Chapter 18
Examples on Stability for Infinite-Dimensional Systems

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Abstract By means of examples, we study stability of infinite-dimensional linear and nonlinear systems. First we show that having a (strict) Lyapunov function does not imply asymptotic stability, even not for linear systems. Second, we show that to conclude (local) exponential stability from the linearization, care must be taken how the linearization is obtained.

18.1 Introduction

I met Arjan for the first time when he was presenting his colloquium for his Ph.D defence. He had already left for Twente, and 4 years later I would follow him. Although we were colleagues for many years, our research did not touch. Arjan worked on nonlinear system described by ordinary differential equations, and I was working on linear systems, described by partial differential equations. This changed when Arjan started to study port-Hamiltoninan systems described by partial differential equations. After some prior discussions, also together with Goran Golo, Arjan, and I joined forces in the Ph.D. project of Javier Villegas. From that time on port-Hamiltonian systems is really one of my research directions. Also inspired by Arjans work is my more recent interest in nonlinear systems. The present paper is a result of this.

For finite-dimensional systems the following two facts are well known and used regularly when studying stability. If there exists a Lyapunov function V such that $\dot{V} < 0$, then the equilibrium point is asymptotically stable. Second, if the linearization of a nonlinear differential equation around a equilibrium point is exponentially stable,

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then the equilibrium point is locally exponentially stable for the original equation. We address these questions for infinite-dimensional systems. That is, we study the following abstract differential equation

$$\dot{x}(t) = Ax(t) + f(x(t)), \qquad x(0) = x_0, \tag{18.1}$$

where A is the infinitesimal generator of a C_0-semigroup on the Hilbert space X, and $f : X \mapsto X$ is a locally Lipschitz continuous function with $f(0) = 0$. Under these conditions, the abstract differential equation possesses for every initial condition x_0 a unique (local) solution, see e.g. [2, Chap. 6], and so we can study the stability of the equilibrium point $x_{eq} = 0$.

In the following section, we show that having a Lyapunov function V satisfying $\dot{V}(x) < 0$ for every $x \neq 0$ does not have to imply that the equilibrium solution is stable. We can even construct a linear counter example.

In Sect. 18.3, we study the question whether the exponential stability of the C_0-semigroup generated by A implies the same for the nonlinear equation (18.1). We recall a positive result, but show by means of a simple example that the conditions in this theorem cannot be weakened.

We end this introduction by introducing some notation. We denote the domain of the operator A by $D(A)$, and the class of bounded, linear operators from X to X by $\mathcal{L}(X)$. Furthermore, the semigroup generated by A is denoted by $(T(t))_{t\geq 0}$. We say that the semigroup $(T(t))_{t\geq 0}$ exponentially stable, when there exists a M and $\omega_0 > 0$ such that $\|T(t)\| \leq Me^{-\omega_0 t}$. It is asymptotically (or strongly) stable when $\lim_{t\to\infty} T(t)x_0 = 0$ for all $x_0 \in X$.

18.2 Strict Lyapunov Function Does Not Imply Asymptotic Stability

Let X be the Hilbert space $L^2(0, \infty)$ equipped with the inner product

$$\langle f, g\rangle := \int_0^\infty f(\zeta)\overline{g(\zeta)}(e^{-\zeta} + 1)d\zeta,$$

and let the operators $T(t) : X \to X$, $t \geq 0$, be defined by

$$(T(t)f)(\zeta) := f(\zeta - t) \text{ for } \zeta > t \text{ and zero otherwise.}$$

Hence $T(t)$ is shifting the function f to the right. It is not hard to show that $(T(t))_{t\geq 0}$ is a C_0-semigroup on X, see e.g. [1].

Since

$$\begin{aligned}\|T(t)f\|^2 &= \int_t^\infty |f(\zeta - t)|^2(e^{-\zeta} + 1)d\zeta \\ &= \int_0^\infty |f(\zeta)|^2(e^{-\zeta+t} + 1)d\zeta \\ &\geq \int_0^\infty |f(\zeta)|^2(e^{-\zeta} + 1)d\zeta = \|f\|^2\end{aligned}$$

we see that $T(t)$ is not asymptotically stable. In fact, for every nonzero f, $T(t)f$ does not converge to zero.

