Chapter 4
Pareto Optimality

In a multi-dimensional Euclidean space there are several ways to classify elements of a given set of vectors. The componentwise order relation introduced in the very beginning of the second chapter seems to be the most appropriate for this classification purpose and leads to the concept of Pareto optimality or efficiency, a cornerstone of multiobjective optimization that we are going to study in the present chapter.

4.1 Pareto Maximal Points

In the space $\mathbb{R}^k$ with $k > 1$ the componentwise order $x \geqq y$ signifies that each component of x is bigger than or equal to the corresponding component of y. Equivalently, $x \geqq y$ if and only if the difference vector $x - y$ has non-negative components only. This order is not complete in the sense that not every couple of vectors is comparable, and hence the usual notion of maximum or minimum does not apply. We recall also that $x > y$ means that all components of the vector $x - y$ are strictly positive, and $x \geq y$ signifies $x \geqq y$ and $x \neq y$. The following definition lays the basis for our study of multiobjective optimization problems.

Definition 4.1.1 Let Q be a nonempty set in $\mathbb{R}^k$. A point $y \in Q$ is said to be a *(Pareto) maximal point* of the set Q if there is no point $y' \in Q$ such that $y' \geqq y$ and $y' \neq y$. And it is said to be a *(Pareto) weakly maximal point* if there is no $y' \in Q$ such that $y' > y$.

The sets of maximal points and weakly maximal points of Q are respectively denoted $\text{Max}(Q)$ and $\text{WMax}(Q)$ (Figs. 4.1 and 4.2). They are traditionally called the *efficient* and *weakly efficient sets* or the *non-dominated* and *weakly non-dominated sets* of Q. The set of minimal points $\text{Min}(Q)$ and weakly minimal points $\text{WMin}(Q)$ are defined in a similar manner. When no confusion likely occurs between maximal and minimal elements, the set $\text{Min}(Q)$ and $\text{WMin}(Q)$ are called the *efficient* and *weakly efficient sets* of Q too. The terminology of efficiency is advantageous in certain circumstances

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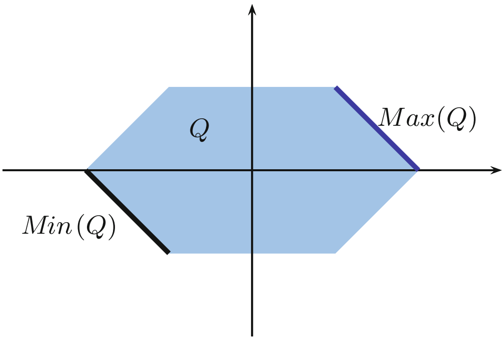

Fig. 4.1 Max and Min

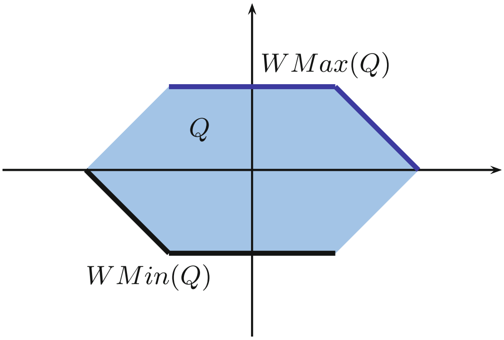

Fig. 4.2 WMax and WMin

in which we deal simultaneously with maximal points of a set as introduced above and maximal elements of a family of subsets which are defined to be maximal with respect to inclusion. Thus, given a convex polyhedron, a face of it is efficient if it consists of maximal points only. When we refer to a maximal efficient face, it is understood that that face is efficient and maximal by inclusion which means that no efficient face of the polyhedron contains it as a proper subset. In some situations one is interested in an *ideal maximal point* (called also a *utopia* point), which is defined to be a point $y \in Q$ that satisfies

$$y \geqq y' \text{ for all } y' \in Q.$$

Such a point is generally unattainable, and if it exists it is unique and denoted by $\text{IMax}(Q)$ (Fig. 4.3).

Geometrically, a point y of Q is an efficient (maximal) point if the intersection of the set Q with the positive orthant shifted at y consists of y only, that is,

$$Q \cap (y + \mathbb{R}^k_+) = \{y\}$$

and it is weakly maximal if the intersection of Q with the interior of the positive orthant shifted at y is empty, that is,

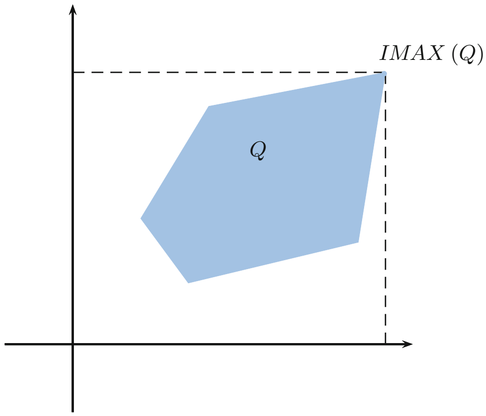

Fig. 4.3 IMax

$$Q \cap (y + \mathrm{int}(\mathbb{R}^k_+)) = \emptyset.$$

Of course, maximal points are weakly maximal, and the converse is not true in general. Here are some examples in $\mathbb{R}^2$.

Example 4.1.2 Let Q be the triangle of vertices $a = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, b = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $c = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ in $\mathbb{R}^2$. Then $\mathrm{Max}(Q) = \mathrm{WMax}(Q) = [b, c]$, $\mathrm{Min}(Q) = \{a\}$ and $\mathrm{WMin}(Q) = [a, b] \cup [a, c]$.

Example 4.1.3 Let Q be the polytope in the space $\mathbb{R}^3$, determined by two inequalities

$$\begin{aligned} y_2 + y_3 &\geqq 0 \\ y_3 &\geqq 0. \end{aligned}$$

Then $\mathrm{Max}(Q) = \mathrm{WMax}(Q) = \emptyset$, $\mathrm{Min}(Q) = \emptyset$ and $\mathrm{WMin}(Q) = Q \setminus \mathrm{int}(Q)$.

Existence of pareto maximal points

As we have already seen in Example 4.1.3, a polyhedron may have no weakly maximal points. This happens when some components of elements of the set are unbounded above. Positive functionals provide an easy test for such situations.

Theorem 4.1.4 *Let Q be a nonempty set and let λ be a nonzero vector in $\mathbb{R}^k$. Assume that $y \in Q$ is a maximizer of the functional $\langle \lambda, . \rangle$ on Q. Then*

(i) *y is a weakly maximal point of Q if λ is a positive vector;*
(ii) *y is a maximal point of Q if either λ is a strictly positive vector, or λ is a positive vector and y is the unique maximizer.*

In particular, if Q is a nonempty compact set, then it has a maximal point.

Proof Assume λ is a nonzero positive vector. If y were not weakly maximal, then there would exist another vector y' in Q such that the vector $y' - y$ is strictly positive. This would yield $\langle \lambda, y' \rangle > \langle \lambda, y \rangle$, a contradiction.

Now, if λ is strictly positive, then for any $y' \geqq y$ and $y' \neq y$, one has $\langle \lambda, y' \rangle > \langle \lambda, y \rangle$ as well. Hence y is a Pareto maximal point of Q.

When λ is positive (not necessarily strictly positive) and not zero, the above inequality is not strict. Actually, we have equality because y is a maximizer. But, in that case y' is also a maximizer of the functional $\langle \lambda, . \rangle$ on Q, which contradicts the hypothesis.

When Q is compact, any strictly positive vector λ produces a maximizer on Q, hence a Pareto maximal point too. □

Maximizers of the functional $\langle \lambda, . \rangle$ with λ positive, but not strictly positive, may produce no maximal points as seen in the following example.

Example 4.1.5 Consider the set Q in $\mathbb{R}^3$ consisting of the vectors $x = (x_1, x_2, x_3)^T$ with $x_3 \leqq 0$. Choose $\lambda = (0, 0, 1)^T$. Then every element x of Q with $x_3 = 0$ is a maximizer of the functional $\langle \lambda, . \rangle$ on Q, hence it is weakly maximal, but not maximal, for the set Q has no maximal element.

Given a reference point a in the space, the set of all elements of a set Q that are bigger than the point a forms a dominant subset, called a *section* of Q at a. The lemma below shows that maximal elements of a section are also maximal elements of the given set.

Lemma 4.1.6 *Let Q be a nonempty set in $\mathbb{R}^k$. Then for every point a in $\mathbb{R}^k$ one has*

$$\mathrm{Max}\left(Q \cap (a + \mathbb{R}^k_+)\right) \subseteq \mathrm{Max}(Q)$$
$$\mathrm{WMax}\left(Q \cap (a + \mathbb{R}^k_+)\right) \subseteq \mathrm{WMax}(Q).$$

Proof Let y be a Pareto maximal point of the section $Q \cap (a + \mathbb{R}^k_+)$. If y were not maximal, then one would find some y' in Q such that $y' \geqq y$ and $y' \neq y$. It would follow that y' belongs to the section $Q \cap (a + \mathbb{R}^k_+)$ and yield a contradiction. The second inclusion is proven by the same argument. □

For convex polyhedra existence of maximal points is characterized by position of asymptotic directions with respect to the positive orthant of the space.

Theorem 4.1.7 *Let Q be a convex polyhedron in $\mathbb{R}^k$. The following assertions hold.*

(i) *Q has maximal points if and only if*

$$Q_\infty \cap \mathbb{R}^k_+ = \{0\}.$$

(ii) *Q has weakly maximal points if and only if*

$$Q_\infty \cap \operatorname{int}(\mathbb{R}^k_+) = \emptyset.$$

In particular, every polytope has a maximal vertex.

Proof Let y be a maximal point of Q and let v be any nonzero asymptotic direction of Q. Since $y + v$ belongs to Q and $Q \cap (y + \mathbb{R}^k_+) = \{y\}$, we deduce that v does not belong to $\mathbb{R}^k_+$. Conversely, assume Q has no nonzero asymptotic direction. Then for a fixed vector y in Q the section $Q \cap (y + \mathbb{R}^k_+)$ is bounded; otherwise any nonzero asymptotic direction of that closed convex intersection, which exists due to Corollary 2.3.16, should be a positive asymptotic vector of Q. In view of Theorem 4.1.4 the compact section $Q \cap (y + \mathbb{R}^k_+)$ possesses a maximal point, hence, in view of Lemma 4.1.6, so does Q.

For the second assertion, the same argument as above shows that when Q has a weakly maximal point, no asymptotic direction of it is strictly positive. For the converse part, by the hypothesis we know that Q_∞ and $\mathbb{R}^k_+$ are two convex polyhedra without relative interior points in common. Hence, in view of Theorem 2.3.10 there is a nonzero vector $\lambda \in \mathbb{R}^k$ separating them, that is

$$\langle \lambda, v \rangle \leqq \langle \lambda, d \rangle \text{ for all } v \in Q_\infty \text{ and } d \in \mathbb{R}^k_+.$$

In particular, for $v = 0$ and for d being usual coordinate unit vectors, we deduce from the above relation that λ is positive. Moreover, the linear function $\langle \lambda, . \rangle$ is then non-positive on every asymptotic direction of Q. We apply Theorem 3.1.1 to obtain a maximum of $\langle \lambda, . \rangle$ on Q. In view of Theorem 4.1.4 that maximum is a weakly maximal point of Q.

