Chapter 14
Online Frequency Estimation of Periodic Signals

Riccardo Marino and Patrizio Tomei

Abstract The problem of estimating online the unknown period of a periodic signal is considered, with no a priori information on the period: this is a crucial problem in the design of learning and synchronizing controls, in fault detection, and for the attenuation of periodic disturbances. Given a measurable continuous, bounded periodic signal, with nonzero first harmonic in its Fourier series expansion, a dynamic algorithm is proposed which provides an online globally exponentially convergent estimate of the unknown period. The period estimate converges from any initial condition to a neighborhood of the true period whose size is explicitly characterized in terms of the higher order harmonics contained in the signal. The accuracy of the frequency estimation can be arbitrarily improved by increasing the order of a prefilter which is incorporated in the estimation algorithm, at the expense of reducing the rate of the exponential convergence. This online frequency estimation algorithm can be used to design hybrid disturbance attenuation controllers for periodic disturbances with unknown period.

14.1 Introduction

Arjan van der Schaft visited the University of Rome Tor Vergata during the summer 1990. We had the pleasure of collaborating with him and Witold Respondek on several aspects of almost disturbance decoupling for nonlinear systems [19] and on more theoretical issues involving transformations of nonlinear systems into prime forms [20]. Our collaboration started in 1986 during a summer visit of the first author at Twente University: at that time, high-gain feedback was investigated to solve almost input-output decoupling and almost disturbance decoupling problems [17, 18]. The

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first author first met Arjan at a conference on Differential Geometric Control Theory in Michigan in 1982 and still remembers with pleasure an adventurous car trip from northern Michigan to St. Louis, Missouri, where he was completing his Ph.D.

This paper is related to the design of feedback controls to attenuate the influence of disturbances. While the strategy in [18, 19] is to reduce the L_2-gain from the disturbance to the output possibly by high-gain feedback since the disturbance is totally unknown, in this paper we explore what can be done if the disturbance is known to be periodic even though its period is unknown. The key step is clearly the online estimation of the unknown period, according to the internal model principle.

Online frequency estimation of a periodic signal is a fundamental problem in several engineering and scientific disciplines. The classical Fourier analysis estimates the frequencies and the amplitudes of a periodic signal provided that the signal can be stored and processed off-line. As far as feedback control is concerned, learning [32] and synchronizing [27] control design and the attenuation of unmeasured periodic disturbances require online frequency estimation. If the period is known, learning controls can track periodic references for classes of linear and nonlinear systems [14, 32]. According to the internal model principle, which was formulated in [6] for linear systems, an error feedback control which is capable of tracking and/or rejecting unknown sinusoidal signals must necessarily be able to reproduce such signals: hence it should estimate their frequencies online. Fault detectors which are based on frequency estimators require online algorithms as well.

Online frequency estimation algorithms can be divided into two classes: the local ones, which converge for sufficiently close initial frequency estimates and the global ones, which converge for any initial frequency estimate. Their convergence may be either exponential or only asymptotic [10] and their domain of attraction may be either global or only local. In addition, the convergence may occur for a suitable tuning of the algorithm parameters or for any parameter value. These differences are apparent in the comparison of several algorithms which are now available for the online frequency estimation of a single sinusoidal signal with no a priori information on the frequency. In [3] a continuous time version of the discrete-time notch filter proposed by [28], which was inspired by commonly used Phase Locked Loop (PLL) algorithms in signal processing, is shown to be locally asymptotically convergent. The adaptation strategy for the frequency estimator presented in [3] was normalized in [9] in order to obtain a globally asymptotically convergent algorithm, provided that the adaptation gain is chosen to be sufficiently small depending on a known bound on the amplitude of the sinusoidal signal. The adaptive notch filter proposed in [9] was extensively analyzed in [4] and was further modified in [24, 25] to show, by means of averaging theory (see for instance [10, 30]) that the frequency estimate will asymptotically converge to a neighborhood of the fundamental frequency even when the signal is periodic but not purely sinusoidal, provided that the adaptation gain and the higher order harmonics are sufficiently small. A similar result was presented in [33] using a different algorithm based on gradient descent methods. Globally exponentially convergent frequency estimation algorithms were obtained both for a single sinusoid and for biased multiple sinusoids in 2002 by [22, 26, 31] without any restriction on the algorithm design parameters. The key techniques are adaptive observers (see [21]) in [22, 26] and adaptive identifiers (see [29]) in [31], while the state space representation of the measured signal allows for a linear parameterization of the unknown frequencies; amplitudes and phases can be recovered as well (see [7, 8] for a detailed analysis) from the estimation of the state variables. Different frequency estimation techniques for sinusoidal signals are still actively studied with the aim of exploring the robustness with respect to unaccounted disturbances (see [1, 2, 5]).

We will follow the adaptive observer approach introduced in [22] in order to address the global frequency estimation of periodic signals. Only local asymptotic frequency estimators for periodic signals have been so far obtained by using an adaptive notch filter with sufficiently small adaptation gain in [24, 25]: the stability analysis has been carried out interpreting the adaptation gain as a small parameter and applying the averaging theorems [10, 30].

In this paper, given a measurable continuous, bounded periodic signal, with nonzero first harmonic in its Fourier series expansion, a dynamic adaptive algorithm is proposed which provides an online globally exponentially convergent estimate of the unknown frequency for any tuning of its parameters, including the adaptation gain. No a priori information on the period is required. The frequency estimate converges from any initial condition to a neighborhood of the true frequency whose size is explicitly characterized in terms of the higher order harmonics contained in the periodic signal. By increasing the order of a prefilter which is incorporated in the estimation algorithm, the accuracy of the frequency estimation can be arbitrarily improved, at the expense of reducing the rate of the exponential convergence. The global stability analysis is carried out using Lyapunov functions and the property of persistency of excitation which lead to a robust exponential convergence of the estimation algorithm. When the periodic signal is a biased sinusoid, the unknown frequency is exactly estimated from any initial condition and for any value of the prefilter order, thus recovering a well-known result with improved robustness. Two examples are carried out and simulated. In the first one, the proposed method is tested on a complex signal and compared to the adaptive notch filter in [25]. In the second one, the frequency estimator is used in conjunction with a disturbance rejection compensator to attenuate a periodic disturbances with unknown frequency. Since the frequency of the disturbance compensator is updated at every predefined time interval, the overall disturbance compensator is of hybrid type. Preliminary results have been presented in [15, 16] for robust compensation of periodic disturbances.

