Chapter 19
Model Reduction by Generalized Differential Balancing

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Abstract In this chapter, we give a generalization of differential balancing method for model reduction of nonlinear systems in the direction to computation. We generalize concepts of differential controllability and observability functions, then use them for model reduction. We show some stability properties are preserved under the model reduction and estimate the error bound by the model reduction.

19.1 Introduction

For the second author, the work in this paper finds its roots in early work, [15], which I did as a Ph.D. student under the supervision of Arjan at the University of Twenty. It is my pleasure to write in the book of my teacher and mentor at the occasion of his 60th birthday. During my Ph.D. research Arjan was an inspiring researcher, teacher, and supervisor, allowing me to pursue a research direction different from the original plan. Even though I was impressed by his knowledge and ideas, I felt he was always available for questions and open discussions, with or without the many (international) visitors who came to spent time in the group in Twenty. Being one of the leaders in the field of nonlinear control, Arjan contributed significantly to the bustling scientific atmosphere in the group, greatly influencing my perspective on scientific life. After years at different universities, we are now colleagues in Groningen. Ever since I started in Groningen, we have been collaborating again, we share ideas and have jointly supervised a few Ph.D. students. I very much appreciate these encounters,

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and I am honored to organize the workshop, edit the book, and contribute a chapter for Arjan's Festschrift with a topic that finds its roots in my Ph.D. work. *Congratulations Arjan!*

Model order reduction problems have been widely studied because the reduced order models are useful for analysis, design, control, and simulation. In both linear and nonlinear control theory, a balanced realization is a useful state-space representation when studying model reduction problems [2, 7, 8, 15, 19], measuring importance of state variables based on how much energy is minimally needed to reach that state variable, and how much energy is obtained starting in that state variable. Besides balancing, also moment matching [2] is a useful tool for model reduction for control, in general computationally stronger than balanced order reduction, but not having a priori error bound, and less intuition. For nonlinear systems, this method has only been recently developed, see [3, 9]. Balancing for nonlinear systems has a longer history [15], but there are still many recent developments, i.e., there are various other types of nonlinear balancing such as a flow balancing [17, 18], incremental balancing [4], and dynamic balancing [14]. These methods are developed to take into account different properties of importance, such as incremental stability, for example [4]. In general, it depends on the system analysis and the control goal which method is best. In this paper we focus on balancing.

Recently, the authors presented a new balancing method based on contraction theory [10]. Contraction theory has been studied in recent decades, and deals with trajectories of nonlinear systems with respect to one another. One of the interesting ideas of contraction theory is considering the infinitesimal metric instead of a feasible distance function. In this setting, for instance, stability [1, 6, 11], optimal and H^∞ control [12, 13], and dissipativity [5, 16] have been studied. However, if the system order becomes large, the analysis and control becomes difficult, which motivates the study of balancing in the contraction framework, called differential balancing theory. Differential balancing theory is based on two energy functions, the so-called differential controllability and observability functions. In [10], it is shown that these two energy functions have close relationships with solutions to types of Lyapunov equations in contraction theory. That is, well-known results on controllability and observability Gramians in linear systems and control theory have partly been generalized. Moreover, a new model reduction method has been established based on the differential balancing, and this model reduction method is demonstrated for a system for which we cannot apply the incremental balancing method of [4].

As with most of the nonlinear balancing methods, computation of the differential energy functions is still not straightforward. Therefore, in this chapter, we generalize differential balancing into a direction that facilitates computations for obtaining a reduced order model based on generalized differential balancing. This generalized method relies on so-called generalized differential energy functions, which give bounds on the original differential energy functions, following similar principles as in [4, 14]. The existence of these generalized differential functions guarantees boundedness of trajectories of the variational system of the nonlinear system, which property is preserved under model reduction based on generalized differential balancing. In addition, generalized differential balancing has several advantages over other computationally feasible methods as in [4, 14]. First, generalized differential balancing does not require that the vector field of the system is an odd function in contrast to the generalized incremental balancing [4]. Second, an error bound for model reduction is estimated differently from the dynamic balancing in [14]. Moreover, generalized differential balancing can be directly applied to time-varying systems.

The remainder of this paper is organized as follows. In Sect. 19.2, we review results on differential balancing such as the differential energy functions and the differential balanced realization. In Sect. 19.3, we develop generalized differential balancing and present a model reduction method based on generalized differential balancing, which is illustrated by a system composed of 100 mass-spring-damper systems with nonlinear springs. Finally in Sect. 19.4 we conclude the paper.

Notations Let $\mathbb{R}$ be the field of real numbers. Denote $\mathbb{R}_{\geq 0} := [0, \infty) \subset \mathbb{R}$. It is said that $u : [a, b] \to \mathbb{R}^m$ is in $L_2^m[a, b]$ if $||u(t)||_{L_2^m[a,b]} := \sqrt{\int_a^b ||u(t)||^2 dt} < \infty$, where $||u(t)|| := \sqrt{u^{\mathrm{T}}(t)u(t)}$. A curve γ on $\mathbb{R}^n$ is a class C^2 mapping $\gamma : \mathbb{R} \supset [0, 1] \to \mathbb{R}^n$. For matrix $A(x, t) = (a_{ij})$, denote $\delta_f(A) := (\partial a_{ij}/\partial t + (\partial a_{ij}/\partial x) f)$. If A is invertible, we use the notation $A^{-\mathrm{T}}$ to denote $(A^{-1})^{\mathrm{T}}$. For the vector valued function $F : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$, denote $\partial F(x, t)/\partial x := [\partial F(x, t)/\partial x_1, \ldots, \partial F(x, t)/\partial x_n]$, and $\partial^{\mathrm{T}} F(x, t)/\partial x := (\partial F(x, t)/\partial x)^{\mathrm{T}}$.

19.2 The Differential Balanced Realization

In this section, we review results on differential balancing [10] for nonlinear systems.

