Chapter 8
Normal Cone Method

An important characterization of efficient faces of a polyhedral set is the fact that their normal cones contain strictly positive vectors. This will be utilized to develop an algorithm to find the efficient solution set and the efficient value set of a multiobjective linear problem.

8.1 Normal Index Sets

We consider a finite system of linear inequalities

$$\langle a^i, x\rangle \leqq b_i, \quad i = 1, \cdots, m, \tag{8.1}$$

where $a^1, \cdots, a^m$ are n-dimensional column vectors and $b_1, \cdots, b_m$ are real numbers. The solution set of this system is denoted X. Throughout this chapter we assume the following hypothesis:

(A1) The system (8.1) is non-redundant and consistent.

Recall that the set of active indices at a point $x^0 \in X$ is denoted $I(x^0)$, which consists of indices $i \in \{1, \cdots, m\}$ such that $\langle a^i, x^0\rangle = b_i$ for $i \in I(x^0)$ and $\langle a^i, x^0\rangle < b_i$ for $i \notin I(x^0)$. We also recall that given a face F of X, the index set $I(F)$ (sometimes denoted I_F) of F is the active index set at a relative interior point of F, and pos(X) is the positive hull of X.

Definition 8.1.1 A nonempty index set $I \subseteq \{1, \cdots, m\}$ is said to be normal if there is some point $x \in X$ such that

$$N_X(x) = \text{pos}\{a^i : i \in I\}.$$

It is clear that when X has a boundary point, that is, at least one vector among $a^i, i = 1, \cdots, m$ is nonzero, then normal index sets exist. Moreover, not every subset of the index set $\{1, \cdots, m\}$ is normal.

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Example 8.1.2 Consider a system of four inequalities in $\mathbb{R}^2$:

$$\begin{aligned} x_1 &\leqq 2 \\ -x_1 &\leqq -1 \\ x_1 + x_2 &\leqq 3 \\ -x_1 - x_2 &\leqq -2. \end{aligned}$$

This system consists of four inequalities that we enumerate from one to four. It is clear that the index sets {1}, {2}, {3}, {4}, {1, 3}, {1, 4}, {2, 3} and {2, 4} are all normal, while the remaining subsets of the index set {1, 2, 3, 4} are not. For instance $I = \{1, 2\}$ is not normal because

$$\text{pos}\{a^1, a^2\} = \text{pos}\left\{\begin{pmatrix}1\\0\end{pmatrix}, \begin{pmatrix}-1\\0\end{pmatrix}\right\} = \text{pos}\left\{\begin{pmatrix}t\\0\end{pmatrix}, t \in \mathbb{R}\right\}$$

is a normal cone to X at no point.

Lemma 8.1.3 *An index set $I \subseteq \{1, \cdots, m\}$ is normal if and only if the following system has a solution*

$$\begin{aligned} &\langle a^i, x\rangle = b_i, i \in I \\ &\langle a^j, x\rangle < b_j, j \in \{1, \cdots, m\}\backslash I. \end{aligned} \tag{8.2}$$

Proof Assume that the system (8.2) has a solution, denoted x. Then x is a boundary point of X and the active index set at x is I. In view of Theorem 2.3.24, we have $N_X(x) = \text{pos}\{a^i : i \in I\}$. By definition I is a normal index set.

Conversely, let I be a normal index set and let $x \in X$ be a point such that $N_X(x) = \text{pos}\{a^i : i \in I\}$. Since x is an element of X, it satisfies the system

$$\begin{aligned} &\langle a^i, x\rangle = b_i, i \in I(x) \\ &\langle a^j, x\rangle < b_j, j \in \{1, \cdots, m\}\backslash I(x). \end{aligned}$$

In view of Theorem 2.3.24, we have $N_X(x) = \text{pos}\{a^i : i \in I(x)\}$ and deduce

$$\text{pos}\{a^i : i \in I(x)\} = \text{pos}\{a^i : i \in I\}.$$

We claim that every vector a^i for $i \in I(x)$ is an extreme ray of the cone $\text{pos}\{a^i : i \in I(x)\}$. Indeed, assume to the contrary, that for some index $i_0 \in I(x)$ one finds $t_i \geqq 0, i \in I(x) \setminus \{i_0\}$ such that $a^{i_0} = \sum_{i\in I(x)\backslash\{i_0\}} t_i a^i$. Then

$$b_{i_0} = \langle a^{i_0}, x\rangle = \sum_{i\in I(x)\backslash\{i_0\}} t_i\langle a^i, x\rangle = \sum_{i\in I(x)\backslash\{i_0\}} t_i b_i.$$

Let $y \in \mathbb{R}^n$ satisfy

$$\langle a^i, y\rangle \leqq b_i \text{ for } i \in I(x) \setminus \{i_0\}.$$

We deduce

$$\langle a^{i_0}, y\rangle = \sum_{i \in I(x)\setminus\{i_0\}} t_i \langle a^i, y\rangle \leqq \sum_{i \in I(x)\setminus\{i_0\}} t_i b_i = b_{i_0}.$$

This shows that inequality $\langle a^{i_0}, y\rangle \leqq b_{i_0}$ is redundant in (8.1), a contradiction to (A1). By this, $I(x) \subseteq I$ and each of vectors a^j, $j \in I$ can be expressed as a positive combination of the vectors a^i, $i \in I(x)$. Again, by a similar argument as above, one proves that $I = I(x)$ due to the non-redundancy hypothesis (A1). Consequently the system (8.2) is consistent. □

There is a close relation between normal index sets and faces of X. We remember that an index set $I \subseteq \{1, \cdots, m\}$ is said to determine a face F of X if F is the solution set to the system

$$\begin{aligned} \langle a^i, x\rangle &= b_i, i \in I \\ \langle a^j, x\rangle &\leqq b_j, j \in \{1, \cdots, m\}\setminus I \end{aligned} \tag{8.3}$$

and if no inequality can be replaced by equality. We know from Corollary 2.3.5 that I, denoted also $I(F)$, coincides with the active index set of a relative interior point of F. Moreover, if F is the solution set of another system corresponding to another index set $I' \subseteq \{1, \cdots, m\}$, then $I' \subseteq I(F)$.

Theorem 8.1.4 *A nonempty index set $I \subseteq \{1, \cdots, m\}$ is normal if and only if the solution set to the system (8.3) is determined by I. Moreover, a nonempty subset F of X is a face if and only if it is determined by a normal index set.*

Proof Let $I \subseteq \{1, \cdots, m\}$ be a normal index set and let F be the solution set to the system (8.3). By definition there is some point $\bar{x} \in X$ such that $N_X(\bar{x}) = \text{pos}\{a^i : i \in I\}$. We wish to prove that I determines F. Towards this end, we first show that F is nonempty, namely it contains $\bar{x}$, that is

$$\langle a^i, \bar{x}\rangle = b_i \text{ for } i \in I.$$

Suppose, to the contrary, that there is some index $i_0 \in I$ such that $\langle a^{i_0}, \bar{x}\rangle < b_{i_0}$. It follows from the definition of the normal cone that

$$\langle a^{i_0}, y\rangle \leqq \langle a^{i_0}, \bar{x}\rangle < b_{i_0} \tag{8.4}$$

for every $y \in X$. Consider the system

$$\langle a^i, y\rangle \leqq b_i \text{ for } i \in \{1, \cdots, m\} \setminus \{i_0\}.$$

We claim that every solution y of this system lies in X. This is because if for some solution y one has $\langle a^{i_0}, y\rangle > b_{i_0}$, then there is a real number $t \in [0, 1]$ such that $\langle a^{i_0}, t\bar{x} + (1-t)y\rangle = b_{i_0}$. As the point $t\bar{x} + (1-t)y$ belongs to X, the latter equality contradicts (8.4). Thus, inequality $\langle a^{i_0}, y\rangle \leqq b_{i_0}$ is redundant, which contradicts (A1). Next, we prove $I = I(\bar{x})$. Inclusion $I \subseteq I(\bar{x})$ is evident. If there is some $j \in I(\bar{x}) \setminus I$, then in view of Theorem 2.3.24, a^j is a normal vector at $\bar{x}$, and hence there are $t_i \geqq 0, i \in I$ such that $a^j = \sum_{i\in I} t_i a^i$. This expression of a^j leads to a contradiction that inequality $\langle a^j, y\rangle \leqq b_j$ is redundant. Hence we conclude that I determines the face F.

To show the second part of the theorem, let F be a face of X. Pick any relative interior point $\bar{x}$ of F. Then $\langle a^j, \bar{x}\rangle < b_j$ for $j \in \{1, \cdots, m\}\backslash I(\bar{x})$ and $\langle a^i, \bar{x}\rangle = b_i$ for $i \in I(\bar{x})$. In view of Lemma 8.1.3, $I(\bar{x})$ is a normal index set that determines F. The converse statement is clear because if F is nonempty and given by the system (8.3), then it is a face of X by Theorem 2.3.3. □

Let $\bar{x}$ be a vertex of X. It is a zero-dimensional face, hence the index set $I(\bar{x})$ has at least n elements. We recall that a point $\bar{x}$ is a non-degenerate vertex of X if there are exactly n linearly independent inequalities in (8.1) that are satisfied as equalities at $\bar{x}$. It follows that the active index set at a non-degenerate vertex has n elements. The next result tells us when a subset $I \subseteq I(\bar{x})$ is a normal set.