14.2 Main Results

Consider the bounded periodic signal $y(t)$, $y \in \mathbb{R}$, of unknown period T, which is available for measurements. Assume that $y(t)$ is continuous so that it can be represented by its Fourier series expansion

$$y(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left[a_k \cos\left(\frac{2\pi k t}{T}\right) + b_k \sin\left(\frac{2\pi k t}{T}\right)\right]$$
$$\triangleq \frac{a_0}{2} + a_1 \cos\left(\frac{2\pi t}{T}\right) + b_1 \sin\left(\frac{2\pi t}{T}\right) + r_y(t) \tag{14.1}$$

in which $r_y(t)$ contains the higher order harmonics of $y(t)$; the first harmonic of unknown frequency $1/T$ is assumed to be different from zero, i.e., $(a_1^2 + b_1^2) > 0$. Let us consider the signal $y_{fl}(t)$ obtained by filtering $y(t)$ through the stable linear filter of order $l \geq 0$

$$\begin{aligned} \dot{y}_{f1} &= -\lambda_f y_{f1} + \lambda_f y, \ \ y_{f1} \in \mathbb{R} \\ \dot{y}_{fk} &= -\lambda_f y_{fk} + \lambda_f y_{f,k-1}, \ \ y_{fk} \in \mathbb{R}, \ 2 \leq k \leq l \end{aligned} \tag{14.2}$$

in which λ_f is an arbitrary positive real. Let y_{fp} be the steady-state periodic component of $y_{fl}(t)$ in (14.2) which is obtained as the solution of

$$\begin{aligned} \dot{y}_{fp1} &= -\lambda_f y_{fp1} + \lambda_f y, \ \ y_{fp1} \in \mathbb{R} \\ \dot{y}_{fpk} &= -\lambda_f y_{fpk} + \lambda_f y_{fp,k-1}, \ \ y_{fpk} \in \mathbb{R}, \ 2 \leq k \leq l \\ y_{fp} &= y_{fpl} \end{aligned} \tag{14.3}$$

with suitable initial conditions. The signal y_{fp} may be rewritten as

$$y_{fp}(t) = \eta_1(t) + r(t) \tag{14.4}$$

where $\eta_1(t)$ is the sum of the bias (if any) and of the first harmonic component of frequency ω (which is different from zero by assumption) while $r(t)$ contains all remaining harmonics at higher frequencies $k\omega$, $k \geq 2$. The differences

$$\begin{aligned} \tilde{y}_{fi} &= y_{fi} - y_{fpi}, \ \ 1 \leq i \leq l \\ \tilde{y}_f &= \tilde{y}_{fl} \end{aligned} \tag{14.5}$$

converge exponentially to zero. The signal $\eta_1(t)$ may be equivalently generated by the exogenous system (exosystem)

$$\begin{aligned} \dot{\eta}_1 &= \eta_2 \\ \dot{\eta}_2 &= -\theta\eta_1 + \eta_3 \\ \dot{\eta}_3 &= 0 \end{aligned} \tag{14.6}$$

with suitable initial conditions, in which the parameter $\theta = (2\pi/T)^2 = \omega^2$ is defined and $\eta = [\eta_1, \eta_2, \eta_3]^T \in \mathbb{R}^3$. Let us introduce the unitary gain first-order stable filter

$$\dot{\chi} = -\lambda\chi + \lambda y_{fl}, \ \ \chi \in \mathbb{R} \tag{14.7}$$

in which λ is an arbitrary positive real and y_{fl} is generated by (14.2). From (14.4), (14.6), (14.2) and (14.7), in the new state coordinates $\eta_E = [\chi, \lambda\eta^T]^T \in \mathbb{R}^4$, we have

$$\begin{aligned}\dot{\eta}_E &= A_c\eta_E - E_1\lambda\chi + r\lambda(E_1 + \theta E_3) \\ &\quad +\lambda E_1\tilde{y}_f - \lambda y_{fp}\theta E_3 \\ \chi &= C_c\eta_E \end{aligned} \tag{14.8}$$

in which E_i denotes the ith column of an identity matrix of suitable dimension and

$$A_c = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Note that the unknown parameter $\theta = \omega^2$ now appears linearly in the dynamic equations (14.8). Make the time-varying change of coordinates (which is called 'filtered transformation' in [21])

$$\begin{aligned} z &= \eta_E - \begin{bmatrix} 0 \\ \theta\xi(t) \end{bmatrix} \\ \dot{\xi} &= D\xi - \lambda E_2 y_{fl},\ \xi \in \mathbb{R}^3 \\ \mu &= C_c\xi,\ \mu \in \mathbb{R} \end{aligned} \tag{14.9}$$

in which

$$D = \begin{bmatrix} -d_2 & 1 & 0 \\ -d_3 & 0 & 1 \\ -d_4 & 0 & 0 \end{bmatrix} \tag{14.10}$$

is an arbitrary Hurwitz matrix, and

$$d_A = \begin{bmatrix} d_3 - d_2^2 + d_2\lambda \\ d_4 - d_2 d_3 + d_3\lambda \\ -d_2 d_4 + d_4\lambda \end{bmatrix}, \quad C_c = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}.$$

From (14.8) and (14.9), we obtain

$$\begin{aligned}\dot{z} &= A_c z - E_1\lambda\chi + r\lambda(E_1 + \theta E_3) \\ &\quad +d\mu\theta + \lambda(E_1 + \theta E_3)\tilde{y}_f \\ \chi &= C_c z. \end{aligned} \tag{14.11}$$

The further change of coordinates

$$w_i = z_{i+1} - d_{i+1} z_1, \quad 1 \le i \le 3 \tag{14.12}$$

transforms (14.11) into ($w = [w_1, w_2, w_3]^T$)

$$\begin{aligned}\dot{w} &= Dw + d_A \chi + d_B r + r\lambda\theta E_2 + (d_B + \lambda\theta E_2)\tilde{y}_f \\ \dot{\chi} &= w_1 + (d_2 - \lambda)\chi + r\lambda + \theta\mu + \lambda\tilde{y}_f\end{aligned} \tag{14.13}$$

in which $d_B = [d_2, d_3, d_4]^T$. Note that in (14.13) the unknown parameter θ appears in the dynamics of the known signal χ multiplied by the known signal μ. The parameter θ also appears in the w-dynamics where it is multiplied by the exponentially decaying term $\tilde{y}_f$ and by $r(t)$, which is viewed as a disturbance. Let us introduce the adaptive observer for (w, χ, θ) in (14.13)

$$\begin{aligned}\dot{\hat{w}} &= D\hat{w} + d_A\chi, \ \hat{w} \in \mathbb{R}^3 \\ \dot{\hat{\chi}} &= C_c\hat{w} + (d_2 - \lambda)\chi + \hat{\theta}\mu + k_o(\chi - \hat{\chi}), \ \hat{\chi} \in \mathbb{R} \\ \dot{\hat{\theta}} &= \gamma\mu(\chi - \hat{\chi}), \ \theta \in \mathbb{R}\end{aligned} \tag{14.14}$$