Consider the nonlinear time-varying system and its associated system of differential dynamics

$$\Sigma_{BC} : \begin{cases} \dot{x}(t) := dx(t)/dt = f(x(t), t) + B(t)u(t), \\ y(t) = C(t)x(t), \end{cases}$$

$$d\Sigma_{BC} : \begin{cases} \delta\dot{x}(t) := \dfrac{d}{dt}\delta x(t) = \dfrac{\partial(f(x(t), t) + B(t)u(t))}{\partial x}\delta x(t) + B(t)\delta u(t), \\ \delta y(t) = C(t)\delta x(t), \end{cases}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^m$ and $y(t) \in \mathbb{R}^p$ are, respectively, the state, input and output of Σ_{BC}; $\delta x(t) \in \mathbb{R}^n$, $\delta u(t) \in \mathbb{R}^m$ and $\delta y(t) \in \mathbb{R}^p$ are, respectively, the state, input, and output of $d\Sigma_{BC}$; $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$, $B : \mathbb{R} \to \mathbb{R}^{n\times m}$ and $C : \mathbb{R} \to \mathbb{R}^{p\times n}$ are class C^2. When $u(t) \equiv 0$ and $\delta u(t) \equiv 0$, we denote Σ_{BC} and $d\Sigma_{BC}$ by Σ_C and $d\Sigma_C$, respectively.

Remark 19.1 For each $s \in [0, 1]$, let curve $\gamma(s)$ be an initial condition for Σ_{BC} and $u(\cdot, s)$ be an input signal. If $u(\cdot, \cdot)$ is class C^2, then $x(\cdot, s)$ is a solution to the system Σ_{BC}. Define $\delta x(t) := \partial x(t, s)/\partial s$ and $\delta u(t) := \partial u(t, s)/\partial s$. Then, $\delta x(\cdot, s)$

is a solution to $d\Sigma_{BC}$ from the initial condition $\partial\gamma(s)/\partial s$. Also, the output signal is given by $\delta y(t,s)$.

For differential balancing, the following two energy functions play important roles [10].

Definition 19.2 The differential controllability function of the system Σ_{BC} is defined as

$$L_{\mathcal{C}}(x_0, \delta x_0, t_0) := \inf_{\delta u \in L_2^m(-\infty, t_0]} \frac{1}{2}\int_{-\infty}^{t_0} ||\delta u(t)||^2 dt,$$

for all feasible trajectories $(x(t), u(t))$ of Σ_{BC}, where $x(t_0) = x_0 \in \mathbb{R}^n$, $\delta x(t_0) = \delta x_0 \in \mathbb{R}^n$ and $\delta x(-\infty) = 0$.

Definition 19.3 The differential observability function of the system Σ_C is defined as

$$L_{\mathcal{O}}(x_0, \delta x_0, t_0) := \frac{1}{2}\int_{t_0}^{\infty} ||\delta y(t)||^2 dt,$$

for all feasible trajectories $x(t)$ of Σ_C, where $x(t_0) = x_0 \in \mathbb{R}^n$, $\delta x(t_0) = \delta x_0 \in \mathbb{R}^n$, $\delta x(\infty) = 0$.

It is not guaranteed that these two differential energy functions always exist. Note that these energy functions are the controllability and observability functions for $d\Sigma_{BC}$ and $d\Sigma_C$, respectively. In the linear case, these two functions are nothing but the controllability and observability functions, respectively. Similar to the linear case, differential controllability and observability functions are characterized by Lyapunov type of equations (note that hereafter we leave out arguments when clear from the context for ease of notation) [10].

Theorem 19.4 *Suppose that there exists a nonsingular, real symmetric, and class* C^1 *solution* $-\infty < P(x,t) < \infty$ $(\forall x \in \mathbb{R}^n, \forall t \in \mathbb{R})$ *to*

$$-\delta_f(P(x,t)) + \frac{\partial f(x,t)}{\partial x}P(x,t) + P(x,t)\frac{\partial^{\mathrm{T}} f(x,t)}{\partial x} = -B(t)B^{\mathrm{T}}(t), \quad (19.1)$$

$$-\delta_B(P(x,t)) = 0. \quad (19.2)$$

Also, suppose that for all feasible trajectories $(\hat{x}(t), \hat{u}(t))$ *of* $\dot{\hat{x}}(t) = -f(\hat{x}(t)) - g(\hat{x}(t))\hat{u}(t)$*, the trajectory* $\delta\hat{x}(t)$ *of the following system is bounded for all* $t \geq t_0$ *and* $\lim_{t\to\infty} \delta\hat{x}(t) = 0$.

$$\frac{d}{dt}\delta\hat{x}(t) = -\frac{\partial f(\hat{x}(t),t)}{\partial x}\delta\hat{x}(t) - B(t)B^{\mathrm{T}}(t)P^{-1}(\hat{x}(t),t)\delta\hat{x}(t). \quad (19.3)$$

Then, $L_{\mathcal{C}}(x_0, \delta x_0, t_0) = \frac{1}{2}\delta x_0^{\mathrm{T}} P^{-1}(x_0, t_0)\delta x_0$. □

Theorem 19.5 *Suppose that for all feasible trajectories $x(t)$ of Σ_C, the trajectory $\delta x(t)$ of $d\Sigma_C$ is bounded for all $t \geq 0$ and $\lim_{t\to\infty} \delta x(t) = 0$. If there exists a real symmetric and class C^1 solution $-\infty < Q(x,t) < \infty$ ($\forall x \in \mathbb{R}^n, \forall t \in \mathbb{R}$) to*

$$\delta_f(Q(x,t)) + \frac{\partial^{\mathrm{T}} f(x,t)}{\partial x} Q(x,t) + Q(x,t)\frac{\partial f(x,t)}{\partial x} = -C^{\mathrm{T}}(t)C(t), \quad (19.4)$$

then $L_{\mathcal{O}}(x_0, \delta x_0, t_0) = \frac{1}{2}\delta x_0^{\mathrm{T}} Q(x_0,t_0)\delta x_0$. □

In terms of the differential controllability and observability functions, we define a differentially balanced realization for the system Σ_{BC} [10].

