

Chapter 2

Dirac Structures and Control by Interconnection for Distributed Port-Hamiltonian Systems

Alessandro Macchelli

Abstract The aim of this work is to show how the Dirac structure properties can be exploited in the development of energy-based boundary control laws for distributed port-Hamiltonian systems. Stabilisation of non-zero equilibria has been achieved by looking at, or generating, a set of structural invariants, namely Casimir functions, in closed-loop, and geometric conditions for the problem to be solved are determined. However, it is well known that this method fails when an infinite amount of energy is required at the equilibrium (dissipation obstacle). So, a novel approach that enlarges the class of stabilising controllers within the control by interconnection paradigm is also discussed. In this respect, it is shown how to determine a different control port that is instrumental for removing the intrinsic constraints imposed by the dissipative structure of the system. The general theory is illustrated with the help of two related examples, namely the boundary stabilisation of the shallow water equation with and without distributed dissipation.

2.1 Introduction

Port-Hamiltonian systems have been introduced about 20 years ago to describe lumped parameter physical systems in a unified manner, [4, 25, 26]. For these systems, the dynamic results from the power conserving interconnection of a limited set of components, each characterised by a particular “energetic behaviour,” i.e. storage, dissipation, generation and conversion. The generalisation to the infinite dimensional scenario leads to the definition of distributed port-Hamiltonian systems [13, 27], that have proved to represent a powerful framework for modelling, simulation and control physical systems described by PDEs. Distributed port-Hamiltonian systems share analogous geometric properties with their finite dimensional counterpart, and also the control development follows the same rationale.

A. Macchelli (✉)

Department of Electrical, Electronic, and Information Engineering,
University of Bologna, Viale del Risorgimento 2, 40136 Bologna, Italy
e-mail: alessandro.macchelli@unibo.it

This first paragraph well summarises the scientific scenario at the time I had the luck to meet Arjan, and to start collaborating with him. It was in 2001, I have to say a life ago for me, from a scientific and personal point of view. I had been staying for 6 months at the Mathematical Department of the University of Twente as a visiting Ph.D. student, with the initial idea of working on some fancy connection between sliding-mode control and port-Hamiltonian systems. After some time spent discussing with Arjan, I completely changed the topic, and I started to look at these distributed port-Hamiltonian systems, a new line of research that Arjan and Bernhard Maschke were starting to develop at those times. Everything was so intriguing to me that I continued to work on it during a second period in Twente for a Post-Doc in 2003, and until now. What I actually am professionally, I owe it also to Arjan, to his patience and clearness in teaching, and to his support and precious suggestions. The motivating idea behind this chapter is then to frame some new results on the control of distributed port-Hamiltonian systems within the classical theory and core properties of port-Hamiltonian systems, topics that Arjan thought to me and to many other PhD students during these years, and on which he is still contributing a lot. In fact, some of the results presented here are based on some recent results by him and his students for lumped parameter systems.

Since the first time I heard about distributed port-Hamiltonian systems, the general theory has been developed a lot, and most of the current research on control and stabilisation deals with the development of boundary controllers. For example, in [14, 15, 20, 23, 24], this task has been accomplished by generating a set of Casimir functions in closed-loop that independently from the Hamiltonian function relates the state of the plant with the state of the controller, a finite dimensional port-Hamiltonian system interconnected to the boundary of the distributed parameter one. The shape of the closed-loop energy function is changed by acting on the Hamiltonian of the controller. This procedure is the generalisation of the control by interconnection via Casimir generation (energy-Casimir method) developed for finite dimensional systems [19, 25], and the result is an energy-balancing passivity-based controller that is not able to deal with equilibria that require an infinite amount of supplied energy in steady state, i.e. with the so-called “dissipation obstacle.”

In finite dimensions, the dissipation obstacle has been solved within the control by interconnection paradigm by defining a *new* passive output for the original system in such a way that, in closed-loop, a new set of Casimir functions that can be employed with success in the energy-shaping procedure is present, [8, 18, 28]. More precisely, in [28], a constructive way to modify the Dirac structure of the system in order to obtain a new interconnection structure that is associated to the same state evolution, but with potentially different Casimir functions is provided. Among such larger set of structural invariants, it is then possible to find the “right” Casimir functions to be employed in the control by interconnection synthesis.

Even if inspired by [28], the approach proposed here is quite different. Starting from the geometrical properties of those energy-shaping control techniques that are not limited by the dissipation obstacle [11, 12], the conditions that the Casimir functions *should* respect to obtain the same results within the control by interconnection paradigm are deduced. Then, for the given plant, new Dirac and resistive structures

that allow to have not only the same state evolution, but also the previously determined Casimir functions in closed-loop are computed. At the end, the result is a new control port and, similarly to [28], the final closed-loop system is characterised by the desired set of invariants, and the limits of the “classical” control by interconnection are clearly removed. It is worth noting that, for distributed port-Hamiltonian systems, the key point is the formulation of the interconnection structure in infinite dimensions in terms of a Dirac structure on a Hilbert space, [6, 7].

