Multiple Criteria Decision Making

Minwir Al-Shammari
Hatem Masri *Editors*

# Multiple Criteria Decision Making in Finance, Insurance and Investment

Springer

# Multiple Criteria Decision Making

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# Multiple Criteria Decision Making

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Editors

# Multiple Criteria Decision Making in Finance, Insurance and Investment

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*To our families*

# Preface

This Volume contains 12 selected papers of the post conference proceedings of the 2013 International Conference on Multidimensional Finance, Insurance and Investment (ICMFII'2013) that was held in the College of Business Administration at the University of Bahrain from 25 to 27 November 2013, with the cosponsorship of the International Society on Multiple Criteria Decision Making and the Institute for Operations Research and the Management Sciences—MCDM section. The ICMFII'13 is the fifth of the series of conferences and provides an international forum for presentations and discussion of research in all areas of Finance, Insurance, and Investment. The first conference was held in Helsinki in 2005 and chaired by Pekka Korhonen, Jaap Spronk, and Ralph Steuer. The second edition of this scientific event took place in Montreal and was organized by Belaid Aouni in 2007. The third edition was organized by Alejandro Balbas in Madrid in 2009. The 2011 edition of the ICMFII held on April 14–16, 2011 in Hammamet Tunisia and chaired by Fouad Ben Abdelaziz. The 2013 edition held in the college of Business Administration at the University of Bahrain and chaired by Minwir Al-Shammari. The ICMFII'13 received 128 manuscripts and after a peer-review process, by national and international reviewers, 64 were accepted to be presented in the conference program. The conference program features a broad international representation from 23 countries, including: Algeria, Australia, Bahrain, China, Czech Republic, France, Germany, India, Indonesia, Iraq, Italy, KSA, Lebanon, Malaysia, Oman, Pakistan, Spain, Taiwan, Tunisia, Turkey, UAE, UK, and USA. The accepted papers include both theoretical and application research works from very diverse areas related to quantitative modeling, multiple criteria decision making, finance, insurance, and investment.

We were very impressed by the high quality of most of the research papers included in this editing volume. We hope that the book has a well-balanced portfolio of fundamental science papers, applied research papers, and large-scale industry projects. We wish this volume will serve as a useful source for the readers who have an interest in the research and practice of multiple criteria decision making in Finance, Insurance, and Investment.

Sakhir, Bahrain

Minwir Al-Shammari
Hatem Masri

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# Multiple Criteria in Islamic Portfolio Selection

**Minwir Al-Shammari, Mohammad Omar Farooq, and Hatem Masri**

**Abstract** Among all topics discussed during the International Conference on Multidimensional Finance, Insurance and Investment (ICMFII'2013) held at the College of Business Administration of University of Bahrain from 23 to 25 November 2013, Islamic finance generated a lot of interest from conference participants. One of the main concerns was to identify the criteria that should be considered to construct a Shariah-compliant portfolio. In this chapter, we report the debate on the Islamic approach for portfolio selection and we present a set of potential criteria for Shariah-compliant investment. The chapter ends with a short discussion about the concept of risk and uncertainty in an Islamic portfolio selection model.

**Keywords** Portfolio Selection • Multiple Criteria Decision Aid • Islamic Finance • Shariah screening

## 1 Introduction

The recent financial crisis and the huge increase in Islamic funds induced financial markets to develop new products to fulfill the demand from investors seeking Shariah-compliant choices. Dow Jones, Financial Times, Morgan Stanley Capital International and HSBC established indices and defined investment guidelines that are compliant to the Shariah parameters.

During the International Conference on Multidimensional Finance, Insurance and Investment (ICMFII'2013), participants were invited to attend a session on Islamic finance and discuss about Shariah-compliant portfolio selection process. The aim of the session was to explore the difference between conventional portfolio selection and Islamic portfolio selection.

The first step of the portfolio selection process is to determine a sample of the most attractive securities. In conventional portfolio selection, these securities may belong to a specific index/market or from around the world. The second step is to

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obtain data about the past performance of these securities in order to predict their future performance. Finally, we should evaluate each of these securities based on well-defined criteria and allocate the amount of the investment among the best securities. The portfolio selection problem can be viewed as a multiple criteria decision problem, where investors maximize the portfolio return, and minimize the risk of their investment (Markowitz 1952). Steuer et al. (2005) summarized from the literature some of the most important criteria that are considered in conventional portfolio selection:

- Maximize dividends
- Maximize liquidity
- Maximize the amount invested in R&D
- Maximize social responsibility
- Minimize deviation from asset allocation percentage
- Minimize the number of securities in the portfolio
- Minimize the cost of the portfolio adjustment
- Minimize the number of securities sold short

Most of the above cited objective functions are stochastic because they depend upon random variables associated with each of the *n* securities. In Islamic finance literature, multiple criteria in portfolio selection are rarely dealt with, neglecting also the stochastic-deterministic distinction (Spronk and Chammas 2008). This distinction is important because portfolio section has both deterministic and stochastic dimension and any model ignoring this fact would be rather weak and inadequate. While the awareness about Islamic portfolio selection process in general and applying multiple criteria in particular is growing, notably there is also greater complexity due to additional screening parameters.

In this chapter, we present a short overview of the Shariah-compliant portfolio selection process. In the next section, we summarize the basics of the investment rules that are mainly taken from Shariah standards derived from the main sources: Quran, Hadith and Ijtihad. In Sect. 3, we focus on the first step of the Shariah-compliant portfolio selection process and we present characteristics of such compliant securities. The second step of this portfolio selection process is to measure the return and the risk of a Shariah-compliant security. In Sect. 4, we report on some criteria used to build a Shariah-compliant portfolio and we focus on quantitative models that have been developed to measure the attractiveness of Shariah-compliant securities. We end this chapter with a discussion about the main challenges and perspectives in Shariah-compliant portfolio selection.

## 2 Shariah-Compliant Investment

Shariah-compliant investment is a form of socially responsible investment where investors should comply with the Shariah standards as developed and/or applied by Shariah boards (Garas and Pierce 2010). A Shariah board is a group of Islamic jurists and experts that ensures compliance with Shariah standards and regulations and supervises the relevant compliance of a financial or investment product or service. Sources of Islamic Law are the Qur'an, the Holy Book of Islam, the Hadith, which is the teachings and sayings of Prophet Muhammad, and Ijtihad, comprising scholarly legal deductions. Within the perspective of seeking salvation in the life hereafter, Muslim investors aim to augment wealth through investments that are interfaced with the real economy within Islamic value framework.

Better portfolio selection has become more important as global Islamic finance assets have already grown to $1.6 trillion and expected to exceed $2 trillion in the next few years (Bellalah 2014, p. ix) by the end of 2014, while global Islamic fund sector reached a new milestone in the same year with more than US$60 billion (Lewis et al. 2014).

All general prudential imperatives are applicable to Islamic investments. These are: (a) Knowledge and research: Investment in only what one knows or through those who are experts/professionals; (b) Diversification and portfolio building: Appropriate level of diversification, since efficient markets generally do not reward unnecessary risk; (c) Patience, discipline and long-term perspective: Being rational in terms of expected returns and having a solid game plan informed by sound research and supported by requisite discipline; (d) Moderation and appropriate risk appetite: Islam teaches moderation in everything, including approaching risk. Also, with strong aversion against gambling and excessive speculation, risk-seeking behavior is to be avoided (Skrainka 2014).

## 3 Shariah-Compliant Security and Portfolio

The first step of the Shariah-compliant portfolio selection process is to ensure the compliance of securities with the Shariah. Basically, there are three aspects of Shariah-compliance in developing an Islamic asset/investment portfolio. First, investment in risk free assets is prohibited as there should be no *Riba* (generally blanketly equated with interest). Also, investor must avoid investing in assets related to things that are expressly prohibited, such as intoxicants, pork, immoralities, etc. Finally, contracts should be without deficiencies by having them written, while meeting all Shariah requirements to minimize the scope of potential disputes and protect the rights and duties of the parties to the contract.

While Shariah standards are issued by various standard-setting bodies such as the Accounting and Auditing Organization for Islamic Financial Institutions (AAOIFI), the Islamic Financial Services Board (IFSB) and the Islamic Fiqh Academy, different banks, companies and funds have their own norms for Shariah screening. These organizations generally do not disclose their norms and many of those who do disclose do not publicize the rationale of their screening methodology. Among these institutions that provide information on their screening criteria are Dow Jones, SEC Malaysia and Meezan Bank in Pakistan (Khatkhatay and Nisar 2007).

The difference between conventional and Shariah-compliant securities is the application of sector screens and financial screens by which the asset universe is reduced to the Shariah-compliant assets. Dow Jones Islamic Market Index applies Shariah screening methodology consisting of two parts (Jacque 2014; Kamso 2013):

1. Sector screens: Alcohol, Pork-related products, Conventional financial services, Entertainment, Tobacco, Weapons and defense
2. Financial screens: All of the following must be less than 33 %:
   - Total debt divided by trailing 24-month average market capitalization (Highly leverage companies)
   - The sum of a company's cash and interest-bearing securities divided by trailing 24-month average market capitalization
   - Accounts receivables divided by trailing 24-month average market capitalization

Quarterly reviews are carried out to ensure criteria are met on an ongoing basis and also to cleanse the fund of a forbidden income by donating to charity. The main risk behind Islamic investment is investing in a Shariah non-compliant security. The non-compliance risk is understood as the risk of occurrence of legal sanctions, financial losses and the loss of reputation and credibility in the market (Lahsasna 2014).

The Shariah-compliant portfolio selection process adds limits to the set of admissible investments which may lower the performance of the Shariah optimal portfolio compared to the conventional optimal portfolio (Derigs and Marzban 2009). However, Shariah restrictions have some beneficial consequences as it reduces excessive risk (Basov and Bhatti 2014). Beck et al. (2013) demonstrated that during the recent crisis, better stock performance of Islamic financial institutions was in part due to better asset quality. This is also consistent with empirical studies that the performance of socially responsible or ethical portfolios is not necessarily inferior to their traditional counterparts (Garcia-Bernabeu et al. 2015).

In the next section, we focus on the two main criteria, risk and return, and illustrate different models used to quantify the performance of a Shariah-compliant portfolio.

## 4 Criteria for a Shariah-Compliant Portfolio Selection

One of the most commonly used model to estimate conventional securities rate of return is the Capital Asset Pricing Model (CAPM) (Sharpe 1964). The CAPM prices securities based on the security's systematic risk represented by the beta (β), the market return ($R_M$) and the return of the risk-free security ($R_f$):

$$r = R_f + \beta(R_M - R_f)$$

The risk-free rate $R_f$ is a predetermined return that is considered by Shariah scholars as interest or *Riba*. Therefore, securities with risk-free component are considered Shariah non-compliant (Sadaf and Andaleeb 2014). Tomkins and Karim (1987) suggested the following adjustment to CAPM, where $R_f$ is removed from the equation:

$$r = \beta R_M$$

The idea to remove the risk-free rate seems to be a simplistic solution, since there are some theoretical and practical reasons for having that component as an anchor of original CAPM and its variations. Therefore, instead of simply removing the risk-free component to render the model Shariah-compliant, others have suggested an alternative to $R_f$. El-Ashker (1987) has proposed replacing $R_f$ by the minimum return that an investor would expect from an investment to cover Zakat

$$Z = \frac{\text{Zakat rate}}{1 - \text{Zakat rate}} = 0.0256$$

Shaikh (2010) suggested replacing $R_f$ with Nominal Gross Domestic Product (NGDP) growth rate and the resulting model is as follows:

$$r = \text{NGDP} + \beta(R_M - \text{NGDP})$$

We recall that the NGDP is the GDP while taking into account the country inflation. Also, the $R_f$ consists of two parts: the real return and the inflation premium (Bhatti and Hanif 2010). While the traditional Shariah view is that it cannot accept the real part of $R_f$, which is regarded as a rent of money for use, the inflation premium $N$ is a debatable issue under Shariah. Notably, excessive inflation is quite common in Muslim-majority countries. Inflation reduces the wealth of the investor and therefore it is argued that compensation equal to inflation premium should be given. Hanif (2011) introduced the following equation for Shariah-compliant asset pricing model:

$$r = N + \beta(R_M - N)$$

All the above described models may fail to provide a robust pricing model for securities as the main concern was to substitute the risk free asset with an alternative component that is Shariah-compliant. Related to this is a traditional view that in an interest-free Islamic economy, the speculative activity is not allowed and the inflation rate should be equal to zero (Zarqa 1983). However, so far there is no empirical evidence that inflation can be sustainably managed to be kept at zero level. Therefore, avoiding the inflation premium by eliminating inflation might not be a viable proposition.

## 5 Conclusions and Perspectives

Islamic finance industry in general and the Islamic capital market in particular face a number of challenges. The global financial market is dominated by the conventional finance. Even in the Muslim majority countries, Islamic finance competes with its dominant conventional counterpart. In most of these countries, capital market is not well developed and the financial markets for investment are not as robust. This is further constrained by the fact that Islamic investment is expected to be linked with the real economy, while current Islamic investment activities are mostly involved with the financial market, instead of the real economy.

The main challenges are in defining Shariah-compliant securities rather than in defining the selection criteria. Too many rules may restrict the investment freedom and efficiency, especially in constructing portfolios based on multiple criteria. An important constraining aspect is that risk and uncertainty need to be operationally better defined from Shariah perspective. Several practical challenges are faced in Islamic portfolio selection. Among these are: (a) inconsistency of published financial statements from Shariah perspective; (b) lack of uniformity in rules/fatwas across jurisdictions; and (c) inconsistent monitoring of change in stock's compliance status. Solutions must include better standardization of such screening methodology and the input used. On the theoretical front, the conventional theories are interest-based and almost invariably with a risk-free component, which is considered *riba* from orthodox viewpoint. This risk-free component is equivalent to the US Treasury Bills. Having a theoretical anchor (risk-free rate as part of a structure to determine expected returns) and a practical equivalent (US Treasury Bill) have made the system quite formidable to come up with a suitable alternative.

Enhanced credibility, standardization, regulation and interface with the real economy, Shariah-compliant investment can provide meaningful alternatives to conventional portfolios that use multiple criteria to optimize for better risk-adjusted performance.

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# Multiple Criteria Decision Making and Goal Programming for Optimal Venture Capital Investments and Portfolio Management

**Cinzia Colapinto and Davide La Torre**

**Abstract** The venture capital market plays a significant role in providing capital to a new feasible business idea (new product, service, or retail concept) and businesses of different type. This chapter focuses on the way venture capitalists make their investment decision, a process involving several conflicting and imprecise criteria. We propose three different models to solve these complex decision making contexts, namely a deterministic goal programming model with satisfaction function, a scenario-based stochastic goal programming model with satisfaction function, and a fuzzy goal programming formulation. The three models have been applied to three concrete examples using real data obtained from some Italian venture capital funds. It turns out that these models are easy and simple to be implemented and analyzed, and represent an implementable approach for both scientists and practitioners.

**Keywords** Multiple criteria decision making • Goal Programming • Venture Capital • Portfolio optimization

## 1 Introduction

In the entrepreneurial setting, Venture Capital (VC) firms are financial intermediaries able to select potential and help to realize it. Their core business is to provide equity capital (usually minority interests) to unlisted companies or startups with the objective of maximize capital gains/goodwill. The VC firms also provide human and social capital, and management expertise or access to other capabilities that bolster the competitive advantage and increase the value of startups that they fund (Piol 2004). Indeed, the VC is an essential resource for economic growth, as it has

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been proven in some technological clusters (e.g. Silicon Valley or MIT in the US, Waterloo region in Canada). We can refer to the *Quadruple-Helix model* which involves financing organizations able to foster revenue growth and commercialization: financing organizations plays a relevant role in the interrelations with universities, government and industry (Colapinto 2007, 2011a, b; Colapinto and Porlezza 2012).

Because startups encounter many hazards and because they have short-track records by which outsiders can evaluate their potential, there is considerable uncertainty about their value. Venture capitalists (VCs) spend a great deal of time and effort seeking and assessing signals of a startup's promise and quality (Hall and Hofer 1993). The decision process is time-consuming and labor-intensive and goes through two main stages, namely screening and due diligence (Sahlman 1990; Tyebjee and Bruno 1984). When a deal arrive to a VC firm, an initial screening based on business plans selects those deals deem interesting enough for a thorough evaluation. During the due diligence process, venture characteristics are further scrutinized and venture team potential is judged. VCs evaluate business plans for funding based on some criteria, the most common are: startup experience, industry experience, leadership experience, management experience, market size, customer adoption, revenue generated, entry timing, competition, strategy, intellectual property rights, value added and profit margins. The VCs typically consider a certain number of investments because time and money, in particular to face due diligence costs, are scarce resources. This leads to an integer model able to include the minimization of the number of investments. The VC firms generally place upper and lower limits on the size of their investments, which are closely related to the overall size of the managed fund.

VCs face some specific business risk factors, such as technological or market development or the development stage of the company. As the target companies often have no history (or limited track record), they are highly risky. VCs required return rates are often affected by many other factors, such as the individual characteristics of the managers (Laughun et al. 1980), organizational culture (Morgan 1986), the age and the size of VCs, national culture (Hofstede 1984) and institutional environment (Tyebjee and Vickery 1988).

Moreover, the investments are illiquid and their success relies on the quality and the skills of the company's management team, thus it is crucial for the Financial Decision Maker (FDM) to rely on mathematical models and tools able to guide him/her in a risky scenario. VCs supply capital for initiatives differentiated by risk, their portfolio approach allows them to balance successful and unsuccessful cases; in other words, through the diversification of the financial portfolio, the overall risk of a diversified VC portfolio will not be as high as the average of its individual investments (Manigart et al. 1994). It is evident that the motivations for constructing a VC portfolio are quite similar to that of constructing a financial portfolio (Markowitz 1952).

This chapter is organized as follows: next section illustrates the main criteria used in the VC decision making process; then we provide an introduction to MCDA and GP while the following section illustrate the Multiple Criteria Venture Capital

Decision Making model in both stochastic and fuzzy environments. The last two sections illustrate two numerical examples of application of these models to real data.

## 2 Venture Capital Decision Making: The Criteria

In the deal evaluation, VCs assess perceived risk and expected return on the basis of a weighting of several characteristics of the prospective venture and decide whether or not to invest as determined by the relative levels of perceived risk and expected return. Several studies of VC investment have been conducted previously where different tools have been applied such as: descriptive methods, linear statistical techniques and multicriteria evaluation. The descriptive studies proposed by Wells (1974), and Poindexter (1976) attempted to ascertain the relative importance of various criteria. Tyebjee and Bruno (1984) point out five dimensions: (1) Market Attractiveness (size, growth, and access to customers), (2) Product Differentiation (uniqueness, patents, technical edge, profit margin), (3) Managerial Capabilities (skills in marketing, management, finance and the references of the entrepreneur), (4) Environmental Threat Resistance (technology life cycle, barriers to competitive entry, insensitivity to business cycles and down-side risk protection), and (5) Cash-Out Potential (future opportunities to realize capital gains by merger, acquisition or public offering). Similarly Hisrich and Jankowicz (1990) identified three basic constructs, namely concept, management, and returns. The concept must rely on a feasible business idea (new product, service, or retail concept) able to offer a substantial competitive advantage or be in a relatively non-competitive industry. The criterion of the management team is quite predominant. The required characteristics range from personal integrity, to good track records or the capability of flexibility (especially for early-stage ventures) and leadership. Additionally, it is crucial the venture is a high-growth one with potential for earnings growth. Moreover, we can distinguish among specialist or generalist investors, the formers have specific criteria on investment size, industries in which they invest, geographic location of the investment, and stage of financing/development. The specialization exploits the ability of investors to influence nature and performances of the venture.

All these are broad generic criteria and the specifics of each criterion may vary from VCs to another. Each VCs evaluates the country in which the company is set up, the industry in which it operates and the availability and the reliability of data on which the choice is based. The country selection problem for business venturing consider four perspectives: economic (access to financial capital, growth of real gross domestic product,...), legal (business law, labor regulations, risks for intellectual property), political (bureaucracy, lack of corruption) and cultural. Different industries have different attractiveness values. Among the financial criteria, the FDM should consider needed time to attend the break-even, the expected rate of return and the needed time to payback. The profitability is a reasonable objective for the VCs (Schaffer 1989; Gilbert et al. 2006). However, Storey (2000) highlighted that entrepreneurs involved in startups worry about the likelihood of survival of their new ventures. Startup owners are concerned about remaining in the market and trying to avoid bankruptcy. Consequently, VCs simultaneously consider both survival and profit maximization. Another relevant criterion is the intellectual capital that refers to knowledge which must be an asset able to be used to create wealth.

From the above discussion, our model will consider the following conflicting criteria: a) the portfolio return, b) the survival rate of the entire portfolio, c) the intellectual capital (we mean patents, copyrights, methods, procedures and archives) and, d) the portfolio risk in terms of country impact.

The Multiple Criteria Decision Aid is driven by the need to deal with so-called complex decision making situations. We can mention some previous works related to VC decision making. Siskos and Zopounidis (1987) model a VC decision making using a multiple criteria decision support system (the Minora system) based on the interactive use of the Uta ordinal regression model. Zacharakisa and Dale Meyerb (2000) demonstrate that VC decision making can be improved through the aid of actuarial models, especially in information laden environments. Zhang (2012) presents a fuzzy optimal decision making model.

Also Goal Programming (GP) models have been extensively used for financial applications (see f.i. Aouni et al. 2014; Ben Abdelaziz et al. 2007, 2009). Recently, Aouni et al. (2013) formulate the VC investment problem through a cardinality constrained stochastic GP model where the Financial Decision Maker's preferences are explicitly incorporated through the concept of satisfaction functions.

## 3 Multiple Criteria Decision Making and Goal Programming

The multiple criteria decision making process involves several dimensions that are simultaneously optimized. These dimensions are usually conflicting and incommensurable. Hence the obtained solution can be considered as the recommendation of the best compromise that satisfies the FDM's preferences. The general formulation of the Multiple Objective Programming (MOP) is as follows (Sawaragi et al. 1985): Optimize $[f_1(x), f_2(x), \ldots, f_p(x)]$, under the condition that $x \in D \subseteq R^n$ where $f_i(x)$ represents the *i-th* objective function and $D$ designates the set of feasible solutions. Let us define a vector function $f(x) := [f_1(x), f_2(x), \ldots, f_p(x)]$; according to this, a classical MOP problem can be formulated as follows (let us assume that all objectives have to be minimized):

$$\text{Min } f(x) \tag{1}$$

Subject to

$$x \in D$$

We say that a point $\hat{x} \in D$ is a global Pareto optimal solution or global Pareto efficient solution if $f(x) \subseteq f(\hat{x}) + (-R_+^p \backslash \{0\})^c$ for all $x \in D$. Practically speaking, a Pareto optimal solution describes a state in which goods and resources are distributed in such a way that it is not possible to improve a single criterion without also causing at least one other criterion to become worse off than before the change. In other words, a state is not Pareto efficient if there exists a certain change in allocation of goods and resources that may result in some criteria being in a better position with no criterion being in a worse position than before the change. If a point $x \in D$ is not Pareto efficient, there is potential for a Pareto improvement and an increase in Pareto efficiency.

The Goal Programming model is a well-known aggregating methodology for solving multiple objective programming decision aid processes. GP model takes into account simultaneously several conflicting objectives and its solution represents the best compromise that can be made by the decision maker (DM). The GP model, based on a satisfying philosophy, is a distance function where the deviation, between the achievement and aspiration levels, is to be minimized. Indeed, both positive and negative deviations are unwanted. Initially formulated by Charnes et al. (1955) and Charnes and Cooper (1952, 1959), GP model is widely applied in several fields such as: accounting, finance, marketing, quality control, human resources, production and operations management (Romero 1991). The popularity of the GP is due to the fact that is a simple and easy model to understand and to apply. Moreover, the GP formulation can be solved through some powerful mathematical programming software such as Lindo and CPLEX. The standard mathematical formulation of the GP model is as follows:

$$\text{Min Z} = \sum\nolimits_{i=1}^{p} \delta_i^+ + \delta_i^- \tag{2}$$

Subject to:

$$\begin{cases} f_i(x) + \delta_i^- - \delta_i^+ = g_i, & i = 1 \dots p \\ x \in D \\ \delta_i^-, \delta_i^+ \geq 0, & i = 1 \dots p \end{cases}$$

where $\delta_i^+$ and $\delta_i^-$ are, respectively, the positive and the negative deviations with respect to the aspiration levels (goals) $g_i$, $i = 1, \dots, p$. An alternative definition of GP Model is the so called Weighted Goal Programming:

$$\text{Min Z} = \sum\nolimits_{i=1}^{p} w_i^+ \delta_i^+ + w_i^- \delta_i^- \tag{3}$$

Subject to:

$$\begin{cases} f_i(x) + \delta_i^- - \delta_i^+ = g_i, & i = 1 \dots p \\ x \in D \\ \delta_i^-, \delta_i^+ \geq 0, & i = 1 \dots p \end{cases}$$

Sometimes the DM wishes to weight each deviation through a satisfaction function. As a result, one can define a GP Model with satisfaction function as follows:

$$\text{Max Z} = \sum\nolimits_{i=1}^{p} w_i^+ F_i^+(\delta_i^+) + w_i^- F_i^-(\delta_i^-) \tag{4}$$

Subject to:

$$\begin{cases} f_i(x) + \delta_i^- - \delta_i^+ = g_i, & i = 1 \dots p \\ x \in D \\ 0 \leq \delta_i^- \leq \alpha_{iv}^-, \ i = 1 \dots p \\ 0 \leq \delta_i^+ \leq \alpha_{iv}^+, \ i = 1 \dots p \end{cases}$$

Martel and Aouni (1990) have introduced the concept of satisfaction functions in the GP model because it allows the DM to explicitly express his/her preferences for any deviation of the achievement from the aspiration level of each objective. Figure 1 illustrates the general shape of a satisfaction function.

where:

- $F_i$ are the satisfaction functions.
- $\alpha_{id}$ are the indifference thresholds: there is total satisfaction when the deviations are less than these values.
- $\alpha_{io}$ is the dissatisfaction threshold: these is no satisfaction when the deviations reach this threshold but the solution is not rejected.
- $\alpha_{iv}$ is the veto threshold: any solution that lead to deviations larger than this threshold is rejected.

The satisfaction functions are taking values in the interval [0, 1]. Therefore, the satisfaction functions have a value of 1 when the DM is totally satisfied; otherwise they are monotonically decreasing and can take values between 0 and 1 (see Fig. 1).

In many real financial contexts, the DM has to take decisions under uncertainty. Hence the objective functions and the corresponding goals are, in general, random variables. The Stochastic Goal Programming (SGP) model deals with the uncertainty related to the decision making situation. Mainly, in the SGP we assume that the goal values are stochastic and follow a specific probability distribution. The general formulation of the SGP is as follows:

$$\text{Min Z} = \sum\nolimits_{i=1}^{p} w_i^+ \widetilde{\delta}_i^+ + w_i^- \widetilde{\delta}_i^- \tag{5}$$

Subject to:

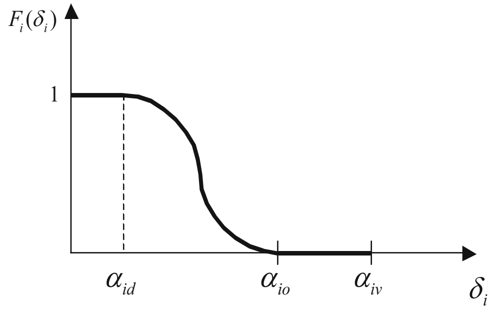


**Fig. 1** Satisfaction function

$$\begin{cases} f_i(x) + \widetilde{\delta}_i^- - \widetilde{\delta}_i^+ = \widetilde{g}_i, \ i = 1 \ldots p \\ x \in D \\ \widetilde{\delta}_i^-, \widetilde{\delta}_i^+ \geq 0, \ i = 1 \ldots p \end{cases}$$

where $\widetilde{g}_i \in N\left(\mu_i; \sigma_i^2\right)$.

Another alternative way to include randomness in the GP model is to consider the so-called scenario-based models. If we assume that the space of all possible events or scenarios $\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}$ with associated probabilities $p(\omega_s) = p_s$ is finite and the objective functions and the corresponding goals are depending on the scenario $\omega_s$, the above SGP model with satisfaction function can be extended to

$$\text{Max Z} = \sum\nolimits_{i=1}^{p} w_i^+ F_i^+(\omega_s, \delta_i^+) + w_i^- F_i^-(\omega_s, \delta_i^-) \tag{6}$$

Subject to:

$$\begin{cases} f(\omega_s, x_i) + \delta_i^- - \delta_i^+ = g_i(\omega_s), \quad i = 1 \ldots p \\ x \in D \\ 0 \leq \delta_i^- \leq \alpha_{iv}^-, \quad i = 1 \ldots p \\ 0 \leq \delta_i^+ \leq \alpha_{iv}^+, \quad i = 1 \ldots p \end{cases}$$

where $\omega_s \in \Omega$ is fixed. This approach has been introduced in Aouni and La Torre (2010) to analyze portfolio optimization model and then extended in Aouni et al. (2013) in the context of VC decision making.

The Fuzzy Goal Programming (FGP) model was developed to deal with some decisional situations where the DM can only give vague and imprecise goal values; in other words aspiration levels are not known precisely. Watada (1997) points out that Markowitz's approach to portfolio management is not suitable in resolving situations in which the aspiration level and utility given by the FDM cannot be defined exactly. And he proposes a fuzzy portfolio selection model able to obtain a solution which realizes the best compromise within a vague aspiration level and a fuzzy number as a goal. The FGP is based on the fuzzy sets theory developed by Zadeh (1965) and Bellman and Zadeh (1970). The concept of membership functions, based on the fuzzy set theory, has been introduced and used by Zimmerman

(1976, 1978, 1983) and Freeling (1980) for modelling the fuzziness related to decision making context parameters. The Narasimhan (1980) and Hannan's (1981) FGP formulations also use the concept of membership functions to deal with the fuzziness of the goal values. The general formulation of the membership function requires two acceptability degrees (lower and upper) (Zimmerman 1990) and the functions are assumed to be linear. Dhingra et al. (1992), Rao (1987) and Zimmerman (1978, 1988) have developed an approximation procedure for the non-linear membership functions. In their papers, Narasimhan (1980) and Hannan (1981) have developed triangular membership functions. The use of such triangular functions is questioned by Ignizio (1982) and Chen and Tsai (2001) as they consider that it can lead, in some cases, to undesired results. The FGP model with integer variables we consider in the following paragraph can be formulated as follows (see also Yang et al. 1991):

$$\text{Max } \lambda \tag{7}$$

Subject to

$$\begin{cases} \lambda \leq \dfrac{f_i(x) - f_i^{MIN}}{f_i^{GOAL} - f_i^{MIN}} \\ \lambda \leq \dfrac{f_i^{MAX} - f_i(x)}{f_i^{MAX} - f_i^{GOAL}} & i = 1 \ldots p \\ x \in D \end{cases}$$

Arenas-Parra et al. (2001) have utilized a FGP for portfolio selection. They have considered three criteria which are expected return, risk level (the variance return of the portfolio) and the portfolio's liquidity as fuzzy terms. Bilbao-Terol et al. (2006) integrate the knowledge of the expert and the preferences of the FDM. They made an extension of Sharpe model where the data are fuzzy and the betas are estimated on basis of the historical data. Bilbao-Terol et al. (2007) have designed flexible decision making models for portfolio selection including expert's knowledge and imprecise preferences and included them in a GP decision making model for portfolio selection. Mansour et al. (2007) developed an imprecise GP model for portfolio selection based on the satisfaction functions. The FDM's intuition, experience and judgment were expressed explicitly through the satisfaction functions. Three objectives were considered: rate of return, the liquidity and the risk. Inuiguchi and Ramik (2000) paper emphasizes that real world problems are not usually so easily formulated as fuzzy models. Moreover, Ignizio (1982), Wang and Zhu (2002), Sharma et al. (2009) and Gupta and Bhattacharjee (2010) developed a FGP approach for portfolio management in different contexts.

# 4 Goal Programming and Venture Capital Investments

In the following pages we introduced three different GP models, namely a GP model with Satisfaction Function, a Stochastic GP model with Satisfaction Function and a Fuzzy GP model.

## 4.1 Model I: A Goal Programming Model with Satisfaction Function

The GP model for VC decision making considers the four following objectives:

- $f_1$ provides the return of the investment,
- $f_2$ assigns the survival rate of the investment,
- $f_3$ gives the intellectual capital rate,
- $f_4$ is the investment risk.

We propose the following GP model with the Satisfaction Function to deal with such a decision making context:

$$\text{Max } Z = \sum_{i=1}^{4} w_i F_i^+(\delta_i^+) + w_i F_i^-(\delta_i^-) \tag{8}$$

Subject to

$$\begin{cases} f_i(x) - \delta_i^+ + \delta_i^- = g_i, i = 1...4 \\ x \in D \\ \delta_i^+, \delta_i^- \le \alpha_{iv}, i = 1...4 \\ \delta_i^+, \delta_i^- \ge 0, i = 1...4 \end{cases}$$

where $F_i^+\left(\delta_i^+\right)$ and $F_i^-\left(\delta_i^-\right)$ are functions having thresholds such that $\delta_i^+$ and $\delta_i^-$ defined the deviation. The $w_i$ represent the intrinsic component of the objective relative importance. The veto threshold $\alpha_{iv}$ is specified by the FDM and $g_i$ are the goal values. As discussed in the previous section, a VC in interested in minimizing the number of investments or at least to keep it less or equal than a fixed number. In order to formulate this decision making model, we will introduce the function $\text{supp}(x) = \{i : x_i \neq 0\}$ which counts the number of nonzero components of the vector $x$ and imposes a limit on the number of investments. Hence the previous GP model with satisfaction function can be rewritten as follows:

$$\text{Max } Z = \sum_{i=1}^{4} w_i F_i^+(\delta_i^+) + w_i F_i^-(\delta_i^-) \tag{9}$$

Subject to

$$\begin{cases} f_i(x) - \delta_i^+ + \delta_i^- = g_i, \quad i = 1, \ldots, 4 \\ x \in D \\ \delta_i^+, \delta_i^- \le \alpha_{iv}, \quad i = 1, \ldots, 4 \\ \delta_i^+, \delta_i^- \ge 0, \quad i = 1, \ldots, 4 \\ \text{supp}(\text{x}) \le \text{M} \end{cases}$$

where $M$ is the fixed number of investments to be considered for the financial portfolio. The presence of the function supp($x$) makes this model more complex to be analyzed. It can be classified as a nonlinear mixed-integer optimization problem and represent a natural extension of a quadratic mixed-integer optimization problem. This kind of models has been extensively analyzed in literature from both computational and complexity perspectives and it has been shown to belong to the class of NP-hard problems (see Bienstock 1996; Chang et al. 2000a, b; Anagnostopoulos, and Mamanis 2011; Bertsimas and Shioda 2009; Fieldsend et al. 2004; Li et al. 2006; Maringer and Kellerer 2003; Shaw et al. 2008; Soleimani et al. 2009). A multiple criteria model involving the function supp($x$) was analyzed in La Torre (2003) in which the author proposed an approximation based on $C^{1,1}$ (differentiable with locally lipschitzian gradient) function.

## 4.2 Model II: A Stochastic Goal Programming Model with Satisfaction Function

As said above, a VCs has to take decisions under uncertainty. Hence the objective functions and the corresponding goals are, in general, random variables and this leads to consider stochastic or scenario-based GP models. If we assume that the space of all possible events or scenarios $\Omega = \{\omega_1, \omega_2, \ldots, \omega_N\}$ with associated probabilities $p(\omega_i) = p_i$ is finite and the objective functions $f_i$ and the corresponding goals $g_i$ are depending on the scenario $\omega_i$, the above model can be extended to

$$\text{Max } Z = \sum_{i=1}^{4} w_i F_i^+(\delta_i^+) + w_i F_i^-(\delta_i^-) \tag{10}$$

Subject to

$$\begin{cases} f_i(x,\omega_s) - \delta_i^+ + \delta_i^- = g_i(\omega_s), \quad i = 1...4 \\ x \in D \\ \delta_i^+, \delta_i^- \le \alpha_{iv}, \quad i = 1...4 \\ \delta_i^+, \delta_i^- \ge 0, \quad i = 1...4 \\ \text{supp}(x) \le M \end{cases}$$

where $\omega_s \in \Omega$. As discussed in Aouni et al. (2010), the most natural way to solve this scenario-based model is to find the solution to the problem for any fixed scenario $\omega_s \in \Omega$ and the optimal solution will correspond to the one which possesses the highest probability.

#### 4.2.1 Model III: A Fuzzy GP Model

In the VC investment decision making process the DM has no sufficient information related to the different criteria: this uncertainty and lack of information can be efficiently described using fuzzy sets and the Fuzzy GP model. We propose the following formulation with integer variables:

$$\text{Max } \lambda \tag{11}$$

Subject to

$$\begin{cases} \lambda \le \mu_{[F_1(x)]} \\ \lambda \le \mu_{[F_2(x)]} \\ \lambda \le \mu_{[F_3(x)]} \\ \lambda \le \mu_{[F_4(x)]} \\ x \in D \\ \text{supp}(x) \le M \\ x_i \in \{0,1\} \end{cases}$$

The equation $x \in D$ describes all possible financial constraints related to the specific decision making context, including a budget constraint, while the inequality $\text{supp}(x) \le M$ limits the number of investments that can be activated in the financial portfolio.

Now we illustrate the GP models by examples.

**Example 1** In order to illustrate the proposed model I, we will consider some data from an anonymous Italian venture capital fund operating in information technology and communication (a specialist investor). In the sequel let us name this company as United Ventures (UV). The company manages a 1000 million euro fund, and the size of investment is usually between one and ten million euros, and typically it holds minority shares between 10 and 30 %. Table 1 shows the ten selected business plans.

**Table 1** UV potential portfolio

| | Company | Focus |
|---|---|---|
| 1 | Invest Newco SA | Reseller of hosting space and domain registration |
| 2 | Egrocery Newco | Offer both mortgage quotes and links to developers of buy-to-let property investment |
| 3 | Mphone Newco S. p.A. | On line financial information |
| 4 | Adv Newco S.r.l. | Internet advertising |
| 5 | Mmania Newco Ltd. | M-Commerce and E-commerce for the UK mobile market |
| 6 | E-Finance Newco S.p.A. | Web design services and Internet financial information |
| 7 | Together Newco | On line group buying in Europe |
| 8 | Info NewCo Ltd. | Distributors of mobile phone in Germany |
| 9 | Mortgage Newco S.A. | On line trading service |
| 10 | Mobile Newco Inc. | New technology into web-enabled or SMS-enabled mobile phones |

**Table 2** UV Investment data

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Investment (mln) | 6 | 3 | 1.05 | 1.78 | 3.18 | 0.51 | 5.24 | 6.64 | 2 | 0.79 |

**Table 3** Investment criteria

| Company | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| Investment return rate | 0.15 | 0.33 | 0.2 | 0.1 | 0.18 | 0.2 | 0.3 | 0.15 | 0.15 | 0.09 |
| Survival rate (1 year) | 0.84 | 0.95 | 0.93 | 0.94 | 0.93 | 0.94 | 0.95 | 0.9 | 0.94 | 0.93 |
| Intellectual capital | 0.1 | 0.1 | 0.1 | 0 | 0.2 | 0.2 | 0.1 | 0.1 | 0.2 | 0.5 |
| Investment risk rate | 0.07 | 0.05 | 0.03 | 0.07 | 0.03 | 0.07 | 0.05 | 0.04 | 0.07 | 0.03 |

Table 2 shows the fixed amount to be invested in each company and Table 3 illustrates the criteria value for each venture-backed company.

Our model involves the investment return, the survival index, and the intellectual capital and the minimization of the investment risk. We suppose that the number of investments M is equal to 7 (as we said the VCs typically consider a small number of investments because time and money are scarce resources), and the available budget is equal to ten millions of euro. The weights are assumed to be equal to $w_1^+ = w_1^- = 0.3, w_2^+ = w_2^- = 0.2, w_3^+ = w_3^- = 0.1, w_4^+ = w_4^- = 0.4$ and the goal levels $g_i$ for each criterion are $g_1 = 2.82$, $g_2 = 5.63$, $g_3 = 1.8$, and $g_4 = 0.5$. Then the GP model with Satisfaction Function is formulated as follows:

$$
\begin{aligned}
Max \quad & w_1^+F_1^+\left(\delta_1^+\right) + w_1^-F_1^-\left(\delta_1^-\right) + w_2^+F_2^+\left(\delta_2^+\right) + w_2^-F_2^-\left(\delta_2^-\right) + \\
& w_3^+F_3^+\left(\delta_3^+\right) + w_3^-F_3^-\left(\delta_3^-\right) + w_4^+F_4^+\left(\delta_4^+\right) + w_4^-F_4^-\left(\delta_4^-\right)
\end{aligned}
\tag{12}
$$

Subject to:

$$
\begin{gathered}
0.9x_1 + 0.99x_2 + 0.21x_3 + 0.178x_4 + 0.5724x_5 + 0.102x_6 + \\
1.572x_7 + 0.996x_8 + 0.3x_9 + 0.0711x_{10} + \delta_1^+ - \delta_1^- = 2.82 \\
0.84x_1 + 0.95x_2 + 0.93x_3 + 0.94x_4 + 0.93x_5 + 0.94x_6 + \\
0.95x_7 + 0.9x_8 + 0.94x_9 + 0.93x_{10} + \delta_2^+ - \delta_2^- = 5.63 \\
0.1x_1 + 0.1x_2 + 0.1x_3 + 0x_4 + 0.2x_5 + 0.2x_6 + \\
0.1x_7 + 0.1x_8 + 0.2x_9 + 0.5x_{10} + \delta_3^+ - \delta_3^- = 1.8 \\
0.42x_1 + 0.15x_2 + 0.0315x_3 + 0.1246x_4 + 0.0954x_5 + 0.0357x_6 + \\
0.262x_7 + 0.2656x_8 + 0.14x_9 + 0.0237x_{10} + \delta_4^+ - \delta_4^- = 0.5 \\
6x_1 + 3x_2 + 1.05x_3 + 1.78x_4 + 3.18x_5 + \\
0.51x_6 + 5.24x_7 + 6.64x_8 + 2x_9 + 0.79x_{10} \leq 10 \\
\sum_{i=1}^{15} x_i \leq 7 \\
x_i \in \{0, 1\} \\
\delta_i^+, \delta_i^- \leq \alpha_{iv} \\
\delta_i^+, \delta_i^- \geq 0
\end{gathered}
$$

Aouni et al. (2010) propose the following expression for the satisfaction function: $F_\alpha(x) = \left(1 + \alpha^2 x^2\right)^{-1}$ and the veto threshold equal to $3\alpha^{-1}$. This function shows a level of satisfaction less than 0.1 when $\delta \geq 3\alpha^{-1}$. With $\alpha = 0.1$, LINGO 12 provides the following solution: $x_1 = 0$, $x_2 = 1$, $x_3 = 1$, $x_4 = 1$, $x_5 = 0$, $x_6 = 1$, $x_7 = 0$, $x_8 = 0$, $x_9 = 1$, $x_{10} = 1$.

**Example 2** Model II presents three different scenarios, $\Omega = \{\omega_1, \omega_2, \omega_3\}$ with probabilities $p_1 = 0.30$, $p_2 = 0.35$ and $p_3 = 0.35$, respectively. The financial portfolio as well as the investment data are reported in Tables 1 and 2 of Example 1. In this stochastic context the investment criteria depend on the realization of different scenario instead: the returns for each investment are reported in Table 4a–c.

As in Example 1, model II involves four criteria, namely the investment return, the survival index, and the intellectual capital and the minimization of the investment risk. We still suppose that the number of investments $M$ is equal to 7, and the available budget is equal to ten millions of euro. Let us assume that the weights are $w_1^+ = w_1^- = 0.3$, $w_2^+ = w_2^- = 0.2$, $w_3^+ = w_3^- = 0.1$, $w_4^+ = w_4^- = 0.4$. We now conduct the numerical simulation when the first scenario $\omega_1$ is realized, the other cases can be treated analogously. Let us suppose that the VCs establishes for the

**Table 4** Scenario-based investment criteria

| (a) | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\omega_1$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| Investment return rate | 0.15 | 0.33 | 0.2 | 0.1 | 0.18 | 0.2 | 0.3 | 0.15 | 0.15 | 0.09 |
| Survival rate (1 year) | 0.84 | 0.95 | 0.93 | 0.94 | 0.93 | 0.94 | 0.95 | 0.9 | 0.94 | 0.93 |
| Intellectual capital | 0.1 | 0.1 | 0.1 | 0 | 0.2 | 0.2 | 0.1 | 0.1 | 0.2 | 0.5 |
| Investment risk rate | 0.07 | 0.05 | 0.03 | 0.07 | 0.03 | 0.07 | 0.05 | 0.04 | 0.07 | 0.03 |
| (b) | | | | | | | | | | |
| $\omega_2$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| Investment return rate | 0.2 | 0.33 | 0.2 | 0.2 | 0.18 | 0.4 | 0.3 | 0.15 | 0.15 | 0.09 |
| Survival rate (1 year) | 0.84 | 0.95 | 0.93 | 0.94 | 0.93 | 0.94 | 0.95 | 0.9 | 0.94 | 0.93 |
| Intellectual capital | 0.05 | 0.04 | 0.1 | 0.2 | 0.2 | 0.2 | 0.1 | 0.1 | 0.2 | 0.5 |
| Investment risk rate | 0.07 | 0.05 | 0.03 | 0.07 | 0.03 | 0.07 | 0.05 | 0.04 | 0.07 | 0.03 |
| (c) | | | | | | | | | | |
| $\omega_3$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| Investment return rate | 0.2 | 0.33 | 0.2 | 0.1 | 0.18 | 0.4 | 0.3 | 0.15 | 0.15 | 0.09 |
| Survival rate (1 year) | 0.84 | 0.95 | 0.93 | 0.94 | 0.93 | 0.94 | 0.95 | 0.9 | 0.94 | 0.93 |
| Intellectual capital | 0.05 | 0.04 | 0.1 | 0.2 | 0.2 | 0.2 | 0.1 | 0.1 | 0.2 | 0.5 |
| Investment risk rate | 0.14 | 0.1 | 0.06 | 0.14 | 0.06 | 0.07 | 0.05 | 0.04 | 0.07 | 0.03 |

scenario $\omega_1$ the goal levels $g_i$ for each criterion as follows: $g_1 = 2.82$, $g_2 = 5.63$, $g_3 = 1.8$, and $g_4 = 0.5$. Model II is the following:

$$\begin{aligned} \text{Max} \quad & w_1^+ F_1^+ \left(\delta_1^+\right) + w_1^- F_1^- \left(\delta_1^-\right) + w_2^+ F_2^+ \left(\delta_2^+\right) + w_2^- F_2^- \left(\delta_2^-\right) + \\ & w_3^+ F_3^+ \left(\delta_3^+\right) + w_3^- F_3^- \left(\delta_3^-\right) + w_4^+ F_4^+ \left(\delta_4^+\right) + w_4^- F_4^- \left(\delta_4^-\right) \end{aligned} \tag{13}$$

Subject to:

$$\begin{aligned} & 0.9x_1 + 0.99x_2 + 0.21x_3 + 0.178x_4 + 0.5724x_5 + 0.102x_6 + \\ & 1.572x_7 + 0.996x_8 + 0.3x_9 + 0.0711x_{10} + \delta_1^+ - \delta_1^- = 2.82 \end{aligned}$$

$$0.84x_1 + 0.95x_2 + 0.93x_3 + 0.94x_4 + 0.93x_5 + 0.94x_6 + \\ 0.95x_7 + 0.9x_8 + 0.94x_9 + 0.93x_{10} + \delta_2^+ - \delta_2^- = 5.63$$

$$0.1x_1 + 0.1x_2 + 0.1x_3 + 0x_4 + 0.2x_5 + 0.2x_6 + \\ 0.1x_7 + 0.1x_8 + 0.2x_9 + 0.5x_{10} + \delta_3^+ - \delta_3^- = 1.8$$

$$0.42x_1 + 0.15x_2 + 0.0315x_3 + 0.1246x_4 + 0.0954x_5 + 0.0357x_6 + \\ 0.262x_7 + 0.2656x_8 + 0.14x_9 + 0.0237x_{10} + \delta_4^+ - \delta_4^- = 0.5$$

$$6x_1 + 3x_2 + 1.05x_3 + 1.78x_4 + 3.18x_5 + \\ 0.51x_6 + 5.24x_7 + 6.64x_8 + 2x_9 + 0.79x_{10} \leq 10$$

$$\sum_{i=1}^{15} x_i \leq 7,$$

$$x_i \in \{0, 1\}$$

$$\delta_i^+, \delta_i^- \leq \alpha_{iv}$$

$$\delta_i^+, \delta_i^- \geq 0$$

As in Example 1, let us choose $F_\alpha(x) = (1 + \alpha^2 x^2)^{-1}$ and $\alpha = 0.1$. LINGO provides the following solution: $x_1 = 0$, $x_2 = 1$, $x_3 = 1$, $x_4 = 1$, $x_5 = 0$, $x_6 = 1$, $x_7 = 0$, $x_8 = 0$, $x_9 = 1$, $x_{10} = 1$. The probability associated with this solution is $p_1 = 0.3$. In a similar manner, we can conduct the numerical experiments for the other two scenarios by using the data provided in Table 4b, c and the same goals as before. When the second scenario $\omega_2$ is realized with probability $p_2 = 0.35$, we get the solution: $x_1 = 0$, $x_2 = 1$, $x_3 = 1$, $x_4 = 1$, $x_5 = 0$, $x_6 = 1$, $x_7 = 0$, $x_8 = 0$, $x_9 = 1$, $x_{10} = 1$, while when the third scenario $\omega_3$ is realized with $p_3 = 0.35$, we have: $x_1 = 0$, $x_2 = 0$, $x_3 = 1$, $x_4 = 0$, $x_5 = 0$, $x_6 = 1$, $x_7 = 1$, $x_8 = 1$, $x_9 = 1$, $x_{10} = 1$. According to the highest probability criterion the optimal strategy is the following $x_1 = 0$, $x_2 = 1$, $x_3 = 1$, $x_4 = 1$, $x_5 = 0$, $x_6 = 1$, $x_7 = 0$, $x_8 = 0$, $x_9 = 1$, $x_{10} = 1$ with probability equal to 0.65. Table 5 summarizes the results we have obtained.

**Example 3** In order to illustrate the proposed model III, we will consider empirical data from another anonymous Italian venture capital fund. Let us name this company by Venture Capital Partners (VCP) whose activity sector is mainly related to marketing and media. The fund manages a 500 million euro fund. The size of investment is usually between one and seven million euros (Table 6), and typically it holds minority shares between 15 and 35 %.

**Table 5** The optimal solution corresponding to each scenario

| Scenario | $p_j, x_i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\boldsymbol{\omega_1}$ | 0.3 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| $\boldsymbol{\omega_2}$ | 0.35 | 0 | 0 | 1 | 0 | 0 | 1 | 1 | 1 | 1 | 1 |
| $\boldsymbol{\omega_3}$ | 0.35 | 0 | 1 | 1 | 1 | 0 | 1 | 0 | 0 | 1 | 1 |

**Table 6** VCP Investment data

| $x_i$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Investment (mln) | 4 | 2 | 3 | 4 | 1.18 | 0.51 | 5.24 | 6.64 | 2 | 1.79 | 3 | 2.17 | 2.7 | 3.36 | 1.56 |

**Table 7** VCP portfolio

| Company | Focus |
|---|---|
| phg | Developer of leading-edge performance marketing solutions |
| eyeka | Online platform to sell user generated content |
| Sonico | Leading destination site for applications and services serving the global Hispanic on-line community |
| IGA | Operates a global network delivering advertisements into video games |
| Shazam | Digital finger printing technology used for music recognition for consumers and airplay monitoring |
| MoreMagic | Mobile commerce transaction platform software provider |
| Digital Chocolate | One of the leading social and mobile social game companies in the world |
| Air Sense | Smart data offload solutions for mobile and tablet devices |
| APSalar | Next generation smartphone application analytics and behavioural targeting solution |
| Mobile Roadie | Leading turn-key platform for mobile application building and management |
| JustBook | Leading app in Europe specialising in same-day hotel bookings for the mobile generation |
| Kana | Leading customer service software provider |
| Tbricks | State of the art automated trading platform designed for executing automated trading strategies on the financial markets |
| Green Leads | Global provider of performance based outbound sales solutions |
| Mister Spex | Germany's largest online retailer of eyewear |

The VCs deal quite frequently with the complex problem of capital budgeting in the case of a high technology company that lacks a sufficient number of comparables/peers, thus the degree of uncertainty is high. This is the case of this fund. Table 7 shows the potential portfolio and Table 8 reports the investment criteria for each venture-backed company.

$$\text{Max } \lambda \tag{14}$$

Subject to:

$$\lambda \leq -\big(0.4x_1 + 0.6x_2 + 0.22x_3 + 0.12x_4 + 0.6x_5 + 0.02x_6 + 1.2x_7 + 0.85x_8 \\ +0.5x_9 + 0.061x_{10} + 0.7x_{11} + 1.0x_{12} + 1.2x_{13} + 0.4x_{14} + 0.3x_{15}\big) + 3.82$$

$$\lambda \leq -\big(0.4x_1 + 0.6x_2 + 0.22x_3 + 0.12x_4 + 0.6x_5 + 0.02x_6 + 1.2x_7 + 0.85x_8 \\ +0.5x_9 + 0.061x_{10} + 0.7x_{11} + 1.0x_{12} + 1.2x_{13} + 0.4x_{14} + 0.3x_{15}\big) - 1.82$$

$$\lambda \leq -\big(0.4x_1 + 0.73x_2 + 0.56x_3 + 0.56x_4 + 0.553x_5 + 0.493x_6 + 0.7x_7 + 0.61x_8 \\ +0.56x_9 + 0.433x_{10} + 0.733x_{11} + 0.933x_{12} + 0.23x_{13} + 0.55x_{14} + 0.64x_{15}\big) + 4.7$$

**Table 8** Investment criteria

| Criteria | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Investment rate | 0.4 | 0.6 | 0.22 | 0.12 | 0.6 | 0.02 | 1.2 | 0.85 | 0.5 | 0.061 | 0.7 | 1.0 | 1.2 | 0.4 | 0.3 |
| Survival Rate | 0.84 | 0.95 | 0.93 | 0.94 | 0.93 | 0.94 | 0.95 | 0.9 | 0.94 | 0.93 | 0.94 | 0.94 | 0.94 | 0.93 | 0.9 |
| Intellectual capital | 0.1 | 0.1 | 0.1 | 0 | 0.2 | 0.2 | 0.1 | 0.1 | 0.2 | 0.5 | 0.4 | 0.5 | 0.2 | 0.2 | 0.4 |
| Investment risk rate | 0.07 | 0.05 | 0.03 | 0.07 | 0.03 | 0.07 | 0.05 | 0.04 | 0.07 | 0.03 | 0.07 | 0.07 | 0.07 | 0.03 | 0.04 |

$$\lambda \leq -\big(0.4x_1 + 0.73x_2 + 0.56x_3 + 0.56x_4 + 0.553x_5 + 0.493x_6 + 0.7x_7 + 0.61x_8 + 0.56x_9 + 0.433x_{10} + 0.733x_{11} + 0.933x_{12} + 0.23x_{13} + 0.55x_{14} + 0.64x_{15}\big) - 2.75$$

$$\lambda \leq -\big(0.15x_1 + 0.125x_2 + 0.125x_3 + 0.x_4 + 0.15x_5 + 0.125x_6 + 0.25x_7 + 0.375x_8 + 0.5x_9 + 0.1875x_{10} + 0.3x_{11} + 0.5375x_{12} + 0.2625x_{13} + 0.15x_{14} + 0.3x_{15}\big) + 3.25$$

$$\lambda \leq -\big(0.15x_1 + 0.125x_2 + 0.125x_3 + 0.x_4 + 0.15x_5 + 0.125x_6 + 0.25x_7 + 0.375x_8 + 0.5x_9 + 0.1875x_{10} + 0.3x_{11} + 0.5375x_{12} + 0.2625x_{13} + 0.15x_{14} + 0.3x_{15}\big) - 1.25$$

$$\lambda \leq -\big(1.71x_1 + 1.71x_2 + 0.35x_3 + 1.857x_4 + 0.286x_5 + 0.224x_6 + 4.0x_7 + 8.57x_8 + 0.571x_9 + 0.357x_{10} + 4.43x_{11} + 4.71x_{12} + 6.0x_{13} + 2.86x_{14} + 2.286x_{15}\big) + 8.14$$

$$\lambda \leq -\big(1.71x_1 + 1.71x_2 + 0.35x_3 + 1.857x_4 + 0.286x_5 + 0.224x_6 + 4.0x_7 + 8.57x_8 + 0.571x_9 + 0.357x_{10} + 4.43x_{11} + 4.71x_{12} + 6.0x_{13} + 2.86x_{14} + 2.286x_{15}\big) - 6.142$$

$$4x_1 + 2x_2 + 3x_3 + 4x_4 + 1.18x_5 + 0.51x_6 + 5.24x_7 + 6.64x_8 + 2x_9 + 1.79x_{10} + 3x_{11} + 2.17x_{12} + 2.7x_{13} + 3.36x_{14} + 2.56x_{15} \leq 10$$

$$\sum_{i=1}^{15} x_i \leq 7$$

$$x_i \in \{0, 1\}, \quad i = 1...15$$

The number of investments M = 7, and the Budget = 10 Millions of euros. LINDO provides the following solution $x_1 = 0$, $x_2 = 1$, $x_3 = 0$, $x_4 = 0$, $x_5 = 1$, $x_6 = 1$, $x_7 = 0$, $x_8 = 0$, $x_9 = 1$, $x_{10} = 1$, $x_{11} = 0$, $x_{12} = 1$, $x_{13} = 0$, $x_{14} = 0$, $x_{15} = 0$. From the above result we get that the number of investments is equal to six companies, and the VC company should invest 9.56 million of euros.

## 5 Conclusions

In this chapter we have focused on the decision making process related to venture capital investments. The literature review points out that this process mainly involves the performance of the following criteria: the return of the investment, the survival rate of the investment, the intellectual capital rate, and the investment risk. We have proposed three different models, namely a deterministic GP model with Satisfaction Function, a Stochastic GP model with Satisfaction Function, and a Fuzzy GP formulation, to cope with this complex decision making situation in which the presence of uncertainty plays a fundamental role. The above models have been applied to specific case studies in the areas of telecommunication and media/marketing. A possible extension of this research involves the inclusion of more variables related to the management team or the extension of the presented models into a dynamic framework.

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# On Dynamic Multiple Criteria Decision Making Models: A Goal Programming Approach

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**Abstract** Dynamic multiple criteria decision making (DMCDM) represents an extension of classical multiple criteria decision making to a context in which all variables are depending on time. This complex decision making problem requires the development of methodologies able to incorporate different and conflicting goals in a satisfying design of policies. We formulate two different goal programming models, namely a weighted goal programming model and a goal programming model with satisfaction functions, for solving DMCDM models. We present an application of this methodology to analyze the trade-off between consumption and investment in a traditional Ramsey-type macroeconomic model with heterogeneous agents. For a specific realistic parameterization, such a model is solved by means of the proposed goal programming formulations.

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## 1 Introduction

As society and the world as a whole become more and more complex, making good decisions becomes harder and harder. More frequently a decision maker (DM) is called to evaluate a set of alternatives in terms of a certain number of conflicting criteria. This is even more critical in an economic setup where the DM tries to allocate resources to his/her best use (going beyond the traditional resource availability constraints) and also needs to take into account several economic goals. Several authors in the literature have highlighted that human needs are incommensurable and thus economic benefits cannot be measured by a mere scalar number. Keeney and Howard (1976: 19) state that "*in complex value problems consequences cannot be adequately described objectively by a single attribute*". Just to provide a simple example, we can mention the recent concern for environmental issues; the DM has to plan the use of natural resources (such as water, land, and forestry) coping with several conflicting objectives (as, for instance, the decrease of the level of emissions of a power plant against the benefits of the power plant itself). Along this direction, André et al. (2009) have recently pointed out that policymakers do not seek to maximize a single function, but they are typically concerned about a bundle of economic, social and environmental variables or indicators, thus they try to design their policies to improve the performance of the economy as measured by multiple indicators.

In reality, all problems and especially those related to economic issues cannot be summarized in a static framework, but carefully require to consider the evolution of the decision making process over time. For example, think about the simplest economic problem that every individual needs to face in his daily life, like the choice between consuming in order to achieve higher current utility or saving in order to invest his resources and possibly achieve higher utility levels in the future. This clearly requires taking into account the dynamic evolution of income over a certain horizon (possibly the whole lifespan) in order to determine the best allocation of a scarce resource (income) between its possible usages (consumption and investment). Even more complex it is the task for economic policymakers, since whenever trying to determine the best policy in order to pursue alternative economic goals, they need to account for the implications of such a policy on every single agent in the society. In addition, the presence of agents' heterogeneity makes the decision making process particularly difficult. Given such a dynamic nature of real world problems and the need to account for agents' heterogeneity, it is quite natural to rely upon dynamic multiple criteria decision making (DMCDM) in which for any feasible path the objective function provides a vector of values, representing the individual utility of every economic agent.

DMCDM models need to be understood in the Pareto sense, meaning that they search for optimal solutions with respect to the Pareto ordering cone. In the literature some results have been proved to characterize the optimal solutions of such models (see for instance, Khanh and Nuong 1988, 1989; Ginchev et al. 2012). In this chapter we do not provide any alternative optimality condition but we focus on practical approaches for solving these models instead. We propose two different goal programming (GP) models, namely a weighted GP (WGP) model and a GP model with satisfaction functions (GPSF), for approximating and solving a DMCDM program. We then illustrate this approach through a specific example in the context of macroeconomic policies in which we extend the classical Ramsey model analyzing the trade-off between consumption and investment choices by introducing a vector-valued utility to take into account agents' heterogeneity. Focusing on the Ramsey model allows us to exemplify the potential implications of the usage of DMCDM models to study economic problems. However, the proposed models can be straightforwardly adapted in order to analyze several other issues in which the problem is dynamic in nature and agents are heterogeneous with respect to certain characteristics. Apart from macroeconomic applications, others may include environmental policies, climate change agreements, and more broadly speaking differential and cooperative games (see Engwerda 2007) for a discussion of a dynamic multiple criteria approach applied to game theory).

The chapter is organized as follows. First we present the basic formulation of a DMCDM model, and then we recall some well-known GP models. After introducing two alternative specifications of the GP model, we present an illustrative example dealing with a multi agent macroeconomic model. Finally, we present some concluding remarks and propose directions for future research.

## 2 Multiple Criteria Decision Making

The general formulation of the multi-criteria model can be specified as follows: optimize $[J_1(x), J_2(x), \ldots, J_p(x)]$ under the condition that $x \in D$ where $J_i(x)$ represents the $i$-th objective function and D designates the set of feasible solutions (typically a compact subset of a normed vector space $X$). Let us define a vector function $J(x) := \left[J_1(x), J_2(x), \ldots, J_p(x)\right]$; according to this definition and by assuming that all objectives have to be minimized, a classical MCDM problem can be formulated as:

$$\text{Min}\, J(x) \tag{1}$$

Subject to:

$$x \in D$$

As usual in multi-criteria optimization, a point $\hat{x} \in D$ is a global Pareto solution or efficient solution if $J(x) \subseteq J(\hat{x}) + (-\mathrm{R}_+^p \backslash \{0\})^c$ for all $x \in D$. In other words, a point $\hat{x} \in D$ is a Pareto solution if there is no $x \in D$ such that $J_i(x) \geq J_i(\hat{x})$ for all $i = 1 \ldots p$ and $J_{i^*}(x) > J_{i^*}(\hat{x})$ for at least one $i^* \in \{1 \ldots p\}$. The set of all Pareto solutions is called the Pareto frontier. Then a Pareto solution is never dominated by another feasible solution and for this reason it is called an undominated solution. The following results provide two conditions that characterize Pareto solutions (Sawaragi et al. 1985).

**Theorem 1** Let $\alpha_i \in (0, 1)$, $\sum_{i=1}^{p} \alpha_i = 1$. Assume that $\hat{x} \in D$ is such that:

$$\hat{x} \in \underset{x \in D}{\operatorname{argmin}} \left\{ \sum_{i=1}^{p} \alpha_i J_i(x) \right\}$$

Then $\hat{x}$ is a Pareto optimal solution.

**Theorem 2** Suppose that $D$ is convex and $J_i(x)$ are convex for all $i = 1 \ldots p$. Then for all Pareto optimal solutions $\hat{x}$ there exists $\alpha \in \mathrm{R}^p$, $\alpha_i \in [0, 1]$, $\sum_{i=1}^{p} \alpha_i = 1$, such that:

$$\hat{x} \in \underset{x \in D}{\operatorname{argmin}} \left\{ \sum_{i=1}^{p} \alpha_i J_i(x) \right\}$$

The above Theorems 1 and 2 provide basic structure on the objective functions and the feasible solution set to characterize Pareto solutions in a multiple criteria problem. In particular Theorem 1 provides a sufficient optimality condition for Pareto optimality while Theorem 2 a sufficient one. It is worth noting that Theorem 2 is valid under the assumption of convexity.

## 3 Goal Programming

Within the multi-criteria decision aid paradigm, several usually conflicting criteria are considered simultaneously. The GP model is a well-known aggregating methodology for solving multi-objective programming problems allowing to take into account simultaneously several conflicting objectives. Thus the obtained solution through the GP model represents the best compromise that can be achieved by the DM. The GP model is a distance function where the unwanted positive and negative deviations, between the achievement and aspiration levels, are to be minimized. The GP model, first proposed by Charnes and Cooper (1952), Charnes and Cooper

(1959), and Charnes et al. (1955), has been widely applied in several fields such as accounting, marketing, quality control, human resources, production, economics and operations management (Lee 1973; Aouni et al. 1997; Romero 1991; Aouni and La Torre 2010; Aouni et al. 2012, 2013, 2014). The standard mathematical formulation of the GP model (Charnes and Cooper 1952) is as follows:

$$\text{Min Z} = \sum_{i=1}^{p} \delta_i^+ + \delta_i^- \tag{2}$$

Subject to:

$$\begin{aligned} &J_i(x) + \delta_i^- - \delta_i^+ = g_i, \quad i = 1 \ldots p \\ &x \in D \\ &\delta_i^-, \delta_i^+ \geq 0, \quad i = 1 \ldots p \end{aligned}$$

where $\delta_i^+$ and $\delta_i^-$ are, respectively, the positive and the negative deviations with respect to the aspiration levels (goals) $g_i$, $i = 1 \ldots p$. The DM's appreciation of the positive and the negative deviations can be different based on the relative importance of the objective which can be expressed through the weights $w_i^+$ and $w_i^-$ respectively. The mathematical formulation of the weighted GP (WGP) is as follows:

$$\text{Min Z} = \sum_{i=1}^{p} w_i^+ \delta_i^+ + w_i^- \delta_i^- \tag{3}$$

Subject to:

$$\begin{aligned} &J_i(x) + \delta_i^- - \delta_i^+ = g_i, \quad i = 1 \ldots p \\ &x \in D \\ &\delta_i^-, \delta_i^+ \geq 0, \quad i = 1 \ldots p \end{aligned}$$

In decision making the role of the DM is crucial, and both how he thinks and decides, and what are his own values can significantly affect the decision making process. Usually a DM has a specific set of preferences which can be described through the notion of satisfaction functions. When such a system of preferences is introduced the GP model takes the following form:

$$\text{Max Z} = \sum_{i=1}^{p} w_i^+ F_i^+ \left(\delta_i^+\right) + w_i^- F_i^- \left(\delta_i^-\right) \tag{4}$$

Subject to:

$$\begin{aligned} &J_i(x) + \delta_i^- - \delta_i^+ = g_i, \quad i = 1 \ldots p \\ &x \in D \\ &0 \leq \delta_i^- \leq \alpha_{iv}^-, \quad i = 1 \ldots p \\ &0 \leq \delta_i^+ \leq \alpha_{iv}^+, \quad i = 1 \ldots p \end{aligned}$$

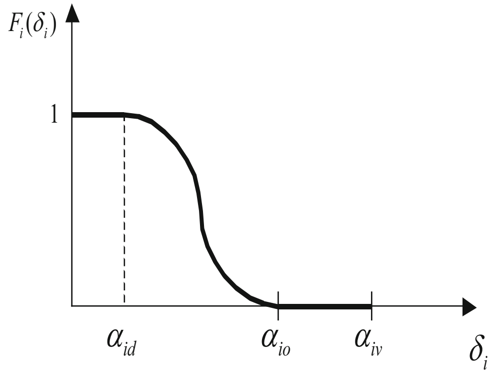


**Fig. 1** General form of the satisfaction function

The satisfaction functions $F_i(\delta_i)$ allow the DM to express explicitly his preferences for any deviation between the achievement and aspiration levels of each objective: the general shape of the satisfaction functions is shown in Fig. 1 (Martel and Aouni 1990), where $F_i(\delta_i)$ is the satisfaction function associated with the deviation $\delta_i$, $\alpha_{id}$ the indifference threshold, $\alpha_{io}$ the dissatisfaction threshold and $\alpha_{iv}$ the *veto* threshold.

## 4 Dynamic Multiple Criteria Decision Making and Goal Programming

Denote with $X = C^1([a,b])$ the space of all differentiable paths defined on $[a,b]$, with $U$ a set of controls, and with $f: \mathrm{R} \times \mathrm{R}^n \times \mathrm{R}^n \to \mathrm{R}^p$ and $h: \mathrm{R} \times \mathrm{R}^n \times \mathrm{R}^n \to \mathrm{R}^n$ two vector-valued smooth functions. Let us consider the following dynamic multi-criteria problem:

$$\text{Min} \int_a^b f(t, x(t), u(t))dt \tag{5}$$

Subject to:

$$\begin{aligned}
&\dot{x}(t) = h(t, x(t), u(t)) \\
&x(a) = x_a \\
&x(b) = x_b \\
&u \in U
\end{aligned}$$

This represents a natural extension of classical optimal control problems to vector problems. We do not formulate any hypothesis on $b$, meaning that the above model can be assumed to be over either a finite or a infinite horizon.

Optimality conditions for (4) can be found, for instance, in Ginchev et al. (2012) and Engwerda (2007). They can be stated by using the optimality conditions presented in the previous case for an abstract optimization model and by defining

$$\mathrm{J_i}(x,u) = \int_a^b f_i(t,x(t),u(t))dt.$$

We are now ready to formulate a GP model for solving (5). Let $g = \left(g_1, g_2, \ldots, g_p\right) \in \mathrm{R}^p$ be a set of $p$ goals corresponding to $p$ functionals $\mathrm{J_i}(x,u)$. For this model, the standard mathematical formulation of the GP model is as follows:

$$\text{Min Z} = \sum\nolimits_{i=1}^{p} \delta_i^+ + \delta_i^- \tag{6}$$

Subject to:

$$\begin{aligned}
&\int_a^b f_i(t,x(t),u(t))dt + \delta_i^- - \delta_i^+ = g_i, \quad i = 1 \ldots p \\
&\dot{x}(t) = h(t,x(t),u(t)) \\
&x(a) = x_a \\
&x(b) = x_b \\
&u \in U \\
&\delta_i^-, \delta_i^+ \geq 0, \quad i = 1 \ldots p
\end{aligned}$$

where $\delta_i^+$ and $\delta_i^-$ are, respectively, the positive and the negative deviations with respect to the aspiration levels (goals) $g_i$, (i = 1,. . ., p). An alternative model is the weighted goal programming that in this context can be formulated as follows:

$$\text{Min Z} = \sum\nolimits_{i=1}^{p} w_i^+ \delta_i^+ + w_i^- \delta_i^- \tag{7}$$

Subject to:

$$\begin{aligned}
&\int_a^b f_i(t,x(t),u(t))dt + \delta_i^- - \delta_i^+ = g_i, \quad i = 1 \ldots p \\
&\dot{x}(t) = h(t,x(t),u(t)) \\
&x(a) = x_a \\
&x(b) = x_b \\
&u \in U \\
&\delta_i^-, \delta_i^+ \geq 0, \quad i = 1 \ldots p
\end{aligned}$$

where $w_i^+$ and $w_i^-$ are the weights corresponding to positive and negative deviations, respectively. The DM can express the relative importance of the objectives by providing a level of satisfaction of each positive and negative deviation. As a result, the GP model with satisfaction function is

$$\text{Max Z} = \sum\nolimits_{i=1}^{p} w_i^+ F_i^+ \left(\delta_i^+\right) + w_i^- F_i^- \left(\delta_i^-\right) \tag{8}$$

Subject to:

$$\begin{aligned}
&\int_a^b f_i(t, x(t), u(t))dt + \delta_i^- - \delta_i^+ = g_i, \quad i = 1 \dots p \\
&\dot{x}(t) = h(t, x(t), u(t)) \\
&x(a) = x_a \\
&x(b) = x_b \\
&u \in U \\
&0 \le \delta_i^- \le \alpha_{iv}^- \;\; (\forall i \in I), \\
&0 \le \delta_i^+ \le \alpha_{iv}^+ \;\; (\forall i \in I).
\end{aligned}$$

The above two alternative formulations (7) and (8), after discretization of both integrals and differential equations, can be solved as static optimization problems.