in which γ is the positive adaptation gain and k_o is the positive observer gain. The dynamics for the estimate $\hat{\theta}$ of the parameter $\theta = \omega^2$ is defined (see Fig. 14.1) in terms of the signal μ generated by the linear filters (14.2) and (14.9) and of the error $\chi - \hat{\chi}$ generated by (14.2), (14.7) and (14.14). Defining the error signals $\tilde{\chi} = \chi - \hat{\chi}$, $\tilde{w} = w - \hat{w}$, $\tilde{\theta} = \theta - \hat{\theta}$, from (14.5), (14.6), (14.9), (14.13) and (14.14), we obtain the error dynamics

$$\begin{aligned}\dot{\tilde{w}} &= D\tilde{w} + \bar{d}_B r + \bar{d}_B\tilde{y}_f \\ \dot{\tilde{\chi}} &= -k_o\tilde{\chi} + r\lambda + \tilde{w}_1 + \tilde{\theta}\mu + \lambda\tilde{y}_f \\ \dot{\tilde{\theta}} &= -\gamma\mu\tilde{\chi} \\ \dot{\xi} &= D\xi - \lambda E_2\tilde{y}_f - \lambda E_2 y_{fp} = D\xi - \lambda E_2 y_{fl} \\ \mu &= C_c\xi\end{aligned} \tag{14.15}$$

in which $\bar{d}_B = d_B + \lambda\theta E_2$. Since (14.9) is a linear dynamic system driven by y_{fl}, the signal μ in (14.9) may be decomposed as

$$\mu = \mu_p + \tilde{\mu} \tag{14.16}$$

in which μ_p is the periodic output of the system

$$\begin{aligned}\dot{\xi}_p &= D\xi_p - \lambda E_2 y_{fp} \\ \mu_p &= C_c\xi_p\end{aligned} \tag{14.17}$$

with proper initial condition $\xi_p(0)$. Now, we are able to state and prove the following theorem which characterizes the convergence properties of the estimation error $\tilde{\theta}$.

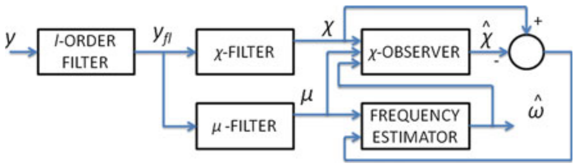


Fig. 14.1 Block diagram for the frequency estimator (14.18)

Theorem 14.1 *Let $y(t)$ be a measurable continuous, bounded periodic signal of unknown period T, with $a_1^2 + b_1^2 > 0$ in its Fourier series expansion (14.1). The following online frequency estimator of order $l + 9$ in which $\hat{\omega}$ denotes the estimate of $\omega = 2\pi/T$ (see the block diagram in Fig. 14.1):*

$$\begin{aligned}
\dot{y}_{f1} &= -\lambda_f y_{f1} + \lambda_f y, \;\; y_{f1} \in \mathbb{R} \\
\dot{y}_{fk} &= -\lambda_f y_{fk} + \lambda_f y_{f,k-1}, \;\; y_{fk} \in \mathbb{R}, \; 2 \le k \le l \\
\dot{\chi} &= -\lambda \chi + \lambda y_{fl}, \;\; \chi \in \mathbb{R} \\
\dot{\xi} &= D\xi - \lambda E_2 y_{fl}, \;\; \xi \in \mathbb{R}^3 \\
\mu &= C_c \xi, \;\; \mu \in \mathbb{R} \\
\dot{\hat{w}} &= D\hat{w} + d_A \chi, \;\; \hat{w} \in \mathbb{R}^3 \\
\dot{\hat{\chi}} &= C_c \hat{w} + (d_2 - \lambda)\chi + \hat{\theta}\mu + k_o(\chi - \hat{\chi}), \;\; \hat{\chi} \in \mathbb{R} \\
\dot{\hat{\theta}} &= \gamma \mu (\chi - \hat{\chi}), \;\; \theta \in \mathbb{R} \\
\hat{\omega} &= \begin{cases} \sqrt{\hat{\theta}} & \text{if } \hat{\theta} > 0 \\ 0 & \text{otherwise} \end{cases}
\end{aligned} \tag{14.18}$$

is such that for any initial condition $y_{f1}(0), \ldots, y_{fl}(0), \chi(0), \xi(0), \hat{w}(0), \hat{\chi}(0), \hat{\theta}(0)$, for any integer $l \ge 0$, for any $\lambda_f > 0$, $\lambda > 0$, $k_o > 3$, $\gamma > 0$ and for any Hurwitz matrix D:

(i) all signals are bounded for any $t \ge 0$;
(ii)

$$|\tilde{\theta}(t)| \le f(\|\tilde{x}(0)\|)e^{-\beta_1 t} + \beta_2 \left[\frac{1}{T}\int_0^T r_y^2(\tau)d\tau\right]^{1/2}, \; \forall t \ge 0$$

in which f is a class-k function [10] of $\tilde{x} = [\tilde{w}^T, \tilde{\chi}, \tilde{\theta}, \xi^T - \xi_p^T, \tilde{y}_{f1}, \ldots, \tilde{y}_{fl}]^T$ and β_1, β_2 are positive reals which tend to zero as l tends to infinity with

$$\beta_1 = O\left[\left(\frac{\lambda_f^2}{\lambda_f^2 + \omega^2}\right)^l\right]$$

$$\beta_2 = O\left[\left(\frac{\lambda_f^2 + \omega^2}{\lambda_f^2 + 4\omega^2}\right)^{l/2}\right].$$

Proof The signal $\mu_p(t)$ in (14.17) is unbiased since the transfer function of the linear system in (14.17) has a zero in the origin, while the signal $\tilde{\mu}(t)$ is exponentially decaying and given by ($\tilde{\xi} = \xi - \xi_p$)

$$\begin{aligned}\dot{\tilde{\xi}} &= D\tilde{\xi} - \lambda E_2 \tilde{y}_f, \ \tilde{\xi}(0) = \xi(0) - \xi_p(0) \\ \tilde{\mu} &= C_c \tilde{\xi}.\end{aligned} \tag{14.19}$$

