Chapter 15
Power-Based Methods for Infinite-Dimensional Systems

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Abstract In this chapter we aim to extend the Brayton Moser (BM) framework for modeling infinite-dimensional systems. Starting with an infinite-dimensional port-Hamiltonian system we derive a BM equivalent which can be defined with respect to a non-canonical Dirac structure. Based on this model we derive stability and new passivity properties for the system. The state variables in this case are the "effort" variables and the storage function is a "power-like" function called the mixed potential. The new property is derived by "differentiating" one of the port variables. We present our results with the Maxwell's equations, and the transmission line with non-zero boundary conditions as examples.

15.1 Introduction

I (the second author) got exposed to Arjan's work for the first time during my master's in Systems and Control at VJTI, Mumbai. It was though a course on non-linear control, where we followed one of his books titled "Non-linear Dynamical Control Systems" which he had coauthored with Henk Nijmeijer. In the final year of my masters, I came across a paper by him on port-Hamiltonian system. Back then, I never imagined to have to Arjan as my Ph.D. advisor. I feel extremely fortunate that I could learn under his supervision. Arjan's immense knowledge and contributions in Systems Theory have always motivated and played a significant part in shaping my career. I am delighted to dedicate this piece of work to Arjan, on his birthday, to recognize him for his monumental research in Systems and Control theory. Happy Birthday, Arjan!

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Energy-based methods for modeling and control of complex physical systems has been an active area of research for the past two decades. In particular, the Hamiltonian-based formulation has proven to be an effective tool in modeling and control of complex physical systems from several physical domains, both finite and infinite-dimensional cases [7]. These systems are inherently passive with the Hamiltonian when bounded from below, serving as the storage function and the input and output pair are power conjugate. This resulted in development of so-called "Energy-Shaping" methods for control of physical systems. In some cases the natural power conjugate port variables do not necessarily help in achieving the control objectives due to the *dissipation obstacle* [13], motivating the search for alternate passive maps. One possible alternative which has been explored extensively in the finite-dimensional case is the "Brayton–Moser" (BM) framework for modeling of electrical networks [2, 5, 6], which has been successfully adapted towards analyzing passivity of RLC circuits [8] and for control of physical systems by "power shaping" [7]. For further details on various energy and power-based modeling techniques we refer to [9].

Most of the literature for control on the BM framework restricts to finite-dimensional case only. One of the first results, in the infinite dimensional case, appeared in [4], in which the authors present a stability theory in the BM framework for a transmission line connected to the non-linear load. However, the proposed Lyapunov functional does not preserve the *pseudogradient*-like structure of the system, which is essential for boundary control, and to derive passive maps is not very obvious. Later, in [10] the authors describe a electromagnetic fields analogue of the Brayton Moser formulation of Maxwell equations, again mostly for zero boundary conditions. In an earlier work [12], we have presented results on control by interconnection of a transmission line by "power shaping" in the BM framework.

In this chapter we present a BM analogue of an infinite-dimensional port-Hamiltonian systems, defined with respect to a constant Stokes Dirac structure [16]. The main results are deriving a new passivity property for mixed finite and infinite-dimensional systems by "differentiating" one of the port variables (possibly the boundary port) and a storage function directly related to the power of the system, while preserving the structure of the system. This new storage function is instrumental in analyzing the stability of the system. We present our results for a general Hamiltonian system, with Maxwell's equations and the transmission line with non-zero boundary conditions, as examples.

This chapter is organized as follows. In Sect. 15.2, we defined the Stokes Dirac structure and its Brayton Moser formulation. In Sect. 15.3, we use Brayton Moser framework to analyze stability and give admissible pairs for Maxwell's equation of electromagnetic fields and telegraphers equations of transmission line with zero energy flow trough boundary. In Sect. 15.4, we present the admissible pairs and stability for transmission line with non-zero energy flows through the boundary and derive new passivity properties. Finally in Sect. 15.5, we derive conservation laws and Casimirs in the BM framework.

Part of the results presented here have appeared in [11].

Notations and Math Preliminaries

Let Z be an n dimensional Riemannian manifold with a smooth $(n-1)$ dimensional boundary ∂Z. $\Omega^k(Z), k = 0, 1, \ldots, n$ denotes the space of all exterior $k-$ forms on Z. The dual space $\left(\Omega^k(Z)\right)^*$ of $\Omega^k(Z)$ can be identified with space of $n-k$ forms $\Omega^{n-k}(Z)$, the space of $(n-k)$ forms on Z. There exists a natural pairing between $\alpha \in \Omega^k(Z)$ and $\beta \in \left(\Omega^k(Z)\right)^*$ given by $\langle\beta|\alpha\rangle = \int_Z \beta \wedge \alpha$, were $\wedge$ is the usual wedge product of differential forms, resulting in the n form $\beta \wedge \alpha$. Similar pairing can be established between the boundary variables.

d denotes the exterior derivative and maps k forms on Z to $k+1$ forms on Z. The Hodge star operator $*$ (corresponding to Riemannian metric on Z) converts p forms to $(n-p)$ forms. Given $\alpha, \beta \in \Omega^k(Z)$ and $\gamma \in \Omega^l(Z)$, the wedge product $\alpha \wedge \gamma \in \Omega^{k+l}(Z)$. We additionally have the following properties (for details on theory of differential forms we refer to [1]).

$$\alpha \wedge \gamma = (-1)^{kl}\gamma \wedge \alpha\ , \quad **\alpha = (-1)^{k(n-k)}\alpha \tag{15.1}$$

$$\int_z \alpha \wedge *\beta = \int_z \beta \wedge *\alpha \tag{15.2}$$

$$\mathrm{d}\,(\alpha \wedge \gamma) = \mathrm{d}\alpha \wedge \gamma + (-1)^k \alpha \wedge \mathrm{d}\gamma \tag{15.3}$$

Given a functional $H(\alpha_p, \alpha_q)$, we compute its variation as

$$\begin{aligned} \delta H &= H(\alpha_p + \partial\alpha_p, \alpha_q + \partial\alpha_q) - H(\alpha_p, \alpha_q) \\ &= \int_z \left[\delta_p H \wedge \partial\alpha_p + \delta_q H \wedge \partial\alpha_q\right], \end{aligned} \tag{15.4}$$

where $\alpha_p,\ \partial\alpha_p \in \Omega^p(Z)$ and $\alpha_q,\ \partial\alpha_q \in \Omega^q(Z)$ and $\delta_p H \in \Omega^{n-p}(Z)$ and $\delta_q H \in \Omega^{n-q}(Z)$ are variational derivatives of $H(\alpha_p, \alpha_q)$ with respective to α_p and α_q. Further, the time derivative of $H(\alpha_p, \alpha_q)$ is

$$\frac{dH}{dt} = \int_z \left(\delta_p H \wedge \frac{\partial\alpha_p}{\partial t} + \delta_q H \wedge \frac{\partial\alpha_q}{\partial t}\right).$$

Let $G : \Omega^{n-p}(Z) \to \Omega^{n-p}(Z)$ and $R : \Omega^{n-q}(Z) \to \Omega^{n-q}(Z)$, we call $G \geq 0$, if and only if $\forall\alpha_p \in \Omega^p(Z)$

$$\int_Z \left(\alpha_p \wedge *G\alpha_p\right) \geq 0$$

G is said to be symmetric if $\langle\alpha_p|G\alpha_p\rangle = \langle G\alpha_p|\alpha_p\rangle$.

Lastly, for $Z \subset \mathbb{R}^n$, given $f(z,t) : Z \times \mathbb{R} \to \mathbb{R}$, we denote $\dfrac{\partial f}{\partial t}(z,t)$ as f_t, similarly $\dfrac{\partial f}{\partial z}(z,t)$ as f_z.

15.2 From Port-Hamiltonian to Brayton Moser Equations

The basic concept needed in the formulation of a port-Hamiltonian system is that of a Dirac structure, which is a geometric object formalizing general power conserving interconnections [15].

Definition 15.1 Let V be a an infinite-dimensional linear space. There exists on $V \times V^*$ a canonically defined symmetric bilinear form

$$\ll (f_1, e_1), (f_2, e_2) \gg := < e_1 \mid f_2 > + < e_2 \mid f_1 > \tag{15.5}$$

with $f_i \in V, e_i \in V^*, i = 1, 2$ and $<|>$ denoting the duality product between V and its dual subspace V^*. A constant Dirac structure on V is a linear subspace $D \subset V \times V^*$ such that

$$D = D^{\perp}, \tag{15.6}$$

where $\perp$ denotes the orthogonal complement with respect to the bilinear form $\ll, \gg$.

