Chapter 17
Network Topology and Synchronization of Systems with Linear Time-Delayed Coupling

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Abstract We consider networks of square input–output systems that interact via linear, time-delayed coupling functions. For given system dynamics, we give conditions for the construction of a (local, global) synchronization diagram. We show that a condition for (local, global) synchronization is that the coupling strength and time-delay are contained in the intersection of scaled copies of the (local, global) synchronization diagram, where the scaling factors are the nonzero eigenvalues of the symmetric Laplacian matrix.

17.1 Introduction

There are many examples of networks of interacting dynamical systems that exhibit collective behavior: Fireflies emit their light pulses at the same instants in time; crickets chirp in unison for extended periods of time; and the electrons move coherently in (arrays of) superconductive Josephson junctions, cf. [22, 30]. The most unambiguous form of collective behavior is that of *synchronization*, which refers to the state in which all systems in the network behave identically. Whether or not a network of systems will synchronize depends on, besides the specific systems' dynamics and coupling functions, the network topology. In this chapter, we consider networks of systems that interact via linear time-delay coupling functions of the form

$$u_i(t) = \sigma \sum_j a_{ij}[y_j(t-\tau) - y_i(t-\tau)] \tag{17.1}$$

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and we relate conditions for synchronization of the systems to the topology of the network. In (17.1) $u_i(t)$ is the input of system i, $y_i(t-\tau)$ and $y_j(t-\tau)$ are the time-delayed outputs of systems i and j, respectively, positive constant σ is the coupling strength, and positive constants a_{ij} are defined by the network. The time-delay τ accounts for sensor and actuator dynamics, in particular, sensor and actuator delays. Such coupling functions appear in, e.g., car-following models [26], where the time-delay, which correlates with the reaction time of the driver, typically takes values between 0.6 and 2 s.

In the delay-free case, i.e., $\tau = 0$, the influence of network topology on synchronization has been studied in [2, 3, 21, 33]. In [33] a conjecture was posed that states that systems in network $\mathcal{G}_1$ synchronize for coupling strength σ_1 if and only if systems in network $\mathcal{G}_2$ synchronize for coupling strength σ_2 and the following relation holds:

$$\sigma_1 \lambda_2(\mathcal{G}_1) = \sigma_2 \lambda_2(\mathcal{G}_2),$$

where constant $\lambda_2(\mathcal{G})$ is the algebraic connectivity of network $\mathcal{G}$ (i.e., the Fiedler eigenvalue of the Laplacian matrix of $\mathcal{G}$) [9]. Although this conjecture was shown to be wrong [20], there is a rich class of systems for which the conjecture seems to hold true, i.e., for those systems that do not show a desynchronizing bifurcation as the coupling strength is increased. A somewhat similar method was proposed in [21], in which the concept of a *Master Stability Function* (MSF) was introduced. In this approach, the coupling parameters (i.e., coupling strength and network topology) are lumped into a single (possibly complex) parameter κ, and subsequently the stability of a linear time-varying system that describes the local dynamics around a synchronous solution is assessed as function of this parameter κ. Then if there exists a nonempty set $\mathcal{K}$ such that for $\kappa \in \mathcal{K}$ the zero solution of this linear system is stable, the condition for synchronization of a network $\mathcal{G}$ is that $\sigma\lambda_j(\mathcal{G}) \in \mathcal{K}$ for all nonzero eigenvalues λ_j of the Laplacian matrix of $\mathcal{G}$. However, it is shown in [15] that the MSF approach might fail if the isolated system (i.e., a single system without coupling) does not have an attractor. Assuming the isolated system to have an attractor might even not be sufficient to conclude that the systems synchronize; It is known that with negative Lyapunov exponents, the criteria used for stability of the MSF, a linear time-varying system may be unstable [14]. In particular, it is shown in [1, 31] that the dynamics of coupled chaotic systems might produce a specific type of intermittent behavior associated with a temporal loss of synchrony; This phenomenon, called attractor bubbling, may occur despite the Lyapunov exponents of the MSF all being negative.

In this chapter we develop a MSF-like approach, which allows the construction of a *local synchronization diagram* $\mathcal{S}$; This local synchronization diagram is the set of coupling strengths σ and time-delays τ for which the zero solution of a particular linear time-varying system is uniformly asymptotically stable. Under the assumption that the isolated system has an attractor with a neighborhood with inflowing boundary, we show that the condition for local synchronization, that is, synchronization of systems whose mutual distance in initial data is small, is that the coupling strength σ and time-delay τ are in the intersection of scaled copies of $\mathcal{S}$. Here the scaling factors are the nonzero eigenvalues of the Laplacian matrix of the network $\mathcal{G}$. See Fig. 17.1

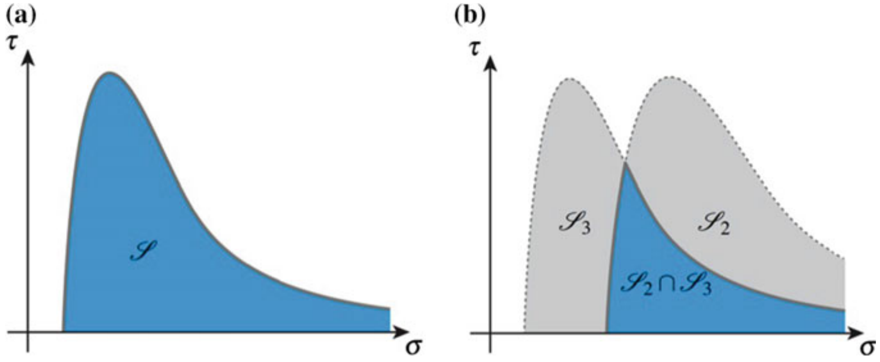

Fig. 17.1 **a** Synchronization diagram $\mathcal{S}$. **b** Two scaled copies of $\mathcal{S}$, denoted by $\mathcal{S}_2$ and $\mathcal{S}_3$, and their intersection

for a graphical example for a network of three systems, where we have assumed the network to be connected and the eigenvalues of the Laplacian matrix of that network to be real. (Under the assumption that a network is connected its Laplacian matrix has a simple zero eigenvalue.) In addition, we present a class of systems for which we are able to construct a global synchronization diagram. The intersection of scaled copies of this global synchronization diagram gives the conditions on σ and τ for which a network of systems synchronizes without requiring the mutual distances in initial data to be small.

The results we present in this chapter are, in part, reported in [27].

Notation We let $\mathbb{R} = (-\infty, \infty)$ denote the real numbers, $\mathbb{R}_+ := \{x \in \mathbb{R} \,|\, x > 0\}$ and $\overline{\mathbb{R}}_+ := \mathbb{R}_+ \cup \{0\}$. For a positive integer n, $\mathbb{R}^n$ is the n-fold Cartesian product $\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}$. We let $|\cdot|$ be the Euclidean norm in $\mathbb{R}^n$: for $x \in \mathbb{R}^n$, $|x| = \sqrt{x^\top x}$ where $^\top$ denotes transposition. We denote by $\otimes$ the Kronecker (tensor) product of two matrices (cf. [13]). We let I_n be the $n \times n$ identity matrix, and $\mathbf{1}_n$ (respectively, $\mathbf{0}_n$) the n-dimensional vector with all entries equal to 1 (respectively, 0). For an $n \times n$-dimensional matrix A we let $\|A\| := \max_{|x|=1} |Ax|$ be the matrix norm induced by $|\cdot|$. Given two sets $\mathcal{X}$ and $\mathcal{Y}$, $\mathcal{C}(\mathcal{X}, \mathcal{Y})$ denotes the set of continuous functions that map $\mathcal{X}$ into $\mathcal{Y}$.