Note that (see [12], p. 494 and Abel's Lemma [11])

$$\begin{aligned}\sup_{\tau\in[0,T]}|\mu_p(\tau)| &\le c_1 \frac{\lambda_f^l}{(\lambda_f^2+\omega^2)^{l/2}} y_M \\ \sup_{\tau\in[0,T]}|\dot{\mu}_p(\tau)| &\le c_2 \frac{\lambda_f^l}{(\lambda_f^2+\omega^2)^{l/2}} y_M\end{aligned} \tag{14.20}$$

where

$$y_M = \sup_{\tau\in[0,T]}|y(\tau)|$$

and c_1, c_2 are positive constants independent on the filter parameters l and λ_f. The signal $\mu_p(t)$ may, in turn, be decomposed as

$$\mu_p(t) = \mu_{p1}(t) + \mu_{pr}(t) \tag{14.21}$$

where μ_{p1} is the first unbiased harmonic at frequency ω and μ_{pr} contains all other higher order harmonics. We can write

$$\begin{aligned}\mu_{p1}(t) = |H_1(j\omega)| &\frac{\lambda_f^l}{(\lambda_f^2+\omega^2)^{l/2}} \\ &\cdots (a_1^2+b_1^2)^{1/2}\cos(\omega t+\varphi_0)\end{aligned} \tag{14.22}$$

in which $H_1(s) = C_c(sI - D)^{-1}E_2\lambda$ and φ_0 is a suitable angle. From (14.21) and (14.22), we have

$$\begin{aligned}&\int_t^{t+T} \mu_p^2(\tau)d\tau = \int_0^T [\mu_{p1}^2(\tau) + \mu_{pr}^2(\tau)]d\tau \\ &\ge \int_0^T \mu_{p1}^2(\tau)d\tau = |H_1(j\omega)|^2 \\ &\times (a_1^2+b_1^2)\frac{T}{2}\frac{\lambda_f^{2l}}{(\lambda_f^2+\omega^2)^l}.\end{aligned} \tag{14.23}$$

Define

$$\bar{\mu}_p = \mu_p/\alpha,\ \alpha = \frac{\lambda_f^l}{(\lambda_f^2 + \omega^2)^{l/2}}. \tag{14.24}$$

From (14.20), we have

$$\begin{aligned}\sup_{\tau\in[0,T]}|\bar{\mu}_p(\tau)| &\le c_1 y_M \triangleq \bar{\mu}_{pM}\\ \sup_{\tau\in[0,T]}|\dot{\bar{\mu}}_p(\tau)| &\le c_2 y_M \triangleq \dot{\bar{\mu}}_{pM}.\end{aligned} \tag{14.25}$$

Since by assumption $a_1^2 + b_1^2 > 0$, from (14.23) and (14.24) we have

$$\begin{aligned}\int_t^{t+T} \bar{\mu}_p^2(\tau)d\tau \ &\ge |H_1(j\omega)|^2(a_1^2 + b_1^2)\frac{T}{2}\\ &\triangleq k_p > 0,\ \forall t \ge 0.\end{aligned} \tag{14.26}$$

Define $q(t)$ as the solution of the scalar differential equation

$$\dot{q} = -q + \bar{\mu}_p^2, \quad q(0) = e^{-T}k_p \tag{14.27}$$

so that, by construction,

$$\sup_{\tau\in[0,T]}\bar{\mu}_p^2(\tau) \ge q(t) \ge k_p e^{-2T}, \quad \forall t \ge 0. \tag{14.28}$$

With reference to the first three equations in the error system (14.15), consider the Lyapunov function

$$V = \frac{1}{2}\left[\tilde{\chi}^2 + \frac{\tilde{\theta}^2}{\gamma} + \tilde{w}^T P\tilde{w} + \gamma_0(\alpha q\tilde{\theta} - \bar{\mu}_p\tilde{\chi})^2\right] \tag{14.29}$$

where γ_0, γ are positive reals and $P > 0$ satisfies the Lyapunov matrix equation $D^T P + PD = -2I$, in which D given by (14.10) is a Hurwitz matrix. From (14.29) and (14.15), differentiating V with respect to time along the solutions of (14.15), we obtain

$$\begin{aligned}\dot{V} = &-k_o\tilde{\chi}^2 + r\lambda\tilde{\chi} + \tilde{w}_1\tilde{\chi} + \tilde{\chi}\tilde{\theta}\mu + \lambda\tilde{y}_f\tilde{\chi}\\ &-\tilde{\theta}\mu\tilde{\chi} - \tilde{w}^T\tilde{w} + \tilde{w}^T P\bar{d}_B r + \tilde{w}^T P\bar{d}_B\tilde{y}_f\\ &+\gamma_0(\alpha q\tilde{\theta} - \bar{\mu}_p\tilde{\chi})[-\alpha q\tilde{\theta} + \alpha\tilde{\theta}\bar{\mu}_p^2 - \alpha^2 q\gamma\bar{\mu}_p\tilde{\chi}\\ &-\alpha q\gamma\tilde{\mu}\tilde{\chi} - \dot{\bar{\mu}}_p\tilde{\chi} - \bar{\mu}_p(-k_o\tilde{\chi} + r\lambda + \tilde{w}_1\\ &+\alpha\tilde{\theta}\bar{\mu}_p + \tilde{\theta}\tilde{\mu} + \lambda\tilde{y}_f)]\\ = &-k_o\tilde{\chi}^2 - \tilde{w}^T\tilde{w} - \gamma_0\alpha^2 q^2\tilde{\theta}^2\end{aligned}$$

$$
\begin{aligned}
&+\tilde{w}_1\tilde{\chi} + \gamma_0\alpha q\tilde{\theta}(-\alpha^2 q\gamma\bar{\mu}_p - \dot{\bar{\mu}}_p + \bar{\mu}_p k_o)\tilde{\chi}\\
&+\gamma_0\tilde{\chi}^2(\alpha^2 q\bar{\mu}_p^2\gamma + \bar{\mu}_p\dot{\bar{\mu}}_p - \bar{\mu}_p^2 k_o) - \gamma_0\alpha q\bar{\mu}_p\tilde{\theta}\tilde{w}_1\\
&+\gamma_0\bar{\mu}_p^2\tilde{\chi}\tilde{w}_1 + \gamma_0(\alpha q\tilde{\theta} - \bar{\mu}_p\tilde{\chi})(-\alpha q\gamma\tilde{\chi} - \bar{\mu}_p\tilde{\theta})\tilde{\mu}\\
&+[\gamma_0(\alpha q\tilde{\theta} - \bar{\mu}_p\tilde{\chi})(-\bar{\mu}_p\lambda) + \lambda\tilde{\chi} + \tilde{w}^T P\bar{d}_B]\tilde{y}_f\\
&+r[\lambda\tilde{\chi} + \tilde{w}^T P\bar{d}_B - \gamma_0(\alpha q\tilde{\theta} - \bar{\mu}_p\tilde{\chi})\bar{\mu}_p\lambda].
\end{aligned}
\tag{14.30}
$$