Definition 19.6 A realization of the associated system $d\Sigma_{BC}$ is said to be a differentially balanced realization on an open subset $D \subset \mathbb{R}^n \times \mathbb{R}$ if there exists a diagonal matrix

$$\Lambda(x,t) = \mathrm{diag}\{\sigma_1(x,t), \sigma_2(x,t), \ldots, \sigma_n(x,t)\}, \quad (19.5)$$

where $\sigma_1(x,t) \geq \sigma_2(x,t) \geq \cdots \geq \sigma_n(x,t) > 0$ holds on D, and $P(x,t) = \Lambda(x,t)$ and $Q(x,t) = \Lambda(x,t)$, respectively, satisfy (19.1), (19.2) and (19.4).

Theorem 19.7 *Let $P(x,t)$ and $Q(x,t)$ be, respectively, real symmetric and class C^1 solutions to (19.1), (19.2) and (19.4), where $0 < P(x,t) < \infty$ and $0 < Q(x,t) < \infty$ for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. The system $d\Sigma_{BC}$ can be transformed into a differentially balanced realization on an open subset $D \subset \mathbb{R}^n \times \mathbb{R}$ by a differential coordinate transformation $\delta z = T(x,t)\delta x$. Moreover, $\sigma_i^2(x,t)$ ($i = 1, \ldots, n$) in (19.5) are the eigenvalues of the product $P(x,t)Q(x,t)$.* □

19.3 Generalized Differential Balancing

19.3.1 Generalized Differential Energy Functions

In the previous section, balancing theory based on the contraction framework is presented, which is a natural extension of linear balancing theory. From an application perspective, it is worth constructing a computationally more feasible method. Here, we present generalized differential balancing, inspired by generalized incremental balancing as in [4].

We generalize concepts of differential energy functions as follows:

Definition 19.8 If there exists a uniformly positive definite matrix $\bar{P}(t) = \bar{P}^{\mathrm{T}}(t)$ such that

$$-\frac{d\bar{P}(t)}{dt} + \frac{\partial f(x,t)}{\partial x}\bar{P}(t) + \bar{P}(t)\frac{\partial^{\mathrm{T}} f(x,t)}{\partial x} \leq -B(t)B^{\mathrm{T}}(t) \quad (19.6)$$

for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ then the function $L_{\mathcal{C}}(\delta x_0, t_0) := \frac{1}{2}\delta x_0^{\mathrm{T}} \bar{P}^{-1}(t_0)\delta x_0$, is said to be a generalized differential controllability function.

Definition 19.9 If there exists a uniformly positive definite matrix $\bar{Q}(t) = \bar{Q}^{\mathrm{T}}(t)$ such that

$$\frac{d\bar{Q}(t)}{dt} + \bar{Q}(t)\frac{\partial f(x,t)}{\partial x} + \frac{\partial^{\mathrm{T}} f(x,t)}{\partial x}\bar{Q}(t) \le -C^{\mathrm{T}}(t)C(t) \tag{19.7}$$

for all $x \in \mathbb{R}^n$, $t \in \mathbb{R}$ then the function $L_{\mathcal{O}}(\delta x_0, t_0) := \frac{1}{2}\delta x_0^{\mathrm{T}} \bar{Q}(t_0)\delta x_0$, is said to be a generalized differential observability function.

Remark 19.10 If we compare (19.1) and (19.4) with (19.6) and (19.7), respectively, we notice that equalities are relaxed into inequalities.

Note that these energy functions are the generalized controllability and observability functions for $d\Sigma_{BC}$, respectively. Also, in the linear case, these two functions are nothing but the generalized controllability and observability functions, respectively. Similar to the linear case, generalized controllability and observability functions are not unique, but they provide a lower bound for the differential controllability function and an upper bound for the differential observability function.

Theorem 19.11 *Suppose that the differential controllability function $L_{\mathcal{C}}(x_0, \delta x_0, t_0)$ and a generalized differential controllability function $\bar{L}_{\mathcal{C}}(\delta x_0, t_0)$ exist. Then,*

$$\bar{L}_{\mathcal{C}}(\delta x_0, t_0) \le L_{\mathcal{C}}(x_0, \delta x_0, t_0)$$

for all $x_0 \in \mathbb{R}^n$, $\delta x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$. □

Theorem 19.12 *Suppose that the differential observability function $L_{\mathcal{O}}(x_0, \delta x_0, t_0)$ and a generalized differential observability function $\bar{L}_{\mathcal{O}}(\delta x_0, t_0)$ exist. Then,*

$$\bar{L}_{\mathcal{O}}(\delta x_0, t_0) \ge L_{\mathcal{O}}(x_0, \delta x_0, t_0)$$

for all $x_0 \in \mathbb{R}^n$, $\delta x_0 \in \mathbb{R}^n$, $t_0 \in \mathbb{R}$. □

19.3.2 Boundedness of Trajectories

Existence of the differential controllability and observability functions is not directly related to controllability and observability, which is the case for linear systems. However, existence of these differential energy functions implies boundedness of trajectories of $d\Sigma_{BC}$.