## 5 An Example: The Ramsey Model with Vector-Valued Utility

For our purpose of exemplifying the usage of the two proposed GP models for solving DMCDM problems, the well-known Ramsey (1928) model may be useful. Indeed, it well fits formulation (2), since it summarizes the investment problem from a macroeconomic point of view as a traditional optimal control problem. The Ramsey (1928) model basically describes how a benevolent social planner (i.e., the DM) might decide what is the optimal level of consumption for the whole society by taking into account the fact that a larger consumption level tends to crowd out resources from investment opportunities: the more we consume today, the less we save and invest, thus the less resources we will have in the future to allow further consumption possibilities. The model is nowadays still the benchmark for assessing the impact of alternative macroeconomic policies on the long run development process of different economies. It has been extended along several directions in order to take into account also issues related to demography (Marsiglio 2014), environment (Marsiglio 2011), technological progress (La Torre and Marsiglio 2010), human capital (Marsiglio and La Torre 2012a, b), and many other aspects relevant for macroeconomic goals.

The standard Ramsey model is a scalar problem in which the DM determines the best choice for the society as a whole which is totally summarized by the characteristics of the so-called “representative agent”. In such a framework the society is totally homogeneous, in the sense that its members have all the same characteristics (preferences, endowments, and even relevant parameters), thus the optimization with respect to the representative agent coincides with the optimization for the whole society. Such a homogeneity in the characteristics of agents is clearly a strong simplification of reality, since in every society individuals differ in several ways. A more sensible description of the problem would thus require to allow for some heterogeneity in the characteristics of agents, and this can be straightforwardly done with the GP models we introduced in the previous section. Indeed, a simple way to account for agents’ heterogeneity is assuming that the instantaneous utility function does not take a scalar form but a vector-valued one. This means that agents are identical for some aspect (capital endowments) but not for others (preferences). A vector-valued extension of the Ramsey model in Banach spaces has been recently discussed in Ginchev et al. (2012) where the authors also provide necessary and sufficient optimality conditions.

In a Ramsey-type (1928) model, the social planner seeks to maximize social welfare by choosing the level of consumption and taking into account the dynamic evolution of capital. For the sake of simplicity we abstract from population growth and we normalize the population size to unity. The dynamic evolution of capital, coinciding with investments, depends on the difference between net (of replacement investments, with $\eta$ being the depreciation rate of capital) output, $Y(t)$, and consumption, $C(t)$. Output is produced according to a Cobb-Douglas production function, $Y(t) = AK^{\alpha}(t)$, where $A$ is a technological scale parameter and $0 < \alpha < 1$ represents the capital share of output. Whenever agents are homogeneous, the social welfare is defined as the discounted ($\rho$ is the pure rate of time preference) sum of the instantaneous utilities of the representative agent; the instantaneous utility function is assumed to take a constant elasticity of substitution form, $\mathrm{U(C(t))} = \frac{C(t)^{1-\varphi}-1}{1-\varphi}$, where $\varphi$ denotes the inverse of the intertemporal elasticity of substitution. However, when agents are heterogeneous and differ for their preferences, focusing on the representative agent is no longer possible. In such a framework we need to take into account the specific preferences of each single agent, and thus the social welfare function needs to reflect this, by attaching some weight to the utility of each agent. Denoting with $U_i(C(t))$ the instantaneous utility for agent $i$, in order to allow for some heterogeneity we assume that the parameter denoting the rate of time preference, $\rho_i$, and the intertemporal elasticity of substitution, $\varphi_i$, can differ from agent to agent. Thus, our vector-valued Ramsey model takes the form:

$$\mathrm{Max}\left(\int_a^b U_1(C(t))e^{-\rho_1 t}dt, \ldots, \int_a^b U_p(C(t))e^{-\rho_p t}dt\right) \tag{9}$$

Subject to

$$\dot{K}(\mathrm{t}) = \mathrm{AK}^{\alpha}(\mathrm{t}) - \eta\mathrm{K}(\mathrm{t}) - \mathrm{C}(\mathrm{t})$$

The objective function in (9), describing the social welfare, takes values in $R^p$ and depends on $p$ instantaneous utility functions $U_i$, $i = 1 \ldots p$ and different discount factors $\rho_i$. The dynamic constraint describes the evolution of physical capital over time, stating that for each $t$, output, $Y(t)$, is either consumed ($C(t)$) or invested $\left(\dot{K}(t) + \eta K(t)\right)$. We assume that capital endowments are the same for each individual and since the capital market is the same for each agent, the dynamic evolution of capital is not agent-specific.

In order to solve the problem above, we rely on the two GP formulations earlier described. For the sake of simplicity we focus on the case in which we only have two agents, that is $p = 2$, meaning that the objective function takes values in $R^2$. The problem we are interested in can thus be formulated as follows:

$$\underset{c(t)}{Max} \left( \int_0^{+\infty} \frac{\mathrm{C}(\mathrm{t})^{1-\varphi_1}}{1-\varphi_1} \mathrm{e}^{-\rho_1 t}\, \mathrm{dt}, \int_0^{+\infty} \frac{\mathrm{C}(\mathrm{t})^{1-\varphi_2}}{1-\varphi_2} \mathrm{e}^{-\rho_2 t}\, \mathrm{dt} \right) \tag{10}$$

Subject to:

$$\dot{K}(\mathrm{t}) = \mathrm{AK}^{\alpha}(\mathrm{t}) - \eta\mathrm{K}(\mathrm{t}) - \mathrm{C}(\mathrm{t})$$

In order to apply our GP models, we need first of all to determine the goal for each criterion. In order to do so, we proceed by solving the two single criterion problems separately, and using the value of the associated optimal objective functions to determine the goal to attach to the relevant criterion. Thus we consider two maximization problems separately, namely one for agent 1:

$$g_1 = \underset{c(t)}{Max} \int_0^{+\infty} \frac{\mathrm{C}(\mathrm{t})^{1-\varphi_1}}{1-\varphi_1} \mathrm{e}^{-\rho_1 t}\, \mathrm{dt} \tag{11}$$

Subject to:

$$\dot{K}(\mathrm{t}) = \mathrm{AK}^{\alpha}(\mathrm{t}) - \eta\mathrm{K}(\mathrm{t}) - \mathrm{C}(\mathrm{t})$$

and one for agent 2:

$$g_2 = \underset{c(t)}{Max} \int_0^{+\infty} \frac{\mathrm{C}(\mathrm{t})^{1-\varphi_2}}{1-\varphi_2} \mathrm{e}^{-\rho_2 t}\, \mathrm{dt} \tag{12}$$

Subject to:

$$\dot{K}(t) = AK^{\alpha}(t) - \eta K(t) - C(t)$$

The problems (11) and (12) can be analytically solved in order to determine the value of the goals $g_1$ and $g_2$ (see Smith 2007). Once $g_1$ and $g_2$ have been determined, our two different specifications of the GP model can applied. Specifically, the WGP model can be constructed as follows:

$$\text{Min Z} = w_1^+\delta_1^+ + w_1^-\delta_1^- + w_2^+\delta_2^+ + w_2^-\delta_2^- \tag{13}$$

Subject to:

$$\begin{aligned}
&\int_0^{+\infty} \frac{C(t)^{1-\varphi_1}}{1-\varphi_1} e^{-\rho_1 t}\, dt + \delta_1^- - \delta_1^+ = g_1 \\
&\int_0^{+\infty} \frac{C(t)^{1-\varphi_2}}{1-\varphi_2} e^{-\rho_2 t}\, dt + \delta_2^- - \delta_2^+ = g_2 \\
&\dot{K}(t) = AK^{\alpha}(t) - \eta K(t) - C(t) \\
&\delta_i^-, \delta_i^+ \geq 0, \quad \forall\, i \in \{1, 2\}
\end{aligned}$$

Instead, the GPSF can be constructed as follows:

$$\text{Max Z} = w_1^+F_1^+\left(\delta_1^+\right) + w_1^-F_1^-\left(\delta_1^-\right) + w_2^+F_2^+\left(\delta_2^+\right) + w_2^-F_2^-\left(\delta_2^-\right) \tag{14}$$

Subject to:

$$\begin{aligned}
&\int_0^{+\infty} \frac{C(t)^{1-\varphi_1}}{1-\varphi_1} e^{-\rho_1 t}\, dt + \delta_1^- - \delta_1^+ = g_1 \\
&\int_0^{+\infty} \frac{C(t)^{1-\varphi_2}}{1-\varphi_2} e^{-\rho_2 t}\, dt + \delta_2^- - \delta_2^+ = g_2 \\
&\dot{K}(t) = AK^{\alpha}(t) - \eta K(t) - C(t) \\
&0 \leq \delta_1^+ \leq \alpha_{1v}^+ \\
&0 \leq \delta_2^+ \leq \alpha_{2v}^+ \\
&0 \leq \delta_1^- \leq \alpha_{1v}^- \\
&0 \leq \delta_2^- \leq \alpha_{2v}^-
\end{aligned}$$

## 5.1 Numerical Simulations

We now provide a numerical solution of our model by applying the GP specification in (13) and (14). For this purpose, we set the values of the parameters as follows: $A = 1$, $\rho_1 = 0.05$, $\rho_2 = 0.06$, $\varphi_1 = 2$, $\varphi_2 = 2.5$, $\eta = 0.05$, $\alpha = 0.33$, and

$K_0 = 1$ (see Barro and Sala-i-Martin 2004, for an economic justification of these parameters' values), and we use LINGO 14 for solving the relevant optimization problem. Under this parameterization, the model (9) reads as:

$$\underset{C(t)}{Max}\left(\int_0^{+\infty}\frac{C(\mathrm{t})^{-1}}{-1}\mathrm{e}^{-0.05t}\,\mathrm{dt},\int_0^{+\infty}\frac{C(\mathrm{t})^{-1.5}}{-1.5}\mathrm{e}^{-0.06t}\,\mathrm{dt}\right) \tag{15}$$

Subject to

$$\dot{K}(\mathrm{t}) = K^{0.33}(t) - 0.05\mathrm{K}(\mathrm{t}) - C(\mathrm{t})$$

We focus first on the WGP specification, where the weights $w_1^+, w_1^-, w_2^+, w_2^-$ are assumed to take different values, in order to assess how attaching a different weight to each different criterion will affect the model's solution. For the sake of simplicity, we assume that positive and negative deviations receive the same weights. This means that if we attach a weight of 0.2 to the first agent $\left(w_1^+ = w_1^- = 0.2\right)$ then we are attaching a weight of 0.8 to the second one $\left(w_2^+ = w_2^- = 0.8\right)$. Specifically, we consider four different weights configurations: 0.2, 0.4, 0.6 and 0.8, meaning that $w_1^+ = w_1^- = 0.2, 0.4, 0.6, 0.8$ whenever $w_2^+ = w_2^- = 0.8, 0.6, 0.4, 0.2$. The WGP model can now formulated as follows:

$$\mathrm{Min\ Z} = w_1^+\delta_1^+ + w_1^-\delta_1^- + w_2^+\delta_2^+ + w_2^-\delta_2^- \tag{16}$$

Subject to:

$$\begin{aligned}
&\int_0^{+\infty}\frac{C(\mathrm{t})^{-1}}{-1}\mathrm{e}^{-0.05t}\,\mathrm{dt} + \delta_1^- - \delta_1^+ = g_1\\
&\int_0^{+\infty}\frac{C(\mathrm{t})^{-1.5}}{-1.5}\mathrm{e}^{-0.06t}\,\mathrm{dt} + \delta_2^- - \delta_2^+ = g_2\\
&\dot{K}(\mathrm{t}) = K^{0.33}(t) - 0.05\mathrm{K}(\mathrm{t}) - C(\mathrm{t})\\
&\delta_i^-, \delta_i^+ \geq 0\ (\forall\, i \in \{1, 2\})
\end{aligned}$$

Since the problem is stated in discrete time, we need to proceed with its discretization in order to perform some numerical simulation. We approximate the infinite horizon integrals with a finite horizon $T$ and the differential equations using classical numerical schemes as follows:

$$\mathrm{Min\ Z} = w_1^+\delta_1^+ + w_1^-\delta_1^- + w_2^+\delta_2^+ + w_2^-\delta_2^-$$

Subject to:

$$\sum_{j=0}^{T}\frac{C(j)^{-1}}{-1}\mathrm{e}^{-0.05j}+\delta_1^- - \delta_1^+ = g_1$$
$$\sum_{i=0}^{T}\frac{C(j)^{-1.5}}{-1.5}\mathrm{e}^{-0.06j}+\delta_2^- - \delta_2^+ = g_2$$
$$K(j+1)-K(j)=K^{0.33}(j)-0.05\mathrm{K}(j)-C(j),\quad j=0\ldots T-1$$
$$\delta_1^-,\delta_1^+,\delta_2^-,\delta_2^+\geq 0$$

where the goals $g_1 = 2.07$ and $g_2 = 1.10$ are determined by the solution of the single criteria problems, as earlier discussed. The optimal dynamics of consumption, *C(t)*, and capital, *K(t)*, for each different values of the weights are shown in Figs. 2 and 3 respectively.

We move now to the GPSF version of the model. Let us consider the following satisfaction function $F_\gamma(\delta)=\frac{1}{1+\gamma^2\delta^2}$. This function presents the desired properties (as in Fig. 4) and it is trivial to verify that $F(0)=1$, $F(+\infty)=0$, $F''(\delta)=0\Leftrightarrow\delta=\frac{1}{2\gamma}$ and that $0.9\leq F(\delta)\leq 1$ if $0\leq\delta\leq\frac{1}{3\gamma}$, $0\leq F(\delta)\leq 0.1$ if $\delta\geq\frac{3}{\gamma}$ and $0\leq F(\delta)\leq 0.01$ if $\delta\geq\frac{3}{\gamma}$. This means that this function shows a level of satisfaction between 90 and 100 % when $0\leq\delta\leq\frac{1}{3\gamma}$ and a level of satisfaction between 0 and 10 % when $\delta\geq\frac{3}{\gamma}$. Natural candidates for the indifference threshold and the dissatisfaction threshold are, respectively, $\gamma_{id}=\frac{1}{3\gamma}$ and $\gamma_{io}=\frac{3}{\gamma}$. Let us assume the veto threshold $\gamma_{iv}=2{*}\gamma_{io}=\frac{6}{\gamma}$.

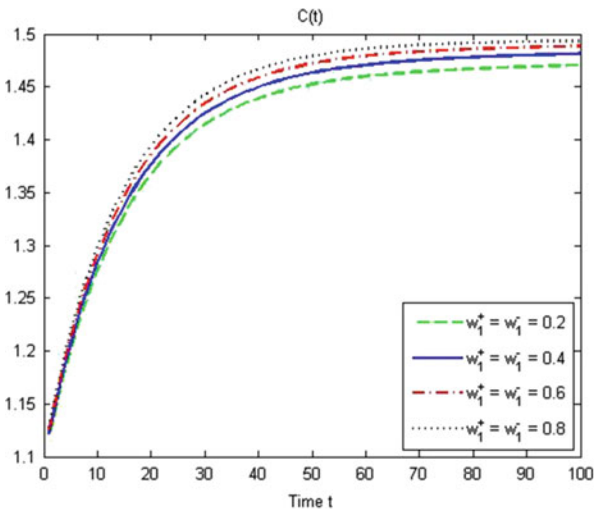


**Fig. 2** Optimal dynamics of consumption (WGP)

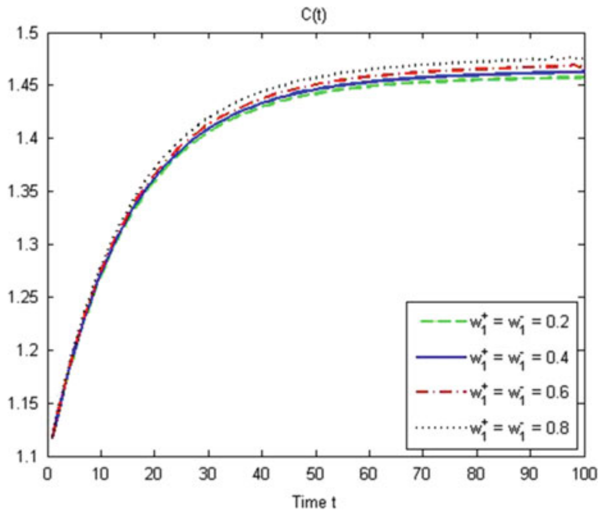


**Fig. 3** Optimal dynamics of capital (WGP)

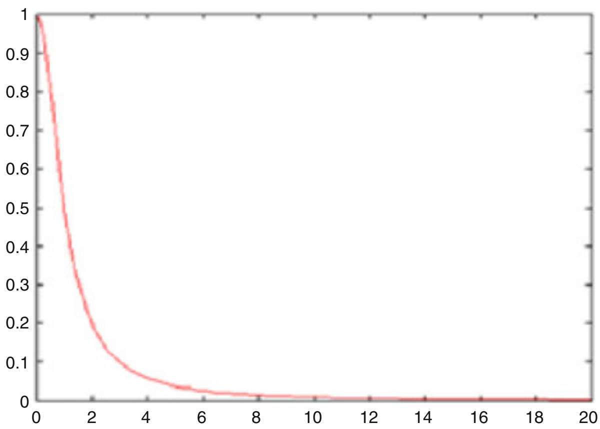


**Fig. 4** The satisfaction function F(δ)

As for the WGP model, we compare the impact of different relative importance in the two goals on the solution, considering exactly the same values of the weights. Using the above set of parameters, the GPSF model can be formulated as follows:

$$\text{Max Z} = \frac{w_1^+}{1 + \left(\gamma\delta_1^+\right)^2} + \frac{w_1^-}{1 + \left(\gamma\delta_1^-\right)^2} + \frac{w_2^+}{1 + \left(\gamma\delta_2^+\right)^2} + \frac{w_2^-}{1 + \left(\gamma\delta_2^-\right)^2} \tag{17}$$

Subject to:

$$\begin{aligned}
&\int_0^{+\infty} \frac{C(\mathrm{t})^{-1}}{-1} \mathrm{e}^{-0.05t}\,\mathrm{dt} + \delta_1^- - \delta_1^+ = g_1 \\
&\int_0^{+\infty} \frac{C(\mathrm{t})^{-1.5}}{-1.5} \mathrm{e}^{-0.06t}\,\mathrm{dt} + \delta_2^- - \delta_2^+ = g_2 \\
&\dot{K}(\mathrm{t}) = K^{0.33}(t) - 0.05\mathrm{K}(\mathrm{t}) - C(\mathrm{t}) \\
&0 \le \delta_i^-, \delta_i^+ \le \frac{6}{\gamma}\,(\forall\, i \in \{1,2\})
\end{aligned}$$

where the goals are determined as earlier and therefore they are set as follows: $g_1 = 2.07$ and $g_2 = 1.10$. We also set $\gamma = 1$. In order to discretize the problem, we proceed as previously by approximating the infinite horizon integrals with a finite horizon $T$ and the differential equations using classical numerical schemes as follows:

$$\text{Max Z} = \frac{w_1^+}{1+\left(\delta_1^+\right)^2} + \frac{w_1^-}{1+\left(\delta_1^-\right)^2} + \frac{w_2^+}{1+\left(\delta_2^+\right)^2} + \frac{w_2^-}{1+\left(\delta_2^-\right)^2} \tag{18}$$

Subject to:

$$\begin{aligned}
&\sum_{j=0}^{T} \frac{C(j)^{-1}}{-1} \mathrm{e}^{-0.05j} + \delta_1^- - \delta_1^+ = 2.07 \\
&\sum_{i=0}^{T} \frac{C(j)^{-1.5}}{-1.5} \mathrm{e}^{-0.06j} + \delta_2^- - \delta_2^+ = 1.10 \\
&K(j+1) - K(j) = K^{0.33}(j) - 0.05\mathrm{K}(j) - C(j), \quad j = 0 \ldots T-1 \\
&0 \le \delta_i^-, \delta_i^+ \le 6\,(\forall\, i \in \{1,2\})
\end{aligned}$$

The optimal dynamics of consumption, *C(t)*, and capital, *K(t)*, for each different values of the weights are shown in Figs. 5 and 6, respectively.

By comparing Figs. 2 and 3 with Figs. 5 and 6, we can see that the results are qualitatively identical. Given the parameter values concerning the rate of time preference and the inverse of the intertemporal elasticity of substitution for the two agents, attaching a higher weight to the welfare of the agent 1 increases the overall consumption and capital stock in the economy.

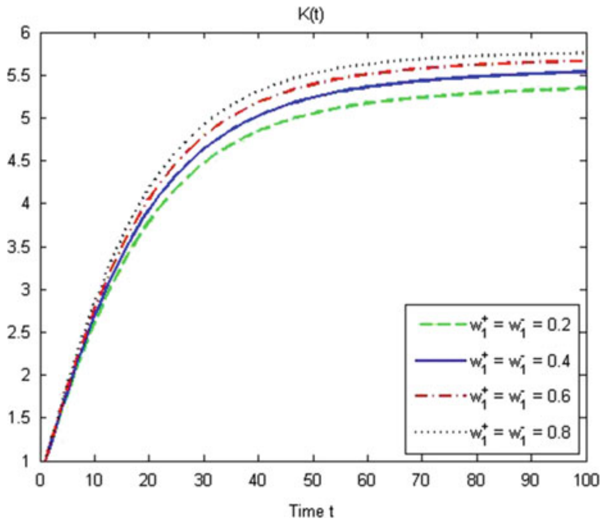


**Fig. 5** Optimal dynamics of consumption (GPSF)

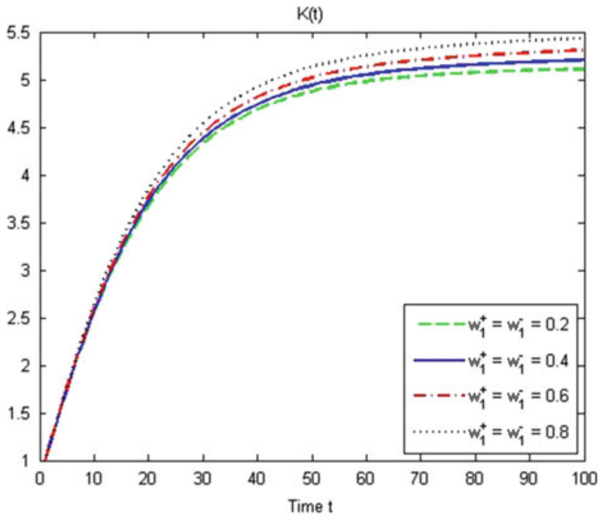


**Fig. 6** Optimal dynamics of capital (GPSF)

## 6 Conclusions

In this chapter we have introduced two different formulations based on the GP philosophy for solving dynamic multi-criteria decision making problems. We have then presented an illustrative example in the area of macroeconomic policy, focusing on consumption and investment decisions, to show how this approach can be implemented when dealing with real world situations. The illustrated multi-criteria philosophy underlying the approach is consistent with the needs of policymakers to deal with dynamic problems with multiple goals to be simultaneously pursued even if they might have different importance. The numerical simulation developed for the vector-valued Ramsey model shows the goodness of this approach in the context of macroeconomic policy with vector-valued utility. This allows us to consider in a simple way how agents' heterogeneity may be encompassed in traditional macroeconomic models and how such a heterogeneity may affect the determination of optimal economic policies. A similar approach may be used to deal with issues which can be modeled as a dynamic problem and in which agents' are heterogeneous. Some specific examples include differential and cooperative games for analyzing environmental policy and climate change negotiations. For future research it might be interesting to combine our approach with a differential game setup in order to consider how agents' heterogeneity affects the potential trade-off between economic development and environmental preservation.

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# Cross-Frontier DEA Methodology to Evaluate the Relative Performance of Stock and Mutual Insurers: Comprehensive Analysis

**María Rubio-Misas and Trinidad Gómez**

**Abstract** This chapter provides an in-depth analysis of the cross-frontier methodology, an innovative approach based on data envelopment analysis (DEA), for estimating the relative efficiency of alternative organizational forms in an industry, and testing hypotheses primarily founded on the agency theory arguments on the coexistence in the insurance industry of two organizational forms—stock insurers, owned by stockholders and mutual insurers, owned by policyholders. The analysis involves estimating the efficiency of the firms in each group not only with respect to a reference frontier consisting only of firms from its own group but also with reference to the other group's frontier. This allows calculating cross-to-own efficiency ratios which measure the distance between the stock and mutual frontiers. These ratios are key statistics to test the superiority of one technology over the other. Linear optimization procedures are used to estimate production, cost and revenue frontiers, both for the standard own-frontiers setups as well as the cross-frontiers models.

**Keywords** Cross-frontier DEA Analysis • Relative Performance • Organizational Forms • Insurance Industry

## 1 Introduction

The insurance industry offers a particularly interesting laboratory for the study of organizational forms as two types coexist in the industry in a large number of countries: stock companies that are owned by stockholders and employ the standard corporate forms, and mutuals that are owned by the customers, the policyholders.

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According to the modern theory of the firm, agency costs provide an explanation for the structure of organizations with the organizations that succeed in a given industry being the ones that minimize costs and maximize revenues, where both cost and revenues are potentially affected by agency cost as well as the firm's production process and operating environment (Jensen and Meckling 1976). The principal advantages of the stock form are more effective control over management and access to capital. The main advantage of the mutual structure results from the merged customer and ownership functions, which yields lower contracting costs to resolve customer-owner conflicts (Mayers and Smith 1988; Cummins et al. 1999).

Based on agency theoretic and non-theoretic arguments about the coexistence of these types of organizational forms, several economic hypotheses have been developed, the two most prominent being the *efficiency structure hypothesis* and the *expense preference hypothesis*. In order to test these hypotheses about the superiority of one organizational form over the other, Cummins et al. (1999) introduced a new and more sophisticated approach—the cross-frontier analysis—for estimating the relative efficiency of the stock and mutual organization forms using nonparametric frontier efficiency analysis and comparing subsets of firms in an industry. An important gap that the cross-frontier methodology comes to solve is that when stocks and mutuals use different technologies and operate with different frontiers, comparing efficiencies based on the pooled frontier (consisting of both stocks and mutuals) is not informative.

The aim of this chapter and its contribution to literature is to present a more comprehensive analysis of the cross-frontier methodology than the extant studies and give the keys to understanding the benefit from using this approach by providing a thorough survey of literature. In doing so, this study is organized as follows. Section 2 presents the theoretical economic background that justifies the use of the methodology. Section 3 provides a revision of the research on the effect of organizational form on the performance of the insurance industry using frontier efficiency analysis, with special focus on studies using the cross-frontier analysis. Section 4 discusses the measurement of inputs, outputs and prices in analyzing efficiency and productivity in the insurance industry. Section 5 provides an overview of the methodology and its formalization as well as discusses several examples. Section 6 summarizes and concludes.

## 2 Theoretical Economic Background

Various economic hypotheses have been developed that address the coexistence of the stock and mutual organizational forms in the insurance industry, the two most prominent being the *efficiency structure hypothesis* and the *expense preference hypothesis*.

The *efficiency structure hypothesis* claims that the two organizational forms are sorted into market segments where they have comparative advantages in minimizing costs and maximizing revenues due to differences in managerial discretion, maturity, and access to capital.[1] The managerial discretion argument states that the degree of managerial discretion required to operate in a given line of insurance is an important determinant of the organizational form likely to succeed in that line. This argument predicts that stock insurers are more successful in complex lines such as industrial insurance which are characterized by more flexibility and high managerial discretion. Mutuals, on the other hand, are expected to be more successful in standardized lines, which are characterized by relatively low managerial discretion such as liability which has standardized policies and good actuarial tables (Mayers and Smith 1988). The maturity argument predicts mutuals to be more successful than stocks in lines such as liability insurance where contracts cover relatively long periods of time. A longer time horizon gives stocks managers more opportunity to behave opportunistically and to reduce the value of policyholder claims of the firm. The relative success of the mutual organizational form in this case is due to the elimination of the owner-policyholder conflict and, consequently, the elimination of the possibility of expropriating value from policyholders to stockholders. The access to capital arguments (which is not an agency theory argument, in contrast with the two previous arguments) posits that stocks have advantages in serving risky lines in which the degree of product innovation is relatively high since they have better access to capital than mutuals (Cummins et al. 2004).[2] Stock firms can raise capital using both equity and debt markets. However, mutuals can raise capital by retaining earnings or through the issuance of surplus notes which is a form of highly subordinated debt.

The *expense preference hypothesis*, in contrast to the *efficient structure hypothesis*, states that mutuals will be less successful than stocks in minimizing costs and maximizing revenues due to unresolved agency conflicts (e.g. higher perquisite consumption by mutual managers) since the available mechanisms for controlling owner-manager conflicts are relatively weak in mutuals (see Mester 1989).

To sum up, the *efficiency structure hypothesis* predicts stocks and mutuals will have equal efficiency after controlling for production technology and business mix, whereas the *expense preference hypothesis* predicts mutuals will be less efficient than stocks. It is important to note that these two hypotheses (*the efficient structure hypothesis and the expense preference hypothesis*) are not mutually exclusive. Mutuals could be more successful in low managerial discretion lines even though mutual managers exhibit expense preference behavior. In fact, empirical studies

[1] Sorting is predicted to occur through the natural operation of the market as firms compete with one another in terms of price, risk management and claims settlement services, product offerings, and another product and service dimensions (Cummins et al. 2004).

[2] Empirical evidence has shown that the access to capital is one of the main motivations for the conversion of mutuals to stocks (e.g. Viswanathan and Cummins 2003; Zanjani 2007; Erhemjamts and Leverty 2010). Several studies (e.g. Mayers and Smith 1988; Lamm-Tennant and Starks 1993) have shown that stocks work in riskier lines than mutuals.

(e.g. Cummins et al. 1999, for USA property-liability insurers) provide evidence of these two hypotheses in the same market.

## 3 Literature Revision

The effect of organizational forms on performance using frontier efficiency analysis is an important field analyzed in literature. Other themes on organizational forms in the insurance industry that have attracted research interest are agency-theoretic considerations (e.g. Lee et al. 1997; Pottier and Sommer 1997; Cole et al. 2011), and changes in the legal form (e.g. Zanjani 2007; Erhemjamts and Phillips 2012). Furthermore—without being exhaustive—researchers have studied how a Pareto-optimal risk allocation can be achieved through mutual insurance in the presence of individual risk (Cass et al. 1996); the dissimilarities concerning capital structure which may result from the cost of raising new capital (Harrington and Niehaus 2002); issues arising from asymmetric information that can restrict the size of the mutuals (Ligon and Thistle 2005); how mutuals can resolve free-rider and commitment issues faced by stock insurers by linking policies to the provision of capital (Laux and Muermann 2010); or have developed a normative theory of the relationship between stock and mutual insurers based on a contingent claims framework (Braun et al. 2015).

In this section we revised the most important studies on the effect of organizational form on performance using frontier efficiency analysis with special focus on papers utilizing the cross-frontier analysis. We have revised 14 papers. Working papers were not included in the revision. We also excluded papers where the organizational form is not an important issue, even though the organizational form is a question which could be controlled in the analysis (as for example with a dummy variable). We are aware of the difficulty to distinguish the relative importance of the issues analyzed in a paper (for a comprehensive review of papers using frontier efficiency analysis in analyzing performance in the insurance industry see Eling and Luhnen 2010; Cummins and Weiss 2013). We present Table 3 in an appendix with a revision of these 14 studies.

Table 3 gives information about: (1) the country or countries analyzed; (2) the industry segment studied (life, non-life); (3) the estimation methodology; (4) the approach used to measure outputs and inputs; (5) the type of frontier estimated; and (6) the principal findings. According to this table most studies focus on the US insurance market but, in a national context, the insurance markets of France, Japan, the Netherlands and Spain have been also studied. Bierner and Eling is the only international study that specifically analyzes the issue of organizational form in the insurance industry using frontier efficiency analysis. In terms of estimation methodology the revised studies use a variety of methods (both econometric and mathematical programming). Most of them (5 in both cases) use the cross-frontier DEA or the standard DEA. However, additional approaches are used in the revised studies, both parametric (SFA, DFA, TFA) and non-parametric (FDH, RAM-DEA).

In addition, there are three papers that calculate the total factor productivity change of insurers over time using the Malmquist index approach as an extension of the DEA methodology.[3]

The value-added approach is the most used method to measure insurance outputs and inputs, but there are several papers that compare their results by using two approaches (the value-added approach and the intermediation approach). There are six non-life studies, five life studies and three studies analyze more than one industry segment. There are three studies analyzing only cost frontiers, three studies analyzing only technical frontiers and seven studies analyzing both cost and technical frontiers. Cummins et al. (2004) is the only study that analyzes revenue frontiers in addition to technical and cost frontiers.

According to our knowledge, there are five papers using the cross-frontier analysis to evaluate the relative performance of stocks and mutual insurers. When Cummins et al. (1999) proposed this methodology, they illustrated the analysis by using a sample of USA stock and mutual property-liability insurers. Cross-to-own technical and cost efficiency ratios were estimated providing evidence that stock technology dominates the mutual technology for producing stock outputs and mutual technology dominates the stock technology for producing mutual outputs. However, the stock cost frontier dominated the mutual cost frontier consistent with the *expense preference hypothesis*. Later, Cummins et al. (2004) applied the cross-frontier analysis to a sample of Spanish stock and mutual insurers. They estimated cross-to-own revenue efficiency ratios in addition to cross-to own technical and cost efficiency ratios. Results in general were consistent with the *efficient structure hypothesis* but were generally not consistent with the *expense preference hypothesis*.

Jeng and Lai (2005) applied the cross-frontier analysis in addition to the RAM-DEA analysis to a sample of Japanese non-life insurers. They estimate cross-frontier technical efficiency and cross-frontier cost efficiency. Their results show that efficiency for keiretsu, non-specialized independent firms (NSIF) and specialized independent firms are equal, except keiretsu, which are more cost efficient than NSIF. Erhemjamts and Leverty (2010) applied the cross-frontier

[3] Regarding the estimation methodology, there are two main approaches in efficient frontier analysis: the econometric approach and the mathematical programming approach. The econometric approaches specify a production, cost, revenue or profit function with a specific shape and usually make assumptions about the distributions of the inefficiency and error terms. There are three principal types of econometric frontier approaches: the stochastic frontier approach (SFA), the distribution free approach (DFA) and the thick frontier approach (TFA). The mathematical programming approaches put significantly less structure on the specification of the efficient frontier and do not decompose the efficiency and error terms. Data envelopment analysis (DEA) is the most used mathematical programming approach which employs linear programming to measure the relationship of produced outputs to assigned inputs and determines the efficiency score as an optimization result. The free disposal hull (FDH) approach is a special configuration of DEA where the convexity assumption on the efficient frontier is relaxed. The ranged-adjusted measure DEA (RAM-DEA) is non-radial in the sense that it does not preserve the mix between inputs in movements toward the frontier (see e.g. Cummins and Weiss 2013).

methodology to the US life insurers during the period 1995–2004. They estimated cross-to-own technical efficiency ratios and provide evidence that efficiency of stock organizational form dominates mutual structure during the sample period. Recently, Biener and Eling (2012) applied this approach to a sample of stock and mutual insurers from 21 countries from northern America and the European Union. They estimated cross-to-own technical efficiency ratios as well as cross-to-own cost efficiency ratios, finding evidence for the *efficient structure hypothesis* in selected segment markets, but no evidence for the *expense preference hypothesis*.

The results are mixed regarding the two most important hypothesis analyzed. Most of the revised studies (e.g. Gardner and Grace 1993; Cummins and Zi 1998; Greene and Segal 2004; Bikker and Gorter 2011) find that stocks are as efficient as mutuals, providing support to the *efficiency structure hypothesis*. Some of the revised studies (e.g. Brockett et al. 2005; Erhemjamts and Leverty 2010) show that stocks are more efficient than mutuals, providing support to the *expense preference hypothesis*. However, there are studies (e.g. Cummins et al. 1999) that show that stocks and mutuals have different technologies ( supporting the *efficiency structure hypothesis*) but that stocks are more successful than mutuals at minimizing costs (supporting the *expense preference hypothesis*). These mixed findings suggest that more research is needed in this field with methodologies like the cross-frontier analysis that take into account the possibilities that mutuals and stocks could have different technologies.

## 4 Measuring Outputs, Inputs and Prices

In the interest of providing all the keys to applying the cross-frontier methodology to the insurance industry we will discuss the measurement of outputs, inputs and prices in this section. There are three principal approaches to measure outputs and inputs in financial services—the intermediation approach, the user-cost approach and the value-added approach (Berger and Humphrey 1992). The *intermediation approach* views insurers as pure financial intermediaries that borrow funds from policyholders, invest then on capital markets and pay out claims, taxes, and costs. Several papers have used this approach (e.g. Fukuyama 1997; Brockett et al. 2005). However, some authors (e.g. Cummins and Weiss 2013) claim that it is not likely to be appropriate for either the non-life and life insurers since it could ignore other services (e.g. insurance services) apart from intermediation services.

The *user-cost approach* differentiates between inputs and outputs based on the net contribution to revenues. If a financial product yields a return that exceeds the opportunity cost of funds or if the financial cost of a liability is less than the opportunity cost, then the product is considered to be a financial output. Otherwise, it is considered a financial input (Hancock 1985). Although this method is considered theoretically sound, it is problematical for the insurance industry since insurance policies bundle together many services, which are priced implicitly (Cummins and Weiss 2013). In this sense Eling and Luhnen (2010) review 95 studies on frontier efficiency in the insurance industry and none of them use the *user-cost approach*.

The *value-added approach* employs as important outputs all categories that have substantial value-added, as judged by operating cost allocations (Berger and Humphrey 1992). Most studies analyzing performance in the insurance industry using frontier efficiency and productivity methods use a modified version of the *value-added approach* and, in general, it is considered the most appropriate method for studying insurance efficiency (Cummins and Weiss 2013). In a recent paper, Leverty and Grace (2010) empirically examined the *intermediation approach* and the *value-added approach* for measuring output in property-liability insurer efficiency studies. They find that the *value-added approach* is closely related to traditional measures of firm performance, but the *intermediation approach* is not. Furthermore, they find that firms being efficient with the *value-added approach* are less likely to fail, while firms characterized as efficient by the *intermediation approach* are generally more likely to become insolvent. In addition, their results show that the theoretical concern regarding the use of losses as a measure of output in the *value-added approach* is not validated empirically.

Studies using a modified version of the value-added approach to measure insurance outputs and inputs recognize that risk-pooling and risk bearing services, real financial services related to insured losses and intermediation services are the three main services in creating value for insurers (Cummins and Weiss 2013).

A satisfactory proxy for the amount of risk pooling/bearing and real insurance services is the value of real losses incurred for the non-life insurance segment and incurred benefits plus addition to reserves for the life insurance segment. Usually, in both cases (non-life segment and life segment) several output variables are used for the major lines of business offered by insurers. The output variable that usually proxies for the intermediation function is the real value of invested assets. In line with the unit price of insurance, the prices of the output variables are defined as premiums minus output divided by output. The price for the intermediation output is given by a measure of the expected rate of return on the insurer’s assets.

According to the value-added approach, insurers use three primary inputs: labor, material and business services, and capital. Usually, physical capital expenditures are included along with business services and materials. Sometimes two types of capital are considered: equity capital and debt capital (e.g. Cummins and Rubio-Misas 2006; Biener and Eling 2012), but using at least equity capital as input is advisable given that financial equity capital is quantitatively quite important for insurers. Several indices are used as input prices: for instance wage rate for labor, business services deflator for material and business services, the expected market return on equity capital for equity capital or a Treasury bill rate for debt capital. As physical measures of input quantities are usually not publicly available the way to approach quantity of physical inputs is by dividing the expense item by a corresponding price index (Cummins and Weiss 2013).

# 5 Methodology and Hypotheses Tests

## *5.1 Distance Functions, Efficiency and Cross-to-Own Efficiency Ratios*

Testing the *efficient structure hypothesis* and the *expense preference hypothesis* in the context of frontier efficiency analysis implies estimating "best practice" efficient production, cost and revenue frontiers, providing measures of technical, input-allocative, cost, revenue and output-allocative efficiency for each firm in the sample.[4] As, according to the microeconomic theory, the objective of the firm is profit maximization, it is important to estimate both cost and revenue efficiency because to be profit efficient, the firm must be both cost efficient and revenue efficient. Data envelopment analysis (DEA), a non-parametric technique, is used to estimate production, cost and revenue frontiers.

In estimating efficiency using DEA it is necessary to adopt an orientation (input, output). The cross-frontier analysis uses input-oriented DEA to estimate cost, technical and input allocative efficiency and output-oriented DEA to estimate revenue and output-allocative efficiency. The choice of input versus output orientation is based on the microeconomic theory establishing the firm's objective in maximizing profits by minimizing costs and maximizing revenues. Cost minimization involves choosing the optimal quantities of inputs to produce a given output vector (i.e. minimizing inputs conditional on outputs), and revenue maximization involves choosing the optimal quantities of outputs conditional on the input vector (i.e., maximizing revenues conditional on inputs).

The fundamental idea behind the cross-frontier analysis and the hypothesis tests is that the stock and mutual organizational forms represent different technologies for producing insurance and, consequently, firms are hypothesized to design their technologies, management structure and contracting relationships to attend their market segments and operational objectives optimally.[5] Thus, if the *efficient structure*

[4] "Best practice" efficient frontier consists of the dominant firms of a reference set. The efficiency values of each firm are measured relative to best practice efficient frontiers. Technical efficiency is defined as the ratio of the input usage of a fully efficient firm producing the same output vector to the input usage by the analyzed firm. Cost efficiency for a specific firm is calculated as the ratio of the costs of a fully efficient firm with the same output quantities and input prices to the specific firm's actual costs. Cost efficiency is the product of technical and input allocative efficiency. Thus, input allocative efficiency is the ratio of cost efficiency to technical efficiency and gives information on whether the firm uses the optimal mix of inputs. Revenue efficiency is defined as the ratio of the revenues of a specific firm to the revenues of a fully efficient firm with the same input vector and the same output prices. Revenue efficiency is the product of the output technical efficiency to the output allocative efficiency. Therefore, the output allocative efficiency can be calculated by the ratio revenue efficiency to output technical efficiency and gives information on whether the firm uses the optimal combination of outputs.

[5] Technology is defined as "including the contractual relationships comprising the firm, organizational, management, and hierarchical structures, and physical technologies" (Cummins et al. 2004, p. 3116).

*hypothesis* is true, stocks and mutuals should operate in different frontiers, the stock technology should dominate the mutual technology for producing stock outputs and the mutual technology should dominate the stock technology for producing mutual outputs. If the *expense preference hypothesis* is true, mutuals are expected to be less successful than stocks in minimizing costs and/or maximizing revenues.

Thus, the analysis of the *efficient structure hypothesis* should start by testing the null hypothesis that stock and mutual insurers are operating on the same frontier against the alternative hypothesis that they operate on different frontiers. This implies estimating by year frontiers with all stock and mutual firms (the pooled efficient frontiers) as well as frontiers for the specific group of firms (stocks or mutuals). Rejecting the null hypothesis would be consistent with the *efficient structure hypothesis* and implies that stocks and mutuals employ different production technologies and that comparison of efficiencies should be based on separate stock and mutual frontiers rather than on the pooled frontier.[6]

After testing that stocks and mutuals are operating on different frontiers, to estimate the relative efficiency of alternative organizational forms and to test hypothesis that firms are sorted into groups with comparative efficiency advantages, Cummins et al. (1999) proposed the cross-frontier analysis. The null hypothesis here is that each group's output vectors could be produced with equal efficiency using the other group's production technology. This involves estimating the efficiency of the firms in each group with reference to the other group's frontier. Rejection of this null hypothesis for both groups would imply that stocks and mutuals have developed dominant technologies for producing their respective output vectors.

Consequently, the cross-frontier analysis, for instance at the production level, implies estimating an input-oriented distance function for a specific decision making unit (e.g. a stock insurer) with respect to a reference frontier consisting only of firms from its own group (stock insurers in this case), as well as a cross-frontier distance function for this specific decision-making unit with respect to the reference set for the other group (mutual insurers in this case). Estimating cross-frontier distance functions allows estimating the efficiency of the firm with a specific organizational form relative to a best practice frontier based on the alternative organizational form. Whereas the distance function values for firms relative to their own group must be $\geq 1$, the distances with respect to the other group's frontier could be $>$, $=$, or $<1$. In the last case, since firms are not included in the group used to construct the frontier, they can perform better than the efficient frontier firms of the alternative organizational form, their cross-frontier distance values may be $<1$ and their cross-frontier efficiency values may be $>1$.

In order to test hypotheses about the superiority of one technology over the other, this approach uses a ratio named the cross-to-own efficiency ratio, which

[6] Studies using cross-frontier analysis (e.g. Cummins et al. 1999, 2004; Biener and Eling 2012) provide evidence that the two groups of firms (stocks and mutuals) use different technologies and operate with different frontiers. Thus, comparing efficiencies based on the pooled frontier is not informative.

measures the distance between the stock and mutual frontiers at each operating point. Because Farrell's technical efficiency is the reciprocal of the distance function value, the cross-to-own technical efficiency ratio at each operating point is calculated as the ratio of the cross-frontier technical efficiency to the own-frontier technical efficiency. Cross-to-own technical efficiency ratios larger than 1 indicate that the own production technology dominates the opposing technology at the considered operating point. We can determine whether the own technology dominates the opposing technology by performing this calculation for all insurers in the sample. The same cross-to-own efficiency ratios are calculated at the cost and revenue frontiers levels.

Before formalizing the cross-frontier analysis we present several examples that can help to better understand this approach. We first illustrate the cross-frontier DEA analysis using two simple examples involving ten firms (five stocks and five mutuals) which use one input to produce a single output following Biener and Eling (2012). In these examples (presented in Table 1) we first calculate input/output ratios and then the efficiency scores on the pooled frontier (consisting of both stocks and mutuals), on the own frontier (consisting only of insurers belonging to its own group) as well as on the cross frontier (consisting only of insurers belonging to its alternative group). The efficiency of insurer *i* on the pooled frontier is the minimum input/output ratio of all insurers in the sample divided by the input/output ratio of insurer *i*. The efficiency of insurer *i* on the own-frontier is the minimum input/output ratio of all firms belonging to its own group divided by the input/output ratio of insurer *i*. And the efficiency of insurer *i* on the cross-frontier is the minimum input/output ratio of all firms belonging to the alternative group divided by the input/output ratio of insurer *i*. The cross-to-own efficiency score of insurer *i* would be obtained by dividing the cross-frontier efficiency score by its own-frontier efficiency score.

Pooled and own frontiers from examples I and II are plotted at the bottom of Table 1. From example I, we can see that in this case the stock frontier dominates the mutual frontier for all the operating points and the cross-to-own efficiency scores are consistently larger than 1 for stocks and consistently lower than 1 for mutuals. However, when we plot pooled and own frontiers from example II, the mutual frontier dominates the stock frontier and the cross-to-own efficiency scores are consistently larger than 1 for mutuals and consistently lower than 1 for stocks.

We also present an additional example (example III) in Table 2, for eight hypothetical insurers (four stocks and four mutuals) which use two inputs to produce a single output. This table shows the amount of output produced by any insurer and the corresponding amount of inputs used. Under the constant returns to scale assumption, these input values are normalized to represent the amounts of resources needed per unit of output (this is shown on the rows named "Input1/output" and "Input2/Output" of Table 2). Table 2 also presents the efficiency scores for any decision-making unit with respect to the pooled frontier, own frontier as well as with respect to the cross frontier. The cross-to-own technical efficiency ratios are calculated by dividing the cross-frontier efficiency score by its respective own frontier efficiency score. Pooled and own frontiers from example III are plotted in Fig. 1.

**Table 1** Examples I and II using one input to produce one output

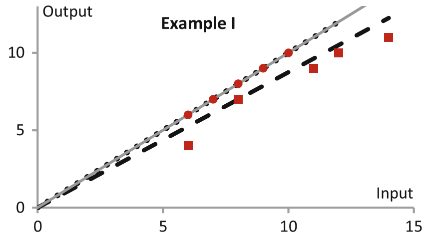
| Insurer ($i$) | Type | Input (I) | Output (O) | I/O | Technical efficiency | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | Pooled | Own (Ow) | Cross (Cr) | Cr/Ow |
| *Example I* | | | | | | | | |
| 1 | Stock | 6.00 | 6.00 | 1.00 | 1.00 | 1.00 | 1.14 | 1.14 |
| 2 | Stock | 7.00 | 7.00 | 1.00 | 1.00 | 1.00 | 1.14 | 1.14 |
| 3 | Stock | 8.00 | 8.00 | 1.00 | 1.00 | 1.00 | 1.14 | 1.14 |
| 4 | Stock | 9.00 | 9.00 | 1.00 | 1.00 | 1.00 | 1.14 | 1.14 |
| 5 | Stock | 10.00 | 10.00 | 1.00 | 1.00 | 1.00 | 1.14 | 1.14 |
| 6 | Mutual | 6.00 | 4.00 | 1.50 | 0.67 | 0.76 | 0.67 | 0.88 |
| 7 | Mutual | 8.00 | 7.00 | 1.14 | 0.88 | 1.00 | 0.88 | 0.88 |
| 8 | Mutual | 11.00 | 9.00 | 1.22 | 0.82 | 0.94 | 0.82 | 0.88 |
| 9 | Mutual | 12.00 | 10.00 | 1.20 | 0.83 | 0.95 | 0.83 | 0.88 |
| 10 | Mutual | 14.00 | 11.00 | 1.27 | 0.79 | 0.90 | 0.79 | 0.88 |
| *Example II* | | | | | | | | |
| 1 | Stock | 6.00 | 6.00 | 1.00 | 0.86 | 1.00 | 0.86 | 0.86 |
| 2 | Stock | 7.00 | 7.00 | 1.00 | 0.86 | 1.00 | 0.86 | 0.86 |
| 3 | Stock | 8.00 | 8.00 | 1.00 | 0.86 | 1.00 | 0.86 | 0.86 |
| 4 | Stock | 9.00 | 9.00 | 1.00 | 0.86 | 1.00 | 0.86 | 0.86 |
| 5 | Stock | 10.00 | 10.00 | 1.00 | 0.86 | 1.00 | 0.86 | 0.86 |
| 6 | Mutual | 6.00 | 4.00 | 1.50 | 0.57 | 0.57 | 0.67 | 1.17 |
| 7 | Mutual | 6.00 | 7.00 | 0.86 | 1.00 | 1.00 | 1.17 | 1.17 |
| 8 | Mutual | 11.00 | 9.00 | 1.22 | 0.70 | 0.70 | 0.82 | 1.17 |
| 9 | Mutual | 12.00 | 10.00 | 1.20 | 0.71 | 0.71 | 0.83 | 1.17 |
| 10 | Mutual | 14.00 | 11.00 | 1.27 | 0.67 | 0.67 | 0.79 | 1.17 |

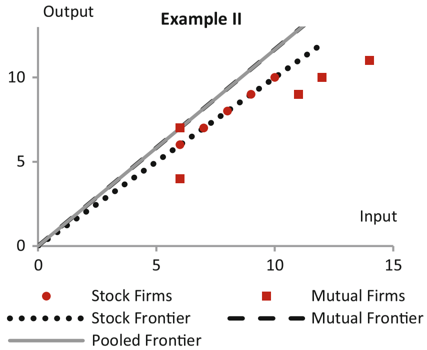
(continued)

**Table 1** (continued)

| Insurer ($i$) | Type | Input (I) | Output (O) | I/O | Technical efficiency | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | Pooled | Own (Ow) | Cross (Cr) | Cr/Ow |

*Efficient frontiers*



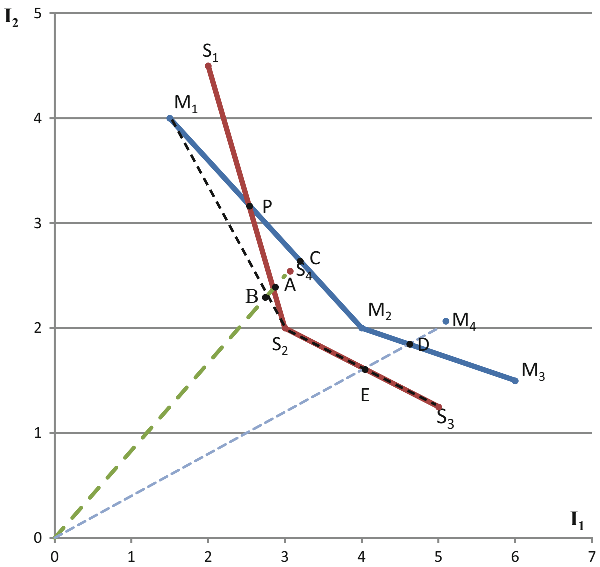




Note: This table presents own-frontier efficiencies (Own), cross-frontier efficiencies (Cross) as well as cross-to-own efficiency ratios (Cr/Ow)

**Table 2** Example III using two inputs to produce a single output

| Insurer (i) | M1 | M2 | M3 | M4 | S1 | S2 | S3 | S4 |
|---|---|---|---|---|---|---|---|---|
| Type | Mutual | Mutual | Mutual | Mutual | Stock | Stock | Stock | Stock |
| Input 1 | 3 | 12 | 24 | 25 | 4 | 9 | 20 | 12 |
| Input 2 | 8 | 6 | 6 | 10 | 9 | 6 | 5 | 10 |
| Output | 2 | 3 | 4 | 5 | 2 | 3 | 4 | 4 |
| Input 1/Output ($I_1$) | 1.5 | 4 | 6 | 5 | 2 | 3 | 5 | 3 |
| Input 2/Output ($I_2$) | 4 | 2 | 1.5 | 2 | 4.5 | 2 | 1.25 | 2.5 |
| *Technical efficiency* | | | | | | | | |
| Pooled | 1 | 0.89 | 0.83 | 0.81 | 0.84 | 1 | 1 | 0.92 |
| Own | 1 | 1 | 1 | 0.92 | 1 | 1 | 1 | 0.95 |
| Cross | 1.33 | 0.89 | 0.83 | 0.81 | 0.85 | 1.18 | 1.2 | 1.06 |
| Cross/Own | 1.33 | 0.89 | 0.83 | 0,87 | 0.85 | 1.18 | 1.2 | 1.12 |

**Fig. 1** Pooled, own and cross technical efficiency for example III




In Fig. 1, the dashed lines connecting $M_1$, $S_2$ and $S_3$ represent the resulting pooled isoquant when all firms (stocks and mutuals) are considered simultaneously. Firms operating on the isoquants are on the production frontier and fully efficient. Consequently, the pooled technical efficiency of $M_1$, $S_2$ and $S_3$ is equal to 1 (as we can see in the row named "*Pooled*" in Table 2). Figure 1 also shows the own isoquants for each group of firms (stocks and mutuals), which represent the best available technology for the respective group. The solid lines connecting $S_1$, $S_2$ and $S_3$ constitute the isoquant (own frontier) for stock firms, while the linear segments connecting $M_1$, $M_2$ and $M_3$ represent the isoquant (own frontier) for mutual firms. Therefore, the own technical efficiency for both $S_1$,$S_2$, $S_3$ and $M_1$, $M_2$, $M_3$ is equal to 1 (as we can see on the row named "*Own*" in Table 2). Figure 1 shows that the isoquants representing the respective own frontiers intersect indicating that the stock technology is optimal for some operating points and the mutual technology is optimal for other operating points. That is, left of point P, mutual technology dominates stock technology, and right of point P, stock technology dominates mutual technology. This is an example in which stock firms dominate mutual firms in areas with comparative advantages and mutual firms dominate stock firms in areas with comparative advantages which would be consistent with the *efficient structure hypothesis*.

Consider a stock firm operating at point $S_4$. Let $\overline{0S_4}$ be the line from zero to $S_4$. This line crosses the pooled frontier at point B, the stock-frontier at point A, and can be projected to the mutual frontier at point C. This implies that this firm could reduce its inputs usage by moving to the pooled isoquant and operating at point B (which is a combination of $M_1$ and $S_2$), or by moving to the stock isoquant and operating at point A (combination of $S_1$ and $S_2$). However, this firm would have to increase its inputs usage when it is measured with respect to the mutual isoquant, in order to attain point C. This indicates that the stock technology dominates the mutual technology at point $S_4$. In detail, the pooled-frontier input distance value for $S_4$ is obtained by ratio $0S_4/0B = \sqrt{3^2 + 2.5^2}/\sqrt{2.77^2 + 2.31^2} = 1.08 > 1$, and its pooled-frontier technical efficiency by $0B/0S_4 = 0.92 < 1$, where $0S_4$ is the euclidean distance from the origin to point $S_4$, and 0B represents the euclidean distance from the origin to point B.

Evaluating this stock firm operating at point $S_4$ with respect to its own frontier (stock frontier) means that its own-frontier input distance value is $0S4/0A = \sqrt{3^2 + 2.5^2}/\sqrt{2.85^2 + 2.38^2} = 1.05 > 1$, and its own-frontier technical efficiency $0A/0S_4 = 0.95 < 1$. While evaluating it with respect to the mutual frontier means that its cross-frontier input distance value is $0S_4/0C = \sqrt{3^2 + 2.5^2}/\sqrt{3.18^2 + 2.65^2} = 0.94 < 1$, and its cross-frontier technical efficiency $0C/0S_4 = 1.06 > 1$. Cross-frontier efficiency greater than 1 implies, in this case, that it would be unfeasible for a mutual to achieve the input/output combination represented by point $S_4$ and, consequently, the mutual technology is dominated by the stock technology at this operating point.

Now consider a mutual firm operating at point $M_4$. Figure 1 shows that the dotted line from zero to $M_4$ crosses the mutual isoquant at D (4.62, 1.84), and the stock isoquant at E (4.03, 1.61). Therefore, its own-frontier input distance value is $0M_4/0D = 1.08 > 1$ and its own-frontier efficiency $0D/0M_4 = 0.92 < 1$. Its cross-frontier input distance and efficiency values are $0M_4/0E = 1.24 > 1$ and $0E/0M_4 = 0.81 < 1$, respectively, indicating that the stock technology dominates the mutual technology at this operating point since its own-frontier efficiency is larger than its cross-frontier efficiency.

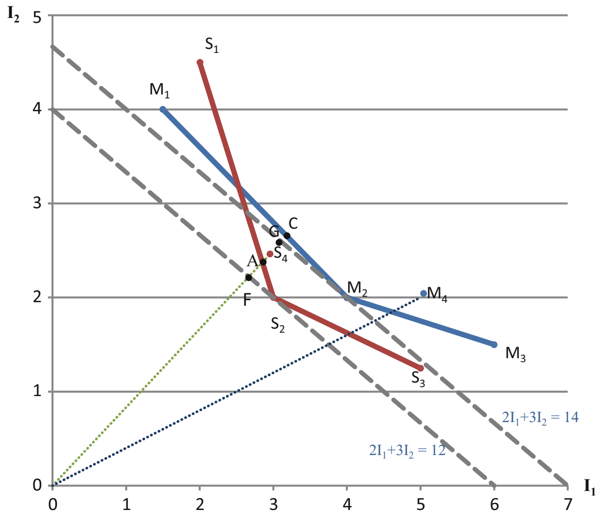
In Fig. 1, we also illustrate the distance between the production frontiers at each operating point by projecting each firm's operating point to its own frontier and then measuring the distance between the frontiers for a fully efficient firm with the same output vector. That is, the distance between the frontiers at operating point $S_4$ would be $0C/0A = 1.12 > 1$, which is equal to the own-frontier input distance function value divided by the cross-frontier input distance function value. In other words, the distance between the frontiers at operating point $S_4$ is the ratio of the cross-frontier technical efficiency to the own-frontier technical efficiency $0C/0A = (0S_4/0A)/(0S_4/0C) = (0C/0S_4)/(0A/0S_4)$ (see row named "*Cross*/*Own*" in Table 2). For this reason, the distance between the frontiers, at each operating point, is referred to as cross-to-own efficiency ratio and is a key statistic to analyze the superiority of one technology over the other.

Cross-to own efficiency ratio larger than 1 indicates that the own-frontier dominates the alternative frontier at this operating point (for example, at point $S_4$) and, conversely, for cross-to-own efficiency ratio lower than 1 (for example, at point $M_4$). The intuition is the following: a cross-to-own efficiency ratio larger than 1 means that the own-frontier input distance value is larger than the cross-frontier input distance value and implies that the own frontier is closer to the origin than the alternative frontier at this operating point.

We extend the previous analysis and study cost efficiency by adding information on input prices to example III (presented in Table 2). Suppose a common unit price for input 1 of 2 monetary units and a common unit price for input 2 of 3 monetary units. The analysis of cost efficiency is presented in Fig. 2 where the broken line passing through $S_2$ represents the isocost line for stocks ($2I_1 + 3I_2 = 12$, combinations of inputs 1 and input 2 with the same total cost), while the other broken line passing through $M_2$ represents the isocost line for mutuals ($2I_1 + 3I_2 = 14$), $S_2$ being for stocks and $M_2$ for mutuals, the optimal operating points resulting of tangency of the isoquants and the respective isocost lines.

Returning to the stock firm operating at point $S_4$, the dotted line $\overline{0S_4}$ (shown in Fig. 2) crosses at point F (2.67, 2.22) and at point G (3.11, 2.59) the isocost lines for stocks and mutuals, respectively. Therefore, the own-frontier cost efficiency for $S_4$ is $0F/0S_4 = 0.89 < 1$, and its cross-frontier cost efficiency is $0G/0S_4 = 1.04 > 1$. The own-frontier cost efficiency (0.89) is the product of its own-frontier technical

**Fig. 2** **O**wn-cost efficiency and **c**ross-cost **e**fficiency for example III



efficiency, $0A/0S_4 = 0.95 < 1$, and its own-frontier input allocative efficiency, $0F/0A = 0.94 < 1$. While the cross-frontier cost efficiency at operating point $S_4$ (1.04) is the product of its cross-frontier technical efficiency, $0C/0S_4 = 1.06 > 1$, and its cross-frontier input allocative efficiency, $0G/0C = 0.98 < 1$.

The distance between the cost frontiers, at operating point $S_4$, would be $0G/0F = 1.17$, the cross-to-own cost efficiency ratio. That is, the ratio of the cross-frontier cost efficiency, $0G/0S_4 = 1.04$, to the own-frontier cost efficiency, $0F/0S_4 = 0.89$. Taking into account that in the ordinary case, cost efficiency is the product of technical efficiency and input allocative efficiency, the cross-to-own cost efficiency ratio can be also expressed as the product of the cross-to-own technical efficiency ratio to the cross-to-own input allocative efficiency ratio. So, in the case of the stock firm operating at point $S_4$, the cross-to-own cost efficiency ratio, $0G/0F = 1.17 > 1$, would be the product of the cross-to-own technical efficiency ratio, $0C/0A = 1.12$, to the cross-to-own input allocative efficiency ratio $(0G/0C)/(0F/0A) = 1.04$. As in the ordinary case of cost efficiency, the analysis of cross-frontier cost efficiency gives more information than the only analysis of cross-frontier technical efficiency, because it also provides information on cross-frontier input allocative efficiencies.

Finally, the revenue efficiency analysis would be analogous to the cost efficiency analysis, but in the case of the revenue analysis the optimal operating point is determined by the tangency of iso-output-price lines and production possibilities curves (Lovell 1993). The distance between the revenues frontiers at a hypothetical operating point would be given by its cross-frontier revenue efficiency to its own-frontier revenue efficiency. That is the product of the cross-to-own output technical efficiency ratio to the cross-to-own output allocative efficiency ratio.[7] Analogously to the cost efficiency problem, measuring the distance between revenue frontiers provides more information than measuring the distance on the technical frontiers alone, because it also provides information on revenue allocative inefficiencies.

To formalize cross-frontier analysis for multiple inputs and outputs the concept of distance function introduced by Shepard (1970) is used. For analyzing production frontiers this approach employs input-oriented distance functions with respect to frontiers characterized by constant returns to scale (CRS).[8] Following Cummins et al. (2004) and Biener and Eling (2012), an input distance function of a specific insurer producing outputs $y = (y_1, y_2, \ldots, y_n)^{\mathrm{T}} \in \mathbb{R}_+^n$ by using input vector $x = (x_1, x_2, \ldots, x_k)^{\mathrm{T}} \in \mathbb{R}_+^k$, is defined as

[7] Output allocative efficiency gives information on the success of the firm in choosing the revenue maximization output combination.

[8] The constant returns to scale approach (CRS) is used most commonly in literature and measures departures from optimal scale as inefficiency. It represents the optimal outcome from an economic perspective. That is, with CRS, firms are not consuming unnecessary resources because they are too large or too small (see e.g. Aly et al. 1990).

$$\mathrm{D}(y,x) = \ sup\{\theta : \ \theta^{-1}x \ \in \mathrm{V}(y)\} = \left(inf\{\theta : \ \theta x \in \mathrm{V}(y)\}\right)^{-1}$$

The distance function estimates the largest $\theta$ for which ( $x/\theta$ ) is in the attainable set V( $y$). $\theta$ can be interpreted as the distance between the operating point and the efficient frontier. Thus, it is assumed a production technology that transforms inputs into outputs and the relation $y \to \mathrm{V}(y) \subseteq \mathbb{R}_+^k$ models this approach because V( $y$) constitutes the subset of all input vectors $x \in \mathbb{R}_+^k$ yielding at least $y$, for any $y \in \mathbb{R}_+^n$. The input distance function is the reciprocal of the Farrell's measure of input technical efficiency T($y$, $x$). That is, the minimum equi-proportional contraction of the input vector $x$ given outputs i.e. T($y$, $x$) = 1/D($y$, $x$). If the insurer, for which the input distance function is calculated, belongs to the reference set used to construct the frontier, D($y$, $x$) must be $\geq$1 and, consequently, T($y$, $x$) $\leq$ 1.

As we explained above, to test the *efficient structure* and the *expense preference* hypotheses, distance functions for stocks and mutuals to several reference sets should be estimated. That means that distance functions with respect to the pooled frontier (for all stocks and mutuals), with respect to the own frontier as well as with respect to the opposing frontier are needed.

Suppose an insurer $i$ uses input vector $x_i = (x_{1i}, x_{2i}, \ldots, x_{ki})^t$ to produce output vector $y_i = (y_{1i}, y_{2i}, \ldots, y_{ki})^t, i \ = \ 1, \ 2, \ \ldots, \ \mathrm{S} + \mathrm{M}$, where S and M is the total number of stock and mutual firms, respectively. The pooled-frontier distance function is defined as:

$$\mathrm{D_P}(y_i, x_i) = sup\{\theta : \ \theta^{-1}x_i \in \mathrm{V_P}(\mathrm{y}_i)\}, \ i = 1, \ 2, \ \ldots \mathrm{S} + \mathrm{M}$$

with subscript P indicating the measurement for firm $i$ ($i$ = 1, 2,. . ., S + M) against the pooled frontier consisting of all stock and mutual insurers.

The own-frontier distance functions for stocks and mutuals are defined, respectively, by:

$$\mathrm{D_S}(y_s, x_s) = sup\{\theta : \ \theta^{-1}x_s \in \mathrm{V_S}(\mathrm{y}_s)\}, \qquad s = 1, \ 2, \ \ldots, \mathrm{S}$$

$$\mathrm{D_M}(y_m, x_m) = sup\{\theta : \ \theta^{-1}x_m \in \mathrm{V_M}(\mathrm{y}_m)\}, \qquad m = \mathrm{S} + 1, \ \mathrm{S} + 2, \ \ldots, \mathrm{S} + \mathrm{M}$$

where the notation ($y_s, x_s$) and ($y_m, x_m$) are employed for designating the output-input vector for stock and mutual firms, respectively. These functions measure the distance of the firm to the own-technology efficiency frontier. For instance $\mathrm{D_S}(y_s, x_s)$ measures the input distance function for stock firm $s$ ($s$ = 1, 2, . . . , S) with respect to a reference set frontier consisting only of stock firms with $\mathrm{V_S}(\mathrm{y}_s)$ the stock firm's input correspondence for the output vector $y_s$. Likewise, $\mathrm{D_M}(y_m, x_m)$ measures the input distance function for mutual firm $m$ ($m$ = S + 1, S + 2, . . . , S + M) with respect to a reference set frontier consisting only of mutual firms, with $\mathrm{V_M}(\mathrm{y}_m)$ being the mutual firm's input correspondence for the output vector $y_m$.

The cross-frontier distance functions, introduced by Cummins et al. (1999), measure the distance of a firm operating point to the opposing technologies' efficient frontier. The respective cross-frontier distance functions for stocks and mutuals are defined by:

$$\mathrm{D_M}(y_s, x_s) = sup\{\theta : \ \theta^{-1}x_s \in \mathrm{V_M}(\mathrm{y}_s)\}, \quad s = 1,\ 2, \ldots, \mathrm{S}$$
$$\mathrm{D_S}(y_m, x_m) = sup\{\theta : \ \theta^{-1}x_m \in \mathrm{V_S}(\mathrm{y}_m)\}, \quad m = \mathrm{S}+1,\ \mathrm{S}+2, \ldots, \mathrm{S}+\mathrm{M}$$

Thus, $\mathrm{D_M}(y_s, x_s)$ measures the distance of stock firm $s$ relative to the mutual frontier, with $\mathrm{V_M}(\mathrm{y}_s)$ being the mutual firm's input correspondence for the output vector $y_s$. $\mathrm{D_S}(y_m, x_m)$ is defined similarly. Accordingly, estimating cross-frontier distance functions enables measuring the performance of the firms with a specific organizational form (stock, mutual) relative to a best practice frontier based on the opposing organizational form.

As we explained above, since firms are evaluated with respect to a subset of firms where they are not included, they can perform better than firms of the efficient frontier and, thus, the cross-frontier distance function values can be $\geq 1$ as well as $<1$. Consequently the cross-frontier efficiencies are not bounded by 1 and range between 0 and infinity.

To measure dominance of one technology over the other the cross-frontier methodology uses the cross-to-own efficiency ratios. These ratios measure the distance between the frontiers for each firm in the sample. Following Cummins et al. (2004) the cross-to-own efficiency ratios are defined as:

$$\mathrm{D_{T\{S:M\}}}(y_s, x_s) = \frac{\mathrm{D_S}(y_s, x_s)}{\mathrm{D_M}(y_s, x_s)} = \frac{\mathrm{T_M}(y_s, x_s)}{\mathrm{T_S}(y_s, x_s)}$$
$$\mathrm{D_{T\{M:S\}}}(y_m, x_m) = \frac{\mathrm{D_M}(y_m, x_m)}{\mathrm{D_S}(y_m, x_m)} = \frac{\mathrm{T_S}(y_m, x_m)}{\mathrm{T_M}(y_m, x_m)}$$

Here, $\mathrm{D_{T\{S:M\}}}(y_s, x_s)$ measures the distance between production frontiers (characterized by T, i.e. technical) at the $(y_s,\ x_s)$ operating point and $\{\mathrm{S:M}\}$ indicates that the stock frontier is measured against the mutual frontier. $\mathrm{D_{T\{M:S\}}}(y_m, x_m)$ is defined similarly.

The distances between the cost frontiers for each firm in the sample are measured by the cross-to-own cost efficiency ratios and are defined by:

$$\mathrm{D_{C\{S:M\}}}(y_s, x_s) = \frac{\mathrm{C_M}(y_s, x_s)}{\mathrm{C_S}(y_s, x_s)} = \frac{\mathrm{T_M}(y_s, x_s)}{\mathrm{T_S}(y_s, x_s)} \times \frac{\mathrm{A_M}(y_s, x_s)}{\mathrm{A_S}(y_s, x_s)}$$
$$\mathrm{D_{C\{M:S\}}}(y_m, x_m) = \frac{\mathrm{C_S}(y_m, x_m)}{\mathrm{C_M}(y_m, x_m)} = \frac{\mathrm{T_S}(y_m, x_m)}{\mathrm{T_M}(y_m, x_m)} \times \frac{\mathrm{A_S}(y_m, x_m)}{\mathrm{A_M}(y_m, x_m)}$$

where $\mathrm{D_{C\{S:M\}}}(y_s, x_s)$ measures the distance between the cost frontiers (C) at the $(y_s,\ x_s)$ operating point, being equal to the ratio of cross-frontier cost efficiency $\mathrm{C_M}(y_s, x_s)$ to the own-frontier cost efficiency $\mathrm{C_S}(y_s, x_s)$. As cost efficiency is the product of technical efficiency and input allocative efficiency, the cross-to-own cost efficiency ratio can be expressed as the product of the cross-to-own technical efficiency ratio to the cross-to-own input allocative efficiency ratio $A_M(y_s, x_s)/A_S(y_s, x_s)$. The cross-to-own input allocative efficiency ratio is measured by the ratio of the cross-frontier input allocative efficiency, $A_M(y_s, x_s)$, to the own-frontier input allocative efficiency, $A_S(y_s, x_s)$. For both the own-frontier and the cross-frontier cases, input allocative efficiency is measured by dividing the cost efficiency by the respective technical efficiency. $D_{C\{M:S\}}(y_m, x_m)$ is defined similarly.