Finally, if Q is a polytope, then its asymptotic cone is trivial. Hence, by the first assertion, it has maximal points. To prove that it has a maximal vertex, choose any strictly positive vector $\lambda \in \mathbb{R}^k$ and consider the linear problem of maximizing $\langle \lambda, . \rangle$ over Q. In view of Theorem 3.1.3 the optimal solution set contains a vertex, which, by Theorem 4.1.4, is also a maximal vertex of Q. □

In Example 4.1.5 a positive functional $\langle \lambda, . \rangle$ was given on a polyhedron having no maximizer that is maximal. This, however, is impossible when the polyhedron has maximal elements.

Corollary 4.1.8 *Assume that Q is a convex polyhedron and λ is a nonzero positive vector in $\mathbb{R}^k$. If Q has a maximal point and the linear functional $\langle \lambda, . \rangle$ has maximizers on Q, then among its maximizers there is a maximal point of Q.*

Proof Let us denote by Q_0 the nonempty intersection of Q with the hyperplane $\{y \in \mathbb{R}^k : \langle \lambda, y \rangle = d\}$ where d is the maximum of $\langle \lambda, . \rangle$ on Q. It is a convex polyhedron. Since Q has maximal elements, in view of Theorem 4.1.7 one has $Q_\infty \cap \mathbb{R}^k_+ = \{0\}$, which implies that $(Q_0)_\infty \cap \mathbb{R}^k_+ = \{0\}$ too. By the same theorem, Q_0 has a maximal element, say y_0. We show that this y_0 is also a maximal element of Q. Indeed, if not, one could find some $y \in Q$ such that $y \geqq y_0$ and $y \neq y_0$. Since λ is positive, we deduce that $\langle \lambda, y \rangle \geqq \langle \lambda, y_0 \rangle = d$. Moreover, as y does not belong to Q_0, this inequality must be strict which is a contradiction. □

We say a set Q in the space $\mathbb{R}^k$ has the *domination property* if its elements are dominated by maximal elements, that is, for every $y \in Q$ there is some maximal element a of Q such that $a \geqq y$. The weak domination property refers to domination by weakly maximal elements.

Corollary 4.1.9 *A convex polyhedron has the domination property (respectively weak domination property) if and only if it has maximal elements (respectively weakly maximal elements).*

Proof The "only if" part is clear. Assume a convex polyhedron Q has maximal elements. In view of Theorem 4.1.7, the asymptotic cone of Q has no nonzero vector in common with the positive orthant $\mathbb{R}^k_+$. Hence so does the section of Q at a given point $a \in Q$. Again by Theorem 4.1.7 that section has maximal points that dominate a and by Lemma 4.1.6 they are maximal points of Q. Hence Q has the domination property. The weak domination property is proven by the same argument. □

We learned in Sect. 2.3 how to compute the normal cone at a given point of a polyhedron. It turns out that by looking at the normal directions it is possible to say whether a given point is maximal or not.

Theorem 4.1.10 *Let Q be a convex polyhedron in $\mathbb{R}^k$. The following assertions hold.*

(i) *$y \in Q$ is a maximal point if and only if the normal cone $N_Q(y)$ to Q at y contains a strictly positive vector.*
(ii) *$y \in Q$ is a weakly maximal point if and only if the normal cone $N_Q(y)$ to Q at y contains a nonzero positive vector.*

Proof Let y be a point in Q. If the normal cone to Q at y contains a strictly positive vector, say λ, then by the definition of normal vectors, the functional $\langle \lambda, . \rangle$ attains its maximum on Q at y. In view of Theorem 4.1.4, y is a maximal point of Q. The proof of the "only if" part of (i) is based on Farkas' theorem. We assume that y is a maximal point of Q and suppose to the contrary that the normal cone to Q at that point has no vector in common with the interior of the positive orthant $\mathbb{R}^k_+$. We may assume that Q is given by a system of inequalities

$$\langle a^i, z \rangle \leqq b_i, \ i = 1, \cdots, m. \tag{4.1}$$

The active index set at y is denoted $I(y)$. By Theorem 2.3.24, the normal cone to Q at y is the positive hull of the vectors $a^i, i \in I(y)$. Its empty intersection with $\mathrm{int}(\mathbb{R}^k_+)$ means that the following system has no solution

$$\begin{aligned} A_{I(y)}\lambda &\geqq e \\ \lambda &\geqq 0, \end{aligned}$$

where $A_{I(y)}$ denotes the matrix whose columns are a^i, $i \in I(y)$ and e is the vector whose components are all equal to one. By introducing artificial variables z, the above system is equivalent to the system

$$\begin{aligned} \left[A_{I(y)}\ (-I)\right]\begin{pmatrix}\lambda\\ z\end{pmatrix} &= e \\ \lambda &\geqq 0 \\ z &\geqq 0. \end{aligned}$$

Apply Farkas' theorem (Theorem 2.2.3) to obtain a nonzero positive vector v such that

$$\langle a^i, v\rangle \leqq 0 \ \text{ for all } i \in I(y).$$

The inequalities (4.1) corresponding to the inactive indices at y being strict, we may find a strictly positive number t such that

$$\langle a^i, y + tv\rangle \leqq b_i \ \text{ for all } i = 1, \cdots, m.$$

In other words, the point $y + tv$ belongs to Q. Moreover, $y + tv \geqq y$ and $y + tv \neq y$ which contradicts the hypothesis. This proves (i).

As to the second assertion, the "if" part is clear, again, Theorem 4.1.4 is in use. For the converse part, we proceed the same way as in (i). The fact that the intersection of $N_Q(y)$ with the positive orthant $\mathbb{R}^k_+$ consists of the zero vector only, means that the system

$$\begin{aligned} A_{I(y)}\lambda &\geqq 0 \\ \lambda &\geqq 0 \end{aligned}$$

has no nonzero solution. Applying Corollary 2.2.5 we deduce the existence of a strictly positive vector v such that

$$\langle a^i, v\rangle \leqq 0 \ \text{ for all } i \in I(y).$$

Then, as before, the vector $y + tv$ with $t > 0$ sufficiently small, belongs to Q and $y + tv > y$, which is a contradiction. □

Example 4.1.11 Consider a convex polyhedron Q in $\mathbb{R}^3$ determined by the system

$$\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \leqq \begin{pmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 1 \end{pmatrix}.$$

We analyze the point $y = (1/3, 1/3, 1/3)^T \in Q$. Its active index set is $I(y) = \{1\}$. By Theorem 2.3.24 the normal cone to Q at that point is generated by the vector $(1, 1, 1)^T$. According to Theorem 4.1.10 the point y is a maximal point of Q. Now, take another point of Q, say $z = (-1, 0, 1)^T$. Its active index set consists of two indices 2 and 5. The normal cone to Q at z is generated by two directions $(0, 1, 1)^T$ and $(0, 0, 1)^T$. It is clear that this normal cone contains no strictly positive vector, hence the point z is not a maximal point of Q because $z^T \leq (0, 0, 1)^T$. It is a weakly maximal point, however, because normal directions at z are positive. Finally, we choose a point $w = (0, 0, 0)^T$ in Q. Its active index set is $I(w) = \{4\}$. The normal cone to Q at w is the cone generated by the direction $(0, 0, -1)^T$. This cone contains no positive vector, hence the point w is not weakly maximal. This can also be seen from the fact that w is strictly dominated by y.

Scalarizing vectors

In remaining of this section we shall use the terminology of efficient points instead of (Pareto) maximal points in order to avoid possible confusion with the concept of maximal element of a family of sets by inclusion. Given a family $\{A_i : i \in I\}$ of sets, we say that A_{i_0} is maximal (respectively minimal) if there is no element A_i of the family such that $A_i \neq A_{i_0}$ and $A_{i_0} \subset A_i$ (respectively $A_{i_0} \supset A_i$). Another formulation of Theorem 4.1.10 is viewed by maximizing linear functionals on the set Q.

Corollary 4.1.12 *Let Q be a convex polyhedron in $\mathbb{R}^k$. Then the following statements hold.*

(i) *$y \in Q$ is an efficient point if and only if there is a strictly positive vector $\lambda \in \mathbb{R}^k$ such that y maximizes the functional $\langle \lambda, . \rangle$ on Q.*
(ii) *$y \in Q$ is a weakly efficient point if and only if there is a nonzero positive vector $\lambda \in \mathbb{R}^k$ such that y maximizes the functional $\langle \lambda, . \rangle$ on Q.*

Proof This is immediate from the definition of normal cones and from Theorem 4.1.10. □

The vector λ mentioned in this corollary is called a *scalarizing vector* (or *weakly scalarizing vector* in (ii)) of the set Q. We remark that not every strictly positive vector is a scalarizing vector of Q like not every strictly positive functional attains its maximum on Q. Moreover, an efficient point of Q may maximize a number of scalarizing vectors that are linearly independent, and vice versa, a scalarizing vector may determine several maximizers on Q. For a given polyhedron Q that has efficient elements, the question of how to choose a vector λ so that the functional associated with it furnishes a maximizer is not evident. Analytical choice of positive directions such as the one discussed in Example 4.1.11 is conceivable and will be given in details later. Random generating methods or uniform divisions of the standard simplex do not work in many instances. In fact, look at a simple problem of finding efficient points of the convex polyhedral set given by the inequality

$$x_1 + \sqrt{2}x_2 \leqq 1$$

in the two-dimensional space $\mathbb{R}^2$. Except for one direction, every positive vector λ leads to a linear problem of maximizing $\langle \lambda, x \rangle$ over that polyhedron with unbounded objective. Hence, using positive vectors λ_i of a uniform partition

$$\lambda_i = \frac{i}{p}\begin{pmatrix}1\\0\end{pmatrix} + \frac{p-i}{p}\begin{pmatrix}0\\1\end{pmatrix}$$

of the simplex $[(1,0)^T, (0,1)^T]$ of the space $\mathbb{R}^2$ for whatever the positive integer p be, will never generate efficient points of the set.