Corollary 8.1.5 *Let $\bar{x}$ be a non-degenerate vertex of X. Then every nonempty subset $I \subseteq I(\bar{x})$ is normal.*

Proof Without loss of generality we may assume that $I(\bar{x}) = \{1, \cdots, n\}$. Let I be a nonempty subset of $I(\bar{x})$. Consider the system

$$\begin{aligned}\langle a^i, x\rangle &= b_i,\ \ i \in I\\ \langle a^j, x\rangle &\leqq b_j,\ j \in \{1, \cdots, m\} \setminus I.\end{aligned}$$

This system is consistent, for instance $\bar{x}$ is a solution. Hence the solution set, denoted F, is a face of X. Since the vectors $a^i, i = 1, \cdots, n$ are linearly independent, the dimension of F is equal to $n - |I|$. Let I_F be the index set that determines the face F and is the active index set at a relative interior point of F. Then $I \subseteq I_F$ and $\dim F = n - \text{rank}\{a^i : i \in I_F\} = n - |I|$. We conclude $I = I_F$. By Theorem 8.1.4, I is a normal index set. □

Corollary 8.1.6 *Let I^1 and I^2 be two normal index sets. Then the intersection $I^1 \cap I^2$ is normal if it is nonempty.*

Proof Since I^1 and I^2 are normal, when $I = I^1$ (respectively $I = I^2$) the system (8.2) has at least one solution, say x (respectively y). Set $z = (x + y)/2$. Then for $i \in I^1 \cap I^2$ we have

$$\langle a^i, z\rangle = \frac{1}{2}\big(\langle a^i, x\rangle + \langle a^i, y\rangle\big) = \frac{1}{2}(b_i + b_i) = b_i\ .$$

For $j \in \{1, \cdots, m\} \setminus (I^1 \cap I^2)$ we have either $j \notin I^1$ which implies $\langle a^j, x\rangle < b_j$, or $j \notin I^2$ which implies $\langle a^j, y\rangle < b_j$. This implies

$$\langle a^j, z\rangle = \frac{1}{2}\big(\langle a^j, x\rangle + \langle a^j, y\rangle\big) < \frac{1}{2}(b_j + b_j) = b_j$$

for all $j \in \{1, \cdots, m\} \setminus (I^1 \cap I^2)$. Consequently, z is a solution to the system (8.2), in which $I = I^1 \cap I^2$. In view of Lemma 8.1.3, the index set $I^1 \cap I^2$ is normal. □

Assume that there exist ℓ edges $F_1, \cdots, F_\ell$ emanating from a vertex $\bar{x}$. Then, each of $I(F_1), \cdots, I(F_\ell)$ has at least $(n-1)$ elements (remember that $I(F_i)$ denotes the active index set of a relative interior point of F_i). Let $J \subseteq \{1, \cdots, \ell\}$ with $|J| = r \leq \min\{\ell, n-1\}$. Take $x^i \in F_i \setminus \{\bar{x}\}, i = 1, \cdots, \ell$ and set

$$x^J = \frac{\bar{x}}{r+1} + \sum_{j\in J} \frac{x^j}{r+1}.$$

The next result allows us to determine the largest face that contains x^J as a relative interior point.

Proposition 8.1.7 *Assume that the active index set $I(x^J)$ is nonempty. Then it is a normal set and the face F determined by the system*

$$\begin{aligned} &\langle a^i, x\rangle = b_i,\ i \in I(x^J) \\ &\langle a^j, x\rangle \leqq b_j,\ j \in \{1, \cdots, m\} \setminus I(x^J), \end{aligned}$$

contains the convex hull of all edges F_j, $j \in \{1, \cdots, m\}$ that satisfy the containment $I(F_j) \supseteq I(x^J)$, including $j \in J$.

Proof Since x^J belongs to X, it is a solution of the system described in the proposition. Hence F is a face of X. It follows from the definition of active index sets, x^J is a solution to the system

$$\begin{aligned} &\langle a^i, x^i\rangle = b_i,\ i \in I(x^J) \\ &\langle a^j, x^i\rangle < b_j,\ j \in \{1, \cdots, m\} \setminus I(x^J). \end{aligned}$$

By Lemma 8.1.3, $I(x^J)$ is a normal index set. Furthermore, if for some index $j \in \{1, \cdots, m\}$ one has $I(F_j) \supseteq I(x^J)$, then $N_X(F_j) \supseteq N_X(F)$, and hence $F_j \subseteq F$. Being convex, the face F contains the convex hull of all such edges. Finally, for $i \in I(\overline{x})$, equality

$$b_i = \langle a^i, x^J\rangle = \langle a^i, \frac{\bar{x}}{r+1} + \sum_{j\in J} \frac{x^j}{r+1}\rangle$$

holds if and only if $\langle a^i, x^j \rangle = b_i$ for all $i \in I(\bar{x})$ and $j \in J$. We conclude that $F_j \subseteq F$ for all $j \in J$. □

8.2 Positive Index Sets

Let C be a $k \times n$-matrix and let the columns of C^T be denoted by $c^1, \cdots, c^k$.

Definition 8.2.1 A vector $v \in \mathbb{R}^n$ is called C-positive if there exist strictly positive numbers $\lambda_1, \cdots, \lambda_k$ such that $v = \sum_{i=1}^{k} \lambda_i c^i$, and it is C-negative if $-v$ is C-positive.

In the matrix form, a column vector v is C-positive if and only if $v = C^T \lambda$ for some strictly positive vector $\lambda \in \mathbb{R}^k$. Throughout this chapter we also assume the following

(A2) the cone $\text{pos}\{c^1, \cdots, c^k\}$ is not a linear subspace.

This assumption is clearly equivalent to the fact that the origin of the space is not C-positive. When the zero vector of $\mathbb{R}^n$ is a strictly positive combination of $c^1, \cdots, c^k$, the problem of vector maximizing Cx over a set $X \subseteq \mathbb{R}^n$ becomes trivial because every feasible solution is maximal. Some more properties of C-positive vectors are given next.

Lemma 8.2.2 *The following properties hold true.*

(i) If C is the identity matrix, then a vector $v \in \mathbb{R}^n$ is C-positive if and only if it is strictly positive.
(ii) The set of C-positive vectors coincides with the relative interior of the cone $\text{pos}\{c^1, \cdots, c^k\}$.
(iii) If there is a vector simultaneously C-positive and C-negative, then the rows of C are linearly dependent.
(iv) For $x \in \mathbb{R}^n$, one has $Cx \geqq 0$ (respectively $Cx > 0$) in $\mathbb{R}^k$ if and only if $\langle v, x \rangle \geqq 0$ (respectively $\langle v, x \rangle > 0$) for every C-positive vector v of $\mathbb{R}^n$.

Proof The first property is immediate from the definition. The second one follows from Lemma 6.4.10. For the third property, we notice that when a vector is simultaneously C-positive and C-negative, then the zero vector is a linear combination of the rows of C. Hence the rows of C are linearly dependant. Let us prove the last property. We have $Cx \geqq 0$ if and only if $0 \leqq \langle Cx, \lambda \rangle = \langle x, C^T \lambda \rangle$ for every $\lambda \in \mathbb{R}^k$, $\lambda > 0$, or equivalently $\langle x, v \rangle \geqq 0$ for every C-positive vector $v \in \mathbb{R}^n$. The strict inequality $Cx > 0$ is proven in a similar way. □

Definition 8.2.3 Let $a^1, \cdots, a^m$ be (column) vectors in $\mathbb{R}^n$. An index set $I \subseteq \{1, \cdots, m\}$ is said to be positive if $\text{pos}\{a^i : i \in I\}$ contains a C-positive vector.

It is clear that if an index set is positive, any index set that contains it is also positive, while a smaller subset is not necessarily positive.

Example 8.2.4 Consider a matrix $C = \begin{pmatrix} 2 & -3 & -1 \\ 3 & 1 & 0 \end{pmatrix}$ and a system of inequalities given by

$$\begin{pmatrix} 1 & 1 & 1 \\ -2 & -2 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leqq \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

The transposes of the row vectors of this system are denoted by $a^1, \cdots, a^5$. Direct calculation shows that the index set $\{2, 4\}$ is not positive. The set $\{1, 4, 5\}$ is positive because the cone $\mathrm{pos}\{a^1, a^4, a^5\}$, where $a^1 = (1, 1, 1)^T$, $a^4 = (0, -1, 0)^T$ and $a^5 = (0, 0, -1)^T$, contains a C-positive vector $v = (7/2, -5/2, -1)^T = (2, -3, -1)^T + \frac{1}{2}(3, 1, 0)^T$.

