mathcal{E}[w] \to \mathcal{E}^A[w-1]$; when $w \neq 1-d/2$, we denote $\frac{1}{d-2w-2}$ times this by $\hat{D}$, where $\hat{D}^A\sigma \stackrel{g}{=} (\sigma,\ \nabla_a\sigma,\ -\frac{1}{d}(\Delta + \mathsf{J})\sigma)$, for the case $\sigma \in \mathcal{E}[1]$, and $2\mathsf{J} = \mathrm{Sc}^g/(d-1)$. For σ a scale, or even a defining density, we shall write $I^A_\sigma := \hat{D}^A\sigma$, which we call the *scale tractor*. Now, $S(\sigma)$ from above is just $S(\sigma) = I^2_\sigma := h_{AB}I^A_\sigma I^B_\sigma$, so Eq. (2) has the nice geometric interpretation $I^2_\sigma = 1$ [5], and this is critical for our treatment.

Note that it is essentially trivial to solve (3) for the case $\ell = 1$. Theorem 2 is then proved inductively via the following lemma. It also yields an algorithm for explicit formulae for the expansion, that we cannot explain fully here, but through this and related results the naturality of $\mathcal{B}$ can be seen.

Lemma 4 *Suppose $I^2_\sigma = S(\sigma)$ satisfies (3) for $\ell = k \geq 1$. Then, if $k \neq d$, there exists $f_k \in \mathcal{E}[-k]$ such that the scale tractor $I_{\sigma'}$ of the new defining density $\sigma' := \sigma + \sigma^{k+1}f_k$ satisfies (3) for $\ell = k+1$. When $k = d$ and $\sigma' := \sigma + \sigma^{d+1}f$, then for any $f \in \mathcal{E}[-d]$,*

$$I^2_{\sigma'} = I^2_\sigma + \mathcal{O}(\sigma^{d+1})\,.$$

Idea of the proof First, because of the scale tractor definition, we have

$$\big(\hat{D}\sigma'\big)^2 = I^2_\sigma + \frac{2}{d}\,I_\sigma.D\big(\sigma^{k+1}f_k\big) + \Big[\hat{D}\big(\sigma^{k+1}f_k\big)\Big]^2.$$

Tractor calculus identities show that the last term is $\mathcal{O}(\sigma^{k+1})$, while $I^2_\sigma = 1 + \sigma^k A_k$. Crucially, the operators σ (acting by multiplication) and $\frac{1}{I^2_\sigma}I_\sigma.D$ generate an $\mathfrak{sl}(2)$, see [7]. Using standard $\mathcal{U}\big(\mathfrak{sl}(2)\big)$ identities, we compute that $f_k := -d\,A_k/(2(d-k)(k+1))$ which deals with the $k \neq d$ cases; the same computation gives the $k = d$ conclusion. □

sketch of the Proof of Proposition 3 The key idea is that, for each t, we can replace σ_t with the corresponding normalised defining density $\bar{\sigma}_t$ which solves $I^2_{\bar{\sigma}_t} = 1 + \bar{\sigma}_t^d\mathcal{B}_{\bar{\sigma}_t}$, via Theorem 2, while maintaining smooth dependence on t. Then, it is easy to prove that $\mathcal{B}_{\bar{\sigma}_0}|_{\mathcal{Z}(\bar{\sigma}_{t=0})} = 0$, while $\partial(I^2_{\bar{\sigma}_t})/\partial t|_{t=0}$ is proportional to $I.D\dot{\bar{\sigma}}$. So, applying $\frac{\partial}{\partial t}\ |_{t=0}$ we get that $\dot{\bar{\sigma}}$ solves a linear $I.D.$ boundary problem up to $\mathcal{O}(\bar{\sigma}^d)$ with obstruction $\dot{\mathcal{B}}_{\bar{\sigma}}$. Using [7, Theorem 4.5] we can easily deduce the conclusion.

□

Acknowledgements Prior to this work, A.R. Gover had discussions about this problem with Fernando Marques and then Pierre Albin and Rafe Mazzeo. We are indebted for the insights so gained. The first author is supported by Marsden grant 10-UOA-113.

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Variation of the Total Q-Prime Curvature in CR Geometry

Kengo Hirachi

Tom Branson introduced the concept of Q-curvature in conformal geometry, in connection with the study of conformal anomaly of determinants of conformally invariant differential operators. The definition can be generalized to CR manifolds via Fefferman's conformal structure on a circle bundle over CR manifolds, see [2]. Using this correspondence, one can translate the properties of conformal Q-curvature to the CR analogue. However, there has been an important missing piece in this correspondence. In conformal geometry, the integral of the Q-curvature, called the total Q-curvature, is a global conformal invariant and its first variation under the deformation of conformal structure is given by the Fefferman–Graham obstruction tensor. On the other hand, the total CR Q-curvature always vanishes for domains in $\mathbb{C}^N$ and has no relation to the obstruction function, which arises in the asymptotic analysis of the complex Monge–Ampère equation. Moreover, CR Q-curvature identically vanishes for a natural choice of contact forms, called pseudo-Einstein contact forms, on the boundary of a domain in $\mathbb{C}^N$.

We claim that the missing piece can be filled by *Q-prime curvature* [3], which was first introduced by Case–Yang [1]. We show that the total Q-prime curvature admits a variational formula that includes the obstruction function. To state the formula, we start with recalling the complex Monge–Ampère equation.

Let $D \subset \mathbb{C}^{n+1}$ be a strictly pseudo-convex bounded domain with smooth boundary $M = \partial D$. We consider the following non-linear PDE with boundary condition:

$$(-1)^{n+1} \det \begin{pmatrix} u & u_j \\ u_{\bar{k}} & u_{j\bar{k}} \end{pmatrix} = 1, \quad u > 0 \text{ in } D, \quad \text{and} \quad u = 0 \text{ on } M.$$

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M. del Mar González et al. (eds.), *Extended Abstracts Fall 2013*,
Trends in Mathematics, DOI 10.1007/978-3-319-21284-5_5

Here, $u_j = \partial u/\partial z^j$, $u_{\overline{k}} = \partial u/\partial \overline{z}^k$ and $u_{j\overline{k}} = \partial^2 u/\partial z^j \partial \overline{z}^k$. Cheng–Yau proved the existence of a unique solution $u \in C^\infty(D) \cap C^{n+1}(\overline{D})$, and Lee–Melrose showed that this solution admits an asymptotic expansion at the boundary

$$u \sim r + r \sum_{j=1}^{\infty} \eta_j \big(r^{n+2} \log r\big)^j,$$

where r is a smooth defining function and $\eta_j \in C^\infty(\overline{D})$. The leading log term coefficient $\mathcal{O} = \eta_1|_{\partial D}$ is called the obstruction function, since $\mathcal{O} = 0$ if and only if $u \in C^\infty(\overline{D})$. Note that $g = -i\partial\overline{\partial} \log u$ is Einstein–Kähler with negative scalar curvature $-n-2$.

We use the smooth defining function r, the smooth part of u, to define the ambient metric on the trivial bundle $\mathbb{C}^* \times \overline{D}$ over $\overline{D}$. Denoting the fiber variable by $z^0 \in \mathbb{C}^*$, we define a Ricci-flat Lorentz–Kähler metric, called *the ambient metric*, on $\mathbb{C}^* \times \overline{D}$ by

$$\tilde{g} = -i\partial\overline{\partial}\left(|z^0|^2 r(z)\right).$$

Let Δ be the Laplacian of $\tilde{g}$. Then the *Q-prime curvature* is defined by

$$Q' = \Delta^{n+1}\big(-\log|z^0|^2\big)^2\big|_{z^0=1, z\in M} \in C^\infty(M).$$

We can see that Q' is a pseudo-Hermitian invariant of the contact form

$$\theta = \frac{i}{2}\big(\partial - \overline{\partial}\big) r|_M,$$

which is pseudo-Einstein (in the sense that the Tanaka–Webster Ricci form is proportional to the Levi form) due to the Ricci-flatness of $\tilde{g}$. This definition of Q' can be generalized to embedded CR manifolds with pseud-Einstein contact forms.

The definition of Q' depends on the embedding of the CR manifold M into $\mathbb{C}^{n+1}$ and is not CR invariant. Since the Kähler potential has the ambiguity of adding pluriharmonic functions, the choice of r also has the ambiguity $\hat{r} = e^{\Upsilon} r$, where Υ is pluriharmonic on $\overline{D}$. Under this change of defining function, we have a transformation rule

$$\hat{Q}' = Q' + P_1 \Upsilon + P_2(\Upsilon^2),$$

where P_1 and P_2 are linear differential operators that are formally self-adjoint, respectively, on the space of pluriharmonic functions and $C^\infty(M)$, and satisfy $P_1 1 = P_2 1 = 0$. It follows that the integral

$$\overline{Q}'(M) = \int_M Q' \theta \wedge (d\theta)^n,$$

called the *total Q-prime curvature*, is a CR invariant of M.

By analogy with the fact that the total Q-curvature in conformal geometry is given by the logarithmic term in the asymptotic expansion of the volume of conformally compact Einstein manifold, we can show that $\overline{Q}'$ appears in the expansion with respect to the volume form weighted by $\|d\log r\|^2$, the squared norm of the 1-form $d\log r$ for g:

$$\int_{r>\epsilon} \|d\log r\|^2 dv_g = \sum_{j=0}^{n} a_j\,\epsilon^{j-n-1} + c_n\,\overline{Q}'\,\log\epsilon + O(1),$$

where $c_n = (-1)^n/(n!)^3$. This formula can be applied to compute the variation of $\overline{Q}'$ under the perturbation of domains. We consider a smooth family of strictly pseudoconvex domains $\{D_t\}_{t\in\mathbb{R}}$ in $\mathbb{C}^{n+1}$ with $D_0 = D$. Let r_t be the defining function of D_t, fixed as above by using the Monge–Ampère equation, and set

$$f = \frac{dr_t}{dt}\Big|_{t=0,z\in M} \in C^\infty(M).$$

Theorem 1 *Let $\overline{Q}'_t = \overline{Q}'(M_t)$ be the total Q-prime curvature of $M_t = \partial D_t$. Then*

$$\frac{d}{dt}\overline{Q}'_t\Big|_{t=0} = c'_n \int_M f\mathcal{O}\theta\wedge(d\theta)^n,$$

where $\mathcal{O}$ is the obstruction function of $M = M_0 \subset \mathbb{C}^{n+1}$ and c'_n is a non-zero constant depending only on n.

The proof of this theorem will appear in our forthcoming paper with Yoshihiko Matsumoto and Taiji Marugame.

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Conformal Invariants from Nullspaces of Conformally Invariant Operators

Dmitry Jakobson

This is an extended abstract of the talk *Conformal invariants from nodal sets*. The talk was based on joint work with Yaiza Canzani, Rod Gover and Raphaël Ponge, and the results appeared in the papers [2, 3]. The current abstract is an abbreviated version of [2].

Let M be a compact Riemannian manifold of dimension $n \geq 3$, and let g be a Riemannian metric on M. We study eigenfunctions of conformally covariant operators constructed in [4], also called the GJMS operators.

For any positive integer k if n is odd, or for any positive integer $k \leq n/2$ if n is even, there is a covariant, formally self-adjoint, differential operator $P_{k,g}$ of order $2k$ such that

(i) $P_k = \Delta_g^k +$ lower order terms;
(ii) if $g_1 = e^{2\omega} g$ is another metric in the conformal class $[g]$, then P_k transforms as follows:

$$P_{k,g_1} = e^{-(\frac{n}{2}+k)\omega} P_{k,g} e^{(\frac{n}{2}-k)\omega}. \tag{1}$$

The operator $P_{1,g} := \Delta_g + \frac{n-2}{4(n-1)} R_g$ is called the *Yamabe operator*; here R_g denotes the scalar curvature of g. The operator $P_{2,g}$ is called the *Paneitz operator*.

The *nullspace* $\ker P_{k,g}$ is the subspace of $L^2(M)$ consisting of eigenfunctions u of $P_{k,g}$ with eigenvalue 0: $\{u \in L^2(M) : P_{k,g} u = 0\}$. It follows easily from (1) that

$$\ker P_{k,g_1} = e^{(k-\frac{n}{2})\omega} \ker P_{k,g_0}. \tag{2}$$

The following results follow from (2).

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M. del Mar González et al. (eds.), *Extended Abstracts Fall 2013*,
Trends in Mathematics, DOI 10.1007/978-3-319-21284-5_6

Proposition 1 *Let M be a compact manifold of dimension $n \geq 3$, and let g be a Riemannian metric on M. Then,*

(i) *the dimension* $\dim\ker P_{k,g}$ *is an invariant of the conformal class* $[g]$*;*
(ii) *if n is even and $k = n/2$, then* $\ker P_{k,g}$ *is itself conformally invariant;*
(iii) *if* $\dim\ker P_{k,g} \geq 1$*, then the nodal set and nodal domains of any nonzero eigenfunction $u \in P_{k,g}$ are invariants of* $[g]$*;*
(iv) *if* $\dim\ker P_{k,g} \geq 2$*, then the (non-empty) intersections of nodal sets of eigenfunctions in* $\ker P_{k,g}$ *are conformally invariant and, hence, so are their complements.*

Now assume that $\dim\ker P_{k,g} = m \geq 2$. Let $u_{1,g}, \ldots, u_{m,g}$ be a basis of $\ker P_{k,g}$. Set $\mathcal{N} := \bigcap_{1\leq j\leq m} u_{j,g}^{-1}(0)$ and define $\Phi : M \setminus \mathcal{N} \to \mathbb{RP}^{m-1}$ by

$$\Phi(x) := (u_{1,g}(x) : \cdots : u_{m,g}(x)) \qquad \forall x \in M \setminus \mathcal{N}.$$

Note that the set $\mathcal{N}$ is independent from the choice of the basis $u_{1,g}, \ldots, u_{m,g}$, but Φ depends on the choice of basis only up to the right action of $\mathrm{PGL}_m(\mathbb{R})$.

Proposition 2 *The class of Φ modulo the right action of* $\mathrm{PGL}_m(\mathbb{R})$ *is an invariant of the conformal class* $[g]$.

For even n and $k = n/2$, the nullspace of $P_{n/2}$ always contains the constant functions, so we may assume that $u_{1,g}(x) = 1$. The counterpart of Φ in that case can be defined by

$$\Psi(x) := \big(u_{2,g}(x), \ldots, u_{m,g}(x)\big) \qquad \forall x \in M.$$

Proposition 3 *The class of Ψ modulo the right action of* $\mathbb{R}^{m-1} \ltimes \mathrm{PGL}_{m-1}(\mathbb{R})$ *is an invariant of* $[g]$.

Denote by $dV_g(x)$ the Riemannian measure defined by g.

Proposition 4 *Assume M is compact, $k < n/2$, and let $u \in$* $\ker P_{k,g}$*. Then the integral* $\int_M |u_g(x)|^{\frac{2n}{n-2k}} dV_g(x)$ *is an invariant of* $[g]$.

We next discuss metrics g for which $P_{k,g}$ has negative eigenvalues. For $m \in \mathbb{N}_0$, denote by $\mathcal{G}_{k,m}$ the set of metrics g on M such that $P_{k,g}$ has at least m negative eigenvalues (counted with multiplicity). One can show that $\mathcal{G}_{k,m}$ is an open set in the C^{2k}-topology; and that if $g \in \mathcal{G}_{k,m}$, then $[g] \subset \mathcal{G}_{k,m}$. It follows from this that the number of negative eigenvalues defines a partition of the set of conformal classes. We also observe that, by results of Kazdan–Warner [5], $\mathcal{G}_{1,0}$ consists of all metrics that are conformally equivalent to a metric with nonnegative scalar curvature.

The following result can be deduced from Lokhamp [6]:

Theorem 5 *Assume M compact. Then, for any m, there is a metric g on M for which the Yamabe operator $P_{1,g}$ has at least m negative eigenvalues.*

It follows that there exist infinitely many conformal classes of metrics on M for which the nullspace of $P_{1,g}$ has dimension ≥ 1, and thus Propositions 1 and 2 all apply.

It would be interesting to obtain similar results for $P_{k,g}$, $k \geq 2$. For $k = 2$, we can prove the following.

Theorem 6 *Assume $M = \Sigma \times \Sigma$, where Σ is a compact surface of genus ≥ 2. Then, for any m, there is a metric g on M for which the Paneitz operator $P_{2,g}$ has at least m negative eigenvalues.*

There is a similar result on compact Heisenberg manifolds. In addition, as an application of Courant's nodal domain theorem, we obtain

Theorem 7 *Let g be a metric such that the Yamabe operator $P_{1,g}$ has exactly m negative eigenvalues. Then any eigenfunction $u \in \ker P_{1,g}$ has at most $m + 1$ nodal domains.*

We end with an application to the *scalar curvature prescription* problem; we refer to [2, 3] for more details.

Theorem 8 *Let $0 \not\equiv u \in \ker P_{1,g}$ and let Ω be a nodal domain of u. Then, for any metric $g_1 \in [g]$, the scalar curvature R_{g_1} cannot be everywhere nonnegative on Ω.*

Related results appear in [1].

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Rigidity of Bach-Flat Manifolds

Seongtag Kim

1 Introduction

Bach-flat metrics were introduced in the study of a conformally invariant gravitational theory and has played important roles in general relativity and geometry. This metric is the most natural generalization of an Einstein metric. Important examples of Bach-flat metrics are Einstein metrics, conformally flat metrics, self-dual Einstein metrics, and K*ä*ler surfaces with zero scalar curvature. Einstein metrics and Bach-flat metrics share many important properties. When the curvature of a given Einstein metric (M, g) is sufficiently close to that of the constant curvature space, in $L_{n/2}$ sense, it is known that (M, g) is isometric to a quotient of the constant curvature space. In this note, we present rigidity phenomena on non-compact complete Bach-flat manifolds and the warped product construction of Bach-flat metrics.

Let (M, g) be a noncompact complete Riemannian 4-manifold with scalar curvature R, Weyl curvature W, Ricci curvature R_{ij}, and curvature tensor *Riem*. Throughout this paper, we assume that (M, g) has finite L_2 Weyl curvature norm. A metric is Bach-flat if it is a critical metric of the functional

$$g \longrightarrow \int_M |W|^2 \, dV_g. \tag{1}$$

Bach-flat condition is equivalent to the vanishing of Bach tensor B_{ij}, which is defined by

$$B_{ij} \equiv \nabla^k \nabla^l W_{kijl} + \frac{1}{2} R^{kl} W_{kijl} \tag{2}$$

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M. del Mar González et al. (eds.), *Extended Abstracts Fall 2013*,
Trends in Mathematics, DOI 10.1007/978-3-319-21284-5_7

(see [2]). There are known rigidity of Bach-flat metrics. For a compact Bach-flat manifold (M, g) with positive Yamabe constant, Chang–Ji–Yang [4] proved that there is only finitely many diffeomorphism classes with an L_2 bound of Weyl tensor, and (M, g) is conformal to the standard sphere if L_2 norm of Weyl tensor is small enough. A conformally invariant sphere theorem is also known, see [3]. For a noncompact complete Bach-flat manifold (M, g) with positive Yamabe constant and zero scalar curvature, Tian–Viaclovsky [9] proved that (M, g) is almost locally Euclidean (ALE) of order 0 with L_2 bounds of curvature, bounded first Betti number and the uniform volume growth for any geodesic ball. This result is extended that (M, g) is ALE of order 2 by Streets [8] and Ache–Viaclovsky [1] .

