Section 2: Vibrations of Strings

A. Transverse Vibrations

36. Modes of Vibration

A *string* can vibrate as a whole or divided into any number of equal segments, separated from one another by nodes of the vibrations. The only difference between these vibrations is that the unity which serves as a measure changes, because, when the string is divided into aliquot parts, each half, each third, etc. makes its motions as if it were a string by itself. The gravest sound is that made when the entire string vibrates and alternately forms the loops represented in Fig. 1 by *ACB* and *ADB*. When it is divided into two parts, one half is on one side of the rest position, while the other is on the opposite side, and the loops are as in Fig. 2, *ADCEB* and *AFCGB*; the sound is more acute by one octave than the fundamental. If the string is separated into three segments, the loops are alternatively placed, as marked in Fig. 5, in two different ways; the sound is more acute by a fifth than that of the second harmonic. If the string is divided into four segments (Fig. 4), the pitch of the sound is increased by a fourth. In general, all of the possible sounds are as the numbers of segments (or as the reciprocals of their lengths); the series will therefore be as the numbers 1, 2, 3, 4, etc. When the gravest is *do*, the series of possible sounds will be:

No. of segments Sounds	1 *do 1*	2 *do 2*	3 *sol 2*	4 *do 3*	5 *mi 3*	6 *sol 3*	7 *si*$^{\flat}$ *3−*	8 *do 4*	
	9 *re 4*	10 *mi 4*	11 *fa 4+*	12 *sol 4*	13 *la 4−*	14 *si 4−*	15 *si*$^{\flat}$ *4−*	16 *do 5*	etc.

In a string of unequal thickness, the vibrations are ordinarily very irregular, except for several special cases; for example, if the lengths of the segments are in inverse order to their diameters.

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37. On the Manner of Producing These Vibrations and Making Them Visible

To produce sound where the string is divided into aliquot segments, one must place a finger very lightly at a point where there is a node of vibration, and apply a violin bow approximately in the middle of the vibrating part. It is not necessary to press the node of vibration very hard, in order to prevent the transmission of the motion of one segment to another; the pressure of the bow should also be much smaller than that for the fundamental sound. The mode of division can be made visible by placing small bits of paper on different points of the string; those which are on the vibrating part will be struck by the vibrations and fall off; but those which are placed on the node of vibrations will remain stationary.

We owe this experiment *to Sauveur* (*Hist. et Mém. de l'Acad. de Paris*, 1701). Wallis (in *Algebra*, vol. 2, p. 466) mentioned sounds of aliquot segments as a discovery made by Noble and Pigot at Oxford, and communicated to him in 1676 by Narcissus Marsh.

The sounds of aliquot parts of a string on the violoncello and on the violin are sometimes used. These are known as *fluted sounds* or *harmonic sounds*. Use is also made of it in the case of an instrument with a single string, which is called a *marine trumpet*. The sounds of the *Aeolian harp* consist of similar vibrations, produced by an air current that acts on the strings in different ways. Ossian and the commentator on Homer, Eustathius, had already mentioned the sound of strings produced by the wind. A. Bale, in the house of a Captain Haas, had very long and very strong strings, exposed to the air, that yielded different sounds, especially during changes in intervals. In the *Annali di Chimica e storia natural*, (Pavia), vol. 18, 1800, similar observations were made by Gaetano Berrettari.

38. Coexistence of Several Vibrations

Several or all kinds of vibrations to which a string is susceptible can exist at the same time. In order to have an idea of the loops of the string, it is not necessary to apply a curve to a straight line, but to the curve which already exists at each moment by other vibrations. Figs. 5–8 represent several examples of similar curves. Sect. 9 of this Part will contain more instruction on this subject.