Proposition 8.2.5 *An index set $I \subseteq \{1, \cdots, m\}$ is positive if and only if the following system is consistent*

$$\begin{aligned} &\sum_{i \in I} \mu_i a^i - \sum_{j=1}^{k} \lambda_j c^j = 0 \\ &\mu_i \geqq 0, i \in I \text{ and } \lambda_j \geqq 1, \ j = 1, \cdots, k. \end{aligned} \tag{8.5}$$

Proof It is clear that if the system has a solution, then $\mathrm{pos}\{a^i : i \in I\}$ contains the C-positive vector $\sum_{j=1}^{k} \lambda_j c^j$. Conversely, if there are strictly positive numbers $\lambda_1, \cdots, \lambda_k$ such that $\sum_{j=1}^{k} \lambda_j c^j$ belongs to the cone $\mathrm{pos}\{a^i : i \in I\}$, then the vector $\sum_{j=1}^{k} \frac{\lambda_j}{\min\{\lambda_1, \cdots, \lambda_k\}} c^j$ belongs to that cone too, by which the system given in the proposition has a solution. □

Given a family of vectors $a^1, \cdots, a^m$ we denote

$$\begin{aligned} I^1 &= \left\{ i \in \{1, \cdots, m\} : a^i \text{ is } C\text{−positive} \right\} \\ I^3 &= \left\{ i \in \{1, \cdots, m\} : a^i \text{ is } C\text{−negative} \right\} \\ I^2 &= \{1, \cdots, m\} \setminus (I^1 \cup I^3). \end{aligned}$$

Under (A2) we have a partition of the index set $\{1, \cdots, k\} = I^1 \cup I^2 \cup I^3$ by disjoint subsets. The next result shows how to find positive normal sets outside I^1.

Theorem 8.2.6 *Assume that $I \subseteq \{1, \cdots, m\}$ is a positive and normal index set such that the cone $\mathrm{pos}\{a^i : i \in I\}$ is not a linear subspace. Then there exists a positive and normal set $I_0 \subseteq I \cap (I^1 \cup I^2)$.*

Proof Let $I = \{i_1, \cdots, i_l\}$ be a positive and normal index set. We prove the theorem by induction on l. If $l = 1$, then ta^{i_1} is C-positive for some $t \geqq 0$ since $\mathrm{pos}\{a^{i_1}\}$ is a positive normal cone. Actually $t > 0$ because otherwise the zero vector would belong to the relative interior of the cone $\mathrm{pos}\{c^1, \cdots, c^k\}$, which contradicts the assumption that $\mathrm{pos}\{c^1, \cdots, c^k\}$ is not a linear subspace. It follows that a^{i_1} is C-positive and $I_0 = I$ satisfies the requirements of the theorem.

Now, let $l > 1$. We claim that $I \cap (I^1 \cup I^2) \neq \emptyset$. Indeed, if not, then one has $I \subseteq I^3$. Let v be a C-positive vector that belongs to the cone $\mathrm{pos}\{a^i : i \in I\}$. As all $a^i, i \in I$ are C-negative, the vector v is C-negative too. We deduce that $0 = v - v$ is a C-positive vector and arrive at the same contradiction as above. Consider two possible cases: $I \cap I^3 = \emptyset$ and $I \cap I^3 \neq \emptyset$, say i_l is a common element of I and I^3. In the first case, the index set $I_0 = I$ will be suitable to achieve the proof. In the second case we claim that the cone $\mathrm{pos}\{a^i : i \in I\}$ does not contain all C-positive vectors in its relative interior. In fact, if not, this relative interior should contain the vector $-a^{i_l}$ because a^{i_l} is C-negative, and hence $\mathrm{pos}\{a^i : i \in I\}$ contains the zero vector $0 = a^{i_l} - a^{i_l}$ in its relative interior, which contradicts the hypothesis. Let u be a C-positive vector outside the relative interior of the cone $\mathrm{pos}\{a^i : i \in I\}$. Joining v and u we find a C-positive vector w on a proper face of the cone $\mathrm{pos}\{a^i : i \in I\}$. Let I' be a proper subset of I such that the face is the cone $\mathrm{pos}\{a^i : i \in I'\}$. In view of Theorem 2.3.26 the index set I' determines a face of P, which contains the face determined by I. In other words, I' is a normal index set. It is positive because it contains the C-positive vector w. By induction, there is a positive normal index set $I_0 \subseteq I' \cap (I^1 \cup I^2) \subseteq I \cap (I^1 \cup I^2)$ as requested. $\square$

Efficient solution faces

Let us consider the following multiobjective linear programming problem (MOLP)

$$\begin{array}{ll} \text{Maximize} & Cx \\ \text{subject to} & Ax \leqq b, \end{array}$$

where C is a real $k \times n$-matrix, A is a real $m \times n$-matrix and b is a column m-vector. The feasible solution set of (MOLP) is denoted X and its efficient (maximal) solution set is denoted $S(MOLP)$. We will assume throughout that X is nonempty. Moreover, if the zero vector of $\mathbb{R}^n$ is C-positive, then every feasible solution is efficient because there is a strictly positive vector $\lambda \in \mathbb{R}^k$ such that $0 = C^T\lambda$ which implies that every element of X is an optimal solution of the scalarized problem (P_λ)

$$\begin{array}{ll} \text{maximize} & \langle \lambda, Cx \rangle \\ \text{subject to} & Ax \leqq b, \end{array}$$

and hence, in view of Theorem 4.3.1, it is an efficient solution of (MOLP). For this reason, we will assume henceforth (A2) as before, that is, the cone $\mathrm{pos}\{c^1, \cdots, c^k\}$ is not a linear subspace.

Theorem 8.2.7 *A feasible solution of (MOLP) is an efficient solution if and only if the active index set at this solution is positive.*

Proof Let $x^0 \in X$ be an efficient solution. In view of Theorem 4.3.1 there is a strictly positive vector λ such that x^0 solves the problem (P_λ). In particular,

$$\langle C^T\lambda, x^0 - x\rangle \geqq 0 \text{ for all } x \in X \tag{8.6}$$

which proves that the vector $C^T\lambda$ is a normal vector to X at x^0. If $I(x^0)$ is empty, then x^0 is an interior point of X. Hence $C^T\lambda$ is the zero vector and contradicts the assumption. In view of Theorem 2.3.24 the normal cone to X at x^0 is the cone $\mathrm{pos}\{a^i : i \in I(x^0)\}$. Hence $I(x^0)$ is a positive and normal index set.

Conversely, assume that $I(x^0)$ is positive. There is a strictly positive vector λ such that $C^T\lambda$ belongs to the cone $\mathrm{pos}\{a^i : i \in I(x^0)\}$. In particular $C^T\lambda$ is normal to the set X at x^0. Consequently, (8.6) is true, and therefore x^0 solves (P_λ). We deduce from Theorem 4.3.1 that x^0 is an efficient solution of (MOLP). □

Corollary 8.2.8 *(MOLP) has an efficient solution if and only if the index set* $\{1, \cdots, m\}$ *is positive, or equivalently, the following system is consistent*

$$\sum_{i=1}^{m} \mu_i a^i - \sum_{j=1}^{k} \lambda_j c^j = 0$$
$$\mu_i \geqq 0, i = 1, \cdots, m \text{ and } \lambda_j \geqq 1, \; j = 1, \cdots, k.$$

Proof Let x^0 be an efficient solution of (MOLP). In view of Theorem 8.2.7 the index set $I(x^0)$ is positive. As the set $\{1, \cdots, m\}$ contains $I(x^0)$, it is positive too. Conversely, if the set $\{1, \cdots, m\}$ is positive, the cone $\mathrm{pos}\{a^1, \cdots, a^m\}$ which is exactly the normal cone of X contains a C-positive vector. Hence there is some point $x^0 \in X$ such that the normal cone to X at x^0 contains that C-positive vector. Again, by Theorem 8.2.7, the point x^0 is efficient, and therefore (MOLP) has efficient solutions. The equivalence between the consistency of the linear system mentioned in the corollary and the positivity of the index set $\{1, \cdots, m\}$ is immediate from the definition. □

Corollary 8.2.9 *Let F be a face of X and $I(F)$ the index set of F. Then F is efficient if and only if $I(F)$ is positive, in which case the dimension of F is equal to* $n - \mathrm{rank}\{a^i : i \in I_F\}$*. In particular, when X is of full dimension, (MOLP) admits an* $(n-1)$*-dimensional efficient face if and only if there is an index* $i_0 \in \{1, \cdots, m\}$ *such that a^{i_0} is C-positive, in which case the* $(n-1)$*-dimensional face determined by the linear system*

$$\langle a^{i_0}, x\rangle = b_{i_0}$$
$$\langle a^j, x\rangle \leqq b_j, \; j \in \{1, \cdots, m\}\backslash\{i_0\},$$

is an efficient face.

Proof Let x be a relative interior point of the face F. Then $I(x) = I(F)$. By Theorem 4.3.8 the face F is efficient if and only if x is efficient. It remains to apply Theorem 8.2.7 to conclude the first part of the corollary. If the dimension of X is equal to n, then a face F is of dimension $n-1$ if and only if it is given by the (non-redundant) system determining X in which only one inequality is equality. In other words the index set of an $(n-1)$-dimensional face consists of one element. Therefore, this face is efficient if and only if that unique index is positive. □

Let $\bar{x}$ be an efficient vertex of (MOLP) and $F_1, \cdots, F_\ell$ the efficient edges emanating from $\bar{x}$. The active index set of each F_i is denoted $I(F_i)$. Below is a condition for an efficient face adjacent to $\bar{x}$ to be maximal, that is, it is not a proper face of any other efficient face of the problem.