Any nonzero positive vector of the space $\mathbb{R}^k$ is a positive multiple of a vector from the standard simplex Δ. This combined with Corollary 4.1.12 yields the following equalities

$$\mathrm{Max}(Q) = \bigcup_{\lambda \in \mathrm{ri}\Delta} \mathrm{argmax}_Q \langle \lambda, . \rangle$$

$$\mathrm{WMax}(Q) = \bigcup_{\lambda \in \Delta} \mathrm{argmax}_Q \langle \lambda, . \rangle$$

where $\mathrm{argmax}_Q\langle \lambda, . \rangle$ is the set of all maximizers of the functional $\langle \lambda, . \rangle$ on Q. Given a point $y \in Q$ denote

$$\Delta_y = \left\{\lambda \in \Delta : y \in \mathrm{argmax}_Q \langle \lambda, . \rangle\right\}$$

$$\Delta_Q = \bigcup_{y \in Q} \Delta_y.$$

The set Δ_Q is called the *weakly scalarizing set* of Q and Δ_y is the *weakly scalarizing set* of Q at y (Fig. 4.4). By Corollary 4.1.12 the set Δ_y is nonempty if and only if the point y is a weakly efficient element of Q. Hence when Q has weakly efficient points, the set Δ_Q can be expressed as

$$\Delta_Q = \bigcup_{y \in \mathrm{WMax(Q)}} \Delta_y, \tag{4.2}$$

in which every set Δ_y is nonempty. By definition a vector $\lambda \in \Delta$ belongs to Δ_y if and only if

$$\langle \lambda, y' - y \rangle \leqq 0 \text{ for all } y' \in Q.$$

The latter inequality signifies that λ is a normal vector to Q at y, and so (4.2) becomes

$$\Delta_Q = \bigcup_{y \in \mathrm{WMax(Q)}} N_Q(y) \cap \Delta.$$

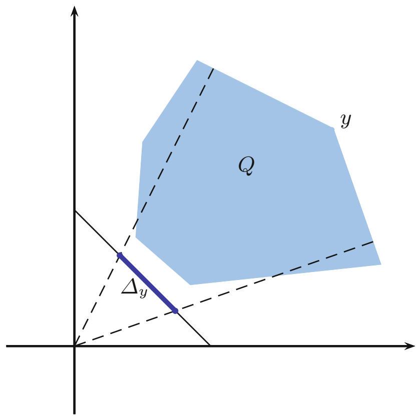

Fig. 4.4 Scalarizing set at y

Let $\mathcal{F} = \{F_1, \cdots, F_q\}$ be the collection of all faces of Q and let $N(F_i)$ be the normal cone to F_i, which, by definition, is the normal cone to Q at a relative interior point of F_i. Since each element of Q is a relative interior point of some face, the decomposition (4.2) produces the following decomposition of Δ_Q:

$$\Delta_Q = \bigcup_{i \in I} \Delta_i, \tag{4.3}$$

where $\Delta_i = N(F_i) \cap \Delta$ and I is the set of those indices i from $\{1, \cdots, q\}$ such that the faces F_i are weakly efficient. We note that when a face is not weakly efficient, the normal cone to it does not meet the simplex Δ. Remember that a face of Q is *weakly efficient* if all elements of it are weakly efficient elements of Q, or equivalently if a relative interior point of it is a weakly efficient element. A face that is not weakly efficient may contain weakly efficient elements on its proper faces.

We say a face of Q is a *maximal weakly efficient face* if it is weakly efficient and no weakly efficient face of Q contains it as a proper subset. It is clear that when a convex polyhedron has weakly efficient elements, it does have maximal weakly efficient faces. Below we present some properties of the decompositions (4.2) and (4.3) of the weakly scalarizing set.

Lemma 4.1.13 *If P and Q are convex polyhedra with $P \cap Q \neq \emptyset$, then there are faces $P' \subseteq P$ and $Q' \subseteq Q$ such that $P \cap Q = P' \cap Q'$ and* $\mathrm{ri}(P') \cap \mathrm{ri}(Q') \neq \emptyset$. *Moreover, if the interior of Q is nonempty and contains some elements of P, then* $\mathrm{ri}(P) \cap \mathrm{int}(Q) \neq \emptyset$ *and* $\mathrm{ri}(P \cap Q) = \mathrm{ri}(P) \cap \mathrm{int}(Q)$.

Proof Let x be a relative interior point of the intersection $P \cap Q$. Let $P' \subseteq P$ and $Q' \subseteq Q$ be faces that contain x in their relative interiors. These faces meet the requirements of the lemma. Indeed, it suffices to show that every point y from $P \cap Q$ belongs to $P' \cap Q'$. Since x is a relative interior point of $P \cap Q$, the segment $[x - \varepsilon(x - y), x + \varepsilon(x - y)]$ belongs to that intersection when $\varepsilon > 0$ is sufficiently small. Moreover, as P' is a face, this segment must lie in P' which implies that y lies in P'. The same argument shows that y lies in Q', proving the first part of the lemma.

For the second part it suffices to observe that P is the closure of its relative interior. Hence it has relative interior points inside the interior of Q. The last equality of the conclusion is then immediate. □

Theorem 4.1.14 *The weakly scalarizing set Δ_Q is a polytope. Moreover, if Δ_Q is nonempty, the elements of the decomposition (4.2) and (4.3) are polytopes and satisfy the following conditions:*

(i) *If $\Delta_y = \Delta_z$ for some weakly efficient elements y and z, then there is $i \in I$ such that $y, z \in F_i$ and $\Delta_y = \Delta_z = \Delta_i$.*
(ii) *If F_i is a maximal weakly efficient face of Q, then Δ_i is a minimal element of the decomposition (4.3). Conversely, if the polytope Δ_i is minimal among the polytopes of the decomposition (4.3), then there is a maximal weakly efficient face F_j such that $\Delta_j = \Delta_i$.*
(iii) *For all $i, j \in I$ with $i \neq j$, one has either $\Delta_i = \Delta_j$ or $\mathrm{ri}(\Delta_i) \cap \mathrm{ri}(\Delta_j) = \emptyset$.*
(iv) *Let F_i and F_j be two weakly efficient adjacent vertices (zero-dimensional faces) of Q. Then the edge joining them is weakly efficient if and only if $\Delta_i \cap \Delta_j \neq \emptyset$.*

Proof Since Δ_y is empty when y is not a weakly efficient point of Q, we may express Δ_Q as

$$\Delta_Q = \bigcup_{y\in Q} \Delta_y = \bigcup_{y\in Q} (N_Q(y) \cap \Delta) = N_Q \cap \Delta$$

which proves that Δ_Q is a bounded polyhedron because the normal cone N_Q is a polyhedral cone. Likewise, the sets $\Delta_y = N_Q(y) \cap \Delta$ and $\Delta_i = N(F_i) \cap \Delta$ are convex polytopes.

To establish (i) we apply Lemma 4.1.13 to the intersections $\Delta_y = N_Q(y) \cap \Delta$ and $\Delta_z = N_Q(z) \cap \Delta$. There exist faces $N \subseteq N_Q(y)$, $M \subseteq N_Q(z)$ and $\Delta^y, \Delta^z \subseteq \Delta$ such that

$$N_Q(y) \cap \Delta = N \cap \Delta^y, \quad \mathrm{ri}(N) \cap \mathrm{ri}(\Delta^y) \neq \emptyset$$
$$N_Q(z) \cap \Delta = M \cap \Delta^z, \quad \mathrm{ri}(M) \cap \mathrm{ri}(\Delta^z) \neq \emptyset.$$

Choose any vector ξ from the relative interior of Δ_y. Then it is also a relative interior vector of the faces N, M, Δ^y and Δ^z. This implies that $N = M$ and $\Delta^y = \Delta^z$. Using Theorem 2.3.26 we find a face F_i of Q such that $N(F_i) = N$. Then F_i contains y and z and satisfies

$$\Delta_i = N(F_i) \cap \Delta = N \cap \Delta = \Delta_y = \Delta_z.$$

For (ii) assume F_i is a maximal weakly efficient face. Assume that Δ_j is a subset of Δ_i for some $j \in I$. We choose any vector ξ from Δ_j and consider the face F'

consisting of all maximizers of $\langle \xi, . \rangle$ on Q. Then F' is a weakly efficient face and contains F_j and F_i. As F_i is maximal, we must have $F' = F_i$. Thus, $F_j \subseteq F_i$ and

$$\Delta_i = N(F_i) \cap \Delta \subseteq N(F_j) \cap \Delta = \Delta_j.$$

Conversely, let Δ_i be a minimal element among the polytopes Δ_j, $j \in I$. If F_i is maximal weakly efficient face, we are done. If it is not, we find a maximal weakly efficient face F_j containing F_i. Then $\Delta_j = N(F_j) \cap \Delta \subseteq N(F_i) \cap \Delta = \Delta_i$ and $\Delta_j = \Delta_i$ by hypothesis.

We proceed to (iii). Assume that the relative interior of Δ_i and the relative interior of Δ_j have a vector ξ in common. In view of Lemma 4.1.13 one can find four faces: N^i of $N(F_i)$, N^j of $N(F_j)$, Δ^i and Δ^j of Δ such that

$$N(F_i) \cap \Delta = N^i \cap \Delta^i,\ \mathrm{ri}(N^i) \cap \mathrm{ri}(\Delta^i) \neq \emptyset$$
$$N(F_j) \cap \Delta = N^j \cap \Delta^j,\ \mathrm{ri}(N^j) \cap \mathrm{ri}(\Delta^j) \neq \emptyset.$$

According to Theorem 2.3.26 there are faces F_ℓ and F_m of Q which respectively contain F_i and F_j with $N(F_\ell) = N^i$ and $N(F_m) = N^j$. Then ξ is a relative interior vector of the faces $N(F_\ell)$, $N(F_m)$, Δ^i and Δ^j. We deduce that the face Δ^i coincides with Δ^j, and F_ℓ coincides with F_m. Consequently, $\Delta_i = \Delta_j$.

To prove the last property we assume F_i and F_j are adjacent vertices (zero-dimensional faces) of Q. Let a one-dimensional face F_l be the edge joining them. According to Corollary 2.3.28 we have $N(F_l) = N(F_i) \cap N(F_j)$. Then $\Delta_l = \Delta_i \cap \Delta_j$ which shows that F_l is weakly efficient if and only if the latter intersection is nonempty. □

Note that two different faces of Q may have the same weakly scalarizing set. For instance the singleton $\{(0, 0, 1)^T\}$ is the weakly scalarizing set for all weakly efficient faces of the polyhedron $\mathbb{R}^2_+ \times \{0\}$ in $\mathbb{R}^3$.

In order to treat efficient elements of Q we need to work with the relative interior of Δ. Corresponding notations will be set as follows

$$\Delta^r_Q = \Delta_Q \cap \mathrm{ri}(\Delta)$$
$$\Delta^r_y = \Delta_y \cap \mathrm{ri}(\Delta)$$
$$\Delta^r_i = \Delta_i \cap \mathrm{ri}(\Delta).$$

The set Δ^r_Q is called the *scalarizing set* of Q. It is clear that $y \in Q$ is efficient if and only if Δ^r_y is nonempty, and it is weakly efficient, but not efficient if and only if Δ_y lies on the border of Δ. The decompositions of the weakly scalarizing set induce the following decompositions of the scalarizing set

$$\Delta^r_Q = \bigcup_{y \in \mathrm{Max(Q)}} \Delta^r_y \tag{4.4}$$

and

$$\Delta_Q^r = \bigcup_{i \in I_0} \Delta_i^r \tag{4.5}$$

where I_0 consists of those indices i from $\{1, \cdots, q\}$ for which F_i are efficient.