This chapter is organised as follows. In Sect. 2.2, a short background on Dirac structures on Hilbert spaces and infinite dimensional port-Hamiltonian systems is given. In Sect. 2.3, the control by interconnection and the control by energy-shaping are discussed from a geometrical point, i.e. the applicability of the methods is related to the properties of the Dirac structure of the system that has to be stabilised. Then, in Sect. 2.4, the problem of defining a new control port that allows to overcome the dissipation obstacle within the control by interconnection paradigm is discussed. Then, in Sect. 2.5, the general methodology is illustrated with the help of an example, namely the shallow water equation with and without dissipation. Conclusions and ideas about future research activities are reported in Sect. 2.6.

2.2 Background

2.2.1 Dirac Structures

A Dirac structure is a linear space which describes internal power flows, and the power exchange between the system and the environment. Denote by $\mathcal{F} \times \mathcal{E}$ the space of power variables, with \mathcal{F} an n -dimensional linear space, the space of flows (e.g., velocities and currents) and $\mathcal{E} \equiv \mathcal{F}^*$ its dual, the space of efforts (e.g., forces and voltages), and by $\langle e, f \rangle$ the power associated to the port $(f, e) \in \mathcal{F} \times \mathcal{E}$, where $\langle \cdot, \cdot \rangle$ is the dual product between f and e .

Definition 2.1 Consider the space of power variables $\mathcal{F} \times \mathcal{E}$. A (constant) Dirac structure on \mathcal{F} is a linear subspace $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ such that $\dim \mathcal{D} = \dim \mathcal{F}$, and $\langle e, f \rangle = 0, \forall (f, e) \in \mathcal{D}$.

A Dirac structure, then, defines a power conserving relation on $\mathcal{F} \times \mathcal{E}$. As discussed in the next Proposition, different representations are possible, [3].

Proposition 2.2 Assume that $\mathcal{F} = \mathcal{E} = \mathbb{R}^n$, which implies that $\langle e, f \rangle = e^T f$. Then, for any Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$, with there exists a pair of $n \times n$ matrices F and E satisfying the conditions

$$EF^T + FE^T = 0 \qquad \text{rank}(F \mid E) = n \qquad (2.1)$$

such that \mathcal{D} can be given in kernel representation as

$$\mathcal{D} = \left\{ (f, e) \in \mathcal{F} \times \mathcal{E} \mid Ff + Ee = 0 \right\} \quad (2.2)$$

or in image representation as

$$\mathcal{D} = \left\{ (f, e) \in \mathcal{F} \times \mathcal{E} \mid f = E^T \lambda, e = F^T \lambda, \lambda \in \mathbb{R}^n \right\} \quad (2.3)$$

The definition of Dirac structure can be generalised to deal with distributed parameter systems. A possible way is to assume that the space of power variables is an Hilbert space. In this respect, Dirac structures on Hilbert spaces have been introduced in [7], while their kernel and image representations in [6]. Here, we assume that the space of flows \mathcal{F} is an Hilbert space, and that the space of efforts is $\mathcal{E} \equiv \mathcal{F}$. Instead of providing their formal definition, which follows the same rationale of the finite dimensional case, their kernel and image representations is directly presented in the next Proposition, [6].

Proposition 2.3 *For any Dirac structure $\mathcal{D} \subset \mathcal{F} \times \mathcal{E}$ on an Hilbert space $\mathcal{F} \equiv \mathcal{E}$, there exists linear maps $F : \mathcal{F} \rightarrow \Lambda$ and $E : \mathcal{E} \rightarrow \Lambda$ satisfying the conditions*

$$FE^* + EF^* = 0 \quad \overline{\text{ran}(F \ E)} = \Lambda$$

being Λ an Hilbert space isometrically isomorphic to \mathcal{F} , such that

$$\mathcal{D} = \left\{ (f, e) \in \mathcal{F} \times \mathcal{E} \mid Ff + Ee = 0 \right\} \quad (2.4)$$

or, equivalently, such that

$$\mathcal{D} = \left\{ (f, e) \in \mathcal{F} \times \mathcal{E} \mid f = E^* \lambda, e = F^* \lambda, \forall \lambda \in \Lambda \right\} \quad (2.5)$$

Here, $\overline{}$ and \cdot^* denote the closure and the adjoint of an operator, respectively, [2].

2.2.2 Port-Hamiltonian Systems

Either in the case of lumped and distributed parameter port-Hamiltonian systems, once the Dirac structure is given, the dynamics follows when the resistive structure and the port behaviour of the energy-storage elements are given. Generally speaking, the Dirac structure defines a power conserving relation between several port variables, e.g. two internal ports $(f_S, e_S) \in \mathcal{F}_S \times \mathcal{E}_S$ and $(f_R, e_R) \in \mathcal{F}_R \times \mathcal{E}_R$, which correspond to energy-storage and dissipation respectively, and an external port $(f_C, e_C) \in \mathcal{F}_C \times \mathcal{E}_C$ which is devoted to an exchange of energy with a controller. As far as the behaviour

at the resistive port is concerned, let us assume that the following linear resistive relation \mathcal{R} holds

$$R_f f_R + R_e e_R = 0 \quad (2.6)$$

where R_f and R_e are $n_R \times n_R$ matrices such that

$$R_f R_e^T = R_e R_f^T > 0 \quad \text{rank}(R_f \mid R_e) = n_R \quad (2.7)$$

Even if most of the results presented here can be applied to a more general class of systems, in this paper we refer to the family of distributed port-Hamiltonian systems that have been studied in [9, 29], i.e. to systems described by