2 Bach-Flat Metrics

In this section, we study noncompact complete Bach-flat manifolds with zero scalar curvature whose L_2 curvature norm is small. By an elliptic estimation for the Laplacian of curvature tensor, we have:

Theorem 1 *Let (M, g) be a noncompact complete Bach-flat Riemannian 4-manifold with zero scalar curvature and $Q(M, g) > 0$. Then there exists a small number c_0 such that if $\int_M |Riem|^2\, dV_g \leq c_0$, then (M, g) is flat, i.e., $Riem = 0$, where Riem is the curvature tensor.*

Sketch of proof We will see that $|Riem| = 0$. The Laplacian of the curvature tensor is

$$\Delta R_{ijkl} = 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) + \nabla_i \nabla_k R_{jl} - \nabla_i \nabla_l R_{jk}$$
$$- \nabla_j \nabla_k R_{il} + \nabla_j \nabla_l R_{ik} + g^{pq}(R_{pjkl} R_{qi} + R_{ipkl} R_{qj}), \tag{3}$$

where $B_{ijkl} = g^{pr} g^{qs} R_{piqj} R_{rksl}$ (see [5]). To simplify notations, we will work in an orthonormal frame. Multiplying R_{ijkl} on (3), we have

$$R_{ijkl} \Delta R_{ijkl} = 2(B_{ijkl} - B_{ijlk} - B_{iljk} + B_{ikjl}) R_{ijkl} + (\nabla_i \nabla_k R_{jl} - \nabla_i \nabla_l R_{jk}) R_{ijkl}$$
$$+ (-\nabla_j \nabla_k R_{il} + \nabla_j \nabla_l R_{ik}) R_{ijkl} + g^{pq}(R_{pjkl} R_{qi} + R_{ipkl} R_{qj}) R_{ijkl}. \tag{4}$$

For a smooth compact supported function ϕ and small $\epsilon > 0$, we integrate the second term in (4)

$$\int_M \phi^2 \big(\nabla_i \nabla_k R_{jl} - \nabla_i \nabla_l R_{jk}\big) R_{ijkl}\, dV_g$$
$$= -\int_M \nabla_i \phi^2 (\nabla_k R_{jl} - \nabla_l R_{jk}) R_{ijkl} + \phi^2 (\nabla_k R_{jl} - \nabla_l R_{jk}) \nabla_i R_{ijkl}\, dV_g$$

$$= -\int_M \nabla_i \phi^2 \nabla_t R_{tjkl} R_{ijkl} + \phi^2 |\nabla_i R_{ijkl}|^2 \, dV_g \tag{5}$$

$$\geq -\int_M \frac{1}{\epsilon} |\nabla \phi|^2 |R_{ijkl}|^2 + (1+\epsilon)\phi^2 |\nabla_i R_{ijkl}|^2 \, dV_g, \tag{6}$$

where the Bianchi identity $\nabla^i R_{ijkl} = \nabla_k R_{jl} - \nabla_l R_{jk}$ is used. We estimate $\triangle|Riem|$,

$$-\int_M \phi^2 |Riem| \triangle |Riem| \, dV_g \tag{7}$$

$$= -\int_M \phi^2 \left(|\nabla Riem|^2 - |\nabla |Riem||^2 + R_{ijkl} \triangle R_{ijkl} \right) dV_g \tag{8}$$

$$\leq \int_M 2 \left(\frac{1}{\epsilon} |\nabla \phi|^2 |R_{ijkl}|^2 + (1+\epsilon)\phi^2 |\nabla_i R_{ijkl}|^2 \right) + c|Riem|^3 \phi^2 \, dV_g, \tag{9}$$

where the Kato inequality $|\nabla Riem|^2 \geq |\nabla |Riem||^2$ is used. Let $E_{ij} = R_{ij} - \frac{1}{4} R g_{ij}$. Multiplying ϕE_{ij} on Bach-flat equation (2), and using $W_{ikj\ell} g_{kl} = 0$, we can show that

$$\int_M \phi^2 \left|\delta W\right|^2 \leq (1-\epsilon_2)^{-1} \int_M \frac{1}{2} \phi^2 W_{ikj\ell} E_{k\ell} E_{ij} + \frac{1}{\epsilon_2} |\nabla \phi|^2 |E_{ij}|^2 \, dV_g. \tag{10}$$

For a general Riemannian n-manifold,

$$|\nabla^i W_{ijkl}|^2 = \left(\frac{n-3}{n-2}\right)^2 \left(|\nabla^i R_{ijkl}|^2 - \frac{1}{6} |\nabla R|^2 \right). \tag{11}$$

Now, from (10) and (11), we can control the second term in (9). For simplicity of notations, we let $u = |Riem|$. Using the Yamabe constant $\Lambda_0 \equiv Q(M, g)$,

$$\Lambda_0 \left(\int_M (\phi u)^4 \, dV_g \right)^{1/2}$$

$$\leq \int_M |u \nabla \phi + \phi \nabla u|^2 \, dV_g + \frac{1}{6} R u^2 \phi^2 \, dV_g \tag{12}$$

$$\leq \int_M (c+1) |\nabla \phi|^2 u^2 \, dV_g + c \left(\int_M (\phi u)^4 \, dV_g \right)^{1/2} \left(\int_M u^2 \, dV_g \right)^{1/2}. \tag{13}$$

Since $\int_M |Riem|^2 \, dV_g$ is sufficiently small, there exists a constant c' such that

$$c' \left(\int_M (\phi u)^4 \, dV_g \right)^{1/2} \leq \int_M |\nabla \phi|^2 u^2 \, dV_g. \tag{14}$$

Now we choose ϕ as

$$\phi = \begin{cases} 1 & \text{on } B_t \\ 0 & \text{on } M - B_{2t} \\ |\nabla \phi| \leq \frac{2}{t} & \text{on } B_{2t} - B_t, \end{cases} \tag{15}$$

with $0 \leq \phi \leq 1$ and $B_t = \{x \in M | d(x, x_0) \leq t\}$ for some fixed $x_0 \in M$. From (14),

$$c' \left(\int_M u^4 \phi^4 \, dV_g \right)^{1/2} \leq \frac{4}{t^2} \int_{B(2t)-B(t)} u^2 \, dV_g. \tag{16}$$

And, by letting t go to infinity, we have $u = 0$. Therefore (M, g) is flat. □

Next, we present rigidity of non-compact complete Bach-flat metric with non constant scalar curvature. We apply a result of the Yamabe problem on noncompact manifold to study rigidity. For a given manifold (M, g), we find a conformal metric $\overline{g} = u^{4/(n-2)} g$ whose scalar curvature is zero. This is equivalent to finding a solution for the following partial differential equation

$$-\Delta_g u + \frac{1}{6} R_g u = 0. \tag{17}$$

The series of works by Tian–Viaclovsky [9], Streets[8] and Ache–Viaclovsky [1] imply that if (M, g) is Bach-flat, scalar-flat and ALE of order 0, then (M, g) is ALE of order 2. Therefore R_g is $O(r^{-4})$, where r is the distance from a fixed point on the manifold (M, g). By the standard elliptic estimations, we can solve u for (17) and obtain a complete metric $u^2 g$, where $|u-1|$ is controlled by the L^2 norm of the scalar curvature.

Theorem 2 *Let (M, g) be a non-compact complete Bach-flat Riemannian 4-manifold with scalar curvature R and $Q(M, g) > 0$. Then there exists a small number c_0 such that if $\int_M |Riem|^2, dV_g \leq c_0$, then (M, g) is conformal to a flat space.*

For details on the proofs of Theorems 1 and 2, we refer the reader to [6, 7].

3 Warped Product Construction of Bach-Flat Metrics

In this section we discuss the construction of Bach-flat metrics. First, we classify the compact Riemannian Bach-flat 4-manifolds, which are warped products of two dimensional Riemannian manifolds. Let (N, g_N) and (F, g_F) be two dimensional Riemannian manifolds, and f be a positive smooth function on N. We denote by π and σ the projections of $N \times F$ onto N and F, respectively. The warped product

$M = N \times_f F$ is the product manifold $M = N \times F$ furnished with the metric g defined by $g = \pi^* g_N + f^2 \sigma^* g_F$, where $*$ denotes the pull-back. We obtain a rigidity of the warped product Bach-flat 4-manifold $M = N \times_f F$.

Theorem 3 *Let $(N \times F, \pi^* g_N + f^2 \sigma^* g_F)$ be a compact Riemannian Bach-flat* 4-*manifold, which is a warped product of two dimensional Riemannian manifolds (N, g_N) and (F, g_F). Then $(N, f^{-2} g_N)$ and (F, g_F) have constant curvature K_N and K_F, respectively, and $K_N = \pm K_F$. Therefore $(N \times F, \pi^* g_N + f^2 \sigma^* g_F)$ is conformal to the product of constant curvature spaces.*

We can also construct non-compact complete Riemannian Bach-flat 4-manifolds with warped products of two dimensional Riemannian manifolds. Some cases share similar rigidity.

Acknowledgements This research was supported by Basic Science Research Program through the National Research Foundation of Korea(NRF) funded by the Ministry of Education (2011-0025674).

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Uniformizing Surfaces with Conical Singularities

Andrea Malchiodi

We consider a class of singular equations, motivated by the problem of prescribing the Gaussian curvature, as well as from some models in theoretical physics such as self-dual Chern–Simons theory or Electroweak theory: our goal is to prove existence results by attacking the problem variationally, using suitable improvements of the Moser–Trudinger inequality.

It is well-known that, considering a compact surface Σ endowed with a metric g, after a conformal change of metric $\tilde{g} = e^{2w} g$ the Gaussian curvature transforms according to the law

$$-\Delta_g w + K_g = K_{\tilde{g}} e^{2w}.$$

In particular, if one wishes to obtain constant Gaussian curvature, namely to solve the *uniformization problem*, one is led to the Liouville equation

$$-\Delta_g w + K_g = \rho e^{2w},$$

where ρ is a constant determined by the Gauss–Bonnet formula.

If one wishes to obtain a given conical structure at a finite set of points $\{p_1, \ldots, p_m\} \subseteq \Sigma$ and to obtain constant Gaussian curvature on $\Sigma \setminus \{p_1, \ldots, p_m\}$, then one is reduced to solving the singular equation

$$-\Delta_g w + K_g = \rho e^{2w} - 2\pi \sum_{i=1}^{m} \alpha_j \delta_{p_j}, \tag{1}$$

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M. del Mar González et al. (eds.), *Extended Abstracts Fall 2013*,
Trends in Mathematics, DOI 10.1007/978-3-319-21284-5_8

where the coefficient α_j is related to the conical angle θ_j at p_j by the formula $\theta_j = 2\pi(1+\alpha_j)$. Normalizing Σ to unit volume, the value of the constant ρ is dictated by a modified Gauss–Bonnet formula, see [14],

$$\rho = 2\pi\left[\chi(\Sigma) + \sum_J \alpha_j\right].$$

For applications in physics other values of the constant ρ may be interesting as well, see e.g., [13, 15].

While some approach to study the above problem relies on Leray–Schauder degree theory, see [10, 11], we will exploit the variational structure of (1): letting $G_p(x)$ denote the Green's function of $-\Delta_g$ on Σ with pole at p, i.e., the solution to

$$-\Delta_g G_p(x) = \delta_p - \frac{1}{|\Sigma|} \quad \text{on } \Sigma, \quad \text{with } \int_\Sigma G_p(x)\, dV_g = 0,$$

by the substitution

$$u \mapsto u + 2\pi \sum_{j=1}^m \alpha_j G_{p_j}(x)$$

(1) transforms into an equation of the type

$$-\Delta_g u + f(x) = \rho\, \tilde{h}(x) e^{2u} \qquad \text{on } \Sigma, \tag{2}$$

where $f(x)$ is a smooth function on Σ and where $\tilde{h}$ is a non-negative function which behaves like $\tilde{h}(x) \simeq d_g(x,p_i)^{2\alpha_i}$ near the singularities.

Equation (2) is the Euler–Lagrange equation associated to the C^1 functional

$$J_\rho(u) = \int_\Sigma |\nabla_g u|^2\, dV_g + 2\int_\Sigma f(x) u\, dV_g - \rho \log \int_\Sigma h(x) e^{2u}\, dV_g \tag{3}$$

defined on the Sobolev space $H^1(\Sigma, g)$.

To study such a functional, a singular variant of the classical Moser–Trudinger inequality was proved in [14] (see also [8]), namely

$$\log \int_\Sigma \tilde{h}\, e^{2(u-\overline{u})}\, dV_g \leq \frac{1}{4\pi \min\{1, 1+\min_i \alpha_i\}} \int_\Sigma |\nabla_g u|^2\, dV_g + C_{\tilde{h},\Sigma,g}. \tag{4}$$

For small values of ρ this inequality implies that J_ρ is coercive, and critical points can be found via direct minimization. Our goal is to use min-max or Morse theory to prove general existence results in non-coercive settings. For applying variational methods some compactness criterion is needed. In [3, 4], extending some previous results by H. Brezis, F. Merle, Y. Li and I. Shafrir, it was shown that solutions to (2) stay compact if $\rho \notin \Lambda_{\underline{\alpha}}$, where $\Lambda_{\underline{\alpha}}$ is a discrete set defined by

$$\Lambda_{\underline{\alpha}} := \left\{ 4k\pi + \sum_{j \in J} 4\pi(1 + \alpha_j) \right\}.$$

To state our main result, we introduce some notation. Given a point $q \in \Sigma$ we define a weighted cardinality as follows:

$$\chi(q) = \begin{cases} 1 + \alpha_j & \text{if } q = p_j \text{ for some } j = 1, \dots, m; \\ 1 & \text{otherwise.} \end{cases}$$

The cardinality of any finite set of points on Σ is obtained extending χ by additivity. We then define a set of *admissible probability measures* by

$$\Sigma_{\rho,\underline{\alpha}} = \left\{ \sum_{q_j \in I} t_j \delta_{q_j} : \sum_{q_j \in I} t_j = 1, \ t_j \geq 0, \ q_j \in \Sigma \quad 4\pi\chi(I) < \rho \right\}. \tag{5}$$

Our main result reads as follows.

Theorem 1 ([7]) *Suppose $\Sigma_{\rho,\underline{\alpha}}$ is defined as in* (5) *and that it is endowed with the topology of weak distributions. Then, if $\rho \notin \Lambda_{\underline{\alpha}}$ and if $\Sigma_{\rho,\underline{\alpha}}$ is not contractible,* (1) *is solvable.*

The above result extends in full generality other previous ones which also used a variational approach, see [1, 2, 5, 6, 12], and especially it allows the weights α_i's to have different signs. The main ideas for the proof of the theorem rely on the construction of suitable test functions parametrized on $\Sigma_{\rho,\underline{\alpha}}$ and on some improvements of the Moser–Trudinger/Troyanov inequality, in the spirit of [9] for the regular case. The main difficulty caused by the presence of singularities is that some scaling-invariant version of these improved inequalities is needed.

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Recent Results and Open Problems on Conformal Metrics on $\mathbb{R}^n$ with Constant Q-Curvature

Luca Martinazzi

1 Constant Q-Curvature Metrics on $\mathbb{R}^{2m}$ and Their Volumes

We consider solutions to the equation

$$(-\Delta)^m u = (2m-1)! e^{2mu} \quad \text{in } \mathbb{R}^{2m}, \tag{1}$$

satisfying

$$V := \int_{\mathbb{R}^{2m}} e^{2mu(x)} dx < +\infty. \tag{2}$$

Geometrically, if u solves (1)–(2), then the conformal metric $g_u := e^{2u}|dx|^2$ has Q-curvature $Q_{g_u} \equiv (2m-1)!$ and volume V (by $|dx|^2$ we denote the Euclidean metric). For the definition of Q-curvature and related remarks, we refer to [2, Chapter 4] or to [6].

Notice that, up to the transformation $\tilde{u} := u + c$, the constant $(2m-1)!$ in (1) can be changed into any positive number, but it is natural to choose $(2m-1)!$ because it is the Q-curvature of the round sphere S^{2m}. This implies that the function $u_1(x) = \log \frac{2}{1+|x|^2}$, which satisfies $e^{2u_1}|dx|^2 = (\pi^{-1})^* g_{S^{2m}}$, is a solution to (1)–(2) with $V = \mathrm{vol}(S^{2m})$ (here, $\pi : S^{2m} \to \mathbb{R}^{2m}$ is the stereographic projection). Translations and dilations of u_1 (i.e., Möbius transformations) then produce a large

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M. del Mar González et al. (eds.), *Extended Abstracts Fall 2013*,
Trends in Mathematics, DOI 10.1007/978-3-319-21284-5_9

family of solutions to (1)–(2) with $V = \mathrm{vol}(S^{2m})$, namely

$$u_{x_0,\lambda}(x) := u_1(\lambda(x-x_0)) + \log\lambda = \log\frac{2\lambda}{1+\lambda^2|x-x_0|^2}, \quad x_0 \in \mathbb{R}^{2m}, \quad \lambda > 0. \tag{3}$$

We shall call the functions $u_{x_0,\lambda}$ *spherical* solutions to (1)–(2).

The question whether the family of spherical solutions in (3) exhausts the set of solutions to (1)–(2) has raised a lot of interest and it is by now well understood. For instance, in dimension 2 we have the following result:

Theorem 1 (Chen-Li [5]) *Every solution to (1)–(2) with $m = 1$ is spherical.*

On the other hand, for every $m > 1$, i.e., in dimension 4 and higher, it was proven by Chang–Chen [3] that the Problem (1)–(2) admits solutions which are non spherical. More precisely:

Theorem 2 (Chang–Chen [3]) *For every $m > 1$ and $V \in (0, \mathrm{vol}(S^{2m}))$ there exists a solution to (1)–(2).*

Several authors have given analytical and geometric conditions under which a solution to (1)–(2) is spherical (see [4, 14, 16]), and have studied properties of non-spherical solutions, such as asymptotic behavior, volume and symmetry (see [9, 11, 15]). In particular Lin proved:

Theorem 3 (Lin [9]) *Let u solve (1)–(2) with $m = 2$. Then either u is spherical (i.e., as in (3)) or $V < \mathrm{vol}(S^4)$.*

Spherical solutions are radially symmetric (i.e., of the form $u(|x-x_0|)$ for some $x_0 \in \mathbb{R}^{2m}$) and the solutions given by Theorem 2 might a priori all be spherically symmetric. The fact that this is not the case was proven by Wei–Ye in dimension 4:

Theorem 4 (Wei–Ye [15]) *For every $V \in (0, \mathrm{vol}(S^4))$ there exist (several) non-radial solutions to (1)–(2) for $m = 2$.*

Remark 5 As recently shown by A. Hyder [7], the proof of Theorem 4 can be extended to higher dimension $2m \geq 4$, yielding several non-symmetric solutions to (1)–(2) for every $V \in (0, \mathrm{vol}(S^{2m}))$, but failing to produce solutions for $V \geq \mathrm{vol}(S^{2m})$. As in the proof of Theorem 2, the condition $V < \mathrm{vol}(S^{2m})$ plays a crucial role.