Let now $(f, e) \in D = D^{\perp}$. Then as an immediate consequence of (15.5)

$$0 = \ll (f, e), (f, e) \gg = 2 < e \mid f > .$$

Thus for all $(f, e) \in D$ we have $< e \mid f >= 0$, expressing power conservation with respect to the dual power variables $f \in V$ and $e \in V^*$
The Stokes Dirac Structure [16]: Define the linear space $\mathcal{F}_{p,q} = \Omega^p(Z) \times \Omega^q(Z) \times \Omega^{n-p}(\partial Z)$ called the space of flows and $\mathcal{E}_{p,q} = \Omega^{n-p}(Z) \times \Omega^{n-q}(Z) \times \Omega^{n-q}(\partial Z)$, the space of efforts, with integers p, q satisfying $p + q = n + 1$. Then, the linear subspace $D \subset \mathcal{F}_{p,q} \times \mathcal{E}_{p,q}$

$$\begin{aligned} D &= \Big\{ \big(f_p, f_q, f_b, e_p, e_q, e_b\big) \in \mathcal{F}_{p,q} \times \mathcal{E}_{p,q} | \\ &\begin{bmatrix} f_p \\ f_q \end{bmatrix} = \begin{bmatrix} *G & (-1)^r \mathrm{d} \\ \mathrm{d} & *R \end{bmatrix} \begin{bmatrix} e_p \\ e_q \end{bmatrix}, \begin{bmatrix} f_b \\ e_b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -(-1)^{n-q} \end{bmatrix} \begin{bmatrix} e_p|_{\partial Z} \\ e_q|_{\partial Z} \end{bmatrix} \Big\} \end{aligned}$$

where $r = pq + 1$, is Stokes Dirac structure with dissipation, [16] with respect to the bilinear form

$$\begin{aligned} &\ll \left(f_p^1, f_q^1, f_b^1, e_p^1, e_q^1, e_b^1\right), \left(f_p^2, f_q^2, f_b^2, e_p^2, e_q^2, e_b^2\right) \gg \\ = &\int_Z (e_p^2 \wedge f_p^1 + e_p^1 \wedge f_p^2 + e_q^2 \wedge f_q^1 + e_q^1 \wedge f_q^2) + \int_{\partial Z} (e_b^2 \wedge f_b^1 + e_b^1 \wedge f_b^2). \end{aligned}$$

Consider a distributed parameter port Hamiltonian system on $\Omega^p(Z) \times \Omega^q(Z) \times \Omega^{n-p}(\partial Z)$, with energy variables $\left(\alpha_p, \alpha_q\right) \in \Omega^p(Z) \times \Omega^q(Z)$ representing two different physical energy domains interacting with each other. The Hamiltonian $H = \int_Z \mathrm{H}$, where H is the Hamiltonian density. Then the below system of equations represent an infinite-dimensional port-Hamiltonian system, with $f_p = -\frac{\partial \alpha_p}{\partial t},\ f_q = -\frac{\partial \alpha_q}{\partial t}$ and the efforts as the co-energy variables, i.e. $e_p = \delta_p H,\ e_q = \delta_q H$.

$$-\frac{\partial}{\partial t}\begin{bmatrix}\alpha_p\\ \alpha_q\end{bmatrix} = \begin{bmatrix}*G & (-1)^r \mathrm{d}\\ \mathrm{d} & *R\end{bmatrix}\begin{bmatrix}\delta_p H\\ \delta_q H\end{bmatrix};\begin{bmatrix}f_b\\ e_b\end{bmatrix} = \begin{bmatrix}1 & 0\\ 0 & -(-1)^{n-q}\end{bmatrix}\begin{bmatrix}\delta_p H\ |_{\partial Z}\\ \delta_q H\ |_{\partial Z}\end{bmatrix} \tag{15.7}$$

The time derivative of the Hamiltonian is computed as

$$\frac{dH}{dt} \leq \int_Z e_b \wedge f_b$$

This means that the increase in energy in the spatial domain is less than or equal to power supplied to the system through its boundary. This implies that the system is passive, with respect to the boundary variables, with the Hamiltonian H, which is assumed to be bounded from below serving as the storage function.

15.2.1 The Brayton Moser Mixed Potential

Brayton and Moser in the early 1960s [5, 6] showed that the dynamics of a class (topologically complete) of non-linear RLC-circuits can be written as

$$A(i_L, v_c)\begin{bmatrix}\frac{di_L}{dt}\\ \frac{dv_c}{dt}\end{bmatrix} = \begin{bmatrix}\frac{\partial P}{\partial i_L}\\ \frac{\partial P}{\partial v_C}\end{bmatrix} + \begin{bmatrix}B_{E_c}^\top E_c\\ -B_{J_c}^\top J_C\end{bmatrix} \tag{15.8}$$

where $A(i_L, v_C) = \mathrm{diag}\{L(i_L), -C(v_C)\}$ and i_L the vector of currents through inductors, v_C vector of capacitor voltages, $L(i_L)$ the inductance matrix, $C(v_C)$ the capacitance matrix, B_{E_c}, B_{J_c} the matrices containing the elements $\{-1, 0, 1\}$ decided by Kirchoff's voltage and current laws. E_C, J_C are respectively the controlled voltage and current sources. P is called the mixed potential function defined by

$$P(i_L, v_C) = F(i_L) - G(v_C) + i_L^\top \gamma\ v_c$$

where $x = (i_L, v_C)$ the system states. Here F is the content of all the current controlled resistors, G is the co-content of all voltage controlled resistors. Matrix γ contains elements $\{-1, 0, 1\}$ depending on the network topology. Computing the time derivative of P along the trajectories of (15.8) we have

$$\dot{P} = \dot{x}^\top \left(A(x) + A^\top(x)\right)\dot{x} + u^T y,$$

where, $u = (E_c,\ J_c)^\top$ and $y = \left(-B_{E_c}\frac{di_L}{dt},\ B_{J_c}\frac{dv_C}{dt}\right)^\top$.

From the above expression we can conclude that the system is passive if $(A(x) + A^\top(x)) \leq 0$, with P the storage functions and $u^\top y$ as the supply rate.

In case $(A(x) + A^\top(x)) \leq 0$ is not satisfied, then it is possible to find new $(\tilde{A}, \tilde{P})$ called an "admissible pair," (refer [3, 13]) satisfying $(\tilde{A}(x) + \tilde{A}^\top(x)) \leq 0$. The dynamics can then be equivalently be written as

$$\tilde{A}\begin{bmatrix}\frac{di_L}{dt}\\ \frac{dv_c}{dt}\end{bmatrix} = \begin{bmatrix}\frac{\partial \tilde{P}}{\partial i_L}\\ \frac{\partial \tilde{P}}{\partial v_C}\end{bmatrix} + \begin{bmatrix}B_{E_c}^\top E_c\\ -B_{J_c}^\top J_C\end{bmatrix} \tag{15.9}$$

Remark 15.2 Contrast to the case where the total energy of the system serves as the storage function and passivity is derived with respect to input–output variables which are power conjugate, for example, the voltage and currents [15]. In this case, making use of the mixed potential function as the storage function we derive passivity either with respect to controlled voltages and the derivatives of currents, or the controlled currents and the derivatives of the voltages.

The Infinite-Dimensional BM Formulation

We aim to write the infinite-dimensional port-Hamiltonian system, defined with respect to a Stokes Dirac structure (15.7) in an equivalent BM form. To begin with, we assume that the mapping from the energy variables (α_p, α_q) to the co-energy variables $(e_p, e_q) = (\delta_p H, \delta_q H)$ is invertible. This means the inverse transformation from the co-energy variables to the energy variables can be written as $(\alpha_p, \alpha_q) = (\delta_{e_p} H^*, \delta_{e_q} H^*)$. H^* is the co-energy of H obtained by $H^*(e_p, e_q) = \int_Z \big((e_p \wedge \alpha_p + e_q \wedge \alpha_q) - H(\alpha_p, \alpha_q)\big)$. Further, assume that the Hamiltonian H splits as $H(\alpha_p, \alpha_q) = H_p(\alpha_p) + H_q(\alpha_q)$, with the co-energy variables given by $e_p = \delta_p H_p,\ e_q = \delta_q H_q$. Consequently the co-Hamiltonian can also be split as $H^*(e_p, e_q) = H_p^*(e_p) + H_q^*(e_q)$. We can now rewrite the spatial dynamics of the infinite-dimensional port-Hamiltonian system, in terms of the co-energy variable as

$$\begin{bmatrix}\delta_p^2 H^* & 0\\ 0 & \delta_p^2 H^*\end{bmatrix}\begin{bmatrix}-\frac{\partial e_p}{\partial t}\\ -\frac{\partial e_q}{\partial t}\end{bmatrix} = \begin{bmatrix}*G & (-1)^r \mathrm{d}\\ \mathrm{d} & *R\end{bmatrix}\begin{bmatrix}\delta_p H\\ \delta_q H\end{bmatrix} \tag{15.10}$$

To begin with, we consider the case of a system which is lossless, that is when R and G are identically equal to zero in (15.7). Define P to be a functional of the form $\int_Z e_q \wedge \mathrm{d}e_p$. Its variation is given as

$$\delta P = P(e_p + \partial e_p, e_q + \partial e_q) - P = e_q \wedge \mathrm{d}\partial e_p + \partial e_q \wedge \mathrm{d}e_p + \cdots$$

Using the relation $e_q \wedge \mathrm{d}\partial e_p = (-1)^{pq} \partial e_p \wedge \mathrm{d}e_q + (-1)^{n-q} \mathrm{d}\left(e_q \wedge \partial e_p\right)$, and the identity (15.4), we have

$$\delta_{e_q} P = \mathrm{d}e_p (-1)^{(n-q)\times q}, \quad \delta_{e_p} P = (-1)^{pq} \mathrm{d}e_q (-1)^{(n-p)\times p}.$$

We can rewrite (15.10) in the following way

$$\begin{bmatrix} \delta_p^2 H^* & 0 \\ 0 & \delta_p^2 H^* \end{bmatrix} \begin{bmatrix} \frac{\partial e_p}{\partial t} \\ \frac{\partial e_q}{\partial t} \end{bmatrix} = \begin{bmatrix} *\delta_{e_q} P \\ *\delta_{e_p} P \end{bmatrix} \tag{15.11}$$

Note that the Hodge star operator in right hand side is necessary, because $(\delta_{e_q} P, \delta_{e_p} P) \in \Omega^q(Z) \times \Omega^p(Z)$, and $(\dot{e}_q, \dot{e}_p) \in \Omega^{n-q}(Z) \times \Omega^{n-p}(Z)$.