The distances between the revenue frontiers for each firm in the sample are measured by the cross-to-own revenue efficiency ratios and are defined as:

$$D_{R\{S:M\}}(y_s, x_s) = \frac{R_M(y_s, x_s)}{R_S(y_s, x_s)} = \frac{T_M(y_s, x_s)}{T_S(y_s, x_s)} \times \frac{A_{RM}(y_s, x_s)}{A_{RS}(y_s, x_s)}$$

$$D_{R\{M:S\}}(y_m, x_m) = \frac{R_S(y_m, x_m)}{R_M(y_m, x_m)} = \frac{T_S(y_m, x_m)}{T_M(y_m, x_m)} \times \frac{A_{RS}(y_m, x_m)}{A_{RM}(y_m, x_m)}$$

where $D_{R\{S:M\}}(y_s, x_s)$ values the distance between revenue frontiers (using R to indicate revenues) at the $(y_s, x_s)$ operating point, being equal to the ratio of the cross-frontier revenue efficiency $R_M(y_s, x_s)$ to the own-frontier revenue efficiency $R_S(y_s, x_s)$. Since revenue efficiency is the product of technical efficiency and output allocative efficiency, the cross-to-own revenue efficiency ratio can be expressed as the product of the cross-to-own technical efficiency ratio to the cross-to-own output allocative efficiency ratio, $A_{RM}(y_s, x_s)/A_{RS}(y_s, x_s)$. The cross-to-own output allocative efficiency ratio is measured by the ratio of cross-frontier output allocative efficiency $A_{RM}(y_s, x_s)$ to the own-frontier output allocative efficiency, $A_{RS}(y_s, x_s)$. Output allocative efficiency is measured as revenue efficiency divided by technical efficiency.[9] $D_{R\{M:S\}}(y_m, x_m)$ is defined similarly.

## 5.2 Estimating Efficiency

DEA efficiency is estimated by solving linear optimization procedures. The standard own-frontier problem setup (which is the same as the pooled frontier problem setup) is discussed for instance in Cooper et al. (2007). The cross-frontier models for technical efficiency and cost efficiency are described in Cummins et al. (1999) and we are the first in describing the cross-frontier model for revenue efficiency. In this subsection we focus on estimating only the cross-frontier models.

For each mutual insurer $m$ ($m = S+1, S+2, \ldots, S+M$), in each time period, cross-frontier technical efficiency of mutual firm ($\theta_m$) with respect to the stock

[9] Since the cross-frontier analysis assumes constant returns to scale (CRS), the output and input orientations will provide equivalent measures of technical efficiency. So, we use the same notation to express technical efficiency although, as we explained above, technical efficiency is calculated input-oriented DEA and revenue efficiency is calculated output-oriented DEA.

reference set is obtained by solving the following linear optimization procedure, for each $m$:

$$\begin{gathered}(\mathrm{D_S}(y_m, x_m))^{-1} = \mathrm{T_S}(y_m, x_m) = min\ \theta_m \\ \text{subject to}\quad \mathrm{Y_S}\lambda_m \geq y_m,\quad \mathrm{X_S}\lambda_m \leq \theta_m x_m,\ \ \lambda_m \geq 0\end{gathered}$$

where $\mathrm{Y_S}$ is an $n \cdot S$ output matrix and $\mathrm{X_S}$ is a $k \cdot S$ input matrix for all stock insurers, $y_m$ is a $n \cdot 1$ output vector and $x_m$ is a $k \cdot 1$ input vector of the evaluated mutual insurer $m$, and $\lambda_m$ is a $S \cdot 1$ intensity vector of stocks with respect to the mutual firm $m$.

The cross-frontier technical efficiency for each stock firm $s$ ($s = 1, 2, \ldots$, S), with respect to the mutual firms, $(\mathrm{D_M}(y_s, x_s))^{-1} = \mathrm{T_M}(y_s, x_s) = \min\theta_s$, is estimated similarly.

To obtain cross-frontier cost efficiency for each stock insurer $s$ ($s = 1, 2, \ldots$, S), with respect to the mutual frontier, this linear optimization problem is solved as the first step:

$$\begin{gathered}Min_{x_s}\quad w_s^t x_s \\ \text{subject to}\quad \mathrm{Y_M}\lambda_s \geq y_s,\quad \mathrm{X_M}\lambda_s \leq x_s,\ \ \lambda_s \geq 0\end{gathered}$$

where $\mathrm{Y_M}$ and $\mathrm{X_M}$ are output and input matrices for all mutual insurers and $y_s, x_s$, and $w_s$ are output, input and input price vectors for the evaluated stock insurer $s$, and $\lambda_s$ is an intensity vector for mutual insurers relative to the stock insurer. The solution $x_s^*$ is the cost-minimizing input vector for insurer $s$, with respect to the mutual reference set. The second step is to calculate cross-frontier cost efficiency $\eta_s = w_s^t x_s^* / w_s^t x_s$. Cross-frontier cost efficiencies of mutual insurers with respect to the stock frontier are calculated similarly.

In order to obtain cross-frontier revenues efficiencies of stock insurers with respect to the mutual frontier the following linear programming problem is solved as the first step:

$$\begin{gathered}Max_{y_s}\quad p_s^t y_s \\ \text{subject to}\quad \mathrm{Y_M}\lambda_s \geq y_s,\quad \mathrm{X_M}\lambda_s \leq x_s,\ \ \lambda_s \geq 0\end{gathered}$$

where now in the revenue model the output price vector for stock insurer $s$, $p_s$, is additionally considered. The solution vector $y_s^*$ is the revenue maximization output vector for stock insurer $s$ with respect to the mutual reference set. Revenue efficiency is then measured by the ratio $\delta_s = p_s^t y_s / p_s^t y_s^*$. Cross-frontier revenue efficiencies of mutual insurers with respect to the stock frontier are calculated similarly.

With estimates of cross-frontier cost efficiency (cross-frontier revenue efficiency) and cross-frontier technical efficiency, estimates of cross-frontier input allocative efficiency (cross-frontier output allocative efficiency) can back out by dividing estimates of cross-frontier cost efficiency (cross-frontier revenue efficiency) by the corresponding estimates of cross-frontier technical efficiency.

## 5.3 Hypotheses Tests

As we explained above, the cross-frontier analysis on the relationship between organizational form and efficiency should start by testing the null hypotheses that stocks and mutuals are characterized by the same frontiers (by taking into account technical, cost and revenue frontiers separately) versus the alternative hypotheses that they are operating on different frontiers. To test these null hypotheses several tests, both parametric and non-parametric, could be applied (see e.g. Elyasiani and Mehdian 1992; Isik and Hassan 2002). Rejecting the null hypotheses signifies that stocks and mutuals are producing their outputs with different technologies. To reinforce this decision, it is also advisable to test hypotheses that the distribution of the own-frontier stock efficiency scores are the same as the distribution of the own-frontier mutual efficiency scores (see e.g. Aly et al. 1990). Rejecting these null hypotheses implies that the stock and mutual efficiencies are not drawn from the same population. As a consequence of the rejection of these two sets of null hypotheses, the implication is that comparing efficiencies based on the pooled frontiers (constructed with all stock and mutual firms) is not informative. Therefore the analysis should focus on separate stock and mutual frontiers and is here when the cross-frontier analysis allows evaluating the relative efficiency of alternative organizational forms.

The cross-frontier analysis implies estimating own-frontier efficiencies (they are calculated with respect to a reference frontier consisting only of firms from its own group) as well as cross-frontier efficiencies (they are calculated with respect to the reference set for the alternative group). These estimations allow calculating the cross-to-own efficiency ratios by dividing the cross-frontier efficiency score by its respective own-frontier efficiency score. The cross-frontier efficiency ratios measure the distance between stock and mutual frontiers at each operating point and are key statistics to evaluate the superiority of one technology over the other since ratios larger than 1 indicate that the own frontier dominates the opposing frontier at the considered operating point. Performing these calculations for all insurers in the sample allow testing the relative performance of every organizational form.

A first step could be to calculate average values of the cross-to-own frontier ratios along the sample period and see the general pattern that emerges to evaluate if these average values are significantly greater than 1 (providing some support to the *efficiency structure hypothesis*) or are significantly less than 1. However, this first step would be an incomplete analysis since in the studies of organizational forms in the insurance industry, the market overview statistics usually suggest that stocks insurers are more successful in certain lines of business and mutuals are more successful in other lines of business. Therefore, in analyzing the relative efficiency of the two organizational forms it is likely to be important to control for line of business participation and other firm characteristics in a multiple regression context.

Consequently, after estimating the cross-to-own frontier efficiency ratios the analysis should be completed with multiple regression analyses where the models would use the cross-to-own frontier ratios (technical, cost, and revenue) as dependent variables and several firm characteristics as independent variables. This analysis provides evidence on whether the differences in cross-to-own frontier efficiencies are maintained when we control for firm characteristics.

An important firm characteristic that the regressions should control is size and in most studies using the cross-frontier analysis this variable is considered by including interaction terms of four size quartile dummy variables with a dummy variable equal to 1 for mutuals and 0 otherwise, as well as interaction terms of the four size quartile dummy variables with a dummy variable equal to 1 for stocks and 0 otherwise.[10] What the regression analysis basically tests is if the coefficients of these interaction terms variables are significantly greater than 1. Coefficients significantly greater than 1 provide support to the *efficiency structure hypothesis*. However, by specially focusing on of the cross-to-own cost frontier ratios as well as cross-to-own revenue frontier ratios, coefficients for the interaction terms variables constructed with the stock dummy variable significantly greater than 1 and coefficients for the interaction terms variables constructed with the mutual dummy variable significantly lower than 1 provide support to the *expense preference hypothesis*.

Furthermore, as we explained above, the regression analyses should control for line participation of the insurers in the sample. Additionally, the regressions usually control for the capital structure (e.g. capitalization ratio), the insurance leverage ratio as well as for differences in cross-to-own frontier ratios across the years included in the analysis by taking into account year dummy variables.

The cross-to-own frontier ratios results measure the distances between frontiers and do not transmit information on allocative efficiency. However, the analysis of allocative efficiency provides additional information of the *expense preference hypothesis* since the cost frontier allocative efficiency measures the success of firms in choosing cost minimizing input combinations and revenue allocative efficiency measures the success of firms in choosing revenue-maximization output combination. Therefore, additional regression analysis where the dependent variables are the cost and revenue allocative efficiencies (both own-frontier and cross-frontier) should be conducted to provide information on allocative efficiency differences between stocks and mutuals after controlling the other firm characteristics.

## 6 Summary and Conclusions

This chapter presents a comprehensive analysis of the cross-frontier methodology (Cummins et al. 1999) and gives the keys to understanding the benefit from using this approach to evaluate the relative performance of stock and mutual insurers by providing a thorough survey of literature. This approach was first configured to test economic hypotheses that address the coexistence of the stock (owned by stockholders) and mutual (owned by policyholders) organizational forms in the

[10] The four size quartile dummy variables could be constructed in the following way: quartiles are formed based on the overall sample including all stocks and mutuals. The measure of size could be total assets, total premiums or total output. Quartile 1 could include the smallest firms and quartile 4 the largest. The quartile 1 size dummy variable takes 1 if the firm is classified by its size in quartile 1and 0 otherwise.

insurance industry, the two most prominent being the *efficiency structure hypothesis* and the *expense preference hypothesis*. The *efficiency structure hypothesis* predicts stocks and mutuals will have equal efficiency after controlling for production technology and business mix, whereas the *expense preference hypothesis* predicts mutuals will be less efficient than stocks. An important gap that this approach comes to solve is that when stocks and mutuals use different technologies and operate with different frontiers, efficiencies based on the pooled frontier is not informative.

The cross-frontier analysis involves estimating the efficiency of the firms in each group not only with respect to a reference frontier consisting only of firms from its own group but also with reference to the other group's frontier. This allows calculating cross-to-own efficiency ratios which measure the distance between the stock and mutual frontiers. These ratios are key statistics to test the superiority of one technology over the other. Linear optimization procedures are used to estimate production, cost and revenue frontiers, both for the standard own-frontiers setups as well as the cross-frontiers models.

There are many potential applications of the cross-frontier analysis that provide avenue for future research. In addition to using this approach to evaluate the relative performance of different organizational forms in the insurance industry in other non-studied national markets, it would be interesting to apply this methodology to analyze the efficiency effects of demutualization in an international context. Furthermore, this analysis could be used to test other economic hypotheses in the insurance industry like the *diversification hypothesis* versus the *specialization hypothesis* as in many countries (e.g. Austria, Belgium, the Netherlands, Spain, the UK) life specialists, non-life specialists and joint insurers, which offer both kinds of insurances (life and non-life) coexist in a market. A further example of potential future research could be to use this approach in other industries, like banking, to evaluate the relative performance of different organizational forms (e.g. commercial banks versus saving banks). Furthermore, from a methodology point of view, it would be interesting to study the similarities between the cross-frontier methodology and the meta-frontier methodology, especially when the meta-frontier is constructed under the non-convexity constraint.

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# Appendix

**Table 3** Studies on organizational forms in the insurance industry using frontier efficiency analysis

| Study | Country/ies | Segment | Method (1) | Output (2) | Frontier | Findings |
|---|---|---|---|---|---|---|
| Fecher et al. (1993) | France | Life<br>Non-life | DEA<br>SFA | Value-added | Production | Mutual non-life insurers more efficient than stocks. Stocks life insurers more efficient than mutuals |
| Gardner and Grace (1993) | US | Life | DFA | Value-added | Cost | Stocks and mutuals equally efficient |
| Fukuyama (1997) | Japan | Life | DEA | F. Intermed. | Production | Stocks and mutuals have identical technologies. Performance differs by organizational form under different economic conditions |
| Cummins and Zi (1998) | US | Life | DEA, DFA | Value-added | Production | Stocks and mutuals equally efficient |
| | | | FDH, SFA | | Cost | |
| Cummins et al. (1999) | US | Non-life | Cross-DEA | Value-added | Production | Stocks and mutuals have different technologies. Stocks more successful at minimizing costs |
| | | | Malmquist | | Cost | |
| Cummins et al. (2004) | Spain | Life<br>Non-life | Cross-DEA | Value-added | Production | Stocks and mutuals operate on separate cost and revenue frontiers. Overall results consistent with the efficient structure hypothesis |
| | | | | | Cost | |
| | | | | | Revenue | |
| Greene and Segal (2004) | US | Life | SFA | Value-added | Cost | Stocks are as efficient and profitable as mutuals |
| Brockett et al. (2005) | US | Non-life | RAM-DEA | F. Intermed. | Production | Stocks are more efficient than mutuals |
| Jeng and Lai (2005) | Japan | Non-life | RAM-DEA<br>Cross-DEA | Value-added<br>F. Intermed | Production<br>Cost | Efficiency for keiretsu, nonspecialized independent firms (NSIF) and specialized independent firms are equal, except keiretsu that are more cost efficient than NSIF |

| | | | | | | |
|---|---|---|---|---|---|---|
| Jeng et al. (2007) | US | Non-life | DEA | Value added | Production | There is not efficiency improvement after demutualization relative to stock control insurers |
| | | | Malmquist | F. Intermed. | Cost | |
| Erhemjamts and Leverty (2010) | US | Life | Cross-DEA | Value-added | Production | Efficiency of stock organizational form dominates mutual structure during the sample period |
| | | | | | Cost | |
| Chen et al. (2011) | US | Non-life | DEA<br>Malmquist | Value-added<br>F. Intermed. | Production<br>Cost | Demutualizing insurers have large gains in cost efficiency and total productivity change than mutuals control group. Efficiency improves after conversion |
| Bikker and Gorter (2011) | Netherlands | Non-life | TFA | Value-added | Cost | Stocks and mutuals have comparative cost averages |
| Biener and Eling (2012) | 21 countries | Life | Cross-DEA | Value-added | Production | Evidence for the efficiency structure hypothesis in selected market segments. No evidence for the expense preference hypothesis |
| | | Non-life | | | Cost | |

Notes: (1) The revised studies use different methodologies: econometric approaches (stochastic frontier approach (SFA), distribution free approach (DFA), thick frontier approach (TFA)) and mathematical programming approaches (standard data envelopment analysis (DEA), range-adjusted DEA (RAM-DEA), cross-frontier DEA (cross-DEA) and free disposal hull (FDH)). Several papers calculate the total factor productivity change of firms over time using the Malmquist index approach, an extension of the DEA methodology. (2) Two output approaches are used: the value-added approach (valued-added) and the financial intermediation approach (F. Intermed.)

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# Canadian Socially Responsible Investment Mutual Funds Performance Evaluation Using Data Envelopment Analysis

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**Abstract** Socially responsible investment (SRI) mutual funds, which rely on social, environmental and ethical considerations in the investment decision-making process, have experienced significant growth over the past 20 years worldwide. This chapter examines the performance, over the 2008–2011 period, of a survivorship bias-free sample of 85 Canadian SRI funds, using a Data Envelopment Analysis (DEA) approach. This technique does not require the specification of benchmarks and allows measuring the relative efficiency of decision making units/funds in the presence of a multiple input-output setting. Various performance indicators or efficiency scores are derived using higher-order moments and tail-risk measures, fee structures, net returns, and fund size. The results confirm the suitability of the DEA-based performance setting and suggest that front-end loads and fund size are the main causes of the inefficiency of Canadian SRI mutual funds. These findings carry important implications for the fund-selection process and performance persistence, and would be of interest to regulators, practitioners, and institutional and individual investors.

**Keywords** Performance evaluation • SRI mutual funds • Data Envelopment Analysis

**JEL Classification** G11 • G32

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# 1 Introduction

Performance measurement and evaluation of actively managed funds continue to receive wide interest among academics and practitioners. The variety of interested parties who are involved in and benefit from the assessment of fund performance suggests the need for a robust measure. At first, it was simply about computing the historical returns without taking into account other factors: the higher the return, the better the performance. With the development of modern portfolio theory by Markowitz (1952), risk was included into the decision-making process, and pioneering measures were proposed by Treynor (1965), Sharpe (1966), and Jensen (1969). Since these contributions, alternative approaches have been proposed in the literature; where they differ is in the way risk is considered, such as the use of multifactor models (Lehmann and Modest 1987; Carhart 1997) and the adoption of the stochastic discount factor-based methodology (Chen and Knez 1996; Farnsworth et al. 2002; Ayadi and Kryzanowski 2005, 2008).

Nevertheless, there is an ample literature arguing that different managerial attributes and other fund characteristics can affect mutual fund performance. These characteristics include the fund size (Indro et al. 1999), management fees (Elton et al. 1993), expenses (Malkiel 1995), and loads (Carhart 1997). Furthermore, Glosten and Jagannathan (1994) provide evidence that actively managed mutual funds have non-normal return distributions with negative skewness and fat tails due to investment restrictions or limitations, such as short-selling restrictions, the use of derivative instruments to hedge risk, and the increasing use of option-like trading or dynamic strategies. They contend that traditional or classical measures of performance would be inappropriate and would lead to a biased assessment of fund managers' true selection ability.[1] This finding was recently corroborated by Ayadi and Kryzanowski (2013) on a sample of Canadian equity mutual funds, and by Agarwal et al. (2014) for several hedge fund portfolios. Both papers advocate the use of nonlinear benchmarks for such investment portfolios.[2]

[1] The asset pricing literature lends strong theoretical and empirical support to the hypothesis that higher moments (co-skewness and co-kurtosis with the market portfolio) are priced by rational risk-averse investors (Harvey and Siddique 2000; Dittmar 2002).

[2] The nonlinear-based benchmarks are used extensively in hedge fund performance measurement. These models are empirically supported by Fung and Hsieh (2001), who show similarity in the payoffs of the trend-following strategies and those of a lookback straddle strategy. Agarwal and Naik (2004) confirm these results for a large number of equity-oriented hedge fund strategies with payoffs resembling a short position in a put option on the market index. Similarly, Chan et al. (2007) develop new measures of hedge fund systematic risks such as illiquidity risk exposure and nonlinear factor models.

The inadequacy of traditional measures in a non-normal world and for portfolios with nonlinear payoffs has led to the development of alternative methods named *frontier analysis methods*. Data Envelopment Analysis is a powerful non-parametric frontier method founded by Charnes et al. (1978) that takes into consideration the dynamics of fund strategies and the various fund characteristics. DEA is suitable to assess and rank mutual fund performance in a (nonlinear) risk-return framework based on several input and output variables, even with a non-parametric relationship for these variables. DEA-based measures of performance offer insights into the level of fund efficiency, given the set of input and output variables. Such information is useful to individual and institutional investors as well to fund managers to uncover the importance of the included variables through their efficiency contributions.

One type of investment portfolio that has experienced tremendous growth in the past 20 years is the socially responsible investment (SRI) mutual fund.[3] The investment strategies of such funds are governed by ethical rules and social screens to select or exclude assets. Advocates of these special investments argue that the inclusion of social and environmental considerations in the investment decision-making process improves investment returns. Therefore, assessing the performance of SRI investments or mutual funds is of interest to various players in the financial system.[4] We build on the previous research by using the DEA method to develop new performance measures for a comprehensive sample of Canadian SRI funds over the 2008–2011 period.[5] Our approach takes into consideration key variables such as net returns, linear and nonlinear risk measures, total assets, and fee structures. In this vein, Basso and Funari (2008) study the performance of ethical mutual funds on the European market and develop new efficiency scores. They find that ethical funds have higher scores only when the employed DEA model considers the ethical level among the output variables. In parallel, Pérez-Gladish et al. (2013) use the same method to examine the performance of a sample of US

[3] The growth in assets under management (AUM) and the number of SRI funds has been rapid over the past 20 years, worldwide. AUMs for Canadian SRI retail mutual funds under SRI guidelines remained unchanged from 2004 to 2011, at 4.4 billion CDN, but are down from 5.5 billion CDN in 2008 (SIO 2013). The corresponding AUMs under SRI guidelines for all Canadian funds are 57.9, 600.9, and 566.7 billion CDN in 2004, 2011, and 2008, respectively. Their estimated share of total AUM in Canada is 3.2 %, 20.1 %, and 20.4 % in 2004, 2011, and 2008, respectively.

[4] Two other related streams of research in SRI fund performance: The first stream focuses on the role of the screening mechanisms adopted by SRI funds, such as negative screening, positive screening and norms-based screening. In particular, various studies test the association between these strategies and performance/risk (see Barnett and Salomon 2006; Lee et al. 2010; Laurel 2011; Humphrey and Lee 2011). The second stream examines the important smart money effect for the relationship between SRI fund performance and money flows (see Renneboog et al. 2007, 2008; Benson and Humphrey 2008).

[5] Mutual funds in Canada are often registered as investment trusts and competition is restricted by not permitting foreign-domiciled funds to register for sale domestically. Fund management services are subject to domestic consumption taxes in Canada and the Canadian distribution model uses financial advisors selling and servicing no-load funds (Alpert et al. 2013).

mutual funds. They conclude that there are no significant performance (efficiency scores) differences between conventional and SRI funds.

The remainder of this chapter is organized as follows. In the next section we introduce socially responsible investments mutual funds and their strategies, with a brief review of the literature on their performance. In Sect. 3, we discuss the DEA approach as an alternative non-parametric performance index. We also highlight the use of DEA in the evaluation of mutual fund performance. Section 4 explains the empirical implementation of the DEA approach, with a description of the data and key variables. It also discusses the obtained results. Finally, the conclusion reviews the major results and identifies possible avenues of future research.

## 2 Socially Responsible Investments Mutual Funds

Socially responsible investments (SRI) mutual funds, also known in the literature as ethical funds, are special investments that aim to harmonize investors' financial and ethical objectives. Instead of relying solely on financial criteria, ethical funds integrate moral and social issues. The Social Investment Organization defines socially responsible investing (SRI) as the inclusion of social, environmental, and governance (ESG) considerations into the management and selection of investments.[6] This organization claims that socially responsible mutual funds, when compared to conventional funds, offer an additional level of analysis and investment by using one SRI strategy or a combination of several. These strategies and previous SRI performance research are presented and discussed in the next sub-section.

### *2.1 SRI Strategies*

Several organizations, such as the US Sustainable and Responsible Investment Forum, the European Sustainable Investment Forum, the Association for Sustainable and Responsible Investment in Asia, the Responsible Investment Association Australasia, and the Canadian Social Investment Organization, recognize five major investment strategies (Social Investment Organization 2013).

**Screening** This is the most adopted strategy and can be divided into three groups: negative screening, positive screening, and standards-based screening. Negative screening is used to exclude companies that are involved in unethical activities, such as tobacco manufacturing, alcohol production, military or weapons-related contracting, gambling, nuclear power, or pornography. Positive screening "is a

[6] Available from the Social Investment Organization http://www.socialinvestment.ca.

proactive process designed to select companies that demonstrate leadership in a variety of environmental, social, and governance issues."[7] This, for example, includes protection of the environment, protection of human rights, ensuring employee standards, or supporting alternative energy. Finally, standards-based screening involves the selection of investments that respect international standards, such as the United Nations Universal Declaration of Human Rights or the UNICEF Convention on the Rights of the Child.

**Integration** This involves the consideration of ESG factors in investment research and in the decision-making process. It differs from screening in the sense that it combines ESG data, research, and analysis, together with financial and other factors, in making investment decisions.

**Sustainability-themed funds** Sustainability-themed investing involves selecting assets on the basis of investment themes such as clean energy, green technology, or sustainable agriculture. Investments are directed at companies or industries that offer innovative solutions to existing problems or that otherwise enhance sustainability practices.

**Impact investing** Impact investing refers to targeted investments that are made in private markets and that aim at solving social or environmental problems while also generating financial returns. Impact investing includes community investing, where capital is specifically directed to traditionally underserving individuals or communities, to businesses with a clear social or environmental purpose, or to revenue-generating non-profits.

**Corporate engagement and shareholder action** This strategy aims at influencing corporate behaviour through various strategies including communicating with senior management and/or boards of directors, filing shareholder proposals, and proxy voting.

## 2.2 *Literature Review*

Viewed from three different perspectives, the phenomenon of ethical funds has been discussed by several researchers. Each of these standpoints is based either on an underperformance, outperformance, or no-effect hypothesis (Hamilton et al. 1993). The first hypothesis claims that ethical funds underperform their conventional peers. The reasons for this underperformance are discussed by Bauer et al. (2007): First, investing in ethical funds limits the diversification of the portfolio because ethical funds can exclude companies with a good financial performance, for ethical considerations. Second, there are costs to developing ethical investment screens and corporate-social-responsibility rankings. Third,

[7] Available from Qtrade Financial Group http://www.qtrade.ca.

irresponsible activities are perceived as more lucrative and recession-proof than are responsible investments. The second hypothesis suggests that ethical funds can outperform conventional funds. The outperformance of ethical funds occurs when "sound social and environmental performance signals high managerial quality, which translates into favourable financial performance" (Renneboog et al. 2008). Outperformance could also be related to the fact that responsible investments avoid paying for the consequences of non-ethical behaviours, for instance, government fees. Finally, the 'no-effect-hypothesis' supposes that there is no significant difference between the performance of ethical and conventional funds. In other words, the social responsibility feature does not affect the stock price (Hamilton et al. 1993).

Several empirical studies have been conducted in various countries to confirm or disconfirm these hypotheses. The majority of the studies focus on the US market, such as Hamilton et al. (1993), Statman (2000), Bauer et al. (2005), Benson et al. (2006), and Renneboog et al. (2008). They all conclude that there is no significant performance difference between ethical and conventional funds. The same conclusion is drawn by Luther et al. (1992), Mallin and Saadouni (1995), Gregory et al. (1997), Kreander et al. (2005), and Renneboog et al. (2008), who examine the performance of ethical funds in the European market. Furthermore, the Australian and Canadian evidence (Bauer et al. 2006, 2007; Humphrey and Lee 2011; Ayadi et al. 2015) supports the no-effect hypothesis. Nevertheless, few studies confirm the underperformance or outperformance hypothesis, such as Chang and Witte (2010), whose findings show a significant underperformance of US SRI funds over 5-, 10- and 15-year periods, but not over the 3-year period.

The above-mentioned studies use classic performance measures, mainly Jensen's alpha, based either on the CAPM, or on the Carhart four-factor-model, Sharpe ratio, and Treynor ratio. However, the comparison is regarded as meaningless if the ethical and conventional funds do not have the same characteristics (age, size, market, investing area, etc).

## 3 Data Envelopment Analysis for Performance Evaluation

### *3.1 Introduction*

Most classical or parametric performance measures rely on the Markowitz portfolio theory (1952). This approach uses the efficient-frontier concept, which is defined as a set of non-dominated portfolios in the mean-variance space; in other words, the efficient frontier consists of portfolios that maximize returns for a given level of risk, or alternatively, minimize risk for a given expected return (Kroll et al. 1984). Similarly, alternative methods of frontier analysis are based on the concept of the production frontier, which illustrates the maximum potential output that a production unit can achieve under a given set of inputs. These methods were initiated by

Farrell (1957) in an attempt to present an efficiency measure that overcomes the problems of index numbers in dealing with multiple inputs. All production units aim at reaching the efficient frontier but may fail due to reasons within or beyond their control. Farrell assumed that a production unit can be inefficient either if it produces less than the maximum output available from a set of inputs (*technical inefficiency*) or if it does not consume the best proportion of inputs in view of their prices (*price* or *allocative inefficiency*).

One non-parametric frontier-analysis approach referred to as *Data Envelopment Analysis* (DEA) was developed by Charnes, Cooper, and Rhodes in 1978 as a solution to the problem introduced by Farrell (1957) in measuring efficiency. It has been a useful tool to evaluate non-profit and public sector organizations. Unlike parametric methods, which require the specification of a functional form of the efficient frontier, the DEA approach is based on mathematical programming to define the efficient frontier and to calculate the efficiency scores. Moreover, it does not assume a precise relation between input and output variables, which would offer flexibility and less susceptibility to specification error. However, DEA does not allow for random error; instead, it attributes all deviation from the frontier to inefficiencies. Further, DEA is sensitive to the choice of input and output variables; adding an important number of inputs and outputs may decrease the model's accuracy. DEA is also vulnerable to the *curse of dimensionality,* which is related to problems associated with a low number of decision-making units (DMU)[8] relative to the number of input-output variables.[9] Finally, the DEA model relies on the following basic assumptions: (1) The positivity of the employed variables; (2) Conditions on the number of DMU to be evaluated; for example, Cooper et al. (2007) claim that if the number of DMUs (n) is less than the combined number of inputs plus outputs (m + s), a large portion of the DMUs will be identified as efficient, and efficiency discrimination among DMUs is lost; (3) The homogeneity of the DMUs.

[8] DEA has several advantages over traditional methods of performance measurement. First, it avoids the benchmark specification problem since there is no need to identify any theoretical model (like CAPM) as a benchmark. Instead, DEA measures the performance of a fund relative to the best-performing ones. Second, DEA is a multidimensional approach that can take into account many inputs and outputs. Hence, it is possible to consider, along with risk and return, other factors that could serve in the evaluation of a fund's performance. Finally, DEA not only measures performance, it also has a powerful ability to identify the reasons behind a fund's poor performance. In fact, slack variables in DEA present the major source of inefficiency and give insight into how a fund can ameliorate its performance (Choi and Murthi 2001).

[9] Charnes et al. (1978) use the term 'decision-making unit' to refer to the unit under evaluation. "Generically a DMU is regarded as the entity responsible for converting inputs into outputs and whose performances are to be evaluated" (see Cooper et al. 2007).

## 3.2 The DEA Model

We adopt the BCC (Banker, Charnes, and Cooper 1984) DEA model based on variable returns to scale, because if one can assume that economies of scale change as fund size increases, then constant-return-to-scale-type DEA models are not an adequate choice. Further, we choose an input-oriented model that emphasizes the reduction of inputs to improve efficiency, as we suppose that mutual-fund managers have more control over inputs than outputs.

The DEA model can be formulated in its dual form as follows:

$$\begin{array}{lll} \text{Min} & z_o - \varepsilon\left(\sum_{i=1}^{m} S_i^- + \sum_{r=1}^{s} S_r^+\right) & \\ \text{Subject to} & \sum_{j=1}^{n} x_{ij}\,\lambda_j + S_i^- = z_o x_{io} & i = 1, 2, \ldots, m \\ & \sum_{j=1}^{n} y_{rj}\,\lambda_j + S_r^+ = y_{ro} & r = 1, 2, \ldots, s \\ & \lambda_j \geq 0 & j = 1, 2, \ldots, n \end{array} \tag{1}$$

where we denote by: $j = 1, 2, \ldots, n$ funds; $r = 1, 2, \ldots, s$ outputs; $i = 1, 2, \ldots, m$ inputs; $y_{rj}$ amount of output $r$ for the fund $j$, $x_{ij}$ amount of input $i$ for fund $j$. $S_i^-$ and $S_r^+$ represent input and output slack variables, respectively. $z_o$ represents the efficiency score for the fund under evaluation. $\lambda_j$ ($j = 1, \ldots, n$) are non-negative scalars. $\varepsilon$ is a non-Archimedean element (a very small positive number).

The dual leads to the same value of the objective function as the primal. However, while the number of constraints of the primal depends on the number of the DMU evaluated, the dual constraints depend on the number of inputs and outputs. Ramanathan (2003) argues and demonstrates that the use of the dual formulation is computationally more efficient because the computational efficiency of linear programing codes depends upon the number of constraints.

According to Cooper et al. (2011), an efficient fund is one that satisfies the following conditions: $z_o{*} = 1$ and all slack variables are equal to zero. When a fund has an efficiency score equal to one and there are some slacks different from zero, it is considered weakly efficient.

Cooper et al. (2007) explain that a DMU can become efficient by reducing its inputs by the ratio $z_o$ and eliminating the negative slacks $S_i^-$. A similar efficiency can be attained if output values are augmented by the positive slacks $S_r^+$. The gross improvements of inputs and outputs are given by the following formulas:

$$\begin{gathered} \Delta x_{io} = x_{io} - \left(z_o x_{io} - S_i^-\right) = (1 - z_o)x_{io} + S_i^- \\ \Delta y_{ro} = S_r^+ \end{gathered} \tag{2}$$

The projection of the inefficient DMU into the frontier is defined by the following formulas:

$$\widetilde{x_{io}} = x_{io} - \Delta x_{io} \qquad (3)$$
$$\widetilde{y_{ro}} = y_{ro} + \Delta y_{ro}$$

While the CCR model relies on two assumptions, namely, the convexity of the efficient frontier and the constant returns to scale, the BCC model (Banker et al. 1984) relaxes the latter assumption in order to handle variable returns to scale. The following constraint was introduced into the envelopment model:

$$\sum\nolimits_{j}^{n} \lambda_j = 1 \qquad (4)$$

### 3.3 Literature Review on DEA Applications to Evaluate Mutual Fund Performance

Murthi et al. (1997) are the first researchers attempting to apply the DEA methodology to assess the performance of mutual funds. Their objective is to overcome the shortcomings of traditional performance measures, especially their inability to consider transaction costs in the analysis. They propose a new performance index called the DEA portfolio efficiency index that takes into account risk, return and transactions costs. The main CCR DEA model is applied to 731 mutual funds in 1993 using the actual return as the output variable, and four input variables: expense ratio, loads, turnover, and standard deviation of returns. As a result, Murthi et al. indicate that mutual funds are approximately mean-variance-efficient and that efficiency is not related to transaction costs. While Murthi et al. (1997) adopt the basic DEA assuming constant returns-to-scale and do not survey the issue of scale effects on mutual funds, McMullen and Strong (1998) estimate a DEA model that assumes variable returns-to-scale to analyze 135 common stock mutual funds. They consider as outputs the returns over different lengths of time and, as inputs the sales charge, expense ratio, minimum initial investment (instead of turnover), and semi-deviation of return measured over 3 years. In addition, seeing that DEA can assign very low weights to some undesirable inputs and outputs in order to increase the efficiency measure, they set constraints upon the weights in order to ensure that not all attributes are disregarded. In a second step, Choi and Murthi (2001) apply a different DEA formulation to the data they used before, and propose a non-oriented additive model that considers the same inputs and outputs of the DPEI index. This approach allows for the control of the scale effects.

Whereas pioneering works focused on return as an output in the DEA model and considered only standard deviation and transaction costs as inputs, subsequent studies include other variables. Basso and Funari (2001) propose a DEA-based performance index taking into account different risk measures and investment costs. They consider both subscription and redemption costs. The risk measures include the return standard deviation, the beta coefficient, and the half-variance risk. Furthermore, they define a new index that reflects an additional output, a stochastic dominance indicator in order to describe the investor's preferences, and the occurrence of returns. In an empirical analysis of the Italian financial market, these authors evaluate the performance of 47 mutual funds and find that redemption costs are an important variable in determining fund rankings. In a subsequent paper, Basso and Funari (2003) develop DEA models that encompass ethical criteria, as in recent decades, investors have become more concerned with satisfying both their financial and their ethical aims. First, they propose a generalization of DEA indexes by adding the ethical measure as a second output. Then, they develop an exogenously fixed output model that contains an ethical level and presents it as a fixed variable. However, these indexes do not take into account the nature of the information available about the ethical level, as in practice, only binary information on the ethical/non-ethical nature, or a ranking of funds according to their ethical level are available. For this reason, these authors present a DEA categorical model with an exogenously fixed output. They test these indices on 50 simulated mutual funds. Subscription and redemption costs, the standard deviation of returns, and beta coefficient are chosen as inputs; and the expected return and an ethical indicator are selected as outputs. Moreover, Basso and Funari (2005) extend their previous indexes so that they can take into account the results of traditional performance measures. Hence, a generalized DEA performance metric, which adds to the outputs the value of the traditional performance indexes, is proposed. Moreover, they present the cross-efficiency matrix, which makes it possible to measure the performance of each fund, using different optimal weights for the other funds.

Considering that the risk measures introduced in previous DEA models do not reflect the characteristics of the funds' return distributions, such as asymmetry and fat-tailedness, Gregoriou et al. (2005) focus on different downside risk measures to examine 614 hedge funds for the period from 1997 to 2001. Chen and Lin (2006) propose a DEA model that considers the value at risk (VaR) and the conditional value at risk (CVaR) as inputs in a test of 22 different input-output specifications. Lamb and Tee (2012) develop a new method with a suitable form of returns to scale and convenient risk measures. Their model directly allows for diversification and employs the mean return as an input, and the maximum between CVaR and zero as an output. Recently, Pérez-Gladish et al. (2013) use the DEA to evaluate the performance of a sample of 46 US domiciled large-cap equity mutual funds. They use the following inputs: turnover ratio, annual report gross, expense ratio, deferred loads, and front loads. As outputs, they use a financial criterion, namely, the mean return, as well as nonfinancial criteria, namely, social and environmental responsibility (SER) level, and quality of the SRI management.

## 4 Data, Implementation, and Results

We first present the sample of SRI funds and the key variables used in different DEA models. A discussion of the descriptive statistics of these variables is also provided. In the second part, we fully show the construction of various DEA

**Table 1** Summary statistics for Canadian SRI mutual funds returns

| Statistics | Mean | Median | Maximum | Minimum | Std. Dev. | Skewness | Kurtosis |
|---|---|---|---|---|---|---|---|
| Mean | −0.002 | 0.002 | 0.079 | −0.124 | 0.043 | −0.718 | 3.974 |
| Median | −0.001 | 0.002 | 0.080 | −0.126 | 0.047 | −0.674 | 3.577 |
| Std. Dev. | 0.005 | 0.005 | 0.034 | 0.067 | 0.018 | 0.531 | 1.614 |
| Minimum | −0.015 | −0.008 | 0.002 | −0.374 | 0.001 | −2.353 | 2.032 |
| Maximum | 0.006 | 0.017 | 0.162 | 0.000 | 0.084 | 1.586 | 10.425 |

This table reports summary statistics (mean, median, maximum, minimum, standard deviation, skewness, and kurtosis) of individual fund returns of 85 Canadian SRI mutual funds, using monthly data from May 2008 through December 2011 (maximum of 44 observations)

models, with a discussion of the efficiency results implied by each model. All the tests and efficiency scores of SRI mutual funds are conducted using the “FxDEA,” software.

## *4.1 Data and Variables*

The sample used in the present chapter is provided from the *Fundata* database and consists of monthly data for 85 Canadian SRI mutual funds over the period of May 2008 to December 2011. To control for selection and survival biases, we include all active and terminated funds in our portfolio tests. Summary statistics of SRI mutual fund returns are provided in Table 1.

We use various input/output variables to assess the performance of our sample of Canadian SRI funds. Most of them are based on earlier studies. The output variables include the net return of the fund and the skewness of fund returns. The return is given by the changes in the net asset values per share (NAVPS), and is adjusted for all distributions. The skewness of fund returns is estimated by the third moment and measures the asymmetry of the return distribution (Joro and Na 2006; Pendaraki 2012). The set of input variables includes the following fund characteristics: (1) The fund size, which is proxied by total net asset (TNA) value (Daraio and Simar 2006); (2) The return standard deviation is given by the second moment and is a measure of fund total risk (Basso and Funari 2003; Daraio and Simar 2006; Chen and Lin 2006; Joro and Na 2006); (3) The value at risk (VaR 95 %), which describes an investment’s possible loss that is not exceeded with a probability of 95 % (Gregoriou et al. 2005; Chen and Lin 2006); (4) The kurtosis of fund returns is given by the fourth moment and measures of the degree of the peakness of the return distribution; (5) The management expense ratio (MER), defined as the mutual fund’s annual fees, which includes the management fees and other operating expenses, expressed as a percentage of the total fund value (Daraio and Simar 2006; Chen and Lin 2006; Ayadi et al. 2015); This is an important variable that differentiate SRI and non-SRI funds. In effect, management expenses are expected to be higher for SRI versus non-SRI mutual funds for one or more of the following reasons: First, SRI funds incur additional monitoring costs of the firms in which they invest to ensure that they maintain socially responsible policies (Gil-Bazo et al. 2010); Second, investors in SRI funds are likely to be less performance sensitive (Gil-Bazo et al. 2010) and studies find that management fees are inversely related with investor performance sensitivity (Christoffersen and Musto 2002; Gil-Bazo and Ruiz-Verdu 2009); and third, SRI funds may have higher management expenses since the smaller size of their sponsors and their assets under management lead to less economies from scale (as reported by Bauer et al. 2005, for German and UK funds, and by Bauer et al. 2006, for Australian funds). (6) Front-end loads and back-end loads, representing sales and deferred sales charges (Daraio and Simar 2006). Our framework is consistent with the axiomatic microeconomic theory suggesting that investors prefer positive skewness and have an aversion to kurtosis (Scott and Horvath 1980; Hwang and Satchell 1999).

Table 2 reports summary statistics of the included input/output variables. It is clear that some of the funds of our sample exhibit negative average returns and negative skewness. In order to satisfy the DEA's non-negative requirement on variables used, we use the translation invariance property of the input-BCC model and normalize returns and skewness through the addition of a constant.

Furthermore, since every variable should be able to bring new information to the analysis, a desired property for each model is the independence of the selected variables. Jenkins and Anderson (2003) reveal that including highly correlated variables in the DEA can significantly affect the efficiency results. Therefore, it is necessary to make sure that the variables are not highly correlated. The input variables' correlation matrix is shown in Table 3. It can be seen that the correlation between back-end loads and front-end loads is equal to 0.75. In order to avoid including the same type of information twice, it is necessary to drop one of the highly correlated variables from the analysis.

## 4.2 Results and Discussion

The DEA program is designed and tested in four different forms that have different combinations of input and output variables. In the first model (DEA-1), standard deviation is considered an input, and net returns, an output (which resembles the mean-variance framework). An extended model, where the management-expense ratio (MER), front-end loads, and total assets are added as inputs, is proposed for DEA-2. The third model, DEA-3, relies on the value at risk as a measure of risk instead of the standard deviation of returns. Finally, model DEA-4 incorporates higher-moment risk variables into the analysis (kurtosis and skewness). Our setup

**Table 2** Summary statistics for employed variables for Canadian SRI mutual funds

| Statistics | Return | Skew. | Std. Dev. | Total assets | Front-end load | Back-end load | MER | VaR 95 % | Kurt. |
|---|---|---|---|---|---|---|---|---|---|
| Mean | −0.002 | −0.718 | 0.043 | 61.846 | 0.018 | 0.025 | 0.019 | 0.069 | 3.974 |
| Median | −0.001 | −0.674 | 0.047 | 9.994 | 0.000 | 0.000 | 0.019 | 0.074 | 3.577 |
| Std. Dev. | 0.005 | 0.531 | 0.018 | 178.571 | 0.025 | 0.029 | 0.008 | 0.031 | 1.614 |
| Skewness | −0.800 | −0.040 | −0.406 | 5.195 | 0.710 | 0.284 | −0.327 | −0.537 | 2.417 |
| Kurtosis | 0.961 | 5.100 | 0.068 | 28.790 | −1.486 | −1.944 | −0.914 | −0.482 | 6.427 |
| Minimum | −0.015 | −2.353 | 0.001 | 0.008 | 0.000 | 0.000 | 0.001 | 0.000 | 2.032 |
| Maximum | 0.006 | 1.586 | 0.084 | 1124.474 | 0.060 | 0.060 | 0.031 | 0.122 | 10.425 |

This table reports summary statistics for the variables employed in the analysis. These variables include the net return and skewness (Skew.) as outputs, and the following inputs: standard deviation (Std. Dev.), total assets, front-end load, back-end load, management expense ratio (MER), value at risk (VaR 95 %), and kurtosis of returns (Kurt.). The data cover the period from May 2008 to December 2011, for a total of 44 observations

**Table 3** Input/output correlation matrix

| | Return | Skew. | Std. Dev. | Total assets | Front-end load | Back-end load | MER | VaR 95 % | Kurt. |
|---|---|---|---|---|---|---|---|---|---|
| Return | 1.000 | | | | | | | | |
| Skew. | 0.554 | 1.000 | | | | | | | |
| Std. Dev. | −0.731 | −0.468 | 1.000 | | | | | | |
| Total assets | −0.043 | 0.030 | 0.178 | 1.000 | | | | | |
| Front-end load | −0.093 | −0.049 | 0.014 | 0.030 | 1.000 | | | | |
| Back-end load | −0.308 | −0.146 | 0.189 | 0.288 | 0.746 | 1.000 | | | |
| MER | −0.534 | −0.365 | 0.418 | 0.223 | 0.462 | 0.722 | 1.000 | | |
| VaR 95 % | −0.583 | −0.247 | 0.893 | 0.194 | 0.066 | 0.162 | 0.363 | 1.000 | |
| Kurt. | −0.624 | −0.819 | 0.442 | −0.042 | 0.065 | 0.241 | 0.338 | 0.143 | 1.000 |

This table presents the correlation matrix of the input/output variables used in the analysis. These variables include the net return and skewness (Skew.) as outputs, and the following inputs: standard deviation (Std. Dev.), total assets, front-end load, back-end load, management expense ratio (MER), value at risk (VaR 95 %), and kurtosis of returns (Kurt.). The data cover the period from May 2008 to December 2011, for a total of 44 observations

in all DEA models is consistent with the rule of thumb suggested by Banker et al. (1989) where $n = 85 > \max(s \times m, 3(s + m)) = \max(2 \times 6, 3 \times (2 + 6)) = 24$.

Table 4 reports the input and output data for each fund and the empirical results of the analysis. In addition, Table 5 compares the efficient set and the minimum and average efficiency scores obtained with each of the employed models.

In the mean-variance framework (DEA-1), only two funds are identified as efficient. However, the other funds have efficiency scores of less than one; thus, they are inefficient. This evidence suggests that not all SRI mutual funds are mean-variance efficient (this result is further confirmed using the Sharpe ratio measure).

By adding the front-end loads, MER and total-assets variables into the standard mean-variance framework (DEA-2), the number of efficient funds and the average efficiency increase significantly. In effect, the average efficiency score is 12.5 % and 49.9 % in the first and second applications, respectively. In addition, the number of efficient funds becomes twelve, representing almost 14 % of our sample. Ten inefficient funds in the mean variance framework turn out to be efficient according to the second application. It is worth noting that adding back-end loads into the analysis did not alter these results.

**Table 4** Empirical results of the analysis of the performance of the Canadian SRI mutual funds

| Fund name | Return | Skew. | Std. Dev. | Total assets | Front-end load | MER | VaR 95 % | Kurt. | DEA-1 | DEA-2 | DEA-3 | DEA-4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Ethical Canadian Dividend Fund Series A | −0.036 | −0.707 | 0.041 | 249.788 | 0.050 | 0.025 | 0.075 | 2.870 | 0.015 | 0.201 | 0.160 | 0.708 |
| Ethical Canadian Dividend Fund Series F | 0.073 | −0.703 | 0.041 | 2.011 | 0.000 | 0.015 | 0.074 | 2.864 | 0.027 | 0.453 | 0.392 | 0.880 |
| GWL Ethics Fund (G) NL | −0.257 | −1.179 | 0.053 | 28.006 | 0.000 | 0.029 | 0.085 | 5.266 | 0.012 | 0.223 | 0.204 | 0.433 |
| GWL Ethics Fund (G) DSC | −0.239 | −1.179 | 0.053 | 30.057 | 0.000 | 0.027 | 0.085 | 5.266 | 0.012 | 0.234 | 0.214 | 0.440 |
| Meritas Jantzi Social Index Fund | −0.227 | −0.520 | 0.055 | 72.407 | 0.050 | 0.020 | 0.083 | 3.412 | 0.011 | 0.246 | 0.234 | 0.652 |
| London Life Ethics Fund (GWLIM) | −0.219 | −1.190 | 0.053 | 83.647 | 0.000 | 0.028 | 0.085 | 5.317 | 0.012 | 0.209 | 0.191 | 0.427 |
| MFS MB Responsible Canadian Equity Fund | −0.109 | −0.593 | 0.054 | 78.630 | 0.000 | 0.018 | 0.091 | 3.577 | 0.011 | 0.270 | 0.243 | 0.642 |
| PH&N Community Values Canadian Equity Fund Ser D | −0.099 | −0.301 | 0.053 | 22.427 | 0.000 | 0.013 | 0.084 | 2.804 | 0.011 | 0.359 | 0.331 | 0.828 |
| PH&N Community Values Canadian Equity Fund Ser O | −0.000 | −0.291 | 0.053 | 35.025 | 0.000 | 0.001 | 0.083 | 2.784 | 0.011 | 1.000 | 1.000 | 1.000 |
| Meritas Monthly Dividend and Income Fund | 0.036 | −0.561 | 0.038 | 22.355 | 0.050 | 0.025 | 0.057 | 2.850 | 0.016 | 0.259 | 0.247 | 0.713 |
| iShares Jantzi Social Index Fund | −0.107 | −0.513 | 0.056 | 20.768 | 0.000 | 0.005 | 0.083 | 3.413 | 0.011 | 0.503 | 0.498 | 0.858 |
| RBC Jantzi Canadian Equity Fund Series A | 0.047 | −0.673 | 0.048 | 22.202 | 0.000 | 0.021 | 0.069 | 3.928 | 0.013 | 0.287 | 0.272 | 0.584 |
| RBC Jantzi Canadian Equity Fund Series F | 0.140 | −0.673 | 0.048 | 0.813 | 0.000 | 0.010 | 0.068 | 3.930 | 0.047 | 0.717 | 0.717 | 0.805 |
| RBC Jantzi Canadian Equity Fund Series D | 0.119 | −0.674 | 0.048 | 1.221 | 0.000 | 0.012 | 0.068 | 3.933 | 0.039 | 0.530 | 0.530 | 0.751 |
| PH&N Community Values Canadian Equity Fund Ser B | −0.151 | −0.304 | 0.053 | 0.292 | 0.000 | 0.019 | 0.085 | 2.810 | 0.011 | 0.525 | 0.528 | 0.912 |

(continued)

**Table 4** (continued)

| Fund name | Return | Skew. | Std. Dev. | Total assets | Front-end load | MER | VaR 95 % | Kurt. | DEA-1 | DEA-2 | DEA-3 | DEA-4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| PH&N Community Values Canadian Equity Fund Ser F | −0.073 | −0.300 | 0.053 | 0.223 | 0.000 | 0.011 | 0.084 | 2.792 | 0.011 | 0.866 | 0.866 | 1.000 |
| Meritas Jantzi Social Index Fund Series F | −0.134 | −0.507 | 0.055 | 4.080 | 0.000 | 0.010 | 0.082 | 3.384 | 0.011 | 0.442 | 0.417 | 0.854 |
| Meritas Monthly Dividend and Income Fund Series F | 0.127 | −0.565 | 0.038 | 0.867 | 0.000 | 0.015 | 0.056 | 2.857 | 0.054 | 0.540 | 0.534 | 0.940 |
| Acuity Social Values Balanced Fund | 0.048 | −1.254 | 0.042 | 62.931 | 0.060 | 0.030 | 0.061 | 5.794 | 0.015 | 0.209 | 0.199 | 0.407 |
| Ethical Balanced Fund Class A | −0.081 | −0.952 | 0.026 | 336.301 | 0.050 | 0.023 | 0.036 | 4.011 | 0.024 | 0.281 | 0.223 | 0.574 |
| Ethical Balanced Fund Class F | 0.010 | −0.951 | 0.026 | 0.873 | 0.000 | 0.012 | 0.035 | 4.003 | 0.024 | 0.644 | 0.639 | 0.826 |
| NEI Canadian Bond Class A | 0.440 | −0.447 | 0.010 | 253.968 | 0.050 | 0.016 | 0.011 | 3.093 | 0.750 | 0.750 | 0.658 | 0.856 |
| Meritas Canadian Bond Fund | 0.389 | −0.026 | 0.009 | 38.494 | 0.050 | 0.020 | 0.011 | 2.108 | 0.762 | 0.796 | 0.643 | 0.990 |
| MFS MB Responsible Fixed Income Fund | 0.574 | −0.156 | 0.011 | 117.395 | 0.000 | 0.015 | 0.010 | 2.636 | 0.943 | 0.960 | 1.000 | 1.000 |
| PH&N Community Values Bond Fund Series D | 0.544 | −0.736 | 0.010 | 18.902 | 0.000 | 0.007 | 0.011 | 3.674 | 0.919 | 0.993 | 0.990 | 0.993 |
| PH&N Community Values Bond Fund Series O | 0.591 | −0.726 | 0.010 | 88.963 | 0.000 | 0.001 | 0.010 | 3.658 | 1,000 | 1.000 | 1.000 | 1.000 |
| NEI Canadian Bond Class F | 0.463 | −0.431 | 0.010 | 49.543 | 0.000 | 0.011 | 0.011 | 3.195 | 0.803 | 0.949 | 0.978 | 0.949 |
| PH&N Community Values Bond Fund Series B | 0.517 | −0.736 | 0.010 | 0.355 | 0.000 | 0.010 | 0.011 | 3.685 | 0.870 | 1.000 | 1.000 | 1.000 |
| PH&N Community Values Bond Fund Series F | 0.544 | −0.720 | 0.010 | 5.493 | 0.000 | 0.007 | 0.011 | 3.624 | 0.920 | 1.000 | 1.000 | 1.000 |
| Meritas Canadian Bond Fund Series F | 0.404 | 0.016 | 0.009 | 5.726 | 0.000 | 0.017 | 0.011 | 2.032 | 0.782 | 1.000 | 1.000 | 1.000 |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Ethical Select Conservative Portfolio Class A | 0.141 | −1.217 | 0.017 | 61.998 | 0.050 | 0.022 | 0.024 | 5.551 | 0.133 | 0.370 | 0.311 | 0.500 |
| Ethical Select Conservative Portfolio Class F | 0.237 | −1.176 | 0.018 | 0.062 | 0.000 | 0.011 | 0.024 | 5.131 | 0.226 | 1.000 | 1.000 | 1.000 |
| Ethical Growth Fund Series A | −0.330 | −0.845 | 0.047 | 307.373 | 0.050 | 0.023 | 0.062 | 3.781 | 0.013 | 0.179 | 0.160 | 0.579 |
| Investors Summa SRI Fund Series C | −0.546 | −0.587 | 0.071 | 1118.804 | 0.000 | 0.029 | 0.121 | 3.798 | 0.009 | 0.142 | 0.086 | 0.547 |
| Acuity Social Values Canadian Equity Fund | −0.209 | −1.123 | 0.064 | 39.500 | 0.060 | 0.030 | 0.100 | 5.153 | 0.010 | 0.195 | 0.183 | 0.431 |
| Investors Summa SRI Class A | −0.521 | −0.559 | 0.070 | 45.480 | 0.000 | 0.027 | 0.120 | 3.760 | 0.009 | 0.207 | 0.185 | 0.560 |
| Acuity Pooled Social Values Canadian Equity Fund | 0.013 | −1.122 | 0.064 | 8.659 | 0.050 | 0.002 | 0.099 | 5.133 | 0.010 | 1.000 | 1.000 | 1.000 |
| Investors Summa SRI Fund Series A | −0.532 | −0.582 | 0.071 | 1124.474 | 0.000 | 0.027 | 0.122 | 3.806 | 0.009 | 0.142 | 0.086 | 0.556 |
| Ethical Growth Fund Series F | −0.239 | −0.844 | 0.047 | 1.093 | 0.000 | 0.013 | 0.062 | 3.777 | 0.013 | 0.534 | 0.537 | 0.770 |
| Alpha Social Values Portfolio | −0.321 | −1.324 | 0.051 | 12.711 | 0.060 | 0.028 | 0.072 | 5.336 | 0.012 | 0.243 | 0.236 | 0.451 |
| Acuity Clean Environment Equity Fund | −0.956 | −1.328 | 0.065 | 80.181 | 0.060 | 0.029 | 0.098 | 5.107 | 0.009 | 0.184 | 0.174 | 0.435 |
| Matrix Sierra Equity Fund | −1.547 | −2.353 | 0.084 | 4.375 | 0.050 | 0.031 | 0.112 | 10.425 | 0.007 | 0.216 | 0.214 | 0.300 |
| Matrix Sierra Equity Fund Class F | −1.531 | −2.289 | 0.084 | 0.008 | 0.000 | 0.021 | 0.112 | 10.086 | 0.007 | 1.000 | 1.000 | 1.000 |
| Meritas Money Market Fund Series A | 0.046 | 1.586 | 0.001 | 6.409 | 0.050 | 0.007 | 0.000 | 4.303 | 1,000 | 1.000 | 1.000 | 1.000 |
| PH&N Community Values Balanced Fund Series D | 0.089 | −0.551 | 0.030 | 10.503 | 0.000 | 0.011 | 0.043 | 3.513 | 0.046 | 0.522 | 0.499 | 0.791 |
| PH&N Community Values Balanced Fund Series O | 0.155 | −0.541 | 0.030 | 1.135 | 0.000 | 0.003 | 0.042 | 3.490 | 0.085 | 1.000 | 1.000 | 1.000 |
| Meritas Balanced Portfolio Series A | −0.010 | −0.748 | 0.022 | 17.653 | 0.050 | 0.026 | 0.033 | 4.005 | 0.027 | 0.318 | 0.275 | 0.552 |

(continued)

**Table 4** (continued)

| Fund name | Return | Skew. | Std. Dev. | Total assets | Front-end load | MER | VaR 95 % | Kurt. | DEA-1 | DEA-2 | DEA-3 | DEA-4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| PH&N Community Values Balanced Fund Series B | 0.037 | −0.543 | 0.030 | 0.153 | 0.000 | 0.017 | 0.043 | 3.494 | 0.020 | 0.621 | 0.621 | 1.000 |
| Meritas Balanced Portfolio Series F | 0.181 | −0.404 | 0.024 | 0.191 | 0.000 | 0.016 | 0.032 | 3.876 | 0.128 | 0.681 | 0.673 | 1.000 |
| Ethical Special Equity Fund Series A | 0.189 | −0.891 | 0.049 | 249.960 | 0.050 | 0.027 | 0.108 | 4.211 | 0.064 | 0.174 | 0.142 | 0.512 |
| Ethical Special Equity Fund Series F | 0.284 | −0.897 | 0.049 | 4.735 | 0.000 | 0.017 | 0.107 | 4.208 | 0.099 | 0.373 | 0.291 | 0.643 |
| Ethical Global Equity Fund Class A | −0.328 | −0.242 | 0.045 | 32.054 | 0.050 | 0.026 | 0.078 | 2.573 | 0.014 | 0.237 | 0.216 | 0.790 |
| Acuity Social Values Global Equity Fund | −0.442 | −0.857 | 0.040 | 16.575 | 0.060 | 0.030 | 0.064 | 3.978 | 0.015 | 0.242 | 0.226 | 0.524 |
| MFS MB Responsible Global Research Fund | −0.275 | −0.402 | 0.040 | 74.033 | 0.000 | 0.019 | 0.063 | 2.572 | 0.015 | 0.288 | 0.265 | 0.815 |
| Mac Universal Sustainable Opportunities Class A | −0.440 | −0.737 | 0.049 | 19.276 | 0.050 | 0.024 | 0.112 | 3.123 | 0.012 | 0.249 | 0.207 | 0.661 |
| PH&N Community Values Global Equity Fund Series D | −0.306 | −0.411 | 0.049 | 6.010 | 0.000 | 0.014 | 0.103 | 3.036 | 0.012 | 0.399 | 0.315 | 0.837 |
| PH&N Community Values Global Equity Fund Series O | −0.218 | −0.386 | 0.049 | 47.560 | 0.000 | 0.002 | 0.102 | 3.051 | 0.012 | 0.785 | 0.725 | 0.938 |
| Ethical Global Equity Fund Class F | −0.230 | −0.248 | 0.045 | 0.943 | 0.000 | 0.016 | 0.077 | 2.570 | 0.014 | 0.497 | 0.492 | 0.951 |
| RBC Jantzi Global Equity Fund | −0.487 | −0.733 | 0.038 | 9.215 | 0.000 | 0.022 | 0.071 | 3.011 | 0.016 | 0.333 | 0.289 | 0.700 |
| RBC Jantzi Global Equity Fund Series D | −0.408 | −0.732 | 0.038 | 0.261 | 0.000 | 0.013 | 0.070 | 3.010 | 0.016 | 0.753 | 0.750 | 0.976 |
| RBC Jantzi Global Equity Fund Series F | −0.370 | −0.744 | 0.038 | 0.268 | 0.000 | 0.010 | 0.070 | 3.020 | 0.016 | 0.892 | 0.891 | 1.000 |

| | | | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| TD Global Sustainability Fund Investor Series | −0.944 | −1.048 | 0.047 | 9.994 | 0.000 | 0.024 | 0.094 | 4.562 | 0.013 | 0.291 | 0.245 | 0.527 |
| TD Global Sustainability Fund Advisor Series | −0.935 | −1.017 | 0.047 | 9.994 | 0.050 | 0.024 | 0.094 | 4.481 | 0.013 | 0.279 | 0.240 | 0.533 |
| TD Global Sustainability Fund—Series F | −0.823 | −1.002 | 0.047 | 9.871 | 0.000 | 0.012 | 0.094 | 4.430 | 0.013 | 0.413 | 0.340 | 0.651 |
| Ethical Global Dividend Fund Series A | −0.114 | −0.563 | 0.050 | 7.291 | 0.050 | 0.025 | 0.091 | 3.218 | 0.012 | 0.280 | 0.248 | 0.656 |
| Ethical Global Dividend Fund Series F | −0.027 | −0.567 | 0.050 | 0.321 | 0.000 | 0.016 | 0.091 | 3.222 | 0.012 | 0.588 | 0.587 | 0.844 |
| Investors Summa Global SRI Fund Series A | −0.507 | −0.443 | 0.049 | 7.433 | 0.000 | 0.027 | 0.081 | 3.434 | 0.012 | 0.282 | 0.256 | 0.620 |
| Investors Summa Global SRI Class Series A | −0.553 | −0.466 | 0.051 | 5.192 | 0.000 | 0.027 | 0.082 | 3.309 | 0.012 | 0.288 | 0.261 | 0.672 |
| Investors Summa Global SRI Fund Series C | −0.521 | −0.442 | 0.049 | 9.854 | 0.000 | 0.029 | 0.082 | 3.424 | 0.012 | 0.264 | 0.239 | 0.594 |
| PH&N Community Values Global Equity Fund Series F | −0.268 | −0.403 | 0.049 | 0.011 | 0.000 | 0.010 | 0.103 | 3.006 | 0.012 | 1.000 | 1.000 | 1.000 |
| Mac Universal Sustainable Opportunities Class T6 | −0.433 | −0.739 | 0.049 | 0.044 | 0.050 | 0.021 | 0.112 | 3.125 | 0.012 | 0.732 | 0.667 | 0.960 |
| Mac Universal Sustainable Opportunities Class T8 | −0.121 | −0.898 | 0.049 | 0.015 | 0.050 | 0.022 | 0.112 | 3.526 | 0.013 | 1.000 | 1.000 | 1.000 |
| RBC Jantzi Balanced Fund Series D | −0.060 | −1.153 | 0.025 | 0.488 | 0.000 | 0.015 | 0.039 | 4.300 | 0.025 | 0.650 | 0.638 | 0.784 |
| MFS MB Responsible Balanced Fund | 0.087 | −0.747 | 0.026 | 88.002 | 0.000 | 0.011 | 0.038 | 3.583 | 0.052 | 0.408 | 0.382 | 0.751 |
| Investors Summa Global Environ Leaders Fund Series A | −1.216 | −1.961 | 0.065 | 9.900 | 0.000 | 0.027 | 0.063 | 8.946 | 0.009 | 0.248 | 0.269 | 0.322 |
| Investors Summa Global Environ Leaders Class Series A | −1.240 | −1.943 | 0.065 | 3.451 | 0.000 | 0.027 | 0.063 | 8.890 | 0.009 | 0.267 | 0.293 | 0.357 |

(continued)

**Table 4** (continued)

| Fund name | Return | Skew. | Std. Dev. | Total assets | Front-end load | MER | VaR 95 % | Kurt. | DEA-1 | DEA-2 | DEA-3 | DEA-4 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Investors Summa Global Environ Leaders Fund Series C | −1.230 | −1.971 | 0.065 | 11.766 | 0.000 | 0.029 | 0.063 | 8.978 | 0.009 | 0.235 | 0.253 | 0.315 |
| Meritas International Equity Fund Series A | −0.592 | −0.472 | 0.040 | 25.087 | 0.050 | 0.029 | 0.068 | 3.066 | 0.015 | 0.230 | 0.211 | 0.663 |
| Ethical International Equity Fund Series A | −0.730 | −0.887 | 0.049 | 45.768 | 0.050 | 0.029 | 0.072 | 4.362 | 0.013 | 0.211 | 0.200 | 0.491 |
| Ethical International Equity Fund Series F | −0.640 | −0.880 | 0.049 | 1.018 | 0.050 | 0.019 | 0.071 | 4.338 | 0.013 | 0.436 | 0.432 | 0.665 |
| Meritas International Equity Fund Series F | −0.488 | −0.438 | 0.040 | 1.697 | 0.000 | 0.019 | 0.066 | 3.042 | 0.015 | 0.406 | 0.375 | 0.845 |
| Ethical American Multi-Strategy Fund Series A | 0.043 | 0.030 | 0.047 | 41.428 | 0.050 | 0.024 | 0.079 | 2.513 | 0.013 | 0.241 | 0.221 | 0.817 |
| Meritas U.S. Equity Fund | −0.319 | −0.332 | 0.042 | 10.519 | 0.050 | 0.029 | 0.083 | 3.056 | 0.015 | 0.263 | 0.229 | 0.665 |
| Ethical American Multi-Strategy Fund Series F | 0.149 | 0.048 | 0.046 | 0.306 | 0.000 | 0.014 | 0.077 | 2.494 | 0.053 | 0.681 | 0.681 | 1.000 |
| Meritas U.S. Equity Fund Series F | −0.217 | −0.322 | 0.042 | 2.026 | 0.000 | 0.012 | 0.082 | 3.052 | 0.015 | 0.499 | 0.424 | 0.897 |

This table lists the efficiency scores (columns 10–13) according to each of the four DEA models for 85 SRI mutual funds. Efficient funds have efficiency scores equal to one. However, inefficient funds are characterized by an efficiency rating of less than one

**Table 5** Comparison of the efficient set and the minimum and average efficiency scores obtained with DEA-1, DEA-2, DEA-3, and DEA-4 models

| Efficient funds | DEA-1 | DEA-2 | DEA-3 | DEA-4 |
|---|---|---|---|---|
| PH&N Community Values Canadian Equity Fund Ser O | | ● | ● | ● |
| PH&N Community Values Canadian Equity Fund Ser F | | | | ● |
| MFS MB Responsible Fixed Income Fund | | | ● | ● |
| PH&N Community Values Bond Fund Series O | ● | ● | ● | ● |
| PH&N Community Values Bond Fund Series B | | ● | ● | ● |
| PH&N Community Values Bond Fund Series F | | ● | ● | ● |
| Meritas Canadian Bond Fund Series F | | ● | ● | ● |
| Ethical Select Conservative Portfolio Class F | | ● | ● | ● |
| Acuity Pooled Social Values Canadian Equity Fund | | ● | ● | ● |
| Matrix Sierra Equity Fund Class F | | ● | ● | ● |
| Meritas Money Market Fund Series A | ● | ● | ● | ● |
| PH&N Community Values Balanced Fund Series O | | ● | ● | ● |
| PH&N Community Values Balanced Fund Series B | | | | ● |
| Meritas Balanced Portfolio Series F | | | | ● |
| RBC Jantzi Global Equity Fund Series F | | | | ● |
| PH&N Community Values Global Equity Fund Series F | | ● | ● | ● |
| Mac Universal Sustainable Opportunities Class T8 | | ● | ● | ● |
| Ethical American Multi-Strategy Fund Series F | | | | ● |
| *Number of efficient funds* | 2 | 12 | 13 | 18 |
| *Minimum score* | 0.007 | 0.142 | 0.086 | 0.300 |
| *Average score* | 0.125 | 0.499 | 0.478 | 0.747 |

This table presents the names and the number of efficient SRI funds according to the four DEA models. It also provides the minimum score and the average efficiency score for each model

In the third application (DEA-3), we introduce the value at risk as a measure of tail risk, instead of the standard deviation. The results show that the efficient set did not change considerably from the second application since only one additional fund is a member of the new efficient set.

In the fourth application (DEA-4), we incorporate higher-moment risk variables (kurtosis and skewness) into the analysis. The results improve in a substantial manner, where six additional funds turn out to be efficient, in comparison with the DEA-2 results. This would suggest the importance and the contribution of these higher moment variables in the assessment of the performance of our sampled funds. This last specification is consistent with the higher moment SDF model of Ayadi and Kryzanowski (2013) for the evaluation of Canadian domestic equity funds. For this extended DEA model, the efficient target values, efficient peer groups, optimal weights, and lambda values for the fourth DEA model are reported in Tables 6, 7, 8, and 9, respectively.