Theorem 19.13 *If there exists a generalized differential controllability function, then $\delta x(t)$ of the system $d\Sigma_{BC}$ is bounded for any $x_0, \delta x_0 \in \mathbb{R}^n$, $u, \delta u \in L_2^m[0, \infty)$.*

Proof By differentiating the generalized differential controllability function $\bar{L}_{\mathcal{C}}(\delta x, t)$ with respect to t, we have

$$\begin{aligned}
&\frac{d\bar{L}_{\mathcal{C}}(\delta x(t), t)}{dt} \\
&= \frac{1}{2}\frac{d}{dt}\left(\delta x^{\mathrm{T}}(t)\bar{P}^{-1}(t)\delta x^{\mathrm{T}}(t)\right) \\
&= \frac{1}{2}\delta x^{\mathrm{T}}(t)\frac{d\bar{P}^{-1}(t)}{dt}\delta x(t) + \frac{1}{2}\left(\delta^{\mathrm{T}}x(t)\frac{\partial^{\mathrm{T}} f(x(t), t)}{\partial x} + \delta^{\mathrm{T}}u(t)B^{\mathrm{T}}(t)\right)\bar{P}^{-1}(t)\delta x(t) \\
&\quad + \frac{1}{2}\delta^{\mathrm{T}}x(t)\bar{P}^{-1}(t)\left(\frac{\partial f(x(t), t)}{\partial x}\delta x(t) + B(t)\delta u(t)\right)
\end{aligned}$$

From (19.6) and $d\bar{P}^{-1}(t)/dt = -\bar{P}^{-1}(t)(d\bar{P}(t)/dt)\bar{P}^{-1}(t)$, we obtain

$$\begin{aligned}
&\frac{d\bar{L}_{\mathcal{C}}(\delta x(t), t)}{dt} \\
&\le -\frac{1}{2}\delta x^{\mathrm{T}}(t)\bar{P}^{-1}(t)B(t)B^{\mathrm{T}}(t)\bar{P}^{-1}(t)\delta x(t) + \frac{1}{2}\delta^{\mathrm{T}}u(t)B^{\mathrm{T}}(t)\bar{P}^{-1}(t)\delta x(t) \\
&\quad + \frac{1}{2}\delta^{\mathrm{T}}x(t)\bar{P}^{-1}(t)B(t)\delta u(t) \\
&= \frac{1}{2}||\delta u(t)||^2 - \frac{1}{2}||\delta u(t) - B(t)\bar{P}^{-1}(t)\delta x(t)||^2 \le \frac{1}{2}||\delta u(t)||^2.
\end{aligned}$$

By integrating this inequality, we have

$$\bar{L}_{\mathcal{C}}(\delta x(t), t) \le \bar{L}_{\mathcal{C}}(\delta x_0, t_0) + \frac{1}{2}\int_{t_0}^{t} ||\delta u(\tau)||^2 d\tau. \tag{19.8}$$

Since the right-hand side is bounded, the left-hand side is also bounded. Moreover, $\bar{P}(t)$ is uniformly positive definite, which implies that $\delta x(t)$ is bounded. □

Theorem 19.14 *If there exists a generalized differential observability function, then there exists a positive real number* α *such that* $||\delta x(t)||^2 \le \alpha||\delta x_0||^2$ *for system* $d\Sigma_C$. *Moreover,* $\lim_{t\to\infty} ||\delta y(t)||^2 = 0$ *holds.*

Proof By differentiating differential observability function $\bar{L}_{\mathcal{O}}(\delta x(t), t)$ with respect to t, from its definition, we have

$$\begin{aligned}
\frac{d\bar{L}_{\mathcal{O}}(\delta x(t), t)}{dt} &= \frac{1}{2}\frac{d}{dt}\left(\delta x^{\mathrm{T}}(t)\bar{Q}(t)\delta x^{\mathrm{T}}(t)\right) \\
&= \frac{1}{2}\delta x^{\mathrm{T}}(t)\frac{d\bar{Q}(t)}{dt}\delta x(t) + \frac{1}{2}\delta^{\mathrm{T}}x(t)\frac{\partial^{\mathrm{T}} f(x(t), t)}{\partial x}\bar{Q}(t)\delta x(t) \\
&\quad + \frac{1}{2}\delta^{\mathrm{T}}x(t)\bar{Q}(t)\frac{\partial f(x(t), t)}{\partial x}\delta x(t) \\
&\le -\frac{1}{2}||\delta y(t)||^2 \le 0.
\end{aligned}$$

By integrating this inequality,

$$\bar{L}_{\mathcal{O}}(\delta x(t), t) \leq \bar{L}_{\mathcal{O}}(\delta x_0, t_0) - \frac{1}{2}\int_{t_0}^{t} ||\delta y(\tau)||^2 d\tau \leq \bar{L}_{\mathcal{O}}(\delta x_0, t_0). \quad (19.9)$$

The uniform positive definiteness of $\bar{Q}(t)$ implies that there exist $\alpha_2 \geq \alpha_1 > 0$ such that

$$\alpha_1 ||\delta x(t)||^2 \leq \bar{L}_{\mathcal{O}}(\delta x(t), t) \leq \bar{L}_{\mathcal{O}}(\delta x_0, t_0) \leq \alpha_2 ||\delta x_0||^2,$$

and consequently $||\delta x(t)||^2 \leq \frac{\alpha_2}{\alpha_1}||\delta x_0||^2$.

On the other hand, (19.9) implies

$$\frac{1}{2}\int_{t_0}^{\infty} ||\delta y(\tau)||^2 d\tau \leq \bar{L}_{\mathcal{O}}(\delta x_0, t_0) - \lim_{t\to\infty} \bar{L}_{\mathcal{O}}(\delta x(t), t) \leq \bar{L}_{\mathcal{O}}(\delta x_0, t_0). \quad (19.10)$$

Since $\bar{L}_{\mathcal{O}}(\delta x_0, t_0)$ is bounded, from Barbalat's lemma $\lim_{t\to\infty} ||\delta y(t)||^2 = 0$. □

Remark 19.15 For a generalized controllability or observability function, if there exists a positive real number α such that

$$-\frac{d\bar{P}(t)}{dt} + \frac{\partial f(x,t)}{\partial x}\bar{P}(t) + \bar{P}(t)\frac{\partial^{\mathrm{T}} f(x,t)}{\partial x} \leq -\alpha I_n$$

or

$$\frac{d\bar{Q}(t)}{dt} + \bar{Q}(t)\frac{\partial f(x,t)}{\partial x} + \frac{\partial^{\mathrm{T}} f(x,t)}{\partial x}\bar{Q}(t) \leq -\alpha I_n, \quad (19.11)$$

then $\mathbb{R}^n$ is a contraction region [11] with respect to the uniformly positive definite metric $\bar{P}(t)$ or $\bar{Q}(t)$, respectively. That is, any trajectory of the system Σ_{BC} is bounded.