$$\frac{\partial x}{\partial t}(t, z) = P_1 \frac{\partial}{\partial z} (\mathcal{L}(z)x(t, z)) + (P_0 - G_0)\mathcal{L}(z)x(t, z) \quad (2.8)$$

with $x \in \mathcal{X}$ and $z \in [a, b]$. Moreover, $P_1 = P_1^T > 0$, $P_0 = -P_0^T$, $G_0 = G_0^T \geq 0$, and $\mathcal{L}(\cdot)$ is a bounded and continuously differentiable matrix-valued function such that $\mathcal{L}(z) = \mathcal{L}^T(z)$ and $\mathcal{L}(z) \geq \kappa I$, with $\kappa > 0$, for all $z \in [a, b]$. For simplicity, $\mathcal{L}(z)x(t, z) \equiv (\mathcal{L}x)(t, z)$. The state space is $\mathcal{X} = L_2(a, b; \mathbb{R}^n)$, and is endowed with the inner product $\langle x_1 \mid x_2 \rangle_{\mathcal{L}} = \langle x_1 \mid \mathcal{L}x_2 \rangle$ and norm $\|x_1\|_{\mathcal{L}}^2 = \langle x_1 \mid x_1 \rangle_{\mathcal{L}}$, where $\langle \cdot \mid \cdot \rangle$ denotes the natural L_2 -inner product. The selection of this space for the state variable is motivated by the fact that $H(\cdot) = \frac{1}{2} \|\cdot\|_{\mathcal{L}}^2$ is the energy function.

To define a distributed port-Hamiltonian system, the PDE (2.8) has to be ‘‘completed’’ by a well-defined boundary port. More precisely, given $\mathcal{L}x \in H^1(a, b; \mathbb{R}^n)$, the boundary port variables are the vectors $f_C, e_C \in \mathbb{R}^n$ given by

$$\begin{pmatrix} e_C \\ f_C \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} W \\ \tilde{W} \end{pmatrix} \begin{pmatrix} P_1 & -P_1 \\ I & I \end{pmatrix} \begin{pmatrix} (\mathcal{L}x)(b) \\ (\mathcal{L}x)(a) \end{pmatrix} \quad (2.9)$$

where W and \tilde{W} are full rank $n \times 2n$ matrices such that $W \Sigma W^T = \tilde{W} \Sigma \tilde{W}^T = 0$, and $W \Sigma \tilde{W}^T = I$, being

$$\Sigma = \begin{pmatrix} 0 & I \\ I & 0 \end{pmatrix}$$

As discussed in [9, 11], it is possible to verify that $\dot{H}(x(t, \cdot)) \leq e_C^T(t) f_C(t)$, and that (2.8) is characterised by a Dirac structure on the space of flows $\mathcal{F}_S \times \mathcal{F}_R \times \mathcal{F}_C$, with $\mathcal{F}_S = L_2(a, b; \mathbb{R}^n)$, $\mathcal{F}_R = L_2(a, b; \mathbb{R}^r)$, and $\mathcal{F}_C = \mathbb{R}^n$, being $r = \text{rank } G_0$. The couple of operators $F : \mathcal{F} \rightarrow \Lambda$ and $E : \mathcal{E} \rightarrow \Lambda$ introduced in Proposition 2.3 are given by

$$F = (F_S \ F_R \ F_C) \quad E = (E_S \ E_R \ E_C) \quad (2.10)$$

where

$$\Lambda = L_2(a, b; \mathbb{R}^n) \times L_2(a, b; \mathbb{R}^r) \times \{0\} \times \mathbb{R}^n \quad (2.11)$$

being $\{0\} \subset \mathbb{R}^n$ the set containing only the origin of \mathbb{R}^n . Moreover, we have that

$$\begin{aligned} F_S &= \begin{pmatrix} I \\ 0 \\ 0 \\ 0 \end{pmatrix} & F_R &= \begin{pmatrix} 0 \\ I \\ 0 \\ 0 \end{pmatrix} & F_C &= \begin{pmatrix} 0 \\ 0 \\ 0 \\ I \end{pmatrix} \\ E_S &= \begin{pmatrix} P_1 \frac{\partial}{\partial z} + P_0 \\ -G_R^T \\ -WR\mathcal{B}_{\mathcal{F}} \\ -\tilde{W}R\mathcal{B}_{\mathcal{F}} \end{pmatrix} & E_R &= \begin{pmatrix} G_R \\ 0 \\ 0 \\ 0 \end{pmatrix} & E_C &= \begin{pmatrix} 0 \\ 0 \\ I \\ 0 \end{pmatrix} \end{aligned} \quad (2.12)$$

where $\mathcal{B}_{\mathcal{F}}(e) = \begin{pmatrix} e(b) \\ e(a) \end{pmatrix}$, with $e \in L_2(a, b; \mathbb{R}^n)$, and

$$\begin{aligned} \text{dom}(F E) &= \left\{ (f, e) \in \mathcal{F} \times \mathcal{E} \mid e_S \text{ abs. continuous,} \right. \\ &\quad \left. \frac{\partial e_S}{\partial z} \in L_2(a, b; \mathbb{R}^n), \text{ and } e_C = WR\mathcal{B}_{\mathcal{F}}(e_S) \right\} \end{aligned} \quad (2.13)$$