Corollary 8.2.10 *Let $J \subseteq \{1, \ldots, \ell\}$ and $I_J = \bigcap_{i \in J} I(F_i)$. Then the face F adjacent to $\bar{x}$ determined by the system*

$$\langle a^i, x\rangle = b_i, i \in I_J$$
$$\langle a^j, x\rangle \leqq b_j, j \in \{1, \cdots, m\}\backslash I_J,$$

is a maximal efficient face if the following conditions hold:

(i) I_J is positive;
(ii) For every $i \notin I_J$ such that $I(F_i) \not\supset I_J$, the index set $I_J \cap I(F_i)$ is either empty or not positive.

Proof It is clear that under (i), F is an efficient face. If it is not maximal, then it is contained in a bigger efficient face, say F'. We may find an edge F_j of F' emanating from $\bar{x}$ which does not belong to F. Then

$$I_J \cap I(F_j) \neq \emptyset.$$

Since this index set contains the positive index set $I(F')$ we conclude that $I_J \cap I(F_j)$ is positive, which contradicts the hypothesis. □

The support of a vector $\mu \in \mathbb{R}^m_+$ is denoted by supp(μ) and defined as

$$\text{supp}(\mu) = \big\{i \in \{1, \cdots, m\} : \mu_i > 0\big\}.$$

We shall use also the following notations: Γ is the solution set to the system formulated in Corollary 8.2.8 and

$$\mathcal{I}_0 = \big\{I \subseteq \{1, \cdots, m\} : I = \text{supp}(\mu) \text{ for some } (\mu, \lambda) \in \Gamma\big\}$$
$$\mathcal{I}_1 = \big\{I \in \mathcal{I}_0 : I = \text{supp}(\mu) \text{ for some vertex } (\mu, \lambda) \in \Gamma\big\}$$

and $\mathcal{I}$ denotes the set of all minimal elements of $\mathcal{I}_1$ with respect to inclusion. We recall also that for a subset $I \subseteq \{1, \cdots, m\}$, the set $F(I)$ consists of feasible solutions

x of (MOLP) such that $\langle a^i, x\rangle = b_i, i \in I$. In particular when I is the empty set, $F(I) = X$.

Corollary 8.2.11 *The following statements hold.*

(i) *Let* $(\mu^i, \lambda^i), i = 1, \cdots, l$ *be vertices of* Γ. *Then for all* $t_i \in (0, 1)$ *with* $\sum_{i=1}^{l} t_i = 1$ *one has*

$$F\big(\mathrm{supp}(\sum_{i=1}^{l} t_i \mu^i)\big) = \bigcap_{i=1}^{l} F\big(\mathrm{supp}(\mu^i)\big).$$

(ii) $S(MOLP) = \bigcup_{I\in\mathcal{I}_0} F(I) = \bigcup_{I\in\mathcal{I}_1} F(I) = \bigcup_{I\in\mathcal{I}} F(I)$

(iii) *Given an index set* $I \subseteq \{1, \cdots, m\}$, *the set* $F(I)$ *is a maximal efficient face if and only if it is nonempty and* $I \in \mathcal{I}$.

Proof It follows from the definition that

$$\mathrm{supp}\big(\sum_{i=1}^{l} t_i \mu^i\big) = \bigcup_{i=1}^{l} \mathrm{supp}(\mu^i).$$

This implies the equality in the first statement.

For the second statement, it is clear that for every $I \in \mathcal{I}_0$, if nonempty, the set $F(I)$ is a face of X. Hence the index set of $F(I)$ that is included in I is positive. By Corollary 8.2.9, $F(I)$ is efficient. Thus, $\bigcup_{I\in\mathcal{I}_0} F(I) \subseteq S(MOLP)$ is true. Conversely, let $\overline{x}$ be an efficient solution of (MOLP). In view of Theorem 8.2.7, the active index set $I(\overline{x})$ is positive, which means that the system

$$\begin{gathered}\sum_{i\in I(\overline{x})} \mu_i a^i - \sum_{j=1}^{k} \lambda_j c^j = 0 \\ \mu_i \geqq 0, i \in I(\overline{x}) \text{ and } \lambda_j \geqq 1,\ \ j = 1, \cdots, k\end{gathered}$$

is solvable. Let (μ, λ) be a solution. Define $\overline{\mu}$ to be the vector the coordinates of which are given by

$$\overline{\mu}_i = \begin{cases} \mu_i & \text{for } i \in I \\ 0 & \text{else.} \end{cases}$$

It is then clear that

$$\overline{x} \in F\big(I(\overline{x})\big) \subseteq F\big(\mathrm{supp}(\overline{\mu})\big)$$

with $\mathrm{supp}(\overline{\mu}) \in \mathcal{I}_0$ and the first equality in (ii) follows. Furthermore, since $\mathcal{I} \subseteq \mathcal{I}_1 \subseteq \mathcal{I}_0$, we deduce inclusions

$$\bigcup_{I \in \mathcal{I}_0} F(I) \supseteq \bigcup_{I \in \mathcal{I}_1} F(I) \supseteq \bigcup_{I \in \mathcal{I}} F(I).$$

For the converse inclusions we notice that for each element $I^1 \in \mathcal{I}_1$, we find an element $I \in \mathcal{I}$ such that $I \subseteq I^1$. Then $F(I^1) \subseteq F(I)$, which proves

$$\bigcup_{I \in \mathcal{I}_1} F(I) \subseteq \bigcup_{I \in \mathcal{I}} F(I).$$

Moreover, for every element $I^0 \in \mathcal{I}_0$, say I =supp(μ) for some $(\mu, \lambda) \in \Gamma$, there exist vertices $(\mu^i, \lambda^i) \in \Gamma$ and positive numbers $t_i, i = 1, \cdots, l$ such that $\sum_{i=1}^{l} t_i = 1$ and $(\mu, \lambda) = \sum_{i=1}^{l} t_i(\mu^i, \lambda^i)$. It follows from the first part that

$$F(I^0) \subseteq F\big(\text{supp}(\mu^i)\big) \quad \text{for every} \;\; i = 1, \cdots, l.$$

We conclude that

$$\bigcup_{I \in \mathcal{I}_0} F(I) \subseteq \bigcup_{I \in \mathcal{I}_1} F(I),$$

by which equalities in the second statement hold.

To prove the last statement we assume that $F(I)$ is a maximal efficient face of (MOLP). By Corollary 8.2.9 the index set I belongs to $\mathcal{I}_0$. According to (ii), there is a minimal index set $I' \in \mathcal{I}$ such that $F(I) \subseteq F(I')$. Since $F(I)$ is maximal, we deduce $F(I) = F(I')$. Under the non-redundancy hypothesis (A1), we obtain $I = I'$. The converse statement is clear because if the efficient face $F(I)$ were not maximal for $I \in \mathcal{I}$, then one would find an efficient face F' that contains $F(I)$ as a proper face. Then the index set of F' is strictly smaller than the index set of $F(I)$, which is a contradiction because the index set of $F(I)$ is equal to I. □

Notice that the family $\mathcal{I}$ as well as the families $\mathcal{I}_0$ and $\mathcal{I}_1$ gathers positive index sets which uniquely depend on the objective matrix C and the constraint matrix A of (MOLP) and do not depend on the second term b of the constraints. This latter term intervenes in the normality of the index sets, that is the nonemptiness of the faces determined by these index sets. Therefore, for a given b, some of subsets in the unions described in (ii) of Corollary 8.2.11 may be empty, which are precisely the case when the corresponding index sets are not normal.

8.3 The Normal Cone Method

In this section we shall give a method for numerically solving the problem (MOLP). The study of normal cones and their relationship with efficient faces that we have developed in the previous sections allow us to construct simple algorithms to determine efficient faces of any dimension. Throughout this section we will make the following assumption.

(A3) The feasible set X is an n-dimensional polyhedral set and contains no lines and there is no redundant inequality in the constraint system of (MOLP).

The method we are going to describe consists of three main procedures:

1—Determine whether (MOLP) has efficient solutions and if it has, find an initial efficient solution.

2—Starting from an efficient vertex, find all efficient edges and efficient rays emanating from it. Since the efficient solution set of (MOLP) is arcwise connected, this procedure allows us to find all efficient vertices and all efficient edges of the problem.

3—Find all efficient faces adjacent to (i.e. containing) a given efficient vertex when all the efficient edges adjacent to this vertex are already known.

Existence of efficient solutions and finding an initial efficient solution for (MOLP)

According to Corollary 8.2.8, (MOLP) has an efficient solution if and only if the system

$$\begin{aligned} A^T\mu - C^T\lambda &= 0 \\ \mu \in \mathbb{R}^m, \mu &\geqq 0 \\ \lambda \in \mathbb{R}^k, \lambda &\geqq e, \end{aligned} \tag{8.7}$$

where e is the vector of ones, has a solution. Remember that the columns of A^T are $a^1, \cdots, a^m$ and the columns of C^T are $c^1, \cdots, c^k$.