Theorem 4.1.15 *Assume that the scalarizing set Δ_Q^r is nonempty. Then*

$$\Delta_Q = \mathrm{cl}(\Delta_Q^r).$$

Moreover the decompositions (4.4) and (4.5) of Δ_Q^r satisfy the following properties:

(i) *If $\Delta_y^r = \Delta_z^r$ for some efficient elements y and z, then there is $s \in I_0$ such that $y, z \in \mathrm{ri}(F_s)$ and $\Delta_y^r = \Delta_z^r = \Delta_s^r$.*
(ii) *For $i \in I_0$ the face F_i is a maximal efficient face if and only if Δ_i^r is a minimal element of the decomposition (4.5).*
(iii) *For all $i, j \in I_0$ with $i \neq j$, one has $\mathrm{ri}(\Delta_i^r) \cap \mathrm{ri}(\Delta_j^r) = \emptyset$.*
(iv) *Let F_i and F_j be two efficient adjacent vertices (zero-dimensional efficient faces) of Q. Then the edge joining them is efficient if and only if $\Delta_i^r \cap \Delta_j^r \neq \emptyset$.*

Proof Since the set Δ_Q^r is nonempty, the set Δ_Q does not lie on the border of Δ. Being a closed convex set, Δ_Q is the closure of its relative interior. Hence the relative interior of Δ_Q and the relative interior of Δ have at least one point in common and we deduce

$$\begin{aligned}\Delta_Q = \Delta_Q \cap \Delta &= \mathrm{cl}\big(\mathrm{ri}(\Delta_Q \cap \Delta)\big)\\ &= \mathrm{cl}\big(\mathrm{ri}\Delta_Q \cap \mathrm{ri}\Delta\big) \subseteq \mathrm{cl}\big(\Delta_Q \cap \mathrm{ri}\Delta\big)\\ &\subseteq \mathrm{cl}\big(\Delta_Q^r\big).\end{aligned}$$

The converse inclusion being evident, we obtain equality $\Delta_Q = \mathrm{cl}(\Delta_Q^r)$.

To prove (i) we apply the second part of Lemma 4.1.13 to have

$$\begin{aligned}\mathrm{ri}\big[\mathrm{cone}(\Delta_y^r)\big] &= \mathrm{ri}\big[N_Q(y) \cap \mathbb{R}_+^k\big] = \mathrm{ri}\big[N_Q(y)\big] \cap \mathrm{int}(\mathbb{R}_+^k)\\ \mathrm{ri}\big[\mathrm{cone}(\Delta_z^r)\big] &= \mathrm{ri}\big[N_Q(z) \cap \mathbb{R}_+^k\big] = \mathrm{ri}\big[N_Q(z)\big] \cap \mathrm{int}(\mathbb{R}_+^k).\end{aligned}$$

If y and z were relative interior points of two different faces, in view of Theorem 2.3.26 we would have $\mathrm{ri}[N_Q(y)] \cap \mathrm{ri}[N_Q(z)] = \emptyset$ that contradicts the hypothesis. Hence they are relative interior points of the same face, say F_s. By definition $N(F_s) = N_Q(y)$ and we deduce $\Delta_s^r = \Delta_y^r$.

For (ii) assume F_i is a maximal efficient face. If for some $j \in I_0$ one has $\Delta_j^r \subseteq \Delta_i^r$, then by Lemma 4.1.13 there is some strictly positive vector that lies in the relative interior of the normal cone $N(F_j)$ and in the normal cone $N(F_i)$. We deduce that either $F_i = F_j$, or F_i is a proper face of F_j. The last case is impossible because F_j is also an efficient face and F_i is maximal. Conversely, if F_i is not maximal, then there is a face F_j that is efficient and contains F_i as a proper face. We have $\Delta_j \subseteq \Delta_i$. This inclusion is strict because the relative interiors of $N(F_i)$ and $N(F_j)$ do not meet each other. Thus, Δ_i is not minimal.

We proceed to (iii). If $\mathrm{ri}(\Delta_i^r) \cap \mathrm{ri}(\Delta_j^r) \neq \emptyset$, in view of Theorem 4.1.14 one has $\Delta_i = \Delta_j$, and hence $\Delta_i^r = \Delta_j^r$. By (i), there is some face that contains relative interior points of F_i and F_j in its relative interior. This implies $F_i = F_j$ a contradiction.

For the last property we know that the normal cone to the edge joining the vertices F_i and F_j satisfies $N([F_i, F_j]) = N(F_i) \cap N(F_j)$. Hence the edge $[F_i, F_j]$ is efficient if and only if the normal cone to it meets the set $\mathrm{ri}(\Delta)$, or equivalently $\Delta_i^r \cap \Delta_j^r$ is nonempty. □

A practical way to compute the weakly scalarizing set is to solve a system of linear equalities when the polyhedron Q is given by a system of linear inequalities.

Corollary 4.1.16 *Assume the polyhedron Q in $\mathbb{R}^k$ is determined by the system*

$$\langle a^i, y \rangle \leqq b_i, i = 1, \cdots, m.$$

Then for every $y \in Q$, the set Δ_y consists of all solutions z to the following system

$$\begin{gathered} z_1 + \cdots + z_k = 1 \\ \sum_{i \in I(y)} \alpha_i a^i = z \\ z_i \geqq 0, i = 1, \cdots, k, \alpha_i \geqq 0, i \in I(y). \end{gathered}$$

In particular the weakly scalarizing set Δ_Q is the solution set to the above system with $I = \{1, \cdots, m\}$.

Proof According to Theorem 2.3.24 the normal cone to Q at y is the positive hull of the vectors $a^i, i \in I(y)$. Hence the set Δ_y is the intersection of the positive hull of these vectors and the simplex Δ, which is exactly the solution set to the system described in the corollary. For the second part of the corollary it suffices to observe that the normal cone of Q is the polar cone of the asymptotic cone of Q (Theorem 2.3.26) which, in view of Theorem 2.3.19, is the positive hull of the vectors $a^i, i = 1, \cdots, m$. □

Example 4.1.17 Consider the polyhedron defined by

$$\begin{aligned} y_1 - y_2 - y_3 &\leqq 1 \\ 2y_1 \qquad + y_3 &\leqq 0. \end{aligned}$$

By Corollary 4.1.16 the weakly scalarizing set Δ_Q is the solution set to the system

$$\begin{aligned} z_1 + z_2 + z_3 &= 1 \\ \alpha_1 + 2\alpha_2 &= z_1 \\ -\alpha_1 &= z_2 \\ -\alpha_1 + \alpha_2 &= z_3 \\ z_1, z_2, z_3, \alpha_1, \alpha_2 &\geqq 0. \end{aligned}$$

This produces a unique solution z with $z_1 = 2/3$, $z_2 = 0$ and $z_3 = 1/3$. Then Δ_Q consists of this solution only. The scalarizing set Δ^r_Q is empty, which shows that Q has no efficient point. Its weakly efficient set is determined by the problem

$$\begin{array}{ll} \text{maximize} & \frac{2}{3}y_1 + \frac{1}{3}y_3 \\ \text{subject to} & y \in Q. \end{array}$$

It follows from the second inequality determining Q that the maximum value of the objective function is zero and attained on the face given by the equations $2y_1 + y_3 = 0$ and $y_1 - y_2 - y_3 \leqq 1$.

In the next example we show how to compute the scalarizing set when the polyhedron is given by a system of equalities (see also Exercise 4.4.13 at the end of this chapter).

Example 4.1.18 Let Q be a polyhedron in $\mathbb{R}^3$ determined by the sytem

$$\begin{aligned} y_1 + y_2 + y_3 &= 1 \\ y_1 - y_2 &= 0 \\ y_3 &\geqq 0. \end{aligned}$$

We consider the solution $\overline{y} = (1/2, 1/2, 0)^T$ and want to compute the scalarizing set at this solution if it exists. As the proof of Theorem 4.1.14 indicates, a vector $\lambda \in \Delta$ in $\mathbb{R}^3$ is a weakly scalarizing vector of Q at $\overline{y}$ if and only if it is normal to Q at that point. Since the last component of $\overline{y}$ is zero, a vector λ is normal to Q at $\overline{y}$ if and only if there are real numbers α, β and a positive number γ such that

$$\lambda = \alpha \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + \beta \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} - \gamma \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

We deduce $\lambda \in \Delta_{\overline{y}}$ if and only if

$$\begin{aligned} \alpha + \beta &\geqq 0 \\ \alpha - \beta &\geqq 0 \end{aligned}$$

$$\alpha - \gamma \geqq 0$$
$$(\alpha + \beta) + (\alpha - \beta) + (\alpha - \gamma) = 1$$

and hence $\Delta_{\bar{y}}$ consists of vectors λ whose components satisfy $0 \leqq \lambda_3 \leqq 1/3$, $\lambda_1 + \lambda_2 = 1 - \lambda_3$ and $\lambda_1, \lambda_2 \geqq 0$.

To obtain the scalarizing vectors, it suffices to choose λ as above with an additional requirement that $\lambda_i > 0$ for $i = 1, 2, 3$.

Structure of the set of efficient points

Given a convex polyhedron Q in the space $\mathbb{R}^k$, the set of its efficient elements is not simple. For instance, it is generally not convex, and an edge of it is not necessarily efficient even if its two extreme end-points are efficient vertices. Despite of this, a number of nice properties of this set can be scrutinized.

Corollary 4.1.19 *Let Q be a convex polyhedron in $\mathbb{R}^k$. The following statements hold.*

(i) *If a relative interior point of a face of Q is efficient or weakly efficient, then so is every point of that face.*
(ii) *If Q has vertices, it has an efficient vertex (respectively a weakly efficient vertex) provided that it has efficient (respectively weakly efficient) elements.*

Proof Since the normal cone to Q at every point of a face contains the normal cone at a relative interior point, the first statement follows directly from Theorem 4.1.10.

For the second statement let y be an efficient point of the polyhedron Q. By Theorem 4.1.10 one can find a strictly positive vector λ such that y is a maximizer of the linear functional $\langle \lambda, . \rangle$ on Q. The face which contains y in its relative interior maximizes the above functional. According to Corollary 2.3.14 there is a vertex of Q inside that face and in view of Theorem 4.1.10 this vertex is an efficient point of Q. The case of weakly efficient points is proven by the same argument. □

A subset P of $\mathbb{R}^k$ is called *arcwise connected* if for any pair of points y and z in P, there are a finite number of points $y^0, \cdots, y^\ell$ in P such that $y^0 = y$, $y^\ell = z$ and the segments $[y^i, y^{i+1}]$, $i = 0, \cdots, \ell - 1$ lie all in P.

Theorem 4.1.20 *The sets of all efficient points and weakly efficient points of a convex polyhedron consist of faces of the polyhedron and are closed and arcwise connected.*

Proof By analogy, it suffices to prove the theorem for the efficient set. According to Corollary 4.1.19, if a point $\bar{y}$ in Q is efficient, then the whole face containing y in its relative interior is a face of efficient elements. Hence, $\mathrm{Max}(Q)$ consists of faces of Q if it is nonempty. Moreover, as faces are closed, their union is a closed set.