17.2 Problem Setting

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E}, A)$ be an undirected weighted graph with $\mathcal{V} = \{1, 2, \ldots, N\}$ the set of vertices and $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ the set of edges. Recall that $\mathcal{G}$ being an undirected graph means that $\mathcal{E}$ is unordered. $A = \left(a_{ij}\right)$ is the $N \times N$ weighted adjacency matrix:

$$a_{ij} = \begin{cases} w_{ij} & \text{if } (i, j) \in \mathcal{E} \\ 0 & \text{otherwise} \end{cases}$$

where w_{ij} is the weight of edge $(i, j) \in \mathcal{E}$. We suppose that $w_{ij} = w_{ji}$ such that A is symmetric. We shall assume that $\mathcal{G}$ contains no self-loops (i.e., $\mathcal{G}$ has no edges of the form (i, i)) and thus $\mathcal{G}$ is a simple graph. In addition, we shall assume that $\mathcal{G}$ is connected, that is, for every two vertices $i, j \in \mathcal{V}$ there exists a path between i and j.

Letting

$$D = \begin{pmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_N \end{pmatrix} = \begin{pmatrix} \sum_j a_{1j} & & & \\ & \sum_j a_{2j} & & \\ & & \ddots & \\ & & & \sum_j a_{Nj} \end{pmatrix}$$

we define

$$L = D - A$$

to be the Laplacian matrix of $\mathcal{G}$. It is well-known that the Laplacian matrix of a connected graph has a simple zero eigenvalue, cf. [4]. Gerschgorin's Disc Theorem [13] implies that all other eigenvalues (which are real as L is symmetric) are positive. We always order the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_N$ of L nondecreasingly

$$0 = \lambda_1 < \lambda_2 \leq \cdots \leq \lambda_N.$$

We assign each vertex $i \in \mathcal{V}$ the dynamics

$$\begin{cases} \dot{x}_i(t) = f(x_i(t)) + Bu_i(t) \\ y_i(t) = Cx_i(t) \end{cases} \tag{17.2}$$

with state $x_i(t) \in \mathbb{R}^n$, input $u_i(t) \in \mathbb{R}^m$ and output $y_i(t) \in \mathbb{R}^m$, $1 \leq m \leq n$, (sufficiently) smooth vectorfield $f : \mathbb{R}^n \to \mathbb{R}^n$, and matrices B and C of appropriate dimensions with CB similar to a positive definite matrix. Systems (17.2) on $\mathcal{G}$ interact via the following linear time-delay coupling law

$$u_i(t) = \sigma \sum_{j \in \mathcal{N}_i} a_{ij}[y_j(t-\tau) - y_i(t-\tau)], \tag{17.3}$$

where positive constant σ is the coupling strength, nonnegative constant τ is a time-delay, and

$$\mathcal{N}_i = \{j \in \mathcal{V} \mid (i, j) \in \mathcal{E}\}$$

is the set of neighbors of system i. Then the dynamics of the coupled systems (17.2) and (17.3) are given by the following delay-differential equation

$$\dot{x}(t) = F(x(t)) - \sigma(L \otimes BC)x(t-\tau) \tag{17.4}$$

where

$$x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_N(t) \end{pmatrix}, \quad F(x(t)) = \begin{pmatrix} f(x_1(t)) \\ f(x_2(t)) \\ \vdots \\ f(x_N(t)) \end{pmatrix}.$$

The state-space of (17.4) is $\mathcal{C} = \mathcal{C}\left([-\tau, 0], \mathbb{R}^{Nn}\right)$, the space of continuous functions that map the interval $[-\tau, 0]$ into $\mathbb{R}^{Nn}$. For $\phi \in \mathcal{C}$ we let $\|\phi\| := \sup_{-\tau \le \theta \le 0} |\phi(\theta)|$. We remark that we also use the notation $\| \cdot \|$ for the induced matrix norm, however, no confusion should arise. Given $t \geq 0$, for $x_t \in \mathcal{C}$ we let $x_t(\theta) := x(t + \theta)$, $-\tau \leq \theta \leq 0$. For given initial data $\phi \in \mathcal{C}$ and a constant $T > 0$, a solution of (17.4) is a function $x_t = x_t(\cdot) = x_t(\cdot; \phi) \in \mathcal{C}$ such that $x_0 = \phi$ and x_t satisfies (17.4) for all $t \in [0, T)$. We assume that the solutions of our coupled systems are uniformly (ultimately) bounded (see [5] for a definition) such that $T = \infty$. Conditions for (ultimate) boundedness expressed at the level of the systems' dynamics can be found in [27, 28]. We shall write $x(t; \phi)$ instead of $x_t(0; \phi)$.

A solution x_t of the coupled systems (17.4) is a *synchronous solution* if and only if

$$x_t(\theta) = I_N \otimes s_t(\theta), \quad \forall \theta \in [-\tau, 0], \quad \forall t \geq 0,$$

where $s_t \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$. Note that, because coupling (17.3) is noninvasive, the asymptotic synchronous solution s_t satisfies the ordinary differential equation

$$\dot{s}(t) = f(s(t)).$$

The coupled systems (17.4) are said to *synchronize* if its solutions converge asymptotically to a synchronous solution:

$$\lim_{t \to \infty} \|x_t - I_N \otimes s_t\| = 0.$$

17.3 Conditions for Local Synchronization

We address first the problem of *local* synchronization, i.e., synchronization of systems with initial data that satisfy

$$\|\phi_i - \phi_j\| < \delta, \quad \phi_i, \phi_j \in \mathcal{C}\left([-\tau, 0], \mathbb{R}^n\right)$$

with δ some small positive constant. We consider the case that the isolated system

$$\dot{s}(t) = f(s(t))$$

has an attractor $\mathcal{A}$ with basin of attraction $\mathcal{B}$. We suppose that there is a neighborhood $\mathcal{U}$ of $\mathcal{A}$ contained in $\mathcal{B}$, and we let $\overline{\mathcal{U}}$ and $\partial\mathcal{U}$ be the closure of $\mathcal{U}$, respectively, the boundary of $\mathcal{U}$. We remark that in general such a neighborhood $\mathcal{U}$ does not need to exist, i.e., when $\mathcal{A}$ is a weak attractor [16]. Furthermore, we assume that $\mathcal{U}$ is *inflowing invariant* with respect to the vectorfield f [8, 32]; That is, there is a positive constant μ such that

$$\langle N(s), f(s)\rangle \leq -\mu, \quad \forall s \in \partial\mathcal{U},$$

where $N(s)$ is the outward normal of $\partial\mathcal{U}$ at point s and $\langle\cdot,\cdot\rangle$ is the innerproduct in $\mathbb{R}^n$. We denote

$$\mathcal{C}_{\mathcal{U}} = \Big\{\phi \in \mathcal{C} \mid \phi(\theta) = \mathrm{col}(\phi_1(\theta), \phi_2(\theta), \ldots, \phi_N(\theta)), \\ \phi_i(\theta) \in \mathcal{U},\ i = 1, 2, \ldots, N,\ -\tau \leq \theta \leq 0\Big\}.$$

