Chapter 1
Introduction

Mathematical optimization studies the problem of finding the best element from a set of feasible alternatives with regard to a criterion or objective function. It is written in the form

$$\begin{array}{ll} \text{optimize} & f(x) \\ \text{subject to} & x \in X, \end{array}$$

where X is a nonempty set, called a feasible set or a set of feasible alternatives, and f is a real function on X, called a criterion or objective function. Here "optimize" stands for either "minimize" or "maximize" which amounts to finding $\bar{x} \in X$ such that either $f(\bar{x}) \leqq f(x)$ for all $x \in X$, or $f(\bar{x}) \geqq f(x)$ for all $x \in X$.

This model offers a general framework for studying a variety of real-world and theoretical problems in the sciences and human activities. However, in many practical situations, we tend to encounter problems that involve not just one criterion, but a number of criteria, which are often in conflict with each other. It then becomes impossible to model such problems in the above-mentioned optimization framework. Here are some instances of such situations.

Automotive design The objective of automotive design is to determine the technical parameters of a vehicle to minimize (1) production costs, (2) fuel consumption, and (3) emissions, while maximizing (4) performance and (5) crash safety. These criteria are not always compatible; for instance a high-performance engine often involves very high production costs, which means that no design can optimally fulfill all criteria.

House purchase Buying property is one of life's weightiest decisions and often requires the help of real estate agencies. An agency suggests a number of houses or apartments which roughly meet the potential buyer's budget and requirements. In order to make a decision, the buyer assesses the available offers on the basis of his or her criteria. The final choice should satisfy the following: minimal cost, minimal maintenance charges, maximal quality and comfort, best environment etc. It is quite

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natural that the higher the quality of the house, the more expensive it is; as such, it is impossible to make the best choice without compromising.

Distributing electrical power In a system of thermal generators the chief problem concerns allocating the output of each generator in the system. The aim is not only to satisfy the demand for electricity, but also to fulfill two main criteria: minimizing the costs of power generation and minimizing emissions. Since the costs and the emissions are measured in different units, we cannot combine the two criteria into one.

Queen Dido's city Queen Dido's famous problem consists of finding a territory bounded by a line which has the maximum area for a given perimeter. According to elementary calculus, the solution is known to be a circle. However, as it is inconceivable to have a city touching the sea without a seashore, Queen Dido set another objective, namely for her territory to have as large a seashore as possible. As a result, a semicircle partly satisfies her two objectives, but fails to maximize either aspect.

As we have seen, even in the simplest situations described above there can be no alternative found that simultaneously satisfies all criteria, which means that the known concepts of optimization do not apply and there is a real need to develop new notions of optimality for problems involving multiple objective functions. Such a concept was introduced by Pareto (1848–1923), an Italian economist who explained the Pareto optimum as follows: "The optimum allocation of the resources of a society is not attained so long as it is possible to make at least one individual better off in his own estimation while keeping others as well off as before in their own estimation." Prior to Pareto, the Irish economist Edgeworth (1845–1926) had defined an optimum for the multiutility problem of two consumers P and Q as "a point (x, y) such that in whatever direction we take an infinitely small step, P and Q do not increase together but that, while one increases, the other decreases." According to the definition put forward by Pareto, among the feasible alternatives, those that can simultaneously be improved with respect to all criteria cannot be optimal. And an alternative is optimal if any alternative better than it with respect to a certain criterion is worse with respect to some other criterion, that is, if a tradeoff takes place when trying to find a better alternative. From the mathematical point of view, if one defines a domination order in the set of feasible alternatives by a set of criteria—an alternative a dominates an alternative b if the value of every criterion function at a is bigger than that at b—then an alternative is optimal in the Pareto sense if it is dominated by no other alternatives. In other words, an alternative is optimal if it is maximal with respect to the above order. This explains the mathematical origin of the theory of multiple objective optimization, which stems from the theory of ordered spaces developed by Cantor (1845–1918) and Hausdorff (1868–1942).

A typical example of ordered spaces, frequently encountered in practice, is the finite dimensional Euclidean space $\mathbb{R}^n$ with $n \geq 2$, in which two vectors a and b are comparable, let's say a is bigger than or equal to b if all coordinates of a are bigger than or equal to the corresponding coordinates of b. A multiple objective optimization problem is then written as

$$\begin{array}{ll} \text{Maximize} & F(x) := (f_1(x), \ldots, f_k(x)) \\ \text{subject to} & x \in X, \end{array}$$

where $f_1, \ldots, f_k$ are real objective functions on X and "Maximize" signifies finding an element $\bar{x} \in X$ such that no value $F(x), x \in X$ is bigger than the value $F(\bar{x})$. It is essential to note that the solution $\bar{x}$ is not worse than any other solution, but in no ways it is the best one, that is, the value $F(\bar{x})$ cannot be bigger than or equal to all values $F(x), x \in X$ in general. A direct consequence of this observation is the fact that the set of "optimal values" is not a singleton, which forces practitioners to find a number of "optimal solutions" before making a final decision. Therefore, solving a multiple objective optimization problem is commonly understood as finding the entire set of "optimal solutions" or "optimal values", or at least a representative portion of them. Indeed, this is the point that makes multiple objective optimization a challenging and fascinating field of theoretical research and application.