Now we prove the connectedness of this set by assuming that Q has efficient elements. Let y and z be any pair of efficient points of Q. We may assume without loss of generality that y is a relative interior point of a face Q_y and z is a relative interior point of a face Q_z. Consider the decomposition (4.5) of the scalarizing set

Δ^r_Q. For a face F of Q, the scalarizing set $N(F) \cap \mathrm{ri}(\Delta)$ is denoted by $\Delta^r_{i(F)}$. Let λ_y be a relative interior point of the set $\Delta^r_{i(Q_y)}$ and λ_z a relative interior point of $\Delta^r_{i(Q_z)}$. Then the segment joining λ_y and λ_z lies in Δ^r_Q. The decomposition of the latter set induces a decomposition of the segment $[\lambda_y, \lambda_z]$ by $[\lambda_i, \lambda_{i+1}]$, $i = 0, \cdots, \ell - 1$ where $\lambda_0 = \lambda_y$, $\lambda_\ell = \lambda_z$. Let $Q_1, \cdots, Q_\ell$ be faces of Q such that

$$[\lambda_j, \lambda_{j+1}] \subseteq \Delta^r_{i(Q_{j+1})} \; j = 0, \cdots, \ell - 1.$$

For every j, we choose a relative interior point y^j of the face Q_j. Then λ_j belongs to the normal cones to Q at y^j and y^{j+1}. Consequently, the points y^j and y^{j+1} lie in the face $\mathrm{argmax}_Q\langle\lambda_j, .\rangle$ and so does the segment joining them. As $\lambda_j \in \Delta^r_Q$, by Theorem 4.1.10 the segment $[y^j, y^{j+1}]$ consists of efficient points of Q. Moreover, as the vector λ_0 belongs to the normal cones to Q at y and at y_1, we conclude that the segment $[y, y^1]$ is composed of efficient points of Q. Similarly we have that $[y^{\ell-1}, z]$ lies in the set $\mathrm{Max}(Q)$. Thus, the union $[y, y^1] \cup [y^1, y^2] \cup \cdots [y^{\ell-1}, z]$ forms a path of efficient elements joining y and z. This completes the proof. □

We know that every efficient point of a convex polyhedron is contained in a maximal efficient face. Hence the set of efficient points is the union of maximal efficient faces. Dimension of a maximal efficient face may vary from zero to $k - 1$.

Corollary 4.1.21 *Let Q be a convex polyhedron in $\mathbb{R}^k$. The following statements hold.*

(i) *Q has a zero-dimensional maximal efficient face if and only if its efficient set is a singleton.*
(ii) *Every $(k-1)$-dimensional efficient face of Q, if any exists, is maximal. In particular in the two dimensional space $\mathbb{R}^2$ every efficient edge of Q is maximal if the efficient set of Q consists of more than two elements.*
(iii) *An efficient face F of Q is maximal if and only if the restriction of the decomposition of Δ^r_Q on Δ_F consists of one element only.*

Proof The first statement follows from the arcwise connectedness of the efficient set of Q. In $\mathbb{R}^k$ a proper face of Q is of dimension at most $k - 1$. Moreover, a k-dimensional polyhedron cannot be efficient, for its interior points are not maximal. Hence, if the dimension of an efficient face is equal to $k - 1$, it is maximal.

The last statement follows immediately from Theorem 4.1.15. □

Example 4.1.22 Let Q be a polyhedron in $\mathbb{R}^3$ defined by the system

$$\begin{aligned} x_1 \quad\quad + x_3 &\leqq 1 \\ x_2 + x_3 &\leqq 1 \\ x_1, x_2, x_3 &\geqq 0. \end{aligned}$$

Since Q is bounded, it is evident that the weakly scalarizing set Δ_Q is the whole standard simplex Δ and the scalarizing set is the relative interior of Δ. Denote

$$q^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix},\ q^2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix},\ q^3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},\ q^4 = \begin{pmatrix} 1/2 \\ 0 \\ 1/2 \end{pmatrix},\ q^5 = \begin{pmatrix} 0 \\ 1/2 \\ 1/2 \end{pmatrix}.$$

Applying Corollary 4.1.16 we obtain the following decomposition of Δ^r_Q:

(i) $\mathrm{ri}[q^4, q^5]$ is the scalarizing set of the face determined by the equalities $x_1 + x_3 = 1$ and $x_2 + x_3 = 1$;
(ii) $\mathrm{ri}(\mathrm{co}([q^3, q^4, q^5]))$ is the scalarizing set of the face determined by the equalities $x_1 + x_3 = 1$, $x_2 + x_3 = 1$ and $x_1 = x_2 = 0$;
(iii) $\mathrm{ri}(\mathrm{co}([q^1, q^2, q^4, q^5]))$ is the scalarizing set of the face determined by the equalities $x_1 + x_3 = 1$, $x_2 + x_3 = 1$ and $x_3 = 0$.

In view of Corollary 4.1.21 the one dimensional face (edge) determined by $x_1 + x_3 = 1$ and $x_2 + x_3 = 1$ is a maximal efficient face.

4.2 Multiobjective Linear Problems

The central multiobjective linear programming problem which we propose to study throughout is denoted (MOLP) and written in the form :

$$\begin{aligned} &\text{Maximize} \quad Cx \\ &\text{subject to } x \in X, \end{aligned}$$

where X is a nonempty convex polyhedron in $\mathbb{R}^n$ and C is a real $k \times n$-matrix. This problem means finding a *Pareto efficient (Pareto maximal) solution* $\bar{x} \in X$ such that $C\bar{x} \in \mathrm{Max}(C(X))$. In other words, a feasible solution $\bar{x}$ solves (MOLP) if there is no feasible solution $x \in X$ such that

$$C\bar{x} \leqq Cx \text{ and } C\bar{x} \neq Cx.$$

The efficient solution set of (MOLP) is denoted $S(MOLP)$. When x is an efficient solution, the vector Cx is called an *efficient or maximal value* of the problem. In a similar manner one defines the set of *weakly efficient solutions* $WS(MOLP)$ to be the set of all feasible solutions whose image by C belong to the weakly efficient set $\mathrm{WMax}(C(X))$. It is clear that an efficient solution is a weakly efficient solution, but not vice versa as we have already discussed in the preceding section. When the feasible set X is given by the system

$$\begin{aligned} Ax &= b \\ x &\geqq 0, \end{aligned}$$

where A is a real $m \times n$-matrix and b is a real m-vector, we say that (MOLP) is given in *standard form*, and it is given in *canonical form* if X is determined by the system

$$Ax \leqq b.$$

The matrix C is also considered as a linear operator from $\mathbb{R}^n$ to $\mathbb{R}^k$, and so its kernel consists of vectors x with $Cx = 0$.

Theorem 4.2.1 *Assume that the problem (MOLP) has feasible solutions. Then the following assertions hold.*

(i) *(MOLP) admits efficient solutions if and only if*

$$C(X_\infty) \cap \mathbb{R}^k_+ = \{0\}.$$

(ii) *(MOLP) admits weakly efficient solutions if and only if*

$$C(X_\infty) \cap \operatorname{int}(\mathbb{R}^k_+) = \emptyset.$$

In particular, if all asymptotic rays of X belong to the kernel of C, then (MOLP) has an efficient solution.

Proof By definition, (MOLP) has an efficient solution if and only if the set $C(X)$ has an efficient point, which, in virtue of Theorem 4.1.7, is equivalent with

$$[C(X)]_\infty \cap \mathbb{R}^k_+ = \{0\}.$$

Now the first assertion is deduced from this equivalence and from the fact that the asymptotic cone of $C(X)$ coincides with the cone $C(X_\infty)$ (Corollary 2.3.17). The second assertion is proven by a similar argument. □

Example 4.2.2 Assume that the feasible set X of the problem (MOLP) is given by the system

$$\begin{aligned} x_1 + x_2 - x_3 &= 5 \\ x_1 - x_2 &= 4 \\ x_1, x_2, x_3 &\geqq 0. \end{aligned}$$

It is nonempty and parametrically presented as

$$X = \left\{ \begin{pmatrix} t+4 \\ t \\ 2t-1 \end{pmatrix} : t \geqq \frac{1}{2} \right\}.$$

Its asymptotic cone is given by

$$X_\infty = \left\{ \begin{pmatrix} t \\ t \\ 2t \end{pmatrix} : t \geqq 0 \right\}.$$

Consider an objective function C with values in $\mathbb{R}^2$ given by the matrix

$$C = \begin{pmatrix} 1 & 0 & 1 \\ -2 & -4 & 0 \end{pmatrix}.$$

Then the image of X_∞ under C is the set

$$C(X_\infty) = \left\{ \begin{pmatrix} 3t \\ -6t \end{pmatrix} : t \geqq 0 \right\},$$

that has only the zero vector in common with the positive orthant. In view of Theorem 4.2.1 the problem has maximal solutions.

Now we choose another objective function C' given by

$$C' = \begin{pmatrix} -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Then the image of X_∞ under C' is the set

$$C'(X_\infty) = \left\{ \begin{pmatrix} 0 \\ 2t \end{pmatrix} : t \geqq 0 \right\},$$

that has no common point with the interior of the positive orthant. Hence the problem admits weakly efficient solutions. It has no efficient solution because the intersection of $C'(X_\infty)$ with the positive orthant does contain positive vectors.

Definition 4.2.3 The objective function of the problem (MOLP) is said to be bounded (respectively weakly bounded) from above if there is no vector $v \in X_\infty$ such that

$$Cv \geq 0 \text{ (respectively } Cv > 0).$$

We shall simply say that (MOLP) is bounded if its objective function is bounded from above. Of course, a bounded problem is weakly bounded and not every weakly bounded problem is bounded. A sufficient condition for a problem to be bounded is given by the inequality

$$Cx \leqq a \text{ for every } x \in X,$$

where a is some vector from $\mathbb{R}^k$. This condition is also necessary when $k = 1$, but not so when $k > 1$.

Example 4.2.4 Consider the bi-objective problem

$$\begin{aligned} \text{Maximize} \quad & \begin{pmatrix} -3 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ \text{subject to} \quad & \begin{pmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \\ & x_1, x_2, x_3 \geqq 0. \end{aligned}$$

The feasible set and its asymptotic cone are given respectively by

$$X = \left\{ \begin{pmatrix} t \\ t \\ 1 \end{pmatrix} \in \mathbb{R}^3 : t \geqq 0 \right\}$$

and

$$X_\infty = \left\{ \begin{pmatrix} t \\ t \\ 0 \end{pmatrix} \in \mathbb{R}^3 : t \geqq 0 \right\}$$

Then for every asymptotic direction $v = (t, t, 0)^T \in X_\infty$ one has

$$Cv = \begin{pmatrix} -2t \\ t \end{pmatrix} \ngeq 0.$$

By definition the objective function is bounded. Nevertheless the value set of the problem consists of vectors

$$C(X) = \left\{ \begin{pmatrix} -2t + 1 \\ t \end{pmatrix} : t \geqq 0 \right\}$$

for which no vector $a \in \mathbb{R}^2$ satisfies $Cx \leqq a$ for all $x \in X$.

Corollary 4.2.5 *The problem (MOLP) has efficient solutions (respectively weakly efficient solutions) if and only if its objective function is bounded (respectively weakly bounded).*

Proof This is immediate from Theorem 4.2.1. □

The following theorem provides a criterion for efficiency in terms of normal directions.