**Table 6** Target values of input and output variables for the fourth DEA model

| Fund name | Return | Skew. | Std. Dev. | Total assets | Front-end load | MER | Kurt. |
|---|---|---|---|---|---|---|---|
| Ethical Canadian Dividend Fund Series A | 0.004 | 0.015 | 0.009 | 5.726 | 0.000 | 0.017 | 2.032 |
| Ethical Canadian Dividend Fund Series F | 0.002 | −0.067 | 0.036 | 1.770 | 0.000 | 0.013 | 2.521 |
| GWL Ethics Fund (G) NL | 0.003 | −0.083 | 0.020 | 12.116 | 0.000 | 0.012 | 2.278 |
| GWL Ethics Fund (G) DSC | 0.003 | −0.099 | 0.022 | 13.238 | 0.000 | 0.012 | 2.319 |
| Meritas Jantzi Social Index Fund | 0.003 | −0.062 | 0.020 | 13.188 | 0.000 | 0.013 | 2.224 |
| London Life Ethics Fund (GWLIM) | 0.003 | −0.082 | 0.022 | 15.166 | 0.000 | 0.012 | 2.271 |
| MFS MB Responsible Canadian Equity Fund | 0.003 | −0.092 | 0.025 | 16.048 | 0.000 | 0.011 | 2.297 |
| PH&N Community Values Canadian Equity Fund Ser D | 0.002 | −0.103 | 0.026 | 17.052 | 0.000 | 0.011 | 2.323 |
| PH&N Community Values Canadian Equity Fund Ser O | 0.000 | −0.291 | 0.053 | 35.025 | 0.000 | 0.001 | 2.784 |
| Meritas Monthly Dividend and Income Fund | 0.004 | 0.015 | 0.009 | 5.726 | 0.000 | 0.017 | 2.032 |
| iShares Jantzi Social Index Fund | 0.001 | −0.335 | 0.038 | 17.809 | 0.000 | 0.004 | 2.926 |
| RBC Jantzi Canadian Equity Fund Series A | 0.003 | −0.089 | 0.021 | 12.956 | 0.000 | 0.012 | 2.292 |
| RBC Jantzi Canadian Equity Fund Series F | 0.001 | −0.403 | 0.036 | 0.654 | 0.000 | 0.008 | 3.163 |
| RBC Jantzi Canadian Equity Fund Series D | 0.002 | −0.240 | 0.036 | 0.917 | 0.000 | 0.009 | 2.953 |
| PH&N Community Values Canadian Equity Fund Ser B | 0.001 | −0.012 | 0.047 | 0.266 | 0.000 | 0.014 | 2.563 |
| PH&N Community Values Canadian Equity Fund Ser F | −0.001 | −0.300 | 0.053 | 0.223 | 0.000 | 0.011 | 2.792 |
| Meritas Jantzi Social Index Fund Series F | 0.003 | −0.313 | 0.022 | 3.486 | 0.000 | 0.008 | 2.891 |
| Meritas Monthly Dividend and Income Fund Series F | 0.002 | −0.110 | 0.036 | 0.815 | 0.000 | 0.014 | 2.687 |
| Acuity Social Values Balanced Fund | 0.004 | −0.126 | 0.017 | 20.737 | 0.000 | 0.012 | 2.358 |
| Ethical Balanced Fund Class A | 0.004 | −0.103 | 0.015 | 18.412 | 0.000 | 0.013 | 2.303 |

(continued)

**Table 6** (continued)

| Fund name | Return | Skew. | Std. Dev. | Total assets | Front-end load | MER | Kurt. |
|---|---|---|---|---|---|---|---|
| Ethical Balanced Fund Class F | 0.004 | −0.494 | 0.021 | 0.721 | 0.000 | 0.010 | 3.308 |
| NEI Canadian Bond Class A | 0.004 | 0.065 | 0.009 | 52.011 | 0.006 | 0.014 | 2.646 |
| Meritas Canadian Bond Fund | 0.004 | 0.053 | 0.009 | 5.742 | 0.001 | 0.016 | 2.087 |
| MFS MB Responsible Fixed Income Fund | 0.006 | −0.156 | 0.011 | 117.395 | 0.000 | 0.015 | 2.636 |
| PH&N Community Values Bond Fund Series D | 0.005 | −0.680 | 0.010 | 18.778 | 0.000 | 0.006 | 3.541 |
| PH&N Community Values Bond Fund Series O | 0.006 | −0.727 | 0.010 | 88.963 | 0.000 | 0.001 | 3.658 |
| NEI Canadian Bond Class F | 0.005 | −0.298 | 0.009 | 40.845 | 0.000 | 0.010 | 2.718 |
| PH&N Community Values Bond Fund Series B | 0.005 | −0.736 | 0.010 | 0.355 | 0.000 | 0.010 | 3.685 |
| PH&N Community Values Bond Fund Series F | 0.005 | −0.720 | 0.010 | 5.493 | 0.000 | 0.007 | 3.624 |
| Meritas Canadian Bond Fund Series F | 0.004 | 0.015 | 0.009 | 5.726 | 0.000 | 0.017 | 2.032 |
| Ethical Select Conservative Portfolio Class A | 0.004 | −0.092 | 0.009 | 30.970 | 0.004 | 0.011 | 2.773 |
| Ethical Select Conservative Portfolio Class F | 0.002 | −1.176 | 0.018 | 0.062 | 0.000 | 0.011 | 5.131 |
| Ethical Growth Fund Series A | 0.003 | −0.049 | 0.018 | 11.882 | 0.000 | 0.013 | 2.190 |
| Investors Summa SRI Fund Series C | 0.004 | −0.003 | 0.012 | 7.506 | 0.000 | 0.016 | 2.078 |
| Acuity Social Values Canadian Equity Fund | 0.003 | −0.061 | 0.020 | 13.093 | 0.000 | 0.013 | 2.221 |
| Investors Summa SRI Class A | 0.004 | −0.014 | 0.013 | 8.550 | 0.000 | 0.015 | 2.105 |
| Acuity Pooled Social Values Canadian Equity Fund | 0.000 | −1.122 | 0.064 | 8.659 | 0.050 | 0.002 | 5.133 |
| Investors Summa SRI Fund Series A | 0.004 | −0.018 | 0.014 | 8.958 | 0.000 | 0.015 | 2.115 |
| Ethical Growth Fund Series F | 0.002 | −0.216 | 0.037 | 0.841 | 0.000 | 0.010 | 2.908 |
| Alpha Social Values Portfolio | 0.003 | −0.128 | 0.016 | 5.734 | 0.000 | 0.013 | 2.407 |

(continued)

**Table 6** (continued)

| Fund name | Return | Skew. | Std. Dev. | Total assets | Front-end load | MER | Kurt. |
|---|---|---|---|---|---|---|---|
| Acuity Clean Environment Equity Fund | 0.003 | −0.061 | 0.020 | 13.031 | 0.000 | 0.013 | 2.220 |
| Matrix Sierra Equity Fund | 0.003 | −0.389 | 0.025 | 1.311 | 0.000 | 0.009 | 3.124 |
| Matrix Sierra Equity Fund Class F | −0.015 | −2.289 | 0.084 | 0.008 | 0.000 | 0.021 | 10.086 |
| Meritas Money Market Fund Series A | 0.000 | 1.586 | 0.001 | 6.409 | 0.050 | 0.007 | 4.302 |
| PH&N Community Values Balanced Fund Series D | 0.002 | −0.273 | 0.024 | 8.307 | 0.000 | 0.008 | 2.778 |
| PH&N Community Values Balanced Fund Series O | 0.002 | −0.541 | 0.030 | 1.135 | 0.000 | 0.003 | 3.490 |
| Meritas Balanced Portfolio Series A | 0.004 | −0.057 | 0.012 | 9.739 | 0.000 | 0.015 | 2.209 |
| PH&N Community Values Balanced Fund Series B | 0.000 | −0.543 | 0.030 | 0.153 | 0.000 | 0.017 | 3.494 |
| Meritas Balanced Portfolio Series F | 0.002 | −0.404 | 0.024 | 0.191 | 0.000 | 0.016 | 3.876 |
| Ethical Special Equity Fund Series A | 0.003 | −0.035 | 0.016 | 10.582 | 0.000 | 0.014 | 2.157 |
| Ethical Special Equity Fund Series F | 0.003 | −0.218 | 0.022 | 3.045 | 0.000 | 0.011 | 2.706 |
| Ethical Global Equity Fund Class A | 0.004 | 0.015 | 0.009 | 5.726 | 0.000 | 0.017 | 2.032 |
| Acuity Social Values Global Equity Fund | 0.004 | −0.005 | 0.012 | 7.699 | 0.000 | 0.016 | 2.083 |
| MFS MB Responsible Global Research Fund | 0.004 | −0.010 | 0.013 | 8.193 | 0.000 | 0.015 | 2.095 |
| Mac Universal Sustainable Opportunities Class A | 0.004 | 0.002 | 0.011 | 7.024 | 0.000 | 0.016 | 2.065 |
| PH&N Community Values Global Equity Fund Series D | 0.003 | −0.179 | 0.017 | 5.030 | 0.000 | 0.012 | 2.541 |
| PH&N Community Values Global Equity Fund Series O | 0.001 | −0.332 | 0.046 | 40.464 | 0.000 | 0.002 | 2.861 |
| Ethical Global Equity Fund Class F | 0.002 | 0.044 | 0.042 | 0.897 | 0.000 | 0.014 | 2.444 |
| RBC Jantzi Global Equity Fund | 0.004 | −0.014 | 0.011 | 6.454 | 0.000 | 0.016 | 2.109 |

(continued)

**Table 6** (continued)

| Fund name | Return | Skew. | Std. Dev. | Total assets | Front-end load | MER | Kurt. |
|---|---|---|---|---|---|---|---|
| RBC Jantzi Global Equity Fund Series D | 0.002 | −0.268 | 0.037 | 0.255 | 0.000 | 0.012 | 2.937 |
| RBC Jantzi Global Equity Fund Series F | −0.004 | −0.744 | 0.038 | 0.268 | 0.000 | 0.010 | 3.020 |
| TD Global Sustainability Fund—Investor Series | 0.003 | −0.126 | 0.015 | 5.264 | 0.000 | 0.013 | 2.403 |
| TD Global Sustainability Fund—Advisor Series | 0.003 | −0.120 | 0.015 | 5.324 | 0.000 | 0.013 | 2.387 |
| TD Global Sustainability Fund—Series F | 0.002 | −0.311 | 0.025 | 6.426 | 0.000 | 0.008 | 2.884 |
| Ethical Global Dividend Fund Series A | 0.004 | 0.021 | 0.015 | 4.786 | 0.000 | 0.016 | 2.112 |
| Ethical Global Dividend Fund Series F | 0.001 | −0.115 | 0.042 | 0.271 | 0.000 | 0.013 | 2.718 |
| Investors Summa Global SRI Fund Series A | 0.004 | 0.022 | 0.017 | 4.606 | 0.000 | 0.016 | 2.128 |
| Investors Summa Global SRI Class Series A | 0.003 | 0.029 | 0.024 | 3.488 | 0.000 | 0.016 | 2.223 |
| Investors Summa Global SRI Fund Series C | 0.004 | 0.015 | 0.009 | 5.726 | 0.000 | 0.017 | 2.032 |
| PH&N Community Values Global Equity Fund Series F | −0.003 | −0.403 | 0.049 | 0.011 | 0.000 | 0.010 | 3.006 |
| Mac Universal Sustainable Opportunities Class T6 | −0.002 | −0.388 | 0.048 | 0.042 | 0.000 | 0.010 | 3.000 |
| Mac Universal Sustainable Opportunities Class T8 | −0.001 | −0.898 | 0.049 | 0.015 | 0.050 | 0.022 | 3.526 |
| RBC Jantzi Balanced Fund Series D | 0.004 | −0.531 | 0.019 | 0.382 | 0.000 | 0.011 | 3.370 |
| MFS MB Responsible Balanced Fund | 0.004 | −0.276 | 0.019 | 37.263 | 0.000 | 0.008 | 2.689 |
| Investors Summa Global Environ Leaders Fund Ser A | 0.003 | −0.310 | 0.021 | 3.186 | 0.000 | 0.009 | 2.879 |
| Investors Summa Global Environ Leaders Class Ser A | 0.003 | −0.426 | 0.023 | 1.233 | 0.000 | 0.010 | 3.177 |
| Investors Summa Global Environ Leaders Fund Ser C | 0.003 | −0.289 | 0.020 | 3.705 | 0.000 | 0.009 | 2.827 |
| Meritas International Equity Fund Series A | 0.004 | 0.015 | 0.009 | 5.726 | 0.000 | 0.017 | 2.032 |

(continued)

**Table 6** (continued)

| Fund name | Return | Skew. | Std. Dev. | Total assets | Front-end load | MER | Kurt. |
|---|---|---|---|---|---|---|---|
| Ethical International Equity Fund Series A | 0.003 | −0.030 | 0.016 | 10.088 | 0.000 | 0.014 | 2.144 |
| Ethical International Equity Fund Series F | 0.003 | −0.224 | 0.032 | 0.677 | 0.000 | 0.012 | 2.884 |
| Meritas International Equity Fund Series F | 0.003 | −0.071 | 0.033 | 1.433 | 0.000 | 0.014 | 2.570 |
| Ethical American Multi-Strategy Fund Series A | 0.004 | 0.030 | 0.009 | 5.732 | 0.000 | 0.017 | 2.052 |
| Meritas U.S. Equity Fund | 0.004 | 0.015 | 0.009 | 5.726 | 0.000 | 0.017 | 2.032 |
| Ethical American Multi-Strategy Fund Series F | 0.001 | 0.048 | 0.046 | 0.306 | 0.000 | 0.014 | 2.495 |
| Meritas U.S. Equity Fund Series F | 0.001 | −0.268 | 0.037 | 1.819 | 0.000 | 0.011 | 2.739 |

This table presents the target values of all input and output variables under the fourth DEA application. The analysis uses the net return and skewness (Skew.) as outputs, and the fund return standard deviation (Std. Dev.), total assets, front-end load, management expense ratio (MER), and kurtosis of returns (Kurt.) as inputs. The data cover 85 SRI funds over the period May 2008 to December 2011

Twelve Canadian SRI funds are efficient under the DEA runs: DEA-2, DEA-3, and DEA-4. This persistency characterizes them as the best-performing SRI funds of the sample under evaluation. On the other hand, 67 funds are found to be inefficient in all DEA runs, which qualify them as the worst-performing funds of the sample.

In order to uncover the reasons for poor fund performance, we compute for each fund the input slack variables, which reflect the improvements needed for an inefficient fund to become efficient (Table 10). The investigation of these slack variables and the relative mean slacks shows that the size of the fund, measured by the total assets, and the loads are the major sources of inefficiency. In this regard, inefficient funds basically need to reduce their loads and size in order to improve their efficiency.

**Table 7** Peer groups of SRI funds for the fourth DEA model

| Fund name | Peer group |
|---|---|
| Ethical Canadian Dividend Fund Series A | Meritas Canadian Bond Fund Series F |
| Ethical Canadian Dividend Fund Series F | PH&N Community Values Canadian Equity Fund Ser F, Meritas Canadian Bond Fund Series F, PH&N Community Values Balanced Fund Series O, Ethical American Multi-Strategy Fund Series F |
| GWL Ethics Fund (G) NL | PH&N Community Values Canadian Equity Fund Ser O, Meritas Canadian Bond Fund Series F, PH&N Community Values Balanced Fund Series O |
| GWL Ethics Fund (G) DSC | PH&N Community Values Canadian Equity Fund Ser O, Meritas Canadian Bond Fund Series F, PH&N Community Values Balanced Fund Series O |
| Meritas Jantzi Social Index Fund | PH&N Community Values Canadian Equity Fund Ser O, Meritas Canadian Bond Fund Series F |
| London Life Ethics Fund (GWLIM) | PH&N Community Values Canadian Equity Fund Ser O, PH&N Community Values Bond Fund Series O, Meritas Canadian Bond Fund Series F |
| MFS MB Responsible Canadian Equity Fund | PH&N Community Values Canadian Equity Fund Ser O, Meritas Canadian Bond Fund Series F |
| PH&N Community Values Canadian Equity Fund Ser D | PH&N Community Values Canadian Equity Fund Ser O, Meritas Canadian Bond Fund Series F |
| PH&N Community Values Canadian Equity Fund Ser O | PH&N Community Values Canadian Equity Fund Ser O |
| Meritas Monthly Dividend and Income Fund | Meritas Canadian Bond Fund Series F |
| iShares Jantzi Social Index Fund | PH&N Community Values Canadian Equity Fund Ser O, Meritas Canadian Bond Fund Series F, PH&N Community Values Balanced Fund Series O |
| RBC Jantzi Canadian Equity Fund Series A | PH&N Community Values Canadian Equity Fund Ser O, Meritas Canadian Bond Fund Series F, PH&N Community Values Balanced Fund Series O |
| RBC Jantzi Canadian Equity Fund Series F | PH&N Community Values Canadian Equity Fund Ser F, PH&N Community Values Bond Fund Series B, PH&N Community Values Balanced Fund Series O, Ethical American Multi-Strategy Fund Series F |
| RBC Jantzi Canadian Equity Fund Series D | PH&N Community Values Bond Fund Series B, Meritas Canadian Bond Fund Series F, PH&N Community Values Balanced Fund Series O, Ethical American Multi-Strategy Fund Series F |
| PH&N Community Values Canadian Equity Fund Ser B | PH&N Community Values Global Equity Fund Series F, Ethical American Multi-Strategy Fund Series F |
| PH&N Community Values Canadian Equity Fund Ser F | PH&N Community Values Canadian Equity Fund Ser F |
| Meritas Jantzi Social Index Fund Series F | PH&N Community Values Canadian Equity Fund Ser O, Meritas Canadian Bond Fund Series F, PH&N Community Values Balanced Fund Series O |

(continued)

**Table 7** (continued)

| Fund name | Peer group |
|---|---|
| Meritas Monthly Dividend and Income Fund Series F | PH&N Community Values Bond Fund Series B, Meritas Canadian Bond Fund Series F, Ethical American Multi-Strategy Fund Series F |
| Acuity Social Values Balanced Fund | PH&N Community Values Canadian Equity Fund Ser O, PH&N Community Values Bond Fund Series O, Meritas Canadian Bond Fund Series F |
| Ethical Balanced Fund Class A | PH&N Community Values Canadian Equity Fund Ser O, PH&N Community Values Bond Fund Series O, Meritas Canadian Bond Fund Series F |
| Ethical Balanced Fund Class F | PH&N Community Values Bond Fund Series B, Meritas Canadian Bond Fund Series F, PH&N Community Values Balanced Fund Series O, Ethical American Multi-Strategy Fund Series F |
| NEI Canadian Bond Class A | MFS MB Responsible Fixed Income Fund, PH&N Community Values Bond Fund Series O, Meritas Canadian Bond Fund Series F, Meritas Money Market Fund Series A |
| Meritas Canadian Bond Fund | Meritas Canadian Bond Fund Series F, Meritas Money Market Fund Series A |
| MFS MB Responsible Fixed Income Fund | MFS MB Responsible Fixed Income Fund |
| PH&N Community Values Bond Fund Series D | PH&N Community Values Bond Fund Series O, PH&N Community Values Bond Fund Series F, Meritas Canadian Bond Fund Series F |
| PH&N Community Values Bond Fund Series O | PH&N Community Values Bond Fund Series O |
| NEI Canadian Bond Class F | PH&N Community Values Bond Fund Series O, Meritas Canadian Bond Fund Series F |
| PH&N Community Values Bond Fund Series B | PH&N Community Values Bond Fund Series B |
| PH&N Community Values Bond Fund Series F | PH&N Community Values Bond Fund Series F |
| Meritas Canadian Bond Fund Series F | Meritas Canadian Bond Fund Series F |
| Ethical Select Conservative Portfolio Class A | PH&N Community Values Bond Fund Series O, PH&N Community Values Bond Fund Series F, Meritas Canadian Bond Fund Series F, Meritas Money Market Fund Series A |
| Ethical Select Conservative Portfolio Class F | Ethical Select Conservative Portfolio Class F |
| Ethical Growth Fund Series A | PH&N Community Values Canadian Equity Fund Ser O, Meritas Canadian Bond Fund Series F |
| Investors Summa SRI Fund Series C | PH&N Community Values Canadian Equity Fund Ser O, Meritas Canadian Bond Fund Series F |
| Acuity Social Values Canadian Equity Fund | PH&N Community Values Canadian Equity Fund Ser O, Meritas Canadian Bond Fund Series F |
| Investors Summa SRI Class A | PH&N Community Values Canadian Equity Fund Ser O, Meritas Canadian Bond Fund Series F |

(continued)

**Table 7** (continued)

| Fund name | Peer group |
|---|---|
| Acuity Pooled Social Values Canadian Equity Fund | Acuity Pooled Social Values Canadian Equity Fund |
| Investors Summa SRI Fund Series A | PH&N Community Values Canadian Equity Fund Ser O, Meritas Canadian Bond Fund Series F |
| Ethical Growth Fund Series F | PH&N Community Values Bond Fund Series B, Meritas Canadian Bond Fund Series F, PH&N Community Values Balanced Fund Series O, Ethical American Multi-Strategy Fund Series F |
| Alpha Social Values Portfolio | PH&N Community Values Canadian Equity Fund Ser O, Meritas Canadian Bond Fund Series F, PH&N Community Values Balanced Fund Series O |
| Acuity Clean Environment Equity Fund | PH&N Community Values Canadian Equity Fund Ser O, Meritas Canadian Bond Fund Series F |
| Matrix Sierra Equity Fund | PH&N Community Values Bond Fund Series B, Meritas Canadian Bond Fund Series F, PH&N Community Values Balanced Fund Series O, Ethical American Multi-Strategy Fund Series F |
| Matrix Sierra Equity Fund Class F | Matrix Sierra Equity Fund Class F |
| Meritas Money Market Fund Series A | Meritas Money Market Fund Series A |
| PH&N Community Values Balanced Fund Series D | PH&N Community Values Canadian Equity Fund Ser O, PH&N Community Values Bond Fund Series O, Meritas Canadian Bond Fund Series F, PH&N Community Values Balanced Fund Series O |
| PH&N Community Values Balanced Fund Series O | PH&N Community Values Balanced Fund Series O |
| Meritas Balanced Portfolio Series A | PH&N Community Values Canadian Equity Fund Ser O, PH&N Community Values Bond Fund Series O, Meritas Canadian Bond Fund Series F, PH&N Community Values Balanced Fund Series O |
| PH&N Community Values Balanced Fund Series B | PH&N Community Values Balanced Fund Series B |
| Meritas Balanced Portfolio Series F | Meritas Balanced Portfolio Series F |
| Ethical Special Equity Fund Series A | PH&N Community Values Canadian Equity Fund Ser O, Meritas Canadian Bond Fund Series F |
| Ethical Special Equity Fund Series F | PH&N Community Values Bond Fund Series B, Meritas Canadian Bond Fund Series F, PH&N Community Values Balanced Fund Series O, Ethical American Multi-Strategy Fund Series F |
| Ethical Global Equity Fund Class A | Meritas Canadian Bond Fund Series F |
| Acuity Social Values Global Equity Fund | PH&N Community Values Canadian Equity Fund Ser O, Meritas Canadian Bond Fund Series F |
| MFS MB Responsible Global Research Fund | PH&N Community Values Canadian Equity Fund Ser O, Meritas Canadian Bond Fund Series F |

(continued)

**Table 7** (continued)

| Fund name | Peer group |
|---|---|
| Mac Universal Sustainable Opportunities Class A | PH&N Community Values Canadian Equity Fund Ser O, Meritas Canadian Bond Fund Series F |
| PH&N Community Values Global Equity Fund Series D | PH&N Community Values Canadian Equity Fund Ser O, Meritas Canadian Bond Fund Series F, PH&N Community Values Balanced Fund Series O |
| PH&N Community Values Global Equity Fund Series O | PH&N Community Values Canadian Equity Fund Ser O, PH&N Community Values Bond Fund Series O, Meritas Canadian Bond Fund Series F |
| Ethical Global Equity Fund Class F | Meritas Canadian Bond Fund Series F, Ethical American Multi-Strategy Fund Series F |
| RBC Jantzi Global Equity Fund | PH&N Community Values Canadian Equity Fund Ser O, Meritas Canadian Bond Fund Series F, PH&N Community Values Balanced Fund Series O |
| RBC Jantzi Global Equity Fund Series D | PH&N Community Values Bond Fund Series B, PH&N Community Values Global Equity Fund Series F, Ethical American Multi-Strategy Fund Series F |
| RBC Jantzi Global Equity Fund Series F | RBC Jantzi Global Equity Fund Series F |
| TD Global Sustainability Fund—Investor Series | PH&N Community Values Canadian Equity Fund Ser O, Meritas Canadian Bond Fund Series F, PH&N Community Values Balanced Fund Series O |
| TD Global Sustainability Fund—Advisor Series | PH&N Community Values Canadian Equity Fund Ser O, Meritas Canadian Bond Fund Series F, PH&N Community Values Balanced Fund Series O |
| TD Global Sustainability Fund—Series F | PH&N Community Values Canadian Equity Fund Ser O, Meritas Canadian Bond Fund Series F, PH&N Community Values Balanced Fund Series O |
| Ethical Global Dividend Fund Series A | Meritas Canadian Bond Fund Series F, Ethical American Multi-Strategy Fund Series F |
| Ethical Global Dividend Fund Series F | PH&N Community Values Bond Fund Series B, PH&N Community Values Global Equity Fund Series F, Ethical American Multi-Strategy Fund Series F |
| Investors Summa Global SRI Fund Series A | Meritas Canadian Bond Fund Series F, Ethical American Multi-Strategy Fund Series F |
| Investors Summa Global SRI Class Series A | Meritas Canadian Bond Fund Series F, Ethical American Multi-Strategy Fund Series F |
| Investors Summa Global SRI Fund Series C | Meritas Canadian Bond Fund Series F |
| PH&N Community Values Global Equity Fund Series F | PH&N Community Values Global Equity Fund Series F |
| Mac Universal Sustainable Opportunities Class T6 | PH&N Community Values Bond Fund Series B, PH&N Community Values Global Equity Fund Series F, Ethical American Multi-Strategy Fund Series F |
| Mac Universal Sustainable Opportunities Class T8 | Ethical Select Conservative Portfolio Class F, PH&N Community Values Global Equity Fund Series F, Mac Universal Sustainable Opportunities Class T8 |

(continued)

**Table 7** (continued)

| Fund name | Peer group |
|---|---|
| RBC Jantzi Balanced Fund Series D | PH&N Community Values Bond Fund Series B, Meritas Canadian Bond Fund Series F, Ethical American Multi-Strategy Fund Series F |
| MFS MB Responsible Balanced Fund | PH&N Community Values Canadian Equity Fund Ser O, PH&N Community Values Bond Fund Series O, Meritas Canadian Bond Fund Series F |
| Investors Summa Global Environ Leaders Fund Ser A | PH&N Community Values Bond Fund Series O, PH&N Community Values Bond Fund Series F, Meritas Canadian Bond Fund Series F, PH&N Community Values Balanced Fund Series O |
| Investors Summa Global Environ Leaders Class Ser A | PH&N Community Values Bond Fund Series B, Meritas Canadian Bond Fund Series F, PH&N Community Values Balanced Fund Series O, Ethical American Multi-Strategy Fund Series F |
| Investors Summa Global Environ Leaders Fund Ser C | PH&N Community Values Bond Fund Series O, PH&N Community Values Bond Fund Series F, Meritas Canadian Bond Fund Series F, PH&N Community Values Balanced Fund Series O |
| Meritas International Equity Fund Series A | Meritas Canadian Bond Fund Series F |
| Ethical International Equity Fund Series A | PH&N Community Values Canadian Equity Fund Ser O, Meritas Canadian Bond Fund Series F |
| Ethical International Equity Fund Series F | PH&N Community Values Bond Fund Series B, Meritas Canadian Bond Fund Series F, PH&N Community Values Balanced Fund Series O, Ethical American Multi-Strategy Fund Series F |
| Meritas International Equity Fund Series F | PH&N Community Values Bond Fund Series B, Meritas Canadian Bond Fund Series F, Ethical American Multi-Strategy Fund Series F |
| Ethical American Multi-Strategy Fund Series A | Meritas Canadian Bond Fund Series F, Meritas Money Market Fund Series A |
| Meritas U.S. Equity Fund | Meritas Canadian Bond Fund Series F |
| Ethical American Multi-Strategy Fund Series F | Ethical American Multi-Strategy Fund Series F |
| Meritas U.S. Equity Fund Series F | PH&N Community Values Canadian Equity Fund Ser F, Meritas Canadian Bond Fund Series F, PH&N Community Values Balanced Fund Series O |

This table presents efficient peers for each SRI funds under the fourth DEA application. The analysis uses the net return and skewness (Skew.) as outputs, and the fund return standard deviation (Std. Dev.), total assets, front-end load, management expense ratio (MER), and kurtosis of fund returns (Kurt.) as inputs. The data cover 85 SRI funds over the period May 2008 to December 2011

**Table 8** Optimal weights in the fourth DEA model

| Fund name | Return | Skew. | Std. dev. | Total assets | Front-end load | MER | Kurt. |
|---|---|---|---|---|---|---|---|
| Ethical Canadian Dividend Fund Series A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.348 |
| Ethical Canadian Dividend Fund Series F | 0.000 | 0.000 | 0.485 | 0.014 | 0.000 | 20.042 | 0.228 |
| GWL Ethics Fund (G) NL | 0.000 | 0.000 | 0.000 | 0.003 | 0.000 | 11.077 | 0.114 |
| GWL Ethics Fund (G) DSC | 0.000 | 0.000 | 0.000 | 0.003 | 0.000 | 11.278 | 0.117 |
| Meritas Jantzi Social Index Fund | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 11.297 | 0.228 |
| London Life Ethics Fund (GWLIM) | 0.000 | 0.000 | 2.087 | 0.000 | 0.000 | 11.411 | 0.108 |
| MFS MB Responsible Canadian Equity Fund | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 11.131 | 0.225 |
| PH&N Community Values Canadian Equity Fund Ser D | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 14.357 | 0.290 |
| PH&N Community Values Canadian Equity Fund Ser O | 0.000 | 0.000 | 0.000 | 0.007 | 0.000 | 25.605 | 0.265 |
| Meritas Monthly Dividend and Income Fund | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.351 |
| iShares Jantzi Social Index Fund | 0.000 | 0.000 | 0.000 | 0.006 | 0.000 | 21.957 | 0.227 |
| RBC Jantzi Canadian Equity Fund Series A | 0.000 | 0.000 | 0.000 | 0.004 | 0.000 | 14.942 | 0.154 |
| RBC Jantzi Canadian Equity Fund Series F | 24.162 | 0.000 | 0.000 | 0.159 | 0.000 | 25.190 | 0.161 |
| RBC Jantzi Canadian Equity Fund Series D | 0.000 | 0.000 | 4.027 | 0.037 | 0.000 | 10.804 | 0.160 |
| PH&N Community Values Canadian Equity Fund Ser B | 0.000 | 0.000 | 0.000 | 0.523 | 0.000 | 0.000 | 0.302 |
| PH&N Community Values Canadian Equity Fund Ser F | 0.333 | 0.000 | 0.479 | 0.016 | 0.000 | 22.914 | 0.260 |
| Meritas Jantzi Social Index Fund Series F | 0.000 | 0.000 | 0.000 | 0.006 | 0.000 | 21.878 | 0.226 |
| Meritas Monthly Dividend and Income Fund Series F | 0.000 | 0.000 | 7.592 | 0.072 | 0.000 | 0.000 | 0.227 |
| Acuity Social Values Balanced Fund | 0.000 | 0.000 | 1.989 | 0.000 | 0.000 | 10.871 | 0.103 |

(continued)

**Table 8** (continued)

| Fund name | Return | Skew. | Std. dev. | Total assets | Front-end load | MER | Kurt. |
|---|---|---|---|---|---|---|---|
| Ethical Balanced Fund Class A | 0.000 | 0.000 | 2.806 | 0.000 | 0.000 | 15.341 | 0.145 |
| Ethical Balanced Fund Class F | 0.000 | 0.000 | 4.432 | 0.040 | 0.000 | 11.890 | 0.176 |
| NEI Canadian Bond Class A | 90.757 | 0.000 | 60.811 | 0.000 | 0.000 | 4.831 | 0.099 |
| Meritas Canadian Bond Fund | 0.000 | 0.000 | 60.580 | 0.000 | 0.000 | 0.000 | 0.221 |
| MFS MB Responsible Fixed Income Fund | 78.533 | 0.000 | 0.000 | 0.000 | 0.000 | 18.883 | 0.272 |
| PH&N Community Values Bond Fund Series D | 94.956 | 0.000 | 96.206 | 0.001 | 9.164 | 0.000 | 0.000 |
| PH&N Community Values Bond Fund Series O | 48.812 | 0.000 | 0.000 | 0.002 | 0.000 | 27.728 | 0.215 |
| NEI Canadian Bond Class F | 0.000 | 0.000 | 91.339 | 0.000 | 16.676 | 8.124 | 0.000 |
| PH&N Community Values Bond Fund Series B | 3.641 | 0.000 | 3.612 | 0.132 | 0.000 | 21.034 | 0.192 |
| PH&N Community Values Bond Fund Series F | 54.413 | 0.000 | 0.000 | 0.020 | 0.000 | 24.239 | 0.201 |
| Meritas Canadian Bond Fund Series F | 0.333 | 0.000 | 0.478 | 0.016 | 0.000 | 22.877 | 0.259 |
| Ethical Select Conservative Portfolio Class A | 0.000 | 0.000 | 7.054 | 0.001 | 0.000 | 15.201 | 0.088 |
| Ethical Select Conservative Portfolio Class F | 0.000 | 0.000 | 4.454 | 0.515 | 0.000 | 67.819 | 0.028 |
| Ethical Growth Fund Series A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 10.039 | 0.203 |
| Investors Summa SRI Fund Series C | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 9.481 | 0.192 |
| Acuity Social Values Canadian Equity Fund | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 7.472 | 0.151 |
| Investors Summa SRI Class A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 9.702 | 0.196 |
| Acuity Pooled Social Values Canadian Equity Fund | 0.000 | 0.000 | 0.000 | 0.013 | 0.000 | 305.185 | 0.046 |
| Investors Summa SRI Fund Series A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 9.632 | 0.195 |

(continued)

**Table 8** (continued)

| Fund name | Return | Skew. | Std. dev. | Total assets | Front-end load | MER | Kurt. |
|---|---|---|---|---|---|---|---|
| Ethical Growth Fund Series F | 0.000 | 0.000 | 4.129 | 0.038 | 0.000 | 11.079 | 0.164 |
| Alpha Social Values Portfolio | 0.000 | 0.000 | 0.000 | 0.003 | 0.000 | 11.550 | 0.119 |
| Acuity Clean Environment Equity Fund | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 7.534 | 0.152 |
| Matrix Sierra Equity Fund | 0.000 | 0.000 | 1.607 | 0.015 | 0.000 | 4.311 | 0.064 |
| Matrix Sierra Equity Fund Class F | 0.000 | 0.000 | 6.291 | 63.162 | 0.000 | 0.000 | 0.000 |
| Meritas Money Market Fund Series A | 0.000 | 0.135 | 0.000 | 0.006 | 0.000 | 23.267 | 0.184 |
| PH&N Community Values Balanced Fund Series D | 0.000 | 0.000 | 4.697 | 0.001 | 0.000 | 24.272 | 0.167 |
| PH&N Community Values Balanced Fund Series O | 0.333 | 0.000 | 0.479 | 0.016 | 0.000 | 22.896 | 0.260 |
| Meritas Balanced Portfolio Series A | 0.000 | 0.000 | 3.276 | 0.001 | 0.000 | 16.929 | 0.116 |
| PH&N Community Values Balanced Fund Series B | 0.000 | 0.000 | 10.804 | 0.920 | 0.000 | 0.000 | 0.153 |
| Meritas Balanced Portfolio Series F | 0.000 | 0.177 | 10.154 | 0.652 | 0.000 | 0.000 | 0.164 |
| Ethical Special Equity Fund Series A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 8.877 | 0.180 |
| Ethical Special Equity Fund Series F | 32.571 | 0.000 | 0.000 | 0.022 | 0.000 | 14.696 | 0.154 |
| Ethical Global Equity Fund Class A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.389 |
| Acuity Social Values Global Equity Fund | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 9.076 | 0.184 |
| MFS MB Responsible Global Research Fund | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 14.123 | 0.286 |
| Mac Universal Sustainable Opportunities Class A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 11.462 | 0.232 |
| PH&N Community Values Global Equity Fund Series D | 0.000 | 0.000 | 0.000 | 0.005 | 0.000 | 21.428 | 0.221 |

(continued)

**Table 8** (continued)

| Fund name | Return | Skew. | Std. dev. | Total assets | Front-end load | MER | Kurt. |
|---|---|---|---|---|---|---|---|
| PH&N Community Values Global Equity Fund Series O | 0.000 | 0.000 | 4.583 | 0.000 | 0.000 | 25.052 | 0.237 |
| Ethical Global Equity Fund Class F | 0.000 | 0.000 | 0.000 | 0.032 | 0.000 | 0.000 | 0.377 |
| RBC Jantzi Global Equity Fund | 0.000 | 0.000 | 0.000 | 0.005 | 0.000 | 17.934 | 0.185 |
| RBC Jantzi Global Equity Fund Series D | 0.000 | 0.000 | 7.298 | 0.422 | 0.000 | 0.000 | 0.204 |
| RBC Jantzi Global Equity Fund Series F | 0.000 | 0.000 | 2.735 | 0.094 | 0.000 | 21.578 | 0.216 |
| TD Global Sustainability Fund—Investor Series | 0.000 | 0.000 | 0.000 | 0.003 | 0.000 | 13.486 | 0.139 |
| TD Global Sustainability Fund—Advisor Series | 0.000 | 0.000 | 0.000 | 0.003 | 0.000 | 13.639 | 0.141 |
| TD Global Sustainability Fund—Series F | 0.000 | 0.000 | 0.000 | 0.004 | 0.000 | 16.668 | 0.172 |
| Ethical Global Dividend Fund Series A | 0.000 | 0.000 | 0.000 | 0.022 | 0.000 | 0.000 | 0.260 |
| Ethical Global Dividend Fund Series F | 0.000 | 0.000 | 6.311 | 0.365 | 0.000 | 0.000 | 0.176 |
| Investors Summa Global SRI Fund Series A | 0.000 | 0.000 | 0.000 | 0.021 | 0.000 | 0.000 | 0.246 |
| Investors Summa Global SRI Class Series A | 0.000 | 0.000 | 0.000 | 0.023 | 0.000 | 0.000 | 0.267 |
| Investors Summa Global SRI Fund Series C | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.292 |
| PH&N Community Values Global Equity Fund Series F | 0.000 | 0.008 | 2.893 | 0.104 | 0.000 | 21.990 | 0.211 |
| Mac Universal Sustainable Opportunities Class T6 | 0.000 | 0.000 | 7.181 | 0.416 | 0.000 | 0.000 | 0.200 |
| Mac Universal Sustainable Opportunities Class T8 | 76.562 | 0.000 | 4.867 | 1.887 | 0.000 | 0.000 | 0.209 |
| RBC Jantzi Balanced Fund Series D | 0.000 | 0.000 | 6.328 | 0.060 | 0.000 | 0.000 | 0.189 |
| MFS MB Responsible Balanced Fund | 0.000 | 0.000 | 3.667 | 0.000 | 0.000 | 20.048 | 0.189 |

(continued)

**Table 8** (continued)

| Fund name | Return | Skew. | Std. dev. | Total assets | Front-end load | MER | Kurt. |
|---|---|---|---|---|---|---|---|
| Investors Summa Global Environ Leaders Fund Ser A | 0.000 | 0.000 | 2.669 | 0.001 | 0.000 | 10.247 | 0.061 |
| Investors Summa Global Environ Leaders Class Ser A | 0.000 | 0.000 | 1.917 | 0.017 | 0.000 | 5.142 | 0.076 |
| Investors Summa Global Environ Leaders Fund Ser C | 0.000 | 0.000 | 2.611 | 0.001 | 0.000 | 10.026 | 0.060 |
| Meritas International Equity Fund Series A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.326 |
| Ethical International Equity Fund Series A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 8.519 | 0.172 |
| Ethical International Equity Fund Series F | 0.000 | 0.000 | 3.566 | 0.033 | 0.000 | 9.567 | 0.142 |
| Meritas International Equity Fund Series F | 0.000 | 0.000 | 6.818 | 0.065 | 0.000 | 0.000 | 0.204 |
| Ethical American Multi-Strategy Fund Series A | 0.000 | 0.575 | 0.000 | 0.000 | 0.000 | 0.000 | 0.398 |
| Meritas U.S. Equity Fund | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.327 |
| Ethical American Multi-Strategy Fund Series F | 0.333 | 0.000 | 0.479 | 0.016 | 0.000 | 22.897 | 0.260 |
| Meritas U.S. Equity Fund Series F | 0.000 | 0.000 | 0.000 | 0.010 | 0.000 | 21.744 | 0.237 |

This table presents the optimal weights in the fourth DEA application. The analysis uses the net return and skewness (Skew.) as outputs, and the fund return standard deviation (Std. Dev.), total assets, front-end load, management expense ratio (MER), and kurtosis of fund returns (Kurt.) as inputs. The data cover 85 SRI funds over the period May 2008 to December 2011

These results present some advantages to either potential investors or mutual fund managers. On the one hand, they help investors identify the best-performing SRI mutual funds and offer insight into the factors they should consider when investing in SRI mutual funds. On the other hand, the results help mutual-fund managers to identify which of their peers are outperforming them, and what are the success factors for SRI funds in order to improve their operational behaviour.

**Table 9** Lambda values in the fourth DEA model

| Fund name | PH&N Community Values Canadian Equity Fund Ser O | PH&N Community Values Canadian Equity Fund Ser F | MFS MB Responsible Fixed Income Fund | PH&N Community Values Bond Fund Series O | PH&N Community Values Bond Fund Series B | PH&N Community Values Bond Fund Series F | Meritas Canadian Bond Fund Series F | Ethical Select Conservative Portfolio Class F | Acuity Pooled Social Values Canadian Equity Fund |
|---|---|---|---|---|---|---|---|---|---|
| Ethical Canadian Dividend Fund Series A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 |
| Ethical Canadian Dividend Fund Series F | 0.000 | 0.122 | 0.000 | 0.000 | 0.000 | 0.000 | 0.255 | 0.000 | 0.000 |
| GWL Ethics Fund (G) NL | 0.226 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.722 | 0.000 | 0.000 |
| GWL Ethics Fund (G) DSC | 0.266 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.674 | 0.000 | 0.000 |
| Meritas Jantzi Social Index Fund | 0.255 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.745 | 0.000 | 0.000 |
| London Life Ethics Fund (GWLIM) | 0.306 | 0.000 | 0.000 | 0.006 | 0.000 | 0.000 | 0.689 | 0.000 | 0.000 |
| MFS MB Responsible Canadian Equity Fund | 0.352 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.648 | 0.000 | 0.000 |
| PH&N Community Values Canadian Equity Fund Ser D | 0.387 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.613 | 0.000 | 0.000 |
| PH&N Community Values Canadian Equity Fund Ser O | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

(continued)

**Table 9** (continued)

| Fund name | PH&N Community Values Canadian Equity Fund Ser O | PH&N Community Values Canadian Equity Fund Ser F | MFS MB Responsible Fixed Income Fund | PH&N Community Values Bond Fund Series O | PH&N Community Values Bond Fund Series B | PH&N Community Values Bond Fund Series F | Meritas Canadian Bond Fund Series F | Ethical Select Conservative Portfolio Class F | Acuity Pooled Social Values Canadian Equity Fund |
|---|---|---|---|---|---|---|---|---|---|
| Meritas Monthly Dividend and Income Fund | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 |
| iShares Jantzi Social Index Fund | 0.470 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.159 | 0.000 | 0.000 |
| RBC Jantzi Canadian Equity Fund Series A | 0.254 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.698 | 0.000 | 0.000 |
| RBC Jantzi Canadian Equity Fund Series F | 0.000 | 0.266 | 0.000 | 0.000 | 0.128 | 0.000 | 0.000 | 0.000 | 0.000 |
| RBC Jantzi Canadian Equity Fund Series D | 0.000 | 0.000 | 0.000 | 0.000 | 0.036 | 0.000 | 0.045 | 0.000 | 0.000 |
| PH&N Community Values Canadian Equity Fund Ser B | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| PH&N Community Values Canadian Equity Fund Ser F | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Meritas Jantzi Social Index Fund Series F | 0.015 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.403 | 0.000 | 0.000 |
| Meritas Monthly Dividend and Income Fund Series F | 0.000 | 0.000 | 0.000 | 0.000 | 0.197 | 0.000 | 0.092 | 0.000 | 0.000 |
| Acuity Social Values Balanced Fund | 0.182 | 0.000 | 0.000 | 0.116 | 0.000 | 0.000 | 0.701 | 0.000 | 0.000 |
| Ethical Balanced Fund Class A | 0.131 | 0.000 | 0.000 | 0.106 | 0.000 | 0.000 | 0.763 | 0.000 | 0.000 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Ethical Balanced Fund Class F | 0.000 | 0.000 | 0.000 | 0.000 | 0.574 | 0.000 | 0.048 | 0.000 | 0.000 |
| NEI Canadian Bond Class A | 0.000 | 0.000 | 0.345 | 0.092 | 0.000 | 0.000 | 0.450 | 0.000 | 0.000 |
| Meritas Canadian Bond Fund | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.976 | 0.000 | 0.000 |
| MFS MB Responsible Fixed Income Fund | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| PH&N Community Values Bond Fund Series D | 0.000 | 0.000 | 0.000 | 0.159 | 0.000 | 0.785 | 0.056 | 0.000 | 0.000 |
| PH&N Community Values Bond Fund Series O | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| NEI Canadian Bond Class F | 0.000 | 0.000 | 0.000 | 0.422 | 0.000 | 0.000 | 0.578 | 0.000 | 0.000 |
| PH&N Community Values Bond Fund Series B | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| PH&N Community Values Bond Fund Series F | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 |
| Meritas Canadian Bond Fund Series F | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 |
| Ethical Select Conservative Portfolio Class A | 0.000 | 0.000 | 0.000 | 0.303 | 0.000 | 0.030 | 0.579 | 0.000 | 0.000 |
| Ethical Select Conservative Portfolio Class F | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 |
| Ethical Growth Fund Series A | 0.210 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.790 | 0.000 | 0.000 |

(continued)

**Table 9** (continued)

| Fund name | PH&N Community Values Canadian Equity Fund Ser O | PH&N Community Values Canadian Equity Fund Ser F | MFS MB Responsible Fixed Income Fund | PH&N Community Values Bond Fund Series O | PH&N Community Values Bond Fund Series B | PH&N Community Values Bond Fund Series F | Meritas Canadian Bond Fund Series F | Ethical Select Conservative Portfolio Class F | Acuity Pooled Social Values Canadian Equity Fund |
|---|---|---|---|---|---|---|---|---|---|
| Investors Summa SRI Fund Series C | 0.061 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.939 | 0.000 | 0.000 |
| Acuity Social Values Canadian Equity Fund | 0.251 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.749 | 0.000 | 0.000 |
| Investors Summa SRI Class A | 0.096 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.904 | 0.000 | 0.000 |
| Acuity Pooled Social Values Canadian Equity Fund | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 |
| Investors Summa SRI Fund Series A | 0.110 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.890 | 0.000 | 0.000 |
| Ethical Growth Fund Series F | 0.000 | 0.000 | 0.000 | 0.000 | 0.068 | 0.000 | 0.044 | 0.000 | 0.000 |
| Alpha Social Values Portfolio | 0.038 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.725 | 0.000 | 0.000 |
| Acuity Clean Environment Equity Fund | 0.249 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.751 | 0.000 | 0.000 |
| Matrix Sierra Equity Fund | 0.000 | 0.000 | 0.000 | 0.000 | 0.306 | 0.000 | 0.133 | 0.000 | 0.000 |
| Matrix Sierra Equity Fund Class F | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Meritas Money Market Fund Series A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| PH&N Community Values Balanced Fund Series D | 0.130 | 0.000 | 0.000 | 0.009 | 0.000 | 0.000 | 0.426 | 0.000 | 0.000 |
| PH&N Community Values Balanced Fund Series O | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Meritas Balanced Portfolio Series A | 0.046 | 0.000 | 0.000 | 0.035 | 0.000 | 0.000 | 0.860 | 0.000 | 0.000 |
| PH&N Community Values Balanced Fund Series B | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Meritas Balanced Portfolio Series F | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Ethical Special Equity Fund Series A | 0.166 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.834 | 0.000 | 0.000 |
| Ethical Special Equity Fund Series F | 0.000 | 0.000 | 0.000 | 0.000 | 0.049 | 0.000 | 0.449 | 0.000 | 0.000 |
| Ethical Global Equity Fund Class A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 |
| Acuity Social Values Global Equity Fund | 0.067 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.933 | 0.000 | 0.000 |
| MFS MB Responsible Global Research Fund | 0.084 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.916 | 0.000 | 0.000 |
| Mac Universal Sustainable Opportunities Class A | 0.044 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.956 | 0.000 | 0.000 |
| PH&N Community Values Global Equity Fund Series D | 0.029 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.637 | 0.000 | 0.000 |
| PH&N Community Values Global Equity Fund Series O | 0.836 | 0.000 | 0.000 | 0.123 | 0.000 | 0.000 | 0.041 | 0.000 | 0.000 |

(continued)

**Table 9** (continued)

| Fund name | PH&N Community Values Canadian Equity Fund Ser O | PH&N Community Values Canadian Equity Fund Ser F | MFS MB Responsible Fixed Income Fund | PH&N Community Values Bond Fund Series O | PH&N Community Values Bond Fund Series B | PH&N Community Values Bond Fund Series F | Meritas Canadian Bond Fund Series F | Ethical Select Conservative Portfolio Class F | Acuity Pooled Social Values Canadian Equity Fund |
|---|---|---|---|---|---|---|---|---|---|
| Ethical Global Equity Fund Class F | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.109 | 0.000 | 0.000 |
| RBC Jantzi Global Equity Fund | 0.031 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.933 | 0.000 | 0.000 |
| RBC Jantzi Global Equity Fund Series D | 0.000 | 0.000 | 0.000 | 0.000 | 0.278 | 0.000 | 0.000 | 0.000 | 0.000 |
| RBC Jantzi Global Equity Fund Series F | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| TD Global Sustainability Fund —Investor Series | 0.022 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.735 | 0.000 | 0.000 |
| TD Global Sustainability Fund —Advisor Series | 0.023 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.746 | 0.000 | 0.000 |
| TD Global Sustainability Fund —Series F | 0.107 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.364 | 0.000 | 0.000 |
| Ethical Global Dividend Fund Series A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.827 | 0.000 | 0.000 |
| Ethical Global Dividend Fund Series F | 0.000 | 0.000 | 0.000 | 0.000 | 0.129 | 0.000 | 0.000 | 0.000 | 0.000 |
| Investors Summa Global SRI Fund Series A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.793 | 0.000 | 0.000 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| Investors Summa Global SRI Class Series A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.587 | 0.000 | 0.000 |
| Investors Summa Global SRI Fund Series C | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 |
| PH&N Community Values Global Equity Fund Series F | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Mac Universal Sustainable Opportunities Class T6 | 0.000 | 0.000 | 0.000 | 0.000 | 0.038 | 0.000 | 0.000 | 0.000 | 0.000 |
| Mac Universal Sustainable Opportunities Class T8 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| RBC Jantzi Balanced Fund Series D | 0.000 | 0.000 | 0.000 | 0.000 | 0.739 | 0.000 | 0.007 | 0.000 | 0.000 |
| MFS MB Responsible Balanced Fund | 0.226 | 0.000 | 0.000 | 0.299 | 0.000 | 0.000 | 0.475 | 0.000 | 0.000 |
| Investors Summa Global Environ Leaders Fund Ser A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.019 | 0.421 | 0.000 | 0.000 |
| Investors Summa Global Environ Leaders Class Ser A | 0.000 | 0.000 | 0.000 | 0.000 | 0.392 | 0.000 | 0.125 | 0.000 | 0.000 |

(continued)

**Table 9** (continued)

| Fund name | PH&N Community Values Canadian Equity Fund Ser O | PH&N Community Values Canadian Equity Fund Ser F | MFS MB Responsible Fixed Income Fund | PH&N Community Values Bond Fund Series O | PH&N Community Values Bond Fund Series B | PH&N Community Values Bond Fund Series F | Meritas Canadian Bond Fund Series F | Ethical Select Conservative Portfolio Class F | Acuity Pooled Social Values Canadian Equity Fund |
|---|---|---|---|---|---|---|---|---|---|
| Investors Summa Global Environ Leaders Fund Ser C | 0.000 | 0.000 | 0.000 | 0.005 | 0.000 | 0.001 | 0.455 | 0.000 | 0.000 |
| Meritas International Equity Fund Series A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 |
| Ethical International Equity Fund Series A | 0.149 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.851 | 0.000 | 0.000 |
| Ethical International Equity Fund Series F | 0.000 | 0.000 | 0.000 | 0.000 | 0.304 | 0.000 | 0.057 | 0.000 | 0.000 |
| Meritas International Equity Fund Series F | 0.000 | 0.000 | 0.000 | 0.000 | 0.143 | 0.000 | 0.207 | 0.000 | 0.000 |
| Ethical American Multi-Strategy Fund Series A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.991 | 0.000 | 0.000 |
| Meritas U.S. Equity Fund | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 |
| Ethical American Multi-Strategy Fund Series F | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Meritas U.S. Equity Fund Series F | 0.000 | 0.540 | 0.000 | 0.000 | 0.000 | 0.000 | 0.256 | 0.000 | 0.000 |

| Fund name | Matrix Sierra Equity Fund Class F | Meritas Money Market Fund Series A | PH&N Community Values Balanced Fund Series O | PH&N Community Values Balanced Fund Series B | Meritas Balanced Portfolio Series F | RBC Jantzi Global Equity Fund Series F | PH&N Community Values Global Equity Fund Series F | Mac Universal Sustainable Opportunities Class T8 | Ethical American Multi-Strategy Fund Series F |
|---|---|---|---|---|---|---|---|---|---|
| Ethical Canadian Dividend Fund Series A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Ethical Canadian Dividend Fund Series F | 0.000 | 0.000 | 0.109 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.513 |
| GWL Ethics Fund (G) NL | 0.000 | 0.000 | 0.052 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| GWL Ethics Fund (G) DSC | 0.000 | 0.000 | 0.060 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Meritas Jantzi Social Index Fund | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| London Life Ethics Fund (GWLIM) | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| MFS MB Responsible Canadian Equity Fund | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| PH&N Community Values Canadian Equity Fund Ser D | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| PH&N Community Values Canadian Equity Fund Ser O | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Meritas Monthly Dividend and Income Fund | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| iShares Jantzi Social Index Fund | 0.000 | 0.000 | 0.371 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| RBC Jantzi Canadian Equity Fund Series A | 0.000 | 0.000 | 0.047 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| RBC Jantzi Canadian Equity Fund Series F | 0.000 | 0.000 | 0.439 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.167 |

(continued)

**Table 9** (continued)

| Fund name | Matrix Sierra Equity Fund Class F | Meritas Money Market Fund Series A | PH&N Community Values Balanced Fund Series O | PH&N Community Values Balanced Fund Series B | Meritas Balanced Portfolio Series F | RBC Jantzi Global Equity Fund Series F | PH&N Community Values Global Equity Fund Series F | Mac Universal Sustainable Opportunities Class T8 | Ethical American Multi-Strategy Fund Series F |
|---|---|---|---|---|---|---|---|---|---|
| RBC Jantzi Canadian Equity Fund Series D | 0.000 | 0.000 | 0.439 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.480 |
| PH&N Community Values Canadian Equity Fund Ser B | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.133 | 0.000 | 0.867 |
| PH&N Community Values Canadian Equity Fund Ser F | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Meritas Jantzi Social Index Fund Series F | 0.000 | 0.000 | 0.582 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Meritas Monthly Dividend and Income Fund Series F | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.710 |
| Acuity Social Values Balanced Fund | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Ethical Balanced Fund Class A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Ethical Balanced Fund Class F | 0.000 | 0.000 | 0.153 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.225 |
| NEI Canadian Bond Class A | 0.000 | 0.113 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Meritas Canadian Bond Fund | 0.000 | 0.024 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| MFS MB Responsible Fixed Income Fund | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| PH&N Community Values Bond Fund Series D | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| PH&N Community Values Bond Fund Series O | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| NEI Canadian Bond Class F | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| PH&N Community Values Bond Fund Series B | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| PH&N Community Values Bond Fund Series F | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Meritas Canadian Bond Fund Series F | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Ethical Select Conservative Portfolio Class A | 0.000 | 0.089 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Ethical Select Conservative Portfolio Class F | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Ethical Growth Fund Series A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Investors Summa SRI Fund Series C | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Acuity Social Values Canadian Equity Fund | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Investors Summa SRI Class A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Acuity Pooled Social Values Canadian Equity Fund | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Investors Summa SRI Fund Series A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Ethical Growth Fund Series F | 0.000 | 0.000 | 0.354 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.533 |
| Alpha Social Values Portfolio | 0.000 | 0.000 | 0.238 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Acuity Clean Environment Equity Fund | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Matrix Sierra Equity Fund | 0.000 | 0.000 | 0.328 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.234 |
| Matrix Sierra Equity Fund Class F | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Meritas Money Market Fund Series A | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

(continued)

**Table 9** (continued)

| Fund name | Matrix Sierra Equity Fund Class F | Meritas Money Market Fund Series A | PH&N Community Values Balanced Fund Series O | PH&N Community Values Balanced Fund Series B | Meritas Balanced Portfolio Series F | RBC Jantzi Global Equity Fund Series F | PH&N Community Values Global Equity Fund Series F | Mac Universal Sustainable Opportunities Class T8 | Ethical American Multi-Strategy Fund Series F |
|---|---|---|---|---|---|---|---|---|---|
| PH&N Community Values Balanced Fund Series D | 0.000 | 0.000 | 0.434 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| PH&N Community Values Balanced Fund Series O | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Meritas Balanced Portfolio Series A | 0.000 | 0.000 | 0.059 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| PH&N Community Values Balanced Fund Series B | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Meritas Balanced Portfolio Series F | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Ethical Special Equity Fund Series A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Ethical Special Equity Fund Series F | 0.000 | 0.000 | 0.363 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.139 |
| Ethical Global Equity Fund Class A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Acuity Social Values Global Equity Fund | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| MFS MB Responsible Global Research Fund | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Mac Universal Sustainable Opportunities Class A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| PH&N Community Values Global Equity Fund Series D | 0.000 | 0.000 | 0.334 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| PH&N Community Values Global Equity Fund Series O | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Ethical Global Equity Fund Class F | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.891 |
| RBC Jantzi Global Equity Fund | 0.000 | 0.000 | 0.037 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| RBC Jantzi Global Equity Fund Series D | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.218 | 0.000 | 0.504 |
| RBC Jantzi Global Equity Fund Series F | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 | 0.000 |
| TD Global Sustainability Fund—Investor Series | 0.000 | 0.000 | 0.243 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| TD Global Sustainability Fund—Advisor Series | 0.000 | 0.000 | 0.232 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| TD Global Sustainability Fund—Series F | 0.000 | 0.000 | 0.529 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Ethical Global Dividend Fund Series A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.173 |
| Ethical Global Dividend Fund Series F | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.138 | 0.000 | 0.733 |
| Investors Summa Global SRI Fund Series A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.207 |
| Investors Summa Global SRI Class Series A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.413 |
| Investors Summa Global SRI Fund Series C | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| PH&N Community Values Global Equity Fund Series F | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 | 0.000 |
| Mac Universal Sustainable Opportunities Class T6 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.901 | 0.000 | 0.061 |
| Mac Universal Sustainable Opportunities Class T8 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 | 0.000 |
| RBC Jantzi Balanced Fund Series D | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.254 |

(continued)

**Table 9** (continued)

| Fund name | Matrix Sierra Equity Fund Class F | Meritas Money Market Fund Series A | PH&N Community Values Balanced Fund Series O | PH&N Community Values Balanced Fund Series B | Meritas Balanced Portfolio Series F | RBC Jantzi Global Equity Fund Series F | PH&N Community Values Global Equity Fund Series F | Mac Universal Sustainable Opportunities Class T8 | Ethical American Multi-Strategy Fund Series F |
|---|---|---|---|---|---|---|---|---|---|
| MFS MB Responsible Balanced Fund | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Investors Summa Global Environ Leaders Fund Ser A | 0.000 | 0.000 | 0.560 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Investors Summa Global Environ Leaders Class Ser A | 0.000 | 0.000 | 0.276 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.207 |
| Investors Summa Global Environ Leaders Fund Ser C | 0.000 | 0.000 | 0.539 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Meritas International Equity Fund Series A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Ethical International Equity Fund Series A | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Ethical International Equity Fund Series F | 0.000 | 0.000 | 0.054 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.584 |
| Meritas International Equity Fund Series F | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.650 |
| Ethical American Multi-Strategy Fund Series A | 0.000 | 0.009 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Meritas U.S. Equity Fund | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |
| Ethical American Multi-Strategy Fund Series F | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 1.000 |
| Meritas U.S. Equity Fund Series F | 0.000 | 0.000 | 0.203 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 | 0.000 |

This table presents the lambdas values in the fourth DEA application where 18 SRI funds are efficient. The analysis uses the net return and skewness (Skew.) as outputs, and the fund return standard deviation (Std. Dev.), total assets, front-end load, management expense ratio (MER), and kurtosis of fund returns (Kurt.) as inputs. The data cover 85 SRI funds over the period May 2008 to December 2011

**Table 10** Slacks of input and output variables for the fourth DEA model

| Fund name | Return | Skew. | Std. Dev. | Total assets | Front-end load | MER | Kurt. |
|---|---|---|---|---|---|---|---|
| Ethical Canadian Dividend Fund Series A | 0.004 | 0.722 | 0.032 | 244.062 | 0.050 | 0.009 | 0.837 |
| Ethical Canadian Dividend Fund Series F | 0.001 | 0.636 | 0.005 | 0.241 | 0.000 | 0.002 | 0.343 |
| GWL Ethics Fund (G) NL | 0.006 | 1.097 | 0.033 | 15.890 | 0.000 | 0.016 | 2.988 |
| GWL Ethics Fund (G) DSC | 0.005 | 1.080 | 0.031 | 16.819 | 0.000 | 0.015 | 2.947 |
| Meritas Jantzi Social Index Fund | 0.005 | 0.457 | 0.035 | 59.220 | 0.050 | 0.007 | 1.188 |
| London Life Ethics Fund (GWLIM) | 0.005 | 1.108 | 0.030 | 68.482 | 0.000 | 0.016 | 3.046 |
| MFS MB Responsible Canadian Equity Fund | 0.004 | 0.501 | 0.030 | 62.581 | 0.000 | 0.006 | 1.280 |
| PH&N Community Values Canadian Equity Fund Ser D | 0.003 | 0.198 | 0.027 | 5.376 | 0.000 | 0.002 | 0.482 |
| Meritas Monthly Dividend and Income Fund | 0.004 | 0.577 | 0.029 | 16.629 | 0.050 | 0.008 | 0.818 |
| iShares Jantzi Social Index Fund | 0.002 | 0.178 | 0.018 | 2.959 | 0.000 | 0.001 | 0.486 |
| RBC Jantzi Canadian Equity Fund Series A | 0.002 | 0.584 | 0.027 | 9.246 | 0.000 | 0.009 | 1.636 |
| RBC Jantzi Canadian Equity Fund Series F | 0.000 | 0.270 | 0.012 | 0.159 | 0.000 | 0.002 | 0.767 |
| RBC Jantzi Canadian Equity Fund Series D | 0.001 | 0.434 | 0.012 | 0.304 | 0.000 | 0.003 | 0.980 |
| PH&N Community Values Canadian Equity Fund Ser B | 0.002 | 0.292 | 0.007 | 0.026 | 0.000 | 0.005 | 0.247 |
| Meritas Jantzi Social Index Fund Series F | 0.004 | 0.194 | 0.033 | 0.594 | 0.000 | 0.001 | 0.493 |
| Meritas Monthly Dividend and Income Fund Series F | 0.001 | 0.455 | 0.002 | 0.052 | 0.000 | 0.001 | 0.170 |
| Acuity Social Values Balanced Fund | 0.003 | 1.127 | 0.025 | 42.194 | 0.060 | 0.017 | 3.436 |
| Ethical Balanced Fund Class A | 0.005 | 0.849 | 0.011 | 317.889 | 0.050 | 0.010 | 1.707 |
| Ethical Balanced Fund Class F | 0.004 | 0.457 | 0.004 | 0.152 | 0.000 | 0.002 | 0.695 |
| NEI Canadian Bond Class A | 0.000 | 0.512 | 0.001 | 201.956 | 0.044 | 0.002 | 0.446 |
| Meritas Canadian Bond Fund | 0.000 | 0.079 | 0.000 | 32.751 | 0.049 | 0.003 | 0.022 |
| PH&N Community Values Bond Fund Series D | 0.000 | 0.055 | 0.000 | 0.124 | 0.000 | 0.000 | 0.133 |

(continued)

**Table 10** (continued)

| Fund name | Return | Skew. | Std. Dev. | Total assets | Front-end load | MER | Kurt. |
|---|---|---|---|---|---|---|---|
| NEI Canadian Bond Class F | 0.000 | 0.133 | 0.001 | 8.698 | 0.000 | 0.001 | 0.477 |
| Ethical Select Conservative Portfolio Class A | 0.003 | 1.125 | 0.009 | 31.028 | 0.046 | 0.011 | 2.778 |
| Ethical Growth Fund Series A | 0.006 | 0.796 | 0.029 | 295.491 | 0.050 | 0.010 | 1.591 |
| Investors Summa SRI Fund Series C | 0.009 | 0.584 | 0.059 | 1111.299 | 0.000 | 0.013 | 1.721 |
| Acuity Social Values Canadian Equity Fund | 0.005 | 1.061 | 0.044 | 26.407 | 0.060 | 0.017 | 2.931 |
| Investors Summa SRI Class A | 0.009 | 0.545 | 0.056 | 36.929 | 0.000 | 0.012 | 1.655 |
| Investors Summa SRI Fund Series A | 0.009 | 0.564 | 0.057 | 1115.516 | 0.000 | 0.012 | 1.691 |
| Ethical Growth Fund Series F | 0.004 | 0.628 | 0.011 | 0.251 | 0.000 | 0.003 | 0.869 |
| Alpha Social Values Portfolio | 0.007 | 1.196 | 0.035 | 6.977 | 0.060 | 0.015 | 2.929 |
| Acuity Clean Environment Equity Fund | 0.013 | 1.267 | 0.045 | 67.150 | 0.060 | 0.017 | 2.887 |
| Matrix Sierra Equity Fund | 0.018 | 1.964 | 0.059 | 3.064 | 0.050 | 0.022 | 7.302 |
| PH&N Community Values Balanced Fund Series D | 0.002 | 0.279 | 0.006 | 2.196 | 0.000 | 0.002 | 0.734 |
| Meritas Balanced Portfolio Series A | 0.004 | 0.691 | 0.010 | 7.914 | 0.050 | 0.012 | 1.795 |
| Ethical Special Equity Fund Series A | 0.001 | 0.856 | 0.033 | 239.378 | 0.050 | 0.013 | 2.054 |
| Ethical Special Equity Fund Series F | 0.000 | 0.679 | 0.027 | 1.690 | 0.000 | 0.006 | 1.502 |
| Ethical Global Equity Fund Class A | 0.007 | 0.257 | 0.036 | 26.328 | 0.050 | 0.009 | 0.541 |
| Acuity Social Values Global Equity Fund | 0.008 | 0.852 | 0.028 | 8.875 | 0.060 | 0.014 | 1.895 |
| MFS MB Responsible Global Research Fund | 0.006 | 0.391 | 0.027 | 65.839 | 0.000 | 0.003 | 0.476 |
| Mac Universal Sustainable Opportunities Class A | 0.008 | 0.739 | 0.039 | 12.252 | 0.050 | 0.008 | 1.058 |
| PH&N Community Values Global Equity Fund Series D | 0.006 | 0.232 | 0.032 | 0.981 | 0.000 | 0.002 | 0.495 |
| PH&N Community Values Global Equity Fund Series O | 0.003 | 0.054 | 0.003 | 7.096 | 0.000 | 0.000 | 0.190 |
| Ethical Global Equity Fund Class F | 0.004 | 0.292 | 0.002 | 0.046 | 0.000 | 0.001 | 0.126 |

(continued)

**Table 10** (continued)

| Fund name | Return | Skew. | Std. Dev. | Total assets | Front-end load | MER | Kurt. |
|---|---|---|---|---|---|---|---|
| RBC Jantzi Global Equity Fund | 0.009 | 0.719 | 0.027 | 2.761 | 0.000 | 0.007 | 0.902 |
| RBC Jantzi Global Equity Fund Series D | 0.006 | 0.464 | 0.001 | 0.006 | 0.000 | 0.001 | 0.073 |
| TD Global Sustainability Fund—Investor Series | 0.013 | 0.922 | 0.032 | 4.731 | 0.000 | 0.012 | 2.159 |
| TD Global Sustainability Fund—Advisor Series | 0.013 | 0.896 | 0.032 | 4.671 | 0.050 | 0.011 | 2.094 |
| TD Global Sustainability Fund—Series F | 0.011 | 0.690 | 0.022 | 3.446 | 0.000 | 0.004 | 1.546 |
| Ethical Global Dividend Fund Series A | 0.005 | 0.585 | 0.034 | 2.505 | 0.050 | 0.009 | 1.106 |
| Ethical Global Dividend Fund Series F | 0.002 | 0.452 | 0.008 | 0.050 | 0.000 | 0.003 | 0.503 |
| Investors Summa Global SRI Fund Series A | 0.009 | 0.465 | 0.033 | 2.827 | 0.000 | 0.011 | 1.306 |
| Investors Summa Global SRI Class Series A | 0.009 | 0.495 | 0.026 | 1.704 | 0.000 | 0.011 | 1.086 |
| Investors Summa Global SRI Fund Series C | 0.009 | 0.458 | 0.040 | 4.128 | 0.000 | 0.012 | 1.392 |
| Mac Universal Sustainable Opportunities Class T6 | 0.002 | 0.351 | 0.002 | 0.002 | 0.050 | 0.010 | 0.125 |
| RBC Jantzi Balanced Fund Series D | 0.005 | 0.622 | 0.005 | 0.105 | 0.000 | 0.004 | 0.929 |
| MFS MB Responsible Balanced Fund | 0.003 | 0.471 | 0.006 | 50.739 | 0.000 | 0.003 | 0.894 |
| Investors Summa Global Environ Leaders Fund Ser A | 0.015 | 1.651 | 0.044 | 6.714 | 0.000 | 0.018 | 6.066 |
| Investors Summa Global Environ Leaders Class Ser A | 0.016 | 1.517 | 0.041 | 2.218 | 0.000 | 0.017 | 5.713 |
| Investors Summa Global Environ Leaders Fund Ser C | 0.015 | 1.682 | 0.044 | 8.061 | 0.000 | 0.020 | 6.151 |
| Meritas International Equity Fund Series A | 0.010 | 0.487 | 0.031 | 19.362 | 0.050 | 0.012 | 1.034 |
| Ethical International Equity Fund Series A | 0.011 | 0.857 | 0.033 | 35.679 | 0.050 | 0.015 | 2.218 |
| Ethical International Equity Fund Series F | 0.009 | 0.656 | 0.016 | 0.341 | 0.050 | 0.006 | 1.454 |
| Meritas International Equity Fund Series F | 0.007 | 0.367 | 0.006 | 0.264 | 0.000 | 0.005 | 0.473 |
| Ethical American Multi-Strategy Fund Series A | 0.004 | 0.000 | 0.038 | 35.696 | 0.050 | 0.008 | 0.460 |

(continued)

**Table 10** (continued)

| Fund name | Return | Skew. | Std. Dev. | Total assets | Front-end load | MER | Kurt. |
|---|---|---|---|---|---|---|---|
| Meritas U.S. Equity Fund | 0.007 | 0.348 | 0.033 | 4.793 | 0.050 | 0.012 | 1.024 |
| Meritas U.S. Equity Fund Series F | 0.003 | 0.054 | 0.005 | 0.208 | 0.000 | 0.001 | 0.313 |

This table presents the slacks values for all input and output variables under the fourth DEA application. The analysis uses the net return and skewness (Skew.) as outputs, and the fund return standard deviation (Std. Dev.), total assets, front-end load, management expense ratio (MER), and kurtosis of returns (Kurt.) as inputs. The data cover 67 inefficient SRI funds over the period May 2008 to December 2011

# 5 Conclusion

The present chapter uses the non-parametric technique of data envelopment analysis (DEA) to investigate the efficiency of 85 Canadian SRI funds during the period of 2008–2011. It extends the previous research in at least two ways. First, and so far as we are aware, it represents the first attempt to apply the DEA to assess the performance of SRI mutual funds in Canada. Moreover, by specifically focusing on the input slacks, measured using the DEA, this chapter offers insights into specific aspects of managerial behaviour that can be improved, rather than merely addressing the summary efficiency score. The evidence suggests that loads and the SRI fund's size are the main sources of inefficiency.

There are at least three ways in which this research could be extended. First, we can use DEA models that can highlight changes in the efficiency of SRI funds over the years. A second extension would be to compare the results of DEA with those of parametric frontier analysis methods. Finally, we can develop advanced models that would include both efficiency and effectiveness components into the performance analysis.

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# Scalarization Methods in Multiobjective Optimization, Robustness, Risk Theory and Finance

**Akhtar A. Khan, Elisabeth Köbis, and Christiane Tammer**

**Abstract** We show that scalarization techniques are very important tools in several fields of mathematics, especially in multiobjective optimization, uncertain optimization, risk theory and finance. Specifically, we consider randomness in scalar optimization problems and explore important connections between a nonlinear scalarization technique, robust optimization and coherent risk measures. Furthermore, we discuss a new model for a Private Equity Fund based on stochastic differential equations. In order to find efficient strategies for the fund manager we formulate a stochastic multiobjective optimization problem for a Private Equity Fund. Using a special case of the nonlinear scalarization technique, the $\varepsilon$-constraint method, we solve this stochastic multiobjective optimization problem.

**Keywords** Multiobjective Optimization • Scalarization • Uncertain Optimization • Robust Optimization • Coherent Risk Measures • Private Equity Fund • Stochastic Differential Equations

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# 1 Introduction

Scalarization techniques are very important tools in many fields of mathematics, especially in multiobjective optimization and for the description of robust counterpart problems in uncertain optimization. Nonlinear scalarizing functionals are very useful in nonlinear functional analysis and optimization theory as separating functionals, especially in the nonconvex case. Furthermore, nonlinear scalarizing functionals are used in many applications like in the definition of coherent risk measures in Mathematical Finance, in Decision Making and in Production Theory (*benefit function* and *shortage function*) associated to the production possibility set.

Many real-world problems require the optimization of conflicting goals. For instance, portfolio optimization (see, e.g., Markowitz 1952) deals with the issue of finding an optimal combination of assets such that the portfolio generates high revenue while the risk is being minimal. Another example for multiple objective optimization is the shortest-path problem (compare, for instance, Martins 1984), which describes the problem of finding a path with minimal distance and minimal costs at the same time. If a planner is not only dealing with one, but several conflicting goals, one speaks of multiobjective optimization, where several objective functions are minimized in parallel. Due to a lack of a total order in $\mathbb{R}^q$ ($q \geq 2$), there usually does not exist just one solution with minimal value, but a whole set, which is sometimes called Pareto-frontier. It is usually quite difficult to generate all minimal solutions of a multiobjective optimization problem if the problem is large or non-convex.

Scalarization describes the process of replacing a multiobjective optimization problem by a family of problems, each with just one objective function, which should be easier to solve. Under convexity assumptions, one can use well-known scalarization techniques like the weighted-sum scalarization to obtain all minimal solutions. If no convexity assumption is provided, it is usually more difficult to retrieve all minimal solutions. In this chapter, we use a nonlinear scalarization method which enables us to obtain all minimal solutions of multiobjective optimization problems under no convexity assumptions, as will be shown in Sect. 2. Certain properties of the nonlinear scalarizing functionals like monotonicity, convexity and continuity are very important for a characterization of solutions of multiobjective optimization problems, for deriving optimality conditions and solution procedures.