Theorem 17.1 *Suppose that the isolated system* (17.2) *has an attractor* $\mathcal{A}$ *with an inflowing invariant neighborhood* $\mathcal{U}$ *contained in* $\mathcal{B}$. *Let there exists a nonempty set* $\mathcal{S} \subset \mathbb{R}_+ \times \overline{\mathbb{R}}_+$ *such that for any* $(\sigma, \tau) \in \mathcal{S}$ *the zero solution of the linear system*

$$\dot{\eta}(t) = J(t)\eta(t) - \sigma BC\eta(t-\tau) \tag{17.5}$$

with

$$J(t) := \frac{\partial f}{\partial x_i}(\xi(t))$$

is uniformly asymptotically stable for all $\xi \in \mathcal{C}(\mathbb{R}, \mathcal{U})$. *Let*

$$\mathcal{S}_j := \Big\{(\sigma, \tau) \in \mathbb{R}_+ \times \overline{\mathbb{R}}_+ \mid (\sigma\lambda_j, \tau) \in \mathcal{S}\Big\}$$

be a scaled copy of $\mathcal{S}$ *with nonzero eigenvalue* λ_j *of* L *as scaling factor. If*

$$(\sigma, \tau) \in \cap_{j=2}^{N} \mathcal{S}_j,$$

then there is a constant $\delta = \delta(\sigma, \tau) > 0$ *such that solutions of the coupled systems* (17.2) *and* (17.3), *with initial data* $\phi \in \mathcal{C}_{\mathcal{U}}$ *for which* $\|\phi_i - \phi_j\| < \delta$ *for all* $i, j = 1, 2, \ldots, N$, *are contained in* $\mathcal{C}_{\mathcal{U}}$. *Moreover, the coupled systems* (17.2) *and* (17.3) *locally synchronize.*

Proof Since L is symmetric there exists a nonsingular $(N-1)\times(N-1)$-dimensional matrix U such that

$$U\begin{pmatrix}\lambda_2 & & \\ & \ddots & \\ & & \lambda_N\end{pmatrix}U^{-1} = L_2,$$

$$\begin{pmatrix}1 & \mathbf{0}_{N-1}^\top \\ \mathbf{1}_{N-1} & -I_{N-1}\end{pmatrix} L \begin{pmatrix}1 & \mathbf{0}_{N-1}^\top \\ \mathbf{1}_{N-1} & -I_{N-1}\end{pmatrix} = \begin{pmatrix}0 & L_1^\top \\ \mathbf{0}_{N-1} & L_2\end{pmatrix}$$

with L_1 a $(N-1)$-dimensional vector. See [24] for details. We remark that L_1 has at least one nonzero entry; If not the network would not be connected. Let the zero solution of the system

$$\begin{pmatrix}\dot{\eta}_2(t) \\ \vdots \\ \dot{\eta}_N(t)\end{pmatrix} = (I_{N-1}\otimes J(t))\begin{pmatrix}\eta_2(t) \\ \vdots \\ \eta_N(t)\end{pmatrix} - \sigma\left(\begin{pmatrix}\lambda_2 & & \\ & \ddots & \\ & & \lambda_N\end{pmatrix}\otimes BC\right)\begin{pmatrix}\eta_2(t-\tau) \\ \vdots \\ \eta_N(t-\tau)\end{pmatrix}$$

be uniformly asymptotically stable for $(\sigma, \tau) \in \cap_{j=2}^{N}\mathcal{S}_j$ such that, for

$$\begin{pmatrix}\zeta_2(t) \\ \vdots \\ \zeta_N(t)\end{pmatrix} = (U\otimes I_n)\begin{pmatrix}\eta_2(t) \\ \vdots \\ \eta_N(t)\end{pmatrix}$$

the zero solution of the system

$$\begin{pmatrix}\dot{\zeta}_2(t) \\ \vdots \\ \dot{\zeta}_N(t)\end{pmatrix} = (I_{N-1}\otimes J(t))\begin{pmatrix}\zeta_2(t) \\ \vdots \\ \zeta_N(t)\end{pmatrix} - \sigma\,(L_2\otimes BC)\begin{pmatrix}\zeta_2(t-\tau) \\ \vdots \\ \zeta_N(t-\tau)\end{pmatrix} \tag{17.6}$$

is uniformly asymptotically stable. We remark that the zero solution of a linear system being uniformly asymptotically stable implies the zero solution of that system to be exponentially stable, cf. Theorem 4.5 of [11]. Thus there exist positive constants α, β such that for any solution $\zeta(\cdot;\psi)$ of (17.6) through $\psi \in \mathcal{C}\left([-\tau, 0], \mathbb{R}^{(N-1)n}\right)$ the following estimate holds:

$$|\zeta(t;\psi)| \leq \beta e^{-\alpha t}\|\psi\|, \quad \forall t \geq 0.$$

Denote

$$\begin{pmatrix}\tilde{x}_1(t) \\ \tilde{x}_2(t) \\ \vdots \\ \tilde{x}_N(t)\end{pmatrix} = \begin{pmatrix}x_1(t) \\ x_1(t) - x_2(t) \\ \vdots \\ x_1(t) - x_N(t)\end{pmatrix},$$

such that

$$\dot{\tilde{x}}_1(t) = f(\tilde{x}_1(t)) - \sigma \left(L_1^\top \otimes BC\right) \begin{pmatrix} \tilde{x}_2(t-\tau) \\ \vdots \\ \tilde{x}_N(t-\tau) \end{pmatrix} \tag{17.7}$$

and

$$\begin{pmatrix} \dot{\tilde{x}}_2(t) \\ \vdots \\ \dot{\tilde{x}}_N(t) \end{pmatrix} = \begin{pmatrix} \tilde{f}(t, \tilde{x}_2(t)) \\ \vdots \\ \tilde{f}(t, \tilde{x}_N(t)) \end{pmatrix} - \sigma \left(L_2 \otimes BC\right) \begin{pmatrix} \tilde{x}_2(t-\tau) \\ \vdots \\ \tilde{x}_N(t-\tau) \end{pmatrix} \tag{17.8}$$

with $\tilde{f}(t, \tilde{x}_i(t)) := f(\tilde{x}_1(t)) - f(\tilde{x}_1(t) - \tilde{x}_i(t))$. It now follows that if $\tilde{x}_1(t) \in \mathcal{U}$ for all $t \geq 0$, then the zero solution of (17.8) is locally exponentially stable, cf. Theorem 4.6 of [11]. In particular, for $\phi \in \mathcal{C}$ with $\|\phi_i - \phi_j\| < \delta_1$, where δ_1 is small enough to ensure that the linear part of (17.8) dominates the nonlinearities, and $K = \left(1 + \frac{1}{2\alpha}\right) \beta^2 e^{2\alpha\tau}$, there is a positive constant γ such that

$$\left| \begin{pmatrix} \tilde{x}_2(t;\phi) \\ \vdots \\ \tilde{x}_N(t;\phi) \end{pmatrix} \right| \leq K e^{-\gamma t} \|\phi\| \leq K\delta_1, \quad \forall t \geq 0.$$