In order to incorporate dissipation we proceed as follows: Consider instead a functional P defined as

$$P(e_p, e_q) = \int_Z \left(e_q \wedge \mathrm{d}e_p + \frac{1}{2} R e_q \wedge *e_q - \frac{1}{2} G e_p \wedge *e_p \right) \tag{15.12}$$

The variation in P is computed as

$$\begin{aligned} P &= e_q \wedge \mathrm{d}\partial e_p + \partial e_q \wedge \mathrm{d}e_p + \frac{1}{2}(e_q \wedge R * \partial e_q + \partial e_q \wedge *e_q) \\ &\quad - \frac{1}{2}(e_p \wedge G * \partial e_p + \partial e_p \wedge *e_p) \\ &= \int_Z \partial e_q \wedge \mathrm{d}e_p + \partial e_p \wedge (-1)^{pq} \mathrm{d}e_q + \frac{1}{2}(e_q \wedge R * \partial e_q + \partial e_q \wedge *e_q) \\ &\quad - \frac{1}{2}(e_p \wedge G * \partial e_p + \partial e_p \wedge *e_p) \\ &= \int_Z \partial e_q \wedge \left(\mathrm{d}e_p + R * e_q\right) + \partial e_p \wedge \left((-1)^{pq} \mathrm{d}e_q - G * e_p\right) \end{aligned}$$

where we have used the relation $e_q \wedge \mathrm{d}\partial e_p = (-1)^{pq} \partial e_p \wedge \mathrm{d}e_q + (-1)^{n-q} \mathrm{d} \left(e_q \wedge \partial e_p\right)$, together with properties of the wedge form and the star operator defined in (15.2) and (15.3). Lastly by making use of (15.4) we can write

$$\begin{bmatrix} \delta_{e_q} P \\ \delta_{e_p} P \end{bmatrix} = \begin{bmatrix} (\mathrm{d}e_p + R * e_q)(-1)^{(n-q)\times q} \\ \left((-1)^{pq} \mathrm{d}e_q - G * e_p\right) (-1)^{(n-p)\times p} \end{bmatrix}, \tag{15.13}$$

The dynamics (15.10) can now be written as

$$\begin{bmatrix} \delta_p^2 H^* & 0 \\ 0 & \delta_p^2 H^* \end{bmatrix} \begin{bmatrix} \frac{\partial e_p}{\partial t} \\ \frac{\partial e_q}{\partial t} \end{bmatrix} = \begin{bmatrix} *\delta_{e_q} P \\ *\delta_{e_p} P \end{bmatrix} \tag{15.14}$$

The dynamics are written as partial differential equations in the co-energy variables (e_p, e_q). The above equations together with the mixed potential functional as defined in (15.12) correspond to system of equations which are usually referred to as the

Brayton Moser equations, [4]. The above system of equations can be written in a concise way as follows,

$$Au_t = *\delta_u P. \tag{15.15}$$

where $u = (e_p, e_q)^\top$ and $A = \begin{bmatrix} \delta_p^2 H^* & 0 \\ 0 & \delta_p^2 H^* \end{bmatrix}$.

Boundary dynamics: The system (15.15) can be interconnected to other systems via the boundary of the infinite-dimensional system, which can either be finite or infinite-dimensional in nature. To include the dynamics arising due to the boundary we need to append the Eq. (15.15) in order to incorporate the boundary dynamics.

$$\begin{bmatrix} A & 0 \\ 0 & A_b \end{bmatrix} \begin{bmatrix} u_t \\ u_t^b \end{bmatrix} = \begin{bmatrix} *\delta_u P \\ *\delta_{u^b} P^b + (-1)^{(n-p)\times p} e_q|_{\partial Z} \end{bmatrix} \tag{15.16}$$

with a new mixed potential function

$$\mathcal{P}(e_p, e_q) = \int_Z P(e_p, e_q) + \int_{\partial Z} P^b(e_p, e_q)$$

with P^b taking into account the mixed potential function arising through the boundary dynamics. u^b represents the states of the systems interconnected at the boundary. The variation in P_d id given by,

$$\begin{aligned} \delta\mathcal{P} &= \int_Z \left(\delta_{e_q} P \wedge \partial e_q + \delta_{e_p} P \wedge \partial e_p\right) \\ &\quad + \int_{\partial Z} \left(\delta_{e_q} P^b \wedge \partial e_p + \left(\delta_{e_p} P^b + (-1)^{(n-p)\times p} e_q\right) \wedge \partial e_p\right) \end{aligned}$$

Now with $U = (u, u^b)^\top$ and

$$\delta_U \mathcal{P} = \begin{bmatrix} \delta_{e_q} P \\ \delta_{e_p} P \\ \delta_{e_q} P^b|_{\partial Z} \\ \left(\delta_{e_p} P^b + (-1)^{(n-p)\times p} e_q\right)|_{\partial Z} \end{bmatrix} \tag{15.17}$$

the Brayton Moser equations incorporating boundary dynamics can be written as

$$\mathcal{A}U_t = *\delta_U \mathcal{P},$$

where $\mathcal{A} = diag(A, A_b)$.

15.2.2 The Dirac Formulation

In this section we aim to find an equivalent Dirac structure formalism of the Brayton Moser equations of infinite-dimensional system. As we shall see such a formulation would result in a non-canonical Dirac structure. For the finite-dimensional version of the Dirac formalism of BM equations we refer to [7]. Denote by $f_s = -u_t$ as the space of flows within the spatial domain and $e_s = \delta_u P$, as the space of effort variables again in the spatial domain. Further denote by $f_b = -u_b$ as the space of boundary flows and $e_b = \delta_{u^b} P$ as the space of boundary efforts. Consider the following subspace

$$\mathcal{D} = ((f_s, e_s, f_b, e_b) \in \mathcal{F}_s \times \mathcal{E}_s \times \mathcal{F}_b \times \mathcal{E}_b : -Af_s = *e_s, \ \ -A_b f_b = *e_b)$$

It can easily be shown that the above defined subspace constitutes a non-canonical Dirac structure, with respect to the bilinear form

$$\begin{aligned}
&\ll (f_s^1, e_s^1, f_b^1, e_b^1), (f_s^2, e_s^2, f_b^2, e_b^2) \gg \\
&= \int_Z \left(e_s^1 \wedge f_s^2 + e_s^2 \wedge f_s^1 + f_s^1 \wedge *(A + A^\top) f_s^2\right) \\
&\quad + \int_{\partial Z} \left(e_b^1 \wedge f_b^2 + e_b^2 \wedge f_b^1 + f_b^1 \wedge *(A_b + A_b^\top) f_b^2\right)
\end{aligned}$$

The above Dirac structure satisfies the power balance equation

$$0 = \int_Z \delta_u P \wedge u_t + \int_{\partial Z} \delta_{u^b} P^b \wedge u_t^b + \int_Z u_t \wedge *(A + A^\top) u_t + \int_{\partial Z} u_t^b \wedge *(A + A^\top) u_t^b$$

Remark 15.3 In the above Dirac structure formalism, we have assumed the case where $** = 1$, where $*$ is the hodge star operator. This is at least true for the case when the spatial domain is of dimension $n = 1$ and $n = 3$, which include respectively the case of the transmission line and the Maxwell's equations, which will be the two examples we will use in the rest of the chapter.

15.3 Admissible Pairs and Stability

Once we have written down the equations in the BM framework (sometimes also referred to as the pseudogradient form) we can pose the following question; does the mixed potential function serve as a storage function (or a Lyaunov function) to infer passivity (or equivalently stability) properties of the system? Below we aim to answer these questions with the aid of two examples.