Theorems 2–4 and Remark 5 strongly suggest that, also in dimension 6 and higher, all non-spherical solutions to (1)–(2) satisfy $V < \mathrm{vol}(S^{2m})$, i.e., (1)–(2) has no solution for $V > \mathrm{vol}(S^{2m})$ and the only solutions with $V = \mathrm{vol}(S^{2m})$ are the spherical ones. Quite surprisingly it turns out that this is not at all the case. In fact, in dimension 6 there are solutions to (1)–(2) with arbitrarily large V:

Theorem 6 (Martinazzi [13]) *For $m = 3$ there exists $V^* > 0$ such that for every $V \geq V^*$ there is a solution u to (1)–(2), i.e., there exists a metric on $\mathbb{R}^6$ of the form $g_u = e^{2u}|dx|^2$ satisfying $Q_{g_u} \equiv 5!$ and $\mathrm{vol}(g_u) = V$.*

The proof of Theorem 6 is based on a ODE argument: one considers radial solutions to (1)–(2), so that (1) reduces to an ODE. Precisely, given $a \in \mathbb{R}$ let $u = u_a(r)$ be the solution of

$$\begin{cases} \Delta^3 u = -120 e^{6u} \quad \text{in } \mathbb{R}^6 \\ u(0) = u'(0) = u'''(0) = u'''''(0) = 0, \quad u''(0) = -a, \quad u''''(0) = 1. \end{cases} \tag{4}$$

Then one shows that

$$\int_{\mathbb{R}^6} e^{6u_a} dx < \infty \text{ for } a \text{ large}, \quad \lim_{a\to\infty} \int_{\mathbb{R}^6} e^{6u_a} dx = \infty.$$

In particular the conformal metric $g_{u_a} = e^{2u_a}|dx|^2$ of constant Q-curvature $Q_{g_{u_a}} \equiv 5!$ satisfies

$$\mathrm{vol}(g_{u_a}) < \infty \text{ for } a \text{ large}, \quad \lim_{a\to\infty} \mathrm{vol}(g_{u_a}) = \infty. \tag{5}$$

Theorem 6 then follows from (5) and the remark that the quantity $\mathrm{vol}(g_{u_a})$ is a continuous function of a when a is sufficiently large (this seems to be false in general if $a > 0$ is not large enough).

The proof of Theorem 2, which is variational and based on the sharpness of Beckner's inequality [1], does not extend to the case $V > \mathrm{vol}(S^{2m})$. On the other hand with the previous ODE approach one can prove that, at least when $m \geq 3$ is odd, Theorem 2 extends as follows.

Theorem 7 (Martinazzi [13]) *Set* $V_m := \frac{(2m)!}{4(m!)^2}\mathrm{vol}(S^{2m}) > \mathrm{vol}(S^{2m})$. *Then, for* $m \geq 3$ *odd and for every* $V \in (0, V_m]$, *there is a non-spherical (but radially symmetric) solution* u *to* (1)–(2), *i.e., there exists a metric on* $\mathbb{R}^{2m}$ *of the form* $g_u = e^{2u}|dx|^2$ *satisfying* $Q_{g_u} \equiv (2m-1)!$ *and* $\mathrm{vol}(g_u) = V$.

The condition $m \geq 3$ odd is (at least in part) necessary in view of Theorems 1 and 3, but the case $m \geq 4$ even is open. Notice also that, when $m = 3$, Theorems 6 and 7 guarantee the existence of solutions to (1)–(2) for

$$V \in (0, V_m] \cup [V^*, \infty),$$

but do not rule out that $V_m < V^*$ and the existence of solutions to (1)–(2) is unknown for $V \in (V_m, V^*)$. Could there be a gap phenomenon?

We remark that the case m even is more difficult to treat since the ODE corresponding to (1), in analogy with (4), becomes

$$\Delta^m u(r) = (2m-1)! e^{2mu(r)}, \quad r > 0,$$

whose solutions can blow up in finite time (i.e., for finite r) if the initial data are not chosen carefully (contrary to what happens when m is odd).

2 Negative Curvature and Odd Dimension

It is natural to investigate how large the volume of a metric $g_u = e^{2u}|dx|^2$ on $\mathbb{R}^{2m}$ can be, also with constant and negative Q-curvature $Q_{g_u} < 0$. Again with no loss of generality we assume $Q_{g_u} \equiv -1$. In other words, consider the problem

$$(-\Delta)^m u = -e^{2mu} \quad \text{on } \mathbb{R}^{2m}, \tag{6}$$

subject to condition (2). Although for $m = 1$ it is easy to see that Problem (6)–(2) admits no solution for any $V > 0$, when $m \geq 2$ we have

Theorem 8 (Martinazzi [10]) *For any $m \geq 2$ Problem (6)–(2) has solutions for some $V > 0$.*

Using the fixed point argument from [15] and a compactness result from [12], Hyder–Martinazzi recently proved:

Theorem 9 (Hyder–Martinazzi [7]) *For any $m \geq 2$ and any $V > 0$ Problem (6)–(2) has solutions.*

Also the odd-dimensional case is interesting, but more delicate since (1) becomes a non-local equation for $m = (k+1)/2$, $k \in \mathbb{N}$. Building upon previous results from [3, 9, 16], we recently proved the following existence result:

Theorem 10 (Jin–Maalaoui–Martinazzi–Xiong [8]) *Fix $m = 3/2$. For every $V \in (0, 2\pi^2]$, Problem (1)–(2) has a solution (where $(-\Delta)^{\frac{3}{2}}$ needs to be suitably defined). Moreover, if u is a non-spherical solution to (1)–(2), then $V < 2\pi^2 = \mathrm{vol}(S^3)$.*

It is interesting to compare the volume restrictions of Theorems 3 and 10 for dimension 3 and 4, with the results of Theorems 6 and 7 for dimension 6 and higher. It is then natural to ask what does the situation look like in dimension 5, i.e., when $m = 5/2$.

Conjecture 11 Problem (1)–(2) for $m = 5/2$ admits solutions for some values of $V > \mathrm{vol}(S^5)$.

In other words, we conjecture that dimension 5 is similar to dimension 6 more than to dimension 4. The intuition behind this is that the kernel of $(-\Delta)^{5/2}$ contains polynomials of degree 4, just as the kernel of $(-\Delta)^3$, while the kernels of $(-\Delta)^{3/2}$ and $(-\Delta)^2$ contain polynomials of degree 2 but not of degree 4, which is crucial in the proofs of Theorems 3 and 10. On the other hand, we remark that there seems to be no chance to extend the proofs of Theorem 6 to dimension 5, since ODE techniques do not fit well in a non-local framework.

Acknowledgements The author is supported by the Swiss National Science Foundation.

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Isoperimetric Inequalities for Complete Proper Minimal Submanifolds in Hyperbolic Space

Sung-Hong Min and Keomkyo Seo

1 Introduction

The classical isoperimetric inequality for a domain $\Sigma \subset \mathbb{R}^k$ with smooth boundary $\partial\Sigma$ is

$$k^k \omega_k \mathrm{Vol}(\Sigma)^{k-1} \leq \mathrm{Vol}(\partial\Sigma)^k, \tag{1}$$

where equality holds if and only if Σ is a ball in $\mathbb{R}^k$. Here, $\mathrm{Vol}(\Sigma)$ and $\mathrm{Vol}(\partial\Sigma)$ denote, respectively, the k and $(k-1)$-dimensional Hausdorff measures, and ω_k is the volume of the k-dimensional unit ball B^k.

It is conjectured that the inequality (1) holds for any k-dimensional compact minimal submanifold Σ of $\mathbb{R}^n$. However, it is still an open problem even for the case of minimal surfaces. See [1, 5, 6, 11, 13, 19] for the historical results.

There are also a few results for minimal submanifolds in hyperbolic space (see [7, 8, 22]). One of that is the following linear isoperimetric inequality: if Σ is a k-dimensional minimal submanifold of $\mathbb{H}^n$, then it satisfies that

$$(k-1)\mathrm{Vol}(\Sigma) \leq \mathrm{Vol}(\partial\Sigma). \tag{2}$$

In this note, based on [17], we deal with the isoperimetric inequality for complete proper minimal submanifolds in the hyperbolic space $\mathbb{H}^n$ (see [2, 3, 14–16] for the existence of complete minimal submanifolds). We identify $\mathbb{H}^n$ with the unit ball

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M. del Mar González et al. (eds.), *Extended Abstracts Fall 2013*,
Trends in Mathematics, DOI 10.1007/978-3-319-21284-5_10

B^n in $\mathbb{R}^n$ using the Poincaré ball model. Then, we have the following conformal equivalence of $\mathbb{H}^n$ with $\mathbb{R}^n$:

$$ds^2_{\mathbb{H}} = \frac{4}{(1-r^2)^2} ds^2_{\mathbb{R}},$$

where $ds^2_{\mathbb{H}}$ and $ds^2_{\mathbb{R}}$ are the hyperbolic and the Euclidean metric on B^n, respectively, and where r is the Euclidean distance from the origin. Let Σ be a k-dimensional complete proper minimal submanifold in the Poincaré ball model B^n. Regarding Σ as a subset of the unit ball B^n in Euclidean space, we can measure the k-dimensional Euclidean volume of Σ and the $(k-1)$-dimensional Euclidean volume of $\partial_\infty \Sigma$, say $\mathrm{Vol}_{\mathbb{R}}(\Sigma)$ and $\mathrm{Vol}_{\mathbb{R}}(\partial_\infty \Sigma)$, respectively. Here, the ideal boundary is $\partial_\infty \Sigma := \overline{\Sigma} \cap \partial_\infty \mathbb{H}^n$.

2 Isoperimetric Inequality

We will give the isoperimetric inequality for complete minimal submanifold Σ in $\mathbb{H}^n$ in terms of $\mathrm{Vol}_{\mathbb{R}}(\Sigma)$ and $\mathrm{Vol}_{\mathbb{R}}(\partial_\infty \Sigma)$.

2.1 Linear Isoperimetric Inequality

Theorem 1 (Linear isoperimetric inequality) *Let Σ be a k-dimensional complete proper minimal submanifold in the Poincaré ball model B^n. Then*

$$\mathrm{Vol}_{\mathbb{R}}(\Sigma) \le \frac{1}{k} \mathrm{Vol}_{\mathbb{R}}(\partial_\infty \Sigma), \tag{3}$$

where equality holds if and only if Σ is a k-dimensional unit ball B^k in B^n.

Proof See [17]. □

This theorem can be regarded as an extension of the inequality (2) in [7, 22]. It should be mentioned that inequality (3) is optimal, in contrast to (2) on which equality holds asymptotically only.

Joining (3) with the additional hypothesis for $\mathrm{Vol}_{\mathbb{R}}(\partial_\infty \Sigma)$, we prove the following isoperimetric inequality.

Theorem 2 (Isoperimetric inequality) *Let Σ be a k-dimensional complete proper minimal submanifold in the Poincaré ball model B^n. If $\mathrm{Vol}_{\mathbb{R}}(\partial_\infty \Sigma) \ge \mathrm{Vol}_{\mathbb{R}}(\mathbb{S}^{k-1}) = k\omega_k$, then*

$$k^k \omega_k \mathrm{Vol}_{\mathbb{R}}(\Sigma)^{k-1} \le \mathrm{Vol}_{\mathbb{R}}(\partial_\infty \Sigma)^k,$$

where equality holds if and only if Σ is a k-dimensional unit ball B^k in B^n.

This conclusion is sharp. Let Σ be totally geodesic in B^n. Then we have the reverse isoperimetric inequality

$$k^k \omega_k \mathrm{Vol}_{\mathbb{R}}(\Sigma)^{k-1} \geq \mathrm{Vol}_{\mathbb{R}}(\partial_\infty \Sigma)^k,$$

because the Euclidean projection of Σ onto the flat hypersurface containing $\partial\Sigma$ is volume-decreasing.

Remark 3 Martin–White [16] recently proved that if S is an open orientable surface with infinite topology, then there exists a proper area-minimizing embedding of S into $\mathbb{H}^3$ such that the limit set in $\partial_\infty \mathbb{H}^3$ is a smooth curve except for one point. If the $(k-1)$-dimensional Euclidean volume (it can be replaced by the $(k-1)$-dimensional Hausdorff measure, more generally) of $\partial_\infty \Sigma$ blows up, then so does the right hand side of the inequality. Therefore the isoperimetric inequality automatically holds.

2.2 Monotonicity

Fraser–Schoen [12] showed that if Σ is a minimal surface in the unit ball $B^n \subset \mathbb{R}^n$ with boundary $\partial\Sigma \subset \partial B^n$, and meeting ∂B^n orthogonally along $\partial\Sigma$, then

$$\mathrm{Area}(\Sigma) \geq \pi.$$

And Brendle [4] proved that if Σ is a k-dimensional minimal submanifold in the unit ball B^n and if Σ meets the boundary ∂B^n orthogonally, then the volume of Σ is bounded from below by the volume of a k-dimensional unit ball B^k in B^n. Recall that complete proper minimal submanifolds meet $\partial_\infty \mathbb{H}^n$ orthogonally. Therefore it is natural to ask whether there is also a sharp lower bound for the Euclidean volume of a complete proper minimal submanifold Σ in the Poincaré ball model, or not.

Theorem 4 *Let Σ be a k-dimensional complete proper minimal submanifold containing the origin in the Poincaré ball model B^n. Then*

$$\mathrm{Vol}_{\mathbb{R}}(\Sigma) \geq \omega_k = \mathrm{Vol}_{\mathbb{R}}(B^k),$$

where equality holds if and only if Σ is a k-dimensional unit ball B^k in B^n.

In order to prove this theorem, we need the monotonicity theorem. By measuring the Euclidean volume rather than the hyperbolic volume in B^n, we have the following result.

Theorem 5 (Monotonicity) *Let Σ be a k-dimensional complete minimal submanifold in B^n. Then the function $(\mathrm{Vol}_{\mathbb{R}}(\Sigma \cap B_r))/r^k$ is nondecreasing in r for $0 < r < 1$. In other words,*

$$\frac{d}{dr}\left(\frac{\mathrm{Vol}_{\mathbb{R}}(\Sigma \cap B_r)}{r^k}\right) \geq 0,$$

which is equivalent to

$$\frac{d}{d\rho}\left(\frac{\mathrm{Vol}_{\mathbb{R}}(\Sigma \cap B_r)}{r^k}\right) \geq 0,$$

where ρ is the hyperbolic distance from the origin.

Proof Use the coarea formula (see [9, 10, 20]). □

We also give the isoperimetric inequality for complete proper minimal submanifolds containing the origin in B^n without the additional hypothesis for $\mathrm{Vol}_{\mathbb{R}}(\partial_\infty \Sigma)$.

Theorem 6 (Isoperimetric inequality) *Let Σ be a k-dimensional complete proper minimal submanifold containing the origin in the Poincaré ball model B^n. Then*

$$k^k \omega_k \mathrm{Vol}_{\mathbb{R}}(\Sigma)^{k-1} \leq \mathrm{Vol}_{\mathbb{R}}(\partial_\infty \Sigma)^k,$$

where equality holds if and only if Σ is a k-dimensional unit ball B^k in B^n.

It would be interesting to find the classes of ambient spaces and submanifolds in them for which the classical isoperimetric inequality (1) remains valid (see [18, 21]). In this point of view, Theorems 2 and 6 give some possible classes of submanifolds of B^n satisfying the classical isoperimetric inequality.

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Total Curvature of Complete Surfaces in Hyperbolic Space

Jun O'Hara and Gil Solanes

We present a Gauss–Bonnet type formula for complete surfaces in n-dimensional hyperbolic space $\mathbb{H}^n$ under some assumptions on their asymptotic behaviour. As in recent results for Euclidean submanifolds (see Dillen–Kühnel [4] and Dutertre [5]), the formula involves an *ideal defect*, i.e., a term involving the geometry of the set of points *at infinity*.

Let S be a complete surface properly embedded in $\mathbb{H}^n$. Assume further that, when we take the Poincaré half-space model of hyperbolic space $\mathbb{H}^n$,

(i) S extends to a compact smoothly embedded surface with boundary $\overline{S} \subset \mathbb{R}^n$,
(ii) $\overline{S}$ meets the ideal boundary $\partial_\infty \mathbb{H}^n = \mathbb{R}^{n-1}$ orthogonally along a curve C.

The second condition guarantees that S is *asymptotically hyperbolic* in the sense that the intrinsic curvature $K_i(x)$ tends to -1 as $x \to \partial_\infty \mathbb{H}^n$. Note also that if S is minimal and fulfills (i), then condition (ii) is also fulfilled.

We are interested in the total *extrinsic* curvature of S, i.e., the integral on the unit normal bundle $N^1 S$ of the Lipschitz–Killing curvature K. Under the above conditions, this converges and

$$\frac{1}{\omega_{n-2}} \int_{N^1 S} K = \int_S (K_i + 1) = \lim_{\varepsilon \to 0} \int_{S_\varepsilon} (K_i + 1),$$

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M. del Mar González et al. (eds.), *Extended Abstracts Fall 2013*,
Trends in Mathematics, DOI 10.1007/978-3-319-21284-5_11

where ω_k is the volume of the unit ball in $\mathbb{R}^k$, and $S_\varepsilon = \{x \in S\colon x_n \geq \varepsilon\}$, still in the half-space model. By the Gauss–Bonnet theorem, one easily gets

$$\frac{1}{\omega_{n-2}} \int_{N^1 S} K = 2\pi\chi(S) + \lim_{\varepsilon\to 0}\left(A(S_\varepsilon) - \frac{L(C)}{\varepsilon}\right), \tag{1}$$

where A denotes the hyperbolic area, and L is the Euclidean length in the model. The previous limit is the well-known *renormalized area* of S (cf., [1]).

Our first result is a variation of (1), motivated by the Crofton formula which states that the volume of a submanifold (of $\mathbb{S}^n$, $\mathbb{R}^n$ or $\mathbb{H}^n$) equals the integral of the number of intersection points with all totally geodesic planes of complementary dimension.

Proposition 1 ([6]) *For a surface $S \subset \mathbb{H}^n$ satisfying (i) and (ii),*

$$\frac{1}{\omega_{n-2}} \int_{N^1 S} K = 2\pi\chi(S) + \lim_{\varepsilon\to 0}\left(\int_{\mathcal{L}_\varepsilon} \#(S\cap\ell)d\ell - \frac{2\omega_{n-2}}{\omega_{n-1}}\frac{L(C)}{\varepsilon}\right),$$

where $d\ell$ is a (suitably normalized) invariant measure on the space $\mathcal{L}$ of totally geodesic planes $\ell \subset \mathbb{H}^n$ of codimension 2, and $\mathcal{L}_\varepsilon \subset \mathcal{L}$ contains those planes represented in the model by a half-sphere of radius $r \geq \varepsilon$.