Procedure 1.

- *Step 1.* Solve the system (8.7).
  (a) If the system has no solution, then stop. (MOLP) has no efficient solution.
  (b) Otherwise, go to Step 2.
- *Step 2.* Let $\lambda > 0$ be a solution and $v = C^T\lambda$. If $v = 0$, then every feasible solution of (MOLP) is an efficient solution. Otherwise, solve the linear programming problem

$$\begin{aligned} &\text{maximize} \quad \langle v, x\rangle \\ &\text{subject to} \quad Ax \leqq b. \end{aligned}$$

This problem has a solution, say $\bar{x}$. Then $\bar{x}$ is an initial efficient solution of (MOLP).

Example 8.3.1 We consider the following (MOLP)

$$\text{Maximize} \qquad \begin{pmatrix} 2 & -3 & 1 \\ 3 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\text{subject to} \begin{pmatrix} 1 & 1 & 1 \\ -2 & -2 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leqq \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}.$$

By solving the system (8.7) we find a solution $\lambda = (2, 1)^T$. The vector v in Step 2 is $v = (7, -5, 2)^T$ and the problem to solve in this step is to maximize the function $7x_1 - 5x_2 + 2x_3$ over the feasible set of (MOLP). The simplex method of Chap. 3 yields a vertex solution $x^0 = (1, 0, 0)^T$, which is also an efficient solution of (MOLP).

Determination of efficient vertices and efficient edges

When (MOLP) has an efficient solution, by solving the linear problem in Step 2 of Procedure 1 one may obtain an efficient vertex. Let $\bar{x}$ be such a vertex. Then the active index set $I(\bar{x})$ has at least n elements. Any edge emanating from x is determined by $n - 1$ linearly independent equations among the m inequality constraints, and of course its index set is a subset of $I(\bar{x})$. An index set $I \subseteq I(\bar{x})$ the cardinality of which is equal to $n - 1$ determines a one-dimensional space that may give rise to an edge of X by the system

$$\langle a^i, v\rangle = 0, \ \ i \in I$$

provided that the vectors $a^i, i \in I$ are linearly independent. Otherwise the solution set of this system would be of higher dimension. Moreover, if I is normal, then there is some real number $t \neq 0$ such that $\bar{x} + tv$ is a solution to the system

$$\begin{aligned} \langle a^i, x\rangle &= b_i, \ \ i \in I \\ \langle a^j, x\rangle &\leqq b_j, \ \ j \in \{1, \cdots, m\} \backslash I. \end{aligned}$$

The edge emanating from $\bar{x}$ in direction v above is efficient if, in addition, I is positive. We are now able to describe the second procedure to solve (MOLP).

Procedure 2.

- *Step 0 (Initialization).* Determine the active index set

$$I(\bar{x}) = \big\{ i \in \{1, \cdots, m\} : \langle a^i, \bar{x}\rangle = b_i \big\}.$$

  Choose $I \subset I(\bar{x})$ with $|I| = n - 1$.

- *Step 1.* Check the linear independence of the family $\{a^i : i \in I\}$.
  If not, choose another $I \subseteq I(\bar{x})$ with $|I| = n - 1$.
  If yes, go further.
- *Step 2. (I is positive?)* Solve

$$\sum_{i \in I} \mu_i a^i - \sum_{j=1}^{k} \lambda_j c^j = 0$$
$$\mu_i \geqq 0, i \in I \text{ and } \lambda_j \geqq 1, \ \ j = 1, \cdots, k.$$

  (a) If it has no solution, pick another $I \subseteq I(\bar{x})$ with $|I| = n - 1$ and return to Step 1.
  (b) Otherwise, I is a positive set, go further.
- *Step 3. (I is normal? If yes, find the corresponding efficient edge)*
  - *Step 3.1.* Find a direction $v \neq 0$ of a possible edge emanating from $\bar{x}$ by solving

$$\langle a^i, v \rangle = 0, i \in I.$$

  - *Step 3.2.* Solve the following system

$$\langle a^i, \bar{x} + tv \rangle \leqq b_i, i = 1, \cdots, m.$$

    Let the solution set be $[t_0, 0]$ or $[0, t_0]$ according to $t_0 < 0$ or $t_0 > 0$. The values $t_0 = -\infty$ and $t_0 = \infty$ are possible.

    (a) If $t_0 = 0$, no edge of X emanating from $\bar{x}$ along v. I is not normal. Pick another $I \subseteq I(\bar{x})$ and go to Step 1.
    (b) If $t_0 \neq 0$ and is finite, then $\bar{x} + t_0 v$ is an efficient vertex and $[\bar{x}, \bar{x} + t_0 v]$ is an efficient edge. Store them if they have not been stored before. Pick another $I \subseteq I(\bar{x})$ and go to Step 1.
    (c) If t_0 is infinite, say $t_0 = \infty$, then the ray $\{\bar{x} + tv : t \geq 0\}$ is efficient. Store the result. Pick another $I \subseteq I(\bar{x})$ and go to Step 1.

We notice that if $\bar{x}$ is a non-degenerate vertex, that is $|I(\bar{x})| = n$, then Step 1 can be skipped because the family of vectors a^i, $i \in I(\bar{x})$ is already linearly independent, and any subset of $I(\bar{x})$ is normal (see Corollary 8.1.5).

Moreover, by solving the system of Step 3.2 we mean finding the solution set of type $[t_0, 0]$ or $[0, t_0]$ with t_0 negative or positive respectively. The infinite values $+\infty$ and $-\infty$ are possible. If $t_0 \neq 0$, then this solution set is a 1-dimensional face of X determined by the system

$$\langle a^i, x \rangle = b_i, i \in I$$
$$\langle a^j, x \rangle \leqq b_j, j \in \{1, \cdots, m\} \setminus I.$$

The active index set of this face contains I, but does not necessarily coincide with I, unless $\bar{x}$ is non-degenerate. It can be found by checking the number of equalities $\langle a^i, \bar{x} + tv\rangle = b_i$ with $i \in I(\bar{x})$ for some $0 < t < t_0$ if $t_0 > 0$, or $t_0 < t < 0$ if $t_0 < 0$.

Example 8.3.2 The aim of this example is to apply Procedure 2 to find all efficient edges adjacent to an initial efficient vertex. We consider the following (MOLP)

$$\begin{array}{ll} \text{Maximize} & \begin{pmatrix} 2 & 3 & 4 \\ 3 & 2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ \text{subject to} & \begin{pmatrix} 1 & 1 & 1 \\ -2 & -2 & -1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leqq \begin{pmatrix} 1 \\ -1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \end{array}$$

To find an initial efficient vertex we solve the system (8.7) that takes the form

$$\begin{aligned} \mu_1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \mu_2 \begin{pmatrix} -2 \\ -2 \\ -1 \end{pmatrix} + \mu_3 \begin{pmatrix} -1 \\ 0 \\ 0 \end{pmatrix} + \mu_4 \begin{pmatrix} 0 \\ -1 \\ 0 \end{pmatrix} + \mu_5 \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \\ = \lambda_1 \begin{pmatrix} 2 \\ 3 \\ 4 \end{pmatrix} + \lambda_2 \begin{pmatrix} 3 \\ 2 \\ 0 \end{pmatrix} \\ \mu_i \geqq 0, i = 1, \cdots, 5, \ \lambda_1 \geqq 1, \lambda_2 \geqq 1. \end{aligned} \tag{8.8}$$

A solution can be given as $\lambda_1 = 3, \lambda_2 = 1, \mu_1 = 12, \mu_2 = 0, \mu_3 = 3, \mu_4 = 1$, and $\mu_5 = 0$. The normal cone of the feasible set contains a C-positive vector $v = \lambda_1 c^{1T} + \lambda_2 c^{2T} = (9, 11, 12)^T$, which generates a denegerate efficient vertex $x^0 = (0, 0, 1)^T$. The active index set of this solution is $I(x^0) = \{1, 2, 3, 4\}$. In order to find efficient edges emanating from x^0 we check the normality and the positivity of each of the 2-element index subsets of $I(x_0)$ and also the linear independence of the corresponding vectors.