Theorem 4.2.6 *Let $\bar{x}$ be a feasible solution of (MOLP). Then*

(i) *$\bar{x}$ is an efficient solution if and only if the normal cone $N_X(\bar{x})$ to X at $\bar{x}$ contains some vector $C^T\lambda$ with λ a strictly positive vector of $\mathbb{R}^k$;*
(ii) *$\bar{x}$ is a weakly efficient point if and only if the normal cone $N_X(\bar{x})$ to X at $\bar{x}$ contains some vector $C^T\lambda$ with λ a nonzero positive vector of $\mathbb{R}^k$.*

Proof If the vector $C^T\lambda$ with λ strictly positive, is normal to X at $\bar{x}$, then

$$\langle C^T\lambda, x - \bar{x}\rangle \leq 0 \text{ for every } x \in X$$

which means that

$$\langle \lambda, Cx\rangle \leq \langle \lambda, C\bar{x}\rangle \text{ for every } x \in X.$$

By Theorem 4.1.4 the vector $C\bar{x}$ is an efficient point of the set $C(X)$. By definition, $\bar{x}$ is an efficient solution of (MOLP).

Conversely, if $C\bar{x}$ is an efficient point of $C(X)$, then by Theorem 4.1.10, the normal cone to $C(X)$ at $C\bar{x}$ contains a strictly positive vector, denoted by λ. We deduce that

$$\langle C^T\lambda, x - \bar{x}\rangle \leq 0 \text{ for all } x \in X.$$

This shows that the vector $C^T\lambda$ is normal to X at $\bar{x}$. The second assertion is proven similarly. □

Example 4.2.7 We reconsider the bi-objective problem given in Example 4.2.2

$$\begin{array}{ll}\text{Maximize} & \begin{pmatrix} 1 & 0 & 1 \\ -2 & -4 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\ & x_1 + x_2 - x_3 = 5 \\ \text{subject to} & x_1 - x_2 = 4 \\ & x_1, x_2, x_3 \geqq 0. \end{array}$$

Choose a feasible solution $\overline{x} = (9/2, 1/2, 0)^T$ corresponding to $t = 1/2$. The normal cone to the feasible set at $\overline{x}$ is the positive hull of the hyperplane of basis $\{(1, 1, -1)^T, (1, -1, 0)^T\}$ (the row vectors of the constraint matrix) and the vector $(0, 0, -1)^T$ (the constraint $x_3 \geqq 0$ is active at this point). In other words, this normal cone is the half-space determined by the inequality

$$x_1 + x_2 + 2x_3 \leqq 0. \tag{4.6}$$

The image of the positive orthant of the value space $\mathbb{R}^2$ under C^T is the positive hull of the vectors

$$v_1 = \begin{pmatrix} 1 & -2 \\ 0 & -4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} \text{ and } v_2 = \begin{pmatrix} 1 & -2 \\ 0 & -4 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} -2 \\ -4 \\ 0 \end{pmatrix}.$$

Using inequality (4.6) we deduce that the vector v_2 lies in the interior of the normal cone to the feasible set at $\overline{x}$. Hence that normal cone does contain a vector $C^T\lambda$ with some strictly positive vector λ. By Theorem 4.2.6 the solution $\overline{x}$ is efficient. It is routine to check that the solution $\overline{x}$ is a vertex of the feasible set.

If we pick another feasible solution, say $\overline{z} = (5, 1, 1)^T$, then the normal cone to the feasible set at $\overline{z}$ is the hyperplane determine by equation

$$x_1 + x_2 + 2x_3 = 0.$$

Direct calculation shows that the vectors v_1 and v_2 lie in different sides of the normal cone at $\overline{z}$. Hence there does exist a strictly positive vector λ in $\mathbb{R}^2$ such that $C^T\lambda$ is contained in that cone. Consequently, the solution $\overline{z}$ is efficient too.

4.3 Scalarization

We associate with a nonzero k-vector λ a scalar linear problem, denoted (LP$_\lambda$)

$$\begin{aligned} &\text{maximize} && \langle \lambda, Cx \rangle \\ &\text{subject to} && x \in X. \end{aligned}$$

This problem is referred to as a *scalarized problem* of (MOLP) and λ is called a *scalarizing vector*. Now we shall see how useful scalarized problems are in solving multiobjective problems.

Theorem 4.3.1 *The following statements hold.*

(i) *A feasible solution $\bar{x}$ of (MOLP) is efficient if and only if there is a strictly positive k-vector λ such that $\bar{x}$ is an optimal solution of the scalarized problem (LP$_\lambda$).*
(ii) *A feasible solution $\bar{x}$ of (MOLP) is weakly efficient if and only if there is a nonzero positive k-vector λ such that $\bar{x}$ is an optimal solution of the scalarized problem (LP$_\lambda$).*

Proof If $\bar{x}$ is an efficient solution of (MOLP), then, in view of Theorem 4.2.6, there is a strictly positive vector λ such that $C^T\lambda$ is a normal vector to X at $\bar{x}$. This implies that $\bar{x}$ maximizes the linear functional $\langle \lambda, C(.)\rangle$ on X, that is, $\bar{x}$ is an optimal solution of (LP$_\lambda$).

Conversely, if $\bar{x}$ solves the problem (LP$_\lambda$) with λ strictly positive, then $C^T\lambda$ is a normal cone to X at $\bar{x}$. Again, in view of Theorem 4.2.6, the point $\bar{x}$ is an efficient solution of (MOLP). The proof of the second statement follows the same line. □

We notice that Theorem 4.3.1 remains valid if the scalarizing vector λ is taken from the standard simplex, that is $\lambda_1 + \cdots + \lambda_k = 1$. Then another formulation of the theorem is given by equalities

$$\mathrm{S(MOLP)} = \bigcup_{\lambda \in \mathrm{ri}\Delta} \mathrm{S}(\mathrm{LP}_\lambda) \tag{4.7}$$

$$\mathrm{WS(MOLP)} = \bigcup_{\lambda \in \Delta} \mathrm{S}(\mathrm{LP}_\lambda) \tag{4.8}$$

where $\mathrm{S}(\mathrm{LP}_\lambda)$ denotes the optimal solution set of (LP_λ). It was already mentioned that a weakly efficient solution is not necessarily an efficient solution. Consequently a positive, but not strictly positive vector λ may produce weakly efficient solutions which are not efficient. Here is an exception.

Corollary 4.3.2 *Assume for a positive vector λ, the set consisting of the values Cx with x being optimal solution of (LP_λ) is a singleton, in particular when (LP_λ) has a unique solution. Then every optimal solution of (LP_λ) is an efficient solution of (MOLP).*

Proof Let x be an optimal solution of (LP_λ) and let y be a feasible solution of (MOLP) such that $Cy \geqq Cx$. Since λ is positive, one has

$$\langle \lambda, Cx \rangle \leqq \langle \lambda, Cy \rangle.$$

Actually we have equality because x solves (LP_λ). Hence y solves (LP_λ) too. By hypothesis $Cx = Cy$ which shows that x is an efficient solution of (MOLP). □

Equalities (4.7) and (4.8) show that efficient and weakly efficient solutions of (MOLP) can be generated by solving a family of scalar problems. It turns out that a finite number of such problems are sufficient to generate the whole efficient and weakly efficient solution sets of (MOLP).

Corollary 4.3.3 *There exists a finite number of strictly positive vectors (respectively positive vectors) λ^i, $i = 1, \cdots, p$ such that*

$$\mathrm{S(MOLP)} = \bigcup_{i=1}^{p} \mathrm{S}(\mathrm{LP}_{\lambda^i})$$

$$\text{(respectively } \ \mathrm{WS(MOLP)} = \bigcup_{i=1}^{p} \mathrm{S}(\mathrm{LP}_{\lambda^i}))$$

Proof It follows from Theorem 3.1.3 that if an efficient solution is a relative interior of a face of the feasible polyhedron and an optimal solution of (LP_λ) for some strictly positive vector λ, then the whole face is optimal for (LP_λ). Since the number of faces is finite, a finite number of such vectors λ is sufficient to generate all efficient solutions of (MOLP). The case of weakly efficient solutions is treated in the same way. □

Corollary 4.3.4 *Assume that (MOLP) has an efficient solution and (LP_λ), where λ is a nonzero positive vector, has an optimal solution. Then there is an efficient solution of (MOLP) among the optimal solutions of (LP_λ).*

Proof Apply Theorem 4.3.1 and the method of Corollary 4.1.8. □

Corollary 4.3.5 *Assume that the scalarized problems*

$$\text{maximize } \langle c^i, x\rangle$$
$$\text{subject to } x \in X$$

where c^i, $i = 1, \cdots, k$ are the columns of the matrix C^T, are solvable. Then (MOLP) has an efficient solution.

Proof The linear problems mentioned in the corollary correspond to the scalarized problems (LP_λ) with $\lambda = (0, \cdots, 1, \cdots, 0)^T$ where the one is on the ith place, $i = 1, \cdots, k$. These problems provide weakly efficient solutions of (MOLP). The linear problem whose objective is the sum $\langle c^1, x\rangle + \cdots + \langle c^k, x\rangle$ is solvable too. It is the scalarized problem with $\lambda = (1, \cdots, 1)^T$, and hence by Theorem 4.3.1, (MOLP) has efficient solutions. □

Decomposition of the scalarizing set

Given a feasible solution x of (MOLP) we denote the set of all vectors $\lambda \in \Delta$ such that x solves (LP_λ) by $\Lambda(x)$, and the union of all these $\Lambda(x)$ over $x \in X$ by $\Lambda(X)$. We denote also

$$\Lambda^r(x) = \Lambda(x) \cap \text{ri}(\Delta)$$
$$\Lambda^r(X) = \Lambda(X) \cap \text{ri}(\Delta).$$

The sets $\Lambda^r(X)$ and $\Lambda(X)$ are respectively called the *scalarizing and weakly scalarizing sets* of (MOLP). The decomposition results for efficient elements (Theorems 4.1.14 and 4.1.15) are easily adapted to decompose the weakly scalarizing and scalarizing sets of the problem (MOLP). We deduce a useful corollary below for computing purposes.

Corollary 4.3.6 *The following assertions hold for (MOLP).*

(i) *A feasible solution $x \in X$ is efficient (respectively weakly efficient) if and only if $\Lambda^r(x)$ (respectively $\Lambda(x)$) is nonempty.*
(ii) *If X has vertices, then the set $\Lambda^r(X)$ (respectively $\Lambda(X)$) is the union of the sets $\Lambda^r(x^i)$ (respectively $\Lambda(x^i)$) where x^i runs over the set of all efficient (respectively weakly efficient) vertices of (MOLP).*

(iii) *If X is given by system*

$$\langle a^i, x\rangle \leqq b_i, i = 1, \cdots, m$$

and x is a feasible solution, then the set $\Lambda(x)$ consists of all solutions λ to the following system

$$\begin{aligned} \lambda_1 + \cdots + \lambda_k &= 1 \\ \sum_{i \in I(x)} \alpha_i a^i &= \lambda_1 c^1 + \cdots + \lambda_k c^k \\ \lambda_i \geqq 0, i = 1, \cdots, k, \alpha_i &\geqq 0, i \in I(x). \end{aligned}$$

In particular the weakly scalarizing set $\Lambda(X)$ is the solution set to the above system with $I = \{1, \cdots, m\}$.