By using Young's inequality ($2ab \le a^2/k^2 + k^2b^2$), we can write

$$
\dot{V} \ \le -\phi^T Q(t)\phi + \|\phi\|^2\rho_1(t) + \rho_2(t) + r^2\rho_3(t) \tag{14.31}
$$

in which

$$
\begin{aligned}
\phi &= \left[\,|\tilde{\chi}|\ |\tilde{\theta}|\ \|\tilde{w}\|\,\right]^T\\
\rho_1 &= \gamma_0\left\|\begin{bmatrix} \alpha q\gamma|\bar{\mu}_p| & (\bar{\mu}_p^2 + \alpha^2 q^2\gamma)/2\\ (\bar{\mu}_p^2 + \alpha^2 q^2\gamma)/2 & |\bar{\mu}_p|\alpha q \end{bmatrix}\right\| |\tilde{\mu}|\\
\rho_2 &= [\alpha^2 q^2\lambda^2\bar{\mu}_p^2 + (\gamma_0\bar{\mu}_p\lambda + \lambda)^2 + 4\|P\bar{d}_B\|^2]\tilde{y}_f^2\\
\rho_3 &= \frac{1}{2}(\lambda^2 + \bar{\mu}_p^4\lambda^2 + q^2\bar{\mu}_p^2\lambda^2 + 2\|P\bar{d}_B\|^2)
\end{aligned}
\tag{14.32}
$$

and $Q(t)$ is a (3×3) symmetric matrix whose elements q_{ij} are given by

$$
\begin{aligned}
q_{11} &= k_o(1 + \gamma_0\bar{\mu}_p^2) - \gamma_0\alpha^2\gamma q\bar{\mu}_p^2 - \gamma_0|\bar{\mu}_p\dot{\bar{\mu}}_p| - \frac{1}{2} - \frac{\gamma_0^2}{2} - \frac{1}{4}\\
q_{22} &= \gamma_0\alpha^2 q^2 - \frac{\gamma_0^2\alpha^2}{2} - \frac{\gamma_0^2}{4}\\
q_{33} &= 1 - \frac{1}{4} - \frac{1}{16}\\
q_{12} &= -\frac{1}{2}\gamma_0\alpha q(\alpha^2 q\gamma|\bar{\mu}_p| + |\dot{\bar{\mu}}_p| + k_o|\bar{\mu}_p|)\\
q_{13} &= -\frac{1}{2}(1 + \gamma_0\bar{\mu}_p^2)\\
q_{23} &= -\frac{1}{2}\gamma_0\alpha q|\bar{\mu}_p|.
\end{aligned}
$$

By using again Young's inequality, we can write

$$
\inf\nolimits_{\tau\in[0,T]}\lambda_{\min}[Q(\tau)] \ge \min_{1\le k\le 3}\bar{q}_{kk} \triangleq Q_m \tag{14.33}
$$

with

$$\bar{q}_{11} = k_o - \frac{5}{4} - \gamma_0\left(\alpha^2\gamma\bar{\mu}_{pM}^4 + \bar{\mu}_{pM}\dot{\bar{\mu}}_{pM} + \frac{\gamma_0}{2}\right)$$
$$\bar{q}_{22} = \gamma_0\alpha^2\{e^{-2T}k_p^2 - \gamma_0[\frac{1}{2} + \bar{\mu}_{pM}^4(\alpha^2\bar{\mu}_{pM}^3\gamma + \dot{\bar{\mu}}_{pM} + k_o\bar{\mu}_{pM})^2 + \bar{\mu}_{pM}^6]\} - \frac{\gamma_0^2}{4}$$
$$\bar{q}_{33} = \frac{3}{16} - \gamma_0^2\bar{\mu}_{pM}^4.$$

Since by definition (14.24) $0 < \alpha < 1$, by choosing

$$k_o \geq 3$$
$$\gamma_0 \leq \min\left\{1, \frac{\sqrt{3}}{4\bar{\mu}_{pM}^2}, c_1, c_2\right\}$$

with

$$c_1 = \frac{e^{-2T}k_p^2}{0.5 + \bar{\mu}_{pM}^4(\alpha^2\bar{\mu}_{pM}^3\gamma + \dot{\bar{\mu}}_{pM} + k_{ob}\bar{\mu}_{pM})^2 + \bar{\mu}_{pM}^6 + 0.25\alpha^{-2}}$$
$$c_2 = \frac{7}{4[\alpha^2\gamma\bar{\mu}_{pM}^4 + \bar{\mu}_{pM}\dot{\bar{\mu}}_{pM} + 0.5]}$$

it follows that $Q(t)$ is positive definite with

$$Q_m = O(\alpha^2). \tag{14.34}$$

Now, note that we can write for $V(t)$

$$V \leq \frac{1}{2}\phi^T \begin{bmatrix} 1 + 2\gamma_0\bar{\mu}_p^2 & 0 & 0 \\ 0 & \frac{1}{\gamma} + \gamma_0\alpha^2 q^2 & 0 \\ 0 & 0 & \|P\| \end{bmatrix} \phi$$
$$\leq \frac{1}{2}\|\phi\|^2 c_{VM} \tag{14.35}$$

with (recall that $0 < \alpha < 1$)

$$c_{VM} = \max\left\{1 + 2\gamma_0\bar{\mu}_{pM}^2, \frac{1}{\gamma} + \gamma_0\bar{\mu}_{pM}^2, \|P\|\right\}. \tag{14.36}$$

Since, from (14.29)

$$V \geq \frac{1}{2} c_{Vm} \|\phi\|^2 \tag{14.37}$$

in which $c_{Vm} = \min\left\{1, \frac{1}{\gamma}, \lambda_{\min}(P)\right\}$, from (14.31) and (14.35) we can write

$$\begin{aligned} \dot{V} &\leq -2\frac{Q_m}{c_{VM}} V + 2c_{Vm} V \rho_1(t) + \rho_2(t) + r^2 \rho_3(t) \\ &\triangleq -cV + \bar{\rho}_1(t) V + \rho_2(t) + r^2 \bar{\rho}_3 \end{aligned} \tag{14.38}$$

where

$$\begin{aligned} c &= 2\frac{Q_m}{c_{VM}} \\ \bar{\rho}_1(t) &= 2c_{Vm} \rho_1(t) \\ \bar{\rho}_3 &= \frac{1}{2}(\lambda^2 + \bar{\mu}_{pM}^6 \lambda^2 + \bar{\mu}_{pM}^4 \lambda^2 + 2\|P\bar{d}_B\|^2). \end{aligned} \tag{14.39}$$

