Chapter 11
Modeling and Analysis of Energy Distribution Networks Using Switched Differential Systems

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Abstract It is a pleasure to dedicate this contribution to Prof. Arjan van der Schaft on the occasion of his 60th birthday. We study the dynamics of energy distribution networks consisting of switching power converters and multiple (dis-)connectable modules. We use parsimonious models that deal effectively with the variant complexity of the network and the inherent switching phenomena induced by power converters. We also present the solution to instability problems caused by devices with negative impedance characteristics such as constant power loads. Elements of the behavioral system theory such as linear differential behaviors and quadratic differential forms are crucial in our analysis.

11.1 Introduction

In recent years, the development of a new paradigm of energy generation and distribution systems has become a pressing research question. Issues such as the urge to reduce CO_2 emissions, the compelling advantages of renewable energy generation, and the undesirable power losses in complex transmission lines, have motivated the development of distributed energy generation systems based on renewable energies [31]. However, the intermittent nature of renewable energies is reflected in the characteristics of the voltages/currents (e.g., amplitude and frequency) provided by transducers, prompting to regulate such variables to satisfy the nominal requirements of the the loads.

In order to achieve voltage/current/frequency regulation and distribution of electricity, interconnections of power converters are implemented; however, their interaction can display unstable behaviors (see [3, 30, 32]). A common example of this

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issue is the *negative impedance instability* produced by current/voltage controlled converters behaving as *constant power loads* (see [17]). In order to address instability problems, we first need to choose a modeling framework that is suitable to describe the network characteristics. We consider the network as a complex switched system whose dynamic modes with variant state space dimension are induced by switching power converters and the arbitrary (dis-)connection of loads.

There exist traditional approaches to switched systems based on *state space-* (see e.g., [7]) and *descriptor form-* (see e.g., [23]) representations, where the dynamic modes share a global state space. However, the fact that the dynamic modes of energy distribution networks do not necessarily share the same state space engenders three main disadvantages in current approaches:

(1) *Loss of parsimony*. The complexity of "lower order modes" needs to be increased by adding fictitious variables and equations, only to satisfy a predefined global structure, see [11, 12].
(2) *State representations are not given a priori*. The modeling of elements of the network as impedances offer considerable computational advantages when dealing with complex scenarios (see e.g., [6]). Such approach leads directly to higher order descriptions and not state space representations. Consequently, additional computations must be performed to derive state space models.
(3) *Loss of modularity*. The incremental modeling of the dynamic modes in the bank is not permitted, i.e., new dynamic modes cannot be added to the underlying bank without altering the existing ones. The need to allow for incremental modeling arises naturally in an energy distribution network when new loads are connected to the network, see [11].

These issues motivated the development of the *switched linear differential systems framework (SLDS)* in [9–12, 18, 19], which is not representation-oriented, and thus permits the use of the type of models that are most natural for each application (e.g., the modeling of impedances). This approach is based on the concepts of behavioral system theory, and allows the modeling of dynamic modes expressed by sets of linear differential equations that do not necessarily share the same state space, as well as the introduction to new dynamic modes to the bank without altering the existing ones. In this chapter, we study the notion of *passivity* in the SLDS framework, using *quadratic differential forms* (see [27]) as a tool to model energy functions of the network. We also derive a systematic procedure to design passive stabilizing filters in terms of standard *bilinear-* and *linear matrix inequalities*, that can be easily constructed from the higher order models.

11.2 Notation

We use the following notation. The space of $\mathtt{n}$ dimensional real vectors is denoted by $\mathbb{R}^{\mathtt{n}}$, and that of $\mathtt{m} \times \mathtt{n}$ real matrices by $\mathbb{R}^{\mathtt{m}\times\mathtt{n}}$. $\mathbb{R}^{\bullet\times\mathtt{m}}$ denotes the space of real matrices with $\mathtt{m}$ columns and an unspecified finite number of rows. Given matrices

$A, B \in \mathbb{R}^{\bullet \times \mathtt{m}}$, $\operatorname{col}(A, B)$ denotes the matrix obtained by stacking A over B. The ring of polynomials with real coefficients in the indeterminate s is denoted by $\mathbb{R}[s]$; the ring of two-variable polynomials with real coefficients in the indeterminates ζ and η is denoted by $\mathbb{R}[\zeta, \eta]$. $\mathbb{R}^{\mathtt{r} \times \mathtt{w}}[s]$ denotes the set of all $\mathtt{r} \times \mathtt{w}$ matrices with entries in s, and $\mathbb{R}^{\mathtt{n} \times \mathtt{m}}[\zeta, \eta]$ that of $\mathtt{n} \times \mathtt{m}$ polynomial matrices in ζ and η. The set of rational $\mathtt{m} \times \mathtt{n}$ matrices is denoted by $\mathbb{R}^{\mathtt{m} \times \mathtt{n}}(s)$. The set of infinitely differentiable functions from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$ is denoted by $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. $\mathfrak{D}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ is the subset of $\mathfrak{C}^{\infty}(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$ consisting of compact support functions. For a function $f : [t - \varepsilon, t) \to \mathbb{R}^{\bullet}$ we set the notation $f(t^-) := \lim_{\tau \nearrow t} f(\tau)$; and similarly for $f : (t, t + \varepsilon] \to \mathbb{R}^{\bullet}$ we set $f(t^+) := \lim_{\tau \searrow t} f(\tau)$, provided that these limits exist.

We also use standard concepts and notation of the behavioral setting, in particular those of *linear differential behaviors*, *state maps* and *quadratic differential forms*. A simplified collection of the theory that is relevant for the presented results can be found in Appendix A, p. 2046 of [12].

11.3 Modeling of Energy Distribution Networks

Consider the energy distribution network in Fig. 11.1, consisting of a switching power converter feeding three types of loads represented by impedances. Z_N represents a nominal load, i.e., the load that is considered during the design stage of the converter and which remains connected in the implementation. Z_k, $k = 1, \ldots, L$, represents a *switched impedance*, i.e., a finite amount of loads that can be connected or disconnected arbitrarily and which are not necessarily known during the design stage, e.g., domestic/commercial (dis-)connectable loads, (dis-)connectable electric vehicles, etc. Finally, Z_{CPL} represents the negative impedance of a switching power converter behaving as a constant power load (CPL), which is a potential destibilizer of the network (see [8]). The CPL is modeled according to [17] as a negative impedance in parallel with a constant current source.

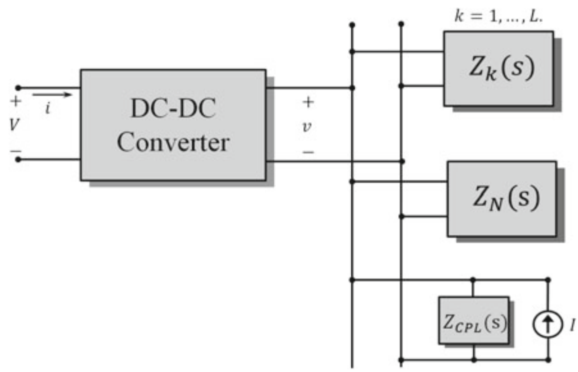


Fig. 11.1 Energy distribution network

Note that the complexity of the network is neither initially bounded nor fixed, i.e., the McMillan degree associated to each impedance depends on their constitutive reactive elements which in the case of Z_k, $k = 1, \ldots, L$, may change depending on the loads that are connected during certain intervals of time. In the following sections, we discuss a natural modeling approach that deals effectively with this type of network.

11.3.1 Modeling of Loads as Impedances

When we study systems consisting of interconnections of port-driven electrical networks, e.g., transmission lines with points of common coupling, filters, loads, etc., we are compelled to adopt the calculus of m-port impedances for simplification of computations, see, e.g., [6, 14, 15, 21, 29]. In the case of energy distribution networks, this is also a common approach for the study of stability, see, e.g., [8, 20, 25, 30].