Proof The first assertion is clear from Theorem 4.3.1. For the second assertion we observe that when X has vertices, every face of X has vertices too (Corollary 2.3.6). Hence the normal cone of X is the union of the normal cones to X at its vertices. Moreover, by writing the objective function $\langle \lambda, C(.)\rangle$ of (LP_λ) in the form $\langle C^T\lambda, .\rangle$, we deduce that

$$\Lambda(x) = \{\lambda \in \mathbb{R}^k : C^T\lambda \in N_X(x) \cap C^T(\Delta)\}. \tag{4.9}$$

Consequently,

$$\begin{aligned} \Lambda(X) &= \bigcup_{x \in X} \Lambda(x) \\ &= \bigcup \left\{\lambda : C^T\lambda \in N_X(x) \cap C^T(\Delta), x \in X\right\} \\ &= \bigcup \left\{\lambda : C^T\lambda \in N_X(x) \cap C^T(\Delta), x \text{ is a vertex of } X\right\} \\ &= \bigcup \left\{\Lambda(x) : x \text{ is a weakly efficient vertex of } X\right\}. \end{aligned}$$

The proof for efficient solutions is similar. The last assertion is derived from (4.9) and Corollary 4.1.16. □

Example 4.3.7 We reconsider the bi-objective problem

$$\begin{array}{ll} \text{Maximize} & \begin{pmatrix} 1 & 1 \\ 2 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \\ \text{subject to} & \begin{aligned} x_1 + \ x_2 &\leqq 1 \\ 3x_1 + 2x_2 &\leqq 2. \end{aligned} \end{array}$$

We wish to find the weakly scalarizing set of this problem. According to the preceding corollary, it consists of positive vectors λ from the standard simplex of $\mathbb{R}^2$, solutions to the following system:

$$\lambda_1 + \lambda_2 = 1$$
$$\alpha_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ 2 \end{pmatrix} = \lambda_1 \begin{pmatrix} 1 \\ 1 \end{pmatrix} + \lambda_2 \begin{pmatrix} 2 \\ -1 \end{pmatrix}$$
$$\alpha_1, \alpha_2 \geqq 0, \lambda_1, \lambda_2 \geqq 0.$$

Solving this system we obtain $\lambda = \begin{pmatrix} t \\ 1-t \end{pmatrix}$ with $7/8 \leqq t \leqq 1$. For $t = 1$, the scalarized problem associated with λ is of the form

$$\begin{array}{ll} \text{maximize} & x_1 + x_2 \\ \text{subject to} & \begin{array}{rl} x_1 + \ \ x_2 & \leqq 1 \\ 3x_1 + 2x_2 & \leqq 2. \end{array} \end{array}$$

It can be seen that x solves this problem if and only if $x_1 + x_2 = 1$ and $x_1 \leqq 0$. These solutions form the set of weakly efficient solutions of the multiobjective problem.

For $t = 7/8$, the scalarized problem associated with $\lambda = (7/8, 1/8)^T$ is of the form

$$\begin{array}{ll} \text{maximize} & \frac{9}{8}x_1 + \frac{3}{4}x_2 \\ \text{subject to} & \begin{array}{rl} x_1 + \ \ x_2 & \leqq 1 \\ 3x_1 + 2x_2 & \leqq 2. \end{array} \end{array}$$

Its optimal solutions are given by $3x_1 + 2x_2 = 2$ and $x_1 \geqq 0$. Since λ is strictly positive, these solutions are efficient solutions of the multiobjective problem. If we choose $\lambda = (1/2, 1/2)^T$ outside of the scalarizing set, then the associated scalarized problem has no optimal solution.

Structure of the efficient solution set

We knew in Chap. 3 that the optimal solution set of a scalar linear problem is a face of the feasible set. This property, unfortunately, is no longer true when the problem is multiobjective. However, a few interesting properties of the efficient set of a polyhedron we established in the first section are still valid for the efficient solution set and exposed in the next theorem.

Theorem 4.3.8 *The efficient solutions of the problem (MOLP) have the following properties.*

(i) *If a relative interior point of a face of X is an efficient or weakly efficient solution, then so is every point of that face.*
(ii) *If X has a vertex and (MOLP) has an efficient (weakly efficient) solution, then it has an efficient (weakly efficient) vertex solution.*
(iii) *The efficient and weakly efficient solution sets of (MOLP) consist of faces of the feasible polyhedron and are closed and arcwise connected.*

Proof Since the normal cone to X at every point of a face contains the normal cone at a relative interior point, the first property follows directly from Theorem 4.2.6.

Further, under the hypothesis of (ii) there is a strictly positive vector $\lambda \in \mathbb{R}^k$ such that the scalarized problem (LP_λ) is solvable. The argument in proving (ii) of Corollary 4.1.19 is applicable to obtain an optimal vertex of (LP_λ) which is also an efficient vertex solution of (MOLP).

The proof of the last property is much similar to the one of Theorem 4.1.15. We first notice that in view of (i) the efficient and weakly efficient solution sets are composed of faces of the feasible set X, and as the number of faces of X is finite, they are closed. We now prove the arcwise connectedness of the weakly efficient solution set, the argument going through for efficient solutions too. Let x and y be two weakly efficient solutions, relative interior points of efficient faces X_x and X_y of X. Let λ_x and λ_y be relative interior vectors of the weakly scalarizing sets $\Lambda(X_x)$ and $\Lambda(X_y)$. The decomposition of the weakly scalarizing set $\Lambda(X)$ induces a decomposition of the segment joining λ_x and λ_y by

$$[\lambda_x, \lambda_y] = [\lambda_1, \lambda_2] \cup [\lambda_2, \lambda_3] \cup \cdots \cup [\lambda_{\ell-1}, \lambda_\ell]$$

with $\lambda_1 = \lambda_x, \lambda_\ell = \lambda_y$ and $[\lambda_i, \lambda_{i+1}] \subseteq \Lambda(X_i)$ for some face X_i of X, $i = 1, ..., \ell-1$. Since λ_i belongs simultaneously to $\Lambda(X_i)$ and $\Lambda(X_{i+1})$, there is some common point $x^i \in X_i \cap X_{i+1}, i = 1, ..., \ell-1$. It is clear that $[x, x_1]\cup[x_1, x_2]\cup\cdots\cup[x_{\ell-1}, y]$ is an arcwise path joining x and y and each member segment $[x_i, x_{i+1}]$ is efficient because being in the face X_i, $i = 1, ..., \ell$ with $x_\ell = y$. □

4.4 Exercises

4.4.1 *Find maximal elements of the sets determined by the following systems*

(a) $$\begin{cases} 2x + y \leqq 15 \\ x + 3y \leqq 20 \\ x, \ y \geqq 0. \end{cases}$$

(b) $$\begin{cases} x + 4y \leqq 12 \\ -2x + y \leqq 0 \\ x, \ y \geqq 0. \end{cases}$$

(c) $$\begin{cases} x + 2y \leqq 20 \\ 7x + z \leqq 6 \\ 3y + 4z \leqq 30 \\ x, \ y, \ z \geqq 0. \end{cases}$$

(d) $$\begin{cases} x + 2y + 3z \leqq 70 \\ x + y + z \leqq 50 \\ - y + z \leqq 0 \\ x, \ y, \ z \geqq 0. \end{cases}$$

4.4.2 *Find maximal and weakly maximal elements of the following sets*

$$Q_1 = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_1 \geqq 0, x_2 \geqq 0, x_3 \geqq 0, x_2^2 + x_3^2 \leqq 1 \right\}$$

$$Q_2 = co(A, B) \text{ with } A = \left\{ \begin{pmatrix} 1 \\ 0 \\ s \end{pmatrix} \in \mathbb{R}^3 : 0 \leqq s \leqq 1 \right\}$$

$$\text{and } B = \left\{ \begin{pmatrix} 0 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_2 \geqq 0, x_3 \geqq 0, x_2^2 + x_3^2 \leqq 1 \right\}.$$

4.4.3 *We say a real function g on $\mathbb{R}^k$ is increasing if $x, y \in \mathbb{R}^k$ and $x \geq y$ imply $g(x) > g(y)$, and it is weakly increasing if $x > y$ implies $g(x) > g(y)$. Prove that g is increasing (respectively weakly increasing) if and only if for every nonempty subset Q of $\mathbb{R}^k$, every maximizer of g on Q is an efficient (respectively weakly maximal) element of Q.*

4.4.4 *Let Q be a closed set in $\mathbb{R}^k$. Prove the following statements.*

(i) *The set $WMax(Q)$ is closed.*
(ii) *The set $Max(Q)$ is closed provided that $k = 2$ and $Q - \mathbb{R}^2_+$ is convex.*
(iii) *$Max(-Q) = -Min(Q)$ and $Max(\alpha Q) = \alpha\, Max(Q)$ for every $\alpha > 0$.*

4.4.5 *Let P and Q be two convex polyhedra in $\mathbb{R}^k$.*

(i) *Prove that $Max(P + Q) \subseteq Max(P) + Max(Q)$.*
(ii) *Find conditions under which equality holds in (i).*

4.4.6 *Prove that the set of maximal elements of a convex polytope is included in the convex hull of the maximal vertices. Is the converse true?*

4.4.7 *An element x of a set P in $\mathbb{R}^k$ is said to be dominated if there is some $x' \in P$ such that $x' \geq x$. Prove that the set of dominated elements of a convex polyhedral set is convex and if a face contains a dominated element, its relative interior points are dominated too.*

4.4.8 A diet problem. *A multiobjective version of the diet problem in hospital consists of finding a combination of foods for a patient to minimize simultaneously the cost of the menu and the number of calories under certain nutritional requirements prescribed by a treating physician. Assume a menu is composed of three main types of foods: meat with potatoes, fish with rice and vegetables. The nutrition facts, calories in foods and price per servings are given below*

	Fats	Carbohydrates	Vitamin	Calories	Prices/serving
Meat + potatoes	0.2	0.2	0.06	400	1.5
Fish + rice	0.1	0.2	0.08	300	1.5
Vegetables	0	0.05	0.8	50	0.8

Using three variables: x= number of servings of meat, y= number of servings of fish and z= number of servings of vegetables, formulate a bi-objective linear problem whose objective functions are the cost and the number of calories of the menu while maintaining the physician's prescription of at least one unit and at most one and half unit for each nutritional substance. Discuss the menus that minimize the cost and the number of calories separately.

4.4.9 An investment problem. *An investor disposes a budget of 20,000 USD and wishes to invest into three product projects with amounts x, y and z respectively. The total profit is given by*

$$P(x, y, z) = 20x + 10y + 100z$$

and the total sale is given by

$$S(x, y, z) = 10x + 2y + z.$$

Find x, y and z to maximize the total profit and total sale.