In this chapter, we study a nonlinear scalarization technique by means of the functional $\varphi_{D,k^0} : Y \to \mathbb{R} \cup \{+\infty\} \cup \{-\infty\} =: \overline{\mathbb{R}}$ defined by

$$\varphi_{D,k^0}(y) := \inf\{t \in \mathbb{R} \mid y \in t\,k^0 - D\}, \tag{1}$$

where $Y$ is a linear topological space, $k^0 \in Y \setminus \{0\}$, $D$ is a proper closed subset of $Y$ with

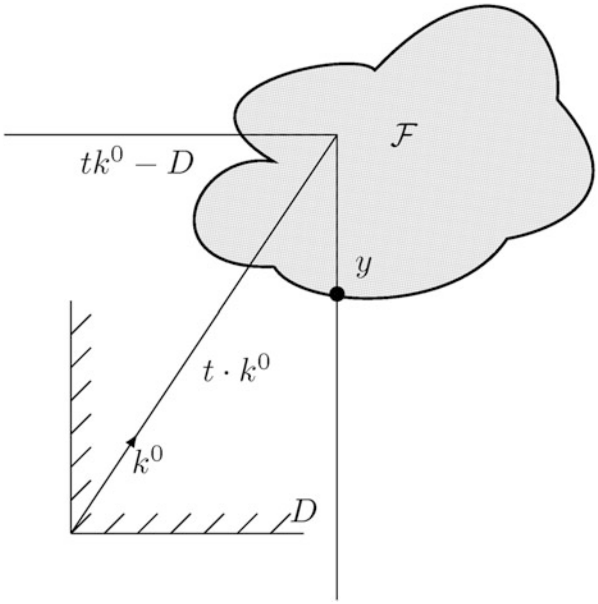


**Fig. 1** Illustration of the functional $\varphi_{D,k^0}(y) := \inf\{t \in \mathbb{R} \mid y \in t\,k^0 - D\}$

$$D + [0, +\infty) \cdot k^0 \subset D. \tag{2}$$

An illustration of the functional $\varphi_{D,k^0}$ for an element $y \in \mathcal{F} \subset Y$ is presented in Fig. 1.

Originally, the scalarizing functional $\varphi_{D,k^0}$ was used in Gerth and Weidner (1990) to prove separation theorems for not necessarily convex sets. Applications of $\varphi_{D,k^0}$ include coherent risk measures in financial mathematics (see, for instance, Heyde 2006), as we will show in detail in Sect. 4.

Monotonicity, translation invariance and continuity properties of the functional $\varphi_{D,k^0}$ were intensely studied by Gerth and Weidner (1990), and later in Weidner (1990) and Göpfert et al. (2003). Further important properties of the functional $\varphi_{D,k^0}$, for example the sublinearity, were shown in Göpfert et al. (2003) under certain additional assumptions concerning the set $D$.

The aim of this chapter is to show that special cases of the functional (1) are useful as scalarization methods in multiobjective optimization (Sect. 2), for deriving concepts of robustness (Sect. 3) and for a description of coherent risk measures (Sect. 4). Furthermore, scalarization methods by means of the functional (1) are used for finding efficient strategies in finance, especially for Private Equity Funds (Sect. 5). This chapter is devoted to highlighting the significant connections between uncertain optimization, nonlinear scalarization, coherent risk measures and the optimization of Private Equity Funds.

## 2 Multiobjective Optimization and $\varepsilon$-Constraint Method

Here we are dealing with multiobjective optimization problems in the following setting. Let $f : X \to Y$ with $X$ being a linear space and $Y$ a linear topological space. Then the minimization problem is denoted by

$$\min_{x \in \mathcal{X}} f(x), \tag{VP}$$

where $\mathcal{X} \subset X$ and the set of **minimal solutions** of (VP) is defined by

$$\mathrm{Min}(\mathcal{F}, D) := \left\{\overline{y} \in \mathcal{F} \;\middle|\; \mathcal{F} \cap (\overline{y} - D) = \{\overline{y}\}\right\},$$

where $\mathcal{F} := f[\mathcal{X}] := \cup_{x\in \mathcal{X}} f(x)$ is a proper subset of $Y$ and $D$ is a proper pointed closed convex cone in $Y$. If additionally $\mathrm{int}D \neq \emptyset$, then the set of **weakly minimal solutions** of (VP) is denoted by

$$\mathrm{Min}(\mathcal{F}, \mathrm{int}D) := \left\{\overline{y} \in \mathcal{F} \;\middle|\; \mathcal{F} \cap (\overline{y} - \mathrm{int}D) = \emptyset\right\}.$$

It is important to mention the monotonicity properties that $\varphi_{D,k^0}$ satisfies and that immediately connect to multiobjective optimization. Therefore, $\varphi_{D,k^0}$ given by (1) is used to scalarize multiobjective optimization problems and the scalarization problem reads

$$\min_{y \in \mathcal{F}} \varphi_{D,k^0}(y). \tag{$P_{k^0,D,\mathcal{F}}$}$$

For instance, weighted sums, Pascoletti-Serafini scalarization, weighted Tschebyscheff and the $\varepsilon$-constraint scalarization can be expressed by the functional $\varphi_{D,k^0}$ and are therefore comprised by the above described scalarization model $\left(P_{k^0,D,\mathcal{F}}\right)$. We show that the $\varepsilon$-constraint method can be described using $\varphi_{D,k^0}$ for the special case $Y = \mathbb{R}^q$, $f : \mathbb{R}^n \to \mathbb{R}^q$: Let some $j \in \{1, \ldots, q\}$ and some real values $\varepsilon_i$, $i = 1, \ldots, q$, $i \neq j$ be given. Then the **$\varepsilon$-constraint scalarization** (see Haimes et al. 1971; Chankong and Haimes 1983; Eichfelder 2008) is given by the functional $\varphi_{D,k^0}$ (see (1)) with

$$D := \mathbb{R}^q_+ - \overline{b}, \quad \text{with} \quad \overline{b} = \left(\overline{b}_1, \ldots, \overline{b}_q\right)^T, \overline{b}_i = \begin{cases} 0 & \text{for} \quad i = j, \\ \varepsilon_i & \text{for} \quad i \neq j, \end{cases} \tag{3}$$

$$k^0 = \left(k^0_1, \ldots, k^0_q\right)^T, \quad \text{where} \quad k^0_i = \begin{cases} 1 & \text{for} \quad i = j, \\ 0 & \text{for} \quad i \neq j. \end{cases} \tag{4}$$

With these parameters $D$ and $k^0$, the $\varepsilon$-constraint problem reads

$$\begin{aligned} &\min \ f_j(x) \\ &\text{subject to} \quad f_i(x) \le \varepsilon_i, \ i = 1, \ldots, q, \ i \ne j, \\ &\qquad\qquad x \in \mathbb{R}^n. \end{aligned} \tag{5}$$

It will be shown in Proposition 2.3 below how optimal solutions of (5) relate to the set of (weakly) minimal solutions of the multiobjective optimization problem ($VP$) (with $Y = \mathbb{R}^q$). To this end, we need to prove the monotonicity properties that the functional $\varphi_{D,k^0}$ (with parameters $D$ and $k^0$ given by (3) and (4)) satisfies.

The $\varepsilon$-constraint scalarization method will be used in Sect. 5 in order to calculate solutions of a Private Equity Fund model.

Below we show that the functional $\varphi_{D,k^0}$ with the above parameters $D$ and $k^0$ is $\mathbb{R}^q_+$-monotone (i.e., $y_1 \in y_2 - \mathbb{R}^q_+ \Rightarrow \varphi_{D,k^0}(y_1) \le \varphi_{D,k^0}(y_2)$).

**Remark 2.1** *$D$ and $k^0$ given by* (3) *and* (4)*, respectively, fulfill property* (2).

*Furthermore, the functional $\varphi_{D,k^0}$ with $D$ and $k^0$ given by* (3) *and* (4)*, respectively, is $\mathbb{R}^q_+$-monotone, since $D + \mathbb{R}^q_+ \subseteq D$ (compare Theorem 5.2.3 (d) in* Khan et al. (2015)*).*

Furthermore, below we show that $\varphi_{D,k^0}$ is strictly (int $\mathbb{R}^q_+$)-monotone $\left(y_1 \in y_2 - \operatorname{int} \mathbb{R}^q_+ \Rightarrow \varphi_{D,k^0}(y_1) < \varphi_{D,k^0}(y_2)\right)$, compare Tannert (2013).

**Proposition 2.2** *The functional $\varphi_{D,k^0}$ with $D$ and $k^0$ given by* (3) *and* (4)*, respectively, is strictly* (int $\mathbb{R}^q_+$)*-monotone.*

*Proof* Consider $t \in \mathbb{R}$, $y \in tk^0 - \operatorname{int} D$. Then $tk^0 - y \in \operatorname{int} D$. Consequently, there exists a value $s > 0$ such that $tk^0 - y - sk^0 \in \operatorname{int} D \subset D$. Using Theorem 2.3.1 in Göpfert et al. (2003), we deduce $\varphi_{D,k^0}(y) \le t - s < t$, and thus

$$tk^0 - \operatorname{int} D \subset \left\{ y \in \mathbb{R}^q \,\middle|\, \varphi_{D,k^0}(\mathrm{y}) < t \right\}. \tag{6}$$

Furthermore, for $y_1 \in y_2 - \operatorname{int} \mathbb{R}^q_+$, it holds

$$\begin{aligned} y_1 &\in y_2 - \operatorname{int} \mathbb{R}^q_+ \\ &\subset \varphi_{D,k^0}(y_2)k^0 - D - \operatorname{int} \mathbb{R}^q_+ \ (\text{because of Theorem 2.3.1 in Göpfert et al. (2003)}) \\ &\subset \varphi_{D,k^0}(y_2)k^0 - \operatorname{int} D \\ &\subset \left\{ y \in \mathbb{R}^q \,\middle|\, \varphi_{D,k^0}(y) < \varphi_{D,k^0}(y) \right\} \ (\text{because of (6)}). \end{aligned}$$

We conclude that $\varphi_{D,k^0}(y_1) < \varphi_{D,k^0}(y_2)$ and thus $\varphi_{D,k^0}$ is strictly (int $\mathbb{R}^q_+$)-monotone. ■

The $\mathbb{R}^q_+$-monotonicity and strict (int $\mathbb{R}^q_+$)-monotonicity äproperty of the functional $\varphi_{D,k^0}$ given by (1) immediately relate to multiobjective optimization, as formulated in the proposition below.

**Proposition 2.3** *Let some* $j \in \{1, \ldots, q\}$ *be given and let* $x^0 \in \mathcal{X}_\varepsilon := \{x \in \mathbb{R}^n | f_i(x) \leq \varepsilon_i,\ i = 1, \ldots, q,\ i \neq j\}$ *be the uniquely optimal solution of problem* (5) *for given values* $\varepsilon_i \in \mathbb{R},\ i = 1, \ldots, q,\ i \neq j$*. Then it holds* $f(x^0) \in \mathrm{Min}(f[\mathbb{R}^n], \mathbb{R}^q_+)$*. If* $x^0 \in \mathcal{X}_\varepsilon$ *is an optimal solution of* (5)*, then* $f(x^0) \in \mathrm{Min}(f[\mathbb{R}^n],\ \mathrm{int}\ \mathbb{R}^q_+)$.

*Proof* We know that problem (5) can be reformulated by using the nonlinear scalarizing functional $\varphi_{D,k^0}$ (see (1)) with parameters $D$ and $k^0$ given by (3) and (4). Moreover, we have already shown that $\varphi_{D,k^0}$ is $\mathbb{R}^q_+$-monotone. Now let a unique solution $x^0 \in \mathcal{X}_\varepsilon$ of (5) be given.

Thus, $y^0 = f(x^0) \in f[\mathbb{R}^n] = \mathcal{F}$ is a unique solution of $(P_{k^0,D,\mathcal{F}})$. Now suppose that $y^0 \notin \mathrm{Min}(f[\mathbb{R}^n], \mathbb{R}^q_+)$. Hence, there exists $y \in f[\mathbb{R}^n] \setminus \{y^0\}$ with $y \in y^0 - \mathbb{R}^q_+$. But since $\varphi_{D,k^0}$ is $\mathbb{R}^q_+$-monotone, this leads to $\varphi_{D,k^0}(y) \leq \varphi_{D,k^0}(y^0)$, in contradiction to $y^0$ being uniquely optimal for $(P_{k^0,D,\mathcal{F}})$. Analogously, one can prove the second part of the proposition. ■

Taking into account Propositions 2.2 and 2.3 we get that (weakly) minimal solutions of $(VP)$ are found by using the $\varepsilon$-constraint scalarization method, and no convexity assumption is needed.

Therefore, this method is convenient to compute representatives of the set of (weakly) minimal solutions of a multiobjective optimization problem for Private Equity Funds in Sect. 5.

## 3 Uncertain Optimization

In this section we are dealing with uncertainties that contaminate data in optimization problems. Uncertain optimization models are very important in many applications ranging from flight scheduling, weather forecasting, facility location and portfolio optimization, among others (compare Goerigk et al. 2011, 2014; Fischetti et al. 2009; Stiller 2009; Carrizosa and Nickel 1998). Since solutions can highly depend on the perturbed data, it is necessary for the optimization process to include the uncertainty in the model and contrive ways to obtain adequate solutions. The results of this section are derived in Klamroth et al. (2013).

We will now formulate a scalar optimization problem with uncertainties. The set of all uncertain parameters is denoted by $\mathcal{U} \subset \mathbb{R}^N$. We call $\mathcal{U}$ the **uncertainty set**, i.e., the set that comprises all possible values of the uncertain parameters.

We assume that the objective function as well as constraints are contaminated with uncertain data. Let the objective function be $f : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R}$, and the constraints are given by means of $F_i : \mathbb{R}^n \times \mathcal{U} \to \mathbb{R},\ i = 1, \ldots, m$. Then an **uncertain** optimization problem is defined as a parametrized optimization problem

$$(Q(\zeta),\ \zeta \in \mathcal{U}), \tag{7}$$

where for a given $\zeta \in \mathcal{U}$ the optimization problem $(Q(\zeta))$ is given by

$$\begin{array}{rl} & \min f(x,\zeta) \\ \text{subject to} & F_i(x,\zeta) \leq 0, \ i=1, \ \ldots, m, \quad (Q(\zeta)) \\ & x \in \mathbb{R}^n. \end{array}$$

When solving the uncertain minimization problem $(Q(\zeta),\ \zeta \in \mathcal{U})$, it is not known which value $\zeta \in \mathcal{U}$ is going to be realized. Now the straightforward question is how one can handle such a family of parametrized optimization problems. Apparently, since we have not specified the structure of the uncertainty set $\mathcal{U}$ yet, there may be infinitely many optimization problems. In the two main approaches to dealing with $(Q(\zeta))$ which are known in the literature, namely stochastic optimization and robust optimization, the objective is to convert the family of parametrized optimization problems $(Q(\zeta),\ \zeta \in \mathcal{U})$ into a single problem which is then solved in order to obtain a solution that is optimal in some sense. Whereas stochastic optimization relies on a probabilistic assumption on the uncertain parameters, robust optimization is rather set-based and worst-case oriented.

We will now show how a robust optimization model fits into this unifying framework using the scalarizing functional $\varphi_{D,k^0}$ (see (1)) for a specific choice of the parameters $D, k^0$ and $\mathcal{F}$ in $\left(P_{k^0,D,\mathcal{F}}\right)$, compare Klamroth et al. (2013).

The robustness concept we introduce here is called **strict robustness**. First mentioned by Soyster (1973), it was later formalized and analyzed by Ben-Tal and Nemirovski (1998) and El Ghaoui and Lebret (1997) and now plays an important part in robust optimization and its applications, compare Kouvelis and Yu (1997). For an extensive collection of results, see Ben-Tal et al. (2009).

The idea is that the worst possible objective function value is minimized in order to get a solution that performs quite well even in the worst case scenario. Furthermore, a solution is required to satisfy the constraints for every possible future scenario $\zeta \in \mathcal{U}$. These requirements lead to the **strictly robust counterpart** of the uncertain optimization problem $(Q(\zeta),\ \zeta \in \mathcal{U})$, which is defined as

$$\begin{array}{rl} & \min \rho_{RC}(\mathrm{x}) = \min \sup\limits_{\zeta \in \mathcal{U}} f(x,\zeta) \\ \text{subject to} & \forall \zeta \in \mathcal{U}: \ F_i(x,\zeta) \leq 0, \ i=1, \ \ldots, m, \\ & x \in \mathbb{R}^n. \end{array} \tag{RC}$$

A feasible solution of $(RC)$ will be called **strictly robust**. The set of strictly robust solutions is denoted by

$$\mathfrak{A} := \left\{x \in \mathbb{R}^n \,\middle|\, \forall \zeta \in \mathcal{U} : F_i(x,\zeta) \leq 0, \ i=1, \ \ldots, m\right\}. \tag{8}$$

We will now show how $(RC)$ can be expressed using the nonlinear scalarizing functional $\varphi_{D,k^0}$ given in (1) by using an adequate selection of $D$ and $k^0$ (cf. Klamroth et al. 2013; Köbis and Tammer 2012) when $\mathcal{U}$ is given by a finite number of uncertain parameter values, i.e., we suppose that $\mathcal{U} := \left\{\zeta_1, \ \ldots, \ \zeta_q\right\}$.

This result will be used in Sect. 4 where we will show that $\max_{\zeta\in\mathcal{U}} -f(x,\zeta)$ (the objective function in (*RC*) with negative values of $f$) is a coherent risk measure.

**Theorem 3.1 (Klamroth et al. 2013, Theorem 3)** *Consider*

$$\mathfrak{A}_1 := \mathfrak{A}, \tag{9}$$

$$D_1 := \mathbb{R}^q_+ \tag{10}$$

$$k^0_1 := 1_q := (1, \ldots, 1)^T \tag{11}$$

$$\mathcal{F}_1 := \left\{ \left( f(x,\zeta_1), \ \ldots, f\left(x,\zeta_q\right) \right)^T \middle| \ x \in \mathfrak{A}_1 \right\}. \tag{12}$$

*For* $k^0 = k^0_1$, $D = D_1$, *condition* (2) *is satisfied and with* $\mathcal{F} = \mathcal{F}_1$, *problem* $\left(P_{k^0, D, \mathcal{F}}\right)$ *is equivalent to problem* (RC) *in the sense that*

$$\min\left\{ \varphi_{D_1,k^0_1}(y) \mid y \in \mathcal{F}_1 \right\} = \min\left\{ \rho_{RC}(x) \mid x \in \mathfrak{A}_1 \right\},$$

*where* $y = \left( f(x,\zeta_1), \ \ldots, \ f\left(x,\zeta_q\right) \right)^T$.

*Proof* Since $D_1 + [0, +\infty) \cdot k^0_1 \subset \mathbb{R}^q_+ + [0, +\infty) \cdot 1_q \subset \mathbb{R}^q_+ = D_1$, condition (2) is satisfied. Since $k^0_1 \in \operatorname{int} \mathbb{R}^q_+$ and $D_1 = \mathbb{R}^q_+$ is closed, the infimum in the definition of $\varphi_{D_1,k^0_1}$ is finite and attained such that we can replace the infimum by a minimum:

$$\begin{aligned}
\min_{y \in \mathcal{F}_1} \varphi_{D_1,k^0_1}(y) &= \min_{y \in \mathcal{F}_1} \min\{t \in \mathbb{R} \mid y \in t\, k^0_1 - D_1\} \\
&= \min_{y \in \mathcal{F}_1} \min\{t \in \mathbb{R} \mid y - t\, k^0_1 \in -D_1\} \\
&= \min_{x \in \mathfrak{A}_1} \min\{t \in \mathbb{R} \mid \left( f(x,\zeta_1), \ldots, f(x,\zeta_q) \right)^T - t\,(1, \ldots, 1)^T \leq 0_q\} \\
&= \min_{x \in \mathfrak{A}_1} \min\{t \in \mathbb{R} \mid \left( f(x,\zeta_1), \ldots, f(x,\zeta_q) \right)^T \leq t\,(1, \ldots, 1)^T\} \\
&= \min\{\max_{\zeta \in \mathcal{U}} f(x,\zeta) \mid x \in \mathfrak{A}_1\} \\
&= \min\{\rho_{RC}(x) \mid x \in \mathfrak{A}_1\}.
\end{aligned}$$

■

Notice that the selection of $k^0_1 = 1_q$ validates that the objective functions $f(x,\zeta)$, $\zeta \in \mathcal{U}$, are considered to be of equivalent interest, i.e., no objective function is preferred to another one. This confirms that no probability distribution is considered.

**Remark 3.2** *Since* $D_1 = \mathbb{R}^q_+$ *is a proper closed convex cone and* $k^0_1 \in \operatorname{int} D_1$, *the functional* $\varphi_{D_1,k^0_1}$ *is continuous, finite-valued,* $\mathbb{R}^q_+$*-monotone, strictly* ($\operatorname{int} \mathbb{R}^q_+$)*-monotone and sublinear.*

**Remark 3.3** *The concept of strict robustness is described by the Tschebyscheff scalarization with the origin as reference point as a special case of functional* (1). *Theorem* 3.1 *shows that (RC) can be interpreted as a max-ordering problem as defined in multiobjective optimization, see* Ehrgott (2005). *This relationship was also observed by* Kouvelis and Sayin (2006)*, and* Sayin and Kouvelis (2005) *where it was used to determine the nondominated set of discrete bicriteria optimization problems.*

The observation that uncertain optimization problems are a special class of the functional $\varphi_{D,k^0}$ (see (1)) serves as a motivation to consider new definitions of robust optimization problems by a different selection of $D$ and $k^0$ as follows.

We will introduce a new approach toward robustness, which we will call **$\boldsymbol{\varepsilon}$-constraint robustness**. In the following we analyze which type of robust counterpart is defined by this scalarization. To this end, let some $j \in \{1, \ldots, q\}$ and some real values $\varepsilon_i$, $i = 1, \ldots, q$, $i \neq j$ be given. Then we use the following components for the $\varepsilon$-constraint scalarization. Let $D_2 := D$, where $D$ is given by (3) and $k_2^0 := k^0$ with $k^0$ defined in (4). Furthermore, let the set of feasible elements be given as $\mathcal{F}_2 := \mathcal{F}_1$ (see (12)). Now the following reformulation holds.

**Theorem 3.4 (Klamroth et al. 2013, Theorem 8)** *Let* $\varepsilon := (\varepsilon_1, \ldots, \varepsilon_q)^T \in \mathbb{R}^q$ *and* $j \in \{1, \ldots, q\}$.

*Then for* $k^0 = k_2^0$, $D = D_2$, *(2) holds and with* $\mathcal{F} = \mathcal{F}_2$, *problem* $(P_{k^0,D,\mathcal{F}})$ *is equivalent to*

$$\begin{aligned} &\min \rho_{RC}(\mathrm{x}) \\ \text{subject to} \quad &\forall \zeta \in \mathcal{U}: \ F_i(x,\zeta) \leq 0, \ i = 1, \ldots, m, \\ &x \in \mathbb{R}^n, \\ &f(x,\zeta_i) \leq \varepsilon_i, \ i \in \{1, \ldots, q\}, \ i \neq j, \end{aligned} \tag{13}$$

where $\rho_{\varepsilon RC}(x) := f(x,\zeta_j)$.

*Proof* Since $D_2 + [0, +\infty) \cdot k_2^0 \subset D_2$, condition (2) is satisfied. Moreover,

$$\begin{aligned} \min_{y \in \mathcal{F}_2} \varphi_{D_2,k_2^0}(y) &= \min_{y \in \mathcal{F}_2} \min\{t \in \mathbb{R} \mid y \in t\, k_2^0 - D_2\} \\ &= \min_{y \in \mathcal{F}_2} \min\{t \in \mathbb{R} \mid y - t\, k_2^0 \in -D_2\} \\ &= \min_{x \in \mathfrak{A}} \min\{t \in \mathbb{R} \mid f(x,\zeta_j) \leq t, \ f(x,\zeta_i) \leq \varepsilon_i, \ i \in \{1, \ldots, q\}, \ i \neq j\} \\ &= \min\{ \rho_{\varepsilon RC}(x) \mid x \in \mathbb{R}^n, \ \forall \zeta \in \mathcal{U}: \ F_i(x,\zeta) \leq 0, \ i = 1, \ldots, m, \\ &\qquad f(x,\zeta_i) \leq \varepsilon_i, \ i \in \{1, \ldots, q\}, \ i \neq j\}. \end{aligned}$$

■

Note that the above suggested analysis can be performed for any possible variation of the parameters $D, k^0$ and $\mathcal{F}$ in order to obtain new concepts for robustness. Such an approach may be beneficial for a decision maker whose attitude has not yet been represented by a given robustness concept. Thus, a new concept may be developed that fits the specific needs of the decision maker, taking his preferences in terms of risk and uncertainty into account.

Theorem 3.4 shows that the problem of minimizing the nonlinear scalarizing functional $\varphi_{D,k^0}$ can be formulated as (13). We call (13) the **$\varepsilon$-constraint robust counterpart** of the uncertain optimization problem $(Q(\zeta),\ \zeta \in \mathcal{U})$ (see (7)). In a next step, we analyze its meaning for robust optimization.

Contrary to the strictly robust counterpart problem (*RC*), the parameter $k_2^0$ symbolizes that only a single objective function is minimized. In particular, the decision maker chooses one specific objective function that he wishes to minimize subject to the constraints that are known from the strictly robust counterpart (*RC*) (although other constraints are entirely possible and the above concept may be adapted to a different set of feasible solutions of $\mathcal{F}$). Furthermore, the former objective functions $f(x,\zeta_i),\ i \in \{1,\ \ldots,\ q\}\ ,\ i \neq j$, are shifted to and treated as constraints. This approach is useful if a solution is required with a given nominal quality for every scenario $\zeta_i,\ i \in \{1,\ \ldots,\ q\}\ ,\ i \neq j$, while finding the best possible objective value for the remaining scenario $j$. When applying this concept, one difficulty is immediately revealed, namely, how to pick the upper bounds $\varepsilon_i$ for the constraints. If they are chosen too small, the set of feasible solutions of (13) may be empty, or the objective function value $f\left(x,\zeta_j\right)$ may not perform well enough. On the other hand, if the bounds $\varepsilon_i$ are chosen too large, the optimality, meaning the value $f(x,\zeta_i),\ i \neq j$, for the other scenarios decreases. Such a concept may be beneficial for a decision maker whose preferences have not yet been represented by any other robustness approach or to provide him with a wider choice of options. In addition, the values $\varepsilon$ could, for instance, represent a company's regulations or safety standards which have to be satisfied.

Further concepts of robustness and of stochastic programming can be described as special cases of the general nonlinear scalarization method in $\left(P_{k^0},_{D,\ \mathcal{F}}\right)$ by choosing the involved parameters and sets appropriately, see Klamroth et al. (2013).

## 4 Coherent Risk Measures

The functional $\varphi_{D,k^0}$ given by (1) is an important tool in the field of financial mathematics. It can be used to describe coherent risk measure associated with investments (see Artzner et al. 1999; Heyde 2006). A risk measure is used as a quantification to describe the risk of an investment. Other types of risks include a company's equity capital, which has to be available in case of a loss in the company's value. In this section we investigate the nonlinear scalarization functional $\varphi_{D,k^0}$ in connection with coherent risk measures.

For a better understanding of the topic, we will now introduce coherent risk measures and their relationship to the strictly robust counterpart ($RC$) of the uncertain optimization problem $(Q(\zeta),\ \zeta\ \in\ \mathcal{U})$.

Let $Y$ be a linear space of random variables, and let $\Omega$ be a set of elementary events (a set of all possible states of the future). Then a future payment of an investment is a random variable $y:\ \Omega\ \rightarrow\ \mathbb{R}$. Positive payments in the future are wins, negative ones are losses. If no investment is being done, then $y$ takes on the value zero. In order to evaluate such an investment, we need to valuate random variables by comparing them. To do that, we introduce an ordering relation that is induced by a set $D\ \subseteq Y$. Artzner et al. (1999) proposed axioms for a nonempty closed set $D\ \subseteq Y$ of random variables that represent acceptable investments:

(A) $\{y\ \in Y\ |\ y(\omega)\geq 0\ (\omega\in\ \Omega)\}\subset D,\ \ D\cap\{y\in Y\ |\ y(\omega)<0\ (\omega\in\ \Omega)\}=\emptyset$,
(B) $D$ is a cone,
(C) $D$ is convex.

In financial terms, axiom (A) means that every investment with almost sure nonnegative results will be accepted and every investment with almost sure negative results is not acceptable. Furthermore, the cone property (B) says that every nonnegative multiple of an acceptable investment is again acceptable.

The convexity property in axiom (C) means that merging two acceptable investments together results again in an acceptable investment. However, in some applications the cone property of $D$ and axiom (C) are not useful, especially, if the investor does not want to lose more than a certain amount of money. In this case Föllmer and Schied (2004) replace the axioms (B) and (C) by a convexity axiom.

Sets $D\subset Y$ satisfying the axioms (A)–(C) of acceptable investments can be used in order to introduce a preference relation on $Y$. The decision maker prefers $y_1$ to $y_2$ (changing from $y_2$ to $y_1$ is an acceptable risk) if and only if $y_1-y_2$ is an element of $D$, i.e.,

$$y_1\geq_D y_2\Leftrightarrow y_1-\ y_2\in D.$$

The smallest set $D$ satisfying the axioms (A)–(C) is $D=\{y\ \in Y\ |\ y(\omega)\geq 0\ (\omega\in\ \Omega)\}$. A decision maker using this particular cone $D$ of acceptable investments is risk-averse, i.e., he only accepts investments with nonnegative payments.

Artzner et al. (1999) axiomatically introduced coherent risk measures. These are functional $\mu:\ \ \mathrm{Y}\ \rightarrow\ \mathbb{R}\cup\{+\infty\}$, where Y is the linear space of random variables, that satisfy the following properties:

(P1) $\mu(y+tk^0)=\ \mu(y)-t$ for $k^0\ \in Y\backslash\{0\}$ (Translation Invariance),
(P2) $\mu(0)=0$ and $\mu(\lambda y)=\ \lambda\mu(y)$ for all $y\in Y$ and $\lambda\ >0$ (Positive Homogeneity),
(P3) $\mu(y_1+y_2)\leq\ \mu(y_1)+\ \mu(y_2)$ for all $y_1,y_2\in Y$ (Subadditivity),
(P4) $\mu(y_1)\ \leq\ \mu(y_2)$ if $y_1\geq_D y_2$ (Monotonicity).

The following interpretation of the properties (P1)–(P4) is to mention: The translation property (P1) means that the risk would be mitigated by an additional safe investment with a corresponding amount, especially, it holds

$$\mu\big(y + \mu(y)k^0\big) = 0.$$

The positive homogeneity of the risk measure in (P2) means that double risk must be secured by double risk capital. For large amounts of money, this assumption would not be appropriate, since doubling an investment should result in a risk that is larger than twice the risk. The subadditivity property in (P3) means that a diversification of risk should be recompensed and finally, the monotonicity property of the risk measure in (P4) means that higher risk needs more risk capital.

It is also possible for a risk measure to be negative. In this case it can be interpreted as a maximal amount of cash that could be given away such that the reduced result remains acceptable.

It can be shown that

$$\mu(y) = \inf\left\{t \in \mathbb{R} \,\middle|\, y + t\,k^0 \in D\right\} \tag{14}$$

is a coherent risk measure if $D$ satisfies assumptions (A)–(C). Obviously, we have (cf. Heyde 2006)

$$\mu(y) = \varphi_{D,k^0}(-y),$$

where $\varphi_{D,k^0}(y)$ is defined by (1). A risk measure induces a set $D_\mu$ of acceptable risks (dependent on $\mu$)

$$D_\mu = \left\{y \in Y \,\middle|\, \mu(y) \le 0\right\}. \tag{15}$$

Furthermore, it holds $D_\mu = D$ (see Artzner et al. 1999).

Moreover, if properties (P1)–(P4) are satisfied by a lower semi continuous functional $\mu$, then $\mu$ takes form (14) where $D = D_\mu$ (see (15)) is closed and fulfills assumptions (A)–(C).

The following interpretation of coherent risk measures in light of robust optimization (see Sect. 3) is to mention: If $Y = \mathbb{R}^q$ (there are $q$ states of the future), $D_1 = \mathbb{R}^q_+$ (that is, indeed, the smallest set that satisfies axioms (A)–(C)) and $k^0_1 = 1_q$, then

$$\mu(y) = \varphi_{D_1,k^0_2}(-y) = \max_{\zeta\in\mathcal{U}}\big(-f(x,\zeta)\big) = -\min_{\zeta\in\mathcal{U}} f(x,\zeta) \text{ (due to Theorem 3.1)}$$

is a coherent risk measure. Specifically, the risk measure $\max_{\zeta\in\mathcal{U}}(-f(x,\zeta))$ is the objective function of the strictly robust counterpart ($RC$) with negative values of $f$. Because $\mu(y) = -\min_{\zeta\in\mathcal{U}} f(x,\zeta)$, negative payments $f$ of an investment in the future result in a positive risk measure, and positive payments result in a negative risk measure. Here negative payments represent losses which, of course, are riskier than investments with only positive payments that are associated with wins.

It was noted in Sect. 3 that the strictly robust problem (*RC*) is a highly risk-averse approach, since the decision maker aims at solutions that are minimal in the worst-case scenario. This can be also be explained by the choice of the set $D_1 = \mathbb{R}^q_+$ (i.e., investments that only yield positive payments in the future) in the functional $\varphi_{D_1,k_1^0}$, as $D_1$ is the smallest set of acceptable investments (i.e., the smallest set satisfying (A)–(C)).

This approach can analogously be used for other concepts of robustness that are known from the literature (assuming that the corresponding set $D$ satisfies (A)–(C)).

Interrelations between robustness and coherent risk measures have also been studied by Quaranta and Zaffaroni (2008): They minimized the conditional value at risk (which is a coherent risk measure) of a portfolio of shares using concepts of robust optimization.

## 5 Optimization of Private Equity Funds

We conclude the chapter with an application of the $\varepsilon$-constraint method (see Sect. 2) on a model for a Private Equity Fund. This model is formulated using stochastic differential equations to describe the development of a private equity fund (compare de Malherbe 2005; Tannert 2013; Tammer and Tannert 2012), and the results presented in this section are derived by Tannert (2013).

The fundament of the model are three differential equations which characterize the draw downs $D(t)$, the returns $R(t)$ and the performance of the fund $U(t)$ at time $t \in [0;T]$, where $T$ is the termination time of the fund.

The change in the portfolio value from time $t$ to $t + dt$ is described by the following stochastic differential equation:

$$dU(t) = U(t)\,(\mu(t)dt + \sigma(t)dW^\mu(t)) + dD(t) - dR(t),\; U(0) = u_0. \tag{16}$$

In this equation $\mu(t)dt$ expresses the development in the period and $\sigma(t)dW^\mu(t)$ the volatility, where $W^\mu$ is a Brownian motion. The term $dD(t)$ characterizes the change of the invested capital and $dR(t)$ denotes the returns to the investors at time $t$. The part in the brackets describes the performance of the assets. Additionally, the value of the portfolio is increased by the investments $D(t)$ and decreased by the returns $R(t)$.

The returns in the model are meant to be before taxes and interests. The stochastic part so far is expressed by the Brownian motion $W^\mu(t)$. If the term $(dD(t) - dR(t))$ is disregarded, $dU(t)/U(t)$ follows a normal distribution with expected value $\mu(t)dt$ and variance $\sigma^2(t)dt$. In this case, $U(t)$ would be approximated by a lognormal distribution. This distribution is often applied in finance to model stock returns.

For the sake of clarity, the committed capital is normed to one. Hence, at time $t$ the fund manager holds capital in the amount of $\overline{D}(t) = 1 - D(t)$. It is assumed that a positive portion $\delta(t)$ of the existing capital is invested at time $t$. At a certain point in time $T^* \in [0;T]$, the existing capital $\overline{D}(T^*)$ is equal to zero. Here we assume that $T^* = T$. Hence, following ordinary differential equation for $t \in [0;T^*]$ describes the change of the invested capital:

$$\begin{aligned} \frac{dD(t)}{dt} &= \delta(t)(1 - D(t)) \\ &= \delta(t)\overline{D}(t). \end{aligned} \tag{17}$$

Therefore, it holds for $D(t)$ and $\overline{D}(t)$ that:

$$D(t) = 1 - \exp\left(-\int_0^t \delta(s)ds\right), \tag{18}$$

$$\overline{D}(t) = \exp\left(-\int_0^t \delta(s)ds\right). \tag{19}$$

At time $t$, the amount $dD(t)$ of the existing capital $\overline{D}(t)$ is invested. Hence:

$$-d\overline{D}(t) = dD(t). \tag{20}$$

As next step, the returns are modeled. It is assumed that a positive portion $\rho(t)$ from the portfolio value is paid back to the investors at time $t$. The change of the returns from time $t$ up to $t + dt$ for $0 \le t < T$ can therefore be described as:

$$dR(t) = \rho(t)U(t)dt. \tag{21}$$

Then, the returns $R(t)$ for $t \in [0;T]$ can be calculated by:

$$R(t) = \begin{cases} \int_0^t \rho(s)U(s)ds & \text{if } 0 \le t < T, \\ \int_0^t \rho(s)U(s)ds + U(T) & \text{if } t = T. \end{cases} \tag{22}$$

In this equation, the term $U(t)$ expresses that the remaining capital at final time $T$ is paid back to the investors. Here we suppose that $U(t) = 0$.

When applying (17) and (21) to (16), it can be concluded that:

$$dU(t) = \delta(t)(1 - D(t))dt + U(t)[(\mu(t) - \rho(t))dt + \sigma(t)dW^{\mu}(t)]. \tag{23}$$

The term in the square brackets, which is proportional to the portfolio value $U(t)$, follows a lognormal distribution with drift $\delta(t)(1 - D(t)) = dD(t)$. The value of the private equity fund is hence described by an inverse Gamma distribution. If $t \geq T^*$, then $\delta(t)(1 - D(t)) = 0$ and as result $U(t)$ would follow a lognormal distribution.

Up to now, the draw downs and returns are still deterministic. However, from the perspective of the investors, the invested capital and the return distribution is not known. The same holds for the fund manager since the development of the portfolio and the changing market conditions cannot be precisely predicted. Therefore, the assumption that (17) and (21) are completely deterministic should be revised.

The stochastic takes part in the calculation of the investment rate $\delta$ and the return rate $\rho$. If a squared Bessel process is used for the modeling, the stochastic differential equations can be described by:

$$d\delta(t) = (c_1 + c_2\delta(t))dt + c_3\sqrt{\delta(t)}\, dW^{\delta}(t), \quad \delta(0) = \delta_0, \tag{24}$$

$$d\rho(t) = (q_1 + q_2\rho(t))dt + q_3\sqrt{\rho(t)}\, dW^{\rho}(t), \quad \rho(0) = \rho_0, \tag{25}$$

Where $W^{\delta}$, $W^{\rho}$ are two Brownian motions related to the investment rate $\delta$ and the return rate $\rho$. Additionally, $c_1, c_3, q_1, q_3 > 0$ and $c_2, q_2 < 0$.

The computation of performance indicators and risk measures is of special interest. They can be obtained from the model equations. For the calculation, the draw downs $D(t)$, the returns $R(t)$, and the portfolio value $U(t)$ at a certain time $t$ are needed. The distributions of the draw downs $D(t)$, returns $R(t)$ and the portfolio value $U(t)$ can be derived out of a Monte Carlo simulation.

The success of the fund can be measured by the internal rate of return or several multiples. The distribution to paid-in (DPI) multiple will be used here. The DPI multiple is the fraction of the cumulative returns to the paid-in capital.

With these notations, the Private Equity Fund Model ($MPE$) can be described by:

$$(MPE) = \begin{cases} dD(t) = \delta(t)\exp\Big(-\int_0^t \delta(s)ds\Big)dt, \quad D(0) = D_0, \\ dR(t) = \rho(t)\; U(t)dt, \quad R(0) = R_0, \\ dU(t) = U(t)(\mu(t)dt + \sigma(t)dW^{\mu}(t)) + dD(t) - dR(t),\; U(0) = U_0, \\ d\delta(t) = (c_1 + c_2\delta(t))dt + c_3\sqrt{\delta(t)}\; dW^{\delta}(t), \quad \delta(0) = \delta_0, \\ d\rho(t) = (q_1 + q_2\rho(t))dt + q_3\sqrt{\rho(t)}\; dW^{\rho}(t), \; \rho(0) = \rho_0. \end{cases}$$

For ($MPE$) the following holds, $t \in [0;T]$, $c_1,\ q_1 \geq 0,\ c_2,\ q_2 < 0,\ c_3,\ q_3 \geq 0$ and $\mu,\ \sigma,\ \delta(0),\ \rho(0),\ D(0),\ R(0),\ U(0) \in \mathbb{R}_+$. Furthermore, $W^{\delta}$, $W^{\rho}$ and $W^{\mu}$ are independent Brownian motions.

### 5.1 A Multiobjective Optimization Problem for Private Equity Funds

Assume $x \in \mathbb{R}^4$ with $x = (c_1, c_3, q_1, q_3)^T$, where $c_1$, $c_3$ satisfy the stochastic differential equation (24) and $q_1$, $q_3$ satisfy the stochastic differential equation (25).

Furthermore, we study in our model the following objective functions:

$$f_i : (\mathbb{R} \times \mathbb{R}^4) \to \mathbb{R} \quad \text{with} \quad i \in [1; n], \ \ n \in \mathbb{N},$$

that describe the success of the investment. The risk (compare Artzner et al. 1999; Föllmer and Schied 2004) of the investment is described by the objective functions

$$g_j : (\mathbb{R} \times \mathbb{R}^4) \to \mathbb{R} \quad \text{with} \quad j \in [1; m], \ \ m \in \mathbb{N}.$$

Especially, as risk measures we use the Variance, the Value at Risk and the Average Value at Risk (see Föllmer and Schied 2004). Moreover, $c_2$, $q_2 < 0$, $\mu$, $\sigma > 0$ are real parameters in each objective function $f_i$, $i = 1, \ldots, n$, $g_j$, $j = 1, \ldots, m$ and the moments $t \in \mathbb{R}^{m+n}$, where $t := [t_1, \ldots, t_{n+m}]^T$. So we get a multiobjective stochastic optimization problem for a Private Equity Fund $(MOP - PE)$ in order to find an effective strategy for the fund manager:

$$\begin{aligned}(MOP - PE) \quad & \min_{x \in \mathbb{R}^4} \quad [f_1(t_1, x), \ \ldots, \ f_n(t_n, x), \ g_1(t_{n+1}, x), \ \ldots, \ g_m(t_{m+n}, x)]^T \\ & \text{s.t. } (0,0,0,0)^T < x \leq (3,3,3,3)^T.\end{aligned}$$

The restrictions are chosen by the fund manager.

The objective functions $f_i$ and $g_j$ with $i \in [1; n]$ and $j \in [1; m]$ describe the success and the risk of the Private Equity Fund. These objective functions are derived using draw downs, returns and the value of the portfolio from the model $(MPE)$ and an approximation based on stochastic differential equations. The variable $t \in [0; T]$ describes the period, the components $c_1$, $c_3$ of $x$ characterize the quota of the investment $\delta$ and the components $q_1$, $q_3$ characterize the quota of the distribute $\rho$.

### 5.2 Numerical Methods

Using the new model $(MPE)$ based on stochastic differential equations for a Private Equity Fund and suitable scalarization methods of multiobjective optimization (especially the $\varepsilon$-constraint method introduced in Sect. 2) it is possible to give a decision support for the fund manager. The approaches are implemented in a MATLAB program for solving the stochastic differential equations from $(MPE)$ and the multiobjective optimization problem $(MOP - PE)$ (see Tannert 2013).

The stochastic differential equations from ($MPE$) are solved using a Euler scheme. The portfolio value, the draw downs, the returns and the performance indicators are computed.

In order to solve the multiobjective stochastic optimization problem for the Private Equity Fund ($MOP - PE$) we use the $\varepsilon$-constraint method (see Sect. 2).

Taking into account the strict (int $\mathbb{R}^q_+$)-monotonicity (for $q = m + n$) of the $\varepsilon$-constraint method (Proposition 2.2) it follows that the generated solutions are weakly minimal solutions of the problem ($MOP - PE$) (Proposition 2.3). Furthermore, we apply a genetic algorithm (see Tannert 2013) in order to approximate the minimal frontier of ($MOP - PE$).

**The Expectation-Variance-Problem**

A special case of *(MOP-PE)* with the variables $x \in \mathbb{R}^4, x = [c_1, c_{3,}, q_1, q_3], t = T$ as well as constants $c_2 < 0,\ q_2 < 0,\ \mu > 0,\ \sigma > 0$ is the bi-criteria optimization problem

$$(EVP) \quad \min_{x \in \mathbb{R}^4} \left[-\mathbb{E}\left[DPI\left(T,x\right)\right], Var[DPI(T,x)]\right)\right]^T$$
$$s.t. \quad (0,0,0,0)^T < x \le (3,3,3,3)^T.$$

Figure 2 shows the approximation of the Pareto frontier of *(EVP)*.

**Expectation-Value at Risk-Problem**

In the next model we replace the variance of the DPI multiple by another risk measure, namely by the Value at Risk.

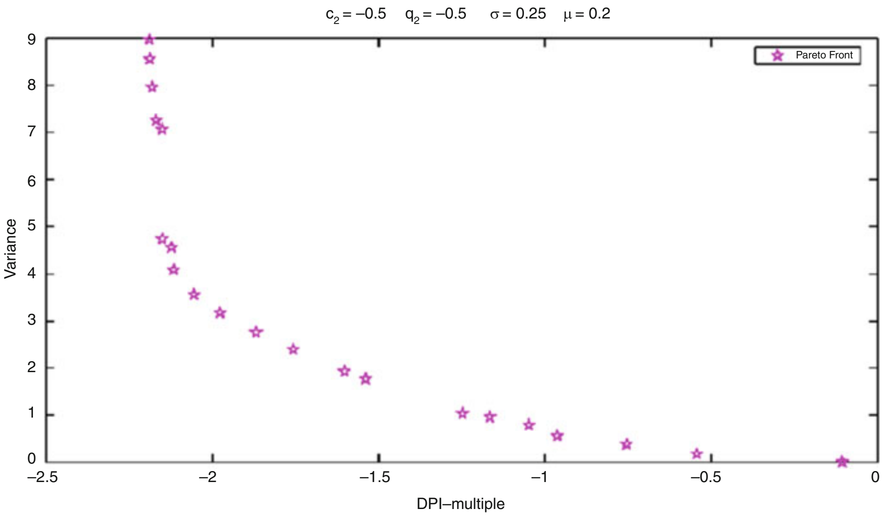


**Fig. 2** The minimal frontier of the expectation-variance-problem

**Definition 5.1 (Value at Risk)** *Let $\Omega$ be a fixed set of scenarios. A financial position is described by a mapping $x : \Omega \to \mathbb{R}$ and $x$ belongs to a given class $\mathcal{X}$ of financial positions. Assume that $\mathcal{X}$ is the linear space of bounded measurable functions containing the constants on some measurable space $(\Omega, A)$. Furthermore, let $P$ be a probability measure on $(\Omega, A)$. A position $x$ is considered to be acceptable if the probability of a loss is bounded by a given level $\varepsilon \in (0,1)$, i.e., if $P[x < 0] \leq \varepsilon$. The corresponding monetary risk measure $V@R_\varepsilon(x)$, defined by*

$$V@R_\varepsilon(x) := \inf\left\{m \in \mathbb{R} \;\middle|\; P[\,m + x < 0] \leq \varepsilon\right\}$$

*is called* Value at Risk.

**Remark 5.2** *$V@R_\varepsilon$ is the smallest amount of capital which, if added to $x$ and invested in the risk-free asset, keeps the probability of a negative outcome below the level $\varepsilon$. $V@R_\varepsilon$ is positively homogeneous but in general it is not convex (cf.* Föllmer and Schied 2004, *Example 4.11), this means that $V@R_\varepsilon$ is not a coherent risk measure.*

The Expectation-Value at Risk-Problem (*EVaR*) with variables $x \in \mathbb{R}^4$, $x = [c_1,\ c_3,, q_1,\ q_3]$, $\varepsilon = 0.05$, $t = T$ and constants $c_2 < 0$, $q_2 < 0$, $\mu > 0$, $\sigma > 0$ is given by:

$$\begin{aligned} (EVaR) \quad & \min_{x \in \mathbb{R}^4} \quad \left[-\mathbb{E}\left[DPI\left(T,x\right)\right],\ -V@R_\varepsilon[DPI(T,x)]\right)\right]^T \\ & s.t. \quad (0,0,0,0)^T < x \leq (3,3,3,3)^T. \end{aligned}$$

The approximation of the Pareto frontier of *(EVaR)* is shown in Fig. 3.

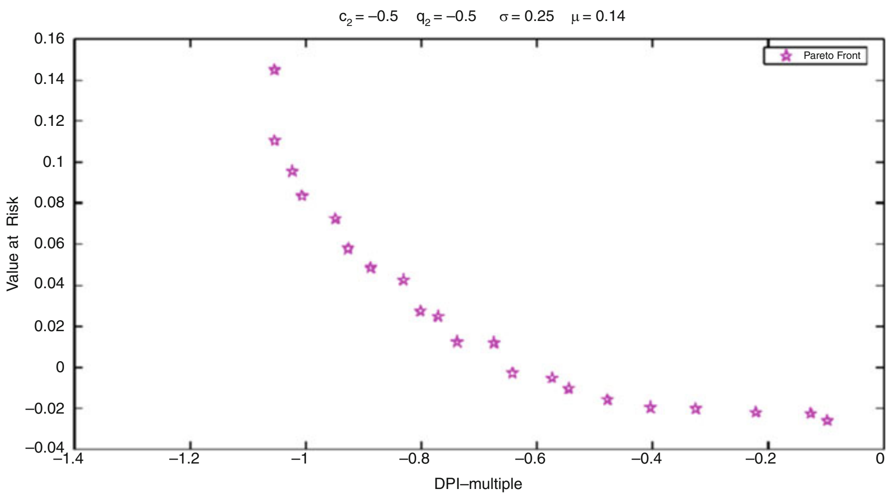


**Fig. 3** The minimal frontier of the expectation-value at risk-problem

**Expectation-Average Value at Risk-Problem**

Now, we replace the Value at Risk by a coherent risk measure, namely by the Average Value at Risk (compare Definition 5.1 for notations).

**Definition 5.3 (Average Value at Risk)** *The Average Value at Risk at level* $\lambda \in (0, 1]$ *of a position* $x \in \mathcal{X}$ *is defined by*

$$AV@R_\lambda\,(x) := \frac{1}{\lambda}\int_0^\lambda V@R_\gamma(x)d(\gamma)$$

**Remark 5.4** *The Average Value at Risk is a coherent risk measure whereas the Value at Risk is not a coherent risk measure (see* Föllmer and Schied 2004 *and* Sect. 4*).*

This leads to the Expectation-Average Value at Risk-Problem ($EAVaR$) as a special case of $(MOP - PE)$ with $x \in \mathbb{R}^4$, $x = [c_1, c_{3,}, q_1, q_3]$, $t = T$ as well as constants $c_2 < 0$, $q_2 < 0$, $\mu > 0$, $\sigma > 0$:

$$\begin{aligned}(EAVaR)\quad &\min_{x\in\,\mathbb{R}^4}\quad \left[-\mathbb{E}\left[DPI\left(T,x\right)\right],\ -AV@R_\varepsilon[DPI(T,x)]\right)\Big]^T\\ &s.t.\quad (0,0,0,0)^T < x \ \leq\ (3,3,3,3)^T.\end{aligned}$$

Figure 4 presents the approximation of the Pareto frontier of the Expectation-Average Value at Risk-Problem ($EAVaR$).

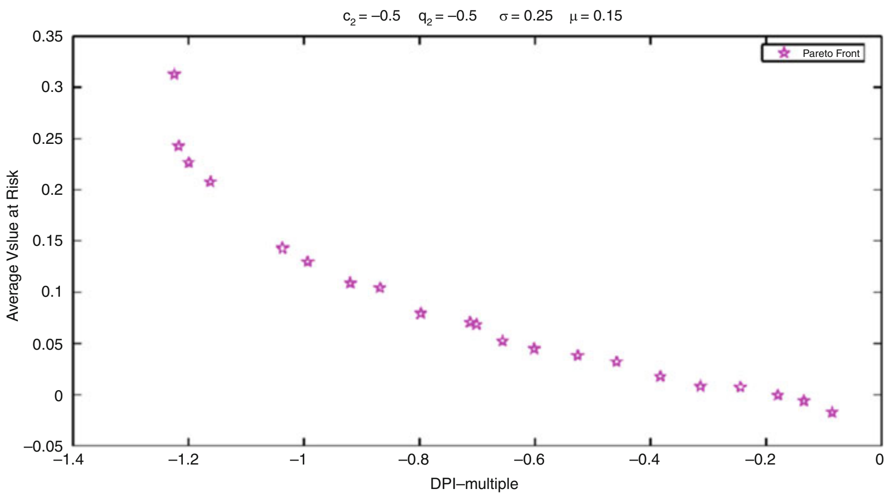


**Fig. 4** The minimal frontier of the expectation-average value at risk-problem

**Expectation-Expectation-Average Value at Risk-Problem**
In order to evaluate Private Equity Funds it is important to take into account the short-term performance as well as the long-term performance. The random variable *DPI* will describe the success of the fund using the DPI multiple. Furthermore, we will study a coherent risk measure to evaluate the fund. Taking into account these criteria we consider as a special case of $(MOP-PE)$ an optimization problem with three objective functions $(n=2,\ m=1)$. In the first objective function $(f_1)$, the expectation of *DPI* for $t \ll T$ characterizes the short-term performance. In the second objective function $(f_2)$, the expectation of *DPI* for $t=T$ characterizes the long-term performance. The Average Value at Risk is taken as the third objective function $(g_1)$ measuring the risk.

This leads to the multiobjective optimization problem (*EAVaR*) for $x \in \mathbb{R}^4$, $x=[c_1,\ c_{3,},q_1,\ q_3]$, $t_1 \ \in (0,T)$, $t_2=T$ as well as real constants $c_2<0,\ \ q_2<0,$ $\mu>0,\ \ \sigma>0$:

$$(EAVaR) \quad \min_{x\in\ \mathbb{R}^4} \ \left[-\mathbb{E}\left[DPI(t_1,x\right],, -\mathbb{E}\left[DPI(t_2,x\right], \ -AV@R_\varepsilon[DPI(T,x)]\right)\Big]^T$$
$$s.t. \quad (0,0,0,0)^T < x \ \le \ (3,3,3,3)^T.$$

Solving the multiobjective optimization problem (*EAVaR*) the fund manager gets a strategy for the choice of the parameters $c_1$, $c_3$, $q_1$ and $q_3$ in the Private Equity Fund Model.

Figure 5 shows the short-term performance as well as the long-term performance for approximately minimal solutions of problem (*EAVaR*).

In Fig. 6, the minimal frontier of problem (*EAVaR*) is illustrated.

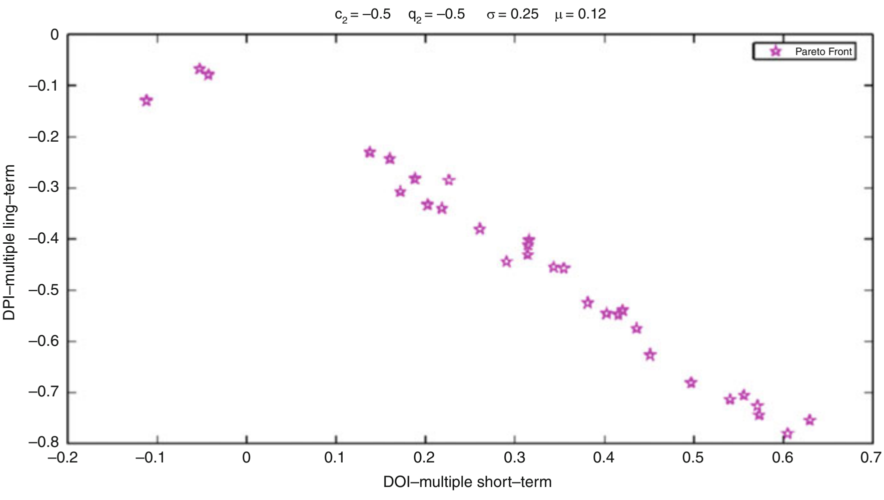


**Fig. 5** Short-term performance as well as the long-term performance in problem (*EAVaR*)

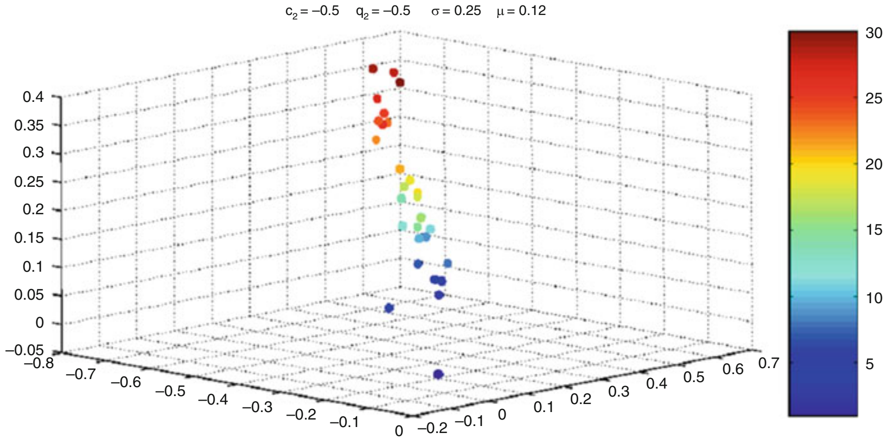


**Fig. 6** The minimal frontier of the expectation-average value at risk-problem

## 6 Conclusions

In this chapter, we point out the connections that exist between a nonlinear scalarization method, uncertain optimization (in particular, robustness approaches), $\varepsilon$-constraint scalarization and coherent risk measures.

As an application of our findings, we introduce a Private Equity Fund model based on stochastic differential equations. We solve several special cases of this problem by means of the $\varepsilon$-constraint method.

Further research could include a robust approach to dealing with such a Private Equity Fund model, where the uncertain parameters involved in the returns and risk values are assumed to belong to some kind of uncertainty set and one searches for solutions that are robust, i.e., solutions which are optimal in the worst cases. Then a multiobjective optimization problem, for all realizations of the uncertain parameter, leads to a set optimization problem. Since set optimization problems can also be solved using scalarization techniques (compare, for instance, Hernández and Rodríguez-Marín 2007; Ide et al. 2014), our approaches can be adapted to solve such problems as well.

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# The Multiobjective Nature of Bonus-Malus Systems in Insurance Companies

**Antonio Heras, Alejandro Balbás, Beatriz Balbás, and Raquel Balbás**

**Abstract** The so-called Problem of *Optimal Premium Calculation* deals with the selection of the appropriate premiums to be paid by the insurance policies. At first sight, this seems to be a statistical estimation problem: we should estimate the mean claim amount, which in actuarial terms is known as the net premium. Nevertheless, several extensions of this problem are clearly multi-objective decision problems. For example, when we allow the company to modify the premiums paid by the policyholders according to their past claim experience, there are several ways of designing the resulting *Bonus-Malus System* (BMS), and they usually involve several different objectives.

Optimal BMS design can thus be considered as a multi-objective problem, since it involves three conflicting objectives, which we have called *Fairness*, *Toughness* and *Equilibrium* (or *Disequilibrium*). Other researchers do not consider the multi-objective nature of this problem, since they always deal with a single objective, taking one of the objectives (Fairness) as the most important. In this chapter we apply a multi-objective approach. We represent in mathematical terms the three objectives, and we formulate the corresponding multi-objective program. Then we discuss several possible ways for solving the problem, and we apply the methodology to the improvement of a real BMS example.

**Keywords** Premium Calculation • Bonus-Malus Systems • Multi-Objective Programming • Insurance

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## 1 Introduction

It is well known that many optimization problems faced by real firms have a multi-objective nature. That is to say, there is not a single objective function to be optimized, but instead the companies are faced with several contradictory objectives that cannot be attained all together, because the improvement of one of them usually implies worsening some of the others.

This is also true in the particular case of the insurance companies. Some of the important classical problems faced by an insurance company (calculation of premiums and reserves, design of the reinsurance policy...) have a multi-objective nature. For example, in the calculation of reserves and in the design of the reinsurance policy, the company has to combine the opposite goals of maximizing profits and safety. The multi-objective nature of these problems is quite similar to the well known financial dilemma between risk and profit.

In this chapter, we deal with a different multi-objective problem which is related to the so-called *Problem of Optimal Premium Calculation*. This is a classical problem in Actuarial Mathematics (see, for example, Goovaerts et al. 1984; Kaas et al. 2001; Young 2004). This problem deals with the selection of the appropriate premiums to be paid by the insurance policies. At first sight, this seems to be a statistical estimation problem: we should estimate the mean claim amount, which in actuarial terms is known as the net premium. Nevertheless, several extensions of this problem are clearly multi-objective. For example, when we allow the company to modify the premiums paid by the policyholders according to their past claim experience, there are several ways of designing the resulting *Bonus-Malus System*, and they usually involve several different objectives. Let us explain the multi-objective nature of this problem.

It is well known that insurance companies aim to classify the insured policies into homogeneous subsets, assigning the same premium to all the policies belonging to the same set. The classification of the policies is based in the selection of the so-called *risk factors*, which are features of the policies that help the companies to predict their claim amounts in a given period of time (usually one year). The usual approaches to select the risk factors and to calculate the premiums are based on sophisticated statistical techniques (see De Jong and Heller 2008; Denuit et al. 2007; Feldblum 2004; Ohlsson and Johansson 2010).

Nevertheless, these a priori rating techniques often do not eliminate the risk heterogeneity within the subsets, because some of the most important risk factors may be unobservable. This fact forces many insurance companies to adopt bonus-malus rating systems, in order to adjust the premium to the policyholder's past claims experience.

Bonus-Malus systems (BMS) methodology performs another division of the policyholders into classes. All the policyholders in the same class pay the same premium. The claims experience during one year determines the next year class, according to a certain set of transition rules. Those policyholders with no claims will be transferred to better classes, paying a lower premium (bonus). Those who have claims will be transferred to worse classes, paying greater premiums (malus) (see Lemaire 1985, 1995, 1998).

BMS design requires the selection of the number of classes, the transition rules between them and the scale of premiums. It is of great interest for the company to build an optimal BMS, but in order to define optimality we previously have to investigate what are the objectives of the company.

As said above real tariff classes are heterogeneous, and therefore in every class one can find good risks and bad risks paying the same premium. The first objective of a BMS is, no doubt, to reduce such heterogeneity, approaching the premium paid by every policyholder to its real mean claim amount. We can call *Fairness* to this objective. This is the most important objective, since the original purpose of BMS is precisely to increase the fairness of rating systems. In other words, we need BMS because real rating systems are unfair. In fact, this is the only objective traditionally addressed in the literature about optimal BMS design: see Pesonen (1963), Norberg (1976), Borgan et al. (1981), Gilde and Sundt (1989), Heras et al. (2002, 2004).

But fairness is not the only objective in BMS design. Common sense tells us that in order to increase fairness we need tougher punishments. In fact, mild BMS are unfair because almost all the policies will eventually end up in the bonus classes, and therefore they do not discriminate between good and bad risks. But tough BMS are not very popular amongst policyholders. The reactions of these to such heavy punishments include leaving the company, "hit and run" and the "bonus hunger" phenomenon. These considerations allow us to define a new objective in BMS design, the *Degree of Toughness*. Real BMS should be very careful with this degree of toughness, avoiding it to take big values.

BMS designers should maximize fairness while minimizing toughness. This can be done only as a compromise, because we have said that both objectives go in opposite directions: in order to maximize fairness we should maximize also toughness. Nevertheless, sometimes we can find miraculous BMS that seem to be both fair and mild. But this is due to the fact that they make the company lose money. These BMS do not guarantee the equivalence between expected premiums and expected claims in the whole portfolio, and therefore they jeopardize the survival of the company in the long run. In order to survive the company is forced to increase every year the initial or base level of the BMS, thus making wrong the system of punishments and discounts. BMS designers should avoid this behaviour, although it is very common in practice (see Verico 2002). This gives us a third objective, the so-called *Financial Disequilibrium*, which has to be minimized.

The multi-objective nature of BMS design has not been acknowledged in the previous literature about the subject. But we have shown that optimal BMS design is a multi-objective problem, since there are three contradictory involved objectives, which we have called *Fairness*, *Toughness* and *Equilibrium* or *Disequilibrium*. In Sect. 2 we represent in mathematical terms these conflicting objectives and we formulate the corresponding multi-objective program. In Sect. 3 we discuss several multi-objective approaches that help to deal with this problem, and in Sect. 4 we show a numerical practical application. Section 5 concludes the chapter.

## 2 Mathematical Formulation of the Problem

In order to build the mathematical model we will assume some additional hypothesis. As usual in the literature (see, for example, Lemaire 1985, 1995; Denuit et al. 2007), we will assume that the risk characteristics of each policy are summarized in the value of a parameter $\Lambda$, and that the claim numbers from different years are conditionally independent and identically distributed given the risk parameter of the policy. We will also assume that the individual claim amounts are independent of the claim numbers and the risk parameter, and mutually independent and identically distributed. As it is also usual in the literature, we will identify the value of the risk parameter of the policy with its mean claim frequency, which is assumed to be stationary in time.

In this case, taking the mean claim cost as one monetary unit, the fairness objective can be attained by calculating a premium for every insured as close as possible to the (unknown) true value of his parameter. Notice that the calculations are based on the number of claims and not on their amount. In fact, many BMS around the world are exclusively based on the number of claims.

Finally, we will assume that the risk parameter $\Lambda$ is a random variable with known cumulative function $U(\lambda)$ (the *structure function*). This distribution is not a subjective distribution in the pure Bayesian sense. It has a frequency interpretation as different policies will have different values of their risk parameters.

The key idea of the mathematical model is that the evolution of every policy in the BMS can be represented as a Markov chain. This chain will be homogeneous, because we have assumed that each claim frequency $\lambda$ is stationary in time. The transition matrix of the chain can be easily defined from the transition rules of the BMS.

Under very general conditions the Markov chain can be assumed to be regular, and in this case it is well known that there exists a *stationary (conditional) probability distribution* $(\pi_1(\lambda), \ldots, \pi_n(\lambda))$, where $\pi_i(\lambda)$ is defined as the limit value (when the number of periods tends to infinity) of the conditional probability that a policy belongs to the class $C_i$ given that $\Lambda = \lambda$. It is also easy to define the stationary (unconditional) probability distribution $(\pi_1, \ldots, \pi_n)$ for an arbitrary policy, as the mean value of the stationary conditional distribution previously defined, that is,

$$\pi_i = \int \pi_i(\lambda) dU(\lambda)$$

It is clear that $\pi_i$ and $\pi_i(\lambda)$ can be interpreted as the probabilities that an arbitrary policy and a policy conditioned to $\Lambda = \lambda$, respectively, belong to class $C_i$ when stationarity is reached. The knowledge of these stationary distributions becomes a very useful tool when designing a BMS, because it informs us about the long term distribution of the policies. Although it is not strictly necessary, we will add to our previous set of hypothesis the assumption that the BMS has reached, or at least approached, its steady state. The technical details of the calculations of the stationary distributions can be found in Lemaire (1985, 1995) or Heras et al. (2002, 2004).

Having these definitions in mind, it is not difficult to formulate measures of the three objectives of fairness, toughness and equilibrium in mathematical terms. We have said before that the fairness objective is attained by calculating a premium for every insured as close as possible to the (unknown) true value of his parameter $\lambda$. But we can not speak about a single premium to be charged to a policy in a BMS. Instead, we have to consider a set of possible premiums $P_1, \ldots, P_n$ which are paid with probabilities $\pi_1(\lambda), \ldots, \pi_n(\lambda)$. We should say now that the fairness objective is attained when the mean premium $\sum_{i=1}^{n} P_i \pi_i(\lambda)$ is close to the mean claim amount $\lambda$. Therefore, for a policy with associate parameter $\lambda$, the degree of fairness can be measured by the expression

$$\left| \sum_{i=1}^{n} P_i \pi_i(\lambda) - \lambda \right|$$

Similarly, a *global measure of fairness* for a BMS could be defined as

$$\int \left| \sum_{i=1}^{n} P_i \pi_i(\lambda) - \lambda \right| dU(\lambda) \tag{1}$$

Other different measures of fairness for a BMS have been proposed in the literature (see for example Norberg 1976; Borgan et al. 1981; Gilde and Sundt 1989; Lemaire 1985, 1995; Verico 2002). All of them are defined using a quadratic distance function.[1] The fairness measure (1) was originally proposed in Heras et al. (2002, 2004) and, since it is defined using the absolute difference function, it is possible to calculate its value by means of the linear programming methodology. This is an important advantage over the other measures, which require quadratic programming for their calculations.

As for the toughness objective, it can be easily measured by means of the expected squared or absolute deviation of the premiums. Other sensible possibility would be to consider the mean deviation with respect to the initial premium, which

[1] For example, Norberg (1976) proposes the following quadratic distance function: $\int_0^\infty \sum_{j=1}^{n} (\lambda - P_j)^2 \pi_j(\lambda) dU(\lambda)$, which gives rise to the so-called *Bayes Scale* of Premiums. Verico (2002) proposes $\int_0^\infty \left( \sum_{i=1}^{n} P_i \pi_i(\lambda) - \lambda \right)^2 dU(\lambda)$. The other references also consider quadratic functions for measuring the degree of fairness, with the only exception of the linear measure in Heras et al. (2002, 2004).

is (close to) the premium paid by all the policyholders in absence of BMS. This gives rise to the following *toughness measure* (where $P_{ini}$ is the initial premium):

$$\sum_{i=1}^{n} |P_i - P_{ini}|\pi_i \tag{2}$$

As for the last objective, a *financial equilibrium measure* must compare the global expectations of the premiums and the claim amount. This is easily achieved as

$$\left|\sum_{i=1}^{n} P_i\pi_i - E(\Lambda)\right| \tag{3}$$

We conclude that BMS designers have to find the number of classes, the transition rules and the premium scale that solve the following multi-objective program:

$$MIN\left\{\int\left|\sum_{i=1}^{n} P_i\pi_i(\lambda) - \lambda\right|dU(\lambda), \sum_{i=1}^{n}|P_i - P_{ini}|\pi_i, \left|\sum_{i=1}^{n} P_i\pi_i - E(\Lambda)\right|\right\} \tag{4}$$

## 3 Optimal Solutions of the Multi-objective Problem

We could apply any multi-objective methodology to solve this program, although it can be difficult to manage the objectives, especially the first one that involves integrals and absolute values. Nevertheless, if, as usual in the literature, we consider the simpler case of a discrete random parameter $\Lambda$, say $\Lambda = (\lambda_1, \ldots, \lambda_m)$ with probabilities $(q_1, \ldots, q_m)$, then the objectives become simply enough so as to apply the *Goal Programming* (GP) technique.

In fact, the fairness of a given BMS can be calculated by solving the next linear GP program, where the decision variables are the positive and negative deviations $y_i^{\pm}$. The fairness of the BMS will coincide with the optimal value of the objective function.

*Fairness*:

$$\begin{aligned}
&MIN\sum_{j=1}^{m}\left(y_j^+ + y_j^-\right)q_j \\
&s.t. \\
&P_1\pi_1(\lambda_1) + \ldots + P_n\pi_n(\lambda_1) + y_1^- - y_1^+ = \lambda_1 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
&P_1\pi_1(\lambda_m) + \ldots + P_n\pi_n(\lambda_m) + y_m^- - y_m^+ = \lambda_m \\
&y_j^+, y_j^- \geq 0, \forall j = 1, \ldots, m
\end{aligned} \tag{5}$$

Of course, if we also take the premium scale $(P_1, \ldots, P_n)$ as nonnegative decision variables, then the previous linear GP problem can be also used to find the premium scale with maximum fairness, given the number of classes and transition rules of the BMS. This is the approach taken in Heras et al. (2002, 2004).

As for the other two objectives, Toughness and Equilibrium, they can be also calculated as the optimal values of the following two linear GP problems:
*Toughness*:

$$
\begin{array}{l}
MIN\sum_{j=1}^{n}\pi_j\left(y_j^{+}+y_j^{-}\right)\\
s.t.\\
P_1+y_1^{-}-y_1^{+}=P_{ini}\\
\cdots\cdots\cdots\cdots\cdots\cdots\\
P_n+y_n^{-}-y_n^{+}=P_{ini}\\
y_i^{+},y_i^{-}\geq 0,\forall i=1,\ldots,n
\end{array}
\tag{6}
$$

*Equilibrium*:

$$
\begin{array}{l}
MIN(y^{+}+y^{-})\\
s.t.\\
\sum_{i=1}^{n}P_i\pi_i+y^{-}-y^{+}=E(\Lambda)\\
y^{+},y^{-}\geq 0
\end{array}
\tag{7}
$$

Thus, we have three different objectives which can be calculated by means of three linear GP programs. We can then apply different multi-objective techniques for finding an appropriate multi-objective optimum of the problem. If we know the trade-offs between the objectives, we can use a scalarization technique and directly find the Pareto optima of the problem. If we are not able to define such trade-offs, we can apply a non-compensatory methodology such as the ELECTRE method for finding an appropriate set of satisfactory solutions. In a real application, it can be sensible to start with a real BMS and to modify some of its features in order to get an improved BMS, with better values of the three objectives. This last approach is shown in the following numerical example.