Models based on impedance matrices describe the "input–output dynamics" of the network in terms of the variables $V := \text{col}(v_1, \ldots, v_m)$ and $I := \text{col}(i_1, \ldots, i_m)$, corresponding, respectively, to the voltages across and currents through each port. Let $P\left(\frac{d}{dt}\right) V = Q\left(\frac{d}{dt}\right) I$, with $P, Q \in \mathbb{R}^{m\times m}[s]$, be an input–output representation (see [16]) of the network obtained by applying current and voltage laws. Adopting $\mathfrak{C}^\infty$ as the solution space, the external behavior of the network is defined as

$$\mathfrak{B} := \left\{ \text{col}(V, I) \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{2m}) \;\middle|\; P\left(\frac{d}{dt}\right) V = Q\left(\frac{d}{dt}\right) I \right\}. \tag{11.1}$$

The impedance $Z \in \mathbb{R}^{m\times m}(s)$ associated to the external behavior is defined as $Z(s) := P(s)^{-1}Q(s)$. If the behavior $\mathfrak{B}$ is controllable (see Chap. 5 of [16]), i.e., $R(s) := \begin{bmatrix} Q(s) & -P(s) \end{bmatrix}$ is such that rank $R(s)$ is equal to rank $R(\lambda)$ for all $\lambda \in \mathbb{C}$, then it admits an image representation

$$\begin{bmatrix} I \\ V \end{bmatrix} = \begin{bmatrix} U\left(\frac{d}{dt}\right) \\ Y\left(\frac{d}{dt}\right) \end{bmatrix} z \tag{11.2}$$

where $z \in \mathfrak{C}^\infty(\mathbb{R}, \mathbb{R}^{\mathtt{z}})$ corresponds to a latent variable and $U, Y \in \mathbb{R}^{m\times m}[s]$ are such that $Z(s) = Y(s)U(s)^{-1}$. Moreover, if $M(\lambda)$ is of full column rank for all $\lambda \in \mathbb{C}$, we conclude that the latent variable z is observable from $w := \text{col}(V, I)$ and its number of components corresponds to the number of inputs, i.e., $\mathtt{z} = m$. A controllable behavior always admits an observable image representation (see [28], Sect. VI-A).

Assuming controllability, the dynamic model of a network described as (11.2) can be obtained in a simple way by *series-* and *parallel computations*, since any complex m-port impedance matrix Z consists of the interconnection of impedances

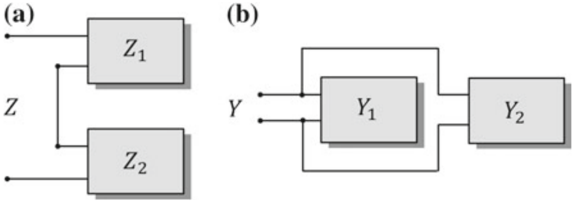


Fig. 11.2 Series/parallel interconnection of impedances/admittances

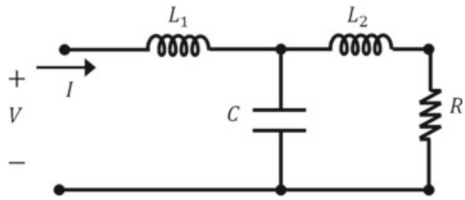


Fig. 11.3 Port-driven electrical circuit

of lower complexity. The simplest components are 1-port impedances corresponding to inductors, resistors, and capacitors, i.e., $Z_L(s) = Ls$, $Z_R(s) = R$, $Z_c(s) = \frac{1}{Cs}$. The inverse of an impedance, if exists, is equal to an admittance denoted by Y, i.e., $Y = Z^{-1}$.

Consider for instance the n-port networks in Fig. 11.2, whose terminals represent an m number of terminal pairs. The resultant m-port impedance/admittance due to series (Fig. 11.2a) and parallel (Fig. 11.2b) interconnections is computed as $Z = Z_1 + Z_2$ and $Y = Y_1 + Y_2$, respectively.

Example 11.1 Consider the 1-port electrical circuit in Fig. 11.3. The 1-port impedance of the circuit can be computed by series and parallel operations as

$$Z(s) = L_1 s + \frac{(L_2 s + R)\left(\frac{1}{C_1 s}\right)}{(L_2 s + R) + \left(\frac{1}{C_1 s}\right)} = \frac{L_1 L_2 C_1 s^3 + R L_1 C_1 s^2 + (L_1 + L_2)s + R}{L_2 C_1 s^2 + R C_1 s + 1}\ , \tag{11.3}$$

which corresponds to the input–output description

$$L_1 L_2 C_1 \frac{\mathrm{d}^3}{\mathrm{d}t^3} I + R L_1 C_1 \frac{\mathrm{d}^2}{\mathrm{d}t^2} I + (L_1 + L_2) \frac{\mathrm{d}}{\mathrm{d}t} I + RI = L_2 C_1 \frac{\mathrm{d}^2}{\mathrm{d}t^2} V + R C_1 \frac{\mathrm{d}}{\mathrm{d}t} V + V\ .$$

Let for simplicity $R = 1\ \Omega$, $L_1 = L_2 = 1\ H$ and $C = 1\ F$, then

$$\underbrace{\begin{bmatrix} V \\ I \end{bmatrix}}_{=:w} = \underbrace{\begin{bmatrix} \frac{\mathrm{d}^3}{\mathrm{d}t^3} + \frac{\mathrm{d}^2}{\mathrm{d}t^2} + 2\frac{\mathrm{d}}{\mathrm{d}t} + 1 \\ \frac{\mathrm{d}^2}{\mathrm{d}t^2} + \frac{\mathrm{d}}{\mathrm{d}t} + 1 \end{bmatrix}}_{=:M\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)} z\ ,$$

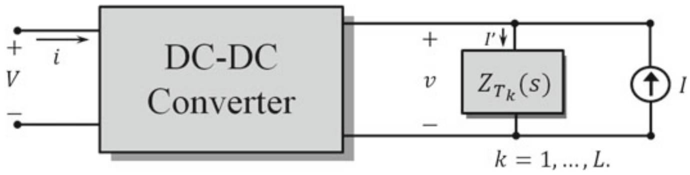


Fig. 11.4 Simplification of the energy distribution network in Fig. 11.1

where z is a latent variable corresponding to the current through the inductor L_2. Since $M(\lambda)$ is of full column rank for all $\lambda \in \mathbb{C}$, we conclude that the latent variable z is observable from w. □

The calculus of impedances facilitates our analysis, for instance the energy distribution network in Fig. 11.1 can be simplified by computing Z_{T_k}, $k = 1, \ldots, L$, as

$$\frac{1}{Z_{T_k}(s)} = \frac{1}{Z_k(s)} + \frac{1}{Z_N(s)} + \frac{1}{Z_{CPL}(s)} \; ; \quad k = 1, \ldots, L \, .$$

The simplified network is depicted in Fig. 11.4.

Remark 11.2 It is important to emphasize that Z_{CPL}, and consequently Z_{T_k}, $k = 1, \ldots, L$, do not necessarily correspond to impedances of passive networks as in traditional circuit theory, since Z_{CPL} corresponds to the local approximation of a constant power load which is by definition nonpassive (i.e., it is not positive-real in the sense of [14]), tipically modeled as a negative resistor [17].

We have illustrated the modeling of loads as impedances, that gives rise in a natural way to higher order descriptions. In the following section, we discuss a modeling approach that permits the study of switching dynamics induced by the DC–DC converter and the switched impedance Z_{T_k}, $k = 1, \ldots, L$, directly in higher order terms.

11.3.2 Switched Linear Differential Systems Framework

We now introduce the SLDS framework. We illustrate the main concepts of this approach by modeling a switching power converter.

Definition 11.3 ([10]) A *switched linear differential system (SLDS)* Σ is a quadruple $\Sigma = \{\mathcal{P}, \mathcal{F}, \mathcal{S}, \mathcal{G}\}$ where

- $\mathcal{P} = \{1, \ldots, N\} \subset \mathbb{N}$, is the set of *indices*;
- $\mathcal{F} = \{\mathfrak{B}_1, \ldots, \mathfrak{B}_N\}$, with $\mathfrak{B}_i$ a linear differential behavior and $i \in \mathcal{P}$, is the *bank of behaviors*;

- $\mathcal{S} = \{s : \mathbb{R} \rightarrow \mathcal{P}\}$, with s piecewise constant and right-continuous, is the set of admissible *switching signals*; and
- $\mathcal{G} = \left\{(G^{-}_{k\rightarrow j}(s), G^{+}_{k\rightarrow j}(s)) \in \mathbb{R}^{\bullet\times \mathtt{w}}[s] \times \mathbb{R}^{\bullet\times \mathtt{w}}[s] \mid 1 \leq k, j \leq N\ ,\ k \neq j\right\}$, is the set of *gluing conditions*.