4.4.10 Bilevel linear programming problem. *A typical bilevel programming problem consists of two problems: the upper level problem of the form*

$$\begin{array}{ll} \text{maximize} & \langle c, x\rangle + \langle d, y\rangle \\ \text{subject to} & A_1 x \leqq b_1 \\ & x \geqq 0 \end{array}$$

and the lower level problem for which y is an optimal solution:

$$\begin{array}{ll} \text{maximize} & \langle p, z\rangle \\ \text{subject to} & A_2 x + A_3 z \leqq b_2 \\ & z \geqq 0. \end{array}$$

Here c, p, d, b_1 and b_2 are vectors of dimension n_1, n_2, n_2, m_1 and m_2 respectively; A_1, A_2 and A_3 are matrices of dimension $m_1 \times n_1$, $m_2 \times n_1$ and $m_2 \times n_2$ correspondingly.

Consider the following multiobjective problem

$$\begin{array}{ll} \text{Maximize} & \begin{pmatrix} x \\ -\langle e, x\rangle \\ \langle p, y\rangle \end{pmatrix} \\ \text{subject to} & A_1 x \leqq b_1 \\ & A_2 x + A_3 y \leqq b_2 \\ & x \geqq 0, y \geqq 0 \end{array}$$

where e is the vector whose components are all equal to one. Prove that $(\overline{x}, \overline{y})$ *is an efficient solution of this latter problem if and only if it is a feasible solution of the upper level problem described above.*

4.4.11 *Apply Theorem 4.1.15 to find a decomposition of the scalarizing set for the polyhedron defined by the system*

$$\begin{aligned} 2x_1 + x_2 + 2x_3 &\leqq 5 \\ x_1 + 2x_2 + 2x_3 &\leqq 5 \\ x_1, x_2, x_3 &\geqq 0. \end{aligned}$$

4.4.12 *Find the weakly scalarizing set of a polyhedron in* $\mathbb{R}^k$ *determined by the system*

$$\begin{aligned} \langle a^i, y\rangle &= b_i,\ i = 1, \cdots, m \\ y &\geqq 0, \end{aligned}$$

and apply it to find the weakly scalarizing set of a multiobjective problem given in standard form.

4.4.13 Scalarizing set at a vertex solution. *Consider the problem (MOLP) in standard form*

$$\begin{aligned} \text{Maximize} \quad & Cx \\ \text{subject to} \quad & Ax = b \\ & x \geqq 0, \end{aligned}$$

where C *is a* $k \times n$*-matrix,* A *is an* $m \times n$*-matrix and* b *is an* m*-vector. Assume* $\overline{x}$ *is a feasible solution associated with a non-degenerate basis* B*. The non-basic part of* A *is denoted* N*, the basic and non-basic parts of* C *are denoted* C_B *and* C_N *respectively. Prove the following statements.*

(a) *A vector* λ *belongs to* $\Lambda(\overline{x})$ *if and only if it belongs to* Δ *and solves the following system*

$$[C_N^T - (B^{-1}N)^T C_B^T]\lambda \leqq 0.$$

(b) *If the vector on the left hand side of the system in (a) is strictly negative for some* λ*, then* $\overline{x}$ *is a unique solution of the scalarized problem*

$$\begin{aligned} \text{maximize} \quad & \langle \lambda, Cx\rangle \\ \text{subject to} \quad & Ax = b \\ & x \geqq 0. \end{aligned}$$

In particular, if in addition λ *is positive, then* $\overline{x}$ *is an efficient solution of (MOLP).*

4.4.14 Pascoletti-Serafini's method. *Let $\lambda \in \mathbb{R}^k$ be a strictly positive vector and $Cx \geqq 0$ for every feasible solution $x \in X$ of (MOLP). Show that if $(\overline{x}, \overline{\alpha})$ is an optimal solution of the problem*

$$\begin{array}{ll} \text{maximize} & \alpha \\ \text{subject to} & \langle c^j, x\rangle \geqq \alpha, \ \ j = 1, ..., k \\ & x \in X, \end{array}$$

then $\overline{x}$ is a weakly efficient solution of (MOLP).

4.4.15 Weighted constraint method. *Prove that a feasible solution $\overline{x} \in X$ is a weakly efficient solution of (MOLP) if and only if there is some strictly positive vector $\lambda \in \mathbb{R}^k$ such that $\overline{x}$ solves*

$$\begin{array}{ll} \text{maximize} & \lambda_\ell\langle c^\ell, x\rangle \\ \text{subject to} & \lambda_j\langle c^j, x\rangle \geqq \lambda_\ell\langle c^\ell, x\rangle, \ \ j = 1, \cdots, k, j \neq \ell \\ & x \in X \end{array}$$

for $\ell = 1, \cdots, k$.

4.4.16 Constraint method. *Choose $\ell \in \{1, \cdots, k\}$, $L_j \in \mathbb{R}$, $j = 1, \cdots, k$, $j \neq \ell$, and solve the scalar problem (P_ℓ):*

$$\begin{array}{ll} \text{maximize} & \langle c^\ell, x\rangle \\ \text{subject to} & \langle c^j, x\rangle \geq L_j, \ j = 1, \cdots, k, \ j \neq \ell \\ & Ax = b, x \geqq 0. \end{array}$$

Note that if L_j are big, then (P_ℓ) may have no feasible solution. A constraint $\langle c^j, x\rangle \geq L_j$ is called binding if equality $\langle c^j, x\rangle = L_j$ is satisfied at every optimal solution of (P_ℓ). Prove that

(a) every optimal solution of (P_ℓ) is a weakly efficient solution of (MOLP);
(b) if an optimal solution of (P_ℓ) is unique or all constraints of (P_ℓ) are binding, then it is an efficient solution of (MOLP);
(c) a feasible solution x^0 of (MOLP) is efficient if and only if it is optimal for all (P_ℓ), $\ell = 1, ..., k$ and

$$L_\ell = (\langle c^1, x^0\rangle, \cdots, \langle c^{\ell-1}, x^0\rangle, \langle c^{\ell+1}, x^0\rangle, \cdots, \langle c^k, x^0\rangle).$$

4.4.17 *Let d be a k-vector such that $Cx \geqq d$ for some feasible solution x of (MOLP). Consider the problem (P)*

$$\begin{array}{ll} \text{maximize} & \langle e, y\rangle \\ \text{subject to} & Cx = d + y \\ & Ax = b, x \geqq 0, y \geqq 0, \end{array}$$

where e is the vector of ones in $\mathbb{R}^k$*. Show that*

(a) a feasible solution x^0 *of (MOLP) is efficient if and only if the optimal value of (P) with* $d = Cx^0$ *is equal to zero;*
(b) (MOLP) has efficient solutions if and only if the optimal value of (P) is finite.

4.4.18 *Let* $\bar{x}$ *be a feasible solution of the problem*

$$\begin{array}{ll} \text{Maximize} & Cx \\ \text{subject to} & Ax \leqq b. \end{array}$$

Show that the following statements are equivalent.

(i) $\bar{x}$ *is a weak Pareto maximal solution.*
(ii) The system

$$\begin{cases} Ax \leqq b \\ Cx > C\overline{x} \end{cases}$$

is inconsistent.
(iii) For every $t > 0$*, the system*

$$\begin{cases} Ax \leqq b - A\overline{x} \\ Cx \geqq te \end{cases}$$

is inconsistent, where e is the vector of ones.
(iv) For every $t > 0$*, the system*

$$\begin{cases} C^T\lambda - A^T\mu = 0 \\ \langle A\bar{x} - b, \mu\rangle + t\langle e, \lambda\rangle = 1 \\ \lambda, \mu \geqq 0 \end{cases}$$

is consistent.

4.4.19 *Consider the multiobjective problem described in the preceding exercise. Assume that the cone* $\text{pos}\{c^1, \cdots, c^k\}$ *contains the origin in its relative interior. Prove that if the interior of the feasible set is nonempty, then every feasible solution of (MOLP) is an efficient solution.*

4.4.20 *Let X denote the feasible set of the problem (MOLP) given in Exercise 4.4.18. Consider the following function*

$$h(x) = \max_{x' \in X} \min_{\lambda \in \Delta} \langle \lambda, Cx' - Cx\rangle.$$

Prove that $\overline{x}$ is a weakly maximal solution of (MOLP) if and only if $h(\overline{x}) = 0$.

4.4.21 Geoffrion's proper efficient solutions. *Let X be a nonempty set in $\mathbb{R}^n$ and let f be a vector-valued function from $\mathbb{R}^n$ to $\mathbb{R}^k$. Consider the following multiobjective problem*

$$\begin{aligned} &\text{Maximize} \quad f(x) \\ &\text{subject to } x \in X. \end{aligned}$$

A feasible solution $\overline{x}$ of this problem is said to be a proper efficient solution if there exists a constant $\alpha > 0$ such that for every $i \in \{1, \cdots, k\}$ and $x \in X$ satisfying $f_i(x) > f_i(\overline{x})$ there exists some $j \in \{1, \cdots, k\}$ for which $f_j(x) < f_j(\overline{x})$ and

$$\frac{f_i(x) - f_i(\overline{x})}{f_j(\overline{x}) - f_j(x)} \leqq \alpha.$$

(i) Justify that every proper efficient solution is efficient. Give an example of efficient solutions that are not proper.
(ii) Prove that when f is linear and X is a polyhedral set, every efficient solution is proper.

4.4.22 Maximality with respect to a convex cone. *Let C be a convex cone in $\mathbb{R}^k$ with $C \cap (-C) = \{0\}$ (one says C is pointed). For $y, z \in \mathbb{R}^k$ define $y \geqq_C z$ by $y - z \in C$. A point z of a set A is called C-maximal if there is no $y \in A$ such that $y \geqq_C z$ and $y \neq z$. Prove the following properties:*

(i) A point $z \in A$ is C-maximal if and only if $(A - a) \cap \mathbb{R}^k = \{0\}$;
(ii) If $\mathbb{R}^k_+ \subseteq C$, then every C-maximal point is Pareto maximal, and if $\mathbb{R}^k_+ \supseteq C$, then every Pareto maximal point is C-maximal;
(iii) If A is a polyhedral set, then there is a polyhedral cone C satisfying $\mathbb{R}^k_+ \subseteq \text{int}(C) \cup \{0\}$ such that a point of A is C-maximal if and only if it is Pareto maximal. Find such a cone for the sets in Exercise 4.4.1 (a) and (b).

4.4.23 Lexicographical order. *The lexicographical order $\geqq_{lex}$ in $\mathbb{R}^k$ is defined as $y \geqq_{lex} z$ for $y, z \in \mathbb{R}^k$ if and only if either $y = z$ or there is some $j \in \{1, \cdots, k\}$ such that $y_i = z_i$ for $i < j$ and $y_j > z_j$. A point z of a nonempty set A in $\mathbb{R}^k$ is called lex-maximal if there is no $y \in A$ such that $y \geqq_{lex} z$ and $y \neq z$.*

(i) Show that the lexicographical order is total in the sense that for every $y, z \in \mathbb{R}^k$ one has either $y \geqq_{lex} z$ or $z \geqq_{lex} y$.
(ii) Find a convex cone C such that $y \geqq_{lex} z$ if and only if $y - z \in C$.
(iii) Prove that every lex-maximal element of a set is Pareto maximal.

Do the same for the colexicographical order: $y \geqq_{colex} z$ if and only if either $y = z$ or there is some $j \in \{1, \cdots, k\}$ such that $y_i = z_i$ for $i > j$ and $y_j > z_j$.