It is easy to verify that the port-Hamiltonian system (2.8) is a consequence of the following port behaviour at the storage and resistive ports:

$$f_S = -\frac{\partial x}{\partial t} \quad e_S = \frac{\delta H}{\delta x}(x) = \mathcal{L}x \quad e_R = -\bar{G}f_R \quad (2.14)$$

where δ denotes the variational derivative, and G_R in (2.12) and \bar{G} are such that $G_0 = G_R \bar{G} G_R^T$, [27]. Note that the resistive relation is in the form (2.6) with $R_f = \bar{G}$, $R_e = I$ and $n_R = r$. Finally, simple calculations show that $F_S^* = F_S^T$, $F_R^* = F_R^T$, $F_C^* = F_C^T$, $E_S^* = E_S^T$, and

$$E_S^* = (-P_1 \frac{\partial}{\partial z} - P_0 - G_R \ 0 \ 0) \quad E_C^* = (\tilde{W}R\mathcal{B}_{\mathcal{F}} \ 0 \ 0 \ 0)$$

with $\lambda = (\lambda_S, \lambda_R, 0, \lambda_u)$, and

$$\text{dom} \begin{pmatrix} F^* \\ E^* \end{pmatrix} = \left\{ \lambda \in \Lambda \mid \lambda_u = WR\mathcal{B}_{\mathcal{F}}(\lambda_S) \right\} \quad (2.15)$$

2.3 Control by Interconnection and Energy-Shaping

If a port-Hamiltonian control system with Hamiltonian H_C is interconnected in power conserving way to the control port (f_C, e_C) of (2.8), the closed-loop system is again in port-Hamiltonian form, and with Hamiltonian given by the sum of the two, i.e. by $H_{cl}(x, x_C) = H(x) + H_C(x_C)$, being x_C the state variable of the controller. To use this closed-loop Hamiltonian as Lyapunov function, one has first to guarantee that this function has a minimum at the desired equilibrium with a proper choice of H_C . In both the finite and infinite dimensional cases, if it is possible to find structural invariants (i.e., that do not depend on the Hamiltonian, but only on the Dirac structure) named Casimir functions of the form

$$C(x, x_C) = x_C - \mathcal{E}(x) \quad (2.16)$$

with $\mathcal{E}(x)$ some smooth well-defined functional of x , then on every invariant manifold defined by $x_C - \mathcal{E}(x) = \kappa$, with $\kappa \in \mathbb{R}$ a constant which depends on the initial plant and controller state, the closed-loop Hamiltonian may be written as, [19, 25]:

$$H_{cl}(x) = H(x) + H_C(\mathcal{E}(x) + \kappa) \quad (2.17)$$

Hence, the closed-loop equilibrium now depends on the choice of H_C , and on the invariant manifold defined by the Casimir functions the Hamiltonian H_{cl} depends on the state variable x of the plant only.

Definition 2.4 Consider a closed-loop system obtained from the power conserving interconnection at (f_C, e_C) between a couple of port-Hamiltonian systems, namely a plant with state space \mathcal{X} , and a (finite dimensional) controller with state space $\mathcal{X}_C \equiv \mathbb{R}^{m_C}$ for some m_C . Then, a function $C : \mathcal{X} \times \mathbb{R}^{m_C} \rightarrow \mathbb{R}$ is a Casimir function if $\dot{C} = 0$ along the trajectories of the closed-loop system for every possible choice of $H(\cdot)$ and $H_C(\cdot)$.

The applicability of the control by interconnection methodology relies then on the existence of a proper set of Casimir functions. Such property is fundamental to be able to properly shape the open-loop Hamiltonian function H , and achieve desired stability properties in closed-loop. Unfortunately, the dissipative structure of the plant may limit the number or even the existence of such structural invariants. It is well known, in fact, that a Casimir function cannot depend on the coordinates on which dissipation is present, and this implies that it is not possible to shape the closed-loop energy function along these directions. This limitation is also known as dissipation obstacle, [19].

In [1, 28], an effective way to determine the achievable Casimir functions for the closed-loop system when the plant is finite dimensional and without knowing the controller and by relying only on the Dirac and resistive structures of the plant is proposed. Such result can be generalised to infinite dimensions, [11].

Proposition 2.5 Denote by $x \in \mathcal{X}$ the state of the plant, and by $x_C \in \mathbb{R}^{n_C}$ the state of the to-be-designed controller. Then, the achievable Casimir functions $C(x, x_C)$ associated to the Dirac structure on Hilbert space with kernel representation (2.4) and operators F and E given as in (2.12) for any kind of power conserving interconnection with the controller are such that

$$\begin{pmatrix} 0 & 0 & f_C^T & \frac{\delta^T C}{\delta x}(x, x_C) & 0 & e_C^T \end{pmatrix}^T \in \mathcal{D} \quad (2.18)$$

for some $(f_C, e_C) \in \mathcal{F}_C \times \mathcal{E}_C$.

Corollary 2.6 Condition (2.18) with $C(x, x_C)$ given as in (2.16) is equivalent to

$$-\begin{pmatrix} 0 \\ 0 \\ \frac{\delta C}{\delta x}(x, x_C) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \frac{\delta \mathcal{E}}{\delta x}(x) \\ 0 \end{pmatrix} \in \text{ran} \begin{pmatrix} E_S^* \\ E_R^* \\ F_S^* \\ F_R^* \end{pmatrix} \quad (2.19)$$

Proof The result follows from the image representation of a Dirac structure (2.5).