15.3.1 Example: Maxwell Equations

The spatial domain $Z \subset \mathbb{R}^3$ is a three-dimensional boundary with a smooth two-dimensional boundary ∂Z. The energy variables are the electric field induction $\mathcal{D}$ and magnetic field induction $\mathcal{B}$. $\mathcal{D} = \frac{1}{2}\mathcal{D}_{ij} z_i \wedge z_j$ and $\mathcal{B} = \frac{1}{2}\mathcal{B}_{ij} z_i \wedge z_j$ are 2−forms on Z. The co-energy variables are electric field intensity $\mathcal{E}$ and Magnetic field intensity $\mathcal{H}$, their relationship with energy variables are given by,

$$* \mathcal{D} = \varepsilon \mathcal{E} \quad , \quad * \mathcal{B} = \mu \mathcal{H}, \tag{15.18}$$

where $\varepsilon(t, z)$ denotes the electric permittivity and $\mu(t, z)$ the magnetic permeability. The co-energy variables are one-forms, linearly related to energy variables. The Hamiltonian H is written as

$$H(\mathcal{D}, \mathcal{B}) = \int_Z \frac{1}{2}(\mathcal{E} \wedge \mathcal{D} + \mathcal{H} \wedge \mathcal{B}) = \int_Z \left(\frac{1}{2\varepsilon} * \mathcal{D} \wedge \mathcal{D} + \frac{1}{2\mu} * \mathcal{B} \wedge \mathcal{B} \right) \tag{15.19}$$

Therefore $\delta_{\mathcal{D}} H = \mathcal{E}$ and $\delta_{\mathcal{B}} H = \mathcal{H}$. Taking into account dissipation term in the system, the dynamics can be written in the port-Hamiltonian form as

$$-\frac{\partial}{\partial t}\begin{bmatrix} \mathcal{D} \\ \mathcal{B} \end{bmatrix} = \begin{bmatrix} 0 & -\mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} \delta_{\mathcal{D}} H \\ \delta_{\mathcal{B}} H \end{bmatrix} + \begin{bmatrix} J_d \\ 0 \end{bmatrix} = \begin{bmatrix} *\sigma & -\mathrm{d} \\ \mathrm{d} & 0 \end{bmatrix} \begin{bmatrix} \delta_{\mathcal{D}} H \\ \delta_{\mathcal{B}} H \end{bmatrix}. \tag{15.20}$$

where $*J_d = \sigma \mathcal{E}$, J_d denotes the current density and $\sigma(z, t)$ is the specific conductivity of the material. In addition we define the boundary variables as $f_b = \delta_D H \mid_{\partial Z}$, $e_b = \delta_B H \mid_{\partial Z}$. The rate of the Hamiltonian is given as

$$\frac{d}{dt} H \leq \int_{\partial Z} \mathcal{H} \wedge \mathcal{E}$$

The Brayton Moser form of Maxwell's equations:
In order to write the Maxwell's equations in the BM form, we proceed as follows: The aim is to rewrite the equations in terms of the co-energy variables, i.e. $\mathcal{H}$ and $\mathcal{E}$.

Define the mixed potential functional corresponding to the Maxwell's equations as

$$P = \int_Z \left(\mathcal{H} \wedge \mathrm{d}\mathcal{E} - \frac{1}{2} \sigma \mathcal{E} \wedge *\mathcal{E} \right), \tag{15.21}$$

which gives us the following form of Maxwell's equations in terms of the mixed potential

$$\begin{bmatrix} -\mu I_3 & 0 \\ 0 & \varepsilon I_3 \end{bmatrix} \begin{bmatrix} \mathcal{H}_t \\ \mathcal{E}_t \end{bmatrix} = \begin{bmatrix} *\mathrm{d}\mathcal{E} \\ -\sigma \mathcal{E} + *\mathrm{d}\mathcal{H} \end{bmatrix} = \begin{bmatrix} *\delta_{\mathcal{H}} P \\ *\delta_{\mathcal{E}} P \end{bmatrix} \tag{15.22}$$

15.3.1.1 Stability Analysis

To infer stability properties of the system (15.22) let us begin with the case of zero energy flow through the boundary of the system. The mixed potential function (15.21) obtained via (15.12) is not positive definite. Hence we cannot use it as Lyapunov/ storage functional. Moreover, the rate of this function is computed as

$$\frac{\partial P}{\partial t} = \int_Z (-\mu \mathcal{H}_t \wedge *\mathcal{H}_t + \varepsilon \mathcal{E}_t \wedge *\mathcal{E}_t)$$

It can be easily seen that the right-hand side of the above equation is not sign definite, and hence P does not serve as a Lyapunov functional to infer any kind of stability (or for that matter passivity) properties of the system. We thus need to look for other possible Lyapunov functionals $\tilde{P}$, or in other words admissible pairs $\tilde{A}$, $\tilde{P}$ as in the case of finite-dimensional systems [8] which can prove stability of the system. Moreover, in order to conclude stability, the admissible pair should be such that the symmetric part of $\tilde{A}$ is negative semidefinite. This can be achieved in the following way, [4, 10]. Let

$$\tilde{P} = \lambda P + \frac{1}{2} \int_Z (\delta_{\mathcal{H}} P \wedge M_1 * \delta_{\mathcal{H}} P + \delta_{\mathcal{E}} P \wedge M_2 * \delta_{\mathcal{E}} P)$$

with λ be a arbitrary constant and symmetric M_1 and M_2 mapping from $\Omega^2(Z) \to \Omega^2(Z)$. Here the aim is to find λ, M_1 and M_2 such that

$$\frac{\partial}{\partial t} \tilde{P} = u_t^{\top} \tilde{A} u_t \leq -K ||u_t||^2 \leq 0 \tag{15.23}$$

where $K \geq 0$ is a constant determined by the $\tilde{A}$. If we can find such a $(\tilde{P}, \tilde{A})$, which satisfies the above condition, then we can conclude stability of the system, by invoking the stability theorem in [4].

Below we present a constructive process to obtain new admissible pairs. The variation in $\tilde{P}$ defined in (15.23) is computed as

$$\begin{bmatrix} \delta_{\mathcal{H}} \tilde{P} \\ \delta_{\mathcal{E}} \tilde{P} \end{bmatrix} = \begin{bmatrix} \lambda I & M_2 \mathrm{d}* \\ M_1 \mathrm{d}* & (\lambda I - \sigma M_2) \end{bmatrix} \begin{bmatrix} \delta_{\mathcal{H}} P \\ \delta_{\mathcal{E}} P \end{bmatrix},$$

applying Hodge star on both sides and using (15.22) we get

$$*\begin{bmatrix}\delta_{\mathcal{H}}\tilde{P}\\ \delta_{\mathcal{E}}\tilde{P}\end{bmatrix} = \begin{bmatrix}-\mu\lambda I & \varepsilon M_2 * \mathrm{d}\\ -\mu M_1 * \mathrm{d} & \varepsilon\,(\lambda I - \sigma M_2)\end{bmatrix}\begin{bmatrix}\mathcal{H}_t\\ \mathcal{E}_t\end{bmatrix}.$$

Further, if we let

$$\tilde{A} = \begin{bmatrix}-\mu\lambda I & \varepsilon M_2 * \mathrm{d}\\ -\mu M_1 * \mathrm{d} & \varepsilon\,(\lambda I - \sigma M_2)\end{bmatrix}$$

we arrive at the following relationship

$$\tilde{A}u_t = *\delta_u \tilde{P}. \tag{15.24}$$

Next we show that $\tilde{P}$ and $\tilde{A}$ are admissible pairs if λ, M_1 and M_2 satisfy $\varepsilon M_2 = \mu M_1 \overset{\triangle}{=} \theta$ and $0 \leq \lambda \leq \sigma\|M_2\|_s$, where $\|\cdot\|_s$ is spectral norm. Some calculations show that the symmetric part of $\tilde{A} = diag(-\mu\lambda I, -\varepsilon\,(\sigma M_2 - \lambda I))$ is negative definite.

We note that P can be simplified to

$$\begin{aligned} P &= \int_z \mathcal{H} \wedge d\mathcal{E} - \frac{1}{2}\sigma\mathcal{E} \wedge *\mathcal{E}\\ &= \int_z -\frac{1}{2\sigma}\left[\delta_{\mathcal{E}}\mathcal{P} \wedge *\delta_{\mathcal{E}}\mathcal{P}\right] + \frac{1}{2\sigma} d\mathcal{H} \wedge *d\mathcal{H}, \end{aligned}$$

resulting in

$$\begin{aligned} \tilde{P} = \int_z \delta_{\mathcal{E}} P \wedge \frac{\sigma M_2 - \lambda I}{2\sigma} * \delta_{\mathcal{E}} P + \frac{1}{2\sigma} d\mathcal{H} \wedge *d\mathcal{H}\\ + \frac{1}{2}\left(\delta_{\mathcal{H}} P \wedge M_1 * \delta_{\mathcal{H}} P\right) \geq 0 \end{aligned} \tag{15.25}$$

Lastly, we choose $M_1 > 0$ and $M_2 > 0$ such that $\varepsilon M_2 = \mu M_1$. The time derivative of $\tilde{\mathcal{P}}$ is

$$\dot{\tilde{P}} = -\int_Z \left(\mu\lambda\mathcal{H}_t \wedge *\mathcal{H}_t + \mathcal{E}_t \wedge *(\lambda I - \sigma M_2)\mathcal{E}_t\right) \leq 0$$

thus implying stability.

15.3.2 Example: The Transmission Line

In this section we first derive the Brayton Moser equivalent of the dynamics of a transmission line modeled by the telegraphers equations. Similar to the case of Maxwell's equations we find the admissible pairs under zero boundary energy flow conditions and infer stability of the system.