Motivated by Banchoff–Pohl's definition of the *area* enclosed by a space curve (see [2]), we introduce the following functional defined on closed curves $C \subset \mathbb{R}^{n-1} \equiv \partial_\infty \mathbb{H}^n$,

$$I(C) := \lim_{\varepsilon\to 0}\left(\int_{\mathcal{L}_\varepsilon} \lambda^2(C,\ell)d\ell - \frac{2\omega_{n-2}}{\omega_{n-1}}\frac{L(C)}{\varepsilon}\right),$$

where $\lambda(C,\ell)$ denotes the linking number between C and the ideal boundary of ℓ.

Combining this definition with Proposition 1 yields the following Gauss–Bonnet type formula.

Theorem 2 ([6]) *For a surface $S \subset \mathbb{H}^n$ satisfying (i) and (ii),*

$$\frac{1}{\omega_{n-2}} \int_{N^1 S} K = 2\pi\chi(S) + \int_{\mathcal{L}} (\#(S\cap\ell) - \lambda^2(C,\ell))d\ell + I(C),$$

where $d\ell$ is an invariant measure on the space $\mathcal{L}$ of totally geodesic planes $\ell \subset \mathbb{H}^n$ of codimension 2.

The equation above involves no limit as $I(C)$ can be represented by

$$I(C) = \frac{2}{\pi}\int_{C\times C} \cos\tau \sin\theta_p \sin\theta_q \frac{dpdq}{|p-q|^2}, \tag{2}$$

where θ_p (resp. θ_q) is the angle between $p-q$ and C at p (resp. at q), and τ denotes the angle between the two planes through p, q tangent at C in p and q respectively.

Theorem 2 shows in particular that $I(C)$ is invariant under Möbius transformations of C. It is interesting to recall another Möbius invariant for closed space curves: the *writhe* (see [3]). It can be expressed as

$$W(C) = \frac{1}{4\pi} \int_{C \times C} \sin\tau \sin\theta_p \sin\theta_q \frac{dpdq}{|p-q|^2}. \tag{3}$$

This suggests that some connection should exist between I and W. For the moment, this is not known.

It would be nice to have integral representations like (2) where the integrand is Möbius invariant (the same applies to (3)). So far, this is only possible for plane curves.

Theorem 3 ([7]) *For a simple closed curve $C \subset \mathbb{R}^2$,*

$$-I(C) = 2\pi + \frac{1}{\pi} \int_{C \times C} \theta \sin\theta \frac{dpdq}{|p-q|^2} \geq 2\pi,$$

where θ is a continuous determination of the angle between the two circles through p, q that are tangent to C at p and q respectively.

It is not hard to see that the integrand above is invariant under the Möbius group.

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Constant Scalar Curvature Metrics on Hirzebruch Surfaces

Nobuhiko Otoba

For each natural number $m \geq 0$, a complex surface Σ_m called *Hirzebruch surface* is defined in [8]. This surface Σ_m has a structure of $\mathbb{CP}^1$ bundle over $\mathbb{CP}^1$ for each m, and the zero-th and first surfaces Σ_0 and Σ_1 are biholomorphically equivalent to $\mathbb{CP}^1 \times \mathbb{CP}^1$ and $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, respectively. The author constructed a one-parameter family of constant scalar curvature metrics on each Hirzebruch surface. These metrics provide a certain generalization of the natural product metrics on $\Sigma_0 = \mathbb{CP}^1 \times \mathbb{CP}^1$.

1 Constant Scalar Curvature Metrics on Hirzebruch Surfaces

After recalling the definition of Hirzebruch surfaces (Sect. 1.1), we get straight to the most fundamental properties of the metrics we will be dealing with (Sect. 1.2).

1.1 The Hirzebruch Surfaces Σ_m*,* $m \geq 0$

Let $p\colon S^3 \to \mathbb{CP}^1$ be the projection of Hopf fibration, which assigns a point (z, w) of the 3-sphere $S^3 = \{(z, w) \in \mathbb{C}^2 \mid |z|^2 + |w|^2 = 1\}$ to the point $[z : w]$ of the complex projective line $\mathbb{CP}^1$. The circle group $S^1 = \{e^{2\pi i\theta} \in \mathbb{C} \mid \theta \in \mathbb{R}\}$ acts freely on S^3 on the right by $(z, w) \cdot e^{2\pi i\theta} = (ze^{-2\pi i\theta}, we^{-2\pi i\theta})$, and the orbits of this action coincide

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M. del Mar González et al. (eds.), *Extended Abstracts Fall 2013*,
Trends in Mathematics, DOI 10.1007/978-3-319-21284-5_12

with the fibers of the projection p; the Hopf fibration is a principal S^1 bundle over $\mathbb{CP}^1$.

The isomorphism classes of all principal S^1 bundles over $\mathbb{CP}^1$ form an abelian group isomorphic to the integers $\mathbb{Z}$, see [10], and the Hopf fibration S^3 corresponds to $1 \in \mathbb{Z}$ in this group. The principal S^1 bundle corresponding to an integer m greater than or equal to 2 is the lens space $S^3/(\mathbb{Z}/m\mathbb{Z})$, which is obtained by making the quotient of S^3 by the cyclic subgroup $\{e^{2\pi i(l/m)} \mid l = 0, 1, \ldots, m-1\} \cong \mathbb{Z}/m\mathbb{Z}$ of its structure group S^1. We denote this lens space $S^3/(\mathbb{Z}/m\mathbb{Z})$ by mS^3, and the trivial bundle $\mathbb{CP}^1 \times S^1$ by $0S^3$; this last element plays the role of 0 in the abelian group.

Suppose the circle group S^1 acts effectively on S^2 on the left by rotations, and consider the S^2 bundle $\Sigma_m := \left(mS^3\right) \times_{S^1} S^2$ associated with mS^3 with respect to this action $S^1 \curvearrowright S^2$, $m \geq 0$. The complex structures on the base space $\mathbb{CP}^1$ and the fiber $S^2 \cong \mathbb{CP}^1$ define a complex structure J_m on the total space Σ_m. (Note that a S^1 connection on Σ_m is required to define J_m; however, J_m does not depend on the S^1 connection chosen.) The complex surface (Σ_m, J_m) obtained in this way is called the m-th *Hirzebruch surface*, and its projection is denoted by $\pi_m\colon \Sigma_m \to \mathbb{CP}^1$. It is easy to see that the zero-th surface is $\mathbb{CP}^1 \times \mathbb{CP}^1$, and the first surface is known to be biholomorphically equivalent to $\mathbb{CP}^2 \# \overline{\mathbb{CP}}^2$, the connected sum of two complex projective planes with the usual and inverse orientations, see [3].

1.2 Constant Scalar Curvature Metrics $g_m(R)$ on Σ_m

There are natural product metrics $g_{FS} + r^2 g_{FS}$ on the zero-th Hirzebruch surface $\mathbb{CP}^1 \times \mathbb{CP}^1$, where g_{FS} stands for the Fubini–Study metric and $r > 0$. These metrics are Kähler, and have the following two properties as well: (i) they have constant scalar curvature; and (ii) they respect the fiber bundle structure on Σ_0 in the sense that they are product metrics. Paying attention to these properties, we construct the following Riemannian metrics on Hirzebruch surfaces.

Theorem 1 (Existence of the metrics) *For each natural number $m \geq 1$ and each real number $R \in \mathbb{R}$, there exists a conformally Kähler metric $g_m(R)$ on the m-th Hirzebruch surface Σ_m with the following properties:*

(i) the scalar curvature of $g_m(R)$ is constant on Σ_m and equal to R;
(ii) the projection of Hirzebruch surface $\pi_m\colon (\Sigma_m, g_m(R)) \to (\mathbb{CP}^1, g_{FS})$ is a Riemannian submersion with totally geodesic fibers.

We remark that these metrics are neither Kähler nor Einstein.

2 Construction of the Metrics $g_m(R)$

After the physicist Page constructed an Einstein metric on $\mathbb{CP}^2\#\overline{\mathbb{CP}}^2$, see [13], Bérard-Bergery not only translated his construction into mathematics, but also characterized and generalized the Page metric from the perspective of cohomogeneity one Riemannian geometry, see [1]. Imposing high symmetry on certain geometric structures and making use of his result, Einstein–Weyl structures [12] and extremal Kähler metrics [9] were constructed.

The constant scalar curvature metrics $g_m(R)$ from Theorem 1 can also be described within the realm of Bérard-Bergery. While the constructions of the two geometric structures mentioned above are reduced to complicated systems of ODE's, for the construction of $g_m(R)$ we have only to solve the following comparatively simple free-boundary value problem of an ODE

$$\frac{d^2f}{dt^2} = -m^2f^3 - \frac{R-8}{2}f,$$
$$f(\pm T) = 0, \quad \frac{df}{dt}(-T) = 1, \quad \frac{df}{dt}(T) = -1.$$

Here, $f(t)$ is a function which defines a rotationally symmetric metric $dt^2 + f^2(t)d\theta^2$ on the typical fiber S^2 of Σ_m. This boundary value problem possesses a unique solution for each $m > 0$ and $R \in \mathbb{R}$; moreover, definite integrals $\int_{-T}^{T} f^l(t)dt$ of any nonnegative integer power of its solution can be written in terms of elementary functions and elliptic integrals of m and R. For example, the first two integrals are

$$2T = \frac{2}{\sqrt[4]{2m^2+\beta^2}}K(k) \qquad \text{and} \qquad \int_{-T}^{T} f(t)dt = \frac{2\sqrt{2}}{m}\arcsin(k), \tag{1}$$

where

$$\beta = -\frac{R-8}{2}, \quad k = \sqrt{\frac{1}{2}\left(1+\frac{\beta}{\sqrt{2m^2+\beta^2}}\right)}, \quad K(k) = \int_0^1 \frac{dx}{\sqrt{1-x^2}\sqrt{1-k^2x^2}}. \tag{2}$$

These integral formulas will prove important to analyze the metrics $g_m(R)$.

3 Further Properties of $g_m(R)$ in Conformal Geometry

An original motivation to consider these constant scalar curvature metrics was to construct new nontrivial conformal classes where the Obata-type uniqueness of Yamabe minimizers holds (namely, uniqueness up to homothety and conformal transformations). This is not yet to be achieved, but the speaker has been aware of the following.

Proposition 2 (Behavior of the Yamabe functional) *The Yamabe functional Y takes the following value at the metric* $g_m(R)$:

$$Y\left(g_m(R)\right) = R\sqrt{Vol\left(g_m(R)\right)} = 2\sqrt[4]{2}\pi R\sqrt{\frac{\arcsin(k)}{m}}.$$

There exists $\varepsilon > 0$ *depending on m so that, if* $R < \varepsilon$ *then* $g_m(R)$ *is a unique Yamabe minimizer in its conformal homothety class. Moreover, for large m, if* $R < 5$ *then* $g_m(R)$ *is strictly stable with respect to the Yamabe functional. On the other hand, if* $R > 24$ *(and regardless of m), then* $g_m(R)$ *is not stable and thus not a Yamabe minimizer.*

Sketch of proof For the computation of $Y\left(g_m(R)\right)$ it suffices to compute the volume of $g_m(R)$ since they have constant scalar curvature. Property (ii) in Theorem 1 enables us to *integrate along the fibers*, and thus it follows that Vol $\left(g_m(R)\right)$ is just the product of the areas of the base space $(\mathbb{CP}^1, g_{FS})$ and the fiber $(S^2, dt^2 + f^2(t)d\theta^2)$. Therefore, thanks to the integration formula in (1), we have

$$\text{Vol}\left(g_m(R)\right) = \pi \cdot 2\pi \int_{-T}^{T} f(t)dt = 4\sqrt{2}\pi^2 \arcsin(k)/m,$$

whence the claim follows.

A slight modification of Böhm–Wang–Ziller [4, Theorem 5.1] ensures the existence of the $\varepsilon > 0$. The last assertions concerning stability with respect to the Yamabe functional are consequences of the following estimates for the first eigenvalue of its Laplacian.[1] From Theorem 1-(ii) again, it follows that the first eigenvalues

$$\lambda_1 = \lambda_1\left(\Sigma_m, g_m(R)\right), \quad \check{\lambda}_1 = \check{\lambda}_1\left(\mathbb{CP}^1, g_{FS}\right) = 8, \quad \hat{\lambda}_1 = \hat{\lambda}_1\left(S^2, dt^2 + f^2(t)d\theta^2\right)$$

of the total space, base space and fiber, respectively, are related by

$$\min\{8, \hat{\lambda}_1\} \leq \lambda_1 \leq 8 \tag{3}$$

(cf., [2, 5]). On the other hand, through Cheeger's isoperimetric inequality [6] and Hersch's inequality [10], $\hat{\lambda}_1$ is estimated as $h^2/4 \leq \hat{\lambda}_1 \leq 8\pi/a$, where h and a are, respectively, the isoperimetric constant and the area of the rotationally symmetric metric $dt^2 + f^2(t)d\theta^2$. Moreover, results from Ritoré [14] show that the value h is

[1] We note that in the fourth dimension, a constant scalar curvature metric g is stable with respect to the Yamabe functional if and only if its scalar curvature R and the first eigenvalue λ_1 of its Laplacian $\Delta = -\text{trace } \nabla^2$ satisfy the inequality $\lambda_1 \geq R/3$, see [11].

attained by a domain of area $a/2$ whose boundary is a *nodoid* of length $4T$; thus h is equal to $8T/a$. Since T and $a = 2\pi \int_{-T}^{T} f(t)dt$ can be written in terms of m and R (integration formulas (1)), we finally obtain

$$\frac{m^2}{2\pi^2\sqrt{2m^2+\beta^2}}\frac{K^2(k)}{\arcsin^2(k)} \leq \hat{\lambda}_1 \leq \frac{\sqrt{2}m}{\arcsin(k)}. \tag{4}$$

Combining the inequalities (3) and (4), the stability and unstability assertions hold (for the stability result, we performed numerical computations using a computer).

□

Finally, we collect some properties of $g_m(R)$ concerning the Weyl functional.

Proposition 3 (Behavior of the Weyl functional) *The Weyl functional takes the value*

$$\int_{\Sigma_m} |W|^2 dVol = \frac{2\pi^2}{m}\left(72m^2 + \frac{59}{3}R^2 - 272R + 960\right)\sqrt{2}\arcsin(k) - 4\pi^2(19R - 120)$$

at the metric $g_m(R)$, where $|W|$ refers to the tensor norm of the Weyl curvature. Some metrics have vanishing derivative with respect to the Weyl functional in the direction of the one-parameter family $\{g_m(R)\}_{R\in\mathbb{R}}$ (for fixed m), but none of them is Bach flat. In fact, any $g_m(R)$ is not B^t-flat for any $t \in \mathbb{R}$ in the sense of Gursky–Viaclovsky [7].

Additionally, the Q-curvature of $g_m(R)$ is not constant on Σ_m, while some conformal classes $[g_m(R)]$ contain other metrics than $g_m(R)$ with constant Q-curvature.

Remark 4 The existence of constant Q-curvature metrics in some conformal classes $[g_m(R)]$ is a consequence of the positivity of total Q-curvature. The Weyl functional takes as small values as the total Q-curvatures are positive. According to computer experiments, the correct minimum value of the Weyl functional along the families $\{g_m(R)\}$ is approximately 1,000.

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Isoperimetric Inequalities for Extremal Sobolev Functions

Jesse Ratzkin and Tom Carroll

1 Introduction

Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with boundary of class $\mathcal{C}^1$. One can measure various geometric and physical quantities attached to Ω, such as volume, perimeter, diameter, in-radius, torsional rigidity, and principal frequency. The first chapter of [16] contains a long list of such interesting quantities, as well as their values for standard shapes such as disks, rectangles, strips, and triangles. We are most interested in *isoperimetric inequalities* in the style of Pólya and Szegö, as described in their monograph [16]. In these inequalities, one seeks to minimize (or maximize) one quantity, such as perimeter or principal frequency, while holding another quantity, such as volume, fixed. There is a huge literature attached to this subject, which we make no attempt to survey here.

The specific quantity we investigate is

$$\mathcal{C}_p(\Omega) = \inf \left\{ \frac{\int_\Omega |\nabla u|^2 d\mu}{\left(\int_\Omega |u|^p d\mu\right)^{2/p}} : u \in W_0^{1,2}(\Omega), u \not\equiv 0 \right\}, \tag{1}$$

which gives the best constant in the Sobolev inequality

$$u \in W_0^{1,2}(\Omega) \Rightarrow \|u\|_{L^p(\Omega)} \leq \mathcal{C}_p(\Omega)^{-1/2} \|\nabla u\|_{L^2(\Omega)}. \tag{2}$$

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M. del Mar González et al. (eds.), *Extended Abstracts Fall 2013*,
Trends in Mathematics, DOI 10.1007/978-3-319-21284-5_13

Here we must take $1 \le p < 2n/(n-2)$, and for this range of exponents Rellich compactness tells us that $\mathcal{C}_p(\Omega)$ is realized by a nontrivial function $\phi > 0$ satisfying

$$\Delta\phi + \Lambda\phi^{p-1} = 0, \qquad \Lambda = \mathcal{C}_p(\Omega)\left(\int_\Omega \phi^p d\mu\right)^{\frac{2-p}{p}}. \tag{3}$$

It is useful to point out the two instances when (3) is a linear PDE: $\mathcal{C}_1(\Omega) = 4/P(\Omega)$, where $P(\Omega)$ is the torsional rigidity of Ω, and $\mathcal{C}_2(\Omega) = \lambda(\Omega)$ is the principal frequency, i.e., the first eigenvalue of the Laplacian with Dirichlet boundary conditions. We are interested in two central questions:

(i) Which of the properties of $P(\Omega)$ and $\lambda(\Omega)$ also hold for $\mathcal{C}_p(\Omega)$, at least for $1 \le p \le 2$?
(ii) Can we track the behavior of $\mathcal{C}_p(\Omega)$ and its extremal function ϕ as p varies?

Many other people have also obtained interesting results which complement our theorems stated below. For a small sample of the literature, please consult [1, 7, 8, 13] (which is, by no means, an exhaustive list).

2 Theorems

In this section we state some of the results we obtain in the direction of the two central questions outlined above. In [3] we proved

Theorem 1 *Let $\Omega \subset \mathbf{R}^n$ be a bounded domain with boundary of class $\mathcal{C}^1$, and let $1 \le p < q < 2n/(n-2)$. Then*

$$|\Omega|^{2/p}\mathcal{C}_p(\Omega) > |\Omega|^{2/q}\mathcal{C}_q(\Omega).$$

In the case $n = 2$, $p = 1$, and $q = 2$ we recover $\lambda(\Omega)P(\Omega) < 4|\Omega|$, which one can find in [16, Section 5.4]. In [3] we also prove

Theorem 2 *Let Ω be a convex domain with in-radius R, and let $u > 0$ solve $\Delta u + \Lambda u^{p-1} = 0$ with Dirichlet boundary data. Then,*

$$u_M^{2-p} := (\sup_{x\in\Omega}\{u(x)\})^{2-p} \le \frac{2\Lambda R^2}{pA_p^2}, \quad A_p = \int_0^1 \frac{dt}{\sqrt{1-t^p}} = \sqrt{\pi}\frac{\Gamma\left(1+\frac{1}{p}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{p}\right)}.$$

Equality occurs if Ω is a slab, e.g., $\Omega = \{x \in \mathbf{R}^n : 0 < x_n < R\}$.