39. On the Curve Formed by a String in Its Transverse Vibrations

The opinions of geometricians differ on the nature of the curves into which a string can convert itself in transverse vibrations. Taylor, D. Bernoulli, and Giordano Riccati have found that these curves have the shape of a very elongated trochoid and that L signifies the length of the string, the ratio of the periphery of the circle to its diameter. And if one expresses the greater order at the middle of a segment of vibrations for the first kind of vibration by A, for the second by B, for the third by C, and so on, any abscissa by x and the number ordered to this abscissa by y for the fundamental type of vibrations, y is estimated to be $= A \sin \frac{\pi x}{L}$, for the second $y = B \sin \frac{2\pi x}{L}$, for the third $y = C \sin \frac{3\pi x}{L}$, etc. But Euler presumed that the wave is arbitrary and that it depends on the first impressions that are made on the string, in the manner that there will not always be continuity of the different segments of the curve, but that each vibrating part takes on the same curve as the other alternatively in the opposite sense. Lagrange has proposed the same opinion as Euler. D'Alembert also attributes still other curves to the strings, such as the trochoidal curves of Taylor, but he did not agree that the string could take on curves which do not conform to any law of continuity.

40. Laws of These Vibrations

If L expresses the length of the string, G the weight, P the tension (which can be expressed by a suspended weight), n the number of segments into which the string is divided and S the relative number of vibrations, or the sound of the string, S will be $= n\sqrt{\frac{P}{LG}}$. In strings that are made of the same material, if D expresses the diameter or thickness, G is $= D^2L$ and $S = n\sqrt{\frac{P}{L^2D^2}}$, or $\frac{\sqrt{P}}{LD}$. Consequently, homogeneous sounds (where n is the same) of strings of the same material will be:

1. When the thickness and the tension are the same, the sounds will be as the *reciprocal lengths* of the strings; that is why we can use a monochord for demonstration of the ratios of the sounds.
2. When the length and the tension are the same, m, the sounds will be as the reciprocal of the *diameters* (or *the square root of the weight*); in such a way that if, for example, the thickness of one string is to that of another as 1–2, the sound of the thicker string will be lower by an octave.

3. When the thickness and the length are the same, the sounds will be as *the square roots of the tension*. If one wishes, for example, for the sound of a string to differ from that of another by an octave, it is necessary that the tensions be as 1–4.

Differences in material have no affect on the determination of sound; a string of catgut and a string of any metal whatever will give the same sound if the length, the weight, and the tension are all the same. The duration of each vibration being reciprocal to the number of vibrations, it will be $\frac{1}{n}\sqrt{\frac{LG}{P}}$. The absolute number of vibrations that the string makes in a second of time can be found by comparing it to a seconds pendulum, where the duration of a vibration is expressed by π (or the ratio of the circumference of the circle to the diameter) multiplied by the square root of the length. The length of a seconds pendulum being *f*; a second, or the duration of an oscillation of the pendulum, will be at *t*, or to the duration of a vibration of the string as $\pi\sqrt{f}$ is to $\frac{1}{n}\sqrt{\frac{LG}{P}}$; *t* will be equal to $\frac{1}{\pi n}\sqrt{\frac{LG}{f\,\mathrm{P}}}$, and the number of vibrations that are made in a second $\pi n\sqrt{\frac{fP}{LG}}$.

41. Authors Consulted

To learn more about transverse vibrations of strings, research on the vibrations of strings can be found in the work of Brook Taylor (*Methodius incrementorum directa et inversa*, London, 1715) where one finds the best analytic research on the vibrations of strings; Johann Bernoulli (*de chordis vibrantibus*, *Comment. Acad. Petrop.,* vol. 3); Leonhard Euler (*Mém. de l'Acad. Berlin*, 1748, 1753, and 1765; *Nov. Comment. Acad. Petrop.*, vol. 9, 17, and 19; *Acta Ac. Petrop.*, 1779, p. 2; 1780, p. 2 and 1781, p. 1; *Mélanges de Philosophie et de Mathématiques de la Société de Turin*, vol. 3.); Daniel Bernoulli (*Mém. de l'Acad. Berlin*, 1753 and 1765; *Nov. Comment. Acad. Petrop.,* vol. 16.); J. Lagrange (*Mélanges de Philosophie et de Mathématiques de la Société de Turin*, vol. 1, 2, and 3); D'Alembert (*Mém. de l'Acad. Berlin*, 1747, 1750, and 1763; *Opuscul.* vols. 1 and 4); Giordano Riccati (*delle corde ovvero fibre elastiche*, Bologna, 1767); Matthew Young (*Enquiry into the principal Phenomena of sounds and musical strings*, Dublin, 1784); Zanotti (*de vi elastica*, in *Comment, Bonon.*, vol. 4).