To prove the theorem we are left to show that $\tilde{x}_1(t) \in \mathcal{U}$ for all $t \geq 0$. Pick

$$\delta_2 < \frac{\mu}{\sigma K |L_1| \|BC\|}$$

and

$$\delta = \min\left(\delta_1, \delta_2\right).$$

Suppose that there is a positive constant t_1 such that $\tilde{x}_1(t_1) \in \partial\mathcal{U}$ and $\tilde{x}_1(t) \notin \overline{\mathcal{U}}$ for some $t > t_1$. Because f is inflowing invariant with constant μ, the $\tilde{x}_1$-dynamics (17.7) can only cross the boundary $\partial\mathcal{U}$ at $t = t_1$ if

$$\left| \sigma \left(L_1^\top \otimes BC\right) \begin{pmatrix} \tilde{x}_2(t-\tau) \\ \vdots \\ \tilde{x}_N(t-\tau) \end{pmatrix} \right| \geq \mu.$$

But

$$\begin{aligned} \left| \sigma \left(L_1^\top \otimes BC\right) \begin{pmatrix} \tilde{x}_2(t-\tau) \\ \vdots \\ \tilde{x}_N(t-\tau) \end{pmatrix} \right| &\leq \sigma |L_1| \|BC\| K \left| \begin{pmatrix} \tilde{x}_2(t-\tau) \\ \vdots \\ \tilde{x}_N(t-\tau) \end{pmatrix} \right| \\ &\leq \sigma |L_1| \|BC\| K < \delta\mu. \end{aligned}$$

hence $t_1 = \infty$. □

Equation (17.5) is a MSF for the time-delay coupled systems (17.2) and (17.3). However, contrary to the MSF approach for the delay-free case presented in [21], we do assume that the isolated system has an attractor $\mathcal{A}$ with inflowing invariant neighborhood $\mathcal{U}$. In addition, we evaluate (17.5) along all possible solutions in $\mathcal{U}$ instead of a single solution on $\mathcal{A}$. However, to verify uniform asymptotic stability of the zero solution (17.5) for all possible solutions in $\mathcal{U}$, one usually has to construct a Lyapunov functional on $\mathcal{U}$. See [27] for an example. We remark that a synchronization diagram computed using the Lyapunov functional approach tends to be conservative in the sense that it is contained, but not equal to the true synchronization diagram. In case the isolated system has a fixed point or periodic orbit as attractor, we can obtain a better estimate of the true synchronization diagram $\mathcal{S}$.

Corollary 17.2 *Assume that the attractor $\mathcal{A}$ defined in Theorem 17.1 is an asymptotically stable fixed point or an orbitally stable period orbit. Let $\xi(\cdot)$ be a solution of $\dot{\xi}(t) = f(\xi(t))$ with $\xi(-\tau) \in \mathcal{A}$, i.e., $\xi(\cdot)$ is a solution of the isolated system on $\mathcal{A}$. Suppose that there exists a nonempty set $\mathcal{S} \subset \mathbb{R}_+ \times \overline{\mathbb{R}}_+$ such that for any $(\sigma, \tau) \in \mathcal{S}$ the zero solution of the linear system*

$$\dot{\eta}(t) = J(t)\eta(t) - \sigma BC\eta(t - \tau)$$

with

$$J(t) := \frac{\partial f}{\partial x_i}(\xi(t))$$

is uniformly asymptotically stable. If

$$(\sigma, \tau) \in \cap_{j=2}^{N} \mathcal{S}_j,$$

then the conclusions of Theorem 17.1 hold.

Proof Consider the linearization of (17.7) and (17.8) around the synchronous solution on $\mathcal{A}$:

$$\dot{\zeta}(t) = (I_N \otimes J(t))\zeta(t) - \left(\begin{pmatrix} 0 & L_1^\top \\ \mathbf{0}_{N-1} & L_2 \end{pmatrix} \otimes BC\right)\zeta(t - \tau).$$

As shown in the proof of Theorem 17.1, one can find new coordinates such that the matrix L_2 is the matrix above becomes diagonal. Denote this diagonal matrix by Λ_2. Thus in these new coordinates the system has a block-triangular structure. If $\mathcal{A}$ is an equilibrium, then $J(t) = J$ is a stable matrix, and it is easy to see that the conditions of the corollary imply that the characteristic equation

$$\Delta(\rho; \sigma, \tau) = \det\left(\rho I_{Nn} - (I_N \otimes J) - \sigma\left(\begin{pmatrix} 0 & L_1^\top \\ \mathbf{0}_{N-1} & \Lambda_2 \end{pmatrix} \otimes BC\right)\exp(-\rho\tau)\right)$$

has no roots in the closed right half of the complex plane. If $\mathcal{A}$ is a periodic orbit, then $J(t) = J(t+T)$ for some nonzero constant T, i.e., $J(t)$ is T-periodic. We now use Floquet theory (cf. [12]) to conclude the proof. First, we observe that the monodromy matrix of the block-triangular system has a block-triangular structure. Then our conditions imply that all Floquet multiplier except one are contained in the open unit disk in the complex plane. Moreover, as the Floquet multipliers are independent of t (cf. [12], Sect. 8.1, Lemma 1.3) it suffices to linearize around a single periodic synchronous solution. □

17.4 Example: Local Synchronization of FitzHugh-Nagumo Neurons

We consider the network shown in Fig. 17.2 with dynamics

$$f(x_i(t)) = \begin{pmatrix} \frac{2}{25}\left(x_{i,2}(t) - \frac{4}{5}x_{i,1}(t)\right) \\ x_{i,2}(t) - \frac{1}{3}x_{i,2}^3(t) - x_{i,1}(t) \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{pmatrix} 0 & 1 \end{pmatrix}.$$

The system above is the FitzHugh-Nagumo (FHN) neuron [10, 17], which is a model of the excitable membrane dynamics of a neuron.