(1) For $I_1 = \{1, 2\}$ we have the vectors $a^1 = (1, 1, 1)^T$ and $a^2 = (-2, -2, -1)^T$ linearly independent. To check its positivity we solve (8.8) by setting $\mu_3 = \mu_4 = \mu_5 = 0$, which leads to

$$\begin{aligned} \mu_1 - 2\mu_2 &= 2\lambda_1 + 3\lambda_2 = 3\lambda_1 + 2\lambda_2 \\ \mu_1 - \mu_2 &= 4\lambda_1. \end{aligned}$$

In particular $\mu_2 = -\lambda_1$, which contradicts the constraints $\mu_2 \geqq 0$ and $\lambda_1 \geqq 1$. Hence I_1 is not positive.
(2) For $I_2 = \{1, 3\}$, we notice that the vectors $a^1 = (1, 1, 1)^T$ and $a^3 = (-1, 0, 0)^T$

are linearly independent. To check its positivity we solve (8.8) by setting $\mu_2 = \mu_4 = \mu_5 = 0$. A solution of it can be given as $\lambda_1 = 2$, $\lambda_2 = 1$, $\mu_1 = 8$, $\mu_3 = 1$. As consequence, I_2 is positive. To see whether it is normal we solve the constraint inequalities of (MOLP) in which the first and the third inequations are equations. It is easy to check that the solution set is the segment connecting the vertex x^0 and the vertex $x^2 = (0, 1, 0)^T$. Thus, I_2 is positive and normal. By this, the segment $[x^0, x^2]$ is an efficient edge.
(3) For $I_3 = \{1, 4\}$, we notice again that the vectors $a^1 = (1, 1, 1)^T$ and $a^4 = (0, -1, 0)^T$ are linearly independent. We set $\mu_2 = \mu_3 = \mu_5 = 0$ in (8.8) for checking the positivity of I_3. Similarly to the case of I_1, the system yields $\mu_4 = -\lambda_2/2$, which is a contradiction. Hence I_3 is not positive.
(4) The index sets $\{2, 3\}$, $\{2, 4\}$ and $\{3, 4\}$ are evidently not positive because in the corresponding systems obtained from (8.8) by setting at least $\mu_1 = 0$, the vector on the left hand side is negative, while the vector on the right hand side is strictly positive.
We conclude that there is only one efficient edge emanating from the vertex x^0 and ending at the vertex x^2.

Determination of higher dimensional efficient solution faces

Assume that $\bar{x}$ is an efficient vertex of problem (MOLP) and $[\bar{x}, \bar{x} + t_i v_i]$, $i = 1, \cdots, r$ are efficient edges emanating from $\bar{x}$ with $t_i > 0$. Here, for the convenience we use $t_i = \infty$ if the ray edge $\{\bar{x} + t v_i : t \geq 0\}$ is efficient and $[\bar{x}, \bar{x} + t_i v_i]$ denotes this ray. Let $I_i \subseteq I(\bar{x})$, $i = 1, \cdots, r$ be the positive index sets determining these edges.

Observe that except for the pathological case when the entire set X is efficient, the largest dimension that an efficient face adjacent to $\bar{x}$ may have is $\min\{r, n-1\}$. For $1 < l \leqq \min\{r, n-1\}$, we have the following procedure to find l-dimensional efficient faces adjacent to the given efficient vertex $\bar{x}$.

Procedure 3.

- *Step 0 (Initialization).* Pick $J \subseteq \{1, \cdots, r\}$ with $|J| = l$ and determine $I = \bigcap_{j \in J} I_j$.
  Find $\operatorname{rank}\{a^i : i \in I\}$.
  If $\operatorname{rank}\{a^i : i \in I\} \neq n - l$, choose another J and repeat this step.
  Otherwise go to the next step.
- *Step 1. (I is positive?)* Solve the system (8.5).
  (a) If it has no solution, return to Step 0 by choosing another J.
  (b) Otherwise, I is positive, go to Step 2.
- *Step 2.* Determine $J_0 = \{j \in \{1, \cdots, r\} : I_j \supseteq I\}$. (It is evident that $J \subseteq J_0$.)
  The convex hull of the edges $[\bar{x}, \bar{x} + t_j v_j]$, $j \in J_0$ forms an l-dimensional efficient face. Return to Step 0 by picking another not yet explored J that is not contained in J_0 with $|J| = l$ until no such J left.

The index set I obtained in the initialization step is normal whenever it is nonempty because it is the intersection of normal sets I_j, $j \in J$ (Corollary 8.1.6). If in addition the rank of the family $\{a^i : i \in I\}$ is equal to $n - l$, then the face determined by the system

$$\begin{aligned}\langle a^i, x\rangle &= b_i, i \in I\\ \langle a^j, x\rangle &\leqq b_j, j \in \{1, \cdots, m\}\backslash I,\end{aligned}$$

has its dimension equal to l, and its active index set contains I. When the efficient vertex $\bar{x}$ is non-degenerate, the condition $\text{rank}\{a^i : i \in I\} = n - l$ is equivalent to the fact that I has l elements.

Example 8.3.3 The aim of this example is to apply Procedure 3 to find all efficient faces adjacent to an initial efficient vertex. We consider the following (MOLP)

$$\text{Maximize} \quad \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$$

$$\text{subject to} \begin{pmatrix}-2&-2&-1\\2&1&2\\1&2&2\\-1&0&0\\0&-1&0\\0&0&-1\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix} \leqq \begin{pmatrix}-1\\2\\2\\0\\0\\0\end{pmatrix}.$$

Applying Procedure 1 and Procedure 2 one finds an initial vertex $x^0 = (0, 0, 1)^T$ to start and the following efficient edges adjacent to x^0: $F_i = [x^0, x^i]$, $i = 1, 2, 3$, where $x^1 = (1, 0, 0)^T$, $x^2 = (0, 1, 0)^T$ and $x^3 = (2/3, 2/3, 0)^T$. The index sets of x^0 and F_i are respectively given by

$$\begin{aligned}I(x^0) &= \{1, 2, 3, 4, 5\}\\ I_1 &= \{2, 5\}\\ I_2 &= \{3, 4\}\\ I_3 &= \{2, 3\}.\end{aligned}$$

We use Procedure 3 to determine a two-dimensional efficient face adjacent to x^0. At Step 0, we choose for instance $J = \{1, 3\}$ and consider $I = I_1 \cap I_3 = \{2\}$. The rank of $a^2 = (2, 1, 2)^T$ is equal to 1, hence we may go further. In Step 2, we check the positivity of I by solving the system (8.5) applied to our example:

$$\mu\begin{pmatrix}2\\1\\2\end{pmatrix} = \lambda_1\begin{pmatrix}1\\0\\0\end{pmatrix} + \lambda_2\begin{pmatrix}0\\1\\0\end{pmatrix} + \lambda_3\begin{pmatrix}0\\0\\1\end{pmatrix}.$$

A solution is given by $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 2$ and $\mu = 1$. We conclude that the two-dimensional face determined by the system of constraints in which the second inequality is set to equation is efficient. Since I is contained in I_1 and I_3, this face contains the edges F_1 and F_3. The edge F_2 is not included in it because I_2 does not contain I.

Determination of all maximal efficient faces adjacent to an efficient vertex

Let $\bar{x}$ be an efficient vertex of (MOLP) and let $\{p_1, \cdots, p_r\}$ be the collection of all efficient edges (possibly rays) emanating from $\bar{x}$ which have been obtained by Procedure 2. The positive index sets determining these edges are denoted $I_1, \cdots, I_r$. Thus, each edge p_i is the solution to the system

$$\begin{aligned} &\langle a^j, x\rangle = b_j \ , \ j \in I_i \\ &\langle a^j, x\rangle \leqq b_j \ , \ j \in \{1, \cdots, m\} \setminus I_i \ . \end{aligned}$$

The next algorithm determines all maximal efficient faces adjacent to $\bar{x}$. The biggest dimension of these faces does not exceed $\min\{r, n-1\}$ as we have already discussed.

Procedure 4.

- *Step 0.* For $l = 2, \cdots, r$ pick $J \subseteq \{1, \cdots, r\}$ with $|J| = l$ and compute

$$I = \bigcap_{j \in J} I_j .$$

  If either $I = \emptyset$, or I is not positive, then choose another J.
  If I is nonempty positive, go to the next step.
- *Step 1.* For each $j \in \{1, \cdots, r\} \setminus J$, compute $I' = I \cap I_j$.
  (a) If either $I' = \emptyset$ or I' is not positive, proceed for other j. If this is the case for all $j \in \{1, \cdots, r\} \setminus I$, the face F_J generated by the edges p_j with $I \subseteq I_j$ including $j \in J$, is a maximal efficient face to be stocked together with the index set $J = \{i : I \subseteq I_i\}$. Return to Step 0 for other J.
  (b) If I' is nonempty positive, set $J = J \cup \{j\}$ (then $|J| \geqq l+1$) and repeat Step 1 until no J left.
- *Step 2.* Set $l := l + 1$ and return to Step 0 by choosing J not yet exploited or not contained in any index subset already stocked in Step 1.

The positivity of I in Step 0 and of I' in Step 1 is checked by solving the system (8.5). The maximality of the efficient faces stocked in Step 1 is due to Corollary 8.2.10.

Example 8.3.4 We continue Example 8.3.3 by choosing $x^0 = (0, 0, 1)^T$ as an initial vertex. Its efficient edges $F_i = [x^0, x^i], i = 1, 2, 3$ where $x^1 = (1, 0, 0)^T, x^2 = (0, 1, 0)^T, x^3 = (2/3, 2/3, 0)^T$. The index set of x^0 is $I(x^0) = \{1, 2, 3, 4, 5\}$ and the index sets of $F_i, i = 1, 2, 3$ are respectively

$$I(F_1) = \{2, 5\}, I(F_2) = \{3, 4\}, I(F_3) = \{2, 3\}.$$

We apply Procedure 4 to determine all maximal efficient faces adjacent to x^0. In Step 0, for $l = 2$, we choose $J \subseteq \{1, 2, 3\}$ with $|J| = 2$.