The spatial domain in case of the transmission is $Z = [0, 1] \subset \mathrm{R}$ with boundary $\partial Z = \{0, 1\}$. The charge $q(z, t)$ and flux densities $\phi(z, t) \in \Omega^1(Z)$ constitute the energy variables, whereas the co-energy variable are voltage $v(z, t)$ and current $i(z, t) \in \Omega^0(Z)$. For simplicity, the relation between the energy and co-energy variables is assumed to be linear, and is given by

$$*q = Cv, \quad *\phi = Li \tag{15.26}$$

where C and L are, respectively, the spatial capacitance and inductance per unit length, which are assumed to be independent of z. The Hamiltonian H, which is the total energy of the system, is written as

$$H = \frac{1}{2}\int_Z (v \wedge q + i \wedge \phi) \tag{15.27}$$

Taking the dissipation term into account, the telegraphers equations written in port-Hamiltonian form as [16]

$$-\frac{\partial}{\partial t}\begin{bmatrix} q \\ \phi \end{bmatrix} = \begin{bmatrix} *G & \mathrm{d} \\ \mathrm{d} & *R \end{bmatrix}\begin{bmatrix} \delta_q H \\ \delta_\phi H \end{bmatrix} \tag{15.28}$$

where $\delta_q H = v$, $\delta_\phi H = i$ (using (15.26) and (15.27)). $R,\ G$, respectively, denote the distributed resistance and conductance per unit length of the transmission line. Further, we define the boundary variables as $f_b = \delta_q H|_{\partial Z}$ and $e_b = \delta_\phi H|_{\partial Z}$. The rate of Hamiltonian is given by

$$\frac{d}{dt}H = (v.i)|_0^1$$

The Brayton Moser form:
The dynamics of the transmission line (15.27) can be written in an equivalent Brayton Moser form as follows: Define a functional P as

$$P = \int_Z \left(-v \wedge \mathrm{d}i + \frac{1}{2}Ri \wedge *i - \frac{1}{2}Gv \wedge *v\right), \tag{15.29}$$

which will serve as the mixed potential function. Using the line voltage and current as the state variables, we can rewrite the dynamics as follows:

$$\begin{bmatrix} -L & 0 \\ 0 & C \end{bmatrix} \begin{bmatrix} i_t \\ v_t \end{bmatrix} = \begin{bmatrix} *\delta_i P \\ *\delta_v P \end{bmatrix} = \begin{bmatrix} Gv + *\mathrm{d}i \\ -Ri - *\mathrm{d}v \end{bmatrix}. \tag{15.30}$$

with $A \triangleq diag(-L, C)$ and $u = (i(z,t)\ v(z,t))^\top$.

15.3.2.1 Admissible Pairs and Stability

Similar to the case of Maxwell equations, we cannot use P and A directly to infer stability. We, therefore, need to generate new admissible pairs $\tilde{P}$ and $\tilde{A}$ satisfying (15.23) and (15.24) such that $\tilde{P} > 0$ and symmetric part of $\tilde{A} < 0$, resulting in stability. As in the case of Maxwell's equations, we propose a $\tilde{P}$ of the form

$$\tilde{P} = \lambda P + \frac{1}{2}\delta_u P \wedge M * \delta_u P \tag{15.31}$$

We choose $M = \begin{bmatrix} \frac{\alpha}{R} & m_2 \\ m_2 & \frac{\beta}{G} \end{bmatrix}$ where α, β, m_2 are positive constants satisfying $\alpha\frac{L}{R} = \beta\frac{C}{G}$ and λ is a dimensionless constant. Such a choice will be clear in the following discussions, which will eventually lead to a stability criterion. It is easy to check that $\tilde{P}$ has units of power. To simplify the calculations we define new positive constants θ, γ and ζ as follows:

$$\begin{aligned} \theta &\triangleq \alpha\frac{L}{R} = \beta\frac{C}{G}, \qquad m_2 \triangleq \frac{2\gamma}{CR + LG} \\ \zeta &\triangleq \frac{2\gamma}{\sqrt{LC}(\alpha+\beta)} \implies m_2 = \frac{\zeta\theta}{\sqrt{LC}}. \end{aligned} \tag{15.32}$$

To show that $\tilde{P} \geq 0$ we start with simplifying the right hand side of (15.31) in the following way. Define

$$\begin{aligned} \Delta_1 &\triangleq \left(\zeta\sqrt{\frac{L}{2}}(Gv + i_z) - \sqrt{\frac{C}{2}}(Ri + v_z)\right) \\ \Delta_2 &\triangleq \left(\zeta\sqrt{\frac{C}{2}}(Ri + v_z) - \sqrt{\frac{L}{2}}(Gv + i_z)\right). \end{aligned} \tag{15.33}$$

Using (15.32), (15.33), and after some calculations, we can show that

$$\begin{aligned}\frac{1}{2}\langle \delta_u P, M\delta_u P\rangle &= \Delta_1^2 + \frac{\beta}{2G}(1-\zeta^2)(Gv+i_z)^2 \\ &= \Delta_2^2 + \frac{\alpha}{2R}(1-\zeta^2)(Ri+v_z)^2\end{aligned}$$

$\tilde{P}$ as defined in (15.31) can then be written as follows

$$\tilde{P} = \frac{\alpha(1-\zeta^2)-\lambda}{2R}(Ri+v_z)^2 + \Delta_2^2 + \frac{\lambda}{2R}v_z^2 + \frac{\lambda}{2}Gv^2 \tag{15.34}$$

$$= \frac{\beta(1-\zeta^2)+\lambda}{2G}(Gv+i_z)^2 + \Delta_1^2 - \frac{\lambda}{2G}i_z^2 - \frac{\lambda}{2}Ri^2 \tag{15.35}$$

which implies that $\tilde{P} \geq 0$ as long as the following conditions are satisfied

$$\begin{aligned}&0 \leq \lambda \leq \alpha(1-\zeta^2),\ 0 \leq \zeta \leq 1 \\ &-\beta(1-\zeta^2) \leq \lambda \leq 0 \text{ or equilvalently} \\ &-\beta(1-\zeta^2) \leq \lambda \leq \alpha(1-\zeta^2),\ \ 0 \leq \zeta \leq 1\end{aligned} \tag{15.36}$$

Further the variational derivative of $\tilde{P}$ with respect to u is calculated as

$$\begin{aligned}\delta_u \tilde{P} &= \begin{bmatrix}(-\lambda+\alpha)(v_z+Ri) - m_2R(Gv+i_z) \\ (\lambda+\beta)(Gv+i_z) - m_2G(Ri+v_z)\end{bmatrix} - \frac{\partial}{\partial z}\begin{bmatrix}\frac{\beta}{G}(Gv+i_z) - m_2(v_z+Ri) \\ \frac{\alpha}{R}(v_z+Ri) - m_2(Gv+i_z)\end{bmatrix} \\ &= \begin{bmatrix}L(\lambda-\alpha-m_2\frac{\partial}{\partial z}) & C(Rm_2+\frac{\beta}{G}\frac{\partial}{\partial z}) \\ L(Gm_2+\frac{\alpha}{R}\frac{\partial}{\partial z}) & -C(\lambda+\beta+m_2\frac{\partial}{\partial z})\end{bmatrix}\begin{bmatrix}i_t \\ v_t\end{bmatrix}.\end{aligned}$$

Therefore

$$\tilde{A} = \begin{bmatrix}L(\lambda-\alpha-m_2\frac{\partial}{\partial z}) & C(Rm_2+\frac{\beta}{G}\frac{\partial}{\partial z}) \\ L(Gm_2+\frac{\alpha}{R}\frac{\partial}{\partial z}) & -C(\lambda+\beta+m_2\frac{\partial}{\partial z})\end{bmatrix} \tag{15.37}$$

satisfies the gradient form (15.24).

Noting that conjugate of $\frac{\partial}{\partial z}$ is $-\frac{\partial}{\partial z}$ and using $\alpha\frac{L}{R} = \beta\frac{C}{G}$ from (15.32), the symmetric part of $\tilde{A}$ (15.37) is simplified to be

$$\frac{\tilde{A}+\tilde{A}^*}{2} = \begin{bmatrix}L(\lambda-\alpha) & \gamma \\ \gamma & -C(\lambda+\beta)\end{bmatrix}$$

The symmetric part of $\tilde{A}$ is negative semidefinite as long as the following conditions are satisfied,

$$-\beta \leq \lambda \leq \alpha, \text{ and } (\lambda - \alpha)(\lambda + \beta) + \frac{(\alpha + \beta)^2}{4}\zeta^2 \leq 0. \tag{15.38}$$

We now present the following result:

Proposition 15.4 *If there exist non-zero α, β, λ and ζ satisfying* (15.32)*,* (15.36)*, and* (15.38) *then $\tilde{P}$ defined in* (15.31) *and $\tilde{A}$ defined in* (15.37) *with M are admissible pairs for the transmission line dynamics. Additionally if the symmetric part of $\tilde{A}$ is negative semidefinite, i.e. (15.38) holds true, them the system of equations (15.30) is stable.*

Proof From (15.32) we define $\tau \triangleq \frac{\alpha}{\beta} = \frac{RC}{LG}$. Given a transmission line R, C, L, G are fixed, therefore $\tau \geq 0$ is related to system parameters and thus can be treated as one. Let $\lambda^{'} = \frac{\lambda}{\beta}$. Using this in (15.36) and (15.38) we get

$$-(1 - \zeta^2) \leq \lambda^{'} \leq \tau(1 - \zeta^2) \tag{15.39}$$

$$(\lambda^{'} - \tau)(\lambda^{'} + 1) + \frac{(\tau + 1)^2}{4}\zeta^2 \leq 0 \tag{15.40}$$