Let us first show that the isolated FHN neuron has a periodic attractor. Consider the function $V : \mathbb{R}^n \to \overline{\mathbb{R}}_+$

$$V(x_i(t)) = \frac{25}{4}x_{i,1}^2(t) + \frac{1}{2}x_{i,2}^2(t).$$

Then

$$\dot{V}(x_i(t)) = -\frac{4}{5}x_{i,1}^2(t) - \left(\frac{1}{3}x_{i,2}^2(t) - 1\right)x_{i,2}^2(t),$$

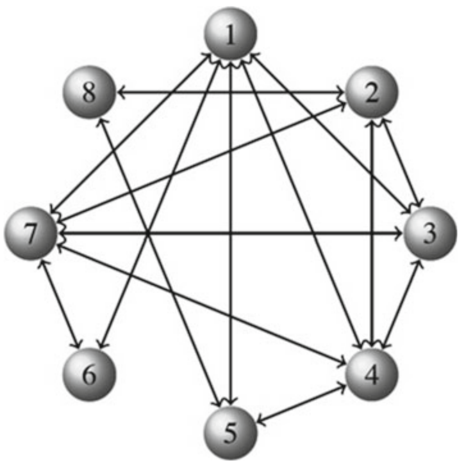

Fig. 17.2 Example network. Each edge has weight 1

and it follows that the set

$$\Omega = \left\{ x_i(t) \in \mathbb{R}^2 \mid V(x_i(t)) \le \tfrac{75}{4} \right\}$$

is positively invariant with respect to the dynamics of the isolated FHN neuron. One easily verifies that this system has a single equilibrium in Ω, the origin, which is unstable. Hence by the Poincaré-Bendixson theorem (cf. [29]) the isolated FHN neuron has a periodic orbit. In fact, applying Liénard's theorem (cf. [29]) to the system obtained after the well-defined change of coordinates

$$x_i(t) \mapsto \begin{pmatrix} v_i(t) \\ w_i(t) \end{pmatrix} = \begin{pmatrix} x_{i,2}(t) \\ x_{2,i}(t) - \tfrac{1}{3}x_{2,i}^3(t) - x_{i,1}(t) \end{pmatrix},$$

i.e.,

$$\begin{pmatrix} \dot{v}_i(t) \\ \dot{w}_i(t) \end{pmatrix} = \begin{pmatrix} w_i(t) \\ -\left(v_i^2(t) - \tfrac{27}{25}\right) w_i(t) - \tfrac{2}{25}\left(\tfrac{4}{15}v_i^3(t) + \tfrac{1}{5}v_i(t)\right) \end{pmatrix},$$

we conclude that Ω contains a unique and orbitally stable period attractor with period time T.

By Corollary 17.2, we may then determine the synchronization diagram $\mathcal{S}$ by computing the Floquet multipliers of the linear T-periodic system

$$\begin{pmatrix} \dot{\eta}_1(t) \\ \dot{\eta}_2(t) \end{pmatrix} = \begin{pmatrix} -\frac{8}{125} & \frac{2}{25} \\ -1 & 1 - \xi_2^2(t) \end{pmatrix} \begin{pmatrix} \eta_1(t) \\ \eta_2(t) \end{pmatrix} - \sigma \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \eta_1(t-\tau) \\ \eta_2(t-\tau) \end{pmatrix},$$

where $\xi_2(t) = \xi_2(t+T)$ satisfies

$$\begin{pmatrix} \dot{\xi}_1(t) \\ \dot{\xi}_2(t) \end{pmatrix} = \begin{pmatrix} \frac{2}{25}\left(\xi_2(t) - \frac{4}{5}\xi_1(t)\right) \\ \xi_2(t) - \frac{1}{3}\xi_2(t) - \xi_1(t) \end{pmatrix}$$

with initial conditions on the unique periodic attractor. The synchronization diagram, which we computed with the numerical software package DDE-Biftool [7, 25], is shown in Fig. 17.3a. The Laplacian matrix of the network shown in Fig. 17.2 is

$$L = \begin{pmatrix} 5 & 0 & -1 & -1 & -1 & -1 & -1 & 0 \\ 0 & 4 & -1 & -1 & 0 & 0 & -1 & -1 \\ -1 & -1 & 4 & -1 & 0 & 0 & -1 & 0 \\ -1 & -1 & -1 & 5 & -1 & 0 & -1 & 0 \\ -1 & 0 & 0 & -1 & 3 & 0 & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 2 & -1 & 0 \\ -1 & -1 & -1 & -1 & 0 & -1 & 5 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 0 & 2 \end{pmatrix}$$

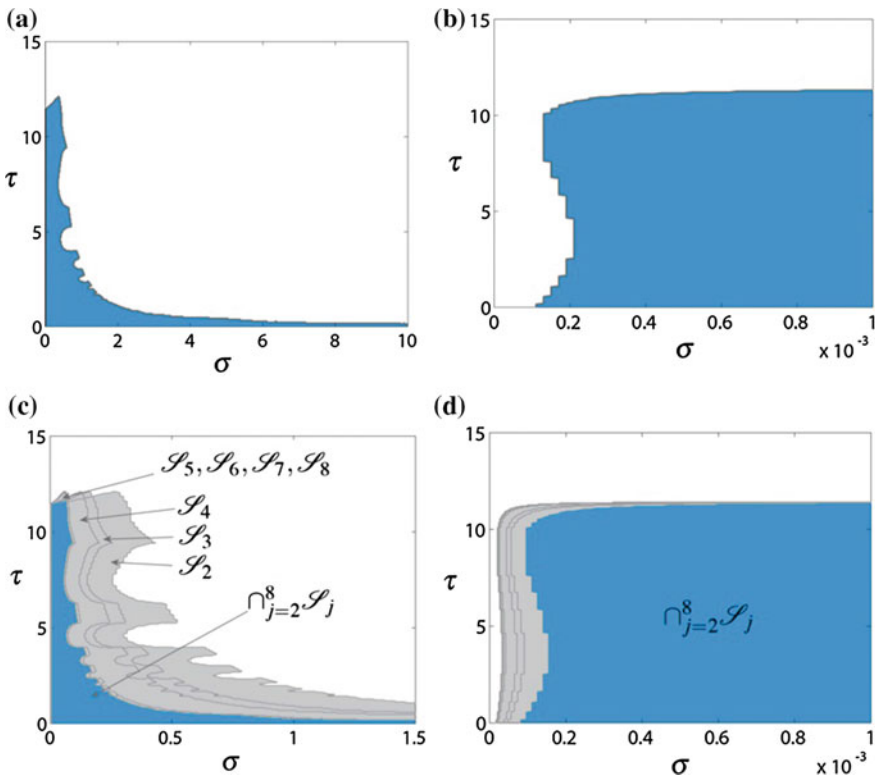

Fig. 17.3 **a** Synchronization diagram $\mathcal{S}$ for the FHN neuron. **b** Zoom of the *left* of (**a**). **c** Seven scaled copies of $\mathcal{S}$ and their intersection. **d** Zoom of the *left* of (**c**)

and has eigenvalues (approximated using Matlab®)

$$\lambda_1 = 0, \quad \lambda_2 = 1.3643, \quad \lambda_3 = 2.3083, \quad \lambda_4 = 2.9266,$$
$$\lambda_5 = 4.9626, \quad \lambda_6 = 5.7110, \quad \lambda_7 = 6.2899, \quad \lambda_8 = 6.4374.$$

The seven scaled copies of $\mathcal{S}$ and their intersection are shown in Fig. 17.3c. By Corollary 17.2, for any values of the coupling strength and time-delay belonging to this intersection, the network of FHN neurons locally synchronizes.