## 4 A Numerical Example

Table 1 shows a real Spanish BMS (see Guillén et al. 2005, p. 83) with fifteen levels: seven “bonus” (levels 1–7), seven “malus” (levels 9–15) and one neutral (level 8), which is also the initial level.

The transition rules are the following:

Every free claims year will improve one position next year.
Every single claim will increase one position next year.

**Table 1** Premium scale in a bonus-malus system

| Level | % |
|---|---|
| 1 | 60 |
| 2 | 65 |
| 3 | 70 |
| 4 | 75 |
| 5 | 80 |
| 6 | 85 |
| 7 | 90 |
| 8 | 100 |
| 9 | 110 |
| 10 | 115 |
| 11 | 125 |
| 12 | 135 |
| 13 | 150 |
| 14 | 180 |
| 15 | 200 |

We have assumed that the claims follow a Poisson distribution mixed with an IG (Inverse Gaussian distribution) with parameters $g = 0.101081$ and $h = 0.062981$. This structure function has been fitted to real data in other research works (Lemaire 1995, pp. 35–37). We have discretized the structure function into 20 classes following the methodology of Vilar (2000), and we have calculated the values of the three objectives. Here we show the results:

The discretized structure function is given in Table 2.

And the values of the objectives are

| | |
|---|---|
| Fairness | 0.05531 |
| Toughness | 0.39156 |
| Equilibrium | 0.03933 |

These results can be used to compare the performance of the given BMS with other alternatives. The decision maker can modify some of the components of the BMS, the transition rules and / or the premium scale, looking for better values of the three objectives. Next, we show how a slight modification of the premium scale can improve the values of the objectives in our example. All the calculations shown in this section have been performed by solving linear programs with the mathematical software MAPLE.

As we said above, the premium scale can be considered as a set of additional decision variables in programs (5), (6) and (7). Here we have solved the linear program (5) with such an enlarged set of decision variables, taking the given values of Toughness and Equilibrium as new constraints of the program. We have also included another constraint forcing level 8 to be the initial level, taking the same value (100). Then the application of GP methodology gives the new premium scale shown in Table 3:

**Table 2** Discretized structure function

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\lambda_i$ | 0.033 | 0.066 | 0.099 | 0.132 | 0.165 | 0.198 | 0.231 | 0.264 | 0.297 | 0.330 |
| $q_i$ | 0.28770 | 0.21179 | 0.23174 | 0.06609 | 0.08872 | 0.02623 | 0.03636 | 0.01126 | 0.01592 | 0.00510 |
| $\lambda_i$ | 0.363 | 0.396 | 0.429 | 0.462 | 0.495 | 0.528 | 0.561 | 0.594 | 0.627 | 0.660 |
| $q_i$ | 0.00732 | 0.00240 | 0.00348 | 0.00116 | 0.00171 | 0.00058 | 0.00085 | 0.00029 | 0.00043 | 0.00078 |

**Table 3** Revised premium scale in the bonus-malus system

| Level | % |
|---|---|
| 1 | 64.83 |
| 2 | 71.32 |
| 3 | 78.35 |
| 4 | 82.27 |
| 5 | 86.38 |
| 6 | 90.70 |
| 7 | 95.23 |
| 8 | 100.00 |
| 9 | 110.00 |
| 10 | 121.00 |
| 11 | 133.10 |
| 12 | 146.41 |
| 13 | 161.05 |
| 14 | 193.26 |
| 15 | 231.91 |

The new values of the objective functions are now

| | |
|---|---|
| Fairness | 0.05303 ($<$0.05531) |
| Toughness | 0.34146 ($<$0.39156) |
| Equilibrium | 0.03421 ($<$0.03933) |

We can see that the three objectives have been improved. The application of GP methodology has produced a better BMS with respect to the three evaluation criteria. The new scale gives a Pareto optimum of the multi-objective problem of BMS design.

## 5 Conclusions

In this chapter we have shown the multi-objective nature of an important classical actuarial problem, the design of an optimal Bonus-Malus System. Many actuarial and financial problems can be also considered as multi-objective, being their objectives usually related to the classical conflict between risk and reward. Nevertheless, in the BMS design problem the conflicting objectives have a different nature. We have outlined three such objectives: fairness, toughness and degree of equilibrium. Ideally, BMS should be fair, that is to say, for every policyholder, in the long run the mean values of premiums and claim amounts should be very close. BMS should also be as mild as possible, without large variations of consecutive premiums. Finally, BMS should also be financially balanced, thus guaranteeing the global financial stability of the company. But real MBS cannot be fair, mild and financially balanced, all at the same time. So they have to reach a compromise between these conflicting objectives.

We have also shown that these objectives can be mathematically represented by means of linear problems. In fact, it is possible to calculate their values with linear Goal Programming techniques. Multi-objective algorithms can then be applied both to the problem of calculating the values of the objectives, and to the problem of choosing the best alternative, given the values of the objectives.

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# A New Fitness Guided Crossover Operator and Its Application for Solving the Constrained Portfolio Selection Problem

**K. Liagkouras and K. Metaxiotis**

**Abstract** This chapter proposes a new fitness guided crossover (FGX) operator that consist a variation of the well-known simulated binary crossover (SBX). The proposed FGX operator is tested for solving the constrained portfolio selection problem (CPSP). The performance of the proposed FGX operator is assessed in comparison with the SBX with the assistance of the Non-dominated Sorting Genetic Algorithm II (NSGAII). The evaluation of the performance is based on three performance metrics, namely hypervolume, spread and epsilon indicator. The experimental results reveal that the proposed FGX operator outperforms with confidence the performance of the SBX operator for the majority of the performance metrics when is applied to the solution of the CPSP.

**Keywords** Multiobjective optimization • Evolutionary algorithms • Crossover • Portfolio optimization • Cardinality constrained

## 1 Introduction

The recombination operator is recognized as one of the key operators for progressing the solutions of an Evolutionary Algorithm (EA) (Deb and Goyal 1996) towards higher fitness regions of the search space. However, the available literature regarding the recombination operators for evolutionary multiobjective optimization remains relatively small (Liagkouras and Metaxiotis 2015). The majority of the contemporary multiobjective evolutionary algorithms (MOEAs) (Zitzler and Thiele 1999; Zitzler et al. 2001) make use of the simulated binary crossover (SBX) operator proposed by Deb and Agrawal (1995). The SBX belongs to the family of real-parameter crossover operators and its basic functionality includes a probability distribution around two parents to create two children

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solutions. The SBX is a well established recombination operator that has proved its usefulness by numerous studies in the field.

Subbaraj et al. (2011) used Tugachi method combined with the SBX operator to improve exploitation capability and robustness of the real coded genetic algorithm. In another study by Basu (2011) the SBX is applied to the NSGAII for handling economic environmental dispatch of fixed head hydrothermal power systems as a true multiobjective optimization problem with competing and noncommensurable objectives. According to Tang et al. (2012) the SBX is found to be particularly useful in problems with multiple optimal solutions and narrow global basin, or in problems where the lower and upper bounds of the global optimum are not known a priori. Ramesh et al. (2012) discuss the application of SBX to a new Modified Non-Dominated Sorting Genetic Algorithm-II (MNSGA-II) for solving the multi-objective Reactive Power Planning (RPP) problem. In another study (Li et al. 2012), the authors apply the SBX operator to a new dynamic neighborhood multi-objective evolutionary algorithm based on hypervolume indicator (DNMOEA/HI). According to the authors the solutions obtained by DNMOEA/HI are close to the Pareto optimal front and at the same time are evenly distributed over the front. Li et al. (2013) formulated the optimal dispatch problem as a constrained multi-objective optimization problem. The authors proposed a novel two-phase multi-objective evolutionary approach to solve the optimal dispatch problem. The algorithm makes use of the SBX operator. The results show that the proposed approach for solving the multi-objective dispatch problem, obtains a set of optimal solutions that allows greater flexibility in decision making.

This chapter proposes a new version of the simulated binary crossover (SBX) named Fitness Guided Crossover (FGX) operator that produces better results. The motivation of this study is to build on the existing SBX and presents a mechanism that allows the better exploration of solution space and is able to generate near optimal solutions that lie very close to the True Efficient Frontier (TEF).

The remainder of the chapter is organized as follows. In Sect. 2, the formulation of the constrained portfolio selection problem (CPSP) is presented. In Sect. 3, a description of the simulated binary crossover (SBX) is given and in Sect. 4 the proposed Fitness Guided Crossover (FGX) operator is analyzed. The experimental environment is presented in Sect. 5. Section 6 presents the performance metrics. In Sect. 7 we benchmark the performance of the proposed FGX against the SBX by using data sets from five different stock markets for the solution of the CPSP. In Sect. 8 the results are analyzed and finally, Sect. 9 concludes the chapter.

## 2 The Constrained Portfolio Selection Problem

We examine the portfolio optimization problem as a bi-objective problem (Liagkouras and Metaxiotis 2013, 2014; Metaxiotis and Liagkouras 2012), where the expected return is maximized and the risk is minimized. In its bi-objective form, the constrained portfolio selection problem (CPSP) can be formulated as follows.

Let $\Omega$ be the search space. Consider 2 objective function $f_1$, $f_2$ where $f_i : \Omega \rightarrow R^m$. and $\Omega \subset R^m$.

$$\begin{array}{ll} \text{Optimize} & f(w) = (f_1(w), f_2(w)) \\ \text{Maximize portfolio return} & f_1(w) = \sum_{i=1}^{m} w_i \bar{r}_i \\ \text{Minimize portfolio risk} & f_2(w) = \sum_{i=1}^{m} \sum_{j=1}^{m} w_i w_j \sigma_i \sigma_j \rho_{ij} \end{array}$$

Subject to the following constraints:

(i) Budget constraint or summation constraint $\sum_{i=1}^{m} w_i = 1$, requires all portfolios to have non-negative weights $(0 \leq w_i \leq 1,\ \ i = 1, 2, \ldots, m)$ that sum to 1.

(ii) Floor and ceiling constraint $l_i \leq w_i \leq u_i$, $\forall i = 1, 2, \ldots, m$.Where $l_i$ is the minimum weighting that can be held of asset i ($i = 1, \ldots, m$), $u_i$ = the maximum weighting that can be held of asset i ($i = 1, \ldots, m$) and $0 \leq l_i \leq u_i \leq 1$, $\forall i = 1, 2, \ldots, m$.

(iii) Cardinality constraint $C_{min} \leq \sum_{i=1}^{m} q_i \leq C_{max}$, where $C_{min}$ is the minimum number of assets that a portfolio can hold, $C_{max}$ is the maximum number of assets that a portfolio can hold, $q_i = 1$, for $w_i > 0$ and $q_i = 0$, for $w_i = 0$.

where:

(a) Decision variables $w = (w_1, \ldots, w_m)$ subject to $w \subset \Omega$ and m equal to the number of stocks.

(b) Rate of return of assets: $r_1, r_2, \ldots, w_m$.

(c) $\rho_{ij}$ is the correlation between asset i and j and $-1 \leq \rho_{ij} \leq 1$.

(d) $\sigma_i, \sigma_j$ represent the standard deviation of stocks returns i and j.

The CPSP is a bi-objective problem (Woodside-Oriakhi et al. 2011, 2013) where the first objective corresponds to the *return of assets* and the second objective corresponds to the portfolio risk. The higher the portfolio's *return* the better and the lower the portfolio's *risk* the better.

For the solution representation we chose the hybrid representation, as implemented by the Streichert et al. (2003). In hybrid representation, two vectors are used for defining a portfolio, a real-valued vector used to compute the proportions of the budget invested in the various securities and a binary vector specifies whether a particular security participates in the portfolio.

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Real-valued vector | $w_1$ | $w_2$ | $w_3$ | $w_4$ | $w_5$ | $w_6$ | $w_7$ | $w_8$ | $w_9$ | $w_{10}$ |

where $W = \{w_1, \ \ldots, w_n\}, \ \ 0.01 \leq w_i \leq 0.99, i = 1, \ldots, \ n$

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| Binary-valued vector | $b_1$ | $b_2$ | $b_3$ | $b_4$ | $b_5$ | $b_6$ | $b_7$ | $b_8$ | $b_9$ | $b_{10}$ |

$$B = \{b_1, \ldots, b_n\}, \ b_i \in \{0, 1\}, i = 1, \ldots, n$$

## 3 Simulated Binary Crossover

According to Deb and Agrawal (1995) the crossover operator is believed to be the main search operator of genetic algorithms (GA) as an optimization tool. Also according to Ortiz-Boyer et al. (2005), the crossover operator should establish an adequate balance between exploration and exploitation, and generate offspring in the exploration and exploitation zones in the correct proportion.

The SBX (Deb and Goyal 1996; Deb and Tiwari 2008) uses a probability distribution around two parents to create two children solutions. In SBX as introduced by Deb and Agrawal (1995) each decision variable $x_i$, can take values in the interval: $x_i^{(L)} \leq x_i \leq x_i^{(U)}, i = 1, 2, \ldots, n$. Where $x_i^{(L)}$ and $x_i^{(U)}$ stand respectively for the lower and upper bounds for the decision variable $i$. In SBX, two parent solutions $p^{(1)}$ and $p^{(2)}$ generate two children solutions $c^{(1)}$ and $c^{(2)}$ as follows:

1. Calculate the spread factor $\beta$:

$$\beta = 1 + \frac{2}{p^{(2)} - p^{(1)}} \min\left[\left(p^{(1)} - p^{(l)}\right), \left(p^{(u)} - p^{(2)}\right)\right]$$

2. Calculate parameter $a$:

$$\alpha = 2 - \beta^{-(\eta_c+1)}$$

3. Create a random number $u$ between 0 and 1.

$$u \rightarrow [0, 1];$$

4. Find a parameter $\beta_q$ with the assistance of the following polynomial probability distribution:

$$\beta_q = \begin{cases} (au)^{1/(\eta_c+1)} & \textit{if} \ \ u \leq \frac{1}{a}, \\ \left(\frac{1}{2-au}\right)^{1/(\eta_c+1)} & \textit{otherwise} \end{cases}$$

   The aforementioned procedure allows a zero probability of creating any children solutions outside the prescribed range $[x^{(L)}, x^{(U)}]$. Where $\eta_c$ is the distribution index for SBX and can take any nonnegative value. In particular, small values of $\eta_c$ allow children solutions to be created far away from parents and large values of $\eta_c$ allow children solutions to be created near the parent solutions.

5. The children solutions are then calculated as follows:

$$c^{(1)} = 0.5\left[\left(p^{(1)} + p^{(2)}\right) - \beta_q\left|p^{(2)} - \ p^{(1)}\right|\right]$$
$$c^{(2)} = 0.5\left[\left(p^{(1)} + p^{(2)}\right) + \beta_q\left|p^{(2)} - \ p^{(1)}\right|\right]$$

The probability distributions as shown in *step 4*, do not create any solution outside the given bounds $[x^{(L)}, x^{(U)}]$ instead they scale up the probability for solutions inside the bounds.

## 4 Fitness Guided Crossover Operator

In this study we present a Fitness Guided Crossover (FGX) operator that utilizes a fitness guided mechanism for moving progressively towards the higher fitness regions of the search space. The proposed FGX operator consist a variation of the well-known SBX operator. Thus, the proposed FGX operator shares some common elements with the SBX operator. In particular, as shown below, first we calculate the spread factor $\beta$ and then the parameter $\alpha$ in the same manner as the SBX.

However, in *step 3* we follow a different strategy. As shown in Sect. 2 that illustrates the SBX operator, a random number $u \in [0, 1]$ is generated. If $u \leq 1/a$, it samples to the left hand side (region between $p^{(L)}$ and $p^{(i)}$), otherwise if $u > 1/a$ it samples to the right hand side (region between $p^{(i)}$ and $p^{(U)}$), where $p^{(i)}$ is the *i*th parent solution.

In FGX at this particular point as shown below we follow a different methodology. Specifically, instead of generating a random number $u \in [0, 1]$, we generate two random numbers, $\mathrm{u}_L \in [0, 1/a]$ to sample the left hand side and a random number $\mathrm{u}_R \in (1/a, 1]$ to sample the right hand side of the probability distribution. From the aforementioned process emerge two values of $\beta_q$, the $\beta_q^L$ that samples the left hand side of the polynomial probability distribution and the $\beta_q^R$ that samples the right hand side of the polynomial probability distribution. Next, as shown below in *step 5* with the assistance of $\beta_q^L$ and $\beta_q^R$ are formulated two variants for each child solution. Specifically, $\mathrm{c}_L^{(1)}$ and $\mathrm{c}_R^{(1)}$ are the two variants that emerge by substituting the $\beta_q^L$ and $\beta_q^R$ to $\mathrm{c}^{(1)}$. Respectively $\mathrm{c}_L^{(2)}$ and $\mathrm{c}_R^{(2)}$ are the two variants that emerge by substituting the $\beta_q^L$ and $\beta_q^R$ to $\mathrm{c}^{(2)}$.

Then, by substituting to the parent solution vector at the position of the selected variable to be crossovered, respectively the $\mathrm{c}_L^{(1)}$ and $\mathrm{c}_R^{(1)}$ we create two different child solution vectors (*csv*), the $\mathrm{csv}_L^{(1)}$ and $\mathrm{csv}_R^{(1)}$. Thanks to the generated $\mathrm{csv}_L^{(1)}$ and $\mathrm{csv}_R^{(1)}$ we are able to perform fitness evaluation for each one of the corresponding cases. As soon as we complete the fitness evaluation process, we select the best child solution between the two variants $\mathrm{c}_L^{(1)}$ and $\mathrm{c}_R^{(1)}$ with the assistance of the Pareto optimality framework. The same procedure is followed for $\mathrm{c}_L^{(2)}$ and $\mathrm{c}_R^{(2)}$. The proposed methodology allows us to probe more efficiently the search space and move progressively towards higher fitness solutions. Whenever, there is not a clear winner i.e. strong or weak dominance, between the $c_L^{(1)}$ and $c_R^{(1)}$, or respectively between the $c_L^{(2)}$ and $c_R^{(2)}$ the generation of a random number allows the random choice of one of the two alternative child solutions.

The procedure of computing children solutions $c^{(1)}$ and $c^{(2)}$ from two parent solutions $p^{(1)}$ and $p^{(2)}$ under the Fitness Guided Crossover (FGX) operator is as follows:

1. Calculate the spread factor $\beta$:

$$\beta = 1 + \frac{2}{p^{(2)} - p^{(1)}} \min\left[\left(p^{(1)} - p^{(l)}\right),\ \left(p^{(u)} - p^{(2)}\right)\right]$$

2. Calculate parameter $a$:

$$\alpha = 2 - \beta^{-(\eta_c+1)}$$

   *Where* $\eta_c$ is the distribution index. The distribution index represents the magnitude of the expected variation from the parent values.
3. Create 2 random numbers $u_L \in [0, 1/a]$ and $u_R \in (1/a, 1]$.

$$\begin{array}{l} u_L \rightarrow [0, 1/\alpha] \\ u_R \rightarrow (1/\alpha, 1]; \end{array}$$

4. Find 2 parameters $\beta_q^L$ and $\beta_q^U$ with the assistance of the following polynomial probability distribution:

$$\begin{array}{ll} \beta_q^L = (au_L)^{1/(\eta_c+1)}, & u_L \in (0, 1/a] \\ \beta_q^R = \left(\frac{1}{2-au_R}\right)^{1/(\eta_c+1)}, & u_R \in (1/a, 1] \end{array}$$

5. Thus, instead of a unique value for $c^{(1)}$ and $c^{(2)}$, we obtain two evaluations for each child solution that correspond to $\beta_q^L$ and $\beta_q^R$ respectively:

$$\begin{aligned} c_L^{(1)} &= 0.5\left[\left(p^{(1)} + p^{(2)}\right) - \beta_q^L\left|p^{(2)} - p^{(1)}\right|\right] \\ c_R^{(1)} &= 0.5\left[\left(p^{(1)} + p^{(2)}\right) - \beta_q^R\left|p^{(2)} - p^{(1)}\right|\right] \\ c_L^{(2)} &= 0.5\left[\left(p^{(1)} + p^{(2)}\right) + \beta_q^L\left|p^{(2)} - p^{(1)}\right|\right] \\ c_R^{(2)} &= 0.5\left[\left(p^{(1)} + p^{(2)}\right) + \beta_q^R\left|p^{(2)} - p^{(1)}\right|\right] \end{aligned}$$

6. We perform fitness evaluation for each variant child solution, by substituting the candidate solutions into the parent solution vector.
7. We select the best variant between the $c_L^{(1)}$ and $c_R^{(1)}$, based on the Pareto optimality framework. In particular, by substituting the candidate solutions $c_L^{(1)}$

and $c_R^{(1)}$ into the parent solution vector we create two different child solution vectors (*csv*), the $csv_L^{(1)}$ and $csv_R^{(1)}$. Thanks to the generated $csv_L^{(1)}$ and $csv_R^{(1)}$ we are able to perform fitness evaluation for each one of the corresponding cases. Thereafter, we select the best alternative based on the Pareto optimality framework.

In particular let $\Omega$ be the search space. Consider $n$ objective functions $f_1, f_2, \ldots, f_n$ where $f_i : \Omega \rightarrow R^n$ and $\Omega \subset R^n$. The multiobjective optimization problem can be described mathematically as follows:

$$\min[f_1(x), f_2(x), \ldots, f_n(x)]$$
$$x \in \Omega$$

A decision vector $x^* \in \Omega$ is **Pareto optimal** if there is no other decision vector $x \in \Omega$ such that $f_i(x) \leq f_i(x^*)$ for all $i \in \{1, \ldots, n\}$ and $f_i(x) < f_i(x^*)$ for at least one $i \in \{1, \ldots, n\}$.

A decision vector $x^* \in \Omega$ is **weakly Pareto optimal** if there is no other decision vector $x \in \Omega$ such that $f_i(x) < f_i(x^*)$ for all $i \in \{1, \ldots, n\}$.

A decision vector $x^* \in \Omega$ **Pareto dominates** a decision vector $x \in \Omega$ if $f_i(x^*) \leq f_i(x)$ for all $i \in \{1, \ldots, n\}$ and $f_i(x^*) < f_i(x)$ for at least one $i \in \{1, \ldots, n\}$.

A decision vector $x^* \in \Omega$ **strictly dominates** a decision vector $x \in \Omega$ if $f_i(x^*) < f_i(x)$ for all $i \in \{1, \ldots, n\}$.

Assuming, $F^L_{Return(1)}$ and $F^L_{Risk(1)}$ stand for the fitness evaluations for the *return* and *risk* objective respectively of the $csv_L^{(1)}$. Likewise, $F^R_{Return(1)}$ and $F^R_{Risk(1)}$ stand for the fitness evaluations for the *return* and *risk* objective respectively of the $csv_R^{(1)}$.

if (*return* and *risk* combination of the $csv_L^{(1)}$ dominates *return* and *risk* combination of the $csv_R^{(1)}$)
then
$c^{(1)} = c_L^{(1)}$
else if (*return* and *risk* combination of the $csv_R^{(1)}$ dominates *return* and *risk* combination of the $csv_L^{(1)}$)
then
$c^{(1)} = c_R^{(1)}$
else

The same procedure is followed for $c_L^{(2)}$ and $c_R^{(2)}$.

8. Whenever, there is not a clear winner i.e. strong or weak dominance, between the $c_L^{(1)}$ and $c_R^{(1)}$, or respectively between the $c_L^{(2)}$ and $c_R^{(2)}$ the generation of a random number allows the random choice of one of the two alternative child solutions. This can be expressed as follows for the case of $c^{(1)}$:

$r \quad \rightarrow [0, 1];$
if (r<= 0.5) then
$\quad c^{(1)} = c_L^{(1)}$
else if (r > 0.5) then
$\quad c^{(1)} = c_R^{(1)}$
end if

# 5 Experimental Environment

All algorithms have been implemented in Java and run on a 2.1GHz Windows Server 2012 machine with 6GB RAM. We compare the performance of the proposed, Fitness Guided Crossover (FGX) operator against the Simulated Binary Crossover (SBX) operator with the assistance of the NSGAII (Deb et al. 2002). In all tests we use, binary tournament and polynomial mutation (PLM) as, selection and mutation operator, respectively. The crossover probability is $P_c = 0.9$ and mutation probability is $P_m = 1/n$, where $n$ is the number of decision variables. The distribution indices for the crossover and mutation operators are $\eta_c = 20$ and $\eta_m = 20$, respectively. Population size is set to *100*, using *25,000* function evaluations with *20* independent runs.

We set the minimum cardinality of the portfolio to two ($K_{min} = 2$) and the maximum cardinality of the portfolio to five ($K_{max} = 5$) for all test problems. The participation of each stock in the portfolio is determined by the lower and upper bounds. We set the lower bound $l_i = 0.01$ and the upper bound $u_i = 0.99$, for each asset $i$, where $i = 1, \ldots, n$.

# 6 Performance Metrics

## 6.1 Hypervolume

The hypervolume (Emmerich et al. 2005; Zitzler et al. 2007) is an indicator of both the convergence and diversity of an approximation set. Thus, given a set $S$ containing $m$ points in $n$ objectives, the hypervolume of $S$ is the size of the portion of objective space that is dominated by at least one point in $S$. The hypervolume of $S$ is calculated relative to a reference point which is worse than (or equal to) every point in $S$ in every objective. The greater the hypervolume of a solution the better considered the solution.

### *6.2 Spread*

The spread of solutions (Δ) (Deb et al. 2002) is an indicator of the quality of the derived set of solutions. Spread indicator examines whether or not the solutions span the entire Pareto optimal region.

### *6.3 Epsilon Indicator* $\mathbf{I}_{\varepsilon}$

The basic usefulness of the unary additive epsilon indicator (Zitzler et al. 2003) of an approximation set $A$ ($I_{\varepsilon+}$) is that it provides the minimum term $\varepsilon$ by which each point of the real front $R$ in the objective space can be shifted by component-wide addition, such that the resulting transformed approximation set is dominated by $A$. The additive epsilon indicator is a good measure of diversity, since it focuses on the worst case distance and reveals whether or not the approximation set has gaps in its trade-off solution set.

## 7 Experimental Results

We performed a number of computational experiments to test the performance of the proposed Fitness Guided Crossover (FGX) operator for the solution of five portfolio optimization problems (port1–5) that correspond to five different capital markets as shown in Table 1. The data sets that have been used in this study have been made publicly available by the OR-Library retained by Beasley (Chang et al. 2000). The performance of the proposed FGX operator is assessed in comparison with the Simulated Binary Crossover (SBX) operator with the assistance of the Non-dominated Sorting Genetic Algorithm II (NSGAII). The evaluation of the performance is based on three metrics, namely hypervolume, spread and epsilon indicator.

Tables 2, 3 and 4, present the results regarding the port1–5 problems in OR-Library. In particular, Table 2 presents the results of FGX and SBX alike for a number of performance metrics. Specifically, it presents the mean, standard deviation (STD), median and interquartile range (IQR) of all the independent runs carried out for Hypervolume (HV), Spread (Δ) and Epsilon indicator respectively.

With regard to HV indicator, the higher the value, the better the computed front. The second indicator the Spread (Δ) examines the spread of solutions across the pareto front. The smaller the value of this indicator, the better the distribution of the solutions. The spread indicator takes a zero value for an ideal distribution of the solutions in the Pareto front. Finally, the Epsilon indicator is a measure of the smaller distance that a solution set A, needs to be changed in such a way that it

**Table 1** The OR-library portfolio optimization problems

| Problem name | Stock market index | Assets |
|---|---|---|
| port1 | Hang Seng | 31 |
| port2 | DAX100 | 85 |
| port3 | FTSE100 | 89 |
| port4 | S&P100 | 98 |
| port5 | Nikkei225 | 225 |

**Table 2** Mean, std, median and Iqr for Hv, spread and epsilon

| | | NSGAII | |
|---|---|---|---|
| Problem | | FGX | SBX |
| PORT1 | HV. Mean and Std | 8.06e–01$_{3.0e–03}$ | 7.60e–01$_{1.1e–02}$ |
| | HV. Median and IQR | 8.06e–01$_{2.5e–03}$ | 7.62e–01$_{1.9e–02}$ |
| | SPREAD. Mean and Std | 7.89e–01$_{7.1e–02}$ | 6.92e–01$_{8.9e–02}$ |
| | SPREAD. Median and IQR | 8.07e–01$_{9.6e–02}$ | 6.96e–01$_{1.4e–01}$ |
| | EPSILON. Mean and Std | 9.76e–05$_{1.9e–05}$ | 2.66e–04$_{5.9e–05}$ |
| | EPSILON. Median and IQR | 9.81e–05$_{2.5e–05}$ | 2.70e–04$_{9.8e–05}$ |
| PORT2 | HV. Mean and Std | 8.13e–01$_{2.5e–03}$ | 7.69e–01$_{1.9e–02}$ |
| | HV. Median and IQR | 8.13e–01$_{5.0e–03}$ | 7.71e–01$_{3.5e–02}$ |
| | SPREAD. Mean and Std | 7.91e–01$_{7.8e–02}$ | 7.98e–01$_{6.9e–02}$ |
| | SPREAD. Median and IQR | 8.18e–01$_{1.4e–01}$ | 8.01e–01$_{1.3e–01}$ |
| | EPSILON. Mean and Std | 4.05e–04$_{2.9e–05}$ | 5.26e–04$_{1.5e–04}$ |
| | EPSILON. Median and IQR | 3.97e–04$_{5.7e–05}$ | 5.17e–04$_{3.1e–04}$ |
| PORT3 | HV. Mean and Std | 7.83e–01$_{6.1e–03}$ | 6.74e–01$_{1.2e–02}$ |
| | HV. Median and IQR | 7.83e–01$_{1.1e–02}$ | 6.74e–01$_{2.3e–02}$ |
| | SPREAD. Mean and Std | 6.44e–01$_{8.6e–02}$ | 5.92e–01$_{3.3e–02}$ |
| | SPREAD. Median and IQR | 5.86e–01$_{1.8e–01}$ | 5.96e–01$_{6.5e–02}$ |
| | EPSILON. Mean and Std | 1.39e–04$_{3.3e–05}$ | 2.82e–04$_{1.6e–05}$ |
| | EPSILON. Median and IQR | 1.56e–04$_{4.9e–05}$ | 2.76e–04$_{2.5e–05}$ |
| PORT4 | HV. Mean and Std | 8.02e–01$_{2.2e–02}$ | 7.33e–01$_{1.2e–02}$ |
| | HV. Median and IQR | 8.08e–01$_{3.5e–02}$ | 7.29e–01$_{2.5e–02}$ |
| | SPREAD. Mean and Std | 8.23e–01$_{6.2e–02}$ | 6.76e–01$_{8.0e–02}$ |
| | SPREAD. Median and IQR | 7.91e–01$_{1.3e–01}$ | 6.42e–01$_{1.4e–01}$ |
| | EPSILON. Mean and Std | 3.87e–04$_{8.2e–05}$ | 5.70e–04$_{5.5e–05}$ |
| | EPSILON. Median and IQR | 3.68e–04$_{1.4e–04}$ | 5.49e–04$_{7.9e–05}$ |
| PORT5 | HV. Mean and Std | 8.56e–01$_{2.2e–02}$ | 7.37e–01$_{1.7e–02}$ |
| | HV. Median and IQR | 8.61e–01$_{3.7e–02}$ | 7.26e–01$_{3.6e–02}$ |
| | SPREAD. Mean and Std | 7.84e–01$_{1.6e–01}$ | 8.11e–01$_{6.9e–02}$ |
| | SPREAD. Median and IQR | 7.25e–01$_{3.0e–01}$ | 7.82e–01$_{1.5e–01}$ |
| | EPSILON. Mean and Std | 2.14e–04$_{4.3e–05}$ | 2.98e–04$_{5.7e–05}$ |
| | EPSILON. Median and IQR | 2.11e–04$_{9.2e–05}$ | 3.13e–04$_{1.2e–04}$ |

**Table 3** Boxplots for Hv, spread and epsilon

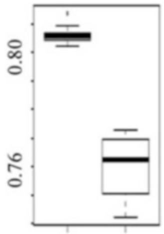
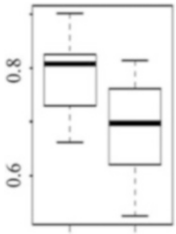
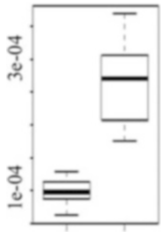
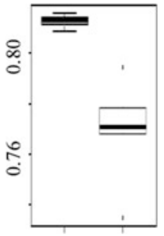
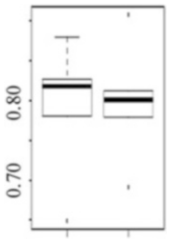
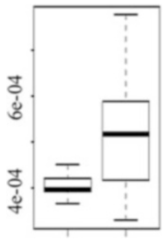
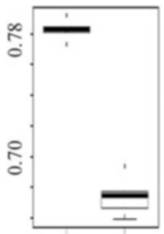
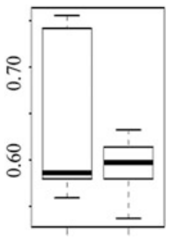
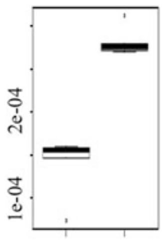
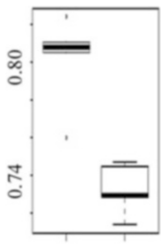
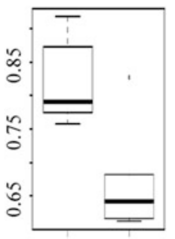
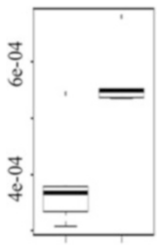
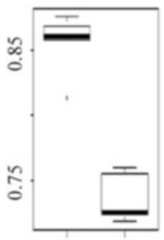
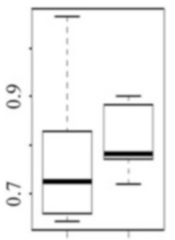
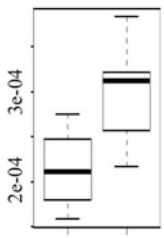
<table>
<tr><th rowspan="3">Problem</th><th colspan="6">NSGAII</th></tr>
<tr><th colspan="2">HV</th><th colspan="2">SPREAD</th><th colspan="2">EPSILON</th></tr>
<tr><th>PGM</th><th>PLM</th><th>PGM</th><th>PLM</th><th>PGM</th><th>PLM</th></tr>
<tr><td>PORT1</td><td colspan="2"></td><td colspan="2"></td><td colspan="2"></td></tr>
<tr><td>PORT2</td><td colspan="2"></td><td colspan="2"></td><td colspan="2"></td></tr>
<tr><td>PORT3</td><td colspan="2"></td><td colspan="2"></td><td colspan="2"></td></tr>
<tr><td>PORT4</td><td colspan="2"></td><td colspan="2"></td><td colspan="2"></td></tr>
<tr><td>PORT5</td><td colspan="2"></td><td colspan="2"></td><td colspan="2"></td></tr>
</table>

**Table 4** Wilcoxon test for Hv, spread and epsilon

| Problem | | | NSGAII with SBX |
|---|---|---|---|
| PORT1 | HV. Mean and Std | NSGAII with FGX | ↑ |
| | HV. Median and IQR | | ↑ |
| | SPREAD. Mean and Std | | ↓ |
| | SPREAD. Median and IQR | | ↓ |
| | EPSILON. Mean and Std | | ↑ |
| | EPSILON. Median and IQR | | ↑ |
| PORT2 | HV. Mean and Std | NSGAII with FGX | ↑ |
| | HV. Median and IQR | | ↑ |
| | SPREAD. Mean and Std | | – |
| | SPREAD. Median and IQR | | – |
| | EPSILON. Mean and Std | | ↑ |
| | EPSILON. Median and IQR | | ↑ |
| PORT3 | HV. Mean and Std | NSGAII with FGX | ↑ |
| | HV. Median and IQR | | ↑ |
| | SPREAD. Mean and Std | | – |
| | SPREAD. Median and IQR | | – |
| | EPSILON. Mean and Std | | ↑ |
| | EPSILON. Median and IQR | | ↑ |
| PORT4 | HV. Mean and Std | NSGAII with FGX | ↑ |
| | HV. Median and IQR | | ↑ |
| | SPREAD. Mean and Std | | ↓ |
| | SPREAD. Median and IQR | | ↓ |
| | EPSILON. Mean and Std | | ↑ |
| | EPSILON. Median and IQR | | ↑ |
| PORT5 | HV. Mean and Std | NSGAII with FGX | ↑ |
| | HV. Median and IQR | | ↑ |
| | SPREAD. Mean and Std | | ↑ |
| | SPREAD. Median and IQR | | ↑ |
| | EPSILON. Mean and Std | | ↑ |
| | EPSILON. Median and IQR | | ↑ |

dominates the optimal Pareto front of this problem. The smaller the value of Epsilon indicator, the better the derived solution set.

Table 3 use boxplots to present graphically the performance of NSGAII under the FGX and SBX respectively, for the three performance indicators, namely: HV, Spread and Epsilon. Boxplots provide a 5-number summary of the data (min, max, Q1, Q3, median) and information about outliers.

Table 4, with the assistance of the Wilcoxon rank-sum test presents, if the results of NSGAII derived under the two different configurations, FGX and SBX respectively are statistically significant or not. In Table 4, three different symbols are used. In particular "–" indicates that there is not statistical significance between the algorithms. "↑" means that the algorithm in the row has yielded better results than

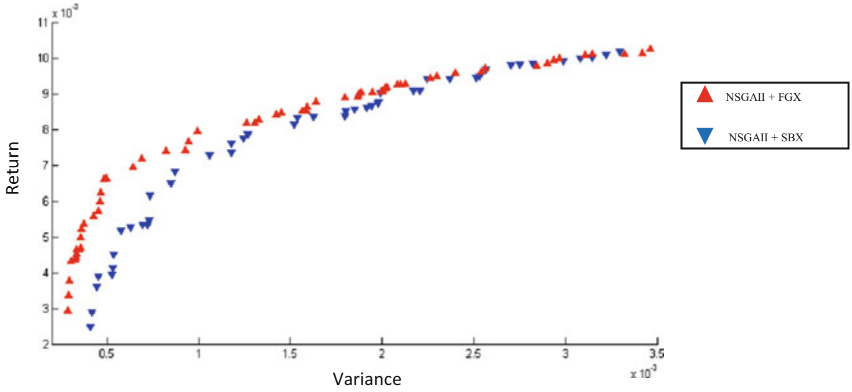



**Fig. 1** Pareto Fronts under the FGX and the SBX for the port1 problem, with n = 31 securities

the algorithm in the column with confidence and "↓" is used when the algorithm in the column is statistically better than the algorithm in the row.

Figure 1 illustrates the approximate efficient frontier derived by the NSGAII under two different configurations: the FGX and the SBX for the port1 test instance. The approximate frontiers that are illustrated in Fig. 1, have been formulated by merging the approximate sets for the 20 independent runs of the algorithm and by removing the dominated solutions. From Fig. 1 it becomes clear that the proposed configuration of the algorithm with the FGX clearly outperforms the typical configuration of the NSGAII with the SBX, as the approximate frontier that is generated by the SBX is dominated by the corresponding approximate frontier that it is generated by the proposed FGX operator. Also, according to Tables 2, 3 and 4 the proposed FGX operator outperforms with confidence the classical SBX in HV and Epsilon performance metrics for the port1 portfolio optimization problem. On the other hand, the SBX operator outperforms the FGX in Spread metric for the port1 problem.

Similarly, Fig. 2 illustrates the approximate efficient frontier derived by the NSGAII under two different configurations: the FGX and the SBX for the port2 test instance. As appeared in Fig. 2 the proposed configuration of the algorithm with the FGX clearly outperforms the typical configuration of the NSGAII with the SBX, as the approximate frontier that is generated by the SBX is dominated by the corresponding approximate frontier that it is generated by the proposed FGX operator. With regard to Tables 2, 3 and 4 the proposed FGX operator outperforms with confidence the SBX in HV and Epsilon performance metrics for the port2 problem. Also, there was not statistical significance between the FGX and SBX for the Spread metric for the port2 problem.

The situation is similar for the port3 problem. The approximate efficient frontier that it is generated by the FGX operator clearly outperforms the corresponding efficient frontier that it is generated by the SBX operator as shown in Fig. 3. Tables 2, 3 and 4 confirm the relevant findings as the FGX operator outperforms

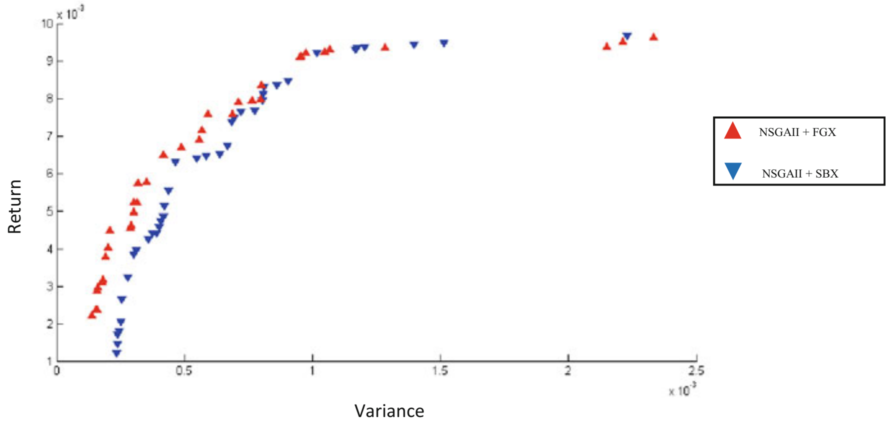



**Fig. 2** Pareto Fronts under the FGX and the SBX for the port2 problem, with n = 85 securities

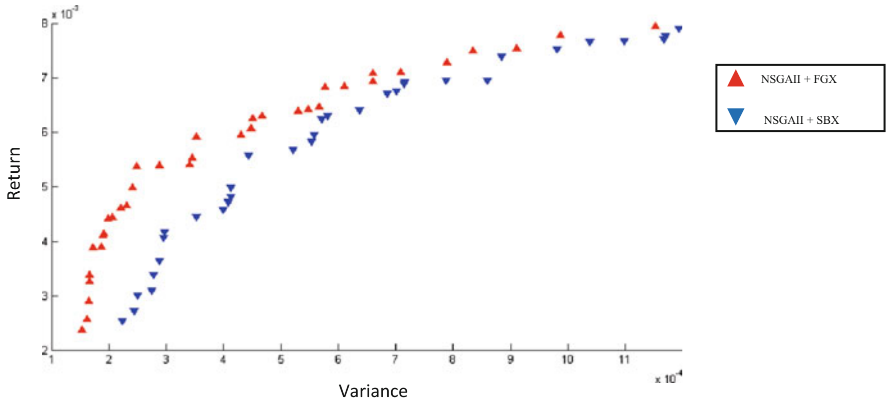



**Fig. 3** Pareto Fronts under the FGX and the SBX for the port3 problem, with n = 89 securities

with confidence the SBX in HV and Epsilon performance metrics for the port3 problem. Also, there was not statistical significance between the FGX and SBX for the Spread metric for the port3 problem.

Regarding the port4 problem as shown in Tables 2, 3 and 4 the proposed FGX operator outperforms with confidence the classical SBX in HV and Epsilon performance metrics. Also, there was not statistical significance between the FGX and SBX for the Spread metric for the port4 problem.

Lastly, the same situation is repeated for the port5 problem. In particular, as shown in Tables 2, 3 and 4 the proposed FGX operator outperforms with confidence the classical SBX in HV and Epsilon performance metrics for the port5 problem. Also, there was not statistical significance between the FGX and SBX for the Spread metric for the port5 problem.

## 8 Analysis of the Results

In this section, we analyze the results obtained by applying the Fitness Guided Crossover (FGX) operator and the Simulated Binary Crossover (SBX) operator respectively to the NSGAII. Three of the most prominent performance indicators of MOEAs namely HV, spread and epsilon have been applied to assess the quality of the proposed recombination operator.

With regard to the results of the HV indicator, as shown in Tables 2, 3 and 4, the proposed FGX operator outperforms with confidence the SBX for all test instances. Figures 1, 2 and 3 provide a graphical representation of the relevant approximate efficient frontiers that are generated by the FGX and the SBX respectively for the port1–3 problems that confirm our findings.

With regard to the spread metric, we notice that there is not statistical significance between the FGX and SBX for the port2–3 test instances. Also, the SBX operator outperforms the FGX in Spread metric for the port1 and 4 problems. Finally, the FGX operator outperforms the SBX in Spread metric for the port5 problem. The performance of the FGX operator in spread metric can be explained by the fitness guided mechanism introduced in the proposed methodology. In particular, sometimes the exploration of the most promising regions of the search space in terms of fitness value can occur at the expense of the spread of solutions.

Lastly, with regard to the epsilon indicator, as shown in Tables 2, 3 and 4 the FGX yields better results with confidence than the conventional configuration of NSGAII with the SBX operator for all test instances port1–5.

## 9 Conclusions

This chapter introduced a new Fitness Guided Crossover (FGX) operator and its application for solving the constrained portfolio selection problem (CPSP). The proposed recombination operator incorporates a fitness guided mechanism that allows the evaluation of the corresponding fitness for the left hand side region and the right hand side region of the parent solution. The selection between the two alternative child solutions is performed with the assistance of the Pareto optimality conditions. Thanks to the fitness guided mechanism introduced in the FGX operator the algorithm is able to move progressively towards the higher fitness regions of the search space and discover near optimal solutions.

The evaluation of the proposed recombination operator is done with the assistance of five portfolio optimization problems (port1–5) obtained from the OR-Library retained by Beasley and correspond to five different capital markets. The performance of the proposed Fitness Guided Crossover (FGX) operator is assessed in comparison with the Simulated Binary Crossover (SBX) operator with the assistance of the Non-dominated Sorting Genetic Algorithm II (NSGAII) for the solution of the CPSP. The evaluation of the performance is based on three performance metrics, namely hypervolume, spread and epsilon.

The experimental results indicate that the proposed FGX operator outperforms with confidence the SBX operator for all test instances, when applied to the NSGAII. Finally, through the use of figures we provide evidence that the approximate efficient frontiers that are generated by the proposed FGX operator dominate the corresponding approximate efficient frontiers that are generated by the SBX operator.

Please, notice that the parameters of the experimental environment in Sect. 5 have been set having in mind the optimum performance of the algorithm. Obviously, the alteration of the experimental parameters affects the search process and subsequently the formulation of the approximate efficient frontier. However, under all possible configurations, the FGX operator outperforms the performance of the SBX operator when is applied to NSGAII for the solution of the CPSP.

In our future work, we will attempt to develop a technique that will update the crossover probability ($P_c$) at run-time according to the performance of the algorithm in hypervolume or another performance metric. Also, we plan to consider some additional constraints like trading or transaction costs constraints.

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# BOCR Analysis: A Framework for Forming Portfolio of Developing Projects

**Ayeley P. Tchangani**

**Abstract** The pressure exerted by citizens (mainly in developing countries) to dispose of high standards in terms of quality of life (modern infrastructures and bearable environment) along with scarce financial resources that countries are facing, render selection and investment in developing projects very challenging for decision makers (national government, regional authorities, municipal councils, etc.). Indeed, modern life requirements are so antagonistic that projects to undertake to satisfy them are so different in nature that, it becomes difficult to evaluate them in order to select the most suitable ones to form projects portfolio with classical decision making tools. These constraints appeal for sound and integrated methods, tools or framework to support addressing developing projects portfolio forming and managing process. In this chapter, recent developments in decision analysis in terms of BOCR (Benefit, Opportunity, Cost, and Risk) analysis, that result from considering the fact that attributes that characterize projects may support or reject decision makers goals and inherent uncertainty that do exist in relationships between different components of such decision problems will be exposed. This framework has the advantage to permit to evaluate projects of different nature (characterized by non-homogeneous attributes or criteria); the important things being the relationships (supporting or rejecting) between these attributes and the pursued objectives.

**Keywords** Decision Analysis • MCDM • BOCR Analysis • Developing Projects • Projects Portfolio Forming

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## 1 Introduction of Developing Projects Issues

There is more and more needs to use analytics in public and policy decision making processes to gain in efficiency, legitimacy and acceptance by stakeholders (see Tsoukias et al. 2013). Decision making in developing projects selection and projects portfolio forming falls into this framework of necessity for efficiency, legitimacy and acceptance. It is well known that, developing processes and citizen's life quality are very dependent on incentives among which infrastructures and mainly those known as strategic or critical infrastructures play a great role. Infrastructures are physical assets such as transport networks (roads, airports, ports, intermodal facilities, railway, mass transit networks and traffic control systems); energy installations and networks; water networks (dams, storage, treatment and networks); production, storage and transport of dangerous goods (e.g. chemical, biological, radiological and nuclear materials); health care networks; etc. that are capable to an intended service delivery as well as flexible assets such as utilities and facilities related to government administration services; communications and information technology; finance and banking; securities and investment; etc. Strategic or critical infrastructures, for an entity such as a country, a region, an organization, or an enterprise, etc. are those assets which destruction, degradation or unavailability for an extended period would significantly impact on its social and economic well-being or affect its ability for defense and security and therefore do have a great impact on developing processes as well as life quality of citizens. Most of developing projects are related to building and managing such infrastructures that are different in nature, are characterized by non-homogenous attributes and must satisfy multiple objectives so that forming a portfolio of developing projects is a multi-attributes/multi-objectives decision making problem.

Most of the time developing projects are considered separately whereas in reality some projects may be redundant so that finally the efficiency of the overall actions is reduced instead of increasing. There is, therefore, necessity to dispose of a framework that permits decision makers forming portfolio of projects instead of financing projects separately.

The purpose of this chapter is to present BOCR analysis framework and apply it to forming developing projects portfolio for a region of a developing country. The chapter is organized as follows: after a review of existing approaches to deal with multi-attributes/multi-objectives decision (selection of projects) making problems, BOCR analysis as a structuring framework will be presented, in a third part results from BOCR analysis will be used in different ways to formulate projects portfolio selection problems, application of this approach to a real world problem will be then presented in a fourth section and a conclusion will end the chapter.

## 2 Review of Existing Approaches

Classically, three main approaches have dominated evaluation process in multi-criteria decision making: value type approach (a value function or an utility measure is derived for each alternative to represent its adequacy with decision goal); outranking methods (a pair comparison of alternatives is realized); and decision rules approach (a set of decision rules are derived from a decision table with possible missing data). All these approaches are supposed to have a single objective to satisfy and a common attributes set for alternatives. These approaches are briefly described below.

### *2.1 Value Type Approach*

Roughly speaking these techniques consider a numerical function $\pi$ known as value or utility function) defined on the alternatives set $U$ such that

$$\pi(u) \geq \pi(v) \Leftrightarrow u \succsim v \tag{1}$$

where $u \succsim v$ stands for "$u$ is at least as good, with regard to decision goal, as $v$" leading to an order on $U$. The evaluation modeling process then consists in building such a function based on attributes measures and decision makers preference (obtained in general by answering some particular questions of the analyst); there are many techniques employed in the literature for constructing such a function where a number of them suppose a particular form for $\pi$ such as an expected utility form or an additive value function (interested reader may consult Bouyssou et al. (2000), Steuer (1986) and references therein).

### *2.2 Outranking Methods*

In this case, a pair-wise comparison of alternatives is carried up under each attribute or criteria to derive a pre-order on the alternatives set $U$ allowing incomparability and/or intransitivity; methods such as ELECTRE procedures and PROMETHEE techniques (Bouyssou et al. 2000; Brans et al. 1986a, b; Vincke 1989) belong to this category.

### 2.3 Decision Rules Approach

A set of decision rules are derived by a learning process from a known decision table with possibly incomplete data (see Greco et al. 2001). One can see that to apply such approach, it is necessary to dispose of previously collected data of similar problems to that being solved; this is generally impossible for some problems such as that of portfolio forming being considered here.

In this chapter we adopt an approach that highlights *bipolarity notion* between attributes and objectives to structure the process and to evaluate alternatives of a decision making problems. We are motivated by the fact that cognitive psychologists have observed for long time that human evaluate alternatives by considering separately their positive aspects and their negative aspects; that is on a bipolar basis (see for instance Caciopo and Berntson 1994; Osgood et al. 1957). To this end, we introduce *supporting* and *rejecting* notions (Tchangani 2010; Tchangani and Pérès 2010; Tchangani et al. 2012) that relate attributes to objectives leading to an evaluation model known as BOCR analysis.

## 3 BOCR Analysis Framework

Cognitive psychologists noticed for long time that humans generally evaluate alternatives in decision process by comparing pros and cons of each alternative with regard to decision goal (see Caciopo and Berntson 1994; Osgood et al. 1957). Building on this observation, we introduce *supporting/rejecting* notions to characterize relationships between attributes and objectives: an objective $o$ is said to be supported (respect. rejected) by an attribute $a$ if and only if its variation is positively (respect. negatively) correlated with the variation of that attribute. Otherwise this attribute is said to be neutral with regard to that objective. It is important to notice that, supporting and rejecting notions are value, culture, and context dependent; that is a supporting attribute in some context or for a stakeholder with particular value or culture may be considered as rejecting in some other circumstances. This framework, therefore, appeals for value and context driven learning instead of data driven learning (generally used in MCDM literature) as suggested by Tsoukias et al. (2013) to tackle policy analytics for decision making.

In order to dispose with a structured framework for the elicitation and evaluation process of attributes, we propose to use BOCR analysis (see Tchangani 2010; Tchangani and Pérès 2010; Tchangani et al. 2012) that comes from the convergence of these supporting and rejecting notions and uncertainty consideration (long range outcomes and/or effects of decision being taken) to give a framework where an alternative (project) is completely described by Table 1 where benefit (B) attributes are certain attributes that support the considered objective; opportunity (O) attributes correspond to uncertain attributes that support the objective; cost

**Table 1** BOCR structured identity card of a project

| Project ($P$): description; necessary amount for this project: $F(P)$ | | |
|---|---|---|
| Objectives | BOCR factors | Attributes |
| …. | …. | …. |
| Objective $i$ | ***B*** | a: description (weight $\omega(a)$, measure $\upsilon(a)$)<br>…. |
| | **O** | a: description (weight $\omega(a)$, measure $\upsilon(a)$)<br>…. |
| | **C** | a: description (weight $\omega(a)$, measure $\upsilon(a)$)<br>…. |
| | **R** | a: description (weight $\omega(a)$, measure $\upsilon(a)$)<br>… |
| …. | …. | …. |

(C) attributes regroup certain attributes that reject that objective and risk (R) attributes are uncertain attributes that reject the objective.

Table 1 is just an indication, indeed, in practice attributes may need to be decomposed until an operational and assessment level is reached. The measure or value $\upsilon(a)$ of attribute $a$ for a given BOCR component, obtained by direct measure or expertise for operational attributes or by aggregating sublevel attributes measures, represents how well this attribute contribute to the corresponding BOCR component achievement; the weight $\omega(a)$ in contrary represents the relative importance degree of this attribute in comparison of other attributes of the same category and level that can easily be determined by methods such as analytic hierarchy process (AHP) (see Saaty 2005) by answering questions like “how important is the contribution of attribute $a$ in comparison of its counterparts in the realization of upper level attribute”.

For certain components, namely benefits and costs, such measures can be assessed using known techniques such as (AHP) or any method that could assign a measure to an attribute with regard to a couple objective-alternative; the advantage of using AHP approach reside in its ability to deal with hierarchy (which allows to decompose attributes from more general statements to more measurable or comparable attributes) and intangible variables.

For uncertain components, opportunity and risk, identification and assessment approach may build on well known risk assessment approach (see Haimes et al. 2002; Tchangani 2011; Bouzarour et al. 2012). The process begin by answering questions of the form “what can go wrong ?” as proposed by Haimes et al. (2002) to identify risk attributes (respectively “what might go better ?” for opportunity attributes). To obtain risk or opportunity value of an attribute, one must consider its likelihood and its severity (in the case of risk) or its importance (in the case of opportunity). Severity and importance depend on some properties of the considered alternative with regards to the attribute such as resilience, robustness, vulnerability, fragility, etc. (see Tchangani 2015).

This model allows alternatives to be characterized by heterogeneous attributes and incomparability between alternatives in terms of Pareto-equilibrium. Indeed, decision making situations where alternatives are characterized by attributes of different nature are pervasive in real world applications. One may think about a government evaluating projects that belong to different domains such as health, infrastructures, social, economics, etc. with the main objective to enhance a country developing process or an enterprise planning to invest in projects of different nature. In these situations, though attributes characterizing projects may be completely different, the important thing is their adequacy with regards to the pursued objectives, so that alternative projects can be ultimately compared on the same basis (decision maker desires).

Finally, each project $P$, will be characterized by four measures, its benefit measure $B(P)$, its opportunity measure $O(P)$, its cost measure, $C(P)$, and its risk measure $R(P)$; these measures are obtained by aggregating the corresponding measures over the set of objectives which are obtained by aggregating attributes measure of each category. Given the homogeneity of attributes obtained from BOCR structuring that creates synergy between measure to aggregate, a synergetic aggregation operator such as Choquet integral associated with a weighted cardinal fuzzy measure (WCFM) (see Tchangani 2013) that leads to a straightforward formula for this integral, is used at each level that needs aggregation.

## 4 Projects Portfolio Forming Problem

The situation of projects portfolio forming is the following: authorities of a community with limited financial resources that we refer to as fund (monetary) which quantity is denoted by $F$ are willing to fund some developing projects to enhance sustainable development and well-being of the community. A set of $n$ projects, where the $i$th project is designated by $P_i$ which needs quantity $F(P_i)$ of fund to be realized, are identified by experts for instance. The purpose is to select a subgroup of projects referred to as a portfolio to be funded in order to enhance as much as possible sustainable development of the community.

Because, any developing project in terms of building and operating infrastructures conveys in general not only positive incentives for users but also negative impacts on environment, modification of population way of living, destruction of culture monuments, etc., analyzing such problems is very challenging.

BOCR analysis offers therefore a sound and flexible framework to analyzing such decision making problem as that of forming projects portfolio.

As stated in previous section, at the end of BOCR analysis process, each project will be characterized by four aggregated attributes namely, its benefit, its opportunity, its cost, its risk and needed fund to realize it; therefore forming a portfolio of developing projects is a multi-objectives decision making problem. Given the structure of BOCR analysis, we propose two approaches to solve this problem: binary multi-objectives programming (BMOP) and bipolar measures approach for short list (BMSL) selection in the framework of satisficing game (Stirling 2003). These approaches are described in the following paragraphs.

### 4.1 BMOP: Binary Multi-objectives Programming

Given the description presented in above paragraph, each project $P_i$ is characterized by its BOCR components namely, $B(P_i)$, $O(P_i)$, $C(P_i)$, and $R(P_i)$; the problem, here, is to form a portfolio that maximize global benefit and opportunity and minimize global cost and risk when respecting limited fund condition. Let us denote by $x_i$ a binary variable indicating whether project $P_i$ is included or not in the portfolio, thus $x_i = 1$ if $P_i$ is included in the portfolio and $x_i = 0$ if not. The problem to solve is therefore a multi-objective programming problem given by following formulation (2) below

$$(BMOP)\begin{cases} \max\sum_{i=1}^{n} B(P_i)x_i \\ \max\sum_{i=1}^{n} O(P_i)x_i \\ \min\sum_{i=1}^{n} C(P_i)x_i \\ \min\sum_{i=1}^{n} R(P_i)x_i \\ \sum_{i=1}^{n} F(P_i)x_i \le F \\ x_i \in \{0, 1\} \end{cases} \tag{2}$$

To solve this problem, different algorithms developed in multi-objective optimization literature such as genetic algorithms, evolutionary algorithms, constraint satisfaction algorithms, etc. (see for instance Coello Coello et al. 2007; Zitzler and Thiele 1999 and references therein) can be used.

Here we propose another possibility, for sake of simplicity and to stay as near as possible to how public decisions are generally made by comparing ratios of positive incentives versus negative impacts, to possibly transform former BMOP problem into binary nonlinear programming problem as given by formulation (3) below

$$\begin{cases} \max\left\{\dfrac{\sum_{i=1}^{n} (\phi B(P_i) + (1-\phi)O(P_i))x_i}{\sum_{i=1}^{n} ((1-\phi)C(P_i) + \phi R(P_i))x_i}\right\} \\ \sum_{i=1}^{n} F(P_i)x_i \le F \\ x_i \in \{0,\ 1\} \end{cases} \tag{3}$$

or a binary linear programming problem as shown by formulation (4)

$$
\begin{cases}
\max\left\{\sum_{\mathrm{i=1}}^{\mathrm{n}}(\phi(B(P_i)-R(P_i))+(1-\phi)(O(P_i)-C(P_i)))x_i\right\} \\
\sum_{i=1}^{n}F(P_i)x_i \leq F \\
x_i \in \{0,\ 1\}
\end{cases}
\tag{4}
$$

where $0 \leq \phi \leq 1$ is the risk averse index of decision maker(s); this index permits to adjust the attitude of decision maker(s) toward uncertainty; for instance a risk averse decision maker, for who $\phi \to 1$, will balance immediate benefit (B) with potential harm (R) regardless of potential gain (O) and immediate cost to pay (C); in the contrary a risky decision maker ($\phi \to 0$) will privilege potential gain (O) compared to immediate cost (C) to pay.

It is possible (a sort of hard optimization drawbacks) that the solution of former programming problems leads to a situation where the remainder of fund can fund a project that is not included in the portfolio; in this case one can repeat the resolution process where selected projects are discarded at each stage.

## *4.2 BMSL: Bipolar Measures for Short List Selection*

Given the way the identity card of each project is structured in terms of positive incentives (benefit and opportunity) and negative impacts (cost and risk), recent bipolar analysis approach being developed by the author and colleagues (see Tchangani 2009, 2010, 2014; Tchangani et al. 2012; Tchangani and Pérès 2010), where an alternative in decision analysis process is characterized by its Selectability degree and its Rejectability degree is well suited for forming a portfolio of projects; it means that one avoid immediate compensation between positive incentives and negative impacts as it was done in BMOP case.

To this end, let us denote by $\mu_S(P_i)$ and $\mu_R(P_i)$ the Selectability measure and Rejectability measure of project $P_i$ respectively; they are obtained by normalized aggregated positive aspects (benefit and opportunity) as given by Eq. (5)

$$
\mu_S(P_i) = \frac{\phi B(P_i) + (1-\phi)O(P_i)}{\sum_{P_j}\left\{\phi B\left(P_j\right) + (1-\phi)O\left(P_j\right)\right\}}
\tag{5}
$$

and normalized aggregated negative aspects (cost and risk) as shown by Eq. (6)

$$
\mu_R(P_i) = \frac{(1-\phi)C(P_i) + \phi R(P_i)}{\sum_{P_j}\left\{(1-\phi)C\left(P_j\right) + \phi R\left(P_j\right)\right\}}
\tag{6}
$$

To select final portfolio, following steps (building on satisficing games theory Stirling (2003)) can be used.

1. Form the satisficing equilibrium set $S_q$ that is given by Eq. (7)

$$S_q = \Sigma_q \cap E \tag{7}$$

where $E$ is the equilibrium set (projects for which there is no other projects having less Rejectability degree and at least the same Selectability degree or having more Selectability and at most the same Rejectability degree; this set is always no empty by construction) and $\Sigma_q$ is the satisficing set (see Stirling 2003), at the boldness or caution index $q$, given by Eq. (8) below

$$\sum_q = \{P_i : \mu_S(P_i) \geq q\mu_R(P_i)\} \tag{8}$$

2. Compute the corresponding fund level $F(S_q)$ as given by Eq. (9)

$$F\left(S_q\right) = \sum_{P_i \in S_q} F(P_i) \tag{9}$$

3. If $F(S_q)$ is less than $F$,
   (a) All satisficing equilibrium projects are included in the portfolio; if the remaining fund $F$-$F(S_q)$ is sufficient to fund at least one project, it will be used in a second selection step where the projects already included in the portfolio are excluded; the process is repeated until no project can be funded with the remaining fund or fund completely used.
   (b) Or decrease caution index in order to include as much as equilibrium projects (if any) in the portfolio until fund is completely used.
4. If $F(S_q)$ is greater than the existing fund $F$, then increase caution index $q$ to reduce satisficing projects and repeat previous steps.

### *4.3 Brief Comparison of BOCR Analysis to Existing Methods*

In terms of (developing) projects selection and portfolio forming, existing methods for decision aid, dominated by classical multiple criteria (or attributes) decision making and multiple objectives decision making paradigms, do lack a structured framework in the earlier process of elicitation and evaluation of attributes. Indeed, first of all objectives and attributes notions are not well defined in these paradigms with even possible confusion of these notions, whereas BOCR analysis considers these two notions to be distinct notions that appear simultaneously in any decision problem. Indeed, objectives are some desires (or preferences) of decision makers that selected alternatives must satisfy as much as possible, whereas attributes are features of alternatives that will be used to evaluate their adequacy with regards to pursued objectives.

In comparison to classical value type evaluation method, the approach presented in this chapter allows hesitation between possibly many alternative projects whereas value type evaluation leads to a total ordering of projects that does not permit hesitation, an important dimension of human behavior when facing a decision situation. Final evaluation by two measures namely Rejectability measure and Selectability measure (allowing incomparability between two alternatives possible) makes BOCR analysis more closed to outranking methods that also allow incomparability between alternatives; but earlier evaluation stages are more flexible in BOCR analysis as this framework allows that alternatives be characterized by heterogeneous attributes. BOCR analysis does not need learning process that necessitates disposing of an initial database as it is required in decision rules approach for decision analysis.

# 5 Application

This section is dedicated to illustrating BOCR analysis mechanism by applying it to a sustainable development program. The application concerns projects funding in a rural area with the ultimate goal to enhance its sustainable development.

## *5.1 Description of the Problem*

Authorities of a rural area in a developing country are confronted to difficulties related to a quick population growing of this area in terms of lack of infrastructure facilities for the well-being of the population. Furthermore, they are sensible to facilities that enhance sustainable development which is considered as the most important issue for the future of this area in particular and national development in general. For this purpose and after largely consulting population and experts, five developing projects have been identified. Because of limited financial resources, these five projects cannot be funded simultaneously so they must be evaluated in order to select projects that must constitute the most sustainable projects portfolio. It is well accepted, from the definition of sustainable development paradigm (development that meets the needs of the present without compromising the ability of future generations to meet their own needs) given by former Norwegian Prime Minister Gro Harlem Brundtland in 1987, that sustainability of a project must be assessed through three main objectives, social objective, economical objective and environmental objective; so these three objectives are those used to evaluate each project. In terms of BOCR analysis, for each objective, attributes corresponding to each BOCR component are elicited and assessed by a panel of experts. Considered projects as well as the attributes along with their measure are given on the following tables (Tables 2, 3, 4, 5, and 6; notice that because of lack of space and the fact that completely describing each project does not add a great value to the analysis nor

**Table 2** BOCR structured identity card of project 1

| **Project 1: Build and operate a waste management facility**<br>As population is quickly growing, the issue of managing waste became important because actually waste is being throwing everywhere causing by the way health problems, soil pollution, sanitary conditions degradation, etc. Necessary amount for this project is $\mathbf{F(P_1) = 75}$ **million MU** | | |
|---|---|---|
| Objectives | BOCR factors | Attributes |
| $o_1$: social objective (1/3) | B<br>0.51 | $a_{01}$: improvement of sanitary conditions (0.5, 0.85)<br>$a_{02}$: reduction of infectious diseases rate (0.1, 0.73)<br>$a_{03}$: permanent jobs creation (0.4, 0.14) |
| | O<br>0.68 | $a_{04}$: improvement of life conditions (0.7, 0.62)<br>$a_{05}$: attraction of high income inhabitants (0.3, 0.81) |
| | C<br>0.62 | $a_{06}$: noxious gas releases (0.4, 0.53)<br>$a_{07}$: noxious surface water releases (0.2, 0.62)<br>$a_{08}$: air pollution (0.4, 0.70) |
| | R<br>0.44 | $a_{09}$: negative impact on health (0.3, 0.75)<br>$a_{10}$: soil pollution by acid rain (0.2, 0.45)<br>$a_{11}$: rise of cancer diseases rate (0.5, 0.10) |
| $o_2$: economical objective (1/3) | B<br>0.55 | $a_{12}$: revenue gained from tax paid by users (0.5, 0.8)<br>$a_{13}$: revenue gained from energy production (0.5, 0.3) |
| | O<br>0.44 | $a_{14}$: attraction of investors (0.3, 0.10)<br>$a_{15}$: improvement of mean revenue (0.4, 0.50)<br>$a_{16}$: improvement of global agriculture productivity (0.3, 0.70) |
| | C<br>0.72 | $a_{17}$: salaries of managers and operators (0.3, 0.80)<br>$a_{18}$: maintenance cost (0.3, 0.50)<br>a: energy and other consumables cost (0.4, 0.90) |
| | R<br>0.65 | $a_{19}$: rising of lands and housing price (0.2, 0.5)<br>$a_{20}$: operational failure cost (0.5, 0.8)<br>$a_{21}$: rising of local tax (0.3, 0.6) |
| $o_3$: environmental objective (1/3) | B<br>0.62 | $a_{22}$: amount of recovered waste (0.4, 0.7)<br>$a_{23}$: reduction of floods frequency and impact (0.4, 0.9)<br>$a_{24}$: reduction of animals (rats) proliferation (0.2, 0.3) |
| | O<br>0.70 | $a_{25}$: reduction of soil pollution by solid waste (0.4, 0.8)<br>$a_{26}$: reduction of ground water pollution (0.6, 0.6) |
| | C<br>0.66 | $a_{27}$: noxious gas releases (0.1, 0.53)<br>$a_{28}$: noxious surface water releases (0.6, 0.80)<br>$a_{29}$: air pollution (0.3, 0.70) |
| | R<br>0.45 | $a_{30}$: negative impact on bio-diversity (0.7, 0.3)<br>$a_{31}$: chemical pollution of soil (0.3, 0.6) |

prevent potential users to assimilate the approach, only attributes of project 1 are completely described). How these measures are obtained either by expertise or by direct measurement will not be detailed here because this is not very important for potential users of the developed structured approach; the only thing is to be able to

**Table 3** BOCR structured identity card of project 2

| | | |
|---|---|---|
| **Project 2: Build and maintain a bridge that connects the area to a main national road** The considered rural area is supplied in different goods from main towns of the country via a rural road that across a river; so that during raining periods, accessibility is rendered difficult causing many troubles for the population of this area; a bridge over this river would be a great advancement towards developing process. Necessary amount for this project is ***F(P$_2$) = 63 million MU*** | | |
| Objectives | BOCR factors | Attributes |
| ... | ... | ... |

**Table 4** BOCR structured identity card of project 3

| | | |
|---|---|---|
| **Project 3: Install and operate a wind farm to supply electricity to this area** At the moment, there is no electricity facility in this area; and its isolated condition render its connection to national power supply infeasible, which power supply, is actually insufficient for that nation. Necessary found of this project is ***F(P$_3$) = 72 million MU*** | | |
| Objectives | BOCR factors | Attributes |
| ... | ... | ... |

**Table 5** BOCR structured identity card of project 4

| | | |
|---|---|---|
| **Project 4: Build and operate a water supply facility** Actually, water used by inhabitants of the considered area for different activities of their domestic life comes from the river where children and women went to fetch water. Therefore having a modern water supply facility in this area will be very important for the population in different viewpoints. The project consists therefore in making a drill in some points to get drinking water from underground. Necessary amount for this project is ***F(P$_4$) = 55 million MU*** | | |
| Objectives | BOCR factors | Attributes |
| ... | ... | ... |

**Table 6** BOCR structured identity card of project 5

| | | |
|---|---|---|
| **Project 5: Build and operate a dam for agriculture irrigation** Principal activity of population of the considered area is agriculture; but it is noticed that the productivity of cultivated lands is very low due in part to insufficient raining. The idea is therefore to build and operate a dam in order to retain rain water to be used for irrigation by farmers. Necessary amount for this project is ***F(P$_5$) = 30 million MU*** | | |
| Objectives | BOCR factors | Attributes |
| ... | ... | ... |

obtain by any method the value of the corresponding parameters that ranges between *0* and *1*; available fund is ***F = 150 million MU*** where *MU* stands for monetary unit.

### 5.2 Results

Necessary fund to finance the five projects is **295 million MU**, whereas actual available fund is **150 million MU** that cannot cover all these projects so that the necessity to select and form a portfolio of projects is inevitable; this selecting process is carried up in BOCR analysis framework.