Now we have to show that for all $\tau \geq 0$ there exists a pair of $\lambda^{'}$ and ζ that satisfies both the above equations. Given a $\zeta \in (0, 1)$ from (15.39) $\lambda^{'}$ lies between a positive value and a negative value $\forall \tau \geq 0$. If we can show that (15.40) has a positive and negative roots, then we can conclude the proof. The roots of (15.40) are

$$r_1 = \frac{1}{2}\left(\tau - 1 + (\tau + 1)\sqrt{1 - \zeta^2}\right)$$
$$r_2 = \frac{1}{2}\left(\tau - 1 - (\tau + 1)\sqrt{1 - \zeta^2}\right)$$

The aim is to find a condition on ζ such that r_1 and r_2 have a different signs, for all $\tau > 0$. For $0 < \tau < 1$ we have $r_2 < 0$. In order to make $r_1 > 0$ we need $\zeta^2 < 4\tau/(1 + \tau)^2$. Further for $\tau > 1$ we have $r_1 > 0$, in which case we require $r_2 < 0$ which leads to the same condition on ζ that is $\zeta^2 < 4\tau/(1 + \tau)^2$. Note that this is a valid condition on ζ since $\forall \tau$, $\frac{4\tau}{(1 + \tau)^2} \leq 1$. Which implies ζ is bounded,

$$0 \leq \zeta^2 \leq \frac{4LCRG}{(LG + RC)^2}. \tag{15.41}$$

Therefore $\forall \zeta \in [0, \frac{4\tau}{(1+\tau)^2}]$ there exists a $\lambda^{'}$ which satisfies (15.39) and (15.40). Finally for any $\beta \in \mathbb{R}^+$, $\alpha = \tau\beta$, $\lambda = \lambda^{'}\beta$ and $\zeta \in [0, \frac{4\tau}{(1+\tau)^2}]$ satisfies (15.32), (15.36) and (15.38).

15.4 Admissible Pairs and Stability for Non-zero Energy Flow Through Boundary

In this section we derive the Brayton Moser formulation of infinite-dimensional systems with non-zero energy flows through boundary. For simplicity, we limit our discussion for systems evolving on spatial domain $Z = (0, 1)$ of dimension $n = 1$ with point boundaries, $\partial Z = \{0, 1\}$. For $z \in Z$, let $u(z, t)$ be the states evolving on the spatial domain Z, further let $u_0(t)$ and $u_1(t)$ denote the states evolving at the boundary $z = 0$ and $z = 1$. Now consider the mixed potential function of the form

$$\mathcal{P}(U) = P(u) + P^0(u_0) + P^1(u_1) \tag{15.42}$$

where $u_0 = u(0, t)$, $u_1 = u(1, t)$ and $U = [u, u_0, u_1]$ with $P(u)$ of the form (15.29). P^0 and P^1 are the contributions to the mixed potential function arising form the boundary dynamics. Similar to infinite-dimensional case, we represent the overall dynamics of finite and infinite-dimensional system in Brayton Moser form. Dynamics evolving on the spatial domain (15.30) are given by (i.e. for $0 < z < 1$)

$$Au_t = \delta_u P,$$

dynamics at boundary $z = 0$ are represented by

$$A_0 u_{0t} = \left.\left(\frac{\partial P^0}{\partial u_0} - P_{u_z}\right)\right|_{z=0} + B_0 E_0$$

with B_0, E_0 representing input matrix and source at $z = 0$ respectively. Further $P_{u_z} = \frac{\partial P}{\partial u_z}$.

The dynamics at boundary $z = 1$ are represented using

$$A_1 u_{1t} = \left.\left(\frac{\partial P^1}{\partial u_0} + P_{u_z}\right)\right|_{z=1} + B_1 E_1$$

where B_1 and E_1 are input matrix and source at $z = 1$. Together they can be written compactly in Brayton Moser form as

$$\mathcal{A}U_t = \delta_U \mathcal{P} + BE \tag{15.43}$$

where $\mathcal{A} = diag\{A, A_0, A_1\}$, A, A_0 and $A_1 \in \mathbb{R}^{2\times 2}$. $B = [B_0, B_1]$ is the input matrix and $E = [E_0\ E_1]^\top$ are the inputs to the system. The variational derivative of $\mathcal{P}$ (15.42) with respect to U is

$$\delta_U \mathcal{P} = \begin{bmatrix} \delta_u P \\ \left(\dfrac{\partial P^0}{\partial u_0} - P_{u_z}\right)\Big|_{z=0} \\ \left(\dfrac{\partial P^1}{\partial u_1} + P_{u_z}\right)\Big|_{z=1} \end{bmatrix}.$$

Further the time derivative of mixed potential function (15.42) is

$$\frac{d}{dt}\mathcal{P} = \int_0^1 (\delta_u P \cdot u_t)\, dz + \left(\frac{\partial P^0}{\partial u_0} - P_{u_z}\right)\Big|_{z=0} \cdot u_{0t} + \left(\frac{\partial P^1}{\partial u_1} + P_{u_z}\right)\Big|_{z=1} \cdot u_{1t} \tag{15.44}$$

where $u_t = \dfrac{\partial u}{\partial t}$, $u_{0t} = \dfrac{\partial u_0}{\partial t}$, $u_{1t} = \dfrac{\partial u_1}{\partial t}$. Using the Brayton Moser form (15.43), $\dot{\mathcal{P}}$ can be written as

$$\begin{aligned} \frac{d}{dt}\mathcal{P} &= \int_0^1 (Au_t \cdot u_t)\, dz + A_0 u_{0t} \cdot u_{0t} + A_1 u_{1t} \cdot u_{1t} - E^\top B^\top U_t \\ &= \int_0^1 \left(u_t^\top \frac{A + A^\top}{2} u_t\right) dz + u_{0t}^\top \frac{A_0 + A_0^\top}{2} u_{0t} + u_{1t} \frac{A_1 + A_1^\top}{2} u_{1t} + E^\top y \end{aligned} \tag{15.45}$$

where $y = -B^\top U_t$. It can be seen that for a positive definite $\mathcal{P}$, and negative definite $\mathcal{A}$ the system is passive with input E and output y. In general $\mathcal{P}$ and $\mathcal{A}$ do not satisfy these conditions. This motivates us to search for new admissible pairs $\mathcal{P} \geq 0$ and $\mathcal{A} \leq 0$ which enables us derive cerain passivity/stability properties.

Definition 15.5 Admissible Pairs: We denote $\tilde{\mathcal{P}} = \tilde{P} + \tilde{P}^0 + \tilde{P}^1$ and $\tilde{\mathcal{A}} = diag\{\tilde{A}, \tilde{A}_0,$
$\tilde{A}_1\}$ Admissible pairs if they satisfy the following:

(a) $\tilde{P} \geq 0$ and $\tilde{A} \leq 0$ such that

$$\tilde{A} u_t = \delta_u \tilde{P} \tag{15.46}$$

(b) $\tilde{P}^0 \geq 0$ and $\tilde{A}_0 \leq 0$ such that

$$\tilde{A}_0 u_{0t} = \left(\frac{\partial \tilde{P}}{\partial u_0} - \tilde{P}_{u_z}\right)\Big|_{z=0} + B_0 E_0 \tag{15.47}$$

(c) $\tilde{P}^1 \geq 0$ and $\tilde{A}_1 \leq 0$ such that

$$\tilde{A}_1 u_{1t} = \left.\left(\frac{\partial \tilde{P}}{\partial u_1} + \tilde{P}_{u_z}\right)\right|_{z=1} + B_1 E_1 \tag{15.48}$$

(d) Together we can write them as $\tilde{\mathcal{P}} \geq 0$ and $\tilde{\mathcal{A}} \leq 0$ such that

$$\begin{aligned} \tilde{\mathcal{A}} U_t &= \delta_U \tilde{\mathcal{P}} + B E_b \\ y_b &= -B^\top U_t. \end{aligned} \tag{15.49}$$

Finally time derivative of $\tilde{\mathcal{P}}$ is

$$\dot{\tilde{\mathcal{P}}} \leq E_b^\top y_b.$$

Which implies that the system is passive with storage function $\tilde{\mathcal{P}}$ and ports E_b and y_b.
We next show how to derive these with the help of an example.

15.4.1 *Example: Transmission Line with Circuit Elements at the Boundary*

Consider a transmission line, whose boundary is interconnected to certain circuit elements as shown in Fig. 15.1. At $z = 0$ is a resistor R_0 in series with inductor L_0 connected to a voltage source E_0. The other end of the transmission line $z = 1$ is terminated with a resistor R_1.