We recover a theorem of Hersch [10] if we set $p = 2$ and a theorem of Sperb [17] if we set $p = 1$.

In [4] we proved two theorems specific to two dimensions. The first of these generalizes the classical Schwarz Lemma from complex analysis, in the same vein as the results of Burckel–Marshall–Minda–Poggi-Corradini–Ransford [2].

Theorem 3 *Let* $\mathbf{D}$ *be the unit disk in the complex plane* $\mathbf{C}$*, let* $f : \mathbf{D} \to \mathbf{C}$ *be conformal, and let* $p \geq 1$*. Then the function*

$$r \longmapsto \frac{\mathcal{C}_p(f(r\mathbf{D}))}{\mathcal{C}_p(r\mathbf{D})} = \frac{r^{4/p}}{\mathcal{C}_p(\mathbf{D})} \mathcal{C}_p(f(r\mathbf{D}))$$

is strictly decreasing, unless f *is linear (in which case the displayed function is constant).*

Setting $p = 2$ we recover a theorem of Laugesen–Morpurgo [12]. In [4] we also prove the following reverse-Hölder inequality.

Theorem 4 *Let* $\Omega \subset \mathbf{R}^2$ *be a bounded domain with* $\mathcal{C}^1$ *boundary, let* $p \geq 1$*, and let* ϕ *be an extremal Sobolev function in* $W_0^{1,2}(\Omega)$*. Then,*

$$\left(\int_\Omega \phi^{p-1} d\mu \right)^2 \geq \frac{8\pi}{p\mathcal{C}_p(\Omega)} \left(\int_\Omega \phi^p d\mu \right)^{\frac{2p-2}{p}} . \tag{4}$$

Moreover, equality only occurs if Ω *is a round disk.*

Setting $p = 2$ we recover a theorem of Payne–Rayner [14]. As a corollary to (4) we obtain the following isoperimetric inequality for a conformally flat metric on Ω.

Corollary 5 *Let* $\Omega \subset \mathbf{R}^2$ *be a bounded domain with* $\mathcal{C}^1$ *boundary, let* $p \geq 1$*, and let* ϕ *be an extremal Sobolev function. Consider the singular, conformally flat metric* $\tilde{ds} = |\nabla\phi| ds$ *on* Ω*, where* ds *is the Euclidean length element. Let* $\tilde{A}$ *be the area of* Ω *with respect to* $\tilde{ds}$*, and let* $\tilde{L}$ *be the length of* $\partial\Omega$ *with respect to* $\tilde{ds}$*. Then,*

$$\tilde{L}^2 \geq \frac{8\pi}{p} \tilde{A}.$$

Moreover, if Ω *is a disk then the above inequality is an equality.*

We remark that, in the case $p = 2$, the Gauss curvature of our conformally flat metric on the disk is not monotone in the radial direction. This is in contrast with the equality condition of the isoperimetric inequality Topping obtained in [18].

In [5] we prove an extension of (4) to n dimensions, but our result is not quite optimal.

Theorem 6 *Let $1 < p < 2n/(n-2)$, let $\Omega \subset \mathbf{R}^n$ be a bounded domain with $\mathcal{C}^1$ boundary, and let Ω^* be the ball with $|\Omega| = |\Omega^*|$. Then*

$$\left(\int_\Omega \phi^{p-1} d\mu\right)^2 \geq n^2 \omega_n^{2/n} |\Omega|^{\frac{n-2}{n}} \left[\frac{2}{p\mathcal{C}_p(\Omega)} - \frac{n-2}{n\mathcal{C}_p(\Omega^*)}\right] \left(\int_\Omega \phi^p d\mu\right)^{\frac{2p-2}{p}}, \qquad (5)$$

where ω_n is the volume of the unit ball. Moreover, equality implies that Ω is a ball.

Notice that the quantity in brackets on the right hand side of (5) can possibly be negative, in which case the statement of the theorem is vacuous. As before, we recover the results from Payne–Rayner [15] by setting $p = 2$. The original inequality of Payne–Rayner has been improved by Kohler-Jobin [11], and again by Chiti [6]; we are currently working towards similar improvements to (5).

3 Questions

In this section we mention two interesting and related open questions.

The first question concerns the level sets of the extremal function ϕ and, roughly, asks if ϕ becomes more "peaked" as p increases. To state this question more precisely, let $\phi_p > 0$ be an extremal function for exponent p, with the normalization $\sup_{x\in\Omega} \phi_p(x) = 1$. Associated to ϕ_p we define the distribution function

$$\mu_p\colon [0,1] \longrightarrow \mathbf{R}, \quad \mu_p(t) = |\{x \in \Omega : \phi_p(x) > t\}|.$$

Is it true that

$$1 \leq p < q < \frac{2n}{n-2} \Longrightarrow \mu_p(t) > \mu_q(t)\ ? \qquad (6)$$

In the case that Ω is a ball, we know (6) is true for the special case of $p = 1$ and $q = 2$. We also know that in the critical case $p = 2n/(n-2)$ one can find extremal functions which are very peaked. Therefore, one should expect that, as p approaches the critical exponent, the extremal function should become more peaked.

For our second question, consider the evolution equation

$$\frac{\partial u}{\partial y} = u^{2-p}\Delta u, \quad u\colon [0,T) \times \Omega \longrightarrow \mathbf{R}. \qquad (7)$$

A fairly standard separation of variables argument shows that stationary states of (7) satisfy the elliptic PDE in (3). We have not encountered (7) elsewhere, but we hope that studying it could lead to new and interesting results for extremal Sobolev functions. Our motivation for this approach is a paper by Graversen–Rao [9] about the principal frequency.

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Part II
Type Theory, Homotopy Theory, and Univalent Foundations

Editors

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Foreword

The seven abstracts appearing in Part II of the present volume have been written by speakers at the conference *Type Theory, Homotopy Theory and Univalent Foundations*, held at the Centre de Recerca Matemàtica (CRM-Barcelona), from September 23th to 27th, 2013. The conference was organised by Nicola Gambino, from the University of Leeds (UK), and Joachim Kock from the Universitat Autònoma de Barcelona (Catalunya, Spain). A total of 65 participants (22 of which from outside Europe) attended to the conference, including 13 participants sponsored by a special conference participation grant coming from the National Science Foundation of the USA, a support that we gratefully acknowledge.

The subject of the conference was the topic of the preceding year's special programme at the Institute for Advanced Study in Princeton, and was timed to serve as a venue for presenting the progress obtained during that year, at the IAS and elsewhere. Approximately half of the talks were given by participants of the IAS programme, the other half by researchers working elsewhere. For related content, including published version of some of the material presented at the conference, we wish to refer to the special issue of *Mathematical Structures in Computer Science*, entitled "*From type theory and homotopy theory to Univalent Foundations of Mathematics*", appearing in 2015.

The conference featured fruitful interaction between different fields, facilitated by the good atmosphere at the CRM-Barcelona. It is a pleasure to thank the CRM direction and staff for smooth organisation of practical matters and for the good working conditions provided – and for accepting to work on a public holiday, La Mercè, the patron saint of the city of Barcelona.

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Univalent Categories and the Rezk Completion

Benedikt Ahrens, Krzysztof Kapulkin, and Michael Shulman

Please note the Erratum to this chapter at the end of the book

When formalizing category theory in traditional set-theoretic foundations, a significant discrepancy between the foundational notion of "sameness" – *equality* – and its practical use arises: most category-theoretic concepts are invariant under weaker notions of sameness than equality, namely isomorphism in a category or equivalence of categories. We show that this discrepancy can be avoided when formalizing category theory in Univalent Foundations.

The *Univalent Foundations* is an extension of Martin-Löf Type Theory (MLTT) recently proposed by V. Voevodsky [4]. Its novelty is the *Univalence Axiom* (UA) which closes a slight insufficiency of MLTT by providing "more equalities between types". This is obtained by identifying equality of types with equivalence of types. To prove that two types are equal, it thus suffices to construct an equivalence between them.

When formalizing category theory in the Univalent Foundations, the idea of Univalence carries over. We define a *precategory* to be given by a type of objects and, for each pair (x, y) of objects, a *set* $\hom(x, y)$ of morphisms, together with identity and composition operations, subject to the usual axioms. In the Univalent Foundations, a type X is called a *set* if it satisfies the principle of Uniqueness of Identity Proofs, that is, for any $x, y : X$ and $p, q : \mathrm{Id}(x, y)$, the type $\mathrm{Id}(p, q)$

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M. del Mar González et al. (eds.), *Extended Abstracts Fall 2013*,
Trends in Mathematics, DOI 10.1007/978-3-319-21284-5_14

is inhabited. This requirement avoids the introduction of coherence axioms for associativity and unitality of categories.

A *univalent* category is then defined to be a category where the type of isomorphisms between any pair of objects is equivalent to the identity type between them. We develop the basic theory of such univalent categories: functors, natural transformations, adjunctions, equivalences, and the Yoneda lemma.

Two categories are called *equivalent* if there is a pair of adjoint functors between them for which the unit and counit are natural isomorphisms. Given two categories, one may ask whether they are equal in the type-theoretic sense – that is, if there is an identity term between them in the type of categories – or whether they are equivalent. One of our main results states that for univalent categories, the notion of (type-theoretic) *equality* and (category-theoretic) *equivalence coincide*. This implies that properties of univalent categories are automatically invariant under equivalence of categories – an important difference to the classical notion of categories in set theory, where this invariance does not hold.

Moreover, we show that any category is weakly equivalent to a univalent category – its *Rezk completion* – in a universal way. It can be considered as a truncated version of the Rezk completion for Segal spaces [3]. The Rezk completion of a category is constructed via the Yoneda embedding of a category into its presheaf category, a construction analogous to the *strictification* of bicategories by the Yoneda embedding into Cat, the 2-category of categories.

Large parts of this development have been formally verified [1] in the proof assistant Coq, building on Voevodsky's *Foundations* library. In particular, the formalization includes the Rezk completion together with its universal property.

A preprint covering the content of this note with more details is available on the arXiv [2].

Acknowledgements The authors thank Vladimir Voevodsky for much assistance during this project. This material is based upon work supported by the National Science Foundation. Any opinions, findings and conclusions or recommendations expressed in this material are those of the authors and do not necessarily reflect the views of the National Science Foundation.

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Covering Spaces in Homotopy Type Theory

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Covering spaces play an important role in classical homotopy theory, whose algebraic characteristics have deep connections with fundamental groups of underlying spaces. It is natural to ask whether these connections can be stated in homotopy type theory (HoTT), an exciting new framework coming with an interpretation in homotopy theory. This note summarizes the author's attempt to recover the classical results (e.g., the classification theorem) so as to explore the expressiveness of the new foundation. Some interesting techniques employed in the current proofs seem applicable to other constructions as well.

1 Introduction

Homotopy type theory (HoTT) is an exciting new interpretation of intensional type theory in terms of ∞-groupoids or topological spaces up to homotopy, which provides an abstract, *synthetic* framework for homotopy theory [2–7, 9, 10]. Under this interpretation, types are spaces, terms are points, sets are discrete spaces (up to homotopy), and functions are continuous maps (Our terminology follows the HoTT book [2]; in particular, *sets* means types of homotopy level zero). It is natural to ask whether we can restate various homotopy invariant concepts and theorems known in classical theories. In this note a fundamental construction will be explored, namely the *covering spaces*. It turns out that we can express covering spaces (up to homotopy) elegantly in HoTT as follows.

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M. del Mar González et al. (eds.), *Extended Abstracts Fall 2013*,
Trends in Mathematics, DOI 10.1007/978-3-319-21284-5_15

Definition 1 A *covering space* of a type (space) A is a family of sets indexed by A.

That is, the type of covering spaces of A is simply $A \to \mathsf{Set}$, where Set is the type of all sets. The key insight here is that *continuity is enforced* by the framework, and thus it is sufficient to specify only the behavior on individual points. In this particular case, it is enough to say that each fiber (without mentioning neighborhoods) of the projection is a discrete set. To verify that this formulation matches the classical one, we proved in HoTT the classification theorem of covering spaces, and that homotopy equivalent classes of paths with one end fixed form a universal covering space. Let $\pi_1(A)$ be the fundamental group of a pointed space A and G-Set be the type of all G-sets, the sets equipped with an action of the given group G. The classification theorem asserts the equivalence between $A \to \mathsf{Set}$ and $\pi_1(A)$-Set for any pointed, path-connected A.

It is worth emphasizing that every proof mentioned in this report has been fully mechanized [1] and checked by the proof assistant Agda [8], thanks to HoTT's ability to express many topological concepts (paths, homotopies, truncation, connectedness, circles, intervals, etc.) fairly easily from its axioms.

Another feature of HoTT is that, being a proof-relevant mathematics, it is able to capture some subtleties that are not immediately visible in the traditional framework. For example, the equivalence between $A \to \mathsf{Set}$ and $\pi_1(A)$-Set in the classification theorem will associate the $\pi_1(A)$ itself (as a $\pi_1(A)$-set) with some universal covering space. However, in general there are more than one such covering space and there is no continuous choice among them, unless we fix one point in the associated covering space. This fact can be elegantly stated in HoTT using truncation, due to the unification of logic and data where theorems themselves are also spaces and they can contain different proofs as different points.

The following sections will outline the two results of covering spaces we reproved in HoTT, namely the classification and the universality. An interesting technical device used for the classification theorem will also be mentioned.

2 Classification

The goal in this section is to show that there is an equivalence between covering spaces of a pointed, path-connected A and $\pi_1(A)$-Set. The definitions of groups and G-sets closely follow the classical ones, with the requirement that the underlying type must be a set. The type A is path-connected if the 0-truncation of A, written $\|A\|_0$, is contractible.

Theorem 2 *For any path-connected, pointed type A,* $(A \to \mathsf{Set}) \simeq \pi_1(A)$-$\mathsf{Set}$.

To establish the equivalence, it is necessary to give a map from all covering spaces to all $\pi_1(A)$-sets, and an inverse map of it. The first map can be easily constructed from the lifting property of the given covering space (as a family of sets). More precisely, suppose $F{:}A \to \mathsf{Set}$ is a covering space of A and a is the distinguished point of A. The transport function $\mathtt{transport}_{x.F(x)}(p)$ associates an automorphism of the set $F(a)$ to each loop p at a. Because F is a family of sets, the type of automorphisms of $F(a)$ is also a set. By the universal property of truncation, an element in the 0-truncated loop space, $\pi_1(A)$, also gives rise to an automorphism of $F(a)$. We then complete the construction of a $\pi_1(A)$-set by considering the set $F(a)$ along with the above process as the action of $\pi_1(A)$ on $F(a)$.

The inverse map, from $\pi_1(A)$-sets to covering spaces of A, is more technically involved. The high-level idea is:

(1) put the given $\pi_1(A)$-set as the fiber over the distinguished point of A;
(2) forge other fibers by introducing a formal transport; and
(3) throw in equations to mimic functoriality of `transport` (so that the formal one behaves as the real one).

We exploit higher-inductive types to achieve the final step. More formally, suppose a is the distinguished point of A. Given a $\pi_1(A)$-set X equipped with an action of type $X \to \pi_1(A) \to X$ (written $x \cdot l$ for $x : A$ and $l : \pi_1(A)$), let $\cdot_0$ be the concatenation of two 0-truncated paths. The higher-inductive type is a family of sets `ribbon` indexed by A with the following two constructors:

$$t\colon \prod_{(a':A)} X \longrightarrow \left\|a =_A a'\right\|_0 \longrightarrow \mathtt{ribbon}(a'),$$

$$\alpha : \prod_{(a':A)} \prod_{(x:X)} \prod_{(l:\pi_1(A))} \prod_{(p:\|a=_A a'\|_0)} t(a')(x \cdot l)(p) =_{\mathtt{ribbon}(a')} t(a')(x)(l \cdot_0 p).$$

The constructor t is the formal transport function to forge other fibers, and α enforces the required functoriality. Note that the formal transport t is taking a 0-truncated path of type $\|a =_A a'\|_0$ so that it goes along with the $\pi_1(A)$-action in the type of α.

Although conceptually similar to a standard argument in classical homotopy theory, the details of this proof are quite different. For example, we do not need to (explicitly) put a topology on the `ribbon` space. Because of these differences, there is only a thin layer between these high-level ideas of the classical proof and the syntactical proof in HoTT. As a consequence, computer-checking becomes practical for HoTT.

The remaining parts are the proof that the two maps are inverse to each other. This is mostly straightforward except one thing: suppose we start from a covering space $F{:}A \to \mathsf{Set}$. We need to show that the associated `ribbon` and F are the same. By functional extensionality and the Univalence Axiom, this reduces to a fiberwise equivalence between `ribbon` and F. The direction from $\mathtt{ribbon}(a')$ to $F(a')$ is to realize the formal transport t by the real `transport`. The other direction from $F(a')$ to $\mathtt{ribbon}(a')$ involves locating a point in the fiber $F(a)$ and a (truncated) path $p : \|a = a'\|_0$, as they are required by the formal transport t. However, the formal transport t needs a 0-truncated path but the path-connectedness condition only gives a (-1)-truncated path. There is still hope because we can show that the α constructor forces different choices for this path to give the same point, and thus in principle a (-1)-truncated path should suffice, which is to say that merely the existence of such path should be sufficient for our construction. The essence of this argument comes down to the following general lemma:

Lemma 3 (Factorization of constant functions) *Let f be a function of type $B \to C$ where C is a set, and let $|-|_{-1}$ be the projection function from B to $\|B\|_{-1}$. If*

$$\prod_{b_1,b_2:B} f(b_1) =_C f(b_2)$$

then there is a function $g\colon \|B\|_{-1} \to C$ such that

$$f \equiv g \circ |-|_{-1}.$$

With $B :\equiv (a = a')$ and the required constancy condition from the α constructor, this lemma enables us to access the path in $\|a = a'\|_{-1}$ even though it is (-1)-truncated, and hence completes the main proof. The proof of this lemma depends on another high-inductive type but is beyond the scope of this note.

The final remark is that, the HoTT proof requires this factorization lemma (while the classical proof does not) because we are actually proving a stronger theorem, in the sense that the proof will associate “equivalent” equivalences to homotopically equivalent pointed spaces. Intuitively, this holds because at each step of the construction, a choice can be made in a continuous way. The factorization lemma is one of the building blocks.

3 Universality

Let A be a path-connected type with a distinguished point a. It is well-known that the homotopy equivalence classes of paths from a in A form a *universal* covering space, in the sense that it is homotopy initial in the category of pointed covering spaces of A (where the morphisms are covering projections). This particular space can be concisely written down in HoTT as follows:

$$U_A :\equiv \lambda(a'{:}A).\ \left\|a =_A a'\right\|_0 .$$

We reproved that every simply-connected, pointed covering space of A is equivalent to U_A, and that this covering space is indeed homotopy initial (in the category mentioned above). The proof is rather simple compared to that of the classification theorem.