The infinitesimal generator A associated to this semigroup is given by

$$Af = -\frac{df}{d\zeta}$$

with domain

$$D(A) = \{f \in X \mid f \text{ is absolutely continuous}, \frac{df}{d\zeta} \in X, \text{ and } f(0) = 0\}.$$

Consider next the standard Lyapunov function $V(x) = \|x\|^2$. Then for $x \in D(A)$,

$$\begin{aligned}\dot{V}(x) &= \langle Ax, x\rangle + \langle x, Ax\rangle \\ &= \int_0^\infty \left[-x(\zeta)'\overline{(\zeta)} - x(\zeta)\overline{x(\zeta)'}\right](e^{-\zeta} + 1)d\zeta \\ &= \left[-|x(\zeta)|^2(e^{-\zeta} + 1)\right]_0^\infty - \int_0^\infty |x(\zeta)|^2 e^{-\zeta} d\zeta \\ &= -\int_0^\infty |x(\zeta)|^2 e^{-\zeta} d\zeta,\end{aligned}$$

where we have used the boundary condition. Since the last expression is nonzero for every $x \neq 0$, we have that

$$\dot{V}(x) < 0, \qquad x \neq 0. \tag{18.2}$$

Hence we have a strict Lyapunov function, whereas the system is not (asymptotically) stable. The reason that this is possible lies in the fact that the trajectories are not precompact. That is, for any $x_0 \neq 0$, the closure of $\{x(t) \mid t \geq 0\}$ is not a compact subset of X. This lack of compactness excludes the use of LaSalle's principle, which is needed to conclude from (18.2) asymptotic stability.

18.3 Linearization and Exponential Stability

One standard technique in finite-dimensional systems to check exponential stability is to check the exponential stability of the linearization. For infinite-dimensional systems, a similar result hold. However, before stating it, we first define two concepts of derivative.

Definition 18.1 For $f : X \mapsto X$ we say that $Df(x)$ is its Fréchet derivative at x if $Df(x)$ is a bounded operator from X to X and

$$\lim_{\|h\|\to 0} \frac{\|f(x+h) - f(x) - (Df)(x)h\|}{\|h\|} = 0.$$

Furthermore, we say that $Df(x)$ is its Gateaux derivative at x if $Df(x)$ is a bounded operator from X to X and if for every $h \in X$ there holds

$$\lim_{\varepsilon\to 0, \varepsilon\in\mathbb{R}} \|\frac{f(x+\varepsilon h) - f(x) - \varepsilon(Df)(x)h}{\varepsilon}\| = 0.$$

Hence the Gateaux derivative calculates the derivative of f by looking at every direction, whereas the Fréchet derivative is uniform. It is easy to see that if f possesses a Fréchet derivative, then it also has a Gateaux derivative, and they are equal.

Using the Fréchet derivative, the linearization result for (18.1) can be formulated. For the proof, we refer to [3].

Theorem 18.2 *Let f have zero Fréchet derivative at zero. If A generates an exponentially stable semigroup on X, then (18.1) is (locally) exponentially stable around zero.*

In the above theorem, we assumed that the Fréchet derivative at the origin was zero. By means of an example, we show that this condition cannot be replaced by the condition that the Gateaux derivative at the origin must be zero.

As state space we take $X = \ell^2(\mathbb{N})$, and we consider the differential equation

$$\dot{x}(t) = -x(t) + f(x(t)), \qquad x(0) = x_0 \tag{18.3}$$

with f given by

$$(f(x))_n = 3\sqrt[n]{(|x_n|)}x_n. \tag{18.4}$$

Hence our system is a diagonal (nonlinear) system with on the diagonal

$$\dot{x}_n(t) = (-1 + 3\sqrt[n]{|x_n(t)|})x_n(t). \tag{18.5}$$

We summarize results of these scalar differential equations in a lemma. The proofs are left to the reader.

Lemma 18.3 *The differential equation (18.5) has the following properties.*

- *The equilibrium's are* $\pm 3^{-n}$ *and zero.*
- *The right-hand side of (18.4) is locally Lipschitz continuous, and for* $|x_n| \leq r$ *the Lipschitz constant can be majorized by* $3(1+\frac{1}{n})\sqrt[n]{r}$.
- *For* $x_n(0) \in (-3^{-n}, 3^{-n})$ *the state converges to zero, and for* $|x_n(0)| > 3^{-n}$ *the state diverges.*
- *For* $|x_n(0)| > 3^{-n}$ *there is a finite escape time.*
- *The linearization of (18.5) around zero is* $\dot{x}_n(t) = -x_n(t)$ *and thus exponentially stable.*

These results are used to characterize the behavior of the nonlinear system (18.3).