19.3.3 The Generalized Differentially Balanced Realization

We are now ready to define a generalized differentially balanced realization in terms of the generalized differential controllability and observability functions.

Definition 19.16 A realization of $d\Sigma_{BC}$ is said to be a generalized differentially balanced realization on an open subset $D \subset \mathbb{R}$ if there exists a diagonal matrix

$$\bar{\Lambda}(t) = \mathrm{diag}\{\bar{\sigma}_1(t), \bar{\sigma}_2(t), \ldots, \bar{\sigma}_n(t)\}, \quad (19.12)$$

where $\bar{\sigma}_1(t) \geq \bar{\sigma}_2(t) \geq \cdots \geq \bar{\sigma}_n(t) > 0$ on D holds, and $\bar{P}(t) = \bar{\Lambda}(t)$ and $\bar{Q}(t) = \bar{\Lambda}(t)$.

Theorem 19.17 *Let $\bar{L}_{\mathcal{C}}(\delta x_0, t_0)$ and $\bar{L}_{\mathcal{O}}(\delta x_0, t_0)$ be generalized differential controllability and observability functions, respectively. For every system Σ_{BC}, there exists a coordinate transformation $z = T(t)x$ which transforms $d\Sigma_{BC}$ into a generalized differentially balanced realization on a domain $D \subset \mathbb{R}$. Also $\bar{\sigma}_i^2(t)$ $(i = 1, \ldots, n)$ in (19.12) are the eigenvalues of $\bar{P}(t)\bar{Q}(t)$.*

Proof In a similar manner as for the linear case, it can be shown that there exists a class C^1 and invertible matrix $T(t) : \mathbb{R} \to \mathbb{R}^{n\times n}$ which achieves $T(t)\bar{P}(t)T^{\mathrm{T}}(t) = \bar{\Lambda}(t)$ and $T^{-\mathrm{T}}(t)\bar{Q}(t)T^{-1}(t) = \bar{\Lambda}(t)$, where $\bar{\Lambda}(t) = \mathrm{diag}\{\bar{\sigma}_1(t), \ldots, \bar{\sigma}_n(t)\}$, and $\bar{\sigma}_i(t) > 0$ $(i = 1, \ldots, n)$. Moreover, $T(t)$ can be chosen such that $\bar{\sigma}_1(t) \geq \cdots \geq \bar{\sigma}_n(t)$ in a sufficiently small open subset $D \subset \mathbb{R}$. Finally, $\bar{P}(t)\bar{Q}(t) = T^{-1}(t)\bar{\Lambda}^2(t)T(t)$ implies that $\bar{\sigma}_i^2(t)$ $(i = 1, \ldots, n)$ are eigenvalues of $\bar{P}(t)\bar{Q}(t)$. □

19.3.4 Model Reduction and Error Bound

Now we can provide a model reduction procedure based on the generalized differentially balanced realization. Moreover, we establish and estimate of the error bound for the model reduction procedure.

In (19.12), suppose that $\bar{\sigma}_k(t) > \bar{\sigma}_{k+1}(t)$ for $k < n$, which implies that z_k is more important than z_{k+1} in the sense of generalized differential energy. Hence, z_1 until z_k are more important than z_{k+1} until z_n. A possibility to reduce the number of states is by truncation, i.e., to put $z_{k+1} = 0, \ldots, z_n = 0$. We partition the system in the z-coordinates correspondingly as follows:

$$\bar{f}(z,t) = \begin{bmatrix} \bar{f}_a(z_a, z_b, t) \\ \bar{f}_b(z_a, z_b, t) \end{bmatrix} := T(t)f(T^{-1}(t)z(t), t),\ \ \bar{B}(t) = \begin{bmatrix} \bar{B}_a(t) \\ \bar{B}_b(t) \end{bmatrix} := T(t)B(t),$$
$$\bar{C}(t) = \begin{bmatrix} \bar{C}_a(t) \ \bar{C}_b(t) \end{bmatrix} := C(t)T^{-1}(t),$$

where $z_a := [z_1, \ldots, z_k]^{\mathrm{T}}$ and $z_b := [z_{k+1}, \ldots, z_n]^{\mathrm{T}}$.

The reduced order system is obtained by simply substituting $z_a = \bar{z}_a$ and $z_b = 0$.

$$\Sigma^r \begin{cases} \dot{\bar{z}}_a(t) = \bar{f}_a(\bar{z}_a(t), 0, t) + \bar{B}_a(t)u(t) \\ \bar{y}_a(t) = \bar{C}_a(t)\bar{z}_a(t) \end{cases}.$$

Theorem 19.18 *The state-space realization of reduced order system Σ_{BC}^r is a generalized differential balanced realization with singular value functions $\bar{\sigma}_1(t) \geq \cdots \geq \bar{\sigma}_k(t)$.*