The constraints imposed at the resistive port (f_R, e_R) in (2.18) or, equivalently, (2.19) imply that if C is a Casimir for a specific resistive relation (2.6) that satisfies (2.7), then C is a Casimir for all the possible resistive relations, i.e. it is independent from the behaviour at the dissipative port. Thanks to this property, the dissipation obstacle is fully characterised from a geometrical point of view both in the finite and infinite dimensional cases, [4, 10–12, 25, 26]. The intrinsic limitations of the control by interconnection paradigm can be removed if the control action is explicitly thought in terms of a state-feedback law that is able to map the initial system into a new one. The target dynamics is characterised by desired Dirac structure, resistive relation, and Hamiltonian $H_d(x) = H(x) + \mathcal{E}(x)$, where now \mathcal{E} is *not* necessarily related to some Casimir function in the form (2.16). In the simplest case, i.e. when only the Hamiltonian function is shaped, in [11] it has been proved that all the admissible functions \mathcal{E} are solution of

$$\begin{pmatrix} 0 \\ \frac{\delta \mathcal{E}}{\delta x}(x) \\ 0 \end{pmatrix} \in \text{ran} \begin{pmatrix} E_S^* \\ F_S^* \\ R_f E_R^* + R_e F_R^* \end{pmatrix} \quad (2.20)$$

It is easy to check that if \mathcal{E} satisfies (2.19), then also (2.20) holds, [10–12].

2.4 Overcoming the Dissipation Obstacle with a New Control Port

The motivating idea of this Section is to determine if there exists a new Dirac structure on Hilbert space $\tilde{\mathcal{D}}$ with operators F and E given as in (2.12), and a resistive relation $\tilde{\mathcal{R}}$ in the form (2.6) such that with the given Hamiltonian $H(x)$:

- The dynamics of the new system is the same of the original one;
- The new system is characterised by a set of Casimir functions that satisfies (2.20).

With (2.17) in mind, the second requirement implies that, for the new system, there exists a set of Casimir functions that can be employed in the control by interconnection procedure and allow to solve the dissipation obstacle. On the other hand, since the dynamics of the new system is the same of the initial one, the only difference between the twos is the behaviour at the control port. This means that a new control port $(\tilde{f}_C, \tilde{e}_C)$ has been determined, and when the interconnection between plant and controller takes place at $(\tilde{f}_C, \tilde{e}_C)$, the resulting closed-loop system is characterised by a new set of Casimir functions, that has been previously determined among the ones that allow to overcome the dissipation obstacle. With the next Proposition, a general expression for the desired Dirac structures $\tilde{\mathcal{D}}$ is provided.

Proposition 2.7 *Let us consider a Dirac structure \mathcal{D} on Hilbert space with kernel representation given in Proposition 2.3, where F and E are given as in (2.10). The set $\tilde{\mathcal{D}} \subset \mathcal{F} \times \mathcal{E}$ defined as*

$$\tilde{\mathcal{D}} = \left\{ (\tilde{f}, \tilde{e}) \in \mathcal{F} \times \mathcal{E} \mid \tilde{F}\tilde{f} + \tilde{E}\tilde{e} = 0 \right\} \quad (2.21)$$

with $\tilde{F} : \Lambda \rightarrow \mathcal{F}$ and $\tilde{E} : \Lambda \rightarrow \mathcal{E}$ a couple of linear operators such that $\tilde{F} = (\tilde{F}_S \tilde{F}_R \tilde{F}_C)$ and $\tilde{E} = (\tilde{E}_S \tilde{E}_R \tilde{E}_C)$, with $\text{dom}(F \ E) = \text{dom}(\tilde{F} \ \tilde{E})$, and where

$$\begin{aligned} \tilde{F}_S &= F_S & \tilde{F}_R &= F_R + \tilde{F}_R & \tilde{F}_C &= F_C \\ \tilde{E}_S &= E_S + \tilde{E}_S & \tilde{E}_R &= E_R + \tilde{E}_R & \tilde{E}_C &= E_C + \tilde{E}_C \end{aligned}$$

is a Dirac structure iff $\overline{\text{ran}(\tilde{F} \mid \tilde{E})} = \Lambda$ and

$$\begin{aligned} \tilde{E}_S F_S^* + F_S \tilde{E}_S^* + E_R \tilde{F}_R^* + \tilde{E}_R (F_R^* + \tilde{F}_R^*) + F_R \tilde{E}_R^* \\ + \tilde{F}_R (E_R^* + \tilde{E}_R^*) + \tilde{E}_C F_C^* + F_C \tilde{E}_C^* = 0 \end{aligned} \quad (2.22)$$

Proof This result follows from Proposition 2.3.

The next Proposition provides necessary and sufficient conditions for the Dirac structure $\tilde{\mathcal{D}}$ to have Casimir functions that satisfy (2.20).