The set of *switching instants* associated with $s \in \mathcal{S}$ is defined by $\mathbb{T}_s := \{t \in \mathbb{R} \mid s(t^-) \neq s(t^+)\} = \{t_1, t_2, \ldots\}$, where $t_i < t_{i+1}$.

The set of all admissible trajectories satisfying the laws of the mode behaviors and the gluing conditions is the *switched behavior*, and is the central object of study in our framework.

Definition 11.4 ([10]) Let $\Sigma = \{\mathcal{P}, \mathcal{F}, \mathcal{S}, \mathcal{G}\}$ be a SLDS, and let $s \in \mathcal{S}$. The s *-switched linear differential behavior* $\mathfrak{B}^s$ is the set of trajectories $w : \mathbb{R} \rightarrow \mathbb{R}^{\mathtt{w}}$ that satisfy the following two conditions:

1. for all $t_i, t_{i+1} \in \mathbb{T}_s$, $w\mid_{[t_i,t_{i+1})} \in \mathfrak{B}_{s(t_i)}\mid_{[t_i,t_{i+1})}$;
2. w satisfies the gluing conditions $\mathcal{G}$ at the switching instants for each $t_i \in \mathbb{T}_s$, i.e.,

$$G^{+}_{s(t_{i-1})\rightarrow s(t_i)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w(t_i^+) = G^{-}_{s(t_{i-1})\rightarrow s(t_i)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w(t_i^-)\ . \tag{11.4}$$

The *switched linear differential behavior (SLDB)* $\mathfrak{B}^{\Sigma}$ of Σ is defined by $\mathfrak{B}^{\Sigma} := \bigcup_{s\in\mathcal{S}} \mathfrak{B}^s$.

The trajectories in $\mathfrak{B}^{\Sigma}$ are *piecewise infinitely differentiable functions* from $\mathbb{R}$ to $\mathbb{R}^{\mathtt{w}}$ denoted by $\mathfrak{C}^{\infty}_p(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$, i.e., smooth when a mode is active and possibly discontinuous at switching instants.

Example 11.5 Consider the *high-voltage switching power converter* presented in [2] and depicted in Fig. 11.5a. For practical purposes such as voltage/current/power regulation, we are particularly interested in the dynamics at the input/output terminals. Consequently, we define the external variable (the set of variables of interest) as $w := \mathrm{col}(E, i_L, v_2, i_o)$.

By means of a switching signal, we can arbitrarily induce two possible electrical configurations that occur when the transistor is in either *closed* (see Fig. 11.5b) or *open* (see Fig. 11.5c) operation. Considering a standard modeling of two-port impedances for each case, we can derive the following physical laws describing the dynamics of the power converter.

$$\text{Mode 1}: \begin{cases} L\frac{\mathrm{d}}{\mathrm{d}t}i_L + R_L i_L - E = 0 \\ (C_1 + C_2)\frac{\mathrm{d}}{\mathrm{d}t}v_2 + \frac{1}{R}v_2 - i_o = 0 \end{cases}.$$

$$\text{Mode 2}: \begin{cases} LC_1\frac{\mathrm{d}^2}{\mathrm{d}t^2}i_L + R_L C_1\frac{\mathrm{d}}{\mathrm{d}t}i_L - C_1\frac{\mathrm{d}}{\mathrm{d}t}E + i_L = 0 \\ C_2\frac{\mathrm{d}}{\mathrm{d}t}v_2 + \frac{1}{R}v_2 - i_o = 0 \end{cases}.$$

The mode behaviors are defined as $\mathfrak{B}_j := \ker\ R_j\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$, $j = 1, 2$, where

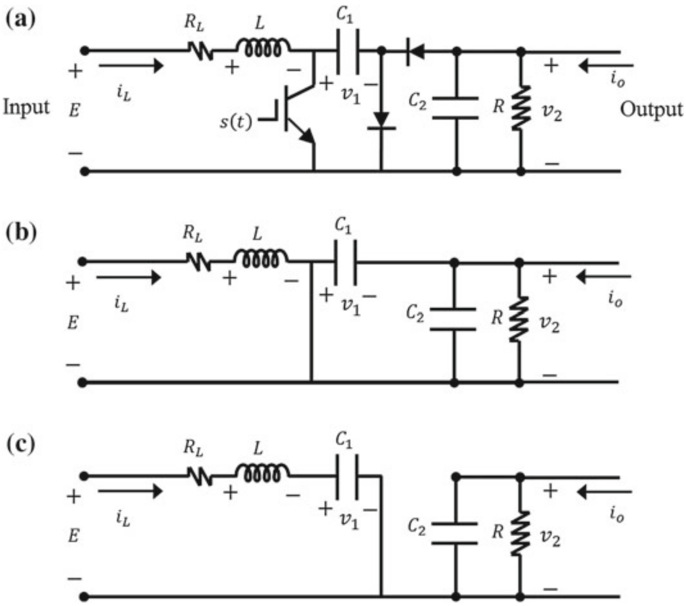


Fig. 11.5 **a** High-voltage switching power converter, **b** electrical configuration when the transistor is closed, **c** electrical configuration when the transistor is open

$$R_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) := \begin{bmatrix} -1 & L\frac{\mathrm{d}}{\mathrm{d}t}+R_L & 0 & 0 \\ 0 & 0 & (C_1+C_2)\frac{\mathrm{d}}{\mathrm{d}t}+\frac{1}{R} & -1 \end{bmatrix};$$

$$R_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) := \begin{bmatrix} -C_1\frac{\mathrm{d}}{\mathrm{d}t} & LC_1\frac{\mathrm{d}^2}{\mathrm{d}t^2}+R_LC_1\frac{\mathrm{d}}{\mathrm{d}t}+1 & 0 & 0 \\ 0 & 0 & C_2\frac{\mathrm{d}}{\mathrm{d}t}+\frac{1}{R} & -1 \end{bmatrix}.$$

As we show later, the physical constraints imposed by physics at switching instants can be modeled using gluing conditions. □

According to Definition 11.3 gluing conditions are algebraic constraints on the trajectories of the dynamical modes at switching instants and in real-life situations their selection is motivated by physical laws. For instance, at switching instants *conservation principles* forbid instantaneous changes in *conserved quantities* (see [13]) such as charge, flux, momentum, molar mass, volume, etc. Another well-known example of this type of constraints is the case of *state reset maps* in multicontroller systems that re-initialize a bank of switched controllers interconnected to a plant.

Example 11.6 (Cont'd from Example 11.5) At switching instants, the physical laws of the circuit impose constraints to the trajectories of the external variable at switching instants. By inspecting the circuits in Fig. 11.5 and using the *principle of conservation of charge* (see [13], Sect. 3.3.3), we find the following conditions at switching instants.

When switching from $\mathfrak{B}_1$ to $\mathfrak{B}_2$ at t_i:

$$\begin{aligned} & i_L(t_i^+) = i_L(t_i^-), \\ & \underbrace{E(t_i^+) - R_L i_L(t_i^+) - L\frac{\mathrm{d}}{\mathrm{d}t} i_L(t_i^+)}_{v_1(t_i^+)} = v_2(t_i^-), \\ & v_2(t_i^+) = v_2(t_i^-). \end{aligned} \tag{11.5}$$

When switching from $\mathfrak{B}_2$ to $\mathfrak{B}_1$ at t_i:

$$\begin{aligned} i_L(t_i^+) &= i_L(t_i^-), \\ (C_1 + C_2)v_2(t_i^+) &= \underbrace{C_1 E(t_i^-) - C_1 R_L i_L(t_i^-) - L C_1 \frac{\mathrm{d}}{\mathrm{d}t} i_L(t_i^-)}_{C_1 v_1(t_i^-)} + C_2 v_2(t_i^-). \end{aligned} \tag{11.6}$$

Consequently, the gluing conditions can be defined as

$$G^+_{1\to 2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) := \begin{bmatrix} 0 \; 0 & 1 & 0 \\ 1 \; 0 & (-R_L - L\frac{\mathrm{d}}{\mathrm{d}t}) & 0 \\ 0 \; 0 & 0 & 1 \end{bmatrix} \; ; \quad G^-_{1\to 2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) := \begin{bmatrix} 0 \; 0 \; 1 \; 0 \\ 0 \; 0 \; 0 \; 1 \\ 0 \; 0 \; 0 \; 1 \end{bmatrix} \; ;$$

$$G^+_{2\to 1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) := \begin{bmatrix} 0 \; 0 \; 1 & 0 \\ 0 \; 0 \; 0 & (C_1 + C_2) \end{bmatrix} \; ; \quad G^-_{2\to 1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) := \begin{bmatrix} 0 & 0 & 1 & 0 \\ C_1 & 0 & (-C_1 R_L - L C_1 \frac{\mathrm{d}}{\mathrm{d}t}) & C_2 \end{bmatrix} .$$

Equations (11.5) and (11.6) can be compactly written as

$$\begin{aligned} G^+_{1\to 2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w(t_i^+) &= G^-_{1\to 2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w(t_i^-) \; ; \\ G^+_{2\to 1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w(t_i^+) &= G^-_{2\to 1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) w(t_i^-) \; . \end{aligned}$$

□

A realistic set of gluing conditions are *well-defined* and *well-posed*. In order to introduce these concepts, we use the notion of *state maps*.