This gives us the following dynamics at the boundary

$$\begin{aligned} v_0 + R_0 i_0 + L_0 \frac{di_0}{dt} &= E_0 \qquad & z = 0 \\ v_1 &= R_1 i_1 \qquad & z = 1 \end{aligned} \tag{15.50}$$

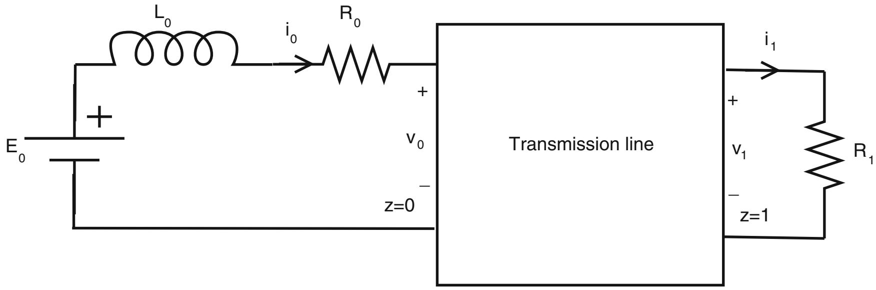

Fig. 15.1 Transmission line with boundary

where $v_0 = v(0,t)$, $i_0 = i(0,t)$ and $v_1 = v(1,t)$, $i_1 = i(1,t)$, let $U = [i, v, i_0, v_0, i_1, v_1]^\top$.

Let $u = [i, v]^\top$, $u_0 = [i_0, v_0]^\top$, $u_1 = [i_1, v_1]^\top$ and $P_{u_z} = \dfrac{\partial P}{\partial u_z}$, $u_z = \frac{\partial u}{\partial z}$.

Next we define the mixed potential function $\mathcal{P} = P + P^0 + P^1$ and $\mathcal{A}$ as follows:

$$P = \int_0^1 \left(-\frac{1}{2} R i^2 + \frac{1}{2} G v^2 + v i_z \right) \mathrm{d}z$$

$$P^0 = -\frac{1}{2} R_0 i_0^2 \qquad P^1 = -\frac{1}{2} R_1 i_1^2$$

$$\mathcal{A} = diag \left\{ \begin{bmatrix} L & -C \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} L_0 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \right\}$$

where P and A are defined in (15.29) and (15.30), respectively. The input matrices $B_0 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}^\top$, $B_1 = 0$ and $E_1 = 0$. The transmission line dynamics governed by Eq. (15.28) together with the boundary dynamics given by (15.50) can be written in a compact form as

$$\begin{bmatrix} L & 0 & 0 & 0 & 0 & 0 \\ 0 & -C & 0 & 0 & 0 & 0 \\ 0 & 0 & L_0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} i_t \\ v_t \\ i_{0t} \\ v_{0t} \\ i_{1t} \\ v_{1t} \end{bmatrix} = \begin{bmatrix} -Ri - \frac{\partial v}{\partial z} \\ -Gv - \frac{\partial i}{\partial z} \\ v_0 + R_0 i_0 \\ -v_0 - R_0 i_0 \\ -v_1 + R_1 i_1 \\ v_1 - R_1 i_1 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} E_0.$$

It can easily be checked that using $\mathcal{P}$ as a storage function does not result in any kind of passivity properties of the system. Therefore, we find new admissible pairs satisfying Definition 15.5. The admissible pairs for spatial domain, found in Sect. 15.3.2.1 for zero energy flow through boundary will satisfy (15.46). Therefore $\tilde{P}$ and $\tilde{A}$ remains same for transmission line with zero or with non-zero energy flow through the boundary. For the rest of the example we choose that $\lambda = -1$, and input matrix $B_0 = \begin{bmatrix} 0 & 0 & -1 & 0 & 0 & 0 \end{bmatrix}^\top$. Next we aim to find $\tilde{A}_0$ and $\tilde{P}^0$ which satisfy (15.47). At $z = 0$ we have

$$\left. \left(\frac{\partial \tilde{P}^0}{\partial u_0} - \tilde{P}_{u_z} \right) \right|_{z=0} + B_0 E_0 = \begin{bmatrix} -m_2 L i_{0t} + \theta v_{0t} + v_0 + \frac{\partial \tilde{P}^0}{\partial i_0} - E_0 \\ \theta i_{0t} - m_2 C v_{0t} + \frac{\partial \tilde{P}^0}{\partial v_0} \end{bmatrix}$$

Let us consider $\tilde{P}^0$ of the form $\dfrac{1}{2} R_0 i_0^2$,

$$\left.\left(\frac{\partial \tilde{P}^0}{\partial u_0} - \tilde{P}_{u_z}\right)\right|_{z=0} + B_0 E_0 = \begin{bmatrix} -m_2 L i_{0t} + \theta v_{0t} + v_0 + R_0 i_0 - E_0 \\ \theta i_{0t} - m_2 C v_{0t} \end{bmatrix}$$
$$= \begin{bmatrix} -m_2 L i_{0t} + \theta v_{0t} - L_0 i_{0t} \\ \theta i_{0t} - m_2 C v_{0t} \end{bmatrix}.$$

In the last step we used the boundary condition at $z = 0$, i.e. $v_0 + R_0 i_0 - E_0 = -L_0 i_{0t}$, further assuming that $\exists \zeta, \theta$ satisfying $\frac{1}{m_2 C}(1-\zeta^2)\theta^2 = L_0$,

$$\left.\left(\frac{\partial \tilde{P}^0}{\partial u_0} - \tilde{P}_{u_z}\right)\right|_{z=0} + B_0 E_0 = \begin{bmatrix} -m_2 L i_{0t} + \theta v_{0t} - \left(\frac{1}{m_2 C}(1-\zeta^2)\theta^2\right) i_{0t} \\ \theta i_{0t} - m_2 C v_{0t} \end{bmatrix}$$
$$= \begin{bmatrix} \theta v_{0t} - \left(m_2 L + \frac{1}{m_2 C}(1-\zeta^2)\theta^2\right) i_{0t} \\ \theta i_{0t} - m_2 C v_{0t} \end{bmatrix}$$
$$= \begin{bmatrix} \theta v_{0t} - \frac{\theta^2}{m_2 C} i_{0t} \\ \theta i_{0t} - m_2 C v_{0t} \end{bmatrix} = \begin{bmatrix} -\frac{\theta^2}{m_2 C} & \theta \\ \theta & -m_2 C \end{bmatrix} \begin{bmatrix} i_{0t} \\ v_{0t} \end{bmatrix}$$

in the last step we used $m_2 = \frac{\zeta\theta}{\sqrt{LC}}$ (15.32). Finally, we denote

$$\tilde{A}_0 = \begin{bmatrix} -\frac{\theta^2}{m_2 C} & \theta \\ \theta & -m_2 C \end{bmatrix} \leq 0, \tag{15.51}$$
$$u_{0t}^\top \tilde{A}_0 u_{0t} \leq 0.$$

Hence $\tilde{P}^0 = \frac{1}{2} R_0 i_0^2$ and $\tilde{A}_0$ (15.52) satisfy (15.47), under the assumption that ζ and θ are chosen such that, $L_0 = \frac{1}{m_2 C}(1-\zeta^2)\theta^2$. Similarly under the assumption that $\frac{\theta}{m_2} = R_1$, we can show that for $\tilde{A}_1 = -\tilde{A}_0$ and $\tilde{P}^1 = \frac{1}{2} R_1 i_1^2$ will satisfy (15.48). But for all (i_{1t}, v_{1t}) satisfying $v_{1t} = R_1 i_{1t} = \frac{\theta}{m_2 C} i_{1t}$ we have

$$\tilde{A}_1 u_{1t} = \begin{bmatrix} \frac{\theta^2}{m_2 C} & -\theta \\ -\theta & m_2 C \end{bmatrix} \begin{bmatrix} i_{1t} \\ v_{1t} \end{bmatrix}$$
$$= \begin{bmatrix} \frac{\theta^2}{m_2 C} & -\theta \\ -\theta & m_2 C \end{bmatrix} \begin{bmatrix} i_{1t} \\ R_1 i_{1t} \end{bmatrix} = \begin{bmatrix} \frac{\theta^2}{m_2 C} & -\theta \\ -\theta & m_2 C \end{bmatrix} \begin{bmatrix} i_{1t} \\ \frac{\theta}{m_2 C} i_{1t} \end{bmatrix} = 0$$

That is we choose θ and m_2 such that u_{1t} is always in the nullspace of $\tilde{A}_1$. Which implies

$$u_{1t}^\top \tilde{A}_1 u_{1t} = 0 \quad \forall (i_{1t}, v_{1t}).$$

Finally for $B_b = \begin{bmatrix} 0 \; 0 \; -1 \; 0 \; 0 \; 0 \end{bmatrix}^\top$ and $E_b = E_0$ we get $y_b = \dfrac{di_0}{dt}$. The time derivative $\tilde{\mathcal{P}} = \tilde{P} + \tilde{P}^0 + \tilde{P}^1$ is computed as

$$\frac{d}{dt}\tilde{\mathcal{P}} \leq E_0 \frac{di_0}{dt},$$

which implies that the system is passive with respect to input E_0 and output $\dfrac{di_0}{dt}$.

Remark 15.6 Note that in Hamiltonian case the storage function is

$$H = \frac{1}{2}\int_0^1 \left(Li^2 + Gv^2\right) + \frac{1}{2}L_0 i_0^2$$

and its time derivative is calculated to be $\dfrac{d}{dt}H \leq E_0 i_0$. The system is passive with port variable E_0 and i_0.