From data of previous tables, BOCR components of each project is computed by aggregating corresponding attributes values; aggregation operator used here is the Choquet integral associated with a weighted cardinal fuzzy measure (wcfm) in order to take into account the synergy realized by organizing attributes in homogeneous classes (see Tchangani 2013). The value of each component, B, O, C, R for each couple project-objective, generically denoted by $X$ is given by following Eq. (10)

$$\mathrm{X} = \sum\nolimits_{k=1}^{|\upsilon|} \left\{ \left\{ \left( \frac{|\upsilon| - (k-1)}{|\upsilon|} \right) \left( \sum\nolimits_{j \in A_k} \omega_j \right) \right\} \left( \upsilon_{\sigma(k)} - \upsilon_{(k-1)} \right) \right\} \tag{10}$$

where $\upsilon$ is the values vector of considered attributes, $|\upsilon|$ stands for the dimension of $\upsilon$; $\omega$ is the relative normalized importance weights vector; $A_k$ is a subset of indices of vector $\upsilon$ given by following Eq. (11)

$$A_k = \{\{\sigma(\mathrm{k}), \sigma(\mathrm{k}+1), \ldots, \sigma(|\upsilon|)\}\} \tag{11}$$

with σ being a permutation over vector $\upsilon$ given by Eq. (12)

$$\upsilon_{\sigma(1)} \leq \upsilon_{\sigma(2)} \leq \ldots \leq \upsilon_{\sigma(|\upsilon|)}\ ; \quad \upsilon_{\sigma(0)} = 0 \tag{12}$$

The overall values for these components are obtained by aggregating over the three objectives considered (here) as equally important to obtain the results given by Table 7.

From, data of Table 7, portfolio forming problem has been considered in two levels, by varying the risk averse index $\phi$ from risky decision making ($\phi = 0$); mean risk taking ($\phi = 0.5$) and risk averse decision making ($\phi = 1$); and then solving the problem by one of methods presented; only BMSL approach is considered here because it does not need programming effort or usage of sophisticated software; nevertheless, we notice that linear programming solution of Eq. (4), using function

**Table 7** Obtained BOCR components values

| Projects | $B(P_i)$ | $O(P_i)$ | $C(P_i)$ | $R(P_i)$ |
|---|---|---|---|---|
| $P_1$ | 0.56 | 0.61 | 0.67 | 0.51 |
| $P_2$ | 0.62 | 0.54 | 0.65 | 0.54 |
| $P_3$ | 0.58 | 0.64 | 0.73 | 0.44 |
| $P_4$ | 0.46 | 0.56 | 0.61 | 0.45 |
| $P_5$ | 0.54 | 0.72 | 0.65 | 0.58 |

**Table 8** BMSL results for different values of $\phi$

| | $\phi = 0$ | | $\phi = 0.5$ | | $\phi = 1$ | |
|---|---|---|---|---|---|---|
| | $\mu_R(P_i)$ | $\mu_S(P_i)$ | $\mu_R(P_i)$ | $\mu_S(P_i)$ | $\mu_R(P_i)$ | $\mu_S(P_i)$ |
| $P_1$ | 0.2024 | 0.1987 | 0.2024 | 0.2007 | 0.2024 | 0.2029 |
| $P_2$ | 0.1964 | 0.1759 | 0.2041 | 0.1990 | 0.2143 | 0.2246 |
| $P_3$ | 0.2205 | 0.2085 | 0.2007 | 0.2093 | 0.1746 | 0.2101 |
| $P_4$ | 0.1843 | 0.1824 | 0.1818 | 0.1750 | 0.1786 | 0.1667 |
| $P_5$ | 0.1964 | 0.2345 | 0.2110 | 0.2161 | 0.2302 | 0.1957 |
| Portfolio | $\{P_2, P_4, P_5\}$ | | $\{P_3, P_5\}$ | | $\{P_2, P_3\}$ | |

*bintprog* (binary integer programming) of Matlab corresponds to selecting the equilibrium set of BMSL approach.

For the first round, analysis is done for a boldness index $q = 1$. From data of Table 8, we see that for $\phi = 0$, at the first round only project 5 is a satisficing equilibrium $(S_1 = \{P_5\})$; this is easily justified by row data of Table 7 as one is comparing opportunity to immediate cost and all projects have their opportunity less than their cost except project $P_5$. As there is no satisficing project that could be included by reducing boldness index $q$, one must repeat equilibrium analysis when discarding project $P_5$; by doing so, projects $P_1$, $P_3$ and $P_4$ become equilibriums for a total of **202 million MU** that cannot simultaneously be funded by the remainder of **150 – 30 = 120 million MU**; furthermore one can easily see that only one of these projects can be funded with the remainder fund; by reasoning on maximum boldness index basis, the next project to fund is project $P_4$ with a remainder fund of **65 million UM** that can cover financial amount of project $P_2$ so that in this case the portfolio consists in $\{P_2, P_4, P_5\}$ with a remainder of **2 million MU**. For $\phi = 0.5$, the satisficing equilibrium set at boldness index $q = 1$ is $S_1 = \{P_3, P_5\}$ which total fund is **102 million UM** and the remainder fund of **48 million UM** cannot finance any of the rest of projects. For risk averse decision making, $\phi = 1$, the satisficing equilibrium projects set is $S_1 = \{P_2, P_3\}$ that is funded for **135 million MU** and the remainder of **15 million MU** cannot fund a further project.

In this application, after presenting the analysis (in terms of risk attitude influence on portfolio structure) to decision makers, they adopted a risk taking attitude mainly because it permits to realize a maximum of projects, to use maximum available fund and importantly they are more sensible to long term effect in terms of opportunity. So finally, the realized portfolio is $\{P_2, P_4, P_5\}$ consisting in project $P_2$ (build and main a bridge with an amount of **63 million MU**), project $P_4$ (build and operate a water supply facility costing **55 million MU**) and project P5 (build and operate a dam with an amount of **30 million MU**), with a remainder fund of **2 million MU**.

## 6 Conclusion

In this chapter, the problem of forming a portfolio of sustainable developing projects has been considered. To structure, assess and evaluate alternative projects in the spirit of decision that cope with efficiency, legitimacy, and acceptance by stakeholders, a framework named BOCR analysis is applied. In this framework, each project is evaluated through four measures in terms of benefit (B), opportunity (O), cost (C), and risk (R), representing aggregation of low levels attributes assessment. Ultimate selection process can be done using different approaches developed by multiple criteria decision making community such as genetic algorithms, evolutionary algorithms, binary programming and so forth. But another interesting thing of structured approach of BOCR analysis and that may constitute the main contribution of this chapter, is that one can avoid compensating too earlier negative aspects and positive ones so to formulate the problem as a satisficing game where negative aspects contribute to a Rejectability measure and positive aspects account for Selectability measure. By so doing ultimate analysis and selection process is carried up in two dimensions space, reducing by the way the complexity of analysis. Application of this approach to a real world problem in sustainable development domain shows great potentialities as a structuring framework for decision analysis.

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# VIKOR Method with Application to Borrowing Terms Selection

**Marija Opricovic**

**Abstract** The main aim of the chapter is presenting the VIKOR method and selecting the best borrowing alternative according to the given criteria, assuming it is decided that borrowing is necessary. The VIKOR method is introduced as one applicable technique to implement within multiple criteria decision making. It focuses on ranking and selecting from a set of alternatives, and determines compromise solutions for a problem with conflicting criteria, which can help the decision makers to reach a final decision. The compromise solution is a feasible solution that is the "closest" to the ideal solution. Here, compromise means an agreement established by mutual concessions. The method compromises conflicting criteria, and group utility with individual regret. To apply the VIKOR method, the borrowing alternatives should be evaluated in terms of established criteria for the stated problem. The alternatives are generated with the values of the following instruments: interest rate, maturity, currency, grace period and repayment schedule, based on the elements offered by potential creditors. The criteria for decision are: borrowing cost, market risk, and liquidity risk. Uncertainties related to this analysis are treated by planning and analyzing scenarios. Vague and imprecise data are treated using fuzzy numbers. The criterion functions are formulated and their numerical values are determined for all alternatives. The alternatives are ranked by the method VIKOR and the compromise solution is determined.

**Keywords** Borrowing terms • Multicriteria decision • Compromise • VIKOR method

## 1 Introduction

Decision making process includes two major approaches: multiattribute utility theory and outranking methods (Keeney and Raiffa 1976; Sawaragi et al. 1985; Vincke 1992). The fundamental assumption in utility theory is that the decision maker chooses the alternative that has maximum value of expected utility.

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However, in many observed problems is not possible to obtain a mathematical representation of the decision maker's utility function $U$. Numerous other aggregating functions are introduced instead of a global utility function (Butler et al. 2001). The question is how to choose a suitable aggregation function. Compromise solution that is introduced by Yu (1973) is based on the idea of finding a feasible solution that is as close as possible to an ideal point. Zeleny (1982) stated that alternatives that are closer to the ideal are preferred to those that are farther away. The rationale human choice is to be as close as possible to a perceived ideal. $L_p$-metric that is introduced by Yu (1973) as an aggregating distances function, called the group regret for a decision, measures a regret that the ideal cannot be chosen.

The VIKOR method focuses on ranking and selecting from a set of alternatives. It determines compromise solutions for a problem with conflicting criteria, which can help the decision makers to reach a final decision. The basic ideas of VIKOR had developed by Serafim Opricovic, in his Ph.D. dissertation in 1979, and an application was published in Duckstein and Opricovic (1980). The name VIKOR appeared in 1990 at national symposium (from Serbian: VIšeKriterijumska Optimizacija i Kompromisno Rešenje, that means Multicriteria Optimization and Compromise Solution, with pronunciation: 〔*vikor*〕). The paper (Opricovic and Tzeng 2004) contributed to the international recognition of the VIKOR method. Thomson Reuters Essential Science IndicatorsSM identified it as the most cited paper in the field of Economics and Management (*Science Watch*, Apr. 2009; http://sciencewatch.com/dr/erf/2009/09aprerf/09aprerfOpriET). The VIKOR method has been compared with three MCDM methods, TOPSIS, PROMETHEE and ELECTRE, providing a contribution to the state of the art of multicriteria decision making—MCDM (Opricovic and Tzeng 2007). Here, the VIKOR method is applied to borrowing terms selection.

This chapter emphasizes the VIKOR method based on the comparative analysis and VIKOR consideration in high cited papers. The paper (Yazdani and Graemi 2014) conducted a state-of-the-art literature review to embody the research on VIKOR and its applications, considering 198 papers from more than 100 journals since 2002.

A recent approach is the BOCR analysis of benefits, opportunities, costs, and risks (Wijnmalen 2007) using AHP/ANP methodology (Saaty 2001). The BOCR introduces the way in which the composite priorities of alternatives on each of the four factors (B, O, C, R) are synthesized into a final BOCR value. Two types of synthesis are discussed in the literature: multiplicative and additive expression. An aggregation operator is proposed as the weighted average of the normalized overall priorities of the alternatives on benefits, opportunities, costs and risks, computed using hierarchic or network models. In a net value oriented analysis, an additive synthesis expression should be used where rescaled cost and risk priorities are subtracted from rescaled benefits and opportunities priorities. Similar weighted average is used in the VIKOR method for one of the ranking measure (symbol $S$, in Sect. 2.2).

Foreign borrowing (external debt) is raising foreign funds for the financing needs of a country and managing the country's financial liabilities. Public foreign debt has increased substantially in last 15 years (in absolute terms) for countries at all income levels, as a result of liberalization of capital movements and financial globalization, as well as current global economic crisis. In relative terms, public debt to GDP has increased, as well (References: Debt to GDP ratios). The theoretical literature suggests that foreign borrowing has a positive impact on investment and economic growth to a certain threshold level; beyond this level, however, its impact is adverse. The thresholds in the relationship between the long-run average public debt to GDP ratio and long-run GDP growth are examined in Caner et al. (2010). The results show that the threshold level of the average long-term public debt to GDP ratio on GDP growth is 77 % for the full sample (101 countries) and 64 % for the subsample (75) of developing countries. The increase in public debt might slow economic growth if debt to GDP ratio surpassed the threshold for an extended period.

The paper (Clements et al. 2003) examines the channels through which external debt affects growth in low-income countries. Debt appears to affect growth via its effect on the efficiency of resource use. External debt also has indirect effects on growth through its effects on public investment. The relationship is nonlinear, with the effect intensifying as the ratio of debt service to GDP rises.

The set of economic and political conditions that are associated with a likely occurrence of a sovereign debt crisis are examined in Manasse and Roubini (2005). If external debt exceeds 50 % of GDP and inflation is above 10.5 %, the likelihood of a crisis is 67 %. The influence of ratio of short-term debt to reserve, exchange rate volatility, and/or real GDP growth is considered, too.

The analysis in Lane and Milesi-Ferretti (2006) highlights the large accumulation of net external liabilities in several countries. The risk profile of a country's international financial liabilities became important for creditor and for debtor. A lot of research is focused on country's net foreign asset position or net international investment position. The net foreign asset position of a country reflects the indebtedness of that country.

The decision to borrow is a part of complex decision making process. The decision maker should handle many economic indicators like growth rate, level of existing debt, deficit, costs, taxes, inflation etc. Multicriteria nature of decision making is recognized and some attempts are made to connect multicriteria techniques with econometric models to give policy recommendations.

The developed model of multicriteria analysis of foreign borrowing terms could be considered as a contribution to the theory and practice of decision making in finance. An original procedure of generating borrowing alternatives is presented. The chapter highlights the importance of three additional criteria, beside the cost of borrowing. The strength resides in the formulation of borrowing alternatives selection problem as multiple criterion decision making problem taking into account risk issues. The increased volatility of international fund flows, the complexity of instruments used and the recent crises highlighted the importance of risk-related criteria, in addition to cost. Uncertainties related to this analysis are treated by planning and analyzing scenarios. Uncertain or unknown economic factors, such as the variable interest rate, are treated using fuzzy numbers. The VIKOR method is applied to determine the compromise solution which is the closest to the ideal. The obtained compromise solution could be accepted by the decision makers because it provides a maximum utility for the majority of criteria, and a minimum individual regret of the opponent. Application is illustrated by the numerical example and the results confirm the applicability of this methodology on decision making problem of foreign borrowing.

In this chapter it is assumed that the decision to borrow is made before, including the borrowing amount. The considering problem is to determine preferred borrowing alternative according to the established criteria. Section 2 considers multiple criteria decision making (MCDM) using VIKOR method. Section 2.1 considers MCDM activities, and Sect. 2.2 presents the VIKOR method for compromise solution which is the closest to the ideal. The compromise solution could be accepted by the decision makers because it provides a maximum utility of the majority, and a minimum individual regret of the opponent. Section 3 introduces the multicriteria model for selecting borrowing terms. Section 3.1 presents the parameters for generating alternatives including: interest rate, maturity, currency, grace period and repayment schedule. Section 3.2 defines the criteria for decision making: borrowing cost, market risk, and liquidity risk, as well as a criterion for particular conditions imposed by the creditor. Section 3.3 introduces evaluation procedures such as: scenario planning, fuzzification, defuzzification, and use of formulas for criteria. An application of the developed model is illustrated in Sect. 4 by the numerical example using data for already concluded borrowings and adjusted to the same loan amount.

## 2 Multiple Criteria Decision Making Using VIKOR Method

### *2.1 MCDM Activities*

Various analytical methods used to predict or solve management problems are characterized by the optimization of a single objective (criterion) function. In practice decision makers are often faced with situations where the system may have multiple, possibly conflicting criteria. These practical problems are characterized by several noncommensurable and competing (conflicting) criteria, and have no feasible solution which simultaneously extremizes all criterion functions. Multiple criteria decision making activities includes selection from a large number of feasible alternatives involving a substantial engineering content That means a creation of particular set of actions which will best accomplish the overall objectives of the decision makers, respecting the constraints of law, morality, economics, resources, political and social pressures. Multicriteria decision making (MCDM) determines the best feasible solution according to the established criteria (representing different effects).

The MCDM may be considered as a complex and dynamic process in which one managerial level and one analytic level can be distinguished. The managerial level defines the goals, and chooses the final "optimal" alternative. The multicriteria nature of decisions is emphasized at this level. The decision makers have the power to accept or reject the solution proposed by analytical level. Preference structure, provided by decision makers are done "off line" of the optimization procedure at the analytical level. The preference structure is often based on political rather than solely on economic criteria. In such cases, system analyst can aid the decision making process by making a comprehensive analysis and by listing the important properties of noninferior and/or compromise solutions. The analytical level defines alternatives and points out consequences of choosing any one of them from the viewpoint of various criteria. This level performs the multicriteria ranking of alternatives.

Multiple criteria decision making consists of performing the following activities:

(a) Problem statement;
(b) Generating alternatives;
(c) Establishing criteria (multiple) that relate capabilities to goals;
(d) Evaluating alternatives in terms of criteria (the values of the criterion functions);
(e) Apply a normative MCDM method (such as VIKOR);
(f) Accept one alternative as "optimal" (preferred);
(g) If the final solution is not accepted, gather new information and go into the next iteration of MCDM.

Activities (a), (c), and (f) are performed at the decision level (upper managerial), where decision makers have a central role. Other steps are mostly technical tasks. The problem in step (a) is determination of the most appropriate borrowing alternative, assuming the decision is made for borrowing certain amount.

Alternatives can be generated and their feasibility can be tested by mathematical or statistical models, and/or by experiments on the existing system or other similar systems. Generating alternatives may be very complex process; there is no general procedure or model, and none mathematical procedure could replace the human creativity in generating and evaluating alternatives. Constraints are seen as high-priority objectives, which must be satisfied in the alternatives generating process.

The main aim of a multicriteria approach is to capture all relevant foreseeable impacts in their most appropriate and representative units. In many management projects the criteria should represent economy, reliability, social environment and natural environment. The set of criteria has to be discussed and accepted by the experts for the stated problem. A systematic way to perceive the entire set of criteria is to establish the hierarchy of goal, objectives, criteria, and of all evaluation measures. The relative importance of each criterion is expressed by weight. These weights do not have a clear economic significance; they give us the opportunity to modelize the real aspects of decision making (the preference structure).

The evaluation of alternatives should be performed according to each criterion from the set of established criteria. The values of criterion functions can be crisp, linguistic, and/or fuzzy. We assume that many criterion functions are crisp (in nature), their values are determined by economic instruments, and/or by measuring. However, the evaluation of alternatives could be implicated with imprecision (or uncertainty) of established criteria, and a fuzzy model is necessary, to deal with "qualitative" (unquantifiable or linguistic) or incomplete information.

The most MCDM methods introduce aggregated function representing total utility, in order to integrate all criteria into one goal function (as the measure $Q$ in the VIKOR method). The main difference between these methods is the type of aggregated function.

## 2.2 VIKOR Method

The VIKOR method is a multicriteria decision analysis method. The background of this method consists of multiple criterion evaluation of alternative solutions, normalization and aggregation of conflicting criteria, and determining compromise solution. The method was developed to solve decision problems with conflicting and noncommensurable (different units) criteria, assuming that compromise is acceptable for conflict resolution, the decision maker wants a solution that is the closest to the ideal, and the alternatives are evaluated according to all established criteria. The use of VIKOR has several desirable properties: the main aim is to capture all relevant foreseeable impacts in their most appropriate and representative units. The VIKOR method focuses on ranking and selecting from a set of feasible alternatives, and determines compromise solutions for a problem with conflicting criteria.

The VIKOR method solves the following problem

$$\underset{j}{mco}\left\{\left(f_{ij}(A_j), j = 1, \ldots, J\right), i = 1, \ldots, n\right\}$$

where: $J$ is the number of feasible alternatives; $A_j = \{x_1, x_2, \ldots\}$ is the $j$-th alternative obtained (generated) with certain values of system variables $x$; $f_{ij}$ is the value of the $i$-th criterion function for the alternative $A_j$; $n$ is the number of criteria; $mco$ denotes the operator of a multicriteria decision making procedure for selecting the best (compromise) alternative in multicriteria sense. The compromise solution $F^c = \left(f_1^c, \ldots, f_n^c\right)$ is a feasible solution that is the "closest" to the ideal solution $F^* = \left(f_1^*, \ldots, f_n^*\right)$ (the best values of criteria). Here, compromise means an agreement established by mutual concessions, represented by $\Delta f_i = f_i^* - f_i^c$, $i = 1, \ldots, n$.

To add values of noncommensurable criteria, first we have to convert them into the same units. Normalization is used to eliminate the units of criterion functions, so that all the criteria are dimensionless. Linear normalization used within VIKOR method is formulated in Eq. (6). The multicriteria merit for compromise solution is an aggregated function Q representing the distance to the ideal. Two distance measures are aggregated: weighted Manhattan distance [Eq. (4)] and weighted Chebychev distance [Eq. (5)]. Aggregation function Q is a weighted average [Eq. (1)]. The input data should be prepared as performance matrix $|f_{ij}|$, and the weights $w_i, i = 1, \ldots, n$, should be given which express the relative importance of the criteria.

The alternatives are ranked by the values $Q_j$, $j = 1, \ldots, J$ in decreasing order.

$$Q_j = v \cdot QS_j + (1 - v)QR_j \tag{1}$$

where:

$$QS_j = \left(S_j - S^*\right) / \left(S^- - S^*\right) \tag{2}$$

$$QR_j = \left(R_j - R^*\right) / \left(R^- - R^*\right) \tag{3}$$

$$S_j = \sum_{i=1}^{n} w_i d_{ij}, \quad \text{weighted and normalized Manhattan distance;} \tag{4}$$

$$R_j = \max_i \left(w_i d_{ij}\right), \quad \text{weighted and normalized Chebychev distance;} \tag{5}$$

$$d_{ij} = \left(f_i^* - f_{ij}\right) / \left(f_i^* - f_i^-\right) \tag{6}$$

$f_i^* = \max_j f_{ij}$, $f_i^- = \min_j f_{ij}$, if the $i$-th function represents a benefit, $i \in I_B$;

$f_i^* = \min_j f_{ij}$, $f_i^- = \max_j f_{ij}$, if the $i$-th function represents a cost, $i \in I_C$ ;

$S^* = \min_j S_j$, $S^- = \max_j S_j$, $R^* = \min_j R_j$, $R^- = \max_j R_j$;

$w_i$ are the weights of criteria, expressing the DM's preference as the relative importance of the criteria; $v$ is introduced as a weight for the strategy of maximum group utility, whereas $1 - v$ is the weight of the individual regret. In VIKOR method $v$ is calculated as $v = (n + 1)/2n$.

The merit $Q$ in Eq. (1) is a weighted arithmetic mean with weights v and (1 – v), $v$ for the strategy of maximum group utility. The "group utility" is used to represent the effects of satisfying the group (majority) of criteria. This chapter does not solve the group decision problem.

The VIKOR method proposes the best ranked alternative $A^{(1)}$ as a compromise solution. If there are several alternatives ($M$) that are "in closeness" (close to the best ranked alternative), VIKOR proposes a set of compromise solutions, that is determined by the distance given by $Q(A^{(M)}) - Q(A^{(1)}) < DQ$. The advantage threshold is introduced as $DQ = 1/(J - 1)$.

The mathematical model (all formulas) is introduced to resolve conflict between criteria. Compromise means an agreement established by mutual concessions.

Compromise is represented by individual deviations of criteria from the ideal, $\Delta f_i = f_i^* - f_i^c$, $i = 1, \ \ldots \ , \ \text{n}$, for the solution of an MCDM problem $F^c = \left(f_1^c, \ldots, f_n^c\right)$.

The solution $A_S^{(1)}$ is obtained by the minimization of the distance $S$ in Eq. (4)

$$A_S^{(1)} = agr \min_j \left[ \sum_{i \in I_B} w_i \left(f_i^* - f_{ij}\right) / \left(f_i^* - f_i^-\right) + \sum_{i \in I_C} w_i \left(f_{ij} - f_i^*\right) / \left(f_i^- - f_i^*\right) \right] =$$
$$= agr \left[ \max_j \sum_{i \in I_B} w_i f_{ij} / \left(f_i^* - f_i^-\right) - \min_j + \sum_{i \in I_C} w_i f_{ij} / \left(f_i^- - f_i^*\right) \right]$$

It provides maximum group utility for the majority represented by maximum total benefit minus minimum total cost.

The solution $A_R^{(1)}$ is obtained by the minimization of the distance $R$ in Eq. (5)

$$A_R^{(1)} = \arg \min_j \max_i \left(w_i d_{ij}\right)$$

It provides minimum of maximal regret of the opponent.

The solution $A_Q^{(1)}$ is obtained by the minimization of the weighted distance $Q$ in Eq. (1). It provides compromise between the above two decision strategies.

Researchers are challenged to provide a guide for choosing the method that is both theoretically well founded and practically operational to solve actual problems. Matching MCDM methods with classes of problems would address the correct applications, and for this reason, the VIKOR characteristics are matched with a class of problems as follows.

- Compromising is acceptable for conflict resolution.
- The decision maker (DM) is willing to approve solution that is the closest to the ideal.
- There exist a linear relationship between each criterion function and a decision maker's utility.
- The criteria are conflicting and noncommensurable (different units).
- The feasible alternatives are evaluated according to all established criteria (performance matrix).
- The DM's preference is expressed by weights, given or simulated.
- The VIKOR method can be started without interactive participation of DM, but the DM is in charge of approving the final solution and his/her preference must be included.
- The solution maximizes group utility and minimizes individual regret of an opponent.
- The VIKOR solution could be a set of solutions "in closeness" with the similarity threshold.

The above information specifies the kind of "behavior" of VIKOR method. The natural goal for such problem is to minimize the difference between the desired output (ideal) and the actual outputs (feasible alternatives). A logical way is to use distance function in n-dimensional space, say L-p metric. However, for practical purpose the parameter $p$ has to be identified. In the context of multicriteria decision making, $L_1$ metric (Manhattan) $L_\infty$ metric (Chebyshev) both have important role. The preference of the decision maker has to be introduced and the weighted distance functions are chosen.

The compromise solution could be the base for negotiation, since it resolves conflict between criteria, and between the maximum group utility of the majority and the minimum individual regret of the opponent. A detailed comparison of TOPSIS and VIKOR is presented in the article by Opricovic and Tzeng (2004). A comparative analysis of the methods VIKOR, TOPSIS, PROMETHEE and ELECTRE is presented in Opricovic and Tzeng (2007).

# 3 Multicriteria Model for Selecting Borrowing Terms

## *3.1 Generating Alternatives for Borrowing*

Alternatives can be generated and their feasibility can be tested by appropriate procedures based on experience. Constraints are seen as high-priority objectives, which must be satisfied in the alternatives generating process. There is no general procedure or model, and none mathematical procedure could replace the human creativity in generating and evaluating alternatives.

Creditors lend funds in return for a certain interest rate, either fixed before or in the moment of borrowing or allowed to vary throughout the life of debt. The variable interest rate can be linked to some commonly accepted interest rate indicator such as the London Interbank Offered Rate (LIBOR). The timing of interest payments (plan) is also a decision variable.

Alternatives are generated with different borrowing instruments offered by the potential creditors. The alternative is defined as the combination of parameters

$a_j = \{x_{1j}, x_{2j}, \ldots, x_{Mj}\}$, $j = 1, \ldots, J$ where: $x_{mj}$—is the value of the $m$-th parameter (variable) for $j$-th alternative. Here, the parameters for generating alternatives could be:

$x_1$—Creditor: banks, financial institutions, various funds, trade partners, etc.
$x_2$—Interest rate: value and type (fixed or variable)
$x_3$—Maturity: long-term, short-term
$x_4$—Currency: EUR, USD, CHF, JPY, CNY or SDR
$x_5$—Repayment schedule: equal rates or negotiable, grace period
$x_6$—Debt refinancing conditions.

The set of alternatives has to be complete, and it should be at least one alternative for each aspect and point of view. The main optimization goal has to be the same within all alternatives. Here, the borrowed amount and the borrowing purpose should be the same for all alternatives. The final solution will be one alternative from given set.

## 3.2 Formulating Criteria for Decision Making

The set of criteria has to be discussed and accepted by the experts for particular decision making problem. The multicriteria model can treat all relevant conflicting effects and impacts in their representative units. The most important concern of debt management had been the cost of borrowing, or even to be able to raise the necessary funds. But, the increased volatility of capital flows, the complexity of instruments used and the recent crises highlighted the importance of risk-related criteria, in addition to cost. The major risks are the market risk, which is defined as the risk of an increase in the cost of debt service as a result of unfavorable movements in market conditions and the liquidity (refunding) risk that indicates the possibility to fail in finding the required funds in order to make debt repayments.

Here, the main objective is determination of the best borrowing alternative according to the given criteria, assuming it is decided before that borrowing is necessary. The multicriteria model includes four criteria: cost of borrowing, market risk, liquidity risk and particular condition(s) imposed by the creditor. The criterion functions are formulated and their numerical values are determined for all alternatives. Uncertainties related to this analysis are treated by planning and analyzing scenarios. Vague and imprecise data are treated using *fuzzy* numbers.

**Cost of Borrowing**

It is possible to define the costs associated with debt in several ways (Balibek 2008). The cost of borrowing can be measured by the market value of debt stock, present or nominal value of future interest cash flows, accrual based interest payments etc. The total cost in the time horizon $T$ could be formulated as following

$$f_1 = \sum_{t=1}^{T} C_t$$

where: $\boldsymbol{C_t}$ is total cost at time $t$.

Here, for cost evaluation the practical procedure is used, that consists of two parts, payment schedule for principal and for interest.

$$C_t = P_t + I_t$$

where: $P_t$ is principal (debt), and $I_t$ is interest.

For borrowing with the grace period of $G$ years the yearly payment $C_{Gt}$ within the grace period is determined by the following formula

$$C_{Gt} = Z \times i$$

where $Z$ is the borrowed amount (loan) – principal, $i$ is the interest rate (absolute value, not %). So, only the yearly interest is paid in the grace period.

The yearly payments $C_{Tt}$ in the period of $T - G$ years (after grace period) is determined as follows

$$C_{Tt} = \frac{Z}{T - G}(1 + (T - t + 1) \times i)$$

Total cost of borrowing is

$$f_1 = \sum_{t=1}^{G} C_{Gt} + \sum_{t=G+1}^{T} C_{Tt}$$

or

$$f_1 = G \times Z \times i + Z + Z \times i \times (T - G + 1)/2 \tag{7}$$

The objective is to minimize the total cost of borrowing $f_1$.

**Market Risk**

Market risk is the risk of an increase in costs, which can be measured in several ways. The borrower might have a target level for the debt service expenditures, and any deviation above this level due to market conditions can be a measure of market risk. Approximating this risk with the standard deviation as in the classical Markowitz model (Markowitz 1952; Heching and King 2009) would result in a quadratic optimization problem, and thus leads to a nonlinear multiobjective model.

Here, the market risk is measured by the worst-case cost which is the highest level of cost that emerges across the entire scenario set. The criterion function for the market risk is formulated as following

$$f_2 = \max_{S} f_1^s \tag{8}$$

where: $f_1^s$ is the total cost in the time horizon $T$ under scenario $s$. The objective is to minimize the criterion function $f_2$.

**Liquidity Risk**

Liquidity or re-financing risk is associated with the actual debt service (or total) cash flows of the borrower. This is the threat that at the time of debt repayment, the borrower will lack the necessary funds and fail in fulfilling its obligations. To avoid such situation, it is common practice to smooth debt repayments and to try to avoid concentration of paybacks in certain periods to control liquidity risk.

The liquidity risk is measured by the maximum liability payment made in a single time step. The criterion function for the liquidity risk could be formulated as following

$$f_3 = \max_{S,t} C_t^s$$

where: $C_t^s$ is the payment tranche under scenario $s$ in a single time step $t$.

The above mentioned practical procedure for cost evaluation with the grace period of $G$ years gives maximum yearly payment at time $t = G + 1$

$$f_3 = \frac{Z}{T - G} + Z \times i \tag{9}$$

which is used to express the liquidity risk.

The objective is to minimize the criterion function $f_3$.

**Particular Conditions**

A creditor may impose particular conditions which may differ between clients. A particular condition may be economical or social, and it is difficult to incorporate its effect in cost of borrowing. Such condition could be one (or few) of the following:

- Restrictions on debt rescheduling,
- Specified usage of borrowed funds,
- Restrictions on assets or incomes,
- Project financed by the loan has to be managed and realized by the creditor.

The objective is to minimize the evaluated effects of the imposed conditions.

## *3.3 Evaluation Procedures*

Two types of evaluation models can be distinguished. Formal models include quantifiable measures with data obtained through monitoring or surveys. As an alternative to formal modeling, judgments can be gathered from people who have knowledge and experience relevant to the particular problem.

The values of criterion functions can be crisp, linguistic, and/or fuzzy. We assume that many criterion functions are crisp (in nature) which values are determined by economic instruments, and/or mathematical model. Such data should be used expressed in original units. However, the evaluation of alternatives could be implicated with imprecision (or uncertainty) of variables and established criteria.

The scenarios planning and analysis could be used when some parameters are not fully known. Using scenarios to see how the different possible decisions behave under different assumptions about the future is the correct way of comparing alternatives.

There are situations when the criterion is "qualitative" (unquantifiable or linguistic). Statements using subjective categories (good, fair, bad) are used for the evaluations. In many cases linguistic variables are converted using conversion scales.

Each alternative should be evaluated according to each criterion function,

**Scenario Planning**

The combinations and permutations of fact and related changes are called "scenarios" (http://en.wikipedia.org/wiki/Scenario_planning). Scenario planning helps decision makers to anticipate hidden weaknesses and inflexibilities. Scenarios planning starts by dividing our knowledge into two broad domains: (1) things we believe we know something about and (2) elements we consider uncertain or unknowable. The first component—trends—is recognizing that our world possesses considerable continuity. The second component—true uncertainties—involves indeterminables such as future interest rates.

Usually, several scenarios are constructed. The current situation does not need to be in the middle of the diagram (interest rate may already be low), and possible scenarios may keep one (or more) of the forces relatively constant, especially if using three or more driving forces. One approach can be to create all positive elements into one scenario and all negative elements (relative to the current situation) in another scenario, and pure best-case and worst-case scenarios are constructed. There is no theoretical reason for reducing to just two or three scenarios, only a practical one. It has been found that the managers who will be asked to use the final scenarios can only cope effectively with a maximum of three versions.

In our model the most uncertain or unknowable economic factor is variable interest rate. Three scenarios are used: best-case, probable or present, and worst-case.

**Fuzzification**

The uncertain or unknowable economic factor such as interest rate could be considered as imprecise value. Imprecision in multicriteria analysis can be modeled using fuzzy set theory. "Much of the decision-making in the real world takes place in an environment in which the goals, the constraints, and consequences of possible actions are not known precisely" (Bellman and Zadeh 1970). Ribeiro provides an overview of the concepts and theories of decision making in a fuzzy environment (Ribeiro 1996).

To express an imprecise value, the *triangular fuzzy number* (TFN) $\widetilde{N} = (l, m, r)$ is used, associated with the membership triangular function defined as follows:

$$\mu_{\widetilde{N}}(x) = \begin{cases} (x-l)/(m-l), & x \le m \\ (r-x)/(r-m), & x \ge m \\ 0, & x \notin [l, r] \end{cases}$$

The membership function $\mu(x)$ denotes the degree of truth that the fuzzy value is equal to $x$ within the real interval $[l, r]$. The fuzzy number $\tilde{N}$ has the core $m$ with $\mu(m) = 1$ and the support $[l, r]$.

In our model the variable interest rate is considered is fuzzy number $\widetilde{i} = (i_l, i_m, i_r)$ where $i_l$ is the best-case value, $i_m$ is the present or probable value, and $i_r$ is the worst-case value. Cost of borrowing is considered as fuzzy number in the similar way, because the actual cost of borrowing will be dependent on the interest rate realizations during the maturity.

**Defuzzification**

Defuzzification is selection of a specific crisp element based on the output fuzzy set, and it also includes converting fuzzy numbers into crisp scores.

There are two approaches to multicriteria analysis in a fuzzy environment, "conventional" and "fuzzy". The conventional approach is based on a nonfuzzy decision model, whereas the fuzziness dissolution (defuzzification) is performed at an early stage. The fuzzy approach is based on processing fuzzy data for decision making, then dissolving the fuzziness at a later stage. In both cases, defuzzification is necessary since the results of the analysis must provide a crisp conclusion. The approach with defuzzification before the multicriteria ranking of alternatives is applied here.

The operation defuzzification can not be defined uniquely, and consequently there are several defuzzification methods (Detyniecki and Yager 2000). Here, the defuzzification presented in Opricovic (2011) is adopted, where the crisp value $Crisp(\tilde{N})$, for the triangular fuzzy number $\widetilde{N} = (l, m, r)$, is determined by the following formula

$$Crisp\left(\widetilde{N}\right) = (2m + l + r)/4$$

This is a practical defuzzification tool for converting a fuzzy number into crisp number.

**Cost Evaluation**

The procedure for cost evaluation is based on the relation (7).

The following short example shows the impact of grace period.

Example: Loan 100,000, maturity 10 years, interest rate 10 %

| | Maximum yearly payment | Total cost |
|---|---|---|
| Borrowing with no grace period, G = 0 | 20,000 | 155,000 |
| Borrowing with grace period, G = 5 years | 30,000 | 180,000 |
| Borrowing with grace period, G = 9 years | 110,000 | 200,000 |

"Grace" is not for free.

The fuzzy cost $\widetilde{C} = (l_C, m_C, r_C)$ is computed with three values of interest rates $\widetilde{i} = (i_l, i_m, i_r)$, respectively (see Sect. 4.2).

**Market Risk Evaluation**

The market risk is evaluated with the worst-case value of total cost, introduced in Eq. (8). Here,

$$f_2 = r_C$$

**Liquidity Risk Evaluation**

The liquidity risk is evaluated with the maximum yearly payment at time $t = G + 1$, determined by the relation (9).

$$f_3 = \frac{Z}{T - G} + Z \times i$$

**Evaluation of Particular Conditions**

The particular condition could be qualitative (unquantifiable or linguistic). The negative effect of this criterion is evaluated by linguistic variables. The linguistic values: very low, low, median, high, very high, could be transformed by scaling into the numbers: 0, 25, 50, 75, 100, respectively. The conversion scaling could be nonlinear function or any mapping with the results as crisp numbers. This criterion function could be included in the set of criteria.

As alternative approach, such condition could be considered twice. First, in generating alternatives, alternative with not acceptable particular condition is rejected. Second, post analysis, the criterion is considered for final decision (selection from set of better ranked alternatives).

**Criteria Weights**

Most multicriteria methods require definition of quantitative weights for the criteria, to assess the relative importance of the different criteria. These weights do not have a clear economic significance. The use of weights gives us the opportunity to modelize the real aspects of decision making. "Equal importance" weights $(w_i = 1/n)$ should be used either when there is no information from the decision maker (DM) or when there is not enough information to differentiate the relative importance of criteria. Within an entropy context this case represents total ignorance about criteria preferences. "Given" weights should be used when the DM has a good knowledge about criteria, in terms of their values and of their relationships. Very often, it is not easy to get the values of weights (preference structure). In this case the preference structure is simulated by assuming different values of weights (with different scenarios of decision making). This approach could help in perceiving the influence of weights on the proposed (compromise) solution. In Sect. 4.3, four scenarios with different values of weights are considered.

# 4 Illustrative Example

An application of the developed model is illustrated by the numerical example using data for already concluded borrowings and adjusted to the same loan amount.

## 4.1 Alternatives

The model assumes alternatives with the same loan amount.

Borrowing alternatives with characteristic data are presented in Table 1. The following notation is used in Table 1:

i.r.—interest rate, SDR—Special Drawing Rights by International Monetary Fund
L-6M EUR—6 Month LIBOR Rate for Euro
L-3M CHF—3 Month LIBOR Rate for Swiss Frank
E-6M EUR—6 Month EURIBIR Rate for Euro
L-6M USD—6 Month LIBOR for USD
Alternatives $A_7$ to $A_{10}$ are with fixed interest rate.

## 4.2 Evaluation and Input Data

The model assumes alternatives with the same loan amount: 300 million EUR, 388 million USD, 363 million CHF, 252 million SDR, 34,509 million RSD (Serbian Dinar), determined by the exchange rates on 30.9.2012.

Payment schedule and the total cost of borrowing $f_1$ is determined by the following relation

**Table 1** Borrowing alternatives

| | Alternatives | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ | $A_8$ | $A_9$ | $A_{10}$ |
| Interest Rate (%) | i.r. SDR | L-6M EUR +0.05 | L-3M CHF +0.56 | E-6M EUR +2.45 | E-6M EUR +1.00 | L-6M USD +2.95 | 4.00 | 6.17 | 3.00 | 7.25 |
| Maturity (year) | 5 | 20 | 12 | 15 | 14 | 12 | 15 | 15 | 18 | 10 |
| Grace period y | 3 | 8 | 3 | 10 | 3 | 2 | 3 | 3 | 3 | 9 |
| Currency | SDR | EUR | CHF | EUR | EUR | USD | EUR | EUR | USD | USD |
| Repayment Schedule | E.Y | E.Y | E.Y | E.Y | E.Y | E.Y | E.Y | E.Y | E.Y | End |

*E.Y.* equal yearly principal paid, *End* total payments at maturity

$$f_1 = \sum_{t=1}^{G} Z \times i + \sum_{t=G+1}^{T} \frac{Z}{T-G}(1 + (T - t + 1) \times i) \tag{10}$$

where: $Z$—is the loan amount, $T$—the maturity (years), $G$—grace period (years), $i$—interest rate (absolute value, not %).

For the alternative with fixed interest rate one value of $f_1$ is computed. For the alternative with variable interest rate three values are computed (fuzzy interest rate $\widetilde{i} = (i_l, i_m, i_r)$):

$l_c$ with the minimum value of variable interest rate $i_l$;
$m_c$ with present value $i_m$;
$r_c$ with maximum value $i_r$.
(see Sect. 3.3, Fuzzification).

The value of criterion function for market risk is

$$f_2 = r_C$$

The value of criterion function for liquidity risk is computed as the maximum payment to be made in a single time step (here yearly)

$$f_3 = \frac{Z}{T-G} + Z \times i$$

or it could be taken from the payment schedule (maximum yearly value) computed by relation (10) with the maximum value of variable interest rate.

The criterion function for particular conditions is not used in this example because of lack of information.

The values for fixed and variable interest rates are presented in Table 2. These values are determined by the data from several sources (References: *Interest rates*). Computing values of maximum yearly payment and total cost of borrowing in original (contracted) currency for each alternative is illustrated in Table 2. For alternatives with variable interest rates, three values of maximum yearly payment and total cost are computed.

The schedule of repayments for alternative $A_7$ is presented here as an illustration.

| T | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Plan $C_t$ | 12 | 12 | 12 | 37 | 36 | 35 | 34 | 33 | 32 | 31 | 30 | 29 | 28 | 27 | 26 |

The values of criterion functions from Table 2 are converted into Euro and presented in Table 3.

The values of criterion functions ($f_1$, $f_2$, $f_3$) in Table 3 are the input data for VIKOR method.

**Table 2** Computing results

| Alter. | Currency | Loan (million) | Interest rate | Maturity T (year) | Grace period (year) | Maximum yearly paym. (million) | Cost of borrowing (million) |
|---|---|---|---|---|---|---|---|
| $A_1$ | SDR | 252 | 0.000700 | 5 | 3 | 126.18 | 252.79 |
| $A_1$ | SDR | 252 | 0.000800 | 5 | 3 | 126.29 | 252.91 |
| $A_1$ | SDR | 252 | 0.034900 | 5 | 3 | 134.79 | 291.58 |
| $A_2$ | EUR | 300 | 0.009312 | 20 | 8 | 27.79 | 340.51 |
| $A_2$ | EUR | 300 | 0.009312 | 20 | 8 | 27.79 | 340.51 |
| $A_2$ | EUR | 300 | 0.052645 | 20 | 8 | 40.79 | 529.01 |
| ... | ... | ... | ... | ... | ... | ... | ... |
| $A_7$ | EUR | 300 | 0.040000 | 15 | 3 | 37.00 | 414.00 |
| ... | ... | ... | ... | ... | ... | ... | ... |
| $A_{10}$ | USD | 388 | 0.072500 | 10 | 9 | 416.13 | 669.30 |

## 4.3 Results by the VIKOR Method

Equal criteria weights should be used if there is no information of DM's preference, or as an additional analysis. Here the following weights are used $w_i = 1/3$, $i = 1$, 2, 3 and $v = 0.667$. Ranking results with characteristic data for ranked alternatives are presented in Table 4.

The results by the VIKOR method are determined with different criteria weights and presented in Table 5.

Preferences of DMs on the total cost of borrowing are expressed by the values of weights $w = \{0.65; 0.15; 0.20\}$. The compromise solution obtained by VIKOR method is the set of alternatives $\{A_3, A_1\}$. The set $\{A_3, A_1\}$ should be presented to the DMs for further decision making process.

Compromise solution with the market risk preference, weights $w = \{0.20, 0.65, 0.15\}$,. is the set of alternatives $\{A_1, A_3\}$.

Compromise solution with the liquidity risk preference, weights $w = \{0.20, 0.15, 0.65\}$. is the set of alternatives $\{A_3, A_9, A_7, A_5\}$.

**Proposing Final Solution**

With four scenarios of preference {1/3, 1/3, 1/3}, {0.65, 0.15, 0.20}, {0.20, 0.65, 0.15}, {0.20, 0.15, 0.65}, the following compromise solutions are determined: $A_3$, $\{A_3, A_1\}$, $\{A_1, A_3\}$, and $\{A_3, A_9, A_7, A_5\}$, respectively. The alternative $A_3$ is the most stable one according to preference and it could be the final solution.

The loan and interest rate are relatively small and liquidity risk is not factor for debt crises, and assuming decision maker is indifferent for currency (SDR or CHF), the final solution could be alternative $A_1$ (with longer maturity).

Ranking results point out alternatives $A_{10}$, $A_4$ and $A_8$ as less favorable.

**Table 3** Value of criterion functions

| | | Alternatives | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|
| | | $A_1$ | $A_2$ | $A_3$ | $A_4$ | $A_5$ | $A_6$ | $A_7$ | $A_8$ | $A_9$ | $A_{10}$ |
| Cost[a] | $l_C$ | 300.96 | 340.51 | 313.53 | 432.02 | 352.25 | 375.10 | 414.00 | 475.85 | 399.00 | 517.50 |
| | $m_C$ | 301.10 | 340.51 | 315.55 | 432.02 | 352.25 | 382.93 | 414.00 | 475.85 | 399.00 | 517.50 |
| | $r_C$ | 347.14 | 529.01 | 385.24 | 599.09 | 467.91 | 453.72 | 414.00 | 475.85 | 399.00 | 517.50 |
| Cost[b] $f_1$ | | 312.58 | 387.84 | 332.47 | 473.79 | 381.17 | 398.67 | 414.00 | 475.85 | 399.00 | 517.50 |
| M.Risk[c] $f_2$ | | 347.1 | 529.0 | 385.2 | 599.0 | 467.9 | 453.7 | 414.0 | 475.8 | 399.0 | 517.5 |
| L.Risk[d] $f_3$ | | 160.5 | 40.79 | 43.99 | 83.01 | 45.93 | 59.48 | 37.00 | 43.51 | 29.00 | 321.8 |

[a]Total cost as fuzzy number ($l_c$, $m_c$, $r_c$), in million EUR
[b]Total cost defuzzified
[c]Market risk evaluated with $r_c$
[d]Liquidity risk is evaluated with maximum yearly payment (tranche)

**Table 4** Ranking list with equal criteria weights

| | Rank | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| Rank by $Q$ | $A_3$ | $A_1$ | $A_9$ | $A_7$ | $A_6$ | $A_5$ | $A_2$ | $A_8$ | $A_4$ | $A_{10}$ |
| $Q$ | 0.000 | 0.159 | 0.198 | 0.272 | 0.288 | 0.289 | 0.457 | 0.550 | 0.802 | 1.000 |
| Currency | CHF | SDR | USD | EUR | USD | EUR | EUR | EUR | EUR | USD |
| Interest Rate (%) | L-3M CHF +0.56 | i.r. SDR | 3.00 | 4.00 | L-6M USD +2.95 | E-6M EUR +1.00 | L-6M EUR +0.05 | 6.17 | E-6M EUR +2.45 | 7.25 |
| Maturity | 12 | 5 | 18 | 15 | 12 | 14 | 20 | 15 | 15 | 10 |
| Grace Period | 3 | 3 | 3 | 3 | 2 | 3 | 8 | 3 | 10 | 9 |
| Cost | 332.47 | 312.58 | 399.00 | 414.00 | 398.67 | 381.17 | 387.84 | 475.85 | 473.79 | 517.50 |
| Market Risk | 385.24 | 347.14 | 399.00 | 414.00 | 453.72 | 467.91 | 529.01 | 475.85 | 599.09 | 517.50 |
| Liquidity risk | 43.99 | 160.48 | 29.00 | 37.00 | 59.48 | 45.93 | 40.79 | 43.51 | 83.01 | 321.75 |
| Rank by $S$ | 1 | 2 | 3 | 4 | 6 | 5 | 7 | 8 | 9 | 10 |
| $S$ | 0.1 | 0.15 | 0.209 | 0.263 | 0.291 | 0.316 | 0.376 | 0.452 | 0.657 | 0.892 |
| Rank by $R$ | 1 | 4 | 2 | 6 | 3 | 5 | 7 | 8 | 9 | 10 |
| $R$ | 0.05 | 0.141 | 0.141 | 0.15 | 0.16 | 0.165 | 0.241 | 0.266 | 0.333 | 0.333 |

The compromise solution is the alternative $A_3$
Alternative $A_1$ is ranked as second
Alternatives $A_{10}$, $A_4$ and $A_8$ are less favorable or unpleasant

**Table 5** Ranking by the VIKOR method with three set of criteria weights

| | Weights | | | Ranks | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| | $w_1$ | $w_2$ | $w_3$ | Ranking | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| | | | | Q ranking | $A_3$ | $A_1$ | $A_5$ | $A_9$ | $A_2$ | $A_6$ | $A_7$ | $A_8$ | $A_4$ | $A_{10}$ |
| *W2* | 0.65 | 0.15 | 0.20 | S ranks | 2 | 1 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| | | | | R ranks | 1 | 2 | 3 | 6 | 4 | 5 | 7 | 9 | 8 | 10 |
| | | | | Q ranking | $A_1$ | $A_3$ | $A_9$ | $A_7$ | $A_6$ | $A_5$ | $A_8$ | $A_2$ | $A_{10}$ | $A_4$ |
| *W3* | 0.2 | 0.65 | 0.15 | S ranks | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| | | | | R ranks | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 9 | 8 | 10 |
| | | | | Q ranking | $A_3$ | $A_9$ | $A_7$ | $A_5$ | $A_6$ | $A_2$ | $A_8$ | $A_1$ | $A_4$ | $A_{10}$ |
| *W4* | 0.2 | 0.15 | 0.65 | S ranks | 1 | 2 | 3 | 4 | 6 | 5 | 7 | 8 | 9 | 10 |
| | | | | R ranks | 1 | 4 | 5 | 2 | 3 | 6 | 8 | 9 | 7 | 10 |

## 5 Conclusion

The decision to borrow is a part of complex decision making to improve the performance of an entity. Here, we assume the decision for borrowing is made before, including the loan amount. The considering problem is to determine borrowing alternative preferred according to the established criteria. This chapter proposes a decision model for selection of borrowing alternative. The alternatives are generated with different borrowing instruments proposed by the creditors.

The multicriteria nature of borrowing has been recognized. The relevant criteria are incorporated in selecting the combination of instruments, including cost and risks. Debtor considers market risk as a criterion that he would like to minimize, and to avoid increasing cost of borrowing. The liquidity risk became important for creditor and for debtor because debt has been increasing substantially in all economies, at all income levels.

In this decision model the most uncertain or unknowable economic factor is variable interest rate, and three scenarios are used: best-case, present (credible), and worst-case. The variable interest rate is considered is fuzzy number $\widetilde{i} = (i_l, i_m, i_r)$ where $i_l$ is the best-case value, $i_m$ is the present or credible value, and $i_r$ is the worst-case value. These three values are determined from the series of historical data. Cost of borrowing is considered as fuzzy number in the similar way, because the actual cost of borrowing will be dependent on the interest rate realizations during the maturity. Fuzzy numbers are defuzzified converting them into crisp scores for ranking alternatives by the VIKOR method.

The VIKOR method is applied to determine the compromise solution which is the closest to the ideal. The obtained compromise solution could be accepted by the decision makers because it provides a maximum utility of the majority, and a minimum individual regret of the opponent. The final solution could be proposed from the ranking results with different criteria weights.

An application of the developed model is illustrated by the numerical example using data for already concluded borrowings. The results confirm the applicability of this model with VIKOR method.

An extension of this model should include currency risk for borrowings in different currencies and a nominal exchange rate risk, as additional factors of uncertainties.

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# Mutual Funds' Socially Responsible Portfolio Selection with Fuzzy Data

**Paz Méndez-Rodríguez, Blanca Pérez-Gladish, José M. Cabello, and Francisco Ruiz**

**Abstract** Socially Responsible Investing (SRI) corresponds to an investment practice that takes into account not only the usual return-risk criteria, but also other non-financial dimensions, namely in terms of environmental, social and governance concerns. However, while a diverse set of models has been developed to support investment decision-making based on classical financial criteria, models including also a socially responsible dimension are rather scarce. In this chapter we present a multicriteria portfolio selection model for mutual funds based on the Reference Point Method which takes into account both a financial and a non-financial dimension. The latter is usually characterized by the imprecise, ambiguous and/or uncertain nature of decision making criteria. This is why fuzzy methodology is used to model social responsibility. The proposed model is intended to be an individual investment decision making tool for mutual funds' portfolio selection, taking into account the subjective and individual preferences of an individual investor under two different scenarios: a low social responsibility degree and a high social responsibility degree scenario. In order to illustrate the suitability and applicability of the investment decision making model proposed, an empirical study on a set of US domiciled equity mutual funds is carried out.

**Keywords** Socially Responsible Investment (SRI) • Portfolio Selection • Equity Mutual Funds • Multicriteria Decision Making • Reference Point method • Fuzzy Numbers

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# 1 Introduction

Socially Responsible Investing (SRI) is broadly defined as an investment process that integrates financial but also social, environmental, and ethical concerns into investment decision making. This investment strategy is gaining popularity. As reported by the Social Investment Forum (SIF) in its 2010 report (SIF 2010): "At the start of 2010, professionally managed assets following SRI strategies stood at $3.07 trillion, a rise of more than 380 percent from $639 billion in 1995 (. . .). Over the same period, the broader universe of assets under professional management increased only 260 percent from $7 trillion to $25.2 trillion".

Mutual funds are the main socially responsible investment tool. The main investment strategy used by socially responsible mutual funds (SRI funds) is screening. Screening, positive and/or negative, is the practice of evaluating mutual funds based on social, environmental, ethical and/or good corporate governance criteria. Positive screening implies investing in profitable companies that make positive contributions to society. Conversely, negative screening implies avoiding investing in companies whose products and business practices are harmful to individuals, communities, or the environment.

SRI funds form a very heterogeneous group in terms of their social, environmental and ethical investment policy; number, type and implementation of non-financial screens applied; engagement degree with shareholder resolutions; voting policy or, even with respect to the degree of transparency and credibility of the non-financial information provided to the investors (SRI research policy, expertise level of the fund managers, communication with companies and investors, external control etc.). However, this heterogeneity is not usually taken into account in the social responsible performance measurement of SRI mutual funds, and according to Muñoz et al. (2004) this lack of harmonization of social criteria among SRI funds is one of the main problems faced by financial managers.

Most of the academic works where a social performance measure is proposed for mutual funds, use a simple binary variable for just two social responsible categories (social responsible/non-social responsible funds), relying on mutual funds' self-classification into one of those categories. Very few studies can be found considering different degrees of social responsibility. These studies usually propose screening intensity as a proxy of mutual funds' social performance degree [some examples are works by Barnett and Salomon (2002, 2006), Lee et al. (2010), Jegourel and Maveyraud (2010), Scholtens (2007) or Renneboog et al. (2008)].

Pérez-Gladish and M'Zali (2010) propose an AHP-based method which allows measurement of social responsibility based on a set of criteria directly related with the quality of the management of socially responsible mutual funds, in terms of its transparency and credibility: investment policy, screening approach, engagement policy, research process, control of companies, external control, competence of fund managers and communication with companies and investors, among others.

Muñoz et al. (2004) evaluate the investment policy of Spanish SRI funds based on the standard "*Ethics. Requirements for ethical and socially responsible financial* *instruments*" (PNE 165001 EX). The main objective of this standard is "to certify that SRI investment products act in accordance with certain parameters and invest in companies also considered socially responsible".

Therefore, from the literature review and existent practice, we can observe the absence of a common basis for measuring mutual funds' social performance (Kaidonis 1999; Van Der Laan 2001; Goodpaster 2003). Investors seeking to invest in mutual funds including socially responsible criteria currently face an important lack of information (Liern et al. 2015). Scoring of mutual funds taking into account socially responsible criteria has an important practical relevance in portfolio selection especially nowadays, given the causes of the 2008 financial crisis, when these concerns became even more relevant for investors. Portfolio Selection models including social and/or environmental criteria are rather scarce and in a large number of cases social and/or environmental performance measurement relies on a crisp or precise real number reflecting the number of applied screens (see Ballestero et al. 2015). Some interesting exceptions are the works by Ballestero et al. (2012, 2015), Gupta et al. (2013), Barracchini (2004), Bilbao-Terol et al. (2012, 2013), Hallerbach et al. (2004), Hirschberger et al. (2012), Steuer et al. (2007), Dorfleitner and Utz (2012), Calvo et al. (2014), Cabello et al. (2014), etc.

The aim of this work is to provide particular investors with an individual tool for mutual funds' portfolio selection taking into account not only the classical financial criteria, risk and return, but also non-financial criteria (socially responsible criteria). In order to do so, first, social performance has been measured relaying not only on screening intensity, but also on the type of the screen and on the transparency and credibility of the social responsible investment strategy and research and control processes.

The social responsibility degree of a mutual fund can be considered, by its own nature, as an imprecise and/or uncertain data which can be handled through a fuzzy number estimated by the individual investor and/or an expert on SRI, based on the investor's personal preferences and on the expert's knowledge.

Secondly, a multicriteria portfolio selection model based on the reference point method is proposed for two different scenarios: low social responsibility and high social responsibility. The proposed optimization model includes constraints on the degree of social responsibility of the portfolio reflecting both scenarios.

Therefore, the model presented in this chapter is an individual investment decision making tool for mutual funds' portfolio selection, taking into account the subjective and individual preferences about different non-financial features, and incorporating the ambiguity and/or imprecision of the social responsibility data obtained from the expert's evaluation.

The structure of the chapter is as follows. In the following section we will propose an approach for the measurement of mutual funds' social responsibility degree; in Sect. 3 will present the portfolio selection model including a set of constraints which impose minimum bounds on the social responsibility of the portfolio; Sect. 4 presents the Multicriteria Decision Making Method proposed in this chapter for the resolution of the portfolio selection problem: the reference point method; in Sect. 5 an empirical study will be carried out in order to illustrate the proposed model and, finally, in Sect. 6 main conclusions will be presented.

## 2 Mutual Funds' Fuzzy Social Responsibility Degree

The definition of socially responsible performance needs a clear understanding of the fundamental criteria. From the review of the literature and current practice, we identify two different main dimensions on Socially Responsible Degree (SRD) measurement: a dimension related to the "Socially Responsible Investment Strategies" followed by the fund manager, and a "Quality of Information" dimension related to transparency and credibility of the information provided by the mutual fund manager.

In this work we will focus on the main Socially Responsible Investment Strategy followed by mutual funds: screening (positive and/or negative). According to the process followed by the extra-financial rating agency Kinder, Lydenberg, Domini & Co (KLD), when rating US companies, a total of 41 screens will be considered which take into account Corporate Social Responsibility across a range of issues that impact a company's various stakeholders: environment, community and society, customers, employees and supply chain, governance and ethics. They are grouped in three different areas of concern: environment, social and governance. The environment concern includes screens related to: climate change and clean technologies, pollution and toxics and other environment issues as recycling questions. Under the social concern we have grouped screens related with community investment, diversity and Equal Employment Opportunities (EEO), human rights and labor relations. The last concern, Governance, relates to board issues. Screens included in a second component "Products and Processes" refer to the exclusion of investments related to production of alcohol, tobacco, or gambling products, known collectively as the "sin" screens, for over 60 years. Other popular negative screens taken into account refer to military weapons production, firearms, and nuclear.

Assessment of mutual funds' social responsibility degree is, due to the ambiguous, imprecise and/or uncertain character of the dimensions and variables considered, a difficult question. A large amount of information is available but data are in most of the cases imprecise, ambiguous and with a high degree of associated uncertainty. It is difficult to verify if the provided information is trustable or not as very few control systems exist in order to guarantee the transparency and credibility of non-financial data.

On the other hand, no clear measures, rules and/or processes exist in order to evaluate the degree of environmental, social, ethical and/or governance responsibility of a mutual fund.

Fuzzy Sets Theory offers some elements which can help decision makers (DMs) to assess the social responsibility degree of mutual funds as it provides suitable tools for dealing with uncertainty and imprecision in data and it facilitates the incorporation of expert knowledge from the DM, which is in most of the cases of subjective character.

The main idea of Fuzzy Set Theory is quite intuitive and natural: instead of determining the exact boundaries as in an ordinary set, a fuzzy set allows for no sharply defined boundaries because of the generalization of a characteristic function to a membership function. By letting $X$ denote a universal set, a fuzzy set $\tilde{A}$ of $X$ can be characterized as a set of ordered pairs of element $x$ and the grade of membership of $x$ in $\tilde{A}$, $\mu_{\tilde{A}}(\mathrm{x})$, and it is often written:

$$\tilde{A} = \left\{ \left( x, \mu_{\tilde{A}}(x) \right) / x \in X \right\} \tag{1}$$

Note that the membership function is an obvious extension of the idea of a characteristic function of an ordinary set because it takes values between 0 and 1, not only 0 and 1. A membership level equal to zero means no membership, a membership value equal to one means Boolean membership and intermediate numbers reflect intermediate membership degrees (see Kauffman and Gil-Aluja 1987; Zimmermann 1996).

A fuzzy number is one of the most common forms of fuzzy set application (Kaufmann and Gupta 1988); it is defined as a fuzzy set defined on the real line with a convex, continuous and normalized membership function.

The problem addressed in this work, the evaluation of the social responsible degree of mutual funds, is similar to that of personnel selection presented by Canós and Liern (2004, 2008), where candidates for a job have to be evaluated on a number of fuzzy competences.

Let us consider $n$ mutual funds $\{F_1, F_2, \cdots, F_n\}$ that will be evaluated with respect to $m$ non-financial screens $\{S_1, S_2, \cdots, S_m\}$. Due to the imprecise description made in linguistic terms of each screen it is difficult for the investor to evaluate each asset with respect to each screen using a single crisp (precise) numerical value. It seems more appropriated to state the imprecise and subjective evaluations in terms of intervals or fuzzy numbers (Slowinski 1998). According to the procedure followed by Canós and Liern (2004) for the prob8lem personnel selection, we will evaluate the social responsibility degree of every screen applied by the *ith* mutual fund assigning to it an interval inside (0,1] (see Gil-Aluja 1996, 1999):

$$\tilde{s}_{ij} = \left\{ \left( s_{ij}, \left[ b^L_{s_{ij}}, b^U_{s_{ij}} \right] \right) : s_{ij} \in S_{ij} \right\},$$
$$\text{where } \left[ b^L_{s_{ij}}, b^U_{s_{ij}} \right] \subseteq (0, 1], \quad \mathrm{i} = 1, \cdots, \mathrm{n}, \quad \mathrm{j} = 1, \cdots, \mathrm{m} \tag{2}$$

Thus, we obtain a discrete fuzzy set for each mutual fund in which the interval $\left[ b^{1i}_{s_j}, b^{2i}_{s_j} \right]$ represents membership function of mutual fund $F_i$ in the screen $s_j$ considered as a tolerance interval. Its membership function is given by:

$$\mu\left(\tilde{s}_{ij}\right) = \left[b^L_{s_{ij}}, b^U_{s_{ij}}\right] \subseteq (0,1] \tag{3}$$

Next step consists of obtaining the weights of each mutual fund in each screen. As we did with the social responsibility degree of every screen applied by the *ith* mutual fund, we will assign each weight an interval inside (0, 1]:

$$\tilde{w}_{ij} = \left\{ \left( w_{ij}, \left[ b^L_{w_{ij}}, b^U_{w_{ij}} \right] \right) : w_{ij} \in W_{ij} \right\},$$
$$\text{where } \left[ b^L_{w_{ij}}, b^U_{w_{ij}} \right] \subseteq (0,1], \quad \mathrm{i} = 1, \cdots, \mathrm{n}, \quad \mathrm{j} = 1, \cdots, \mathrm{m} \tag{4}$$

These weights will be also a discrete set for each mutual fund in which the interval $\left[b^L_{w_{ij}}, b^U_{w_{ij}}\right]$ represents membership function of the weight of mutual fund $F_i$ in the screen $s_j$, considered as a tolerance interval. Its membership function is given by:

$$\mu\left(\hat{w}_{ij}\right) = \left[b^L_{w_{ij}}, b^U_{w_{ij}}\right] \subseteq (0,1] \tag{5}$$

These weights play a correcting role as they represent the degree of transparency and credibility of the information on the screening process provided by the mutual funds. They are given by an expert and they depend on several criteria: quality of the description of the screening process, existence of an external research team composed on experts in SRI, periodical non-financial audits, description of engagement policy and public disclosure of proxy voting practices and education of the fund manager on SRI practices.

Once the weights have been established by the expert, the problem consists of aggregating all the information available to construct a global measure of the social responsibility of each mutual fund.

We will review first some basic ideas of interval arithmetic. Let $[a,b]$ and $[c,d]$ be two closed and bounded intervals. It follows that:

$$[a,b] + [c,d] = [a+c, b+d] \tag{6}$$

$$[a,b] \div [c,d] = [a,b] \times \left[\frac{1}{d}, \frac{1}{c}\right] \tag{7}$$

If zero does not belong to $[c,d]$ then:

$$[a,b] \times [c,d] = [k,v], \tag{8}$$

where $k = \min\{ac, ad, bc, bd\}$, $v = \max\{ac, ad, bc, bd\}$. If $a > 0$ and $c > 0$:

$$[a,b] \times [c,d] = [ac, bd], \tag{9}$$

Then, for each mutual fund $F_i$, its Fuzzy Social Responsible Degree, $\widetilde{SRD}_i$ will be defined as the following fuzzy weighted average mean:

$$\widetilde{SRD}_i = \frac{\sum_{j=1}^{m} \widetilde{w}_{ij}\widetilde{s}_{ij}}{\sum_{j=1}^{m} \widetilde{w}_{ij}}, \qquad i = 1, \cdots, n \tag{10}$$

And taking into account (2) and (4) we will obtain the following Social Responsibility Degree interval for each mutual fund $i$:

$$\widetilde{SRD}_i = \left[SRD_i^L, SRD_i^u\right] = \frac{\sum_{j=1}^{m} \left[b_{w_{ij}}^L, b_{w_{ij}}^U\right] \times \left[b_{s_{ij}}^L, b_{s_{ij}}^U\right]}{\sum_{j=1}^{m} \left[b_{w_{ij}}^L, b_{w_{ij}}^U\right]}, \qquad i = 1, \cdots, n \tag{11}$$

## 3 Mutual Funds' Portfolio Selection Model Taking into Account Socially Responsibility Constraints

**Decision Variables**

We will consider $n$ mutual funds $(i = 1, \cdots, n)$. Let us consider a portfolio $P$ whose composition will be denoted by $\overline{x} = (x_1, \cdots, x_n)$ where $x_i$ denotes the proportion of the investor's budget invested in mutual fund $i$ $(i = 1, \cdots, n)$. Besides, we will consider $n$ instrumental binary variables, $\overline{y} = (y_1, \cdots, y_n)$, which take the value 1 if the corresponding fund is in the portfolio, and 0 otherwise.

**Objectives**

Two objectives are considered:

*Maximization of the Portfolio's Expected Return* The expected return of each fund will be approximated by considering the historical mean of weekly returns of the asset for a given observation period:

$$ER_i = \frac{1}{T}\sum_{t=1}^{T} r_{it}, \quad i = 1, \cdots, n \tag{12}$$

where $r_{it}$ is the return obtained by fund $i$ over the period $t$.

*Minimization of Risk Variance* The covariance between returns of funds $i$ and $k$ which will be approximated as follows:

$$\sigma_{ik} = \frac{1}{T}\sum_{t=1}^{T} (r_{it} - ER_i)(r_{kt} - ER_k), \quad i, k = 1, \cdots, n \tag{13}$$

**Constraints**
The following constraints are considered:

*Minimum Bounds on the Portfolio's Fuzzy Social Responsibility Degree*

$$\widetilde{SRD}_P = \sum_{i=1}^{n} \widetilde{SRD}_i x_i \geq g \tag{14}$$

The investor can take into account a global measure including a weighted average of all the screens applied, without differentiating among social responsibility dimensions and all the screens (41 screens). It is also possible to consider separately each screen or to consider screens grouped in their different dimensions: climate change, board issues, human rights, alcohol, tobacco, animal testing etc.

*Budget Constraint* The sum of the proportions to be invested in the assets should be equal to 1 which means 100 % of the total budget should be invested in the portfolio:

$$\sum_{i=1}^{n} x_i = 1 \tag{15}$$

*Diversification Constraints* This set of constraints includes lower and upper bounds on the investment in each particular mutual fund, if it is part of the portfolio, in order to ensure diversification:

$$0.05 y_i \leq x_i \leq 0.2 y_i, \quad i = 1, \cdots, n \tag{16}$$

Besides, an upper bound is imposed on the total number of funds in the portfolio:

$$\sum_{i=1}^{n} y_i \leq 8 \tag{17}$$

Then, the formulation of the portfolio selection model is:

$$\left\{\begin{array}{ll} \max & ER(\boldsymbol{x}) = \sum_{i=1}^{n} ER_i x_i \\ \min & \sigma^2(\boldsymbol{x}) = \sum_{i=1}^{n} \sum_{k=1}^{n} \sigma_{ik} x_i x_k \\ \text{s.t.} & \sum_{i=1}^{n} \widetilde{SRD}_i x_i \geq g \\ & \sum_{i=1}^{n} x_i = 1 \\ & 0.05 y_i \leq x_i \leq 0.2 y_i \\ & \sum_{i=1}^{n} y_i \leq 8 \end{array}\right. ; \left\{\begin{array}{ll} \max & ER(\boldsymbol{x}) = \sum_{i=1}^{n} ER_i x_i \\ \min & \sigma^2(\boldsymbol{x}) = \sum_{i=1}^{n} \sum_{k=1}^{n} \sigma_{ik} x_i x_k \\ \text{s.t.} & \sum_{i=1}^{n} \left[SRD_i^L, SRD_i^U\right] x_i \geq g \\ & \sum_{i=1}^{n} x_i = 1 \\ & 0.05 y_i \leq x_i \leq 0.2 y_i \\ & \sum_{i=1}^{n} y_i \leq 8 \end{array}\right. \tag{18}$$

We will address the resolution of the above model considering two scenarios: a scenario with low social responsibility degree ($SRD_i^L$) and a scenario with high social responsibility degree ($SRD_i^U$):

$$\begin{cases} \max & ER(\boldsymbol{x}) = \sum_{i=1}^{n} ER_i x_i \\ \min & \sigma^2(\boldsymbol{x}) = \sum_{i=1}^{n}\sum_{k=1}^{n} \sigma_{ik} x_i x_k \\ \text{s.t.} & \sum_{i=1}^{n} SRD_i^L x_i \geq g \\ & \sum_{i=1}^{n} x_i = 1 \\ & 0.05 y_i \leq x_i \leq 0.2 y_i \\ & \sum_{i=1}^{n} y_i \leq 8 \end{cases} ; \begin{cases} \max & ER(\boldsymbol{x}) = \sum_{i=1}^{n} ER_i x_i \\ \min & \sigma^2(\boldsymbol{x}) = \sum_{i=1}^{n}\sum_{k=1}^{n} \sigma_{ik} x_i x_k \\ \text{s.t.} & \sum_{i=1}^{n} SRD_i^U x_i \geq g \\ & \sum_{i=1}^{n} x_i = 1 \\ & 0.05 y_i \leq x_i \leq 0.2 y_i \\ & \sum_{i=1}^{n} y_i \leq 8 \end{cases} \tag{19}$$

Social responsibility degrees are handled by means of confidence intervals. Therefore, if we consider the low bounds of the intervals provided by an expert on SRI, we will be reflecting a situation where, in the expert's opinion, the degree of credibility and transparency of the non-financial information provided by the mutual funds is low. This situation will be characterized by poor information on the social screening process and few guarantees on the quality of the information provided by the mutual funds with regard to their social screening process. On the contrary, a high social responsibility scenario will reflect a highly confident context with regard to the transparency and credibility of the social screening process followed by the mutual funds.

## 4 The Reference Point Method for Multicriteria Decision Making Problems

Many methods exist for solving multiple criteria decision making problems, like the one modeled in this section. Most of them try to find efficient solutions for the multiple criteria problem, understood a feasible solutions such that it is not possible to improve one of the objectives without worsening at least some other one. Some of the methods just generate a set (or all) of efficient solutions of the problem, and the decision maker (DM) chooses one among them (a posteriori methods). Others ask the DM for some preferential information, and then generate the efficient solution that best fits these preferences (a priori methods). Finally, a third group of methods carry out several iterations, where the preferential information is gradually incorporated, and the method stops when a satisfactory enough solution has been found (interactive methods). The reference point based methods (see Wierzbicki 1980) constitute a link between the two latter classes. The decision maker (DM) is asked to give desired (reference) levels for each objective. Then, a single objective problem is solved where a so-called achievement scalarizing function (which measures the closeness of each feasible solution to the reference point) is optimized. Under mild conditions, the optimal solution of this problem is assured to be efficient for the original multiple criteria problem. This formulation can also be complemented with preferential weights that indicate how important is for the DM to achieve each of the reference levels (see Luque et al. 2009). Finally, this scheme can be easily embedded in an interactive framework, where reference levels and weights can be updated after each iteration has been carried out and the corresponding solution has been shown to the decision maker (DM), until he decides to stop. For further information about Multiple Criteria Optimization Methods in general, see Miettinen (1999).