Definition 11.7 Let Σ be a SLDS and let $X_j \in \mathbb{R}^{\mathrm{n}(\mathfrak{B}_j)\times \mathrm{w}}[s]$, induce minimal state maps for $\mathfrak{B}_j$, $j = 1, \ldots, N$. The gluing conditions are *well-defined* if there exist constant matrices $F^-_{j\to k}$ and $F^+_{j\to k}$, with $j, k = 1, \ldots, N$, $j \neq k$, such that $G^-_{j\to k}(s) = F^-_{j\to k} X_j(s)$ and $G^+_{j\to k}(s) = F^+_{j\to k} X_k(s)$, with $j, k = 1, \ldots, N$, $j \neq k$.

If $\mathcal{G} := \{(F^-_{j\to k} X_j(s), F^+_{j\to k} X_k(s))\}_{j,k=1,\ldots,N,\; j\neq k}$. are well-defined, we call them *well-posed* if for all $k, j = 1, \ldots, N$ with $k \neq j$, there exists a *re-initialisation* map

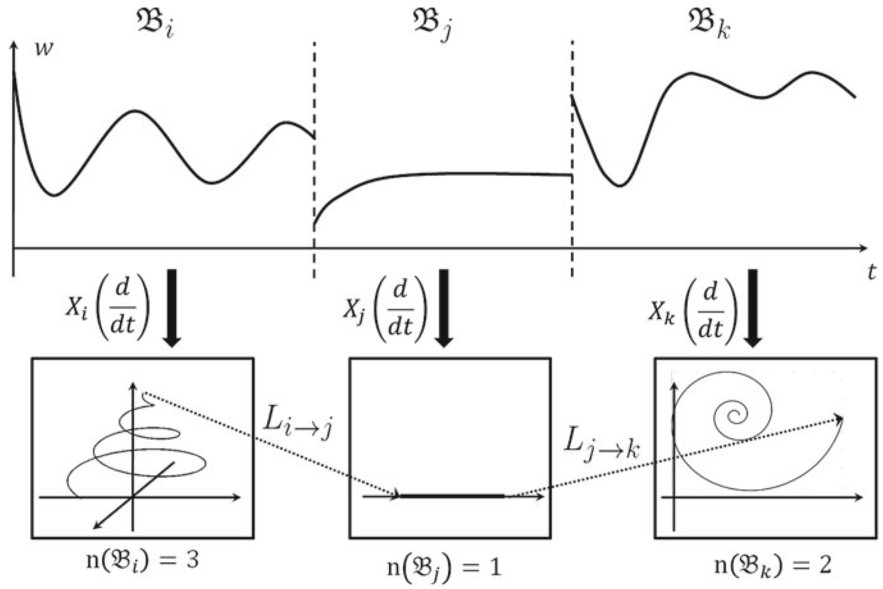


Fig. 11.6 Example: Switching between dynamical modes with different state space and well-posed gluing conditions

$L_{j\to k} : \mathbb{R}^{n(\mathfrak{B}_j)} \to \mathbb{R}^{n(\mathfrak{B}_k)}$ such that given a switching signal $s \in \mathcal{S}$ such that $s(t_{i-1}) = j$ and $s(t_i) = k$; for all $t_i \in \mathbb{T}_s$ and all admissible $w \in \mathfrak{B}^{\Sigma}$ with associated latent variable trajectories, it holds that $X_j\left(\frac{d}{dt}\right) w(t_i^+) = L_{k\to j} X_k\left(\frac{d}{dt}\right) w(t_i^-)$.

Well-defined and well-posed gluing conditions imply that if a transition occurs between $\mathfrak{B}_j$ and $\mathfrak{B}_k$ at t_i, and if an admissible trajectory ends at a "final state" $v_j := X_j\left(\frac{d}{dt}\right) w(t_i^-)$, then there exists at most one "initial state" for $\mathfrak{B}_k$, defined by $X_k\left(\frac{d}{dt}\right) w(t_i^+) =: v_k$, compatible with the gluing conditions. Moreover, the matrix $L_{j\to k}$ determines the reinitialization of the state space of $\mathfrak{B}_k$ as a linear function of that of $\mathfrak{B}_j$, as illustrated in Fig. 11.6.

Example 11.8 (Cont'd Example 11.5) Consider the following state maps for $\mathfrak{B}_1$ and $\mathfrak{B}_2$ respectively.

$$X_1\left(\frac{d}{dt}\right) := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\ ; \quad X_2\left(\frac{d}{dt}\right) := \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & (-R_L - L\frac{d}{dt}) & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\ ;$$

inducing the states $X_1\left(\frac{d}{dt}\right) w = \text{col}(i_L, v_2)$ and $X_2\left(\frac{d}{dt}\right) w = \text{col}(i_L, v_1, v_2)$. The gluing conditions can be written as

$$G^{+}_{1\to 2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) := I_3 X_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \; ; \; G^{-}_{1\to 2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) := \begin{bmatrix}1 & 0\\ 0 & 1\\ 0 & 1\end{bmatrix} X_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \; ;$$

$$G^{+}_{2\to 1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) := \begin{bmatrix}1 & 0\\ 0 & (C_1+C_2)\end{bmatrix} X_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \; ; \; G^{-}_{2\to 1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) := \begin{bmatrix}1 & 0 & 0\\ 0 & C_1 & C_2\end{bmatrix} X_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) .$$

It is thus a matter of straightforward verification to conclude that the gluing conditions are well-defined and well-posed according to Definition 11.7. □

The properties of well-definedness and well-posedness are in general satisfied for common implementations of energy networks, consider for example the following proposition.

Proposition 11.9 *Assume that switching among the dynamical modes of a switched electrical network does not involve short-circuiting of voltage sources or open-circuiting of current sources. Then the gluing conditions are well-defined.*

Proof If switching between modes does not involve short- or open-circuiting sources, no constraints on the input variables of the system are imposed at the switching instants. Consequently, the gluing conditions only impose constraints on the output variables of the modes, which are linear functions of the state variables. The claim follows. □

Well-posed gluing conditions (see Definition 11.7) guarantee that after a switching instant, only one initial state for the new dynamical regime is specified from the final state of the previous one. Such property holds since the switching cannot cause any increase in the total amount of charge or flux stored in the system. On this issue, see [13] where the analysis of a wide variety of physical systems exhibiting discontinuities is presented, and [4, 5, 22]. In the rest of this paper, we assume that *the gluing conditions are well-posed*.

11.3.3 Latent Variables

As discussed in the previous section, controllable mode behaviors can be described using observable image representations $w = M_j\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) z_j$, $j = 1, \dots N$. It follows that every trajectory of the latent variable z_j corresponds to a unique trajectory of the external variable w when the jth mode is active. In the rest of this chapter we adopt the use of image representations, where $w := (u, y)$ has m inputs and m outputs, denoted by u and y, respectively, and corresponding to port-voltages and currents, as discussed in Sect. 11.3.1.