The pointedness condition of covering spaces helps us pin down one equivalence between two universal covering spaces. Without it, there is no canonical choice among possibly many different equivalences, and one can only show the mere existence of such. Let F_1 and F_2 be the two covering spaces in discussion. The mere existence of such equivalence can be stated in HoTT using (-1)-truncation:

$$\left\| \prod_{a':A} F_1(a') \simeq F_2(a') \right\|_{-1} .$$

The intuition is that, even though the choices made in the construction of the equivalences might not all be continuous, the type representing the *mere existence* of it is continuous in the parameters. This matches up with the classical existential quantifier, which only cares about the existence of one element. In fact, one can model much classical reasoning by truncating every theorem down to the (-1)-level. Intuitively, while the interpretation enforces continuity, we can effectively relax that condition by a suitable truncation.

4 Conclusion and Future Work

This note confirms that one can reason about covering spaces in HoTT. There are many interesting future directions, for example: (i) the correspondence between the category of covering spaces and that of $\pi_1(A)$-sets, not just the objects; and (ii) the more general form $A \to n\text{-}\mathtt{Type}$ where covering spaces are the special case where $n = 0$.

Acknowledgements I want to thank Carlo Angiuli, Steve Awodey, Spencer Breiner, Guillaume Brunerie, Daniel Grayson, Robert Harper, Chris Kapulkin, and Ed Morehouse for helping me learn the classical theory, improve the presentation, and revise previous versions. This material is based upon work supported by the National Science Foundation under Grant No. 1116703.

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Towards a Topological Model of Homotopy Type Theory

Paige North

The model of homotopy type theory in simplicial sets [7] has proven to be a grounding and motivating influence in the development of homotopy type theory. The classical theory of topological spaces has also proven to be motivational to the subject. Though the Quillen equivalence between simplicial sets and topological spaces provides, in some weak sense, a model in topological spaces, we explore the extent to which the category of topological spaces may be a more direct and strict model of homotopy type theory. We define a notion of model of homotopy type theory, and show that the category of topological spaces fully embeds into such a model.

Definition 1 A *model of dependent type theory* $(\mathcal{C}, \mathcal{P})$ is a category $\mathcal{C}$ with a replete subcategory $\mathcal{P}$ such that

(i) $\mathcal{C}$ has a terminal object;
(ii) every map of $\mathcal{C}$ whose codomain is terminal is in $\mathcal{P}$;
(iii) $\mathcal{C}$ has all pullbacks along all morphisms of $\mathcal{P}$; and
(iv) for all pairs of morphisms f in $\mathcal{C}$ and p in $\mathcal{P}$, the pullback of p along f is in $\mathcal{P}$.

We call the morphisms of $\mathcal{P}$ *dependent projections* or just *projections*. For $B \in \mathcal{C}$, we consider the usual slice categories $\mathcal{C}/B$ and $\mathcal{P}/B$, but here we will also consider $\mathcal{C}_{\mathcal{P}}/B$, the full subcategory of $\mathcal{C}/B$ spanned by those objects which are morphisms of $\mathcal{P}$.

Our notion of *model of dependent type theory* is equivalent to Joyal's notion of *tribe*, and his notions of π-tribe and h-tribe are slightly more general than our notions of models with Π-types and $\mathtt{Id}$-types introduced below, [6].

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M. del Mar González et al. (eds.), *Extended Abstracts Fall 2013*,
Trends in Mathematics, DOI 10.1007/978-3-319-21284-5_16

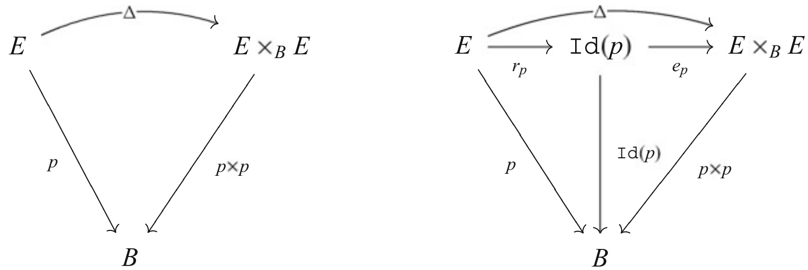
Definition 2 Let us say that a model of dependent type theory $(\mathcal{C}, \mathcal{P})$ has Σ-*types* if for all projections $: A \to B$, the pullback functor $p^*: \mathcal{C}/B \to \mathcal{C}/A$ has a left adjoint $\Sigma_p: \mathcal{C}/A \to \mathcal{C}/B$ which restricts to a functor $\Sigma_p: \mathcal{C}_\mathcal{P}/A \to \mathcal{C}_\mathcal{P}/B$.

Similarly, let us say that a model of dependent type theory $(\mathcal{C}, \mathcal{P})$ has Π-*types* if for all projections $p: A \to B$, the pullback functor $p^*: \mathcal{C}/B \to \mathcal{C}/A$ has a right adjoint $\Pi_p: \mathcal{C}/A \to \mathcal{C}/B$ which restricts to a functor $\Pi_p: \mathcal{C}_\mathcal{P}/A \to \mathcal{C}_\mathcal{P}/B$.

Note that every model of dependent type theory has Σ-types as Σ_p can be given by post-composition with p.

A morphism of any category is called *exponential* if its associated pullback functor f^* exists and has a right adjoint. Thus, to require that $(\mathcal{C}, \mathcal{P})$ has Π-types is to require that the projections are contained in the exponentiable morphisms of $\mathcal{C}$ and are closed under the partial operation given by $(p, f) \mapsto \Pi_p f$ when the codomain of f is the domain of p.

Definition 3 Let us say that a model of dependent type theory $(\mathcal{C}, \mathcal{P})$ has $\mathtt{Id}$-*types* if for every projection $E \xrightarrow{p} B$, there is a factorization of the diagonal



in $\mathcal{C}_\mathcal{P}/B$ such that

(i) e_p is a projection;
(ii) $r: p \to \mathtt{Id}(p)$ has the left lifting property against all projections in $\mathcal{C}_\mathcal{P}/B$, and additionally, for all $\alpha: f \to p$ in $\mathcal{C}_\mathcal{P}/B$, the induced map of pullbacks $\alpha^* r: \alpha^* 1_p \to \alpha^* ev_i$ has the left lifting property against all projections in $\mathcal{C}_\mathcal{P}/B$ (for $i = 0, 1$ and where ev_i stands for either composite $\mathtt{Id}(p) \xrightarrow{e} p \times p \xrightarrow{\pi_i} p$).

Now, in a model $(\mathcal{C}, \mathcal{P})$ of dependent type theory with identity types, any $f: X \to Y$ can be factored as

$$X \xrightarrow{f^* r} f^* ev_0 \xrightarrow{\pi ev_1} Y,$$

where ev_1 is a projection, and $f^* r$ has the left lifting property against all projections. In fact:

Proposition 4 *A model* $(\mathcal{C}, \mathcal{P})$ *of dependent type theory with* Σ-, Π-, *and* `Id`-*types generates a weak factorization system on* $\mathcal{C}$ *whose left class consists of those morphisms of* $\mathcal{C}$ *which have the left lifting property against all projections.*

The proof of this proposition follows the same line of reasoning as Gambino–Garner's analogous result [4, Theorem 10].

So, to find a model of dependent type theory with Σ-, Π-, and `Id`-types is to find a cartesian closed category $\mathcal{C}$ with a weak factorization system $(\mathcal{L}, \mathcal{R})$ and a class of morphisms $\mathcal{P}$ of $\mathcal{C}$ which is contained in the exponentiable maps and $\mathcal{R}$, which is closed under composition, pullback along any map of $\mathcal{C}$, and the partial operation $(p,f) \mapsto \Pi_p f$, and which contains the right factor of $\Delta: E \to E \times_B E$ for all $p: E \to B$ in $\mathcal{P}$.

Perhaps there is a convenient subcategory of topological spaces which models dependent type theory with Σ-, Π-, and `Id`-types, but none in the literature (e.g., [2, 3]) satisfy these requirements. However, if one considers a locally cartesian closed category with a weak factorization system $(\mathcal{L}, \mathcal{R})$ where $\mathcal{R}$ is closed under the partial operation $(p,f) \mapsto \Pi_p f$, then $\mathcal{C}$ is a model of dependent type theory with Σ-, Π-, and `Id`-types by setting $\mathcal{P} := \mathcal{R}$.

Thus, we turn to considering locally cartesian closed categories which contain topological spaces. There are several candidates, but we choose to focus on filter spaces, a description of which can be found in [5]. The following result is folklore.

Proposition 5 *The category of filter spaces is a locally cartesian closed category, and contains topological spaces as a full, reflective subcategory.*

Now, we can imitate the Hurewicz (trivial cofibration, fibration) weak factorization system from the category of topological spaces in the category of filter spaces, and in particular, its algebraic characterization as described in [1]. For any filter space X, we define $\mathtt{Id}(X)$ to be the usual Moore path space of X equipped with two retractions $X \xrightarrow{r} \mathtt{Id}(X) \xrightarrow{ev_i} X$ (for $i = 0, 1$). We factorize any map of filter spaces $f: X \to Y$, as $X \xrightarrow{f^* r} f^* ev_0 \xrightarrow{\pi ev_1} Y$ and call the left factor Lf and the right factor Rf. Then, L and R underlie comonads and monads on the category of morphisms of filter spaces, and we define $\mathcal{R}$ (respectively, $\mathcal{L}$) to be all those morphisms of filter spaces which have (co)algebra structures for the (co)pointed endofunctor R (respectively, L).

Theorem 6 *As defined above,* $(\mathcal{L}, \mathcal{R})$ *is a weak factorization system on the category of filter spaces.*

Using such an algebraic description of the weak factorization system, we can prove the following. This may be understood as extending a classical result from [8].

Theorem 7 *If* p, q *are morphisms in the right class* $\mathcal{R}$ *of the weak factorization system on filter spaces, then* $\Pi_p q$ *is also in* $\mathcal{R}$.

With these two theorems, we immediately find the main point of this note.

Corollary 8 *The category of filter spaces with subcategory of projections $\mathcal{R}$ is a model of dependent type theory with* Σ-, Π-*, and* $\mathtt{Id}$*-types.*

Therefore, the category of topological spaces is a full, reflective subcategory of a model of dependent type theory with Σ-, Π-, and $\mathtt{Id}$-types.

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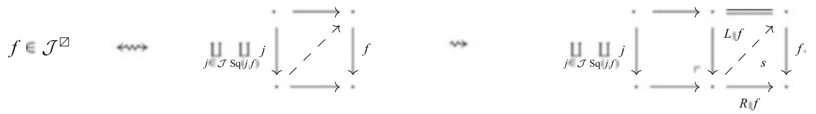
Made-to-Order Weak Factorization Systems

Emily Riehl

1 The Algebraic Small Object Argument

For a cocomplete category **M** which satisfies certain "smallness" condition (such as being locally presentable), the *algebraic small object argument* defines the functorial factorization necessary for a "made-to-order" weak factorization system with right class $\mathcal{J}^{\boxslash}$. For now, $\mathcal{J}$ is an arbitrary set of morphisms of **M** but later we will use this notation to represent something more sophisticated.

The small object argument begins by defining a *generic lifting problem*, a single lifting problem that characterizes the desired right class:



(1)

The diagonal map defines a solution to any lifting problem between $\mathcal{J}$ and f. Taking a pushout transforms the generic lifting problem into the *step-one functorial factorization*, another generic lifting problem that also factors f.

This defines a pointed endofunctor $R_1 \colon \mathbf{M}^{\mathbf{2}} \to \mathbf{M}^{\mathbf{2}}$ of the arrow category. An R_1-*algebra* is a pair (f, s) as displayed. By construction, $L_1 f \in {}^{\boxslash}(\mathcal{J}^{\boxslash})$. However, there is no reason to expect that $R_1 f \in \mathcal{J}^{\boxslash}$: maps in the image of R_1 need not be R_1-algebras – unless R_1 is a monad. The idea of the algebraic small object argument, due to Garner [5], is to freely replace the pointed endofunctor R_1 by a monad. (When all maps in the left class are monomorphisms, the free monad is defined by "iteratively attaching non-redundant cells" until this process converges.)

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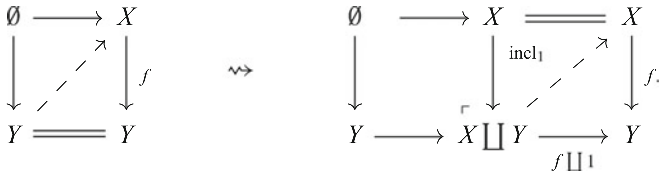
M. del Mar González et al. (eds.), *Extended Abstracts Fall 2013*,
Trends in Mathematics, DOI 10.1007/978-3-319-21284-5_17

Following Kelly [6], and assuming certain "smallness" or "boundedness" conditions, it is possible to construct the free monad $\mathbb{R}$ on a pointed endofunctor R_1 in such a way that the categories of algebras are isomorphic. Garner shows that with sufficient care, Kelly's construction can be performed in a way that preserves the fact that the endofunctor R_1 is the right factor of a functorial factorization whose left factor L_1 is already a comonad. In this way, the algebraic small object produces a functorial factorization $f = \mathbb{R}f \cdot \mathbb{L}f$ in which $\mathbb{L}$ is a comonad, $\mathbb{R}$ is a monad, and $\mathbb{R}$-**Alg** $\cong R_1$-**Alg** $\cong \mathcal{J}^{\boxslash}$.

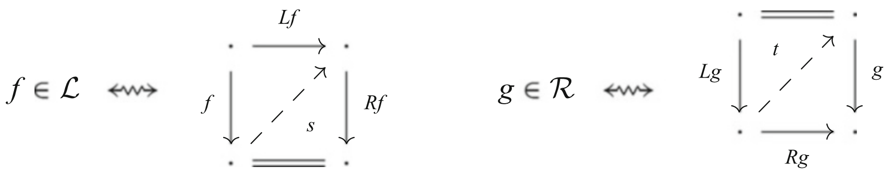
Example 1 Consider $\{\emptyset \to *\}$ on the category of sets. The algebraic small object argument produces the generic lifting problem displayed on the left and the step-one functorial factorization displayed on the right:



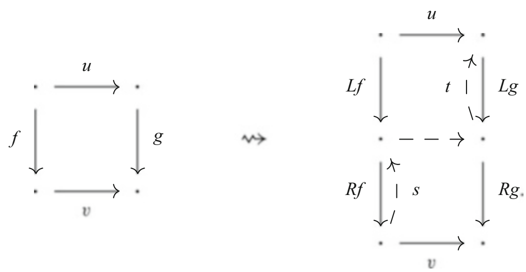
Every lifting problem after step one is redundant. Indeed, $\mathbb{R}f = f \coprod 1$ is already a monad and the construction converges in one step to define the factorization $f = f \coprod 1 \cdot \text{incl}_1$.

Example 2 Consider $\{\partial\Delta^n \hookrightarrow \Delta^n\}_{n\geq 0}$ on the category of simplicial sets. Here we may consider lifting problems against a single generator at a time, inductively by dimension. The step-one factorization of $X \to Y$ attaches the 0-skeleton of Y to X. There are no non-redundant lifting problems involving the generator $\emptyset \hookrightarrow \Delta^0$, so we move up a dimension. The step-two factorization of $X \to Y$ now attaches 1-simplices of Y to all possible boundaries in $X \cup \text{sk}_0 Y$. After doing so, there are no non-redundant lifting problems involving $\partial\Delta^1 \hookrightarrow \Delta^1$. The construction converges at step ω.

The algebraic small object argument produces an *algebraic weak factorization system* $(\mathbb{L}, \mathbb{R})$, a functorial factorization that underlies a comonad $\mathbb{L}$ and a monad $\mathbb{R}$, and in which the canonical map $LR \Rightarrow RL$ defines a distributive law. The functorial factorization $f = \mathbb{R}f \cdot \mathbb{L}f$ characterizes the underlying weak factorization system $(\mathcal{L}, \mathcal{R})$:



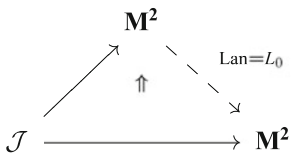
because the specified lifts assemble into a canonical solution to any lifting problem:



2 Generalizations of the Algebraic Small Object Argument

The construction of the generic lifting problem admits a more categorical description which makes it evident that it can be generalized in a number of ways, expanding the class of weak factorization systems whose functorial factorizations can be "made-to-order."

Step zero of the algebraic small object argument forms the *density comonad*, i.e., the left Kan extension along itself, of the inclusion of the generating set of arrows:



$$L_0 f = \int^{j \in \mathcal{J}} \coprod_{\mathrm{Sq}(j,f)} j.$$

When **M** is cocomplete, this construction makes sense for any small *category* of arrows $\mathcal{J}$. The counit of the density comonad defines the generic lifting problem (1), admitting a solution if and only if $f \in \mathcal{J}^{\boxslash}$ – but now $\mathcal{J}^{\boxslash}$ denotes the category in which an object is a map f together with a choice of solution to any lifting problem against $\mathcal{J}$ that is coherent with respect to (i.e., commutes with) morphisms in $\mathcal{J}$. Proceeding as before, the algebraic small object argument produces an algebraic weak factorization system $(\mathbb{L}, \mathbb{R})$ so that $\mathbb{R}$-**Alg** $\cong \mathcal{J}^{\boxslash}$ over $\mathbf{M^2}$, and $\mathbb{L}$-coalgebras lift against $\mathbb{R}$-algebras.

Example 3 In the category of cubical sets, let $\sqcap, \sqsupset, \sqcup, \sqsubset$ suggestively denote four subfunctors of the 2-dimensional representable $\square^2$. For $n > 2$ and $J \subset \{1, \ldots, n\}$ with $|J| = n - 2$, define $\sqcap^J \subset \square^n$ to be $\sqcap \otimes \square^J$, and similarly for the other three shapes. Consider the category whose objects are the inclusions $\sqcap^J \hookrightarrow \square^n$ for each shape and whose morphisms are generated by

- the projections $\begin{array}{ccc} \sqcap^J & \longrightarrow & \sqcap^{J\setminus\{j\}} \\ \downarrow & & \downarrow \\ \square^n & \underset{\pi_j}{\longrightarrow} & \square^{n-1} \end{array}$ for each $j \in J$, and
- the inclusions $\begin{array}{ccc} \sqcap^J & \longrightarrow & \sqcap^{J\cup\{i\}} \\ \downarrow & & \downarrow \\ \square^n & \longrightarrow & \square^{n+1} \end{array}$ embedding $\square^n$ as the face $i = 0$ or $i = 1$.

This generates the fibrant replacement functor, see Bezem–Coquand–Huber [3].

Example 4 ([8, §4.2]) Any algebraic weak factorization system $(\mathbb{L}, \mathbb{R})$ on $\mathbf{M}$ induces a pointwise-defined algebraic weak factorization system $(\mathbb{L}^{\mathbf{A}}, \mathbb{R}^{\mathbf{A}})$ on the category $\mathbf{M}^{\mathbf{A}}$ of diagrams. Moreover, when $(\mathbb{L}, \mathbb{R})$ is generated by $\mathcal{J}$, $(\mathbb{L}^{\mathbf{A}}, \mathbb{R}^{\mathbf{A}})$ is generated by the category $\mathbf{A}^{\mathrm{op}} \times \mathcal{J}$, whose objects are tensors of arrows of $\mathcal{J}$ with covariant representables.