17.5 Conditions for Global Synchronization

In this section, we introduce a class of systems for which there exists a global synchronization diagram. This global synchronization diagram allows for the construction of a set of values of the coupling strength and time-delay for which a network of systems globally synchronizes. First, since we have assumed the matrix CB to be similar to a positive definite matrix, it is possible to find new coordinates

$$x_i(t) \mapsto \begin{pmatrix} z_i(t) \\ y_i(t) \end{pmatrix}$$

with $z_i(t) \in \mathbb{R}^{n-m}$. See [6, 23] for details about this transformation. In these new coordinates the systems' dynamics read as

$$\dot{z}_i(t) = q(z_i(t), y_i(t)) \tag{17.9a}$$

$$\dot{y}_i(t) = a(z_i(t), y_i(t)) + CBu_i(t) \tag{17.9b}$$

where $q : \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^{n-m}$ and $a : \mathbb{R}^{n-m} \times \mathbb{R}^m \to \mathbb{R}^m$ are (sufficiently) smooth vectorfields.

We shall assume that

A1. There exists a nonempty set $\mathcal{S}_B \in \mathbb{R}_+ \times \overline{\mathbb{R}}_+$ such that for $(\sigma, \tau) \in \mathcal{S}_B$ the solutions of the coupled systems are uniformly bounded with bound B that is independent of N.

In addition we assume that

A2. There exists a positive definite matrix $P = P^\top$ and a positive constant κ such that

$$\left[\frac{\partial q}{\partial z_i}(z_i, y_i)\right]^T P + P\left[\frac{\partial q}{\partial z_i}(z_i, y_i)\right] \leq -\kappa I_{n-m}$$

for all $z_i \in \mathbb{R}^{n-m}$ and $y_i \in \mathbb{R}^m$.

The latter assumption implies that the system

$$\dot{z}_i(t) = q(z_i(t), y_i(t))$$

is an exponentially convergent system with respect to input $y_i(t)$ [18, 19]. Interesting is that such an exponentially convergent system has an exponentially stable steady-state solution that is solely determined by the vectorfield q and input signal $y_i(\cdot)$. It then follows that for any two input signals $y_i(\cdot), y_j(\cdot)$ that satisfy

$$\lim_{t\to\infty} |y_i(t) - y_j(t)| = 0,$$

the solutions of the systems

$$\dot{z}_i(t) = q(z_i(t), y_i(t))$$

and

$$\dot{z}_j(t) = q(z_j(t), y_j(t))$$

satisfy

$$\lim_{t\to\infty} |z_i(t) - z_j(t)| = 0,$$

independent of the initial conditions of those systems.

We first give a result about global synchronization of two coupled systems.

Lemma 17.3 *Consider two coupled systems (17.9a) and (17.3) and let $a_{12} = a_{21} = 1$. Suppose that assumptions A1 and A2 hold. Then there exist two positive constants $\bar{\sigma}$ and $\bar{\gamma}$ such that if*

$$(\sigma, \tau) \in \mathcal{S}^* \cap \mathcal{S}_B,$$

where

$$\mathcal{S}^* := \left\{ (\sigma, \tau) \in \mathbb{R}_+ \times \overline{\mathbb{R}}_+ \,|\, \sigma \geq \bar{\sigma} \text{ and } \sigma\tau \leq \bar{\gamma} \right\},$$

then the two coupled systems globally synchronize.

The set $\mathcal{S}^*$ is shown in Fig. 17.4. The proof of the lemma follows from the proof of the next theorem.

Theorem 17.4 *Consider a network of coupled systems (17.9a) and (17.3) and suppose that assumptions A1 and A2 hold. If*

$$(\sigma, \tau) \in \mathcal{S}_2^* \cap \mathcal{S}_N^* \cap \mathcal{S}_B,$$

where

$$\mathcal{S}_j^* := \left\{ (\sigma, \tau) \in \mathbb{R}_+ \times \overline{\mathbb{R}}_+ \,\middle|\, \left(\frac{\lambda_j}{2}\sigma, \tau \right) \in \mathcal{S}^* \right\}, \quad j = 2, N,$$

with $\mathcal{S}^*$ as in Lemma 17.3, then the network of coupled systems globally synchronizes.

Proof Let

$$\tilde{y}_j(t) = y_1(t) - y_j(t), \quad \tilde{z}_j(t) = z_i(t) - z_j(t), \quad j = 2, \dots, N,$$

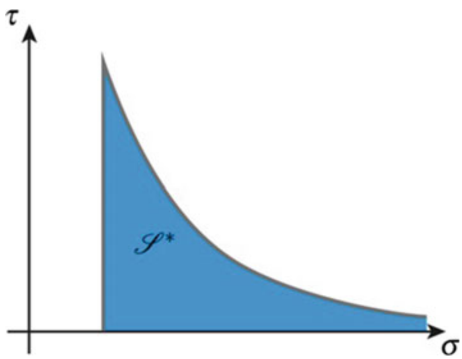

Fig. 17.4 The global synchronization diagram $\mathcal{S}^*$ for two coupled systems with its shape predicted by Lemma 17.3

$\tilde{z}(t) = \mathrm{col}(\tilde{z}_2(t), \ldots, \tilde{z}_N(t))$ and $\tilde{y}(t) = \mathrm{col}(\tilde{y}_2(t), \ldots, \tilde{y}_N(t))$, to obtain

$$\dot{\tilde{z}}(t) = \tilde{q}(z_1(t), y_1(t), \tilde{z}(t), \tilde{y}(t)) \tag{17.10a}$$

$$\dot{\tilde{y}}(t) = \tilde{a}(z_1(t), y_1(t), \tilde{z}(t), \tilde{y}(t)) - \sigma(L_2 \otimes CB)\tilde{y}(t-\tau) \tag{17.10b}$$

with

$$\tilde{q}(z_1(t), y_1(t), \tilde{z}(t), \tilde{y}(t)) := \begin{pmatrix} q(z_1(t), y_1(t)) - q(z_1 - \tilde{z}_2(t), y_1(t)) \\ \vdots \\ q(z_1(t), y_1(t)) - q(z_1 - \tilde{z}_N(t), y_1(t)) \end{pmatrix},$$

$$\tilde{a}(z_1(t), y_1(t), \tilde{z}(t), \tilde{y}(t)) := \begin{pmatrix} a(z_1(t), y_1(t)) - a(z_1 - \tilde{z}_2(t), y_1(t)) \\ \vdots \\ a(z_1(t), y_1(t)) - a(z_1 - \tilde{z}_N(t), y_1(t)) \end{pmatrix},$$

and the $(N-1) \times (N-1)$-dimensional matrix L_2 defined in the proof of Theorem 17.1. Recall that there is a matrix U such that

$$U^{-1} L_2 U = \begin{pmatrix} \lambda_2 & & \\ & \ddots & \\ & & \lambda_N \end{pmatrix}.$$

We assume without loss of generality that $\|U^{-1}\| = 1$. Using the equality

$$\tilde{y}(t-\tau) = \tilde{y}(t) - \int_{-\tau}^{0} \dot{\tilde{y}}(t+s)\mathrm{d}s$$

we obtain

$$\begin{aligned} \dot{\tilde{y}}(t) = {} & \tilde{a}(z_1(t), y_1(t), \tilde{z}(t), \tilde{y}(t)) - \sigma(L_2 \otimes CB)\tilde{y}(t) \\ & + \sigma(L_2 \otimes CB) \int_{-\tau}^{0} [\tilde{a}(z_1(t+s), y_1(t+s), \tilde{z}(t+s), \tilde{y}(t+s)) \\ & \qquad - \sigma(L_2 \otimes CB)\tilde{y}(t+s-\tau)]\mathrm{d}s. \end{aligned} \tag{17.11}$$