Example 11.10 (Cont'd from Example 11.5) Recall that $w := \mathrm{col}(E, i_L, v_2, i_o)$. It can be verified that the mode behaviors $\mathfrak{B}_j$, $i = 1, 2$, are *controllable* and thus can be described by $w = M_j\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) z_j$, $j = 1, 2$, where

$$M_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) := \begin{bmatrix} L\frac{\mathrm{d}}{\mathrm{d}t}+R_L & 0 \\ 0 & (C_1+C_2)\frac{\mathrm{d}}{\mathrm{d}t}+\frac{1}{R} \\ 1 & 0 \\ 0 & 1 \end{bmatrix};$$

$$M_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) := \begin{bmatrix} LC_1\frac{\mathrm{d}^2}{\mathrm{d}t^2}+R_LC_1\frac{\mathrm{d}}{\mathrm{d}t}+1 & 0 \\ 0 & C_2\frac{\mathrm{d}}{\mathrm{d}t}+\frac{1}{R} \\ C_1\frac{\mathrm{d}}{\mathrm{d}t} & 0 \\ 0 & 1 \end{bmatrix};$$

and $z_1 := \mathrm{col}(i_L, v_2)$, $z_2 := \mathrm{col}(v_1, v_2)$. Moreover, since $M_j(\lambda)$, $j = 1, 2$, are full column rank for all $\lambda \in \mathbb{C}$ we conclude that the latent variables z_j, $j = 1, 2$ are observable from w. □

According to Definitions 11.3 and 11.4, the gluing conditions are algebraic constraints acting on the external variables at switching instants; however, they can be rewritten in terms of latent variables z_j, $j = 1, \ldots N$, in the following manner. Define

$$\overline{G}^{+}_{s(t_{i-1})\to s(t_i)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) := \left(G^{+}_{s(t_{i-1})\to s(t_i)}M_{s(t_i)}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\right),$$

$$\overline{G}^{-}_{s(t_{i-1})\to s(t_i)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) := \left(G^{-}_{s(t_{i-1})\to s(t_i)}M_{s(t_i-1)}\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\right),$$

with $s \in \mathcal{S}$. Consequently, if w and z_j are related by an observable image representation $w = M_j\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)z_j$, the gluing conditions in (11.4) can be equivalently written as

$$\overline{G}^{+}_{s(t_{i-1})\to s(t_i)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)z_{s(t_i)}(t_i^{+}) = \overline{G}^{-}_{s(t_{i-1})\to s(t_i)}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)z_{s(t_i-1)}(t_i^{-}).$$

Example 11.11 (Cont'd from Example 11.10) Given the gluing conditions in Example 11.8, we can reformulate them in terms of latent variables using $M_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$ and $M_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$ as follows.

$$\overline{G}^{-}_{1\to 2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) := \left(G^{-}_{1\to 2}M_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 1 \end{bmatrix},$$

$$\overline{G}^{+}_{1\to 2}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) := \left(G^{+}_{1\to 2}M_2\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) = \begin{bmatrix} C_1\frac{\mathrm{d}}{\mathrm{d}t} & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}^{\top},$$

$$\overline{G}^{-}_{2\to 1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) := \left(G^{-}_{2\to 1}M_2\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) = \begin{bmatrix} C_1\frac{\mathrm{d}}{\mathrm{d}t} & 0 \\ C1 & C2 \end{bmatrix},$$

$$\overline{G}^{+}_{2\to 1}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) := \left(G^{+}_{2\to 1}M_1\right)\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) = \begin{bmatrix} 1 & 0 \\ 0 & C_1 + C_2 \end{bmatrix}.$$

□

11.4 Modularity

One of the main features of this framework is its *modularity*; every time a dynamic mode is added to the underlying bank, there is no need to modify the mathematical description of the existing modes. In the case of the energy distribution network in Fig. 11.4, the dynamic modes of the converter and the loads can be individually modeled and linked in a single model by the elimination of auxiliary variable. Consider the following proposition.

Proposition 11.12 *Consider the energy distribution network in Fig. 11.4. Assume that the dynamical modes of the switching power converter can be described in image form* $w = M_j\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) z_j$, *where* $M_j \in \mathbb{R}^{4\times 2}[s]$; $z_j = \mathrm{col}(z_{1,j}, z_{2,j}) \in \mathfrak{C}^{\infty}_{p}(\mathbb{R}, \mathbb{R}^2)$; $j = 1, 2$; *and* $w := \begin{bmatrix} V & I & i & v \end{bmatrix}^{\top}$. *Let* $z_k \in \mathfrak{C}^{\infty}_{p}(\mathbb{R}, \mathbb{R})$, $k = 1, \ldots, L$, *then there exist* $\widehat{M}_{j,k} \in \mathbb{R}^{4\times 2}[s]$ *such that the mode behaviors can be described by image representations*

$$\begin{bmatrix} V \\ I \\ i \\ v \end{bmatrix} = \widehat{M}_{j,k}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)\begin{bmatrix} z_{1,j} \\ z'_k \end{bmatrix}, \tag{11.7}$$

with $j = 1, 2$, *and* $k = 1, \ldots, L$.

Proof The impedance Z_{T_k}, $k = 1, \ldots, L$, is described by a one-port, and consequently can also be represented in observable image representation by $M' \in \mathbb{R}^{2\times 1}[s]$ with external variables $w' := \begin{bmatrix} I' & v \end{bmatrix}^{\top}$ and a one-dimensional latent variable denoted by z'_k. It follows from the elimination theorem (see Sect. 6 of [16]) that after the elimination of the latent variable $z_{2,j}$, $j = 1, 2$, the interconnection of this one-port with the switching power converter has a number $2L$ of dynamic modes that can be described as two-ports, corresponding to the image representations (11.7). □

Example 11.13 Consider the energy distribution network in Fig. 11.4, where the DC-DC converter is that of Fig. 11.5. Let $p_k, q_k \in \mathbb{R}[s]$, $k = 1, \ldots, L$, define $Z_k(s) := \frac{n_k(s)}{d_k(s)}$, $k = 1, \ldots, L$. The mode dynamics with $w := \mathrm{col}(E, I, i_L, v)$ are described by $w = M_{j,k}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) z_k$, where $z_1 := \mathrm{col}(i_1, z'_k)$, $z_2 := \mathrm{col}(v_1, z'_k)$, $k = 1, \ldots, L$, and $j = 1, 2$.

$$M_{1,k}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) := \begin{bmatrix} R_L + L\frac{\mathrm{d}}{\mathrm{d}t} & 0 \\ 0 & d_k\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) + (C_1 + C_2)\frac{\mathrm{d}}{\mathrm{d}t} n_k\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ 1 & 0 \\ 0 & n_k\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{bmatrix};$$

$$M_{2,k}\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) := \begin{bmatrix} LC_1\frac{\mathrm{d}^2}{\mathrm{d}t^2} + R_L C_1\frac{\mathrm{d}}{\mathrm{d}t} + 1 & 0 \\ 0 & d_k\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) + C_2\frac{\mathrm{d}}{\mathrm{d}t} n_k\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ C_1\frac{\mathrm{d}}{\mathrm{d}t} & 0 \\ 0 & n_k\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{bmatrix};$$

with $k = 1, \ldots, L$. The gluing conditions can be obtained by defining the impedances Z_k, $k = 1, \ldots, L$ and following the procedure exemplified in Examples 11.5 and 11.11. □

As illustrated in Example 11.13, each mode can be modeled independently, i.e., we compute the laws of each two-port network that depends on the mode of operation of the converter and the model of the switched impedance Z_k, $1, \ldots, L$. It can be easily verified that the McMillan degree of each mode behavior is not fixed and depends on the degree of the denominator of Z_k, $1, \ldots, L$. However each mode exhibits only the required level of complexity to describe each dynamic mode. This is in sharp contrast with the traditional approach where the dynamic modes are represented by $\frac{\mathrm{d}}{\mathrm{d}t}x = A_i x$, with $A_i \in \mathbb{R}^{n\times n}$, i.e., considering a global state space and where n is the highest possible McMillan degree. The latter approach results in more complex dynamic models (with more variables and more equations), which has an impact also on the complexity of stability analysis, simulation, control, etc. Moreover, there is no compelling reason to resort to such non-parsimonious approach if we can study the dynamic properties of the network directly in higher order terms, as shown in the following section.

11.5 Passivity

The concept of passivity will be crucial for the development of stability conditions and stabilization methods discussed in this chapter. To define passive SLDS, we first introduce the following notation. Since we require the integration of functionals acting on $w \in \mathfrak{B}^{\Sigma}$, we assume that they involve piecewise infinitely differentiable trajectories of compact support whose set is denoted by $\mathfrak{D}_p(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. Thus the trajectories which we will be considering in the following belong to $\mathfrak{B}^{\Sigma} \cap \mathfrak{D}_p(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.