If $\mathbf{M}$ is tensored, cotensored, and enriched over a closed monoidal category $\mathbf{V}$, we may choose to define the generic lifting problem using the $\mathbf{V}$-enriched left Kan extension

$$L_0 f = \int^{j \in \mathcal{J}} \underline{\mathrm{Sq}}(j,f) \otimes j,$$

where $\underline{\mathrm{Sq}}(j,f) \in \mathbf{V}$ is the object of commutative squares. The enriched algebraic small object argument produces an algebraic weak factorization system whose underlying left and right classes satisfy an enriched lifting property, defined internally to $\mathbf{V}$. The classes of an ordinary weak factorization system satisfy this enriched lifting property if and only if tensoring with objects from $\mathbf{V}$ preserves the morphisms in the left class [9, §13].

Example 5 Consider $\{0 \to R\}$ in the category of modules over a commutative ring R with identity. In analogy with Example 1, the unenriched algebraic small object argument produces the left-hand functorial factorization, while the enriched algebraic small object argument produces the factorization on the right:

$$X \xrightarrow{\mathrm{incl}} X \oplus (\oplus_Y R) \xrightarrow{f \oplus \mathrm{ev}} Y, \qquad\qquad X \xrightarrow{\mathrm{incl}} X \oplus Y \xrightarrow{f \oplus 1} Y.$$

Example 6 (Barthel–May–Riehl [1]) On the category of unbounded chain complexes of R-modules, consider the sets $\{0 \to D^n\}_{n \in \mathbb{Z}}$ and $\{S^{n-1} \hookrightarrow D^n\}_{n \in \mathbb{Z}}$, where D^n is the chain complex with R in degrees n and $n-1$ and identity differential, and where S^n has R in degree n and zeroes elsewhere. The enriched algebraic small object argument converges at step one in the former case and at step two in the latter case to produce the natural factorizations through the mapping cocylinder and the mapping cylinder, respectively (see “Mapping (co)cylinder factorizations via the small object argument” on the n-Category Café).

The algebraic weak factorization systems constructed in Examples 4 and 6 are not cofibrantly generated (in the usual sense) [4, 7].

Example 7 (Barthel–Riehl [2]) There are two algebraic weak factorization systems on topological spaces whose right class is the class of Hurewicz fibrations. A map is a *Hurewicz fibration* if it has the homotopy lifting property, i.e., solutions to lifting problems

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle \mathrm{incl}_0}\big\downarrow & \nearrow & \big\downarrow{\scriptstyle f} \\ A \times I & \longrightarrow & Y \end{array} \qquad (2)$$

defined for every topological space A. As there is proper class of generators, it is not possible to form the coproduct in (1). However, the functor $\mathbf{Top}^{\mathrm{op}} \to \mathbf{Set}$ sending A to the set of lifting problems (2) is represented by the mapping cocylinder Nf:

$$\begin{array}{ccc} Nf & \longrightarrow & Y^I \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle \mathrm{ev}_0} \\ X & \xrightarrow[f]{} & Y \end{array} \qquad \rightsquigarrow \qquad \begin{array}{ccc} Nf & \longrightarrow & X \\ {\scriptstyle \mathrm{incl}_0}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ Nf \times I & \longrightarrow & Y \end{array}$$

It follows that any lifting problem (2) factors uniquely through the generic lifting problem displayed on the right. The algebraic small object argument proceeds as usual, though there are some subtleties in the proof of its convergence.

There is another algebraic weak factorization system “found in the wild”: the factorization through the space of Moore paths. The category of algebras for the Moore paths monad admits the structure of a double category in such a way that the forgetful functor to the arrow category becomes a double functor. A recognition criterion due to Garner implies that this defines an algebraic weak factorization system.

Acknowledgements I would like to thank Steve Awodey for sparking my interest in homotopy type theory and encouraging me to keep in touch with recent progress.

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A Descent Property for the Univalent Foundations

Egbert Rijke

We present a version of the descent property [4, 5] which is formulated using families rather than morphisms. By the univalence axiom [3], there is an equivalence $(\sum_{Y:\mathrm{Type}} Y \to X) \simeq (X \to \mathrm{Type})$ for every type X [1]. A similar equivalence will hold for the kind of families over graphs we will study here: the equifibered families. This equivalence can be used to translate our simple version of the descent property back into the usual formulation of it.

In the present note, we use the notation from [1].

1 Equifibered Families of Graphs

Definition 1 A *(directed) graph* Γ is a pair (Γ_0, Γ_1) consisting of a type Γ_0 and a binary relation $\Gamma_1 : \Gamma_0 \to (\Gamma_0 \to \mathrm{Type})$ over it.

Naturally, there are notions of families of graphs, sections thereof and interpretations of the type constructors for those families. If we allow ourselves to denote $\Sigma_{i,j:\Gamma_0}\Gamma_1(i,j)$ by $\tilde{\Gamma}_1$, a family of graphs corresponds to a pair (f_0, f_1) of functions $f_0 : X_0 \to \Gamma_0$ and $f_1 : \tilde{X}_1 \to \tilde{\Gamma}_1$ such that the diagrams

$$\begin{array}{ccc} \tilde{X}_1 & \xrightarrow{f_1} & \tilde{\Gamma}_1 \\ {\scriptstyle s}\downarrow & & \downarrow{\scriptstyle s} \\ X_0 & \xrightarrow[f_0]{} & \Gamma_0 \end{array} \qquad \begin{array}{ccc} \tilde{X} & \xrightarrow{f_1} & \tilde{\Gamma}_0 \\ {\scriptstyle t}\downarrow & & \downarrow{\scriptstyle t} \\ X_0 & \xrightarrow[f_0]{} & \Gamma_0 \end{array} \tag{1}$$

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M. del Mar González et al. (eds.), *Extended Abstracts Fall 2013*,
Trends in Mathematics, DOI 10.1007/978-3-319-21284-5_18

commute. Here, s and t denote the source and target maps.

While equifibered families can be characterized among arbitrary families by a mere proposition, this note takes a shorter route, introducing them as a separate concept. The pertinent verifications such as closure under type constructors are omitted, but it should be noted that the notion of terms of equifibered families introduced below agrees with the notion of terms for arbitrary families.

Definition 2 An *equifibered family E over a graph* Γ is a pair (E_0, E_1) consisting of

$$E_0 \colon \Gamma_0 \to \text{Type}$$
$$E_1 \colon \Pi_{i,j:\Gamma_0} \Gamma_1(i,j) \longrightarrow \big(E_0(i) \simeq E_0(j)\big).$$

The type of equifibered families over Γ is denoted by equiFib(Γ).

Definition 3 A *term x of an equifibered family E over* Γ is a pair (x_0, x_1) consisting of

$$x_0 : \Pi_{(i:\Gamma_0)} E_0(i)$$
$$x_1 : \Pi_{(i,j:\Gamma_0)} \Pi_{(q:\Gamma_1(i,j))} E_1(q, x_0(i)) = x_0(j).$$

The type of terms of an equifibered family is denoted by $\mathcal{T}(E)$.

Equifibered families over graphs correspond to pairs of maps for which both squares in Eq. (1) are homotopy pullback squares. They can therefore be seen as étale maps in the category of graphs, see [2].

We only introduce the type constructor of dependent pair types for equifibered families here:

Definition 4 Let E be an equifibered family over Γ. Then we define the graph $\sigma(\Gamma, E)$ by

$$\Sigma(\Gamma, E)_0 \mathbin{\text{▪}}_0 \equiv \Sigma_{i:\Gamma_0} E_0(i)$$
$$\Sigma(\Gamma, E)_1\big((i,u),(j,v)\big) \mathbin{\text{▪}}_0 \equiv \Sigma_{(q:\Gamma_1(i,j))} E_1(q,u) = v.$$

2 The Descent Property

Colimits of graphs are defined as certain higher inductive types, see [1, Chapter 6]. We stress that, since no homotopy level restrictions are imposed on our graphs, the colimits we introduce here include the familiar examples of higher inductive types such as arbitrary homotopy pushouts.

There is another reason why it is essential that there are no restrictions on the homotopy levels of types: in [5, Example 2.3], Rezk shows that this general version descent property does not hold in the topos of sets.

Definition 5 Let Γ be a graph. The colimit of Γ is a higher inductive type $\mathsf{colim}(\Gamma)$ with basic constructors

$$\alpha_0 \colon \Gamma_0 \longrightarrow \mathsf{colim}(\Gamma)$$
$$\alpha_1 \colon \Pi_{(i,j:\Gamma_0)}\, \Gamma_1(i,j) \longrightarrow \alpha_0(i) = \alpha_0(j).$$

The induction principle for $\mathsf{colim}(\Gamma)$ is that, for any type family $P \colon \mathsf{colim}(\Gamma) \rightarrow$ Type, if there are

$$A_0 : \Pi_{(i:\Gamma_0)} P(\alpha_0(i))$$
$$A_1 : \Pi_{(i,j:\Gamma_0)}\, \Pi_{(q:\Gamma_1(i,j))} \alpha_1(q)_*(A_0(i)) = A_0(j),$$

then there is a section $\sigma : \Pi_{(w:\mathrm{colim}(\Gamma))} P(w)$ for which we have $\sigma(\alpha_0(i))\cdot_0 \equiv A_0(i)$ for each $i : \Gamma_0$, and for which there are paths

$$\beta(q) : \sigma(\alpha_1(q)) = A_1(q)$$

for every $q : \Gamma_1(i,j)$ and every $i,j : \Gamma_0$.

In the following lemma univalence does not play a role yet. It is true simply because transportation along a path (or path lifting) is always an equivalence. We give a sketch of the proof because the function $\mathsf{famToEquifib}$ which is defined in it plays an essential role in the descent theorem.

Lemma 6 *For any graph* Γ*, there is a function*

$$\mathsf{famToEquifib} \colon \big(\mathsf{colim}(\Gamma) \rightarrow \mathit{Type}\big) \rightarrow \mathsf{equiFib}(\Gamma).$$

Proof The function $\mathsf{famToEquifib}$ is defined by substitution. Let $P \colon \mathsf{colim}(\Gamma) \rightarrow$ Type. Then we define the equifibered family $\mathsf{famToEquifib}(P)$ over Γ by

$$\mathsf{famToEquifib}(P)_0(i) :\equiv P(\alpha_0(i))$$
$$\mathsf{famToEquifib}(P)_1(q,u) :\equiv \alpha_1(q)_*(u)$$

for $i,j : \Gamma_0$, $q : \Gamma_1(i,j)$, $u : P(i)$. □

Now the descent property states that colimits are universal:

Theorem 7 (Descent) *The function* $\mathsf{famToEquifib}$ *defined in Lemma 6 is an equivalence for every graph* Γ.

Proof We have to define a function

$$\varphi \colon \mathsf{equiFib}(\Gamma) \rightarrow \big(\mathsf{colim}(\Gamma) \rightarrow \mathrm{Type}\big)$$

which is homotopy inverse to famToEquifib. In this note we will only define the function φ. Let E be an equifibered family over Γ. We will define the type family $\varphi(E)$ over $\mathsf{colim}(\Gamma)$ by induction over $\mathsf{colim}(\Gamma)$. Thus we have to find

$$A_0 \colon \Gamma_0 \to \mathrm{Type}$$

$$A_1 \colon \Pi_{(i,j:\Gamma_0)}\, \Gamma_1(i,j) \longrightarrow \big(A_0(i) = A_0(j)\big).$$

By the univalence axiom we have an equivalence $(A_0(i) = A_0(j)) \simeq (A_0(i) \simeq A_0(j))$, so we see that the data we need is exactly provided by E. □

The previous theorem tells us that we can view colim not only as a functor between the 'categories' Graph and Type, it acts as a functor between the *categories with families* in the sense that it maps equifibered families over a graph Γ to type families over $\mathsf{colim}(\Gamma)$. This motivates the following notation:

Definition 8 Let E be an equifibered family over Γ. We define $\mathsf{colim}(E) \colon \mathsf{colim}(\Gamma) \to \mathrm{Type}$ to be the type family $\varphi(E)$ of the proof of Theorem 7.

Moreover, terms of an equifibered family E over Γ are mapped to sections of $\mathsf{colim}(E)$ (which may be seen as the terms of $\mathsf{colim}(E)$):

Theorem 9 *Let Γ be a graph and let P be a type family over* $\mathsf{colim}(\Gamma)$. *Then there is an equivalence*

$$\big(\Pi_{(w:\mathsf{colim}(\Gamma))}P(w)\big) \simeq \mathcal{T}\big(\mathsf{famToEquifib}(P)\big).$$

We have translated the descent property to a statement relating equifibered families over graphs to type families over $\mathsf{colim}(\Gamma)$, whereas the usual statement of the descent property relates cartesian morphisms of diagrams to functions with codomain $\mathsf{colim}(\Gamma)$.

We end mentioning a theorem which asserts that the total space of the type family $\mathsf{colim}(E)$ is the colimit of the graph $\sigma(\Gamma, E)$.

Theorem 10 *There is an equivalence*

$$\mathsf{colim}\big(\sigma(\Gamma, E)\big) \simeq \sum_{(w:\mathsf{colim}(\Gamma))} \mathsf{colim}(E)(w).$$

The descent property is used to *define* type families over higher inductive types, while the above theorem may then be used to derive pleasant properties of the family under consideration. Therefore, the results presented here play a crucial role in the development of homotopy theoretic results in type theory, see [1, Chapter 8].

Acknowledgements The research leading to these results has received funding from the European Union's 7-th Framework Programme under grant agreement n. 243847 (ForMath).

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Classical Field Theory via Cohesive Homotopy Types

Urs Schreiber

In the year 1900, at the International Congress of Mathematics in Paris, David Hilbert stated his famous list of 23 central open questions of mathematics [7]. Among them, the sixth problem (see [3] for a review) is arguably the one that Hilbert himself regarded as the most valuable: "From all the problems in the list, the sixth is the only one that continually engaged [Hilbert's] efforts over a very long period, at least between 1894 and 1932", see [4]. Hilbert stated the problem as follows:

Hilbert's mathematical problem # 6 *To treat by means of axioms, those physical sciences in which mathematics plays an important part.*

Since then, various aspects of physics have been given a mathematical formulation. The following table, necessarily incomplete, gives a broad idea of central concepts in theoretical physics and the mathematics that captures them.

	Physics	Maths
	Prequantum physics	*Differential geometry*
18xx–19xx	Mechanics	Symplectic geometry
1910s	Gravity	Riemannian geometry
1950s	Gauge theory	Chern–Weil theory
2000s	Higher gauge theory	Differential cohomology
	Quantum physics	*Noncommutative algebra*
1920s	Quantum mechanics	Operator algebra
1960s	Local observables	Co-sheaf theory
1990s–2000s	Local field theory	(∞, n)-category theory

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M. del Mar González et al. (eds.), *Extended Abstracts Fall 2013*,
Trends in Mathematics, DOI 10.1007/978-3-319-21284-5_19

These are traditional solutions to aspects of Hilbert's sixth problem. Two points are noteworthy: on the one hand the items in the list are crown jewels of mathematics; on the other hand their appearance is somewhat unconnected and remains piecemeal.

Towards the end of the 20th century, William Lawvere, the founder of categorical logic and of categorical algebra, aimed for a more encompassing answer that rests the axiomatization of physics on a decent unified foundation. He suggested to

(1) rest the foundations of mathematics itself on topos theory [8];
(2) build the foundations of physics *synthetically* inside topos theory by

  (a) imposing properties on a topos which ensure that the objects have the structure of *differential geometric spaces* [13];
  (b) formalizing classical mechanics on this basis by universal constructions ("Categorical dynamics" [9], "Toposes of laws of motion" [12]).

While this is a grandiose plan, we have to note that it falls short in two respects:

(1) Modern mathematics prefers to refine its foundations from topos theory to *higher topos theory* [15] or *homotopy type theory*, [1].
(2) Modern physics needs to refine classical mechanics to *quantum mechanics* and *quantum field theory* at small length/high energy scales, [5, 18].

Concerning the first point, notice that *homotopy type theory with higher inductive types and univalent type universes weakly à la Tarski (as discussed in* [24]*) can be interpreted in certain Quillen model category presentations of* ∞*-stack* ∞*-toposes.*

Therefore our task is to: refine Lawvere's synthetic approach to Hilbert's sixth problem from classical physics formalized in synthetic differential geometry axiomatized in topos theory to high energy physics formalized in higher differential geometry axiomatized in higher topos theory. Specifically, the task is to add to (univalent) homotopy type theory axioms that make the homotopy types have the interpretation of differential *geometric homotopy types* in a way that admits a formalization of high energy physics.

The canonical way to add such *modalities* on type theories is to add *modal operators* which, in homotopy type theory, are *homotopy modalities*, [23]. We say

Definition 1 ([22]) *Cohesive homotopy type theory* is univalent homotopy type theory equipped with an adjoint triple of homotopy (co-)modalities $\int \dashv \flat \dashv \sharp$, to be called: *shape modality* $\dashv$ *flat co-modality* $\dashv$ *sharp modality*, such that there is a canonical equivalence of the $\flat$-modal types with the $\sharp$-modal types, and such that $\int$ preserves finite product types.

This definition has been formalized in HoTT-Coq by M. Shulman, see [22] for details.

With hindsight one finds that this modal type theory is essentially what Lawvere was envisioning in [10], where it is referred to as encoding "being and becoming", and later more formally in [11, 14], where it is referred to as encoding "cohesion".

While Definition 1 may look simple, its consequences are rich. In [21] we show how cohesive homotopy type theory synthetically captures not just key aspects of differential geometry, but produces the theory of *differential generalized cohomology* [20]. This is the cohomology theory in which physical gauge fields (such as the field of electromagnetism) are cocycles. We show in [21] that cohesion implies the existence of geometric homotopy types **Phases** such that

(1) the dependent homotopy types over **Phases** are *prequantized covariant phase spaces* of physical field theories;
(2) correspondences between these dependent types are *spaces of trajectories equipped with local action functionals*;
(3) group actions on such dependent types encode the Hamilton–de Donder–Weyl equations of motion of local covariant field theory;
(4) the "motivic" linearization of these relations over suitable stable homotopy types yields the corresponding quantum field theories.

An exposition of what all this means is in [21, § 1.2]. See [16] for details on the last point, and [19] for a general overview.