Proof Equations (19.6) and (19.7) in the z-coordinates are

$$
\begin{aligned}
&-\frac{d}{dt}\bar{\Lambda}(t) + \bar{\Lambda}(t)\begin{bmatrix} \frac{\partial \bar{f}_a}{\partial z_a} & \frac{\partial \bar{f}_a}{\partial z_b} \\ \frac{\partial \bar{f}_b}{\partial z_a} & \frac{\partial \bar{f}_b}{\partial z_b} \end{bmatrix}^{\mathrm{T}} (z_a, z_b, t) + \begin{bmatrix} \frac{\partial \bar{f}_a}{\partial z_a} & \frac{\partial \bar{f}_a}{\partial z_b} \\ \frac{\partial \bar{f}_b}{\partial z_a} & \frac{\partial \bar{f}_b}{\partial z_b} \end{bmatrix} (z_a, z_b, t)\bar{\Lambda}(t) \\
&\le -\begin{bmatrix} \bar{B}_a\bar{B}_a^{\mathrm{T}} & \bar{B}_a\bar{B}_b^{\mathrm{T}} \\ \bar{B}_a\bar{B}_b^{\mathrm{T}} & \bar{B}_b\bar{B}_b^{\mathrm{T}} \end{bmatrix}(t), \\
&\frac{d}{dt}\bar{\Lambda}(t) + \begin{bmatrix} \frac{\partial \bar{f}_a}{\partial z_a} & \frac{\partial \bar{f}_a}{\partial z_b} \\ \frac{\partial \bar{f}_b}{\partial z_a} & \frac{\partial \bar{f}_b}{\partial z_b} \end{bmatrix}^{\mathrm{T}} (z_a, z_b, t)\bar{\Lambda}(t) + \bar{\Lambda}(t)\begin{bmatrix} \frac{\partial \bar{f}_a}{\partial z_a} & \frac{\partial \bar{f}_a}{\partial z_b} \\ \frac{\partial \bar{f}_b}{\partial z_a} & \frac{\partial \bar{f}_b}{\partial z_b} \end{bmatrix} (z_a, z_b, t) \\
&\le -\begin{bmatrix} \bar{C}_a^{\mathrm{T}}\bar{C}_a & \bar{C}_b^{\mathrm{T}}\bar{C}_a \\ \bar{C}_b^{\mathrm{T}}\bar{C}_a & \bar{C}_b^{\mathrm{T}}\bar{C}_b \end{bmatrix}(t).
\end{aligned}
$$

Let $\bar{\Lambda}_k(t) := \mathrm{diag}\{\bar{\sigma}_1(t), \dots, \bar{\sigma}_k(t)\}$. For $z_a = \bar{z}_a$ and $z_b = 0$, the upper left $k \times k$ matrix equations become

$$
\begin{aligned}
&-\frac{d}{dt}\bar{\Lambda}_k(t) + \bar{\Lambda}_k(t)\frac{\partial^{\mathrm{T}} \bar{f}_a(\bar{z}_a, 0, t)}{\partial z_a} + \frac{\partial \bar{f}_a(\bar{z}_a, 0, t)}{\partial z_a}\bar{\Lambda}_k(t) \le -\bar{B}_a(t)\bar{B}_a^{\mathrm{T}}(t), \\
&\frac{d}{dt}\bar{\Lambda}_k(t) + \frac{\partial^{\mathrm{T}} \bar{f}_a(\bar{z}_a, 0, t)}{\partial z_a}\bar{\Lambda}_k(t) + \bar{\Lambda}_k(t)\frac{\partial \bar{f}_a(\bar{z}_a, 0, t)}{\partial z_a} \le -\bar{C}_a^{\mathrm{T}}(t)\bar{C}_a(t).
\end{aligned}
$$

Thus, $(1/2)d\bar{z}_a^{\mathrm{T}}(t_0)\bar{\Lambda}_k^{-1}(t_0)d\bar{z}_a(t_0)$ and $(1/2)d\bar{z}_a^{\mathrm{T}}(t_0)\bar{\Lambda}_k(t_0)d\bar{z}_a(t_0)$ are a generalized differential controllability and observability functions for Σ_{BC}^r, respectively. □

Remark 19.19 For the reduced order system Σ_{BC}^r, Theorems 19.13 and 19.14 hold.

Next, we estimate an error bound of the trajectories of the original and reduced system. Consider the dynamics of the error $\xi := z - \bar{z}$,

$$
\begin{cases}
\dot{\xi}_a(t) = \bar{f}_a(\xi_a(t) + \bar{z}_a(t), \xi_b(t), t) - \bar{f}_a(\bar{z}_a(t), 0, t), \\
\dot{\xi}_b(t) = \bar{f}_b(\xi_a(t) + \bar{z}_a(t), \xi_b(t), t) + \bar{B}_b(t)u(t), \\
y_\xi(t) = \bar{C}(t)\xi(t),
\end{cases}
\tag{19.13}
$$

where $\xi_b \equiv z_b$. Since $\bar{z}_a(t) \in \mathbb{R}^k$ can be seen as an external function of time, the associated system of differential dynamics is

$$
\begin{cases}
\delta\dot{\xi}_a(t) = \dfrac{\partial \bar{f}_a(\xi_a(t) + \bar{z}_a(t), \xi_b(t), t)}{\partial \xi_a(t)}\delta\xi_a(t) + \dfrac{\partial \bar{f}_a(\xi_a(t) + \bar{z}_a(t), \xi_b(t), t)}{\partial \xi_b(t)}\delta\xi_b(t), \\
\delta\dot{\xi}_b(t) = \dfrac{\partial \bar{f}_b(\xi_a(t) + \bar{z}_a(t), \xi_b(t), t)}{\partial \xi_a(t)}\delta\xi_a(t) + \dfrac{\partial \bar{f}_b(\xi_a(t) + \bar{z}_a(t), \xi_b(t), t)}{\partial \xi_b(t)}\delta\xi_b(t) \\
\qquad\quad + \bar{B}_b(t)\delta u(t), \\
\delta y_\xi(t) = \bar{C}(t)\delta\xi(t),
\end{cases}
$$

where $\delta\xi_b \equiv \delta z_b$. We can upper bound the effect of δu on δy_ξ as follows:

Theorem 19.20 *Consider the error dynamics* (19.13). *Suppose that* $\bar{\sigma}_1(t) \geq \cdots \geq \bar{\sigma}_k(t) > \bar{\sigma}_{k+1}(t) \geq \cdots \geq \bar{\sigma}_n(t) > 0$ *for all* $t \geq t_0 \in \mathbb{R}^n$; $\delta z(t_0) = \delta \bar{z}(t_0) = 0$. *Then, for all* $t \in [t_0, \infty)$,

$$||\delta y_\xi(\tau)||_{L_2^p[t_0,t]} \leq 2 \sum_{i=k+1}^{n} ||\bar{\sigma}_i(\tau)\delta u(\tau)||_{L_2^m[t_0,t]}. \tag{19.14}$$