42. A Particular Case: Where the Tone from a String Divided into Two Parts Is Lower than That of the Entire String

I will now add a unique phenomenon, in which a string, divided into two parts, produces a lower tone than that corresponding to the ordinary vibrations of the entire string. Hellwag, physician to the reigning Duke of Oldenburg at Eutin, having observed it, was kind enough to communicate it to me. If we place a support under the string, so that it is not fixed but lightly touched, and if we pinch the string in order that it strike vertically in the support, there will be some cases in which one will hear the striking as a perceptible sound lower than its fundamental, but very raucous and disagreeable because of the deformity of the vibrations. This sound can be called the "snoring sound" of the string.[1] If we apply the support to the middle of the string, the snoring sound is graver by a fifth than the ordinary sounds of the entire string. When the string, Fig. 9, is pulled from its rest position *pnq* toward *m* and released, it strikes the support *n* after one half of a vibration; the two halves continue their motions forming the curves *pkn* and *nfq*; then they return and, as soon as they reach the axis *pnq* the entire string makes a half **vibration toward *pmq* and another toward *pnq***, and so on. One therefore hears the shocks on the support in the sum of the following time intervals:

1. The half vibration of each half *pn* and *nq* toward *pkn* and *nfq*, one quarter of an ordinary vibration
2. The return of each half to the axis *pnq*, one quarter of a vibration
3. The motion of the entire string toward *pnq*, half a vibration
4. The return of the entire string to the axis, where it strikes the support, one half a vibration

The time interval, therefore, between two strikings of the support, $\frac{1}{4}+\frac{1}{4}+\frac{1}{2}+\frac{1}{2}=\frac{3}{2}$; it must be that the snoring sound is a fifth lower than the ordinary sound, confirming to the experiment. But, because of the motions of each half, there is always a mixture of the higher sound that belongs to these halves, and finally, when the shocks cease, the more acute sound continues for a bit. There are only two cases which find this sound perceptible, but much less distinct. If the string is divided in the same way into two parts, which are $\frac{2}{5}$ and $\frac{3}{5}$, the snoring sound is a semi-tone more acute than in the previous case. It seems to me that the ratio to ordinary sound is $\frac{18}{25}$ to 1. If the support divides the string into two parts that are $\frac{1}{3}$ and $\frac{2}{3}$, the snoring sound is lower by a ninth than the ordinary sound. The ratio is

[1] The "snoring sound" refers to the raucous and disagreeable sound caused by the string striking its support.—*GB*

therefore $\frac{2}{9}$ to 1. The effect was almost the same if the end where one plucks the string was different, or if the support was not placed exactly in the place mentioned.

B. Longitudinal Vibrations of Strings

43. Different Types of Longitudinal Vibrations

A longitudinal vibration consists of the contraction and expansion of a string, or its aliquot parts, moving alternately between fixed points or a vibrational node. In the simplest longitudinal motion, the entire string moves alternately toward one fixed point and then toward the other (Figs. 34a, b). The second kind of longitudinal motion is that in which the string is divided into two equal parts, which alternate between the vibrational node in the middle and the fixed points at the ends (Figs. 35a, b). In the third type of longitudinal vibrations, the motions of the parts are alternately as Figs. 36a, b, etc. The sounds together have the ratios as those of transverse vibrations, being as the numbers 1, 2, 3, 4, etc.; but there is no fixed relationship for the absolute pitch of the sound between these two types of motion because the laws are very different.

44. Manner of Producing Them

To produce these sounds, we must rub a vibrating part of the string longitudinally with a violin bow, which is held at a very acute angle, or with a finger, or with another flexible body to which one has applied rosin powder. For division of the string into aliquot parts, it is necessary to touch a node of the vibrations lightly at the same time.

45. Laws of These Vibrations

The laws of longitudinal vibrations differ altogether from those of the transverse vibrations. The only resemblance is that the sounds are in the inverse ratio to their lengths; but in longitudinal vibrations, the sound does not depend on the thickness of the string or on the tension, but only on the length and the type of material of which it is made. For example, a brass string gives a more acute sound by about a sixth than that of a catgut string. And the sound of a steel string surpasses the one of the brass string by about a fifth. To do the experiments, it is necessary to use strings of a considerable length, since these sounds are very acute. I even used strings that were 48 ft long. Sect. 5 of this Part will contain more information on longitudinal vibrations.