Recalling (14.2), (14.3), (14.5), (14.19), (14.32) and (14.39), we can write for any $t \geq 0$

$$\begin{aligned} \rho_2(t) &\leq \rho_{20}(\|[\tilde{y}_{f1}(0), \ldots, \tilde{y}_{fl}(0)]\|) e^{-2\lambda_f t} \\ \bar{\rho}_1(t) &\leq \bar{\rho}_{10}(\|[\tilde{y}_{f1}(0), \ldots, \tilde{y}_{fl}(0), \tilde{\xi}^T(0)]\|) e^{-\lambda_m t} \end{aligned} \tag{14.40}$$

in which ρ_{20}, $\bar{\rho}_{10}$ are class-k functions and

$$\lambda_m = \min_{1 \leq i \leq 3} \{-\mathrm{Re}[\lambda_i(D)]\}$$

with λ_i being the ith eigenvalue of matrix D. By applying the comparison principle and the variation of constants formula (see [13, 23]), from (14.38) we can write (recall that $\bar{\rho}_1$ is exponentially decaying)

$$\begin{aligned} V(t) \leq e^{\|\bar{\rho}_1\|_1} \Bigg[& V(0)e^{-ct} + \int_0^t e^{-c(t-\tau)} \rho_2(\tau) d\tau \\ & + \bar{\rho}_3 \int_0^t e^{-c(t-\tau)} r^2(\tau) d\tau \Bigg] \\ \leq e^{\|\bar{\rho}_1\|_1} \Bigg[& V(0)e^{-ct} + \int_0^t e^{-c(t-\tau)} \rho_2(\tau) d\tau \\ & + \bar{\rho}_3 \sum_{k=0}^{N} e^{-ckT} \int_0^T r^2(\tau) d\tau \Bigg] \end{aligned} \tag{14.41}$$

with $\|\bar{\rho}_1\|_1 = \int_0^\infty |\bar{\rho}_1(\tau)|d\tau$ and N such that $0 \le t - NT < T$ and $\|\bar{\rho}_1\|_1 = \int_0^\infty \bar{\rho}_1(\tau)d\tau$. Since $\rho_2(t)$ is exponentially decaying, from (14.41) we can conclude that all signals are bounded. Recalling (14.40) and (14.29), from (14.41) we have

$$\begin{aligned}\tilde{\theta}^2(t) \le 2\gamma e^{\|\bar{\rho}_1\|_1}\left[V(0)e^{-ct} + \frac{\rho_{20}}{c}e^{-2\lambda_f t}\right]\\ +2\gamma e^{\|\bar{\rho}_1\|_1}\bar{\rho}_3\frac{1}{1-e^{-cT}}\int_0^T r^2(\tau)d\tau.\end{aligned} \tag{14.42}$$

Now, note that by defining

$$\begin{aligned}\beta &= \begin{bmatrix} a_2 \\ b_2 \\ \vdots \\ a_k \\ b_k \\ \vdots \end{bmatrix}^T \quad \Phi(t) = \begin{bmatrix} \cos(2\omega t) \\ \sin(2\omega t) \\ \vdots \\ \cos(k\omega t) \\ \sin(k\omega t) \\ \vdots \end{bmatrix}^T\\ R &= \text{block diag}\left[\, R_2 \cdots R_k \cdots \right]\\ R_k &= M_k \begin{bmatrix} \cos\psi_k & -\sin\psi_k \\ \sin\psi_k & \cos\psi_k \end{bmatrix}\\ M_k &= \frac{\lambda_f^l}{(\lambda_f^2 + k^2\omega^2)^{l/2}}, \quad \psi_k = l\arctan\frac{-k\omega}{\lambda_f}\end{aligned}$$

we can write for $r_y(t)$ in (14.1) and $r(t)$ in (14.4),

$$\begin{aligned}r_y(t) &= \Phi^T(t)\beta\\ r(t) &= \Phi^T(t)R\beta.\end{aligned} \tag{14.43}$$

Since

$$\|R\| = (\lambda_{MAX}(R^T R))^{1/2} = \frac{\lambda_f^l}{(\lambda_f^2 + 4\omega^2)^{l/2}}$$

and, by Parseval Theorem,

$$\begin{aligned}\frac{1}{T}\int_0^T r^2(\tau)d\tau &= \frac{1}{2}\beta^T R^T R\beta \le \frac{1}{2}\frac{\lambda_f^{2l}}{(\lambda_f^2 + 4\omega^2)^l}\beta^T\beta\\ &= \frac{\lambda_f^{2l}}{(\lambda_f^2 + 4\omega^2)^l}\frac{1}{T}\int_0^T r_y^2(\tau)d\tau\end{aligned} \tag{14.44}$$

from (14.44) and (14.42), we obtain statement (ii) with

$$
\begin{aligned}
f(\|x(0)\|) &= \left\{2\gamma e^{\|\bar{\rho}_1\|_1}\left[V(0)+\frac{\rho_{20}}{c}\right]\right\}^{1/2} \\
\beta_1 &= \min\left\{\frac{c}{2},\lambda_f\right\} \\
\beta_2 &= \left[2\gamma\bar{\rho}_3 e^{\|\bar{\rho}_1\|_1}\frac{1}{1-e^{-cT}}\right]^{1/2}\frac{\lambda_f^l}{(\lambda_f^2+4\omega^2)^{l/2}}.
\end{aligned}
\tag{14.45}
$$

Since by (14.39) and (14.34), c is $O(\alpha^2)$, for sufficiently small α (and, consequently, for sufficiently high order l) we can write

$$
\frac{1}{1-e^{-cT}} \simeq \frac{1}{cT} \tag{14.46}
$$

which implies that

$$
\beta_2 = O\left[\left(\frac{\lambda_f^2+\omega^2}{\lambda_f^2+4\omega^2}\right)^{l/2}\right] \tag{14.47}
$$

and

$$
\beta_1 = O\left[\left(\frac{\lambda_f^2}{\lambda_f^2+\omega^2}\right)^{l}\right].
$$

The case $l = 0$ can be simply treated by considering $y(t)$ in place of $y_{fl}(t)$ and adjusting, accordingly, the various steps of the proof. □

Corollary 14.2 *If $y(t)$ is a biased sinusoidal signal with no higher order harmonics, then the estimate $\hat{\omega}(t)$ provided by the frequency estimator (14.18) in Theorem 14.1 is such that, for any integer $l \geq 0$, $|\tilde{\theta}(t)| \leq f(\|x(0)\|)e^{-\beta_1 t}$, $\forall t \geq 0$, in which f is a class-k function.*

Proof It follows directly from statement (ii) in Theorem 14.1, since $r_y(t) = 0$ in (14.1). □

Remark 14.3 If in Theorem 14.1, the hypothesis $a_1^2 + b_1^2 > 0$ is not satisfied but the signal $y(t)$ is not constant, then the algorithm (14.18) guarantees properties similar to (i) and (ii) for the first nonzero harmonic in the signal $y(t)$.