For each objective function $f$, let us denote by $f^*$ its optimal value in the feasible set (called ideal value), and by $f_*$ its anti-ideal value, which is the worst value of $f$ in the optimal solutions of the rest of the objective functions. $f_*$ is frequently used as an approximation of the nadir value of $f$, which is the worst value $f$ takes in the efficient set.

For our portfolio selection problem, if the decision maker (DM) sets reference levels $q^{ER}$ and $q^{\sigma}$, for the expected return and risk, respectively, with preferential weights $\omega^{ER}$ and $\omega^{\sigma}$, then the problem to be solved for the low social responsibility scenario is:

$$\begin{cases} \min & d + \rho\left(\dfrac{1}{ER^* - ER_*}\left(q^{ER} - ER(\boldsymbol{x})\right) + \dfrac{1}{\sigma_* - \sigma^*}\left(\sigma^2(\boldsymbol{x}) - q^{\sigma}\right)\right) \\ \text{s.t.} & \sum_{i=1}^{n} SRD_i^L x_i \geq g \\ & \sum_{i=1}^{n} x_i = 1 \\ & 0.05 y_i \leq x_i \leq 0.2 y_i \\ & \sum_{i=1}^{n} y_i \leq 8 \\ & \dfrac{\omega^{ER}}{ER^* - ER_*}(q^{ER} - ER(\boldsymbol{x})) \leq d \\ & \dfrac{\omega^{\sigma}}{\sigma_* - \sigma^*}(\sigma^2(\boldsymbol{x}) - q^{\sigma}) \leq d \end{cases} \tag{20}$$

As can be seen, term $\left(ER^* - ER_*\right)$, or the corresponding one for the risk, is used as a normalizing factor. The objective function that is minimized in (20) is the achievement scalarizing function, which takes a positive optimal value if the reference levels cannot be simultaneously achieved and a negative value otherwise. In the latter case, the use of this function guarantees that the values of the objective functions are improved beyond their reference levels until an efficient solution is achieved. The second term of the achievement function (called augmentation term) is an instrumental term that guarantees that the final solution is efficient. $\rho$ is a small positive number. The problem for the high social responsibility scenario can be built in an analogous way. Further details can be found in Wierzbicki (1980). As mentioned before, this scheme can be used in an interactive fashion, so that the decision maker (DM) gives the reference levels (and weights, if so desired) at each step, problem (20) is solved, the optimal solution is shown to the DM, and the process continues until the DM is satisfied with the current solution.

## 5 Empirical Model: Socially Responsible Portfolio Selection from US Equity Mutual Funds

Our database is composed of 35 large cap conventional and socially responsible mutual funds. The so-called set of socially responsible mutual funds consists of all the 25 large cap equity mutual funds which are members of the US Social Investment Forum (SIF). The other ten funds were chosen among the conventional funds that has a better Sharpe ratio, because, not having any social responsibility degree, they will only enter the portfolio based on their expected return and risk. Due to space limitations, we do not show the covariance matrix of the funds. Instead, the expected return and the Sharpe ratio of the funds are displayed in Table 1.

The decision maker, who in this example is a SRI expert from a non-profit organization, based on her expert knowledge, evaluates the social responsibility degree of each of the screens applied. She has taken into account the type of screening, positive or negative (for example, for a particular decision maker negative screening could be more social responsible than positive screening) and the different issues screened (for example, for a particular decision maker human

**Table 1** Expected return and Sharpe ratio of the funds considered

| Fund | Return | Sharpe | Fund | Return | Sharpe |
|---|---|---|---|---|---|
| **F1** | −0.01 | −0.44 | **F19** | −0.01 | −0.53 |
| **F2** | −0.03 | −1.03 | **F20** | 0.09 | 3.49 |
| **F3** | −0.03 | −0.97 | **F21** | 0.08 | 3.12 |
| **F4** | 0.00 | −0.07 | **F22** | 0.11 | 4.80 |
| **F5** | −0.01 | −0.17 | **F23** | 0.03 | 0.98 |
| **F6** | −0.03 | −1.11 | **F24** | 0.03 | 0.90 |
| **F7** | −0.05 | −1.77 | **F25** | 0.06 | 2.54 |
| **F8** | −0.05 | −1.77 | **F26** | 0.14 | 0.9 |
| **F9** | −0.02 | −0.79 | **F27** | 0.10 | 1.3 |
| **F10** | 0.00 | −0.06 | **F28** | 0.10 | 1.1 |
| **F11** | −0.02 | −0.82 | **F29** | 0.08 | 1.1 |
| **F12** | −0.02 | −0.73 | **F30** | 0.08 | 1.2 |
| **F13** | 0.06 | 2.44 | **F31** | 0.07 | 0 |
| **F14** | 0.05 | 1.80 | **F32** | 0.06 | |
| **F15** | 0.05 | 1.85 | **F33** | 0.05 | |
| **F16** | 0.07 | 2.82 | **F34** | 0.06 | |
| **F17** | −0.01 | −0.24 | **F35** | 0.06 | |
| **F18** | −0.02 | −0.61 | | | |

rights could be *more* social responsible than recycling). The quantitative imprecise and subjective data obtained are incorporated to the model by means of the $\widetilde{s}_{ij}$ fuzzy coefficients. Note that the decision maker can also be an individual investor who incorporates his/her subjective personal preferences about the different social screens into the evaluation processes.

The second step consists of evaluating the transparency and credibility of the screening process. From the information provided by the mutual funds and displayed on the website of the Social Investment Forum, we can observe how all the socially responsible funds indicate to some degree explicit criteria for screening decisions. They apply both positive and negative screening, but all of them allow restricted investments in certain activities i.e. they seek to avoid only poorer performers in one area but they do not totally exclude investments engaged in certain activities (tobacco, alcohol, gambling. . ..). The funds take into account not only direct but also indirect infringement of screens. It is interesting to observe that no fund makes explicit reference to the support of shareholders resolutions, but they all provide proxy voting guidelines or policies and this information is available for the general public upon request or in their websites.

With respect to the socially responsible research process, almost all the funds have their own internal research team analyzing companies' activities in order to identify suitable investments. Some of them complete their internal research process with external experts or databases. Very few funds explicitly describe their research methodology and process. None of the funds makes reference to engagement in an ethical external audit periodically.

Taking into account the previous information, the expert in SRI evaluates the transparency and credibility of the screening process. Then, the quantitative information obtained is incorporated into the portfolio selection model by means of the fuzzy coefficients $\widetilde{w}_{ij}$. Table 2 displays the evaluation of the global Social Responsibility Degree for each mutual fund obtained using the information provided by mutual funds and by the expert, and using expression (11). The Social Responsibility Degree of funds F26-F35 is zero, as they are conventional funds not applying an explicit non-financial social responsible screening process (see Table 2).

**Table 2** Mutual funds' fuzzy social responsibility degree $\widetilde{SRD}_i$

| Fund | $SRD_i^L$ | $SRD_i^U$ | Fund | $SRD_i^L$ | $SRD_i^U$ | Fund | $SRD_i^L$ | $SRD_i^U$ |
|---|---|---|---|---|---|---|---|---|
| **F1** | 0.4 | 1.2 | **F10** | 0.3 | 1.4 | **F19** | 0.4 | 1.1 |
| **F2** | 0.3 | 1.0 | **F11** | 0.3 | 1.2 | **F20** | 0.4 | 1.2 |
| **F3** | 0.3 | 1.2 | **F12** | 0.4 | 1.0 | **F21** | 0.4 | 0.9 |
| **F4** | 0.3 | 1.3 | **F13** | 0.3 | 1.1 | **F22** | 0.3 | 1.3 |
| **F5** | 0.3 | 1.0 | **F14** | 0.3 | 1.5 | **F23** | 0.4 | 1.1 |
| **F6** | 0.3 | 1.3 | **F15** | 0.3 | 1.3 | **F24** | 0.4 | 1.1 |
| **F7** | 0.3 | 1.1 | **F16** | 0.3 | 1.2 | **F25** | 0.4 | 1.2 |
| **F8** | 0.3 | 1.2 | **F17** | 0.4 | 1.1 | **F26–35** | 0 | 0 |
| **F9** | 0.3 | 1.0 | **F18** | 0.4 | 1.2 | | | |

In order to illustrate the construction of the intervals displayed in Table 2, let us consider two different mutual funds, F1 and F26. From the information displayed in the US Social Investment Forum website, and from the mutual funds' prospectus, an expert on SRI evaluates the socially responsible performance of each mutual fund in each of the 41 social, environmental and ethical screens (criteria) considered in this work.

This is done using binary variables (procedure followed by KLD for US companies). Thus, the variable takes value "1" if the mutual fund accomplishes the corresponding screen, and value "0" otherwise. Let us, for instance, consider three screens related with one of the controversial products, alcohol (e.g. j = 31, 32, 33). The binary crisp evaluation of each of the two mutual funds on each of those screens is (see Table 3)

$$s_{1,31} = 1, s_{26,31} = 1, s_{1,32} = 0, s_{26,32} = 0, s_{1,33} = 1, s_{26,33} = 0$$

Precise (crisp) numbers are therefore available for the expert representing the global degree of social responsibility of each mutual fund with respect to each screen.

However, as described in previous sections, social responsibility criteria are by their own nature uncertain, imprecise and vague and therefore, for the expert, it is more realistic to handle social responsibility degrees by means of fuzzy numbers instead of crisp values. Therefore, and based on her expert knowledge, she assigns each crisp value reflecting the social responsibility degree of the fund $i$ with respect to the screen $j$, $s_{ij}$, an interval, $\left[b_{s_{ij}}^{L}, b_{s_{ij}}^{U}\right] \subseteq (0, 1]$. This interval, as explained before, will represent the membership degree of the social characteristic (screen) of the fund (see Table 4). Let us consider one of the previously presented alcohol screens, $s_{i,31}$: "The fund avoids investing in companies which license their company or brand name to alcohol products". Using a binary variable and relaying only on the information provided by the mutual fund, the crisp score obtained by mutual funds

**Table 3** An example of the social responsible criteria considered

| | Alcohol screens | F1 | F26 |
|---|---|---|---|
| $s_{31}$ | The fund avoids investing in companies which license their company or brand name to alcohol products | 1 | 1 |
| $s_{32}$ | The fund avoids investing in companies which manufacture or are involved in manufacturing alcoholic beverages including beer, distilled spirits, or wine | 0 | 0 |
| $s_{33}$ | The fund avoids investing in companies which derive revenues from the distribution (wholesale or retail) of alcohol beverages | 1 | 0 |

Source: US SIF

**Table 4** An example of the fuzzy valuations of funds $i = 1, 26$ for the alcohol screen $j = 31$

| Fund | $s_{ij}$ | $b_{s_{ij}}^{L}$ | $b_{s_{ij}}^{U}$ |
|---|---|---|---|
| F1 | 1 | 0.3 | 0.9 |
| F26 | 1 | 0.5 | 0.6 |

$i = 1, 26$ will be the same $s_{1,31} = s_{26,31} = 1$. However, when the SRI expert valuates these funds on the same screen, she assigns the funds, based on her knowledge, two different intervals, [0.3, 0.9] and [0.5, 0.6] reflecting the imprecision and ambiguous nature of this screen. Thus, the evaluation of fund F1 on the screen $s_{i,31}$ is a value between 0.3 and 0.9 and the evaluation of fund 26 is a value between 0.5 and 0.6. The latter is more imprecise with respect to the screen considered, even being a conventional fund.

It is interesting to observe how at a particular point of time, both mutual funds obtain the same precise (crisp) score. Moreover, the expert evaluation of the screen from the information provided by the fund is less imprecise, in this example, for the conventional mutual fund. However, only mutual fund F1 is a member of the US SIF and therefore, although at the evaluation moment conventional fund F26 obtained a similar socially responsible score, there is no compromise by part of this fund to follow socially responsible guidelines in its investment policy. On the contrary, mutual fund F1, has an explicit ethical compromise with SRI. In order to reflect this, weights acting as correcting factors are introduced in the measurement of the social responsibility degree of the mutual funds. These correcting factors reflect the level of confidence of the expert on the transparency and credibility of the information provided by the mutual funds with respect to the social screens (see (4) and (5), Table 5).

Let us observe that the expert, based on her knowledge, assigns a confidence interval [0.6, 0.9] reflecting the transparency and credibility degree of the information provided with regard to the screen considered. In the case of the conventional fund, F26, the value assigned is zero, as no SRI policy is explicitly followed by this fund. Once each fund has been evaluated with respect to the 41 screens the information is aggregated (see (11) and Table 2).

In this example, we will consider only a global social responsibility constraint. To this end, we have initially used different minimum bounds, $g$, which depend on the scenario (low social responsibility degree or high social responsibility degree), as shown in Table 6. These bounds have been chosen by the expert taking into account the different social responsibility degrees of the mutual funds in each of the

**Table 5** Weights (correcting factors)

| Fund | $b^L_{w_{ij}}$ | $b^U_{w_{ij}}$ |
|---|---|---|
| F1 | 0.6 | 0.9 |
| F26 | 0 | 0 |

**Table 6** Minimum bounds, $g$, on the portfolio's social responsibility degree

| Low scenario | High scenario |
|---|---|
| 0 | 0.9 |
| 0.1 | 1.0 |
| 0.2 | 1.1 |
| 0.3 | 1.2 |
| 0.4 | 1.3 |

scenarios considered. In this situation, an investor would be able to decide whether he/she prefers to follow the expert's advice with regard to the transparency and credibility levels of the information provided by the mutual funds, or to follow his/her own intuition. As social (non-financial) external audits are not available and the information on social screens is directly provided by the mutual funds, the credibility and transparency of the information depends on a high degree on the decision maker's (DM's) opinion.

First, we have calculated the ideal and anti-ideal values of the expected (weekly) return and risk, for each of the bounds and for each of the scenarios. These values are displayed in Table 7.

As can be seen, higher SRD requirements, in both scenarios, produce portfolios with worse expected return and risk values. This gives an idea of the existing tradeoffs existing for these funds between SRD and the classical financial criteria.

As a second step, we have solved two reference point models, one for each SRD scenario. As an example, and with the goal of briefly illustrating the obtained results, we have chosen one bound for each scenario.

For the low SRD scenario with $g = 0.3$, and taking into account the ideal and anti-ideal values displayed in Table 7, we have chosen reference levels of 0.08 for the expected return, and 5 for the risk. We have used equal weights for both objectives. In the optimal solution, the reference levels are satisfied and improved, obtaining an expected return of 0.084 and a risk of 4.733. The value of the (low) SRD is exactly 0.3. The optimal portfolio is formed as shown in Table 8.

For the high SRD scenario with $g = 1.2$, and taking again into account the ideal and anti-ideal values displayed in Table 7, we have chosen reference levels of 0.07 for the expected return, and 5.8 for the risk. Again, we have used equal weights for both objectives. In the optimal solution, the reference levels are satisfied and

**Table 7** Ideal and anti-ideal values for the different SRD bounds

| Low SRD scenario | | | | | High SRD scenario | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $g$ | $ER^*$ | $ER_*$ | $\sigma^*$ | $\sigma_*$ | $g$ | $ER^*$ | $ER_*$ | $\sigma^*$ | $\sigma_*$ |
| 0.0 | 0.109 | 0.077 | 3.849 | 6.163 | 0.9 | 0.098 | 0.071 | 4.029 | 6.277 |
| 0.1 | 0.109 | 0.077 | 3.849 | 6.163 | 1.0 | 0.094 | 0.065 | 4.174 | 6.256 |
| 0.2 | 0.106 | 0.080 | 3.910 | 5.964 | 1.1 | 0.088 | 0.056 | 4.449 | 6.221 |
| 0.3 | 0.096 | 0.067 | 4.257 | 6.109 | 1.2 | 0.081 | 0.054 | 5.092 | 6.170 |
| 0.4 | 0.058 | 0.039 | 6.738 | 7.106 | 1.3 | 0.074 | 0.072 | 6.139 | 6.354 |

**Table 8** Optimal portfolio for the low SRD scenario

| Fund | Percentage |
|---|---|
| 20 | 20 |
| 21 | 20 |
| 22 | 20 |
| 25 | 20 |
| 26 | 6.1 |
| 33 | 13.9 |

**Table 9** Optimal portfolio for the high SRD scenario

| Fund | Percentage |
|---|---|
| 14 | 14.7 |
| 16 | 20 |
| 20 | 20 |
| 22 | 20 |
| 25 | 20 |
| 33 | 5.3 |

improved, obtaining an expected return of 0.076 and a risk of 5.542. The value of the (high) SRD is exactly 1.2. The optimal portfolio can be seen in Table 9.

As we can observe, both the portfolio composition and the levels of return, risk and social responsibility degree achieved by the portfolio, vary depending on the scenario considered and on the minimum social responsibility bound. Nevertheless, under both scenarios the composition of the optimal portfolio will be mainly based on investment in SRI funds. This is due to the existence of funds with high values of SRI, which also achieve high levels of profitability and risk as shown by the corresponding Sharp ratio in Table 1.

Under a low scenario the financial results obtained are better in terms of the return (which is higher) and risk (which is lower). However, the social responsibility degree of the portfolio under the low scenario is small as compared to the one obtained under the high scenario. The most remarkable difference between the two scenarios is that the first one complements the solution with two conventional funds. F26 is chosen due to its high yield and relatively low level of risk, and F33 is chosen because of its lower covariance with the rest of the funds. The second scenario is more demanding on the SRD level, and only adds 5 % of F33 because of its lower covariance. Let us notice that, although social responsibility degrees have been handled in fuzzy, imprecise and ambiguous terms during the resolution of the portfolio selection problem, the investor is provided with a crisp value for each scenario in order to ease the interpretation of the results.

## 6 Conclusions

A portfolio selection model has been proposed for a particular individual investor taking into account financial and social responsibility criteria. First, the uncertainty and vagueness of the SRD data is handled through the use of fuzzy numbers, taking into account evaluations by experts. Next, different efficient portfolios are obtained using the reference point scheme for multicriteria problems, where the classical financial criteria (expected return and risk) are considered as objectives, and the SRD is included as a constraint derived from the previous fuzzy treatment.

The method proposed is illustrated through a real numerical example where different portfolios are obtained for an individual investor with particular subjective evaluations and preferences about social responsible issues. In this particular example, where portfolios are constructed from US domiciled large cap mutual funds considering data from 2007, the portfolios obtained are mostly composed by socially responsible mutual funds, even when this means a small reduction on expected return and sometimes, slightly higher levels of risk.

The model proposed is flexible and can be adapted to the particular preferences of any investor. It incorporates the uncertainty, ambiguity and/or imprecision inherent to the evaluation of the social responsibility degree of any asset, which depends in a high extent on the degree of expertize of the analyst and on the subjective preferences and the personal values of the investor.

Two further steps can be given as the future research lines. On the one hand, we can incorporate behavioral portfolio theory with mental accounting (BPT–MA, see Das et al. 2010) into the proposed model in order to better reflect the preferences and behavior of socially responsible investors. On the other hand, we can develop an algorithm that automatically generates the variance thresholds (lower and upper bounds) of the reference point components as interval values, supporting in this way DM in his/her choice of preferences and of compromise solutions.

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# Auditing and Game Theory: A Survey

**Fouad Ben abdelaziz, Souhir Neifar, and Marc de Bourmont**

**Abstract** A continuing debate in the area of financial accounting and reporting relates to the question of the principal-agent relationship. A common example of the agent-principal relationship arises between those who own a business (shareholders) and those who manage it (the manager). In a firm, the manager, who has private information can use his discretion to preserve his personal interest. The auditor then plays the role of a controller trying to align the interest of the parties.

Generally, accounting studies have focused on analyzing the statistical properties of accounting data, and their implications on decision making. An alternative approach, that takes into consideration the strategic interaction of the different decision makers, has been developed using a "*game-theoretic view*" of accounting data. This chapter presents basic setting that describes Game theory and reviews the main studies that have applied this theory in the audit field. This study can help the non-specialist reader understanding the importance of strategic interaction in the auditing process.

**Keywords** Game Theory • Audit • Principal-Agent Model • Multiple Decision Makers

## 1 Introduction

A principal–agent relationship may arise in various domains of the group decision making situations. In classical financial theory, a principal delegates some tasks to an agent who should react on the best interest of the principal. However, in most situations, this delegation of tasks creates an information gap between the principal and the agent when the latter owns more relevant information.

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The origin of the principal-agent model research can be found in the study by Coase (1937) "*The Nature of the Firm*", who explores the question of why firms exist in the first place. The topic was also introduced in the "*contract theory*" by Arrow (1970). Contract theory examines the conflict arising from delegation of tasks to an agent in the presence of an information asymmetry. In the 1960s and 1970s, a stream of economic researches investigated the concept of delegation of tasks which started to be known as "*Agency theory*".

Two main contributions to the principal–agent relationship were achieved by Ross (1973), completed by Jensen and Meckling (1976). In his study, Ross (1973) argued that the application of the Pareto principle in the principal agent problem assumes that perfect information is held by the participants. Thus, the optimal solution to the principal's problem implies that the different strategies of the agent have to be completely known by the principal.

As underlined by March (1962) and Cyert and March (1963), the major cause of conflict of interest in a firm is the existence of multiple decision makers who have different objectives. Thus, the information that is provided is not "*innocent*" (March 1987). Game theory can help to understand the difficulties of co-operation and information transfer.

Decision makers need to have the appropriate information. However, in a firm, there is always an information gap between the manager who runs the business and the shareholders who own it. As a result, the manager may camouflage the information or transfer manipulated information in order to maximize his own profit.

Due to the importance of financial information in a multiple decision setting, especially regarding the decisions made by the principal (shareholders) and the agent (the manager) in a firm, the accounting literature has generally focused on an answer to the following questions: What makes accounting information persuasive? And how to trust the quality of the information that is disclosed?

Therefore, accounting researches and practices have been interested in studying the role of external audit in guaranteeing the sincerity and the regularity of financial information. In general, the examination of audit quality has been studied by statistical and econometric approaches. However, accounting researches need to emphasis on designing the measurement rules of accounting processes rather than describing them and, therefore, to focus more on the modeling of human behavior than on the analysis of ex post data (Kanodia 2014).

Accounting was largely described as a tool that managers (responsible of the financial statement) can modify. The use of the term "*accounting game*" is not new. Many books and academics articles use this term in order to signal the numerous manipulations in the accounting field.

Chau (1996) addresses this question: "*What are the unique benefits to the auditors in using the game theory model*?" He predicts that the game theory models are more appropriate to describe behaviors in auditing, since there are conflicting behaviors inherent to audit. However, the auditor receives information from the auditee. As a result, the auditor has a complicated mission that needs more investigations to examine the sincerity of financial information.

Good audit quality has been described as a tool to reduce conflicts of interest between multiple decision makers in the firm (especially the one between shareholders and the manager). Therefore, the application of Game theory in audit studies ensures the comprehension of the way auditors and auditees make decisions by taking into account the decisions made by the other players. However, and despite the importance of Game theory in providing analyses of the decision making processes, the use of this theory in the auditing field remains new and limited. Thus, the aim of this chapter is to review the major studies that have used Game theory in the audit field.

The plan of this chapter is as follows: first we will present the theory of Games, second we will review studies that have used Game theory in Auditing, and Finally we will conclude the paper.

## 2 Section 1: Game Theory

Game theory was pioneered by Neumann and Morgenstern (1944). Generally, a game consists of a number of players, a set of rules that describe the game, and the moves that the players are allowed to make. Two main types of games are used in order to classify the game theory: cooperative or non-cooperative games. In cooperative games, there is a "*binding agreement*" between players. In non-cooperative games, there is "*no binding agreement*" between players (Harsanyi 1966). According to Migdalas (2002), Game theory provides mathematical models of conflict and cooperation between different utility optimizers whose decisions influence each other's utility. The importance of the use of Game theory is due to its impact on the comprehension of multiple decision makers' choices. It is the science of interactive decision making by excellence.

### *2.1 Why Do We Use Game Theory?*

According to Williams (2002), Game theory is used for three targets: explanation, prediction and advice or prescription.

- Explanation: when the situation involves an interaction of decision makers with different aims, Game theory supplies the key to understanding the situation and explains why it happened.
- Prediction: when looking ahead to situations where multiple decisions makers will interact strategically, people can use Game theory to foresee what actions they will take and what outcomes will result.
- Advice or Prescription: Game theory can help one participant in the future interaction, and tell him which strategies are likely to yield good results and which ones are liable to lead to disaster. Dixit and Susan (1999)

McMillan (1992) provides some explanations of the benefits of using the game theory in decision making. He states that: "*. . .The game theory can give us a short* *cut to what skilled players have learned intuitively from long and costly experience.... Real-life strategic situations are often extremely complicated. The game theory provides a model of this complexity*" (McMillan 1992, pp. 6–7).

Game theory is also used for modeling as it provides an "*audit trail*" demonstrating a coherent explanation of a phenomenon; it also provides a logic system to eliminate the error in the reasoning in the case of an incorrect analysis (Saloner 1994).

### 2.2 Elements of Games

According to De Nitti (2014), a game is a "*formal model*" of interactive situations in which, by anticipating the responses of the other agents (or players), an agent can maximize his utility. One of the main assumptions of Game theory is that the players are assumed to be strictly self-interested. So they are assumed to act in a rational manner, and to choose actions that maximize their individual's expected utility. Moreover, the players are considered to have complete knowledge of the game including the preferences of the other players (Simonian 2007).

### 2.3 Equilibrium

According to Camerer (1991), equilibrium refers to "*the discovery*" and "*analysis of equilibrium points*".

When players are moving, there exist two types of strategic interdependence: sequential and simultaneous. A sequential game is a game where an Entrant must choose first, and then the follower will know the Entrant's choice and then make his strategy (Williams 2002). In a simultaneous game, all the players make their decisions at the same time (Dixit and Nalebuff 2008). In this type of games, the planning horizon is short. All the players make their decisions in order to maximize their own utility functions. Therefore, the outputs of the game are immediately visible. Simultaneous games are the most suitable for one shot games. In dynamic games, the players move at different points in time. In this type of games, there is an "*explicit time-schedule*" that describes when the players make their decisions (Webb 2007). Thus, the key difference between static (or simultaneous) games and dynamic games is that in the first type of game there is no new information revealed to any of the players before their decision choice. However, in the second type of games, the decision makers can know about others' moves, as they do not play at the same time (Fig. 1).

In a simultaneous game, generally the equilibrium is the Nash equilibrium. And, in a sequential game it is the Stackelberg Equilibrium. Note also the existence of a third type of games: the experimental games.

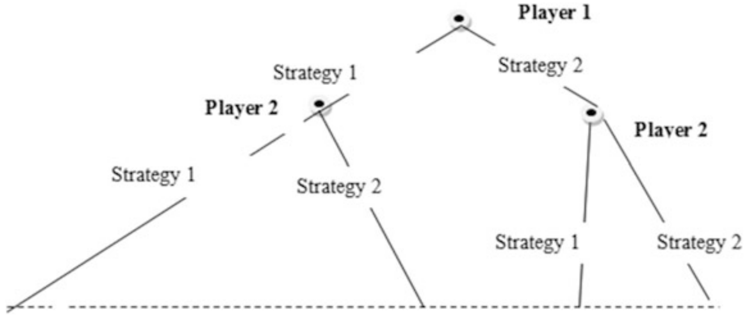


**Fig. 1** Game tree

### 2.3.1 Nash Equilibrium

The Nash equilibrium is a solution concept in which all choices made by several players, knowing each other's strategies, have become stable because no one can change its strategy without weakening his own payoff. In this case, the solution is described by the situation where no player is able to gain by "*individually changing*" his decision (Da Costa et al. 2009).

According to Han et al. (2011) a pure-strategy Nash equilibrium of a non-cooperative game can be defined as follows:

$$s^* \in S \ is \ a \ pure \ Nash \ equilibrium \ of \ G = \left(N, \ (S_i)_{i\in N}, u_{ii\in N}\right) if \ \ for \ \ every$$

$$player \ \ i, \qquad \boldsymbol{u_i}\left(\boldsymbol{s_i^*}, \boldsymbol{s_{-i}^*}\right) \geq \boldsymbol{u_i}\left(\boldsymbol{s_i}, \boldsymbol{s_{-i}^*}\right); \boldsymbol{s_i} \in \boldsymbol{S_i} \tag{1}$$

**With:**

| | |
|---|---|
| $s_i$ | The strategy profile of the player |
| $S_i$ | Set of strategies of the player |
| $S_{-i}$ | A set of all strategy profiles for all players expect i |
| $s_{-i}$ | The strategy profile of the player $j$, with $j \neq i$. |
| $u_i$ | Utility function of the player |
| G | Pure-strategy Nash equilibrium |

**Example 1: Game with no equilibria in pure strategy**

In this section, we will present the example of the matching pennies game. This game is played between two players: player A and player B. Each of these two decision makers has a penny and must simultaneously make secretly turn the penny to "*heads*" or "*tails*". If the decisions made by both players lead to the strategies (heads, heads) or (tails, tails) occur, therefore, the player A wins both pennies. In the remaining two other strategies, the player B wins both pennies.

Figure 2 describes this game.

| | | Player 2 | |
|---|---|---|---|
| | | Heads | Tails |
| Player 1 | Heads | 1, -1 | -1, 1 |
| | Tails | -1, 1 | 1, -1 |

**Fig. 2** The pennies game matrix payoff

| | Quiet | Fink |
|---|---|---|
| Quiet | 2,2 | 0,3 |
| Fink | 3,0 | 1,1 |

**Fig. 3** The Prisoner's Dilemma matrix payoff

In this game, there is no pure strategy Nash equilibrium because there is no pure strategy that represents the best reaction for the two players.

**Example 2: Game with a unique equilibrium: the case of two players' game**
In this section we will examine the example of the Prisoner's Dilemma, which is one of the most famous examples of simultaneous game. The mathematician Tucker has been given credit for discovering this game.

This game can be described as a major crime held by two criminals. The two players have no idea about what the other will say. And they have to make their strategies in a simultaneous way. In addition, the two players have to decide their strategy without knowing what the other player will have chosen. The two prisoners are not allowed to communicate with each other, and therefore, they need to make decisions simultaneously without knowing what the other will choose. In a simultaneous game, one player must try to predict what the other players will decide to do. If both stay quiet, each will be convicted 1 year in prison. If only one of them finks, he will be freed and the other will spend 4 years in prison. If they both fink, each of them will spend 3 years in prison.

Figure 3 describes this game.

In this example, the pair of strategies (Fink, Fink) is a Nash equilibrium because in this situation, the two players 1 and 2 are winners (they are less charged).

**Example 3: Game with multiple equilibria**
There are many game situations with multiple Nash equilibria. In such situations, the problem is to define how to select between the different possible equilibria. In this section we will present the example of the voting game.

Three players make simultaneous decisions about voting to one of three alternatives A, B or C. If a majority of players choose the same policy, this one will be implemented.

We suppose that the different preferences are as follows:

$$\boldsymbol{u}_1(\boldsymbol{A}) > \boldsymbol{u}_1(\boldsymbol{B}) > \boldsymbol{u}_1(\boldsymbol{C}) \tag{2}$$

$$\boldsymbol{u}_2(\boldsymbol{A}) > \boldsymbol{u}_2(\boldsymbol{B}) > \boldsymbol{u}_2(\boldsymbol{C}) \tag{3}$$

$$\boldsymbol{u}_3(\boldsymbol{A}) > \boldsymbol{u}_3(\boldsymbol{B}) > \boldsymbol{u}_3(\boldsymbol{C}) \tag{4}$$

In this type of game, there are multiple Nash equilibria:

– (A, A, A), (B, B, B), (C, C, C): in these three cases of Nash equilibria, each player cannot change his income by a single move. Therefore, the deviation of one of these equilibria is not profitable for all the players.
– (A, B, A), and (A, C, C): in these two cases of Nash equilibria, each of the two A and the two C players, respectively, is "*pivotal*" but still would not deviate because it would lead to a less desirable result.

### 2.3.2 Stackelberg Equilibrium

In a non-cooperative game, the players can play sequentially or simultaneously. In a sequential game, a hierarchy might exist among the players when one or more players make strategies before the other players. In such a hierarchical decision-making scheme, these first players can have more power than the other players and enforce their own strategies upon them (Han et al. 2011). According to Stackelberg (1934), the player who holds the powerful position is called the leader, and the other players who react (rationally) to the leader's decision (strategy) are called the followers. Note that there are situations where there are multiple leaders as well as multiple followers.

**Example 1: Two players' game**
We consider a two-player non-cooperative game between a leader and a follower, with $S_1$ being the set of strategies of the leader and $S_2$ the set of strategies of the follower. The leader first announces his strategy to play $s_1 \in S_1$ and then the follower must choose his strategy $s_2 \in S_2$ taking into account the strategy of the leader.

According to Han et al. (2011), the set $\mathbb{R}_2(s_1)$ is the optimal response of player 2 to the strategy $s_1 \in S_1$ of player 1. $\mathbb{R}_2(s_1)$ is defined for each strategy $s_1 \in S_1$ as follows:

$$\mathbb{R}_2(\boldsymbol{s}_1) = \{\boldsymbol{s}_2 \in \boldsymbol{S}_2 : \boldsymbol{u}_2(\boldsymbol{s}_1, \boldsymbol{s}_2) \geq \boldsymbol{u}_2(\boldsymbol{s}_1, \boldsymbol{t}), \forall \boldsymbol{t} \in \boldsymbol{S}_2\} \tag{5}$$

Thus, a strategy $s_1^* \in S_1$ is called a Stackelberg equilibrium strategy for the leader if:

$$\mathbf{min}_{\mathbf{s}_2 \in \mathbf{R}_2(\mathbf{s}_1^*)} \mathbf{u}_1(\mathbf{s}_1^*, \mathbf{s}_2) = \mathbf{max}\ \mathbf{min}_{\mathbf{s}1 \in \mathbf{S}1,\ \mathbf{s}2 \in \mathbf{R}_2(\mathbf{s}1)} \mathbf{u}_1(\mathbf{s}_1, \mathbf{s}_2) \triangleq \mathbf{u}_1^* \tag{6}$$

**With**:

| | |
|---|---|
| $S_1$ | Set of strategies of the leader |
| $S_2$ | Set of strategies of the follower |
| $s_1$ | Leader (player 1) strategy to player |
| $s_2$ | Follower (player 2) strategy to player |
| $\mathbb{R}_2(s_1)$ | Optimal response of the follower (player 2) |
| $u_i$ | Utility function of the player |

The Stackelberg optimum strategy of the leader $s_1^*$ ensures that the leader does not receive a utility that is lower than $u_1^*$, where $u_1^*$ constitutes a secured utility level for the leader (Han et al. 2011). While the leader's Stackelberg optimum strategy does not need to be unique, the follower's optimal response becomes unique for every strategy of the leader.

**Example 2: More than two players' games**
The Stackelberg solution for a multiplayer game can be between a single leader and multiple followers.

In the multi-follower case, the optimal strategy of the followers is the set of joint strategies that maximizes their utilities, where each follower's utility is a function of the leader's strategy (as he moves first) and the strategies of the other followers (Han et al. 2011).

While the case of the two player Stackelberg equilibrium has been largely defined and described, the use of multiplayers in Stackelberg game equilibrium remains limited.

### 2.3.3 Experimental Game Theory

The last three decades have marked the use of "*experience*" to modify "*game theory*" and therefore the creation of a whole field named experimental game theory.

Ledyard and Palfrey (1995) classify the main use of experimental game theory as follows:

– *Development and testing alternative models of behavior in games*: this includes a combination of theory, experiment, and econometrics. Developing alternative models of behavior in games and contrasting their predictions to overcome the deficiencies of Nash equilibrium (and its refinements) (Ledyard and Palfrey 1995).
– *Learning in games*: generally experimental games are run by having subjects who play the same game many times. Each subject is asked to develop strategies. Consequently, they learn about the "*differential success*" of their strategy and therefore try to improve it during the next experiences. These experiences motivate participants to "*develop general behavioral plans*" (Güth 2000). Thus, the use of experimental games helps to design models and to run

experience in order to involve a better understanding of "*the dynamics of learning in games*" (Ledyard and Palfrey 1995)
– *Bargaining and related applications*: this kind of experimental games is not used very often. In most of the prevailing studies, "*there is a conflict between the equilibrium predictions of non-cooperative equilibrium, when choice behavior is sequentially rational, and the concepts of efficiency and fairness. Experimental research has typically found the greatest divergence from game theoretic predictions when such a conflict arises*" (Ledyard and Palfrey 1995).

## 3 Section 2: Game Theory in Auditing

March (1991) defines decision-making as an act of information processing achieved by transforming information into managerial action. Thus, the information that is obtained must be relevant in order to make suitable decisions. However, the process of decision making suffers from two severe limitations: limited rationality and conflicts of interest in decision making (March and Simon 1958; Cyert and March 1963). Moreover, the separation between property and control that exists in most firms can lead to a situation of information asymmetry between the manager and the other stakeholders of these firms and two types of problems might arise in this case: a moral hazard problem and an adverse selection problem.

A moral hazard problem is a situation where someone (usually the manager of a firm) holds more information than the other persons (usually the investors and the owners of this firm) about the situation of the firm. The manager then might use his position to maximize his utility by decreasing others' utilities, which will engender a "*moral hazard*" problem.

As Holmstrom (1979, p.74) states: "*…a problem of moral hazard may arise when individuals engage in risk sharing under conditions such that their privately taken actions affect the probability distribution of the outcome*."

An adverse selection problem is a mechanism by which some potential investors might consequently refuse to acquire the shares of a company because they do not have the same level of information than the managers (this situation was described by Akerlof—(1970)—in a famous article about the behavior of people when they either want to sale or purchase a car). Less informed than the manager about the situation of the firm, the potential investors might think that the manager (and/or other insiders) knows a lot more about the current conditions and future prospects of the firm than they do. In this case the manager may have opportunistically biased or otherwise managed the information released to investors, and, because of this situation, the investors will not invest in the shares of the company till they are absolutely confident about the situation.

Auditing being one way, if not the way, to check if the information provided by a manager is correct. The application of Game Theory in the context of auditing is of particular interest, in a situation where both a moral hazard problem and an adverse selection problem might exist.

According to Chau (1996), an audit is "*frequently viewed as a single period engagement in which the auditor examines and attests to some financial phenomena after the auditee has taken all the necessary steps to prepare the financial records*". Besides, in the hypothetical model, the auditor's decision has been simplified to a "*true-and-false question*" (error/no-error, fair/unfair, etc.).

Fellingham and Newman (1985) and Fellingham et al. (1989) suggested that the behavior of the auditor will be affected by the strategy of the other "*party game*". Whereas the society expects that an "*independent*" auditor will not cooperate with the auditee, the realities of auditing process involve a "*considerable degree*" of cooperation Cook et al. (1997). Thus, and due to the obvious lack of realism of hypothetical models and to the interactions that exist between auditors and auditees, using Game theory to study audit can be a solution to those limits.

In the next section we review the main studies that have used the theory of Games in auditing while classifying these most important contributions into three major categories: simultaneous, sequential game and experimental game studies.

### *3.1 Simultaneous Games*

Many studies have used Game theory in the auditing field. Fellingham and Newman (1985) focus on the importance of using the game theory concepts in analyzing the auditing process. In their study, Fellingham and Newman (1985) focus on an example in which the auditee decides between using high or low effort in their internal control and the auditor decides between using high or low effort to investigate the internal control.

Their game tree is presented in Fig. 4.

In their model, Fellingham and Newman (1985) consider that the costs of the auditor and the client are as follows:

$C_1$ = the auditor's expected cost of qualifying in case there is no material error;
$C_2$ = the auditor's expected cost of not qualifying in case there is a material error;
$C_A$ = the auditor's direct cost of making audit tests; the direct cost of not extending audit tests is zero;
$C_H$ = the client's direct cost of expending higher effort ($E_2$) is arbitrarily set at zero;
$C_Q$ = the client's expected cost of a qualified opinion;
$CE_Q$ = the client's expected cost of a material error when the opinion is not qualified.

They find that the equilibrium is a Nash equilibrium.

Matsumura and Tucker (1992) investigate the strategic interaction between the manager and the auditor by developing a "*theoretical foundation through game theoretic analysis and economic experimentation*". The game involves one decision by the manager and two by the auditor. The manager moves first, choosing the probability of committing a fraud. The auditor, without knowing the manager's choice, decides whether to perform a test of control and then decides the level of the

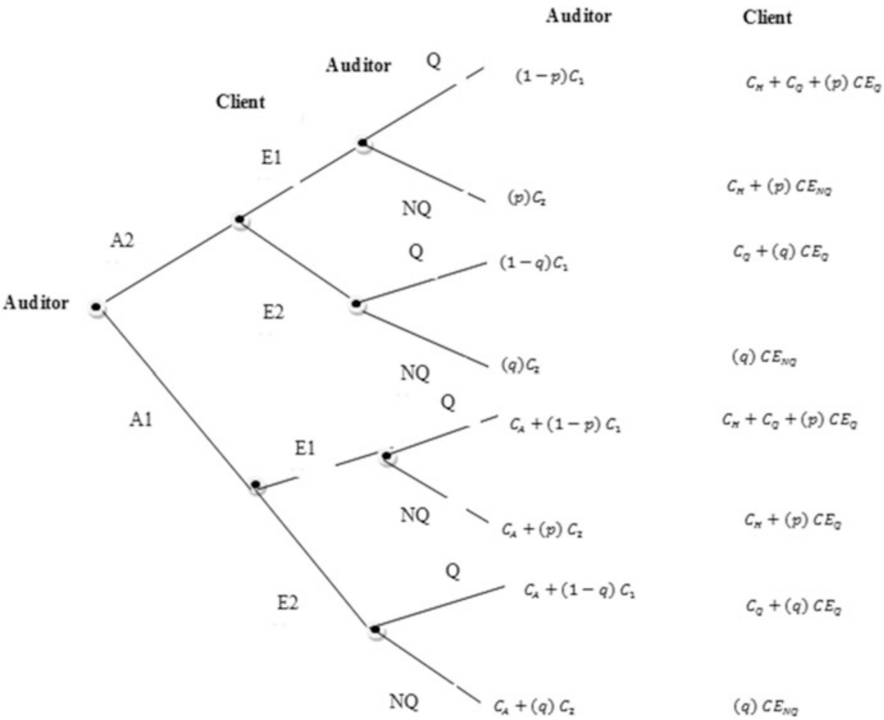


**Fig. 4** Game tree of Fellingham and Newman (1985) game between managers and auditors (p. 639)

detailed tests. At this stage, the probability of detecting fraud increases with the level of testing.

Matsumura and Tucker (1992) describe their game using game tree (Fig. 5).

Matsumura and Tucker (1992) use the following notations:

$t$ = the level of fraud, 0 or $\hat{t} > 0$;
$\alpha_1$ = the number of items from test 1 to detect compliance deviations;
$w$ = the entire population of test 1;
$\varnothing$ = no item has been sampled;
$y$ = rate of errors in the population, $y \sim N(\mu + t, \sigma^2)$;
$\alpha_2$ = the number of items sampled in test 2 to detect the fraud;
L = low sample size for test 2;
H = high sample size for test 2;
NF = no fraud;
FND = fraud exists but is not detected;
FD = fraud is detected;

The equilibrium generated between the manager and the auditor in this game is a Nash equilibrium. Note that this study examines the effect of four variables:

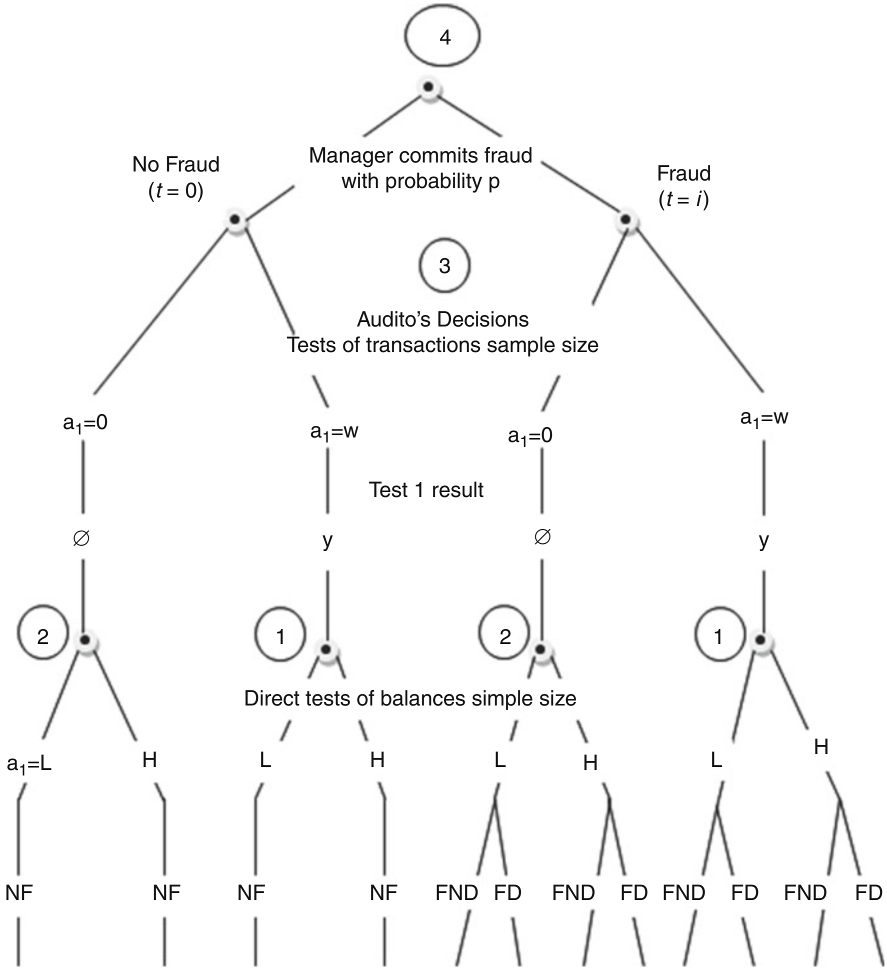


**Fig. 5** Game tree of Matsumura and Tucker (1992) game between managers and auditors (p. 756)

auditor's penalty, auditing standards requirements, quality of internal control structure and audit fees. They find that increasing the auditor's penalty leads to decreased fraud, increased detailed tests of balance, decreased test of transactions, and increased fraud detection. The results also show that with strong internal control, the auditor increases the tests of transactions and detected fraud more frequently and then, managers commit fraud less frequently. And, finally, they find that increasing the audit fees results in less fraud.

Patterson (1993) extended the "*hidden action game*" in Shibano (1990)'s paper. In his game, the auditee chooses the means, $w_j$, $j = 1, 2$, of the sample evidence distribution, $F\left(x \middle| n, w_j\right)$, $j = 1, 2$, where the auditor prefers to reject $w_2$, a material error arising from defalcation, and accepts $w_1$, an immaterial error. The auditee's choice of $w_j$ does not include a reporting choice, and is not observable by the auditor. The auditor's dominant reporting rule is to choose a critical value, $c \in X = [-\infty, +\infty]$ such that he rejects when $x \geq c$ and otherwise accepts (Patterson 1993).

The Nash equilibrium concept is used to characterize equilibria because the game is equivalent to a simultaneous move game where the players' strategy sets are as follows:

$$S_{\text{Auditor}} = \{\, f(c), \text{where} \;\; f(.) \in F \text{ the space of probablity measures over X}\}$$

$$S\_\text{Auditor} = \{t \in [0, 1], \text{ such as } ''t'' \text{and } ''1-t'' \text{are the probabilities } w_1 \text{ and } w_2 \text{ occur}\}$$

The audit evidence depends on the asset values reported by the auditee and the auditor's decisions about the size of sample to use.

Finley (1994) develops a "*strategic internal auditing model*" to examine the use of discovery sampling to detect the manipulation used by the auditee.

The "*strategic variables*" are the level of fraud chosen by the auditee and the level of effort chosen by the auditor.

The study of Finley (1994) develops a game in which the auditee makes decision about the level of fraud to use and the auditor makes decision about the level of effort to do. The goal of the auditee is to maximize his expected benefit from fraud activities "*net*" of sanctions that may underwent in case of fraud detection. The goal of the auditor is to minimize the expected costs generated by the sampling process or by fraud losses. In this study, Finley (1994) analyzes simultaneous play and commitment versions of the game. He reports that, for both versions, pure strategies are optimal. He documents that the commitment version equilibrium results show a greater audit effort and less fraud by the auditee compared to this of the simultaneous version. Finley (1994) analyses also the effects of sanction level and recovery rates on the two versions. He finds that for a simultaneous play, optimal monitoring effort increases with the recovery rate and decreases with the sanction level. However, these effects are reversed for the commitment version. Therefore, the results of this study prove the sensitivity of the audit effort communication arrangement to the audit objective of fraud detection.

Blomfield (1995) explains that in auditing an account balance, an auditor must first assess the inherent risk that the balance is misrepresented. Blomfield (1995) uses a timeline of events to describe his game, as shown in Fig. 6.

Blomfield (1995) identifies some factors that determine the accuracy of the auditor's "*inherent risk assessment*" in a hidden information audit setting, under the assumption of the presence of rational players known as "*rationalization*". In

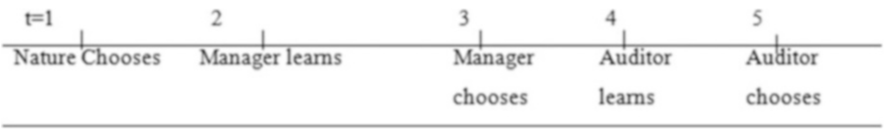


**Fig. 6** Timeline of events of Blomfield (1995) game (p. 73)

this study, the auditor's risk assessment accuracy is proved to be influenced by the risk of unintentional errors, the players' incentives, the precision of the auditor's data, and the regulatory bounds on risk detection (Blomfield 1995). The results suggest that the traditional Nash equilibrium analysis of strategic audit settings may have limited predictive power. Because Nash equilibrium outcomes may arise only when strategic dependence is low, the behavior of auditors and managers may not always change in the ways predicted by the traditional Nash equilibrium analysis.

Cook et al. (1997) developed a game model of the audit process that aims to simulate both the substantive testing involved in auditing and the internal control investigation. Cook et al. (1997) modeled the relation between auditors and auditees in two situations: non-cooperative games and cooperative games.

The game tree of Cook et al. (1997) is described in Fig. 7.

In their non-cooperative model, the auditee has to choose between H and L

H: high level of effort in processing information, and
L: low level of effort in processing information.

Different notation of Cook et al. (1997) game tree is summarized as follows:

*E*: material errors;
*NE*: no material errors;
$A_1$:high effort by the auditor to observe the auditee's choices;
$A_2$: no effort made by the auditor;
$B_1$: testing the accounting information for material errors;
$B_2$: less extensive test to test material errors.

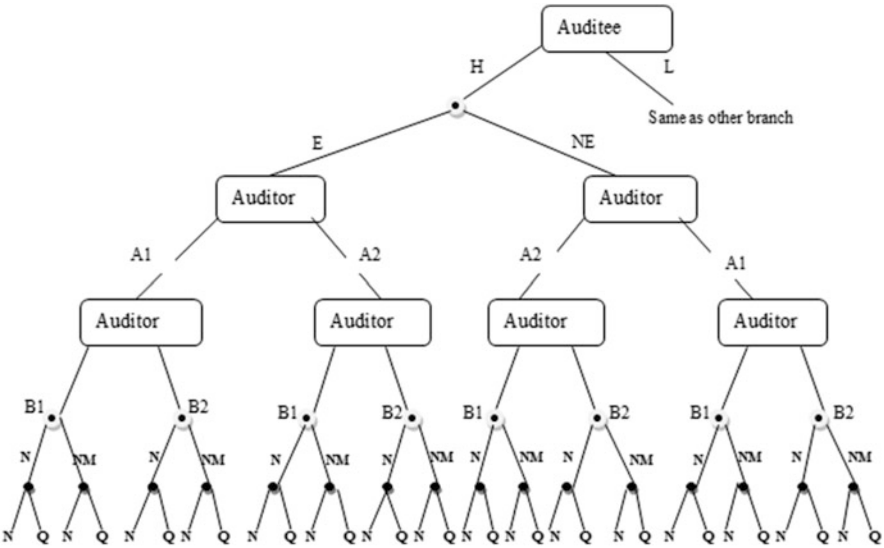


**Fig. 7** Cook et al. (1997) Game tree in a non-cooperative model (p. 473)

In this type of game, there are four strategies:

Q: qualify;
NQ: not qualify;
R: qualify if the B-test reports errors, otherwise do not qualify;
U : do not qualify the accounts if the B-test reports errors otherwise qualify.

As shown in Fig. 7, the auditee has two strategies: to put a high effort in processing accounting information or to put a low effort on it. The auditor has to choose between 72 possible pure strategies. When the auditor chooses $A_2$ (no effort made by the auditor and by consequence any knowledge of the auditee's strategy); he decides on one of his four strategies for qualifying the accounts. In this situation, the auditor has eight pure strategies: 21Q, 21NQ, 21R, 21U, 22Q, 22NQ, 22R and 22 U. The remaining strategies involve when the auditor chooses $A_1$ (high effort by the auditor in observing the auditee's choices). In this situation, the auditor is informed about the strategy used by the auditee (whether the auditee has played *H* or L).

The equilibrium of this game has been divided into regions, in each region there is one or more equilibria (Fig. 8). The different regions are determined according to the "*expected cost to the auditee of not qualifying when there is a material error*" ($D_E^{NQ}$) and "*the expected cost to the auditor of not qualifying when there is a material error*" ($C_E^{NQ}$) (Cook et al. 1997).

If the auditee chooses *L* "*providing a low level of effort in processing accounting information*", the equilibrium is when the auditor decides on the strategy 22Q. In this region (region 1), there is a high level of expected cost to the auditor of not qualifying when there is a material error ($C_E^{NQ}$). In region 2, there are two equilibriums. In the first, the auditee chooses a high effort and the auditor selects the strategy 21R. In the second, the auditee chooses low effort, and the auditor decides on the strategy 21R. In this region, there is less level of expected cost to the auditor of not qualifying when there is a material error than region 1. In region 3, there is only one equilibrium, defined as follows: the auditee chooses high effort and the auditor decides the strategy 21R. Finally, when there is the less $C_E^{NQ}$, the equilibrium is defined according to the expected cost to the auditee of not qualifying when there is a material error ($D_E^{NQ}$). If this type of error is low, the equilibrium is obtained in region 5. In this region, both players play randomized or mixed

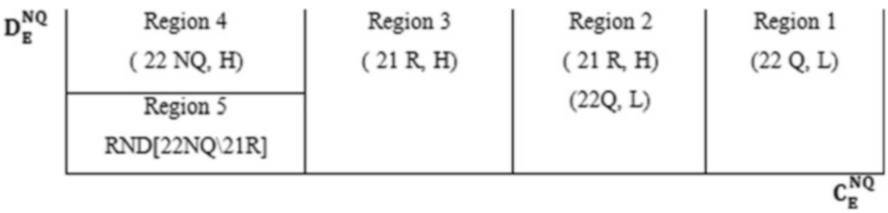


**Fig. 8** Equilibrium non-cooperative model of Cook et al. (1997) (p. 476)

strategies (the auditee a mixture of H and L and the auditor a mixture of 22NQ and 21R). Finally, the region 4 is characterized by more $D_E^{NQ}$ than region 5. In this region, the equilibrium is defined as the auditee chooses H and the auditor decides the strategy 22NQ.

Researchers also studied non-cooperative models with "*penalty discounts*" which led also to equilibrium by region. In their cooperative model game, Cook et al. (1997) find that choosing a penalty regime and a discount factor for evidence of high auditor's effort ensures the ideal strategies for the firm in both a cooperative or non-cooperative game. Basing on the game of chicken, Coates et al. (2002) model the client-auditor financial reporting and audit effort strategies. In their model, they focus on the auditee's misreporting decisions and auditor's effort to detect this misstatement. They use a "*welfare game*" to model the auditee-auditor strategies. Therefore, they extend the "*welfare game*" in order to provide additional comprehension to the ethical and audit effort issues. This "*welfare game*" then provides the equilibrium in mixed strategies. Coates et al. (2002) define four possible outcomes solution from their game:

1. Financial Statements are fairly presented by the client and the auditor performs a normal audit,
2. Financial Statements are fairly presented by the client and the auditor performs an extended audit (over auditing),
3. Financial Statements are misstated by the client and detected by the auditor, and
4. Financial Statements are misstated by the client and not detected by the auditor (audit failure although there is no intended unethical action on the part of the auditor).

Coates et al. (2002), p. 1

Two extensions of the "*welfare game*" have been defined. The first one allows the clients to be ethical or unethical clients. The second one allows the client to unintentionally misstate the financial statements; the client "*strategy*" then becomes at random (a play of nature). Therefore, the auditor must be able to differentiate between the strategic play and the random play of the "*welfare game*".

Ronen et al. (2006) model earnings management using three players: shareholders, directors, and management. They add an auditor as a participant in their game. They also use a time line to describe their game. Ronen et al. (2006) solve their game in two situations: first, if the shareholders design directly the manager's incentives and, second, if the directors design the optimal contract. The timeline of their game is described in Fig. 9.

Ronen and Yaari (2007) also propose a principal agent game between shareholders and manager in the presence of an auditor. In their game, the auditor is considered as a participant. The authors also use a time line to describe their game (see Fig. 10).

Fandel and Trockel (2011) examine the relationship between two players (a manager and a controller). They demonstrate how a high cost deviation and related penalties have an impact on the behavior of both the manager and the controller. The model of Fandel and Trockel (2011) reflects how the Nash

| Date 1 | Date 2 | Date 3 | Date 4 | Date 5 | Date 6 | Date 7 |
|---|---|---|---|---|---|---|
| Shareholders award directors shares and options | The board designs the accounting flexibility and the contract of the manager. The manager signs the contract. | The manager exerts effort. | Nature chooses earnings, and the manager alone observes them. | The manager gives a draft of the reports to the auditor. | The manager is paid. | A new period is started…. |
| | | | | After the audit, the firm publicizes its financial reports | A stock price is set. The directors can sell shares and exercise options. | After a while, the market learns the truth. The directors sell the rest of their holdings. |

**Fig. 9** Timeline of Ronen et al. (2006) model game (p. 365)

| Date 1 | Date 2 | Date 3 | Date 4 | Date 5 | Date 6 | Date 7 |
|---|---|---|---|---|---|---|
| The principal designs the contract T | The agent chooses an effort, α | The outcome x, is realized, and it is observed by the agent alone. | The agent discloses an outcome to an auditor. The monitor audits this disclosure. | The principal receives the audited public report, r. | The agent is paid t and quits. | The principal collects his residual shares (x – t(r)) |

**Fig. 10** Timeline of Ronen and Yaari (2007) model game (p. 130)

equilibrium moves towards the combination of strategies due to the higher value of the penalties.

## 3.2 *Sequential Game Studies*

The use of sequential games in the audit field remains limited. Patterson and Noel (2003) examine the strategies of an auditor toward his audit plan when the auditee has the opportunity to commit various types of fraud. The specifications of their model are the assumptions that the auditee can even misreport financial information, misappropriate assets, or misreport financial information in combination with defalcation. The game tree of Patterson and Noel (2003) model is described in Fig. 11.

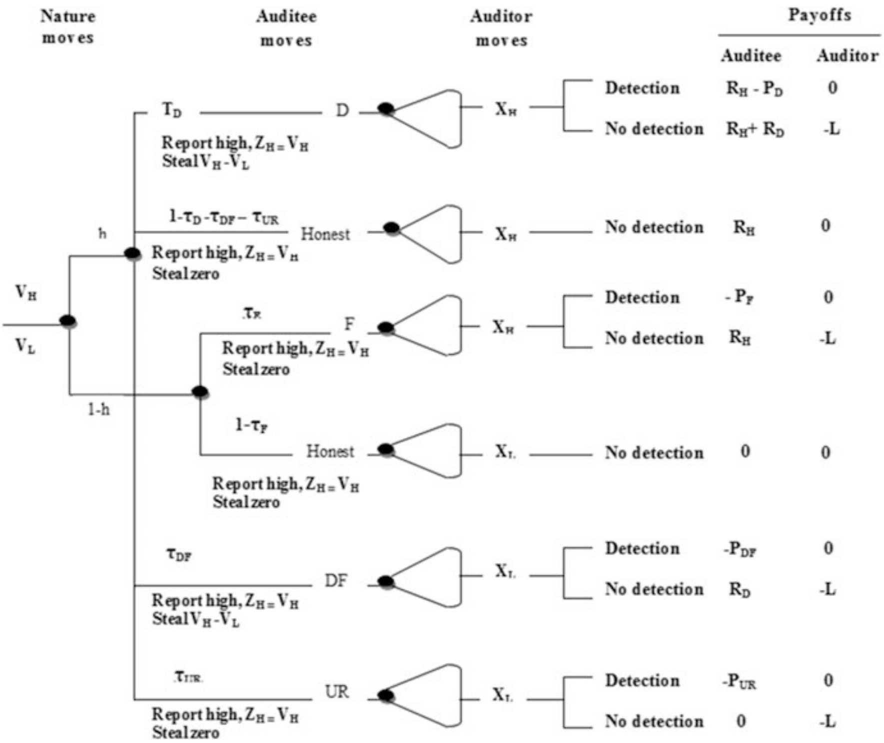


**Fig. 11** Patterson and Noel (2003) model game tree (p. 527)

With:

$h =$ probability of high asset value, $v_L$ occurs;
$\tau_D =$ auditee's randomization of defalcation, D;
$\tau_F =$ the auditee's randomization of a fraudulent financial reporting, F;
$\tau_{DF} =$ the auditee's randomization of defalcation-and-fraud, DF;
$\tau_{UR} =$ the auditee's randomization of underreporting, UR;
$x_H =$ the auditor's effort for observing high reports;
$x_L =$ the auditor's effort for observing low reports;
$M_H =$ the observation of the high $v_H$ asset values;
$M_L =$ the observation of the low $v_L$ asset values.

Patterson and Noel (2003) model is written as follows:

$$\begin{aligned}\mathbf{M_H}(\boldsymbol{\tau}_\mathbf{D}, \boldsymbol{\tau}_\mathbf{DF}) = (1 - \ \boldsymbol{\tau}_\mathbf{D} - \boldsymbol{\tau}_\mathbf{DF})\mathbf{R_H} \\ +\boldsymbol{\tau}_\mathbf{D}\{\mathbf{Exp}\ (-\mathbf{x_H})(\mathbf{R_H} + \mathbf{R_D}) + [1 - \mathbf{Exp}(-\mathbf{x_H})](\mathbf{R_H} - \mathbf{P_D})\} \\ +\boldsymbol{\tau}_\mathbf{DF}\{\mathbf{Exp}\ (-\mathbf{x_L})(\mathbf{R_D}) + [1 - \mathbf{Exp}(-\mathbf{x_L})](-\mathbf{P_{DF}})\}\end{aligned} \tag{7}$$

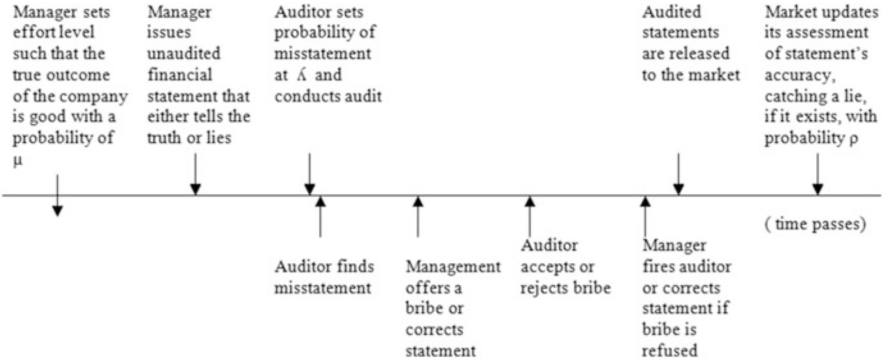


**Fig. 12** Brown (2008) timeline model game (p. 182)

$$\mathbf{M_L}(\boldsymbol{\tau}_\mathbf{F}) = \boldsymbol{\tau}_\mathbf{F}\{\mathbf{Ex\,p}(-\mathbf{x_H})\mathbf{R_H} + [1 - \mathbf{Ex\,p}(-\mathbf{x_H})](-\mathbf{P_F})\} \tag{8}$$

Basing on the auditee's behavior toward rewards and penalties, the cost of audit effort and the expectations about the auditee-firm's performance, Patterson and Noel (2003) identify four possible equilibria. They found that if the cost of an audit effort is considered small, therefore the fraud risk assessment depends on the auditee's rewards and penalties related to each type of fraud. In this game, and in order to deter all other types of fraud, the auditor should develop an audit plan that focuses on the type of fraud the auditee is most motivated to commit. The equilibrium used to resolve this sequential game is a perfect Bayesian equilibrium.

Brown (2008) considers the manager's and auditor's decision making process in a non-cooperative game. His study takes into account the probability of collusion between the manager and the auditor; he adds a scenario of bribing the auditors and uses a timeline to describe his sequential model (Fig. 12).

In Brown's game (2008), if the performance of the firm is good, the manager will tell the truth to the auditor. However, in the case of bad firm performance, the manager has two alternatives: tell the truth or lie. In the case he lies, the auditor might find the misstatement if the audit intensity is greater than zero. When the auditor finds the misstatement, the manager has two possible strategies: whether to offer a bribe to the auditor or not. In this situation, the auditor also has two possible strategies: whether to accept bribe or not. If the auditor refuses the bribe, the manager must choose between giving into the auditor's demands to fix the misstatement or firing the auditor.

Brown's game tree (2008) is described in Fig. 13.

Brown (2008) proves that the audit will be most intense in the collusive auditor scenario because the auditor likes to maximize his payoff. He has used the sub-game perfect information to resolve the equilibrium. Brown (2008) finds three possible scenarios: Compliant manager scenario; opinion Shopping scenario and Collusive auditor scenario. He then estimates the different optimum equilibrium conditions for each scenario.

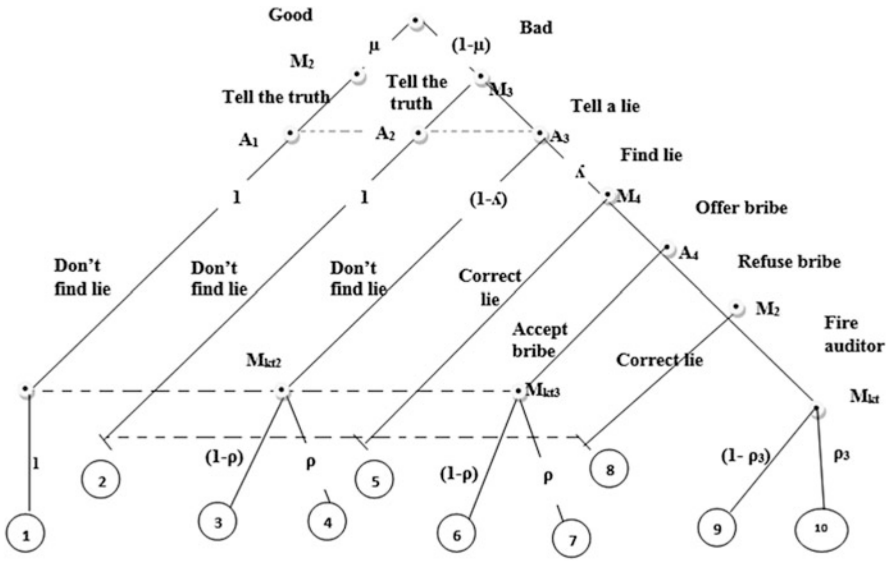


**Fig. 13** Brown (2008) game tree between manager and auditor (p. 184)

### 3.3 Experimental Game Studies

The emergence of experimental game theory is due to two major factors: the need for practical information about the strategic behavior principles and the advantages of experiments in capturing these behaviors. The laboratory (experience) can provide a description of the real behavior of players. Therefore, the game theory predictions associated with laboratory experiments often give a "*decisive advantage*" in detecting the possible relationship that may exist between the strategic behavior and the reality (Crawford 2002).

Blomfield (1997) questions the reliability of experimental detection of fraud risk by auditors, and the usefulness of traditional equilibrium analysis, where accurate fraud risk assessment is needed. He sets four games for 24 MBA students acting as (guessers) auditors and managers (choosers), in pairs, for 100 rounds, with a two by two factorial design. He produces a matrix of liability ratio against error rate and examines the errors in expectations, the predictive power of Nash equilibrium, the welfare effects of strategic dependence, and the time-averaged strategy choices. He finds that strategic dependence reduces the chances of the accurate expectations of subjects "*opponents*" actions, and individual audit outcomes are far from equilibrium values. Consequently, pay-off losses are worse when the risk of liability is high but the risk of unintentional error is small.

Zimbelman and Waller (1999) investigate the interaction between the auditor and the auditee in a "*strategic setting*". In this study, the auditees are assimilated to assert an asset value, knowing the true value. The auditors selected a costly sample of the assets and choose between two strategies: whether to accept an auditee's asserted value or to reject it. Zimbelman and Waller (1999) examine the possibility that an auditee anticipates the effect of the auditors' ambiguity aversion on their decisions to sample and reject the reported balance. The results of this study show that as auditors' ambiguity or auditees' incentive to misstate increases, as the sampling rises. Zimbelman and Waller (1999) report that a higher auditees' misstatement rate presents a strong incentive to the auditee to make misstatement; and that a lower auditees' misstatement rate occurs when there is more auditors' ambiguity. This finding supports the strategic effect of the auditors' ambiguity in reducing the ability of the auditee to make misstatement.

King (2002) investigates an "*audit trust*" game played by two players : a manager and an auditor. In his game, the manager decisions variables are related to the level of fraud. The auditor's main function is to predict the fraud level chosen by the manager. The King's experience (2002) consists in reporting the decisions of 44 manager/auditor pairs. The subjects are students from business-schools. King (2002) uses 11 pairs for each player and tests four settings: "*no-puffery/weak group; yes-puffery/weak group; no-puffery/strong group; yes-puffery/strong group*". The results of this study show that, comparative to the no-puffery settings, the manager's use of puffery settings increases significantly the auditors' willingness to trust the manager. King (2002) found that the auditors in the "*strong group settings*" trust significantly less the manager than did the auditors in the "*weak group settings*".

Fischbacher and Stefani (2007) report "*experimental results*" for a simple bi-matrix game between a manager and an auditor. They analyze the effect of an increase of the proportion of honest auditors on improving the audited financial statements quality. In their experimental game, Fischbacher and Stefani (2007) use subjects who played the role of the manager and of the auditor. They conducted five sessions in which the participants interacted in line with treatment A (situations of purely "*opportunistic*" managers and auditors) and then with treatment B. In each period, they asked the participants who played the role of manager about their order of preference to disclose truth financial statements or to overstate the real income. The participants who played the role of the auditor had to choose whether to exert a low or high audit effort. After each participant decision choice, both managers and auditors were told about their counterparts' decisions and their expected earnings. Fischbacher and Stefani (2007) find that an high audit effort improves the quality of audited reports.

Bowlin et al. (2009) designed an experiment to analyze the effect of "*audit experience*" on the reporting decisions when the auditor plays the role of the manager of the firm. Bowlin et al. (2009) found that the managers who have prior experience as auditors are more responsive about penalties for aggressive reporting than the other managers.

Bowlin (2011) found that the financial statements that are more likely to be misstated are related to an "*under risk-based auditing*" and a more audit resources allocation. By using a "*laboratory experiment*", Bowlin (2011) reports that the participants who assume the role of the auditor are not naturally attuned to strategic risks, however, they focus resources toward financial statements that includes a

**Table 1** Summary of studies using game theory in auditing

| | 2 players | More than 2 players |
|---|---|---|
| Simultaneous games | Fellingham and Newman (1985)<br>Matsumura and Tucker (1992)<br>Patterson (1993)<br>Finley (1994)<br>Cook et al. (1997)<br>Coates et al. (2002)<br>Ronen and Yaari (2007)<br>Fandel and Trockel (2011)<br>Chou et al. (2012) | Ronen et al. (2006) |
| Sequential games | Patterson and Noel (2003)<br>Brown (2008) | |
| Experimental studies | Blomfield (1997)<br>Zimbelman and Waller (1999)<br>King (2002)<br>Fischbacher and Stefani (2007)<br>Bowlin et al. (2009) | |

high non-strategic risk. The participants who play the role of managers exploit these allocations by prevailing more the low-risk the financial statements than the other types of financial statements. However, the auditor who is concerned about to the prediction of the early behavior of the manager and to the audit resource allocations may allocate more resources to the "*low-risk accounts*".

Table 1 summarizes the situations that were studied in previous researches.

## 4 Conclusion

In this chapter, we reviewed main models that have used Game theory in auditing. We first presented the theory of games, its main hypothesis and variants; then we classified the auditing models following the Game theory typology. The existing studies have generally used three types of game: simultaneous games, sequential games and experimental games. But it seems that the use of Game theory in auditing should take into account the many pieces of information which are in partial or full contradiction with each other in principle-agent situations. The generation of different modelling scenarios should also be determined from more practical and real observation of auditors' strategies.

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