Let $s \in \mathcal{S}$ be a fixed switching signal, with associated set of switching instants $\mathbb{T}_s := \{t_1, t_2, \ldots, t_n, \ldots\}$. We denote by $|\mathbb{T}_s|$ the total number of switching instants, possibly infinite, in $\mathbb{T}_s$. Let

$$\Phi := \frac{1}{2}\begin{bmatrix} 0_{m\times m} & I_m \\ I_m & 0_{m\times m} \end{bmatrix}, \tag{11.8}$$

and let $w \in \mathfrak{B}^{\Sigma} \cap \mathfrak{D}_p(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$. If $|\mathbb{T}_s| = \infty$, define

$$\int Q_{\Phi}(w) := \int_{-\infty}^{t_1^-} Q_{\Phi}(w)\,\mathrm{d}t + \int_{t_1^+}^{t_2^-} Q_{\Phi}(w)\,\mathrm{d}t + \cdots + \int_{t_n^+}^{t_{n+1}^-} Q_{\Phi}(w)\,\mathrm{d}t + \cdots \quad .$$

If $0 < |\mathbb{T}_s| < \infty$, then define

$$\int Q_{\Phi}(w) := \int_{-\infty}^{t_1^-} Q_{\Phi}(w)\,\mathrm{d}t + \sum_{k=2}^{|\mathbb{T}_s|} \int_{t_{k-1}^+}^{t_k^-} Q_{\Phi}(w)\,\mathrm{d}t + \int_{t_{|\mathbb{T}_s|}^+}^{\infty} Q_{\Phi}(w)\,\mathrm{d}t.$$

If $|\mathbb{T}_s| = 0$, i.e., no switching takes place, then

$$\int Q_{\Phi}(w) := \int_{-\infty}^{+\infty} Q_{\Phi}(w)\,\mathrm{d}t.$$

The definition of passive SLDS is as follows.

Definition 11.14 Let Σ be a SLDS and define Φ as in (11.8). Σ is *passive* if $\int Q_{\Phi}(w) \geq 0$ for all $w \in \mathfrak{B}^{\Sigma} \cap \mathfrak{D}_p(\mathbb{R}, \mathbb{R}^{\mathtt{w}})$.

In the previous definition, the quadratic differential form Q_{Φ} can be interpreted as the power that is oriented into the system, and consequently its integral over the real line measures the energy that is being supplied to, or flows out from the SLDS. If the net flow of energy is nonnegative, then we call the SLDS passive. Passivity implies input–output stability (see e.g., [26]), in the sense that unbounded output trajectories cannot occur as a consequence of bounded input trajectories (see also Sect. V of [10] for further elaboration).

In the SLDS framework, the concept of storage function arises in a natural way, describing the energy stored in each individual dynamical mode.

Definition 11.15 Let Σ be a SLDS and let $s \in \mathcal{S}$. An N-tuple $\left(Q_{\Psi_1}, \ldots, Q_{\Psi_N}\right)$ is a *multiple storage function* for Σ with respect to Q_{Φ} if

(1) $\frac{\mathrm{d}}{\mathrm{d}t} Q_{\Psi_i} \overset{\mathfrak{B}_i}{\leq} Q_{\Phi}, i = 1, \ldots, N.$
(2) $\forall\, w \in \mathfrak{B}^{\Sigma}$ and $\forall\, t_k \in \mathbb{T}_s$, it holds $Q_{\Psi_{s(t_{k-1})}}(w)(t_k^-) - Q_{\Psi_{s(t_k)}}(w)(t_k^+) \geq 0$.

We now prove that the existence of a multiple storage function implies that the SLDS is passive.

Theorem 11.16 *Let Σ be a SLDS and let $\Phi := \frac{1}{2}\begin{bmatrix} 0_{m\times m} & I_m \\ I_m & 0_{m\times m}\end{bmatrix}$. Assume that there exists a multiple storage function as in Definition 11.15. Then Σ is passive.*

Proof Let $t_0 := -\infty$ and let $s_w \in \mathcal{S}$ denote the switching signal that corresponds to a given trajectory $w \in \mathfrak{B}^{\Sigma}$. We consider the three possible cases, i.e., (1) $|\mathbb{T}_s| = \infty$, (2) $0 < |\mathbb{T}_s| < \infty$ and (3) $|\mathbb{T}_s| = 0$. It follows from Theorem 4.3 of [24], that since there exists Q_{Ψ_i} such that $\frac{d}{dt}Q_{\Psi_i} \overset{\mathfrak{B}_i}{\leq} Q_\Phi$, $i = 1, \ldots, N$, then $\mathfrak{B}_i$, $i = 1, \ldots, N$, is passive (i.e., dissipative with respect to Q_Φ).

Let $a < b$, then for all $w \in \mathfrak{B}^\Sigma$ with $s_w(t) = i$ for $t \in [a, b]$, it holds that $\int_a^b Q_\Phi(w)\ dt \geq Q_{\Psi_i}(w)(b) - Q_{\Psi_i}(w)(a)$, corresponding to the integration over $t \in [a, b]$ of $Q_{\Psi_i} \leq Q_\Phi$, for $w \in \mathfrak{B}_i \in \mathfrak{D}_p(\mathbb{R}, \mathbb{R}^{\mathrm{w}})$.

Since $\lim_{t\to\pm\infty} w(t) = 0$ for all $w \in \mathfrak{B}^\Sigma \cap \mathfrak{D}_p(\mathbb{R}, \mathbb{R}^{\mathrm{w}})$ we obtain the following expressions for cases (1) and (2), where $s = s_w$:

$$(1) \int Q_\Phi(w) \geq (Q_{\Psi_{s(t_0)}}(w)(t_1^-) - Q_{\Psi_{s(t_1)}}(w)(t_1^+)) + \cdots$$
$$+ (Q_{\Psi_{s(t_{n-1})}}(w)(t_n^-) - Q_{\Psi_{s(t_n)}}(w)(t_n^+)) + \cdots .$$
$$(2) \int Q_\Phi(w) \geq (Q_{\Psi_{s(t_0)}}(w)(t_1^-) - Q_{\Psi_{s(t_1)}}(w)(t_1^+))$$
$$+ \Sigma_{k=2}^{|\mathbb{T}_s|-1}(Q_{\Psi_{s(t_{k-1})}}(w)(t_k^-) - Q_{\Psi_{s(t_k)}}(w)(t_k^+))$$
$$+ (Q_{\Psi_{s(|\mathbb{T}_s|-1)}}(w)(t_{|\mathbb{T}_s|}^-) - Q_{\Psi_{s(|\mathbb{T}_s|)}}(w)(t_{|\mathbb{T}_s|}^+)).$$

Since $Q_{\Psi_{s(t_{k-1})}}(w)(t_k^-) - Q_{\Psi_{s(t_k)}}(w)(t_k^+) \geq 0$, $\forall\ t_k \in \mathbb{T}_s$, we conclude that in both cases $\int Q_\Phi(w) \geq 0$.

Finally the claim for (3) when no switching takes place, i.e., $s_w(t) = i$ for all t, follows readily from the existence of a storage function Q_{Ψ_i} and Theorem 4.3 of [24]. □

The conditions for the existence of a multiple storage function can be expressed in terms of linear matrix inequalities according to the following result (see Theorem 4 of [10]) providing an LMI-based test for passivity of SLDS. In the following, the *coefficient matrix* of $F(s) = \sum_{i=0}^N F_i s^i \in \mathbb{R}^{q_1\times q_2}[s]$ is defined by

$$\widetilde{F} := \begin{bmatrix} F_0 & F_1 & \ldots & F_N \end{bmatrix} . \quad (11.9)$$

Note that $F(s) = \widetilde{F}\begin{bmatrix} I_{q_2} & sI_{q_2} & \ldots & I_{q_2}s^N \end{bmatrix}^\top$.