Remark 14.4 The frequency estimator (14.18) may be compared to the adaptive notch filter proposed in [25] in the special case in which $a_0 = 0$ in (14.1):

$$
\begin{aligned}
\dot{x}_1 &= x_2 \\
\dot{x}_2 &= -\hat{\omega}^2 x_1 - 2\zeta\hat{\omega}x_2 + 2\zeta\hat{\omega}^2 y \\
\dot{\hat{\omega}} &= -\gamma x_1(\hat{\omega}^2 y - \hat{\omega}x_2).
\end{aligned} \tag{14.48}
$$

They are both adaptive linear filters whose input is the periodic signal $y(t)$ and whose output is the estimate $\hat{\omega}$: while (14.18) is a $(l+9)$-order adaptive linear filter in which $\hat{\omega}^2$ is the adapted filter parameter, the algorithm (14.48) is a third-order filter in which $\hat{\omega}$ is the adapted filter parameter. They both guarantee the convergence of the estimate $\hat{\omega}$ into a neighborhood of the true value $\omega = 2\pi/T$: while (14.48) guarantees an asymptotic convergence for sufficiently small initial errors, higher order harmonics and adaptation gain γ, the algorithm (14.18) guarantees exponential convergence for any initial condition and any parameters choice. For both algorithms $\hat{\omega}$ converges to the true value ω if there are no higher order harmonics in (14.1).

Remark 14.5 From (14.4) and (14.6), an estimate of the amplitude and phase of the first biased harmonic term can also be obtained. If we write $\eta_1(t)$ as

$$\eta_1(t) = \theta_1 \sin(\omega t) + \theta_2 \cos \omega t + \theta_3$$

the parameters θ_i may be estimated using the gradient method (see [29]) as

$$
\begin{aligned}
\begin{bmatrix} \dot{\hat{\theta}}_1 \\ \dot{\hat{\theta}}_2 \\ \dot{\hat{\theta}}_3 \end{bmatrix} &= \gamma_1(\eta_1 - \eta_I) \begin{bmatrix} \sin \omega t \\ \cos \omega t \\ 1 \end{bmatrix} \\
\eta_I &= \hat{\theta}_1 \sin(\omega t) + \hat{\theta}_2 \cos(\omega t) + \hat{\theta}_3.
\end{aligned} \tag{14.49}
$$

in which $\gamma_1 > 0$. Since, however, η_1 and ω are not known, their estimates provided by the frequency estimator (14.18) are used, so that in place of (14.49) we use

$$
\begin{aligned}
\begin{bmatrix} \dot{\hat{\theta}}_1 \\ \dot{\hat{\theta}}_2 \\ \dot{\hat{\theta}}_3 \end{bmatrix} &= \gamma_1(\hat{\eta}_1 - \hat{\eta}_I) \begin{bmatrix} \sin \hat{\omega} t \\ \cos \hat{\omega} t \\ 1 \end{bmatrix} \\
\hat{\eta}_I &= \hat{\theta}_1 \sin(\hat{\omega} t) + \hat{\theta}_2 \cos(\hat{\omega} t) + \hat{\theta}_3.
\end{aligned}
$$

The recursive least square method could be also used [29].

14.3 Examples

As a first example, we consider the problem of estimating the period of the periodic signal $y(t)$ of frequency $\omega = 3$ given by (see Fig. 14.2)

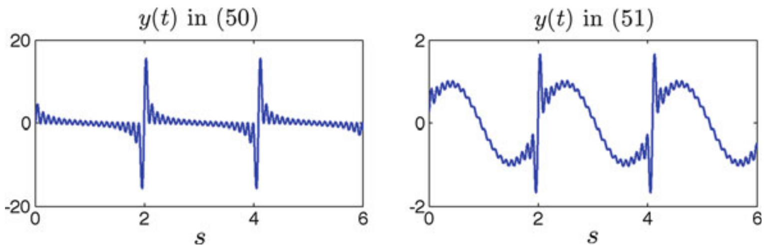


Fig. 14.2 Periodic signals: *left curve*, $y(t)$ in (14.50); *right curve*, $y(t)$ in (14.51)

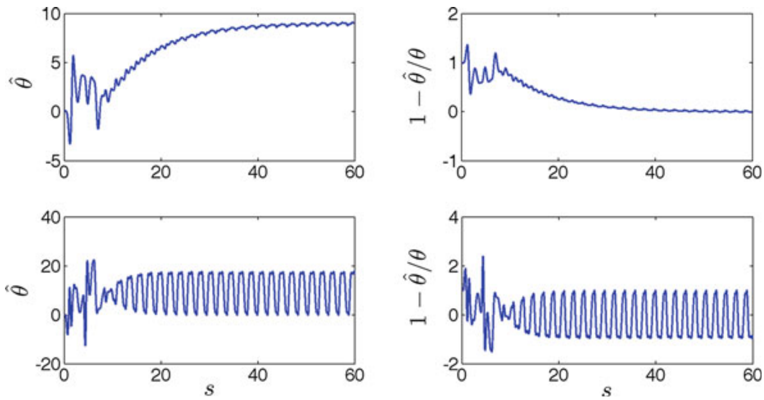


Fig. 14.3 Frequency estimator (14.18): *upper curves*, $l = 2$; *lower curves*, $l = 1$

$$y(t) = \sum_{k=1}^{21} \sin(3kt + 0.3k) \tag{14.50}$$

by means of the frequency estimator (14.18). The results are illustrated in Fig. 14.3 in which the following time histories are reported: the value of $\hat{\theta}(t)$ as obtained by the algorithm (14.18), the relative error $1 - \hat{\theta}(t)/\theta$ between the true and the estimated square of the frequency. The following parameters and initial conditions have been adopted: $\gamma = 30000$, $d_2 = 4$, $d_3 = 5$, $d_4 = 2$, $\lambda_f = 1$, $\lambda = 1$, $k_o = 1$, $\hat{\theta}(0) = 0.1$ and all other initial conditions set to zero. The upper curves refer to the case in which a second-order filter is adopted ($l = 2$) while the lower curves report the results obtained with $l = 1$. Figure 14.3 shows that in the case $l = 1$ the rate of convergence is increased while the accuracy is worse with respect to the case $l = 2$. The previous results may be compared to those obtained by the adaptive notch filter (14.48) illustrated in Remark 14.4 which are reported in Fig. 14.4. As suggested by the authors in [25], the parameters used in the algorithm (14.48) are: $\gamma = 0.1$, $\zeta = 0.35$ while the initial value for $\hat{\omega}$ was $\hat{\omega}(0) = 2.8$ (10 % less than the true value) and null initial conditions. The upper curves in Fig. 14.4 report the time histories of the square of the frequency estimate $\hat{\omega}^2(t)$ and of the relative error $[\omega^2 - \hat{\omega}^2(t)]/\omega^2$