Proposition 2.8 *Let us consider the Dirac structures \mathcal{D} and $\bar{\mathcal{D}}$ presented in Proposition 2.7. A function $C(x, x_C)$ is a Casimir associated to $\bar{\mathcal{D}}$ that satisfies (2.20) iff*

$$\overline{\text{ran} \begin{pmatrix} \tilde{E}_S^* \Phi \\ (E_R^* + \tilde{E}_R^*) \Phi \\ (F_R^* + \tilde{F}_R^*) \Phi \end{pmatrix}} \subseteq \text{ran} \begin{pmatrix} (E_S^* + \tilde{E}_S^*) \Psi \\ (E_R^* + \tilde{E}_R^*) \Psi \\ (F_R^* + \tilde{F}_R^*) \Psi \end{pmatrix} \quad (2.23)$$

where $\Phi : \Lambda_\Phi \rightarrow \Lambda$ and $\Psi : \Lambda_\Psi \rightarrow \Lambda$ are a couple of linear operators such that

$$\text{ran} \Phi = \text{Ker} E_S^* \cap \text{Ker} (R_f E_R^* + R_e F_R^*) \quad \text{ran} \Psi = \text{Ker} F_S^* \quad (2.24)$$

Proof Since C satisfies (2.20), there must exist $\lambda \in \Lambda$ such that $\lambda = \Phi \lambda_\Phi$, with $\lambda_\Phi \in \Lambda_\Phi$, and that $\frac{\delta C}{\delta x} = F_S^* \lambda$. On the other hand, C is required to be a Casimir for $\bar{\mathcal{D}}$, so from (2.19) in Corollary 2.6, there must exist $\bar{\lambda} \in \Lambda$ such that

$$\bar{E}_S^* \bar{\lambda} = 0 \quad \bar{E}_R^* \bar{\lambda} = 0 \quad \bar{F}_R^* \bar{\lambda} = 0 \quad (2.25)$$

and $\frac{\delta C}{\delta x}(x) = F_S^* \bar{\lambda}$. This latter requirement implies that $\bar{\lambda} = \Phi \lambda_\Phi + \Psi \lambda_\Psi$, with $\lambda_\Psi \in \Lambda_\Psi$. The statement is proved once it is verified that for all λ_Φ there exists at least one λ_Ψ such that (2.25) holds, which is equivalent to require that (2.23) holds.

The next Proposition provides necessary and sufficient conditions for the port-Hamiltonian system associated to the Dirac structure $\bar{\mathcal{D}}$, with resistive structure $\bar{\mathcal{R}}$ defined later on, and Hamiltonian H to have the same state evolution of the port-Hamiltonian system with Dirac structure \mathcal{D} and resistive structure \mathcal{R} .

Proposition 2.9 *Let us consider the Dirac structures \mathcal{D} and $\bar{\mathcal{D}}$ presented in Proposition 2.7, and suppose that the resistive structure $\bar{\mathcal{R}}$ defined by*

$$\bar{R}_f \bar{f}_R + \bar{R}_e \bar{e}_R = 0 \quad (2.26)$$

is interconnected at the resistive port (\bar{f}_R, \bar{e}_R) of $\bar{\mathcal{D}}$, where \bar{R}_f and \bar{R}_e are square matrices that satisfy conditions similar to (2.7). If the behaviour at the energy-storage port (\bar{f}_S, \bar{e}_S) is as in (2.14), then the resulting state evolution is the same of the system associated to \mathcal{D} iff

$$\overline{\text{ran} \begin{pmatrix} \tilde{E}_S^* \bar{\Phi} \\ [\bar{R}_f (E_R^* + \tilde{E}_R^*) + \bar{R}_e (F_R^* + \tilde{F}_R^*)] \bar{\Phi} \end{pmatrix}} \subseteq \text{ran} \begin{pmatrix} (E_S^* + \tilde{E}_S^*) \bar{\Psi} \\ [\bar{R}_f (E_R^* + \tilde{E}_R^*) + \bar{R}_e (F_R^* + \tilde{F}_R^*)] \bar{\Psi} \end{pmatrix} \quad (2.27)$$

where $\bar{\Phi} : \Lambda_{\bar{\phi}} \rightarrow \Lambda$ and $\bar{\Psi} : \Lambda_{\bar{\psi}} \rightarrow \Lambda$ are a couple of linear operators such that

$$\text{ran}\bar{\Phi} = \text{Ker} (R_f E_R^* + R_e F_R^*) \quad \text{ran}\bar{\Psi} = \text{Ker} F_S^* \cap \text{Ker} F_C^* \quad (2.28)$$

Proof Without loss of generality, assume an effort-in causality at the control ports (f_C, e_C) and (\bar{f}_C, \bar{e}_C) . Then, from the image representation (2.3) of a Dirac structure, and the behaviours (2.14) and (2.26) imposed at the resistive ports of \mathcal{D} and $\bar{\mathcal{D}}$, respectively, we have that there must exist $\lambda = \bar{\Phi}\lambda_{\bar{\phi}}$, with $\lambda_{\bar{\phi}} \in \Lambda_{\bar{\phi}}$, and

$$\bar{\lambda} \in \text{Ker} (\bar{R}_f \bar{E}_R^* + \bar{R}_e \bar{F}_R^*), \quad \bar{\lambda} \in \Lambda \quad (2.29)$$

such that

$$-\frac{\partial x}{\partial t} = E_S^* \lambda = \bar{E}_S^* \bar{\lambda} \quad (2.30)$$

and $\frac{\delta H}{\delta x}(x) = F_S^* \lambda = F_S^* \bar{\lambda}$, and $e_C = F_C^* \lambda = F_C^* \bar{\lambda}$. These last two conditions are equivalent to $\bar{\lambda} = \bar{\Phi}\lambda_{\bar{\phi}} + \bar{\Psi}\lambda_{\bar{\psi}}$, with $\lambda_{\bar{\psi}} \in \Lambda_{\bar{\psi}}$. The statement is proved once it is verified that for all $\lambda_{\bar{\phi}}$ there exists at least one $\lambda_{\bar{\psi}}$ such that (2.29) and (2.30) hold, which is equivalent to require that (2.27) holds.