Proof Suppose that $k = n-1$. Consider the dynamics of $\eta := z + \bar{z}$:

$$\begin{cases} \dot{\eta}_a(t) = \bar{f}_a(\eta_a(t) - \bar{z}_a(t), \eta_b(t), t) + \bar{f}_a(\bar{z}_a(t), 0, t) + 2\bar{B}_a(t)u(t), \\ \dot{\eta}_b(t) = \bar{f}_b(\eta_a(t) - \bar{z}_a(t), \eta_b(t), t) + \bar{B}_b(t)u(t), \\ y_\eta(t) = \bar{C}(t)\eta(t), \end{cases}$$

where $z_b \equiv \eta_b$, and its associated system of differential dynamics is

$$\begin{cases} \delta\dot{\eta}_a(t) = \dfrac{\partial \bar{f}_a(\eta_a(t) - \bar{z}_a(t), \eta_b(t), t)}{\partial \eta_a(t)}\delta\eta_a(t) + \dfrac{\partial \bar{f}_a(\eta_a(t) - \bar{z}_a(t), \eta_b(t), t)}{\partial \eta_b(t)}\delta\eta_b(t) \\ \qquad +2\bar{B}_a(t)\delta u(t), \\ \delta\dot{\eta}_b(t) = \dfrac{\partial \bar{f}_b(\eta_a(t) - \bar{z}_a(t), \eta_b(t), t)}{\partial \eta_a(t)}\delta\eta_a(t) + \dfrac{\partial \bar{f}_b(\eta_a(t) - \bar{z}_a(t), \eta_b(t), t)}{\partial \eta_b(t)}\delta\eta_b(t) \\ \qquad +\bar{B}_b(t)\delta u(t), \\ \delta y_\eta(t) = C(t)\delta\eta(t). \end{cases}$$

By using $\bar{\Lambda}(t)$ in (19.12), denote two differential energy functions.

$$\begin{aligned} 2\bar{L}_{\mathcal{C}}(\eta(t), \delta\eta(t), t) &:= \delta\eta^{\mathrm{T}}(t)\bar{\Lambda}^{-1}(t)\delta\eta(t), \\ 2\bar{L}_{\mathcal{O}}(\xi(t), \delta\xi(t), t) &:= \delta\xi^{\mathrm{T}}(t)\bar{\Lambda}(t)\delta\xi(t). \end{aligned}$$

Since $\bar{\Lambda}$ satisfies (19.6) and (19.7), we obtain

$$\begin{aligned} 2\dot{\bar{L}}_{\mathcal{C}}(\eta(t), \delta\eta(t), t) \leq{}& -\delta\eta^{\mathrm{T}}\bar{\Lambda}^{-1}\bar{B}\bar{B}^{\mathrm{T}}\bar{\Lambda}^{-1}\delta\eta + 2\delta u^{\mathrm{T}}\bar{B}_a^{\mathrm{T}}\bar{\Lambda}_{n-1}^{-1}\delta\eta_a \\ & +2\delta\eta_a^{\mathrm{T}}\bar{\Lambda}_{n-1}^{-1}\bar{B}_a\delta u + \delta u^{\mathrm{T}}\bar{\sigma}_n^{-1}\bar{B}_b^{\mathrm{T}}\delta\eta_b \\ & + \delta\eta_b^{\mathrm{T}}\bar{B}_b\bar{\sigma}_n^{-1}\delta u, \\ 2\dot{\bar{L}}_{\mathcal{O}}(\xi(t), \delta\xi(t), t) \leq{}& -\delta\xi^{\mathrm{T}}\bar{C}\bar{C}^{\mathrm{T}}\delta\xi + \delta u^{\mathrm{T}}\bar{\sigma}_n\bar{B}_b^{\mathrm{T}}\delta\xi_b + \delta\xi_b^{\mathrm{T}}\bar{B}_b\bar{\sigma}_n\delta u. \end{aligned}$$

Because of $\delta\xi_b \equiv \delta\eta_b \equiv \delta x_b$, we have

$$\begin{aligned} & 2\dot{\bar{L}}_{\mathcal{O}}(\xi(t), \delta\xi(t), t) + 2\bar{\sigma}_n^2(t)\dot{\bar{L}}_{\mathcal{C}}(\eta(t), \delta\eta(t), t) \\ & \leq -\delta\xi^{\mathrm{T}}\bar{C}\bar{C}^{\mathrm{T}}\delta\xi - \bar{\sigma}_n^2\delta\eta^{\mathrm{T}}\bar{\Lambda}^{-1}\bar{B}\bar{B}^{\mathrm{T}}\bar{\Lambda}^{-1}\delta\eta \\ & \qquad + 2\bar{\sigma}_n^2\delta u^{\mathrm{T}}\bar{B}_a^{\mathrm{T}}\bar{\Lambda}_{n-1}^{-1}\delta\eta_a + 2\bar{\sigma}_n^2\delta\eta_a^{\mathrm{T}}\bar{\Lambda}_{n-1}^{-1}\bar{B}_a\delta u \\ & \qquad + 2\bar{\sigma}_n^2\delta u^{\mathrm{T}}\bar{\sigma}_n^{-1}\bar{B}_b^{\mathrm{T}}\delta\eta_b + 2\bar{\sigma}_n^2\delta\eta_b^{\mathrm{T}}\bar{B}_b\bar{\sigma}_n^{-1}\delta u \\ & \leq -||\delta y_\xi||^2 + 4\bar{\sigma}_n^2||\delta u||^2 - \bar{\sigma}_n^2||2\delta u - \bar{B}^{\mathrm{T}}\bar{\Lambda}^{-1}\delta\eta||^2. \end{aligned}$$

Integrating over time we obtain

$$\begin{aligned}&2\bar{L}_{\mathcal{O}}(\xi(t),\delta\xi(t),t)+2\bar{\sigma}_n^2(t)\bar{L}_{\mathcal{C}}(\eta(t),\delta\eta(t),t)-2\bar{L}_{\mathcal{O}}(\xi(t_0),\delta\xi(t_0),t_0)\\&-2\bar{\sigma}_n^2(t_0)\bar{L}_{\mathcal{C}}(\eta(t_0),\delta\eta(t_0),t_0)\\&\leq\int_{t_0}^{t}\left(-||\delta y_\xi||^2+4\bar{\sigma}_n^2||\delta u||^2-\bar{\sigma}_n^2||2\delta u-\bar{B}^{\mathrm{T}}\bar{\Lambda}^{-1}\delta\eta||^2\right)dt.\end{aligned}$$