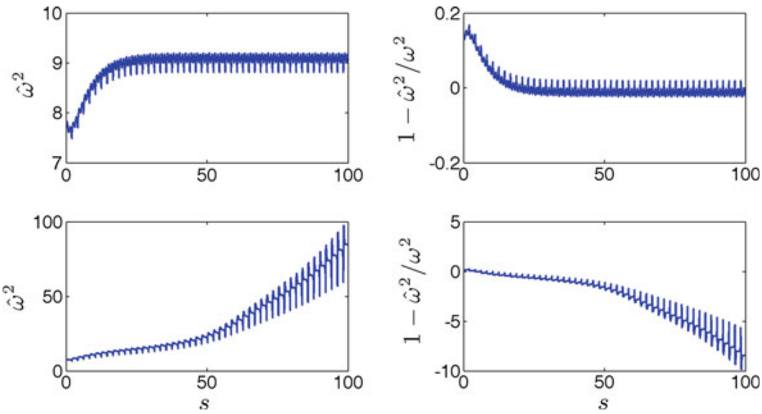


Fig. 14.4 Adaptive notch filter (14.48): *upper curves*, $y(t)$ as in (14.51), *lower curves*, $y(t)$ as in (14.50)

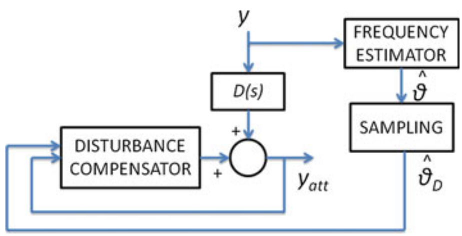
in the case in which the higher order harmonics in (14.50) are reduced by 90 %, i.e., the following signal is applied (see Fig. 14.2)

$$y(t) = \sin(3t + 0.3) + 0.1 \sum_{k=2}^{21} \sin(3kt + 0.3k). \tag{14.51}$$

The lower curves illustrates the performance achieved when the complete signal (14.50) was used. Figure 14.4 shows that while the adaptive notch filter has good performance when both the higher order harmonics and the initial estimate error are small, a divergent behavior occurs when the complete signal (14.50) is applied. Note that the initial frequency estimate error for the adaptive notch filter is much smaller than the corresponding initial error for the frequency estimator (14.18).

As a second example, we consider the problem of attenuating a periodic disturbance assuming that it is the output of an unknown stable system $D(s)$ whose input $y(t)$ is measurable (see Fig. 14.5)

Fig. 14.5 Block diagram for the disturbance compensator



$$y(t) = \sum_{k=1}^{5} \frac{1}{k} \sin(3kt + 0.3k). \tag{14.52}$$

Note that the scheme in Fig. 14.5 applies to active noise cancelation if $D(s)$ is the transfer function between the source of noise and the listener. First of all, assuming that the frequency of the periodic signal is known and given by $\hat{\theta}_D$, the following disturbance compensator is designed

$$\begin{aligned}
\dot{x}_1 &= x_2 - k_c y_{att} \\
\dot{x}_2 &= -\hat{\theta}_D x_1 \\
\dot{x}_3 &= x_4 - k_c y_{att} \\
\dot{x}_4 &= -4\hat{\theta}_D x_3 \\
\dot{x}_5 &= x_6 - k_c y_{att} \\
\dot{x}_6 &= -9\hat{\theta}_D x_5 \\
y_{att} &= y + x_1 + x_3 + x_5
\end{aligned} \tag{14.53}$$

which is capable of cutting the first three harmonics in the periodic signal $D(s)y(s)$ when $\hat{\theta}_D = 3$. We select $D(s) = 1$ for the simulation set-up. Then, the frequency estimator (14.18) is used, with $l = 2$ and the same parameters used in the first example (with the exception of $\gamma = 3000$), to update every time interval $T = 4$ s the value of $\hat{\theta}_D$ in (14.53), so that the overall disturbance compensator is hybrid. The results of the simulation are illustrated by Fig. 14.6 in which are reported the time histories of the disturbance $y(t)$, the attenuated disturbance $y_{att}(t)$, the discrete-time estimate $\hat{\theta}_D(t)$

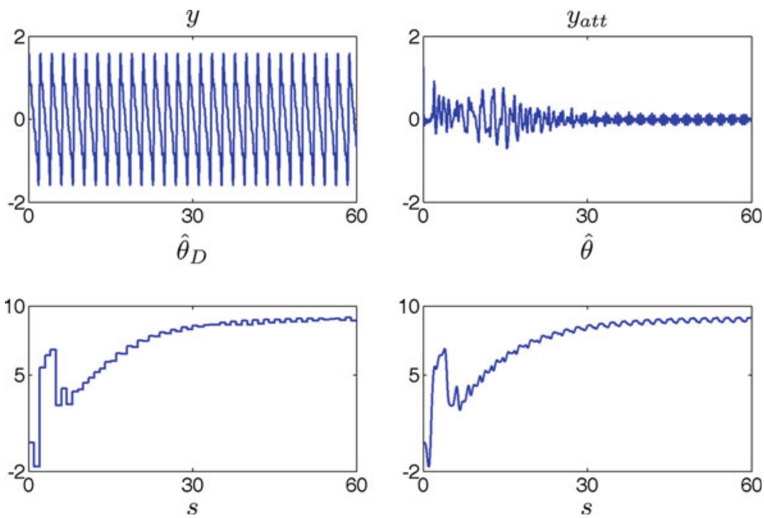


Fig. 14.6 Hybrid disturbance compensator

and the continuous-time estimate $\hat{\theta}(t)$. It can be noted that, even though there is a nonzero error in the frequency estimate, the attenuated signal $y_{att}(t)$ is much smaller than the original disturbance $y(t)$. The residual error is due to two different causes: the mismatch between the true and estimated period, and the remaining fourth and fifth harmonics which are not blocked in (14.53).

14.4 Conclusions

The adaptive $(l+9)$-order frequency estimator (14.18) has been proposed to estimate the period of a measured bounded continuous periodic signal: l denotes the order of the linear prefilter. No a priori information on the period is required. Theorem 14.1 establishes that for any initial condition, the parameter estimation error converges exponentially into a closed interval whose size depends on the higher order harmonics in (14.1). By increasing the order l of the prefilter, the accuracy of the frequency estimation can be arbitrarily improved, at the expense of reducing the rate of the exponential convergence. If there are no higher order harmonics in (14.1), that is (14.1) is a biased sinusoidal signal, then the frequency estimation error converges exponentially to zero for any value of l, including $l = 0$. This result improves the widely studied [3, 4, 9, 24, 25, 28] adaptive notch filter (14.48) whose frequency estimate convergence into a neighborhood of the true value is proved to be asymptotic and local in [25], provided that the adaptation gain is sufficiently small. Moreover, the frequency estimator may be also used to provide, at each predefined time interval, updated frequency estimates to disturbance compensators operating with constant frequency, as it is shown in the included example.

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