- For $J = \{1, 3\}$ one has $I = I(F_1) \cap I(F_3) = \{2, 5\} \cap \{2, 3\} = \{2\}$. This index set I is positive (see Example 8.3.3), we go to Step 1. Let $j \in \{1, 2, 3\}\backslash J = \{2\}$. Compute $I' = I \cap I(F_2) = \{2\} \cap \{3, 4\} = \emptyset$. Thus, the face generated by the index set $\{2\}$ is a maximal efficient face that contains the edges $F_j, j \in J$.
- For $J = \{1, 2\}$ we have $I = I(F_1) \cap I(F_2) = \emptyset$. Return to Step 0.
- For $J = \{2, 3\}$ we have $I = I(F_2) \cap I(F_3) = \{3\}$. It is positive because the system

$$\mu \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + \lambda_2 \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + \lambda_3 \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \ \lambda_1, \lambda_2, \lambda_3 \geqq 1, \ \mu \geqq 0$$

  admits a solution $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = 2$ and $\mu = 1$. We go to Step 1. Let $j \in \{1, 2, 3\}\backslash J = \{1\}$. Compute $I' = I \cap I(F_1) = \{3\} \cap \{2, 5\} = \emptyset$. The face generated by the index set $I = \{1\}$ is maximal efficient and contains the edges $F_j, j \in J = \{2, 3\}$.

The algorithm yields two maximal efficient faces that contain respectively the edges F_1, F_3 and F_2, F_3.

Determination of the entire efficient solution set of (MOLP)

Since every efficient solution of (MOLP) is contained in a maximal efficient face, the efficient solution set of the problem will be completely found if we can identify all maximal efficient faces. The next algorithm for generating all maximal efficient faces is based on Procedures 1, 2 and 4 and on the fact that the solution set of a multiobjective linear problem is arcwise connected, that is any two efficient vertices can be joined by a finite number of efficient edges.

The Algorithm

- *Step 1.* Determine whether (MOLP) has maximal solutions. If yes, find an efficient vertex to start by using Procedure 1.
- *Step 2.* Find all efficient edges adjacent to this efficient vertex by Procedure 2.
- *Step 3.* Determine all maximal efficient faces adjacent to the given vertex by Procedure 4 and stock them together with the active index set of each such a face.

- *Step 4.* Choose a new efficient vertex adjacent to the given vertex and return to Step 2 with this vertex to start unless no such efficient vertex left.

So far the analysis made in this chapter requires three assumptions (A1–A3), under which the algorithm terminates after a finite number of iterations because this is so for the three procedures we apply. There are some simple situations when one or some of the above mentioned assumptions do not hold and there is no need to solve the problem. For instance when (1) the feasible set is empty; or (2) the cone $\text{pos}\{c^1, \cdots c^k\}$ is a linear subspace (the zero vector is C-positive, and hence every feasible solution is maximal).

Particular case 1: Efficient sets in $\mathbb{R}^2$

Sometimes we wish to compute the efficient set of a polyhedron (a bounded polyhedral convex set) which corresponds to the efficient solution set of the problem (MOLP) with C being the identity matrix. Below we provide an effective and direct algorithm to do this in the case $X \subseteq \mathbb{R}^2$.

By renumbering the indices if necessary, we may assume

$$0 < \theta_1 < \theta_2 < \cdots < \theta_l < \frac{1}{2}\pi \leq \theta_{l+1} < \cdots < \theta_m \leq 2\pi,$$

where $\theta_1, \cdots, \theta_m$ are the angular coordinates of $a^1, \cdots, a^m$ in the polar coordinate system of $\mathbb{R}^2$ (Fig. 8.1).

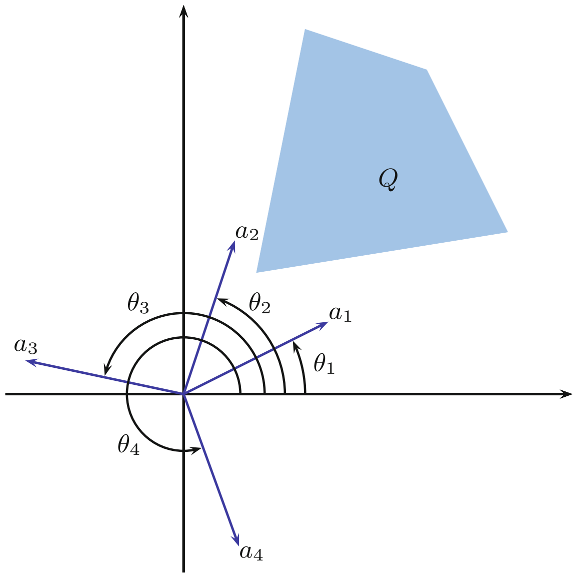

Fig. 8.1 Angular coordinates

It is evident that $a^1, \cdots, a^l$ are positive vectors and hence each of them determines an efficient edge. Moreover, each of the pairs of indices $\{m, 1\}, \{1, 2\}, \cdots, \{l, l+1\}$ is positive and normal. So, by Corollary 8.2.9 they determine all 0-dimensional efficient faces (vertices) of X. Denote by x^i the intersection point of the lines

$$\langle a^i, x\rangle = b_i$$
$$\langle a^{i+1}, x\rangle = b_{i+1},$$

$i = 0, \cdots, l$, where $a^0 = a^m$, $b_0 = b_m$. Then the efficient set of X is given by

$$\bigcup_{i=0}^{l}[x^i, x^{i+1}].$$

Particular case 2: Efficient sets in $\mathbb{R}^3$

If the dimension of X is three, then it may have efficient faces of dimension 0 or 1 or 2. We recall that a point $\bar{x} \in X$ is said to be an ideal efficient point if $\bar{x} \geqq x$ for all $x \in X$. It is easy to see that X does not possess ideal efficient points if and only if it has efficient faces of dimension 1 or 2, or it has no efficient point at all. Now we describe an algorithm to determine the set of all efficient points of $X \subseteq \mathbb{R}^3$. With one exceptional case when $\mathrm{Max}(X)$ consists of only one point, the efficient set $\mathrm{Max}(X)$ can be completely determined if we know all efficient edges.

- *Step 1 (Determine whether X possesses an ideal efficient point).*
  Solve the linear problem

$$\begin{array}{ll}\text{maximize} & \langle e^i, x\rangle \\ \text{subject to} & x \in X.\end{array}$$

  for $i = 1, 2, 3$, where $e^1 = (1, 0, 0)^T$, $e^2 = (0, 1, 0)^T$, $e^3 = (0, 0, 1)^T$. Let x_1^*, x_2^*, x_3^* be the optimal values of these problems. If $x^* = (x_1^*, x_2^*, x_3^*) \in X$, then x^* is an ideal efficient point of X and $\mathrm{Max}(X) = \{x^*\}$;
  Otherwise, go to Step 2.
- *Step 2.* Decompose the index set $\{1, \cdots, m\}$ into I^1, I^2, I^3, where $I^1 = \{i : a^i > 0\}$, $I^3 = \{i : a^i < 0\}$, $I^2 = \{1, \cdots, m\} \setminus (I^1 \cup I^3)$.
  If $I^1 = \emptyset$, then there are no efficient faces of dimension 2. Go to Step 3 to find efficient faces of smaller dimension.
  Otherwise, each $a^i, i \in I^1$ determines an efficient face of dimension 2 by the system

$$\langle a^i, x\rangle = b_i$$
$$\langle a^j, x\rangle \leqq b_j, \ j \in \{1, \cdots, m\}\backslash\{i\}.$$

Go to Step 3 to find efficient faces of smaller dimension, not included in the above 2-dimensional efficient faces.

- *Step 3.* Choose $i, j \in I^2$.
  - *Step 3.1. (Is* $\{i, j\}$ *positive ?)*
    Solve the system

$$\begin{aligned} ta^i + (1-t)a^j &> 0 \\ 1 \geqq t &\geqq 0. \end{aligned}$$

    If it has a solution, then $\{i, j\}$ is positive. Go to Step 3.2.
    Otherwise, $\{i, j\}$ is not positive. Pick other pair $i, j \in I^2$ and return to Step 3.1.
  - *Step 3.2. (Is* $\{i, j\}$ *normal?)*
    Determine the set $\Delta_{ij} := \{x \in X : \langle a^i, x\rangle = b_i, \langle a^j, x\rangle = b_j\}$. If $\Delta_{ij} = \emptyset$ or Δ_{ij} is a point , then either $\{i, j\}$ is not normal or $\dim \Delta_{ij} = 0$. Pick other pair $i, j \in I^2$ and Return to Step 3.1.
    Otherwise Δ_{ij} is a segment. This segment is an efficient edge. Store it. Pick another $i, j \in I^2$ and return Step 3.1.

Remark According to Corollary 8.2.9, Step 2 and Step 3 allow us to generate the entire efficient set of X because other efficient faces are included in those that were found in these steps.