From $\delta z(t_0)=\delta\bar{z}(t_0)=0$, we obtain $\delta\eta(t_0)=\delta\xi(t_0)=0$ and thus

$$\begin{aligned}\bar{L}_{\mathcal{O}}(\xi(t_0),\delta\xi(t_0),t_0)=0,\\\bar{L}_{\mathcal{C}}(\eta(t_0),\delta\eta(t_0),t_0)=0.\end{aligned}$$

Because of $\bar{L}_{\mathcal{O}}(\xi(t),\delta\xi(t),t)>0$, $\bar{L}_{\mathcal{C}}(\eta(t),\delta\eta(t),t)>0$ and $\bar{\sigma}_n||2\delta u-\bar{B}^{\mathrm{T}}\bar{\Lambda}^{-1}\delta\eta||\geq 0$, we have

$$||\delta y_\xi(\tau)||_{L_p^2[t_0,t]}\leq 2||\bar{\sigma}_n(\tau)\delta u(\tau)||_{L_m^2[t_0,t]}.$$

By repeating this procedure for $i=n,\dots,k$, we obtain (19.14). □

19.3.5 Example

We apply model reduction based on generalized differential balancing on a system composed by 100 mass-spring-damper systems with nonlinear springs, see Fig. 19.1, where k_l and k_n are, respectively, spring constants of linear and nonlinear springs, and $m=k_l=d=1$ and $k_n=2$. The characteristic of the nonlinear springs is provided in the state-space description. The original state-space representation has 200 states, f, B and C are given by

$$\begin{aligned}&f_{2i-1}=x_{2i}\ (i=1,\dots,100),\\&f_2=-x_{2i-1}+x_{2i+1}-2(x_{2i-1}-x_{2i+1})^3-x_{2i}+x_{2i+2},\\&f_{2i}=-x_{2i-1}+x_{2i-3}-2(x_{2i-1}-x_{2i-3})^3-x_{2i-1}+x_{2i+1}-2(x_{2i-1}-x_{2i+1})^3\\&\qquad-x_{2i}+x_{2i-2}-x_{2i}+x_{2i+2}\ (i=2,\dots,99),\\&f_{200}=-x_{199}+x_{197}-2(x_{199}-x_{197})^3-x_{200}+x_{198},\\&B=\left[0\cdots 0\ 1\right]^{\mathrm{T}},\ C=\left[0\cdots 0\ 1\ 0\right],\end{aligned}$$

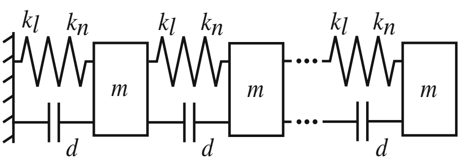

Fig. 19.1 Mass-spring-damper systems with nonlinear springs

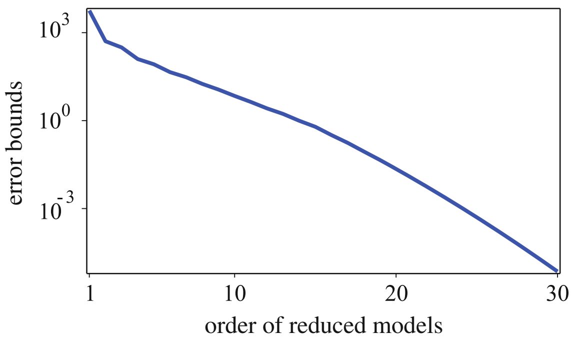

Fig. 19.2 Error bound versus order of reduced model

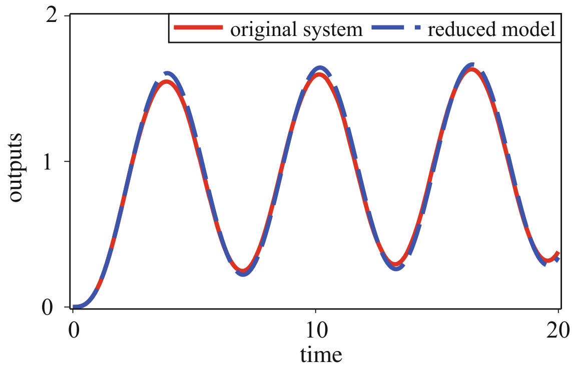

Fig. 19.3 Output trajectories of 200-dimensional original system and 20-dimensional reduced order model

where x_{2i-1} and x_{2i} $(i = 1, \ldots, 100)$ are, respectively, position and velocity of the ith mass-spring-damper subsystems. By solving both (19.6) and (19.7), we obtain positive definite matrices, and consequently the system can be transformed into a generalized differential balanced realization. Thus, we can provide an error bound for model reduction using Theorem 19.20, which is shown in Fig. 19.2. For example, it can be seen that the error bound is less than 2.24×10^{-2} for the 20-dimensional reduced order model. Figure 19.3 shows output trajectories of the original system and reduced order model starting from zero initial states and input $u(t) = \sin t$.

19.4 Conclusion

In this chapter, we have presented results on generalized differential balancing for nonlinear systems, which provides an approximation method for balanced truncation with differential balancing constructed in the contraction framework. Generalized differential balancing is based on two energy functions called generalized differential controllability and observability functions. The existences of these generalized differential energy functions guarantee boundedness of trajectories of variational systems of the nonlinear systems, which is preserved under model reduction. We also provide error bounds for model reduction based on generalized differential balancing. The simulation results for a 20-dimensional reduced order model from a system composed of 100 mass-spring-damper systems show a good approximation of the original 200 order model to a sinusoidal input signal.

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