If it is possible to determine a Dirac structure $\bar{\mathcal{D}}$ and a dissipative structure $\bar{\mathcal{R}}$ such that the conditions of Propositions 2.7, 2.8 and 2.9 hold, we have determined a new control port (f_C, e_C) for the original system such that for some controller in port-Hamiltonian form the closed-loop system is characterised by a set of Casimir functions that are able to overcome the dissipation obstacle. In the next Corollary, a sufficient condition to be checked in order to have (2.23) and (2.23) satisfied is given.

Corollary 2.10 *Under the hypothesis of Propositions 2.8 and 2.9, with the further requirement that $\bar{R}_f = R_f$ and $\bar{R}_e = R_e$, conditions (2.23) and (2.27) hold if*

$$\overline{\text{ran} \begin{pmatrix} \bar{E}_S^* \Phi \\ (E_R^* + \bar{E}_R^*) \Phi \\ (F_R^* + \bar{F}_R^*) \Phi \end{pmatrix}} \subseteq \text{ran} \begin{pmatrix} (E_S^* + \bar{E}_S^*) \bar{\Psi} \\ (E_R^* + \bar{E}_R^*) \bar{\Psi} \\ (F_R^* + \bar{F}_R^*) \bar{\Psi} \end{pmatrix} \quad (2.31)$$

where Φ and $\bar{\Psi}$ are defined in (2.24) and in (2.28), respectively.

2.5 Example: Boundary Stabilisation of the Shallow Water Equation

Let us consider a rectangular open channel with a single flat reach, of length L and unitary width, which is delimited by upstream and downstream gates, and terminated by an hydraulic outfall. Moreover, it is assumed that the fluid has a unitary density;

we are in fact considering a simplified model of [5], even if all the results discussed here can be easily extended to more general cases. The dynamics is described by the shallow water equations, whose port-Hamiltonian formulation has been extensively discussed e.g. in [5, 21].

Denote by $[0, L]$ the spatial domain, and by $q(t, z) > 0$ and $p(t, z)$ the infinitesimal volume and kinetic momentum density, respectively. These are the state (energy) variables. Note that, due to the unitary width and fluid density assumptions, these quantities are numerically equal to the height of the fluid in the channel and to its velocity. Under the hypothesis of linearity in the internal friction forces (if present), the port-Hamiltonian formulation of the shallow water equations is in the form (2.8)

$$\frac{\partial}{\partial t} \begin{pmatrix} q \\ p \end{pmatrix} = \left[\begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \frac{\partial}{\partial z} - \begin{pmatrix} 0 & 0 \\ 0 & D \end{pmatrix} \right] \frac{\delta H}{\delta x}(q, p) \quad (2.32)$$

where $x = (q, p)$, $D \geq 0$ models the dissipative effects, $H(q, p) = \frac{1}{2} \int_0^L (qp^2 + gq^2) dz$ is the total energy of the fluid, and g is the gravity acceleration. Note that the co-energy variables are

$$\frac{\delta H}{\delta q}(q, p) = \frac{1}{2} p^2 + gq =: P(q, p) \quad \frac{\delta H}{\delta p}(q, p) = qp =: Q(q, p)$$

which equal the hydrodynamic pressure, P , and water flow, Q , respectively. It is assumed that the controller is acting on the boundary port (f_C, e_C) defined as

$$e_C(t) = \begin{pmatrix} Q(t, 0) \\ P(t, L) \end{pmatrix} \quad f_C(t) = \begin{pmatrix} P(t, 0) \\ -Q(t, L) \end{pmatrix}$$

The input is e_C . The associated Dirac structure can be written in the kernel representation (2.4), with operators F and E given in (2.10), and space Λ given in (2.11), with $n = 2$ and $r = 1$. Finally, the behaviour at the energy-storage and dissipative ports is (2.14), with $\bar{G} = D \geq 0$.

If dissipation is not present, i.e. if $D = 0$, it is possible to prove that the closed-loop system is characterised by a couple of Casimir functions in the form (2.16) that satisfy (2.18) or, equivalently, (2.19). More precisely, with the controller

$$\begin{cases} \dot{x}_C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \frac{\partial H_C}{\partial x_C}(x_C) + f'_C \\ e'_C = \frac{\partial H_C}{\partial x_C}(x_C) \end{cases}, \quad x_C \in \mathbb{R}^2 \quad (2.33)$$

that is interconnected to the system through (f_C, e_C) , i.e. $f'_C = f_C$ and $e_C = -e'_C$, the resulting closed-loop system is characterised by the following Casimir functions

$$C_1(x_C, q, p) = x_{C1} - \int_0^L p \, dz \quad C_2(x_C, q, p) = x_{C2} - \int_0^L q \, dz$$

Such Casimir functions are useful to select H_C to properly shape the Hamiltonian of the closed-loop system, [11, 17].