Computing weakly efficient solutions

The normal cone method we presented above to compute efficient solutions of (MOLP) is also suitable to find weakly efficient solutions. The only difference is that the C-positivity must be substituted by the weak C-positivity in all procedures. Namely, we say that an index set I is weakly C-positive if there are nonnegative numbers $\lambda_1, \cdots, \lambda_k$, not all zero, such that $v = \sum_{i=1}^{k} \lambda_i c^i$. Here are some results that provide theoretical basis of the normal method for weakly efficient solutions:

- A face F of X is weakly efficient if and only if its active index set $I(F)$ is weakly C-positive.
- (MOLP) has weakly efficient solutions if and only if the following system is consistent

$$\sum_{i=1}^{m} \mu_i a^i - \sum_{j=1}^{k} \lambda_j c^j = 0$$

$$\mu_i \geqq 0, \quad \lambda_j \geqq 0, \sum_{j=1}^{k} \lambda_j = 1.$$

- If the above system has a solution $(\mu_1, \cdots, \mu_m, \lambda_1 \cdots, \lambda_k)$, then the function $x \mapsto \langle \sum_{j=1}^{k} \lambda_j c^j, x\rangle$ attains its maximum on X and every maximum point is a weakly efficient solution of (MOLP).

8.4 Exercises

8.4.1 Find a counter-example to show that without the non-redundancy hypothesis (A1), an index set I may be normal while the system

$$\begin{aligned} \langle a^i, x\rangle &= b_i,\ i \in I \\ \langle a^j, x\rangle &< b_j,\ j \in \{1, \cdots, m\}\backslash I \end{aligned}$$

is inconsistent.

8.4.2 Consider (MOLP) described in Sect. 8.2 and assume that $I \subseteq \{1, \cdots, m\}$ is a positive and normal index set. Prove that the efficient face determined by the system

$$\begin{aligned} \langle a^i, x\rangle &= b_i,\ i \in I \\ \langle a^j, x\rangle &\leqq b_j,\ j \in \{1, \cdots, m\}\backslash I, \end{aligned}$$

is not a maximal efficient face if the system

$$\begin{aligned} &\sum_{i\in I} \mu_i a^i - \sum_{j=1}^{k} \lambda_j c^j = 0 \\ &\mu_i \geqq 1, i \in I \text{ and } \lambda_j \geqq 1,\ j = 1, \cdots, k, \end{aligned}$$

is inconsistent. Show that the converse statement is not always true.

8.4.3 Find all normal index sets of the following systems:

$$(1) \begin{cases} x_1 - x_2 + 2x_3 \leqq 3 \\ x_1 - 2x_2 \leqq -2 \\ -x_2 \leqq 0 \end{cases} \qquad (2) \begin{cases} x_1 + x_2 + x_3 \leqq 6 \\ 5x_1 + 3x_2 + 6x_3 \leqq 15 \\ -x_1 - x_2 - x_3 \leqq 0. \end{cases}$$

8.4.4 Consider (MOLP) described in Sect. 8.2. Let X be the solution set of a linear system $Ax \leqq b$.

(1) Prove that if the polar cone of the asymptotic cone of X contains a C-positive vector, then positive and normal index sets exist. In particular, when X is bounded, there always exists a positive and normal index set.

(2) Assume that the system $\langle a^i, x\rangle \leqq b_i, i = 1 \cdots, m$ is non-redundant. Prove that it has an $(n-1)$-dimensional efficient face if and only if there is some $i_0 \in$

$\{1, \cdots, m\}$ such that a^{i_0} is C-positive, and that such an index is unique for each $(n-1)$-dimensional efficient face.

8.4.5 Solve the following problem by the normal cone method

$$\begin{aligned} \text{Minimize} \quad & \begin{pmatrix} -x_1 - x_2 - 0.25x_3 \\ x_1 + x_2 + 1.5x_3 \end{pmatrix} \\ \text{subject to} \quad & \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ -1 & -1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \geqq \begin{pmatrix} 2 \\ 2 \\ -6 \end{pmatrix} \\ & x_1, x_2, x_3 \geqq 0. \end{aligned}$$

and prove that the index set $I = \{6\}$ determines a 2-dimensional maximal efficient face whose vertices are $x_1 = (0.67, 0.67, 0)^T$, $x_2 = (2, 0, 0)^T$, $x_3 = (0, 2, 0)^T$, $x_4 = (6, 0, 0)^T$, $x_5 = (0, 6, 0)^T$.

8.4.6 Solve the problem

$$\begin{aligned} \text{Minimize} \quad & \begin{pmatrix} -x_1 + 100x_2 + 0x_3 \\ -x_1 - 100x_2 + 0x_3 \\ 0x_1 + 0x_2 - 1x_3 \end{pmatrix} \\ \text{subject to} \quad & \begin{pmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 5 & 5 & 6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leqq \begin{pmatrix} 10 \\ 10 \\ 30 \end{pmatrix} \\ & x_1, x_2, x_3 \geqq 0. \end{aligned}$$

We have obtained the following list of efficient vertices and faces.

(a) 3 two dimensional efficient faces F_1, F_2, F_3 which are determined by $I(F_1) = \{1\}$, $I(F_2) = \{2\}$, $I(F_3) = \{3\}$;
(b) Face F_1 has 3 efficient vertices : $x_2 = (2, 4, 0)$, $x_4 = (0, 0, 5)$, $x_5 = (0, 5, 0)$ and 3 efficient edges : :$[x_2, x_4]$, $[x_2, x_5]$, $[x_4, x_5]$.
(c) Face F_2 has 3 efficient vertices : $x_1 = (4, 2, 0)$, $x_3 = (5, 0, 0)$, $x_4 = (0, 0, 5)$ and 3 efficient edges : :$[x_1, x_3]$, $[x_1, x_4]$, $[x_3, x_4]$.
(d) Face F_3 has 3 efficient vertices : $x_1 = (4, 2, 0)$, $x_2 = (2, 4, 0)$, $x_4 = (0, 0, 5)$ and 3 efficient edges : :$[x_1, x_2]$, $[x_1, x_4]$, $[x_2, x_4]$.

Write problems to determine

- an initial weakly efficient solution;
- all weakly efficient edges emanating from a given weakly efficient vertex;
- all maximal weakly efficient faces adjacent to a given weakly efficient vertex;
- the weakly efficient solution set of (MOLP).

8.4.7 Consider the following problem

$$\begin{aligned}
&\text{Minimize} \quad \begin{pmatrix} -1 & -1 & -0.25 \\ 1 & 1 & 1.5 \\ 0 & 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
&\text{subject to} \quad \begin{pmatrix} -1 & -1 & 1 \\ -1 & -1 & -1 \\ -2 & -2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \geqq \begin{pmatrix} -3 \\ -5 \\ -8 \end{pmatrix} \\
&\qquad\qquad x_1,\ x_2,\ x_3 \geqq 0.
\end{aligned}$$

Using the normal method establish the following result

(a) 5 weakly efficient vertices:

$$v_1 = (3, 0, 0),\ v_2 = (0, 0, 0),\ v_3 = (0, 3, 0),$$
$$v_4 = (3.667, 0, 0.667),\ v_5 = (0, 3.667, 0.667);$$

(b) 6 weakly efficient edges:

$$[v_1, v_2],\ [v_1, v_3],\ [v_1, v_4],$$
$$[v_2, v_3],\ [v_3, v_5],\ [v_4, v_5];$$

(c) 2 maximal weakly efficient faces of dimension 2:

$$F_1 = \mathrm{co}\{v_1, v_3, v_4, v_5\}$$
$$F_2 = \mathrm{co}\{v_1, v_2, v_3\}.$$

8.4.8 Consider the following problem

$$\begin{aligned}
&\text{Minimize} \quad \begin{pmatrix} -1 & 100 & 0 \\ -1 & -100 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
&\text{subject to} \quad \begin{pmatrix} -1 & -2 & -2 \\ -2 & -1 & -2 \\ -5 & -5 & -6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \geqq \begin{pmatrix} -10 \\ -10 \\ -30 \end{pmatrix} \\
&\qquad\qquad x_1,\ x_2,\ x_3 \geqq 0.
\end{aligned}$$

Using the normal cone method to obtain the following efficient vertices and faces.

(a) 6 weakly efficient vertices:

$$v_1 = (5, 0, 0);\ v_2 = (4, 2, 0);\ v_3 = (0, 0, 5)$$
$$v_4 = (2, 4, 0);\ v_5 = (0, 0, 0);\ v_6 = (0, 5, 0).$$

(b) 8 weakly efficient edges:

$$[v_1, v_2], [v_1, v_3], [v_2, v_4], [v_2, v_3],$$
$$[v_3, v_5],\ [v_4, v_6], [v_4, v_3],\ [v_6, v_3],$$

(c) 3 maximal weakly efficient faces of dimension 2:

$$F_1 = \text{co}\{v_1, v_2, v_3\},$$
$$F_2 = \text{co}\{v_2, v_3, v_4\},$$
$$F_3 = \text{co}\{v_3, v_4, v_6\}.$$