We now show that the conditions of the theorem imply that the function

$$V(\tilde{z}(t), \tilde{y}(t)) = \tilde{z}^{\top}(t)(I_{N-1} \otimes P)\tilde{z}(t) + \tfrac{1}{2}\tilde{y}^{\top}(t)(U^{-\top}U^{-1} \otimes I_m)\tilde{y}(t)$$

is a Lyapunov–Razumikhin function [12], that proves uniform asymptotic stability of the origin of (17.10a) and (17.11), hence synchronization of the coupled systems. Assumption A2 implies that there exists a positive constant c_1 such that

$$[q(z_1(t), y_1(t)) - q(z_1 - \tilde{z}_j(t), y_1(t))]^\top P \\ + P[q(z_1(t), y_1(t)) - q(z_1 - \tilde{z}_j(t), y_1(t))] \leq -c_1|\tilde{z}_j(t)|^2.$$

See [24] for details. Moreover, since the solutions of the coupled systems are assumed to be bounded and the functions a and q are sufficiently smooth, there exist positive constants c_2, c_3 and c_4 such that

$$|2P[q(z_1(t) - \tilde{z}_j(t), y_1(t)) - q(z_1(t) - \tilde{z}_j(t), y_1(t) - \tilde{y}_j(t))]| \leq c_2|\tilde{y}_j(t)|,$$

and

$$\begin{aligned} &|a(z_1(t), y_1(t)) - a(z_1(t) - \tilde{z}_j(t), y_1(t) - \tilde{y}_j(t))| \\ &\quad \leq |a(z_1(t), y_1(t)) - a(z_1(t) - \tilde{z}_j(t), y_1(t))| \\ &\quad + |a(z_1(t) - \tilde{z}_j(t), y_1(t)) - a(z_1(t) - \tilde{z}_j(t), y_1(t) - \tilde{y}_j(t))| \\ &\quad \leq c_3|\tilde{z}_j(t)| + c_4|\tilde{y}_j(t)|. \end{aligned}$$

Choose constant $\nu > 1$ such that if

$$\nu|\tilde{y}(t)| \geq |\tilde{y}(t+\theta)|$$

and

$$\nu^2 V(\tilde{z}(t), \tilde{y}(t)) \geq (\tilde{z}(t+\theta), \tilde{y}(t+\theta))$$

for $-2\tau \leq \theta \leq 0$, then

$$\begin{aligned} \dot{V} &\leq - W(\tilde{z}(t), \tilde{y}(t)) \\ &= -\begin{pmatrix} \tilde{z}(t) \\ \tilde{y}(t) \end{pmatrix}^\top \begin{pmatrix} c_1 & -\frac{c_2+c_4+\gamma c_4}{2} \\ -\frac{c_2+c_4+\gamma c_4}{2} & \beta_1\sigma\lambda_2 - c_3 - \gamma(c_3 + \beta_2\sigma\lambda_N) \end{pmatrix} \begin{pmatrix} \tilde{z}(t) \\ \tilde{y}(t) \end{pmatrix}, \end{aligned}$$

where $\gamma = \nu\beta_2\sigma\tau\lambda_N$, with positive constants β_1 and β_2 being the smallest, respectively, largest eigenvalue of CB. For a network of $N = 2$ systems with $a_{12} = a_{21} = 1$ we have $\lambda_2 = \lambda_N = 2$. It follows that whenever σ is sufficiently large and γ sufficiently small, i.e., $\sigma \geq \bar{\sigma}$ and $\gamma \leq \bar{\gamma}$ for some positive constants $\bar{\sigma}$ and $\bar{\gamma}$, then the function W is negative definite. This proves Lemma 17.3. Then we conclude that for any other network the function W negative definite if $(\sigma, \tau) \in \mathcal{S}_2^* \cap \mathcal{S}_N^*$. □

17.6 Example: Global Synchronization of FitzHugh–Nagumo neurons

Let us show that the FHN neurons introduced in Sect. 17.4 satisfy the conditions of Lemma 17.3. Let

$$x_i(t) = \begin{pmatrix} x_{i,1}(t) \\ x_{i,2}(t) \end{pmatrix} = \begin{pmatrix} z_i(t) \\ y_i(t) \end{pmatrix}$$

and

$$f(x_i(t)) = \begin{pmatrix} q(z_i(t), y_i(t)) \\ a(z_i(t), y_i(t)) \end{pmatrix} = \begin{pmatrix} \frac{2}{25}\left(y_i(t) - \frac{4}{5} z_i(t)\right) \\ y_i(t) - \frac{1}{3} y_i^3(t) - z_i(t) \end{pmatrix}.$$

Then one easily verifies that assumption A2 holds with $P = 1$. We will now show that assumption A1 is satisfied as well.

Proposition 17.5 *Consider N time-delay coupled FHN neurons and suppose that*

- $\max_i \sum_{j \in \mathcal{N}_i} a_{ij} = 1$;
- $\sigma\tau\left(6\sigma + \frac{39}{4}\right) \leq \frac{9}{4}$;
- *for each $i = 1, \ldots, N$, $\phi_i \in \mathcal{C}([-\tau, 0], \mathbb{R}^n)$, the initial data for the ith FHN neuron, is Lipschitz continuous on $[-\tau, 0]$ with Lipschitz constant $K \leq 12$.*

Then the set $\Omega^N := \Omega \times \Omega \times \cdots \times \Omega$ with

$$\Omega := \left\{(z_i, y_i) \in \mathbb{R}^2 \mid |z_i| \leq \tfrac{15}{4} \text{ and } |y_i| \leq 3\right\}$$

is a positively invariant set for the coupled FHN neurons.

Proof Let us consider first an isolated FHN neuron. The nulclines of this isolated neuron and the set Ω are shown in Fig. 17.5. From this picture it is clear that the coupling (17.3) can drive the solution $x_i(t) = \text{col}(z_i(t), y_i(t))$ outside of Ω though the boundaries $y_i = \bar{y}$ or $y_i = -\bar{y}$ with $\bar{y} = 3$. Consider an arbitrary solution of the coupled systems and let $t_1 \leq 0$ be such that this solution is contained in Ω^N for $t \leq t_1$. Suppose that at t_1 the solution of the ith is at the boundary $\bar{y}$, i.e. $y_i(t_1) = \bar{y}$. Write