Specifically, cohesive homotopy type theory has semantics in the ∞-topos $\mathbf{H}$ of ∞-stacks over the site of smooth manifolds, see [21, § 4.4]. This contains a canonical line object $\mathbb{A}^1 = \mathbb{R}$, the *continuum*, abstractly characterized by the fact that the shape modality exhibits, in the sense of [23], the corresponding $\mathbb{A}^1$-homotopy localization. Form the quotient type by the type of integers yields the *smooth circle group* $U(1) \simeq \mathbb{R}/\mathbb{Z}$. This being an abelian group type means equivalently that for all $n \in \mathbb{N}$ there is a pointed n-connected type $\mathbf{B}^n U(1)$ such that $U(1) \simeq \Omega^n \mathbf{B}^n U(1)$ is the n-fold loop type. Write then

$$\mathrm{fib}(\epsilon)\colon \flat_{\mathrm{dR}} \mathbf{B}^{n+1}(1) \longrightarrow \mathbf{B}^{n+1} U(1)$$

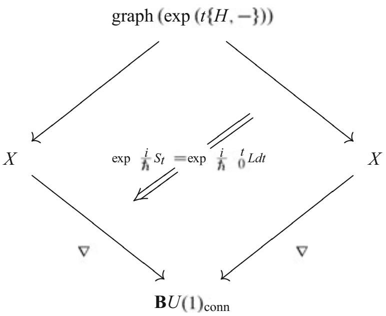
for the homotopy fiber of the co-unit $\epsilon\colon \flat \mathbf{B}^{n+1} U(1) \longrightarrow \mathbf{B}^{n+1} U(1)$ of the flat co-modality, and write

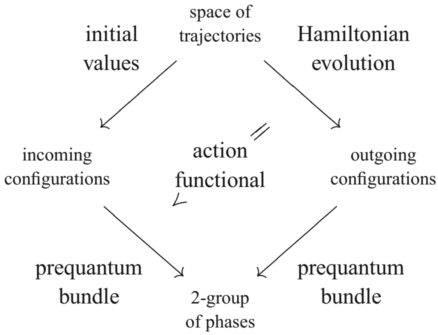
$$\theta_{\mathbf{B}^n U(1)} := \mathrm{fib}(\mathrm{fib}(\epsilon))\colon \mathbf{B}^n U(1) \longrightarrow \flat_{\mathrm{dR}} \mathbf{B}^{n+1} U(1)$$

for the homotopy fiber of that. Cohesion implies that we may think of this as the *universal Chern-character* for ordinary smooth cohomology [21, § 3.9.5]. Hence we write $\mathbf{Phases} := \mathbf{B}^n U(1)_{\mathrm{conn}}$ for the dependent sum of "all" homotopy fibers of $\theta_{\mathbf{B}^n U(1)}$ (for some choice of "all", see [21, § 4.4.16]). Then a dependent type ∇ over $\mathbf{B}U(1)_{\mathrm{conn}}$ is a *prequantized phase space* (see [21, § 3.9.13]) in classical mechanics [2]. An equivalence of dependent types over $\mathbf{B}U(1)_{\mathrm{conn}}$ is a *Hamiltonian symplectomorphism* and a (concrete) function term

$$H\colon \mathbf{B}\mathbb{R} \longrightarrow \prod_{\mathbf{B}U(1)_{\mathrm{conn}}} \mathbf{B}\mathrm{Equiv}(\nabla, \nabla)$$

of the function type from the delooping of $\mathbb{R}$ to the delooping of the dependent product of the type of auto-equivalences of ∇ is equivalently a choice of *Hamiltonian*. It sends the ("time") parameter $t : \mathbb{R}$ to the Hamiltonian evolution $\exp(t\{H, -\})$ with Hamilton–Jacobi action functional $\exp(iS_t/\hbar)$, see [2]. In the ∞-categorical semantics this is given by a diagram in $\mathbf{H}$ of the following form[1]:





Here,

$$X := \sum_{\mathbf{B}U(1)_{\mathrm{conn}}} \nabla$$

is the phase space itself and ∇ is its *pre-quantum bundle* [6].

This statement concisely captures and unifies a great deal of classical Hamilton–Lagrange–Jacobi mechanics, as in [2]. Moreover, when replacing $\mathbf{B}U(1)_{\mathrm{conn}}$ here with $\mathbf{B}^n U(1)_{\mathrm{conn}}$ for general $n \in \mathbb{N}$, then the analogous statement similarly captures n-dimensional classical field theory in its "covariant" Hamilton–de Donder–Weyl formulation on dual jet spaces of the field bundle[2] (see, e.g., [17]). This is shown in [21, § 1.2.11].

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[1] This is a pre-quantization of the *Lagrangian correspondences* of [25].

[2] I am grateful to Igor Khavkine for discussion of this point.

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How Intensional Is Homotopy Type Theory?

Thomas Streicher

Martin-Löf's Extensional Type Theory (ETT) has a straighforward semantics in the category **Set** of sets and functions and actually in any locally cartesian closed category with a natural numbers object (nno), e.g., in any elementary topos with a nno. Dependent products are interpreted by right adjoints to pullback functors, and extensional identity types are interpreted as diagonals in slice categories as explained, e.g., in [4].

Despite its intuitive flavour ETT has the defect that type checking for it is not decidable for the following reason. Since ETT identifies propositional and judgemental equality for closed terms t of type $N{\to}N$ the proposition(al type) $\Pi x{:}N.\mathtt{Id}_N(t(x),0)$ is provably inhabited in ETT if and only if ETT proves the judgemental equality $t = \lambda x{:}N.0 \in N{\to}N$. Moreover, ETT proves $\lambda x{:}N.r_N(0) \in \Pi x{:}N.\mathtt{Id}_N(t(x),0)$ if and only if ETT proves $t = \lambda x{:}N.0 \in N{\to}N$. Thus, if type checking for ETT were decidable one could decide which Π^0_1-sentences are derivable in ETT. But for every consistent recursively enumerable extension $\mathcal{T}$ of primitive recursive arithmetic (PRA) the set of Π^0_1-sentences provable in $\mathcal{T}$ is not decidable.[1] For this reason interactive theorem provers based on type theory (like

[1]Since, otherwise, one could recursively separate the r.e. sets $A_0 = \{n \in \mathbb{N} \mid \{n\}(n) = 0\}$ and $A_1 = \{n \in \mathbb{N} \mid \{n\}(n) = 1\}$ which can be seen as follows. For natural numbers n consider the primitive recursive predicate $P_n(k) \equiv T(n,n,k) \to U(k) = 0$. If $n \in A_0$ then $\mathcal{T} \vdash \forall k.P_n(k)$ and if $n \in A_1$ then $\mathcal{T} \vdash \neg\forall k.P_n(k)$ and thus $\mathcal{T} \nvdash \forall k.P_n(k)$. Now let f be a total recursive function with $f(n) = 0$ if and only if $\mathcal{T} \vdash \forall k.P_n(k)$, which exists since the set of Π^0_1-sentences provable in $\mathcal{T}$ is assumed to be decidable. But then $f(n) = 0$ if $n \in A_0$ and $f(n) \neq 0$ if $n \in A_1$, i.e., f recursively separates the sets A_0 and A_1, which is known to be impossible.

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M. del Mar González et al. (eds.), *Extended Abstracts Fall 2013*,
Trends in Mathematics, DOI 10.1007/978-3-319-21284-5_20

the systems Coq or ALF) are based on Martin-Löf's Intensional Type Theory (ITT) with its characteristic separation of propositional and judgemental equality.

After having investigated the semantics of ETT in the extended version [4] of my PhD Thesis from 1989, it became generally accepted that ITT is the appropriate kind of type theory for computer assisted interactive theorem proving. For this reason in my subsequent Habilitation Thesis [5] I constructed models for ITT validating the following *Criteria of Intensionality*:

(I1) $A\colon \mathrm{Set}, x, y\colon A, z\colon \mathtt{Id}_A(x, y) \not\vdash x = y\colon A$,
(I2) $A\colon \mathrm{Set}, B\colon A \to \mathrm{Set}, x, y\colon A, z\colon \mathtt{Id}_A(x, y) \not\vdash B(x) = B(y) : \mathrm{Set}$,
(I3) $\vdash p\colon \mathtt{Id}_A(t, s)$ implies $\vdash t = s\colon A$,

for some universe Set. Moreover, these models refuted most of those propositions which trivially hold in ETT but cannot be derived in ITT as, e.g., the *function extensionality* principle

$$\Pi x{:}A.\mathtt{Id}_B\big(f(x), g(x)\big) \longrightarrow \mathtt{Id}_{A\to B}(f, g)$$

for $A, B \in \mathrm{Set}$ and $f, g \in A{\to}B$, even when A and B are base types like the type N of natural numbers or the type N_2 of Booleans.

Unfortunately, the models constructed in [5] do not refute the principle UIP

$$A\colon \mathrm{Set}, x, y\colon A, u, v\colon \mathtt{Id}_A(x, y) \vdash \mathtt{Id}_{\mathtt{Id}_A(x,y)}(u, v)$$

of *Uniqueness of Identity Proofs*, which can be easily derived in ETT. To overcome this shortcoming Martin Hofmann and I in 1993 introduced the *groupoid model*, see [3] for a detailed account, within which we identified a universe U of *small discrete* groupoids where $A, B \in U$ were propositionally equal iff they were isomorphic. This observation was the precursor of Voevodsky's *Univalence Axiom* (UA) lying at the heart of *Homotopy Type Theory* (HoTT). An introduction can be found in [1].

In the present note we will (1) give a simplified construction of a model for ITT satisfying the above criteria for intensionality; and (2) discuss to which extent HoTT is intensional.

1 Truly Intensional Models of ITT

Truly Intensional Models of ITT, i.e., models of ITT validating the criteria (I1), (I2) and (I3) above were constructed in [5]. In more modern terminology they may be described as living within the $\neg\neg$-separated objects of the topos $\mathsf{Gl}(\mathcal{E}\!f\!f) = \mathbf{Set}{\downarrow}\Gamma$ obtained by *gluing* the global elements functor $\Gamma = \mathcal{E}\!f\!f(1, -)\colon \mathcal{E}\!f\!f \to \mathbf{Set}$. The book [6] is an excellent reference for all things related to realizability models and, in particular, the effective topos $\mathcal{E}\!f\!f$.

For the sake of simplicity, we replace Γ by the identity functor on **Set** giving rise to the *Sierpiński* topos $\mathcal{S} = \mathbf{Set}{\downarrow}\mathbf{Set} = \mathbf{Set}^{2^{\mathrm{op}}}$. Up to isomorphism $\neg\neg$-separated objects of $\mathcal{S}$ are inclusions of subsets. We write $\mathcal{LP}$ for the ensuing category of *logical predicates*. Its objects are pairs $X = (|X|, P_X)$ where $|X|$ is a set and $P_X \subseteq |X|$. Morphisms from X to Y are functions $f\colon |X| \to |Y|$ such that $f(x) \in P_Y$ whenever $x \in P_X$. It is easy to see that, like **Set**, the category $\mathcal{LP}$ gives rise to a model for ETT. However, for obtaining a truly intensional model of ITT we have to choose an appropriate universe U within $\mathcal{LP}$ which serves the purpose of interpreting the constant Set in (I1), (I2) and (I3). Let $\mathcal{U}$ be a Grothendieck universe. Then U consists of all objects $X \in \mathcal{LP}$ with $|X| \in \mathcal{U}$ and $0 = \emptyset \in |X|$.

The intuition behind this definition of U is that for $X \in U$ the set $|X|$ is the set of *potential* objects of X and P_X is the subset of *actual* objects of X. Elements of $|X| \setminus P_X$ will serve the purpose of *simulating the syntactic notion of free variables on the level of semantics*.

For showing that U in $\mathcal{LP}$ gives rise to a truly intensional model of ITT we next describe the interpretation of identity types within U. As usual, let 2 be the set $\{0, 1\}$. For $X \in U$ we define its identity type as

$$\mathtt{Id}_X(x, y) = (2, \{1\}) \qquad \text{if } x = y,$$

$$\mathtt{Id}_X(x, y) = (2, \emptyset) \qquad \text{if } x \neq y,$$

for $x, y \in |X|$. We interpret $r_X(x)$ as 1 for all $x \in |X|$. For $C \in \Pi x, y{:}X.\ \mathtt{Id}_X(x, y) \to U$ and $d \in \Pi x{:}X.C(x, x, r_X(x))$ we put

$$J((x)d)(x, x, 1) = d(x) \qquad \text{and} \qquad J((x)d)(x, y, 0) = 0 \in C(x, y, 0)$$

for $x, y \in |X|$. Similarly, one may interpret the eliminator K of [5] allowing one to prove UIP.

Theorem 1 *For the above interpretation of identity types the universe U in $\mathcal{LP}$ validates the criteria of intensionality* (I1), (I2) *and* (I3) *and refutes the principle of function extensionality.*

Proof For (I1) and (I2), the reason is that $0 \in \mathtt{Id}_X(x, y)$ even if $x \neq y$. And (I3) holds since the interpretation of $\vdash t \in \mathtt{Id}_X(x, y)$ is necessarily $1 \in \mathtt{Id}_X(x, y)$ (since $(\{0\}, \{0\})$ is terminal in $\mathcal{LP}$) and thus $x = y$.

Notice that for $X, Y \in U$ we have

(1) $x{:}X \vdash f(x) = g(x){:}\ Y$ if and only if $f = g$, and
(2) $x{:}X \vdash \mathtt{Id}_Y(f(x), g(x))$ if and only if $f_{|P_X} = g_{|P_X}$,

for $f, g \in X{\to}Y$. There are types X and Y and different elements f and g in $P_{X\to Y}$ which, however, coincide on P_X. For this reason the principle of function extensionality fails for U in $\mathcal{LP}$. □

Notice that, when interpreting $X{\to}Y$ for $X, Y \in U$, one has to replace $\lambda x.0$ by 0 and redefine the application function appropriately. But otherwise $X{\to}Y$ is interpreted as the full function space in the sense of **Set**. Moreover, elements f of

$|X \to Y|$ are actual, i.e., $f \in P_{X \to Y}$ if and only if it preserves actual elements, i.e., $f(x) \in P_Y$ whenever $x \in P_X$. In [5] this kind of bureaucracy could be avoided since one was working in the category of $\neg\neg$-separated objects of the gluing of $\Gamma: \mathcal{E}\!f\!f \to \mathbf{Set}$. There for U one took those X where $|X|$ is a *modest set* (see [6]) containing an element 0_X realized by 0, and P_X still was an arbitrary subset of (the underlying set of) $|X|$. By appropriate choice of Gödel numbering one has $\{0\}(n) = 0$ for all $n \in \mathbb{N}$ and, accordingly, the function $0_{X \to Y}$ sends all elements of $|X|$ to 0_Y.

Finally, we discuss our interpretation of the types N and N_k in the universe U in $\mathcal{LP}$. The type N of natural numbers is interpreted as $(\mathbb{N}, \mathbb{N} \setminus \{0\})$. We put $0_N = 1$ and the successor operation *succ* is given by $succ_N(0) = 0$ and $succ_N(n+1) = n + 2$. Similarly, one interprets the finite types N_k. Thus, the principle of function extensionality fails already for $X = Y = N_1$ because, if f is the identity on 2 and g is the constant map with value $1 \in 2$, then f and g are different elements of $P_{N_1 \to N_1}$ although $x{:}\, N_1 \vdash \mathtt{Id}_{N_1}(f(x), g(x))$ is witnessed (essentially) by the identity on $2 = \{0, 1\}$.

2 How Intensional Is HoTT?

Since the model of Sect. 1 and the groupoid model were both constructed for the purpose of showing that certain propositions cannot be derived in ITT, one might dream of combining both ideas in order to construct a model of ITT which is truly intensional and at the same time refutes UIP. The most immediate idea is to construct a groupoid model inside one of the models described in Sect. 1. However (as became clear to me in discussion with S. Awodey), for constructing a universe U of small discrete groupoids in (a model of) ITT where $\mathtt{Id}_U(A, B)$ is the set, i.e., discrete groupoid, of isomorphisms from A and B one needs the principle of function extensionality in order to organize U into a groupoid (bijections are equal if and only if they are pointwise equal).

Thus, since the groupoid model validates UA, one might ask whether UA is compatible with our criteria for intensionality. The answer to this question, however, is negative since as shown in [1], the univalence axiom UA allows one to derive from it the principle of function extensionality. The latter, however, is in contradiction with condition (I3) since together with function extensionality it has the consequence that for closed terms t of type $N \to N$ the proposition $\Pi x{:}N.\mathtt{Id}_N(t(x), 0)$ is derivable if and only if $t = \lambda x{:}N.0 \in N \to N$ is derivable, which is impossible since the set of Π^0_1-sentences provable in ITT + UA is not decidable (since ITT + UA extends PRA).

Thus, adding UA to ITT is an extension which is not conservative with respect to Basic Type Theory (BTT), i.e., ITT without universes, since this extension is not even conservative with respect to Π^0_1-sentences.[2] However, we have the following conservation result for ITT extended by function extensionality.

Theorem 2 *If a proposition of* BTT *can be proved in* ITT + UA *then it can be proved in* ITT *with a universe, the principle* $\mathrm{Ext}_{\mathrm{fun}}$ *of function extensionality and UIP in form of the eliminator K.*

Proof In ITT + $\mathrm{Ext}_{\mathrm{fun}}$ + K with a universe one can construct the groupoid model of [3]. Notice that we need $\mathrm{Ext}_{\mathrm{fun}}$ in the meta-theory for

(1) getting exponentials of groupoids right;
(2) defining `Id`-types on the universe of discrete groupoids, since we need extensional equality of isomorphisms between types in the universe.

The eliminator K is needed for avoiding problems with intensional identity types. In ITT + $\mathrm{Ext}_{\mathrm{fun}}$ + K with a universe one can prove that all types of BTT are interpreted by their corresponding discrete groupoid. Accordingly, in this theory one can prove for every type A of BTT that if the interpretation of A is inhabited in the groupoid model by some element a then the type A is inhabited actually by "stripping" the element a from additional information. □

Thus, as far as BTT is concerned, the univalence axiom does not contribute more than the principle $\mathrm{Ext}_{\mathrm{fun}}$ of function extensionality and the eliminator K. In [2], using "setoid" models, M. Hofmann has investigated to which extent extensional concepts can be interpreted within intensional type theory. Actually, he managed to interpret ITT + $\mathrm{Ext}_{\mathrm{fun}}$ + K without universes in ITT (with a universe). If one could also interpret universes this way, Theorem 2 would give a positive answer to Voevodsky's "no junk" conjecture, which claims the following: *for every closed term t of type N (natural numbers) there exists an* $n \in \mathbb{N}$ *such that HoTT proves* $t = \underline{n} \in N$, *where* $\underline{n}$ *stands for the numeral* $\mathsf{succ}^n(0)$. Thus, in light of Theorem 2, the problem rather is to prove this "no junk" conjecture for ITT + $\mathrm{Ext}_{\mathrm{fun}}$ + K.

Summarizing, we observe that the answer to our question is twofold. HoTT is inconsistent with *equality reflection* and thus with ETT; but, on the other hand, it is conservative over ITT + $\mathrm{Ext}_{\mathrm{fun}}$ + K with respect to basic type theory.

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[2]This has to be seen in sharp contrast with the fact that most known non-syntactic models for ITT + UA (as, e.g., the groupoid and the simplicial sets model) validate the same propositions of BTT as the model in **Set** does.

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Erratum to: Univalent Categories and the Rezk Completion

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Erratum to:
'**Univalent Categories and the Rezk Completion**' in : M. del Mar González et al. (eds.), *Extended Abstracts Fall 2013*, Trends in Mathematics,
DOI 10.1007/978-3-319-21284-5_14

The spelling of the author "Benedikt Ahrensm" was incorrect in the Table of Contents and in the opening page of the chapter.

The name of the author has been corrected and it now reads as "Benedikt Ahrens".

The online version of the original chapter can be found under
DOI 10.1007/978-3-319-21284-5_14

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M. del Mar González et al. (eds.), *Extended Abstracts Fall 2013*,
Trends in Mathematics, DOI 10.1007/978-3-319-21284-5_21