Theorem 11.17 *Let Σ be a SLDS with $\mathcal{G}$ well-defined and well-posed. Let $X_k \in \mathbb{R}^{\mathrm{n}(\mathfrak{B}_k)\times \mathrm{z}}[s]$ be a minimal state map for $\mathfrak{B}_k$, acting on the latent variable z_k, $k = 1, \ldots, N$, and let $L_{i\to j} \in \mathbb{R}^{\mathrm{n}(\mathfrak{B}_j)\times\mathrm{n}(\mathfrak{B}_i)}$ for all $i, j \in \mathcal{P}$, $i \neq j$, be the re-initialisations maps of Σ. Denote the coefficient matrix of $M_k(s)$ by $\widetilde{M}_k := \begin{bmatrix} M_{k,0} & \ldots & M_{k,L_k}\end{bmatrix}$; then there exist $X_{k,j} \in \mathbb{R}^{\mathrm{n}(\mathfrak{B}_k)\times m}$, $k = 1, \ldots, N$, $j = 1, \ldots, L_k - 1$ such that $X_k(s)$ can be written as $\widetilde{X}_k := \begin{bmatrix} X_{k,0} & \ldots & X_{k,L_k-1}\end{bmatrix}$.*

If there exist $K_k = K_k^\top \in \mathbb{R}^{\mathrm{n}(\mathfrak{B}_k)\times\mathrm{n}(\mathfrak{B}_k)}$, $k = 1 \ldots, N$, such that

$$\widetilde{M}_k^\top \Phi \widetilde{M}_k - \begin{bmatrix} 0_{m\times \mathrm{n}(\mathfrak{B}_k)} \\ \widetilde{X}_k^\top \end{bmatrix} K_k \begin{bmatrix} \widetilde{X}_k & 0_{\mathrm{n}(\mathfrak{B}_k)\times m}\end{bmatrix} - \begin{bmatrix} \widetilde{X}_k^\top \\ 0_{m\times\mathrm{n}(\mathfrak{B}_k)}\end{bmatrix} K_k \begin{bmatrix} 0_{\mathrm{n}(\mathfrak{B}_k)\times m} & \widetilde{X}_k\end{bmatrix} \geq 0 , \quad (11.10)$$

and moreover, if for $k, j = 1, \ldots, N$, $k \neq j$, it holds that

$$K_k - L_{k\to j}^{\top} K_j L_{k\to j} \geq 0\,, \tag{11.11}$$

then Σ is passive.

Based on this result, in the following section we develop a stabilization technique for energy distribution networks.

11.6 Energy-Based Stabilization

To deal with instability of energy distribution networks, we use *passive damping* (see, e.g., [1]), where a passive load (filter) is interconnected to the system in order to guarantee stability.

We consider the case where the energy distribution network is unstable due to the presence of constant power loads (see [17]). We proceed to design a filter that guarantees stability when interconnected to the converter, see Fig. 11.7.

For ease of exposition, we consider only one impedance $Z_T(s)$ and the filter as an additional load in the array depicted in Fig. 11.7. The impedance function of the filter is given by

$$Z_f(s) = \frac{p(s)}{q(s)}\;; \tag{11.12}$$

with an associated image form representation

$$\begin{bmatrix} i_f \\ v \end{bmatrix} = \begin{bmatrix} p(\frac{\mathrm{d}}{\mathrm{d}t}) \\ q(\frac{\mathrm{d}}{\mathrm{d}t}) \end{bmatrix} z'\,, \tag{11.13}$$

and whose parameters need be computed. The interconnection of impedances (11.12) and $Z_T(s)$ in Fig. 11.7 yields

$$Z_{int}(s) := \frac{Z_T(s)Z_f(s)}{Z_T(s)+Z_f(s)} = \frac{n(s)}{d(s)}\,. \tag{11.14}$$

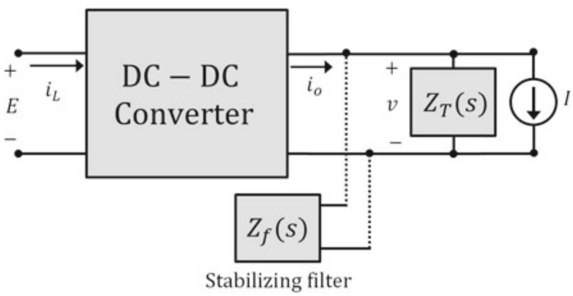


Fig. 11.7 Energy distribution network with a stabilizing filter

The first step in our procedure is to obtain image representations $w = M_k\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) z_k$, $i = 1, \ldots, N$, describing each mode as in Proposition 11.12, and exemplified in Example 11.13. Similarly, we model the corresponding gluing conditions and compute reinitialization maps as in Definition 11.7.

The second step in our procedure is the setting up of a system of matrix inequalities corresponding to the conditions of Theorem 11.17. To make explicit the linear dependence on the parameters of Z_{int}, in the following we write $M_k(s)$ and their corresponding state maps $X_k(s)$, respectively, as $M_{k,\widetilde{n},\widetilde{d}}(s)$, and $X_{k,\widetilde{n},\widetilde{d}}(s)$, where $\widetilde{n}$, $\widetilde{d}$ are the coefficient matrices of the numerator and denominator of Z_{int}, that also involve the coefficients of the passive filter:

$$
\begin{aligned}
&\widetilde{M}_{k,\widetilde{n},\widetilde{d}}^{\top}\Phi\widetilde{M}_{k,\widetilde{n},\widetilde{d}} - \begin{bmatrix} 0_{m\times \mathrm{n}(\mathfrak{B}_k)} \\ \widetilde{X}_{k,\widetilde{n},\widetilde{d}}^{\top} \end{bmatrix} K_k \begin{bmatrix} \widetilde{X}_{k,\widetilde{n},\widetilde{d}} & 0_{\mathrm{n}(\mathfrak{B}_k)\times m} \end{bmatrix} \\
&\qquad - \begin{bmatrix} \widetilde{X}_{k,\widetilde{n},\widetilde{d}}^{\top} \\ 0_{m\times \mathrm{n}(\mathfrak{B}_k)} \end{bmatrix} K_k \begin{bmatrix} 0_{\mathrm{n}(\mathfrak{B}_k)\times m} & \widetilde{X}_{k,\widetilde{n},\widetilde{d}} \end{bmatrix} \geq 0\,, \quad k = 1, \ldots, N\,, \\
&K_k - L_{k\to j}^{\top} K_j L_{k\to j} \geq 0\,, k, j = 1, \ldots, N\,, k \neq j\,.
\end{aligned} \tag{11.15}
$$

The third step is to formalize the requirement that the filter is passive. Define

$$
\Phi' := \frac{1}{2}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\,, \quad M'(s) := \begin{bmatrix} p(s) \\ q(s) \end{bmatrix}\,, \quad X'(s) := \begin{bmatrix} 1 \\ s \\ \vdots \\ s^{\deg(p)-1} \end{bmatrix}\,, \tag{11.16}
$$

and denote the coefficient matrices of M' and X' by $\widetilde{M'}_{\widetilde{p},\widetilde{q}}$ and $\widetilde{X}'$, respectively. With these positions, it follows from the positive-real lemma that $\frac{q}{p}$ is positive-real if and only if there exists $K' = K'^{\top} \in \mathbb{R}^{\deg(p)\times\deg(p)}$ such that

$$
\begin{aligned}
&\widetilde{M}_{\widetilde{p},\widetilde{q}}'^{\top}\Phi'\widetilde{M}_{\widetilde{p},\widetilde{q}}' - \begin{bmatrix} 0_{1\times\deg(p)} \\ \widetilde{X}'^{\top} \end{bmatrix} K' \begin{bmatrix} \widetilde{X}' & 0_{\deg(p)\times 1} \end{bmatrix} \\
&\qquad - \begin{bmatrix} \widetilde{X}'^{\top} \\ 0_{1\times\deg(p)} \end{bmatrix} K' \begin{bmatrix} 0_{\deg(p)\times 1} & \widetilde{X}' \end{bmatrix} \geq 0\,.
\end{aligned} \tag{11.17}
$$

If values of the parameters $\widetilde{p}$ and $\widetilde{q}$ exist such that the matrix inequalities (11.15), (11.17) are satisfied for some K_k, $k = 1, \ldots, N$ and K', then the interconnection of Fig. 11.7 is passive, and consequently i/o stable. Moreover, the filter $\frac{q}{p}$ can be implemented using only resistors, capacitors, inductors, and transformers (see [14]).

We close this section with a numerical example.

Example 11.18 (Cont'd from Example 11.13) We consider the implementation in Fig. 11.8, with $R_L = 0.1\,\Omega$; $L = 880\,\mu\mathrm{H}$; $C_1 = C_2 = 220\,\mu\mathrm{F}$; $R = 500\,\Omega$.

According to (11.12), we define the impedance of the filter $Z_f(s) := \frac{p(s)}{q(s)}$ with $p(s) = a_0 s + a1$ and $q(s) = 1$, for which the a-parameters will be computed.