Theorem 18.4 *For the nonlinear system (18.3) and (18.4) the following holds.*

1. f *is (locally) Lipschitz continuous from* X *to* X.
2. f *is Gateaux differentiable but not Fréchet at the origin. The Gateaux derivative at the origin is zero.*
3. *The origin is an unstable equilibrium point.*

Proof 1. Let x, z be two elements of X with norm bounded by r. Without loss of generality, we may assume that $r > 1$. Since the norms are bounded by r, the same holds for the absolute value of every element, i.e., $|x_n|, |z_n| \leq r$. Hence we find that

$$\begin{aligned} \|f(x) - f(z)\|^2 &= \sum_{n=1}^{\infty} \left(3\sqrt[n]{(|x_n|)}x_n - 3\sqrt[n]{(|z_n|)}z_n\right)^2 \\ &\leq \sum_{n=1}^{\infty} \left(3(1+\frac{1}{n}\sqrt[n]{r})\right)^2 (x_n - z_n)^2 \\ &\leq \ (6r)^2 \|x-z\|^2, \end{aligned}$$

where we have used Lemma 18.3 and the fact that $r > 1$. Thus f is Lipschitz continuous, and so is the right-hand side of (18.3).

2. We show that the Gateaux derivative of f is zero. This implies that the (Gateaux) linearization of (18.3) is $\dot{x}(t) = -x(t)$.

For $x \in X$ and $\varepsilon \in \mathbb{R}\backslash\{0\}$ we have

$$\begin{aligned} \|\frac{f(0+\varepsilon x) - f(0)}{\varepsilon} - 0\|^2 &= \sum_{n=1}^{\infty} 9\sqrt[n]{\varepsilon^2 x_n^2}x_n^2 \qquad (18.6) \\ &= 9\sum_{n=1}^{\infty} \sqrt[n]{\varepsilon^2}\sqrt[n]{x_n^2}x_n^2 \end{aligned}$$

Next take a $\delta \in (0, 1)$ and choose N such that $\sum_{n=N}^{\infty} x_n(t)^2 \leq \delta$. In particular, this implies that $\sqrt[n]{x_n^2} \leq 1$ for $n \geq N$. Now choose ε such that $|\varepsilon| < 1$ and $\sum_{n=1}^{N-1} \sqrt[n]{\varepsilon^2}\sqrt[n]{x_n^2}x_n^2 \leq \delta$. Combining these two gives that for this ε there holds that

$$\|\frac{f(0+\varepsilon x) - f(0)}{\varepsilon} - 0\|^2 \leq 9(\delta + \delta).$$

Since δ is arbitrarily, this show that

$$\lim_{\varepsilon \to 0} \|\frac{f(0+\varepsilon x) - f(0)}{\varepsilon} - 0\|^2 = 0$$

and so 0 is the Gateaux derivative of (18.5).

If f would be Fréchet differentiable, then its derivative would equal the Gateaux derivative, and thus zero. However, by choosing in Eq. (18.6) $\varepsilon = 1$ and $x = (x_n)_{n\in\mathbb{N}}$ with $x_n = 0$ for $n \neq N$ and $x_N = 2^{-N}$, we see that $\lim\sup_{\|x\|\to 0} \|f(x)\|/\|x\| > 0$.

3. We choose $x(0) = (x_{0n})_{n\in\mathbb{N}}$ with $x_{0n} = 0$ for $n \neq N$ and $x_{0N} = 2^{-N}$. By Lemma 18.3 we see that the Nth equation of (18.3) is unstable, and thus the state $x(t)$ diverge. Since for $N \to \infty$, there holds $\|x(0)\| \to 0$, we see that there exists an initial state arbitrarily close to zero which is unstable. Thus the nonlinear system is not stable in the origin. □

The example is this section is not uniformly Lipschitz continuous, and almost every solution of (18.3) will have finite escape time. The following simple adaptation of (18.5) gives a uniformly Lipschitz continuous differential equation on X,

$$\dot{x}_n(t) = \frac{(-1 + 3\sqrt[n]{|x_n(t)|})x_n(t)}{1 + x_n(t)^2}.$$

References

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