15.5 Casimirs and Conservation Laws

We obtain conservation laws which are independent from the mixed potential function as follows [14, 16]: For simplicity, we consider the case of systems without dissipation. We further assume that the energy and the co-energy variables are related via a linear relation, given by

$$\alpha_p = *\varepsilon \; e_p \text{ and } \alpha_q = *\mu \; e_q. \tag{15.52}$$

We can write (15.10) in the following way:

$$\begin{bmatrix} -\mu & 0 \\ 0 & \varepsilon \end{bmatrix} \begin{bmatrix} \dot{e}_q \\ \dot{e}_p \end{bmatrix} = \begin{bmatrix} *\delta_{e_q} P \\ *\delta_{e_p} P \end{bmatrix} \tag{15.53}$$

Consider a function $\mathcal{C} : \Omega^{n-p}(Z) \times \Omega^{n-q}(Z) \times Z \to$ R, which satisfies

$$\mathrm{d}(*\delta_{e_p}\mathcal{C}) = 0, \quad \mathrm{d}(*\delta_{e_q}\mathcal{C}) = 0$$

$$\frac{d}{dt}\mathcal{C}(e_q, e_p) = \int_Z \left(\delta_{e_q} C \wedge \dot{e}_q + \delta_{e_p} C \wedge \dot{e}_p\right)$$

$$
\begin{aligned}
&= \int_Z \left(-\delta_{e_q} C \wedge * \frac{1}{\mu} \mathrm{d} e_p (-1)^{(n-q)\times q} + \delta_{e_p} C \wedge * \frac{1}{\varepsilon} (-1)^{pq} \mathrm{d} e_q (-1)^{(n-p)\times p} \right) \\
&= \int_Z \left(-(-1)^{(n-q)\times q} \frac{1}{\mu} \mathrm{d} e_p \wedge * \delta_{e_q} C + (-1)^p \frac{1}{\varepsilon} \mathrm{d} e_q \wedge * \delta_{e_p} C \right) \\
&= \int_Z \left(-(-1)^{(n-q)\times q} \frac{1}{\mu} [\mathrm{d}(e_p \wedge * \delta_{e_q} C) + (-1)^q e_p \wedge \mathrm{d}(* \delta_{e_q} C)] \right. \\
&\quad \left. + (-1)^p \frac{1}{\varepsilon} [\mathrm{d}(e_q \wedge * \delta_{e_p} C) + (-1)^p e_p \wedge \mathrm{d}(* \delta_{e_p} C)] \right) \\
&= \int_{\partial Z} \left(e_q \wedge * \delta_{e_p} C) \mid_{\partial Z} + (e_p \wedge * \delta_{e_q} C) \mid_{\partial Z} \right)
\end{aligned}
$$

In the particular case when $* \delta_{e_p} C \mid_{\partial Z} = * \delta_{e_q} C \mid_{\partial Z} = 0$, then $\frac{dC}{dt} = 0$, along the system trajectories. Such a function is called a Casimir function.

15.5.1 Example: Transmission Line

In case of the lossless transmission line, the total current

$$C_I = \int_0^1 i(t, z) dz$$

and the line voltage

$$C_v = \int_0^1 v(t, z) dz$$

are the systems conservation laws. This can easily be inferred by the following

$$
\begin{aligned}
\frac{d}{dt} C_I &= -\int_0^1 \frac{1}{l} \frac{\partial v}{\partial z} = \frac{v}{L} \mid_0 - \frac{v}{L} \mid_1 \\
\frac{d}{dt} C_v &= -\int_0^1 \frac{1}{C} \frac{\partial i}{\partial z} = \frac{i}{C} \mid_0 - \frac{i}{C} \mid_1
\end{aligned}
$$

15.5.2 Example: Maxwell's Equations

In case of Maxwell's equations with no dissipation terms, it can easily be checked that the magnetic field intensity $\int_Z \mathcal{H}$ and the electric field intensity $\int_Z \mathcal{B}$ constitute the conserved quantities. This can be seen via the following expressions:

$$\int_Z \frac{d}{dt}\mathcal{H}_t = -\int_{\partial Z} \frac{1}{\mu}\mathcal{E}$$
$$\int_Z \frac{d}{dt}\mathcal{E}_t = \int_{\partial Z} \frac{1}{\varepsilon}\mathcal{H}$$

Another class of conserved quantities can be identified in the following way: Using (15.11), the system of equations (15.53) can be rewritten as

$$\begin{bmatrix} -\mu & 0 \\ 0 & \varepsilon \end{bmatrix}\begin{bmatrix} *\dot{e}_q \\ *\dot{e}_p \end{bmatrix} = \begin{bmatrix} *\mathrm{d}e_p(-1)^{(n-q)\times q} \\ *(-1)^{pq}\mathrm{d}e_q(-1)^{(n-p)\times p} \end{bmatrix} \tag{15.54}$$

Note that

$$-\mathrm{d}\left(\mu * \dot{e}_q\right) = \mathrm{d}(\mathrm{d}e_p(-1)^{(n-q)\times q}) = 0$$
$$\mathrm{d}\left(\mu * \dot{e}_p\right) = \mathrm{d}((-1)^{pq}\mathrm{d}e_q(-1)^{(n-p)\times p}) = 0$$

This means that $\mathrm{d}(\mu * e_q)$, $\mathrm{d}(\varepsilon * e_p)$ are differential forms which do not vary with time.

In terms of *Maxwell's Equations* this would mean $\mathrm{d}(\mu * \mathcal{H})$ is a constant three-form representing the charge density and $\mathrm{d}(\varepsilon * \mathcal{E})$ is actually zero. In standard electromagnetic texts these would mean $\nabla \cdot \mathcal{D} = J$, and $\nabla \cdot \mathcal{B} = 0$, representing respectively the Gauss' electric and magnetic law.

15.6 Conclusions

The main results in this chapter deal with the Brayton Moser formulation of infinite-dimensional systems, starting from the Hamiltonian formulation of infinite-dimensional systems, defined with respect to a Stokes' Dirac structure. This formulation provides a means to generate new passive maps for infinite-dimensional systems, while preserving the pseudogradient-like structure of the Brayton Moser formulation. The preserving of the structure is key for boundary control by interconnection of infinite-dimensional systems.

References

1. R. Abraham, J. Marsden, T. Ratiu, *Manifolds, Tensor Analysis, and Applications*, 2nd edn. (Springer, Berlin, 1988)
2. G. Blankenstein, Geometric modeling of nonlinear RLC circuits. IEEE Trans. Circuits Syst. I: Regul. Pap. **52**(2), 396–404 (2005)
3. G. Blankenstein, Power balancing for a new class of non-linear systems and stabilization of RLC circuits. Int. J. Control **78**(3), 159–171 (2005)

4. R.K. Brayton, W.L. Miranker, A stability theory for nonlinear mixed initial boundary value problems. Arch. Rat. Mech. Anal. **17**(5) (1964)
5. R.K. Brayton, J.K. Moser, A theory of nonlinear networks i. Q. Appl. Math. **22**(1), 1–33 (1964)
6. R.K. Brayton, J.K. Moser, A theory of nonlinear networks ii. Q. Appl. Math. **22**(2), 81–104 (1964)
7. V. Duindam, A. Macchelli, S. Stramigioli, H. Bryuninckx (eds.), *Modeling and Control of Complex Physical Systems: The port-Hamiltonian Approach* (Springer, Berlin, 2009)
8. D. Jeltsema, R. Ortega, J.M.A. Scherpen, On passivity and power-balance inequalities of nonlinear RLC circuits, in *IEEE Transactions on Circuits and Systems-I: Fundamental Theory and Applications*, vol. 50, pp. 1174–1179, September (2003)
9. D. Jeltsema, J.M.A. Scherpen, Multidomain modeling of nonlinear networks and systems. IEEE Control Syst. Mag. (2009)
10. D. Jeltsema, A. van der Schaft, Pseudo-gradient and Lagrangian boundary control system formulation of electromagnetic field. J. Phys. A : Math. Theor. **40**, 1–17 (2007)
11. K.C. Kosaraju, R. Pasumarthy, D. Jeltsema, Alternate Passive Maps and Stability of Infinite-dimensional Systems via Mixed Potential Functions, in *Proceedings of the 5th IFAC Workshop on Lagrangian and Hamiltonian Methods for Non Linear Control*, Lyon, France, 4–7 July (2015)
12. R. Pasumarthy, K.C. Kosaraju, A. Chandrasekar, On Power Balancing and Stabilization for A Class of Infinite-dimensional Systems, in *Proceedings of the Mathematical Theory of Networks and Systems*, Groningen, The Netherlands, July, pp. 7–11 (2014)
13. R. Ortega, D. Jeltsema, J.M.A. Scherpen, Power shaping: a new paradigm for stabilization of nonlinear RLC circuits. IEEE Trans. Autom. Control **48**(10), 1762–1767 (2003)
14. R. Pasumarthy, A. van der Schaft, Achievable Casimirs and its implications on control of port-Hamiltonian systems. Int. J. Control **80**(9), 1421–1438 (2007)
15. A.J. van der Schaft, *L2-Gain and Passivity Techniques in Nonlinear Control* (Springer, London, 2000)
16. A.J. van der Schaft, B.M. Maschke, Hamiltonian formulation of distributed-parameter systems with boundary energy flow. J. Geom. Phys. **42**, 166–194 (2002)