We consider the total impedance as a constant power load, i.e., $Z_T(s) = -R_{CP}$ with $-R_{CP} = -300\,\Omega$. Considering (11.14), we obtain $n(s) = 300(a_0 + a_1 s)$ and $d(s) = 300 - a_0 - a_1 s$. We thus substitute $n\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$ and $d\left(\frac{\mathrm{d}}{\mathrm{d}t}\right)$ in the dynamic models computed in Example 11.13. Define state maps for each dynamical mode acting, respectively, on the latent variables z_1 and z_2 as

$$X_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) := \begin{bmatrix} 1 & 0 \\ 0 & n\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \\ 0 & d\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{bmatrix}, \quad X_2 := \begin{bmatrix} C_1\frac{\mathrm{d}}{\mathrm{d}t} & 0 \\ 1 & 0 \\ 0 & n\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) \end{bmatrix},$$

then for every $t_k \in \mathbb{T}_s$, the gluing conditions can be expressed as $X_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) z_2(t_k^+) = L_{1\to 2} X_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) z_1(t_k^-)$ and $X_1\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) z_1(t_k^+) = L_{2\to 1} X_2\left(\frac{\mathrm{d}}{\mathrm{d}t}\right) z_2(t_k^-)$, where

$$L_{1\to 2} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad L_{2\to 1} := \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{C_1}{C_1+C_2} & \frac{C_2}{C_1+C_2} & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

We now solve simultaneously the bilinear matrix inequalities (11.15) and (11.17) using standard solvers such as `Yalmip`. We thus obtain a solution $a_0 = 377$, $a_1 = 293 \times 10^{-6}$, $b_2 = 377$. Finally, the realization of the filter with impedance $Z_f(s) = 293 \times 10^{-6} s + 377$ is shown in Fig. 11.9. □

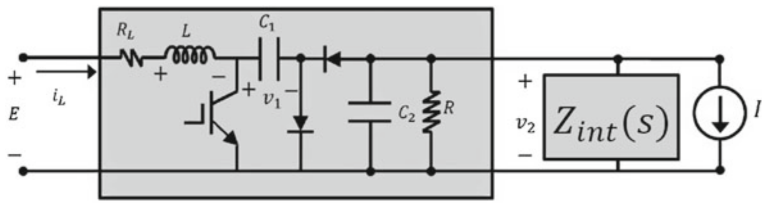
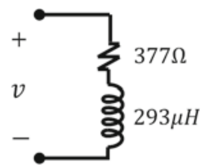


Fig. 11.8 Stable interconnection of a DC–DC converter with a passive filter and a constant power load

Fig. 11.9 Realization of the stabilizing filter



11.7 Conclusions

We introduced a modeling approach for energy distribution networks based on the switched linear differential framework in [12]. We also introduce the concept of passive SLDS and we study its relevance in the study of networks, deriving a stabilization method for switching power converters feeding potential destabilizers such as constant power loads. We have shown that elements of behavioral system theory such as linear differential behaviors and quadratic differential forms provide suitable tools to study the network using higher order differential models obtained directly from first principles.

References

1. M. Cespedes, L. Xing, J. Sun, Constant-power load system stabilization by passive damping. IEEE Trans. Power Electron. **26**(7), 1832–1836 (2011)
2. Z. Dongyan, A. Pietkiewicz, S. Cuk, A three-switch high-voltage converter. IEEE Trans. Power Electron. **14**(1), 177–183 (1999)
3. W. Du, J. Zhang, Y. Zhang, Z. Qian, Stability criterion for cascaded system with constant power load. IEEE Trans. Power Electron. **28**(4), 1843–1851 (2013)
4. R. Frasca, M.K. Çamlibel, I.C. Goknar, L. Iannelli, F. Vasca, Linear passive networks with ideal switches: consistent initial conditions and state discontinuities. IEEE Trans. Circuits Syst. I, Reg. Papers **57**(12), 3138–3151 (2010)
5. I. Goknar, Conservation of energy at initial time for passive RLCM network. IEEE Trans. Circuit Theory **19**(4), 365–367 (1972)
6. S. Krishna, *An Introduction to Modelling of Power System Components* (Springer, New York, 2014)
7. D. Liberzon, *Switching in Systems and Control* (Birkhauser, Boston, 2003)
8. H. Lin, P.J. Antsaklis, Stability and stabilizability of switched linear systems: a survey of recent results. IEEE Trans. Autom. Control **54**(2), 308–322 (2009)
9. J.C. Mayo-Maldonado, P. Rapisarda, On Positive-realness and Stability of Switched Linear Differential Systems, in *Proceedings of the 53nd IEEE CDC* (2013)
10. J.C. Mayo-Maldonado, P. Rapisarda, Dissipative switched linear differential systems. IEEE Trans. Autom. Control (2014), submitted
11. J.C. Mayo-Maldonado, P. Rapisarda, Modelling of Switching Dynamics in Electrical Systems, in *Proceedings of the Mathematical Theory of Networks and Systems Symposium (MTNS)* (2014)
12. J.C. Mayo-Maldonado, P. Rapisarda, P. Rocha, Stability of switched linear differential systems. IEEE Trans. Autom. Control **59**(8), 2038–2051 (2014)
13. P.J. Mosterman, G. Biswas, A theory of discontinuities in physical system models. J. Franklin Inst. **335**(3), 401–439 (1998)
14. R. Newcomb, *Linear Multiport Synthesis* (McGraw-Hill, New York, 1966)
15. L. Peng, B. Lehman, Performance prediction of dc-dc converters with impedances as loads. IEEE Trans. Power Electron. **19**(1), 201–209 (2004)
16. J.W. Polderman, J.C. Willems, *Introduction to Mathematical System Theory: A Behavioral Approach* (Springer, Berlin, 1997)
17. A.M. Rahimi, A. Emadi, Active damping in DC/DC power electronic converters: a novel method to overcome the problems of constant power loads. IEEE Trans. Industr. Electron. **56**(5), 1428–1439 (2009)

18. P. Rapisarda, P. Rocha, Positive Realness and Lyapunov Functions. in *Proceedings of the 20th MTNS*, Melbourne, Australia, 2012
19. P. Rocha, J.C. Willems, P. Rapisarda, D. Napp, On the Stability of Switched Behavioral Systems, in *Proceedings of the 50th IEEE CDC-ECC* (2011), pp. 1534–1538
20. J. Sun, Impedance-based stability criterion for grid-connected inverters. IEEE Trans. Power Electron. **26**(11), 3075–3078 (2011)
21. N. Tleis, *Power Systems Modelling and Fault Analysis: Theory and Practice* (Newnes, Woburn, 2007)
22. J. Tolsa, M. Salichs, Analysis of linear networks with inconsistent initial conditions. IEEE Trans. Circuits Syst. I, Fundam. Theory **40**(12), 885–894 (1993)
23. S. Trenn, in *Switched Differential Algebraic Equations*. eds. by F. Vasca, L. Iannelli. Dynamics and Control of Switched Electronic Systems. Chapter 6 (Springer, New York, 2012)
24. H.L. Trentelman, J.C. Willems, Every storage function is a state function. Syst. Control Lett. **32**(5), 249–259 (1997) (System and Control Theory in the Behavioral Framework)
25. S. Vesti, T. Suntio, J.A. Oliver, R. Prieto, J.A. Cobos, Impedance-based stability and transient-performance assessment applying maximum peak criteria. IEEE Trans. Power Electron. **28**(5), 2099–2104 (2013)
26. J.C. Willems, K. Takaba, Dissipativity and stability of interconnections. Int. J. Robust Nonlin. **17**(5–6), 563–586 (2007)
27. J.C. Willems, H.L. Trentelman, On quadratic differential forms. SIAM J. Control Optim. **36**(5), 1703–1749 (1998)
28. J.C. Willems, H.L. Trentelman, Synthesis of dissipative systems using quadratic differential forms: part I. IEEE Trans. Autom. Control **47**(1), 53–69 (2002)
29. O. Wing, *Classical Circuit Theory* (Springer, New York, 2008)
30. F. Xiaogang, L. Jinjun, F.C. Lee, Impedance specifications for stable DC distributed power systems. IEEE Trans. Power Electron. **17**(2), 157–162 (2002)
31. Y. Xinghuo, C. Cecati, T. Dillon, M.G. Simes, The new frontier of smart grids. IEEE Industr. Electron. Mag. **5**(3), 49–63 (2011)
32. Q.C. Zhong, T. Hornik, *Control of Power Inverters in Renewable Energy and Smart Grid Integration* (Wiley, New York, 2012)