On the other hand, when dissipation is present, i.e. when $D > 0$, no useful Casimir functions in closed-loop exist. But, it has been illustrated in [16, 17] that there exists a boundary state-feedback law thanks to which it is possible to overcome the dissipation obstacle and obtain an energy function $H(q, p) + \mathcal{E}(q, p)$ with the desired stability properties. The function \mathcal{E} satisfies (2.20), that now becomes

$$\frac{\partial}{\partial z} \frac{\delta \mathcal{E}}{\delta p}(q, p) = 0 \quad \frac{\partial}{\partial z} \frac{\delta \mathcal{E}}{\delta q}(q, p) + D \frac{\delta \mathcal{E}}{\delta p}(q, p) = 0 \quad (2.34)$$

The same result can be obtained with the methodology discussed in this paper by relying on Corollary 2.10. In this respect, the operators Φ and $\bar{\Psi}$ are given by

$$\Phi(\lambda_q, \lambda_p) = \begin{pmatrix} D(L-z)\lambda_p + \lambda_q \\ \lambda_p \\ -D\lambda_p \\ 0 \\ 0 \\ \lambda_p \\ \lambda_q \end{pmatrix} \quad \bar{\Psi}(\lambda_R) = \begin{pmatrix} 0 \\ 0 \\ \lambda_R \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

with $\text{dom } \Phi = \mathbb{R}^2$ and $\text{dom } \bar{\Psi} = L_2(0, \ell; \mathbb{R})$. Then, it is possible to prove that conditions (2.23) and (2.27) can be satisfied by selecting $\bar{E}_R = 0$ and

$$\bar{E}_S^* = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & D & 0 \end{pmatrix} \quad \bar{F}_R^* = (0 \ 0 \ 0 \ 0 \ 0 \ -1 \ 0) \quad \bar{E}_C^* = \begin{pmatrix} 0 & D & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

which is equivalent to have

$$\bar{E}_S = \begin{pmatrix} 0 & -\frac{\partial}{\partial z} \\ -\frac{\partial}{\partial z} & 0 \\ 0 & -1 \\ 0 & \cdot|_0 \\ \cdot|_L & 0 \\ \cdot|_0 & D \int_0^L \cdot \\ 0 & \cdot|_L \end{pmatrix} \quad \bar{F}_R = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ -\int_0^L \cdot \\ 0 \end{pmatrix} \quad \bar{E}_C = \begin{pmatrix} 0 & 0 \\ D & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

where $\cdot|_0$ and $\cdot|_L$ denote the value of a function in $z = 0$ and in $z = L$. With this choice, a new control port (\bar{f}_C, \bar{e}_C) is defined, in which $\bar{e}_C = (\bar{e}_{C1}, \bar{e}_{C2}) = e_C$ and

$$\bar{f}_C = \begin{pmatrix} \frac{\delta H}{\delta q}(0) - 2D \int_0^L \frac{\delta H}{\delta p}(\cdot, z) dz + DL \bar{e}_{C1} \\ -\frac{\delta H}{\delta p}(L) \end{pmatrix}$$

is the new passive output that can be used in the control by interconnection strategy to have a closed-loop system characterised by a set of Casimir functions that satisfies (2.23). In this respect, with the controller (2.33) now interconnected to the plant through the new control port, i.e. $f'_C = \bar{f}_C$ and $\bar{e}_C = -e'_C$, the resulting closed-loop system is characterised by the following Casimir functions that clearly satisfy (2.34):

$$C_1(x_C, q, p) = x_{C1} - \int_0^L [D(L-z)q + p] dz \quad C_2(x_C, q, p) = x_{C2} - \int_0^L q dz$$

Thanks to these Casimir functions, H_C can be selected to shape the Hamiltonian of the closed-loop system in the desired manner. It is possible to verify that the same control law obtained by relying on an energy-shaping approach based on trajectory matching between the open-loop system and a target one discussed e.g. in [16, 17] can be obtained within the control by interconnection paradigm.

2.6 Conclusions and Future Work

The motivating idea of the paper has been the development of a general methodology for the definition of a new control port for distributed parameter port-Hamiltonian systems with dissipation that is instrumental for the synthesis of stabilising boundary control laws able to overcome the dissipation obstacle within the control by interconnection via Casimir generation paradigm. When the interconnection between plant and controller takes place at this new control port, the same results provided by the control by energy-shaping, where the control action is explicitly determined as a state-feedback law able to shape the energy function in an appropriate manner, are recovered. Beside having established a link between these two control methodologies (i.e., between the control by interconnection via Casimir generation, and the control by energy-shaping), this result is interesting because it allows to study the properties of the closed-loop system in terms of the “interconnection of sub-systems” paradigm. This is useful, in particular, in the distributed parameter case, because it paves the way for the extension to a wider class of problems the methodologies presented e.g. in [22] that deal with the proof of the existence of solutions of systems of PDEs, and of the asymptotic/exponential stability of interconnected systems. This topic is currently under investigation.

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