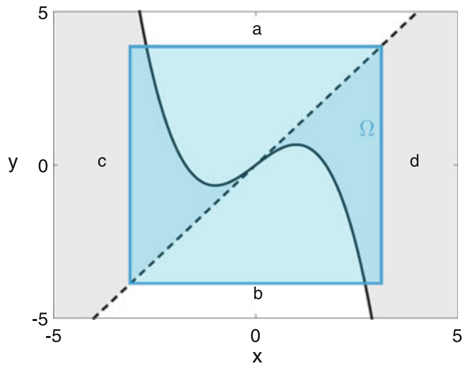

Fig. 17.5 The set Ω (in cyan) and nulclines of the isolated ($u_i = 0$) FHN neuron. Thick black line represents $\dot{y}_i = 0$, dashed black line represents $\dot{z}_i = 0$

$$\begin{aligned} u_i(t) &= \sigma \sum_{j\in\mathcal{N}_i} a_{ij}[y_j(t-\tau) - y_i(t-\tau)] \\ &= \sigma \sum_{j\in\mathcal{N}_i} a_{ij}[(y_j(t-\tau) - y_i(t)) + (y_i(t) - y_i(t-\tau))], \end{aligned}$$

hence,

$$u_i(t_1) = \sigma \sum_{j\in\mathcal{N}_i} a_{ij}[(y_j(t_1-\tau) - \bar{y}) + (\bar{y} - y_i(t_1-\tau))] \le \sigma(\bar{y} - y_i(t_1-\tau))$$

as $|y_j(t_1-\tau)| \le \bar{y}$ for all j and $\sum_{j\in\mathcal{N}_i} a_{ij} \le 1$. It then follows that $y_i(t) > \bar{y}$ for some $t > t_1$ requires

$$0 < \dot{y}_i(t_1) \le a(z_i, \bar{y}) + \sigma(\bar{y} - y_i(t_1-\tau)) \le -\nu + \sigma(\bar{y} - y_i(t_1-\tau)),$$

where

$$\nu = \max_{-\frac{15}{4}\le z_i \le \frac{15}{4}} -a(z_i, \bar{y}) = \min_{-\frac{15}{4}\le z_i \le \frac{15}{4}} a(z_i, -\bar{y}) = \frac{9}{4}.$$

As $|y_i(t_1-\tau)| \le \bar{y}$ we have

$$\dot{y}_i(t_1) \le -\nu + 2\sigma B_1,$$

where $B_1 := \bar{y} = 3$, hence to escape from Ω it is required that $\sigma > \frac{\nu}{2B_1}$. Thus let $\sigma > \frac{\nu}{2B_1}$. For $t_1 > 0$ we have

$$\begin{aligned} y_i(t_1) - y_i(t_1-\tau) &= \int_{t_1-\tau}^{t_1} \left[a(z_i(s), y_i(s)) - \sigma \sum_{j\in\mathcal{N}_i} a_{ij}[y_j(s-\tau) - y_i(s-\tau)] \right] \mathrm{d}s \\ &\le \tau(B_2 + 2\sigma B_1), \end{aligned}$$

where $B_2 := \max_{(z_i,y_i)\in\Omega} |a(z_i, y_i)| = \frac{39}{4}$. Hence

$$\dot{y}(t_1) \le -\nu + \sigma\tau(B_2 + 2\sigma B_1).$$

By assumption, $\sigma\tau(B_2 + 2\sigma B_1) = \sigma\tau(\frac{39}{4} + 6\sigma) \le \frac{9}{4} = \nu$, which gives $\dot{y}(t_1) \le 0$ for $t_1 > 0$. Thus we can only have a crossing of $\bar{y}$ at $t_1 = 0$. If $t_1 = 0$, i.e. $\phi_1(0) = \bar{y}$, then we have

$$\phi_i(0) - \phi_i(-\tau) \le K\tau.$$

But $K \le B_2 + \nu = \frac{39}{4} + \frac{9}{4} = 12$ such that, as $\sigma > \frac{\nu}{2B_1}$ hence

$$K \le B_2 + \nu < B_2 + 2B_1\sigma,$$

we have

$$\sigma\tau K \le \sigma\tau(B_2 + 2B_1\sigma) \le \nu,$$

which implies $\dot{y}(0) \le 0$. The same reasoning gives that, if $y_i(t_3) = -\bar{y}$ for some $t_3 \ge 0$, then $\dot{y}(t_3) \ge 0$, hence solutions cannot escape from Ω^N. □

By Proposition 17.5, assuming the Lipschitz condition on the initial data, we conclude that assumption A1 is satisfied for all

$$(\sigma, \tau) \in \mathcal{S}_B := \left\{(\sigma, \tau) \in \mathbb{R}_+ \times \overline{\mathbb{R}}_+ \mid \sigma\tau\left(6\sigma + \tfrac{39}{4}\right) \le \tfrac{9}{4}\right\}.$$

Then Lemma 17.3 implies the existence of a non-empty set $\mathcal{S}^* \cap \mathcal{S}_B$ such that for $(\sigma, \tau) \in \mathcal{S}^* \cap \mathcal{S}_B$ two time-delay coupled FHN neurons globally synchronize (in $\Omega \times \Omega$). Invoking Theorem 17.4 we derive conditions for global synchronization (in Ω^N) of any network of N time-delay coupled FHN neurons.

17.7 Discussion

We have constructed a (local, global) synchronization diagram for time-delay coupled systems and we have shown that a condition for (local, global) synchronization of a network is that the coupling strength and time-delay belong to the intersection of scaled copies of that (local, global) synchronization diagram. The scaling factors are the nonzero eigenvalues of the Laplacian matrix of the undirected, simple, and connected network. We have demonstrated our results with a network of FHN neurons.

We have assumed the network Laplacian matrix to be symmetric to ensure that the eigenvalues (and thus the scaling factors) are real valued. A natural extension of this work would be to allow for networks with asymmetric network Laplacian matrices, e.g., in case of directed networks.

An other important extension would be to consider coupling functions of the form

$$u_i(t) = \sigma \sum_{j\in\mathcal{N}_i} a_{ij}[y_j(t-\tau) - y_i(t)]. \tag{17.12}$$

There is an important difference between this type of coupling and the coupling functions considered in this chapter, i.e., coupling (17.3); coupling (17.12) is invasive whereas the coupling (17.3) is not. For invasive coupling functions, the synchronized dynamics depend on the values of the coupling strength and time-delay. Thus for coupling (17.12), one has to impose additional conditions to ensure that the synchronization manifold exists. A sufficient condition for existence of the synchronization manifold is that the network adjacency matrix $A = \left(a_{ij}\right)$ has constant row-sums, e.g.,

$$\sum_{j \in \mathcal{N}_i} a_{ij} = 1 \quad \forall i = 1, \ldots, N,$$

cf. [28]. Under the assumption above, one can easily derive that the synchronization diagram depends on σ, τ and $\sigma\lambda_j(A)$, with $\lambda_j(A)$ being any eigenvalue of the network adjacency matrix other than 1. (We remark that in case the network is connected and all rows of A sum up to 1, the matrix A has a simple eigenvalue equal to 1.) Thus for invasive coupling (17.12), the synchronization diagram and its intersections need to be drawn in a three-dimensional space.

Finally, (for both types of coupling functions) it would be valuable to extend our results to the multiple delay case.

17.8 Epilogue

This chapter is a tribute to the 60th birthday of Arjan van der Schaft. Over a period of more than 35 years, the second author has shared many ideas, papers, thoughts, running miles, cigars, and much more with Arjan. It is my expectation that this will continue for the next 35 years; I look forward to that.

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