Robust Optimal Control of Dynamically Decoupled Systems via Distributed Feedbacks

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Abstract. We consider an optimal control problem for a large-scale dynamical system represented by a team of objects with linear time-varying decoupled dynamics subject to disturbances and coupling constraints. It is assumed that centralized control is impossible and a delay in the communication network between systems is present. An algorithm for distributed feedback control is proposed. The algorithm breaks the large scale optimal control problem into sub-problems optimizing only for the inputs of the associated system. Feasibility and suboptimality of distributed control for the overall system is established and relevant data to be exchanged between the systems is analyzed.

Keywords: Optimal control · Large-scale system · Multi-agent system · Distributed feedback · Uncertainty · Algorithm

1 Introduction

Control problems for interacting dynamical systems has received a significant attention over the recent years. This is motivated by permanent progress of control techniques and computing power that allow to tackle complex large-scale problems. In various applications centralized control of such systems is impractical or impossible due to, e.g., communication restrictions. Besides specific properties of the network are not adequately addressed by a general centralized control algorithm. In these cases distributed control techniques are employed.

Many approaches have been proposed for control of linear and nonlinear systems with coupled or decoupled dynamics within distributed model predictive control (DMPC) framework (see, e.g., [1] and the references therein). In particular, in [2] for a class of discrete-time systems with coupled linear time-invariant dynamics sufficient conditions for stability of the closed-loop using stability constraints and assuming one-step communication delay are given. In [3] a distributed control strategy is obtained by solving local min-max optimization problems that treat states of the neighboring systems as disturbances and therefore minimize the worst-case local performance. In [4] an iterative cooperating distributed algorithm for linear discrete-time systems interconnected by their inputs is presented that is equivalent to the centralized controller at the limit of iterations.

A. Plakhov et al. (Eds.): EmC-ONS 2014, CCIS 499, pp. 95–106, 2015.
DOI: 10.1007/978-3-319-20352-2_6

Other than stabilization cooperative tasks such as consensus and synchronization are handled, e.g., by a general DMPC framework reported in [5].

Most DMPC schemes, when defining the local optimal control problems for each system, do not take into account disturbances acting on the dynamical systems. The notable exception is [6] where linear time-invariant systems subject to coupling constraints and bounded disturbances are considered and a robust DMPC scheme is proposed that implies sequential solution of local optimal control problems at each step.

In this paper we consider an optimal control problem for continuous time systems with decoupled linear time-varying dynamics subject to unknown but bounded disturbances. The systems are coupled by state constraints. The control objective on a finite control interval is to minimize the worst-case value of a given terminal penalty, though, as will be shown below, other types of performance index can be handled within the proposed approach. The idea is to incorporate distributed feedback control design into the classical optimal control problem, obtaining suboptimality of some degree and guaranteeing satisfaction of the hard constraints at each time instant. The approach presented here follows the ideas of [7,8], where dynamically coupled systems are considered. In contrast to the latter here we are able to prove recursive feasibility and suboptimality of distributed inputs.

The overall paper is structured as follows. In Sect. 2 we outline the mathematical problem formulation and the control objective for a set of linear time-varying systems subject to coupling constraints and unknown but bounded disturbances. Section 3 reviews centralized solution to this problem that guarantees robust constraint satisfaction and minimizes the worst-case performance for all possible disturbances. Section 4 presents an algorithm for robust distributed control. Feasibility and suboptimality of the distributed inputs with respect to the overall system behavior as well as communication data and requirements for the distributed algorithm are analyzed. The effectiveness of the proposed scheme is demonstrated in Sect. 5 with an illustrative example comparing performance of centralized and distributed controls. Section 6 provides some conclusions.

2 Problem Formulation

We consider a team of q continuous-time linear time-varying systems with decoupled dynamics

$$\dot{x}_i = A_i(t)x_i + B_i(t)u_i + M_i(t)w_i,\ x_i(t_0) = x_{i0},\ t \in [t_0, t_f], \tag{1}$$

where $x_i = x_i(t) \in \mathbb{R}^{n_i}$ denotes the state of the i-th system at time t, $u_i = u_i(t) \in U_i \subset \mathbb{R}^{r_i}$ denotes the bounded control input to system i and $w_i = w_i(t) \in W_i \subset \mathbb{R}^{p_i}$ is the unknown piecewise continuous disturbance acting upon system i, $A_i(t) \in \mathbb{R}^{n_i \times n_i}$, $B_i(t) \in \mathbb{R}^{n_i \times r_i}$, $M_i(t) \in \mathbb{R}^{n_i \times p_i}$, $t \in [t_0, t_f]$, are piecewise continuous matrix functions, $i \in I = \{1, 2, \dots, q\}$. The input constraint set U_i and the disturbance set W_i are given convex polytopes containing the origin and independent across the systems.

The input u_i is a sampled-data control that changes its value at fixed sampling instants and is constant in between. Sampling instants are in the following denoted by τ, where $\tau \in T_h = \{t_0 + kh, k = \overline{0, N-1}\}$. Here h denotes the constant sampling time defined in terms of the discretization $N \in \mathbb{N}$ of the finite control interval $[t_0, t_f]$: $h = (t_f - t_0)/N$. Thus, the input u_i in (1) is given by:

$$u_i(t) \equiv u_i(\tau), \quad t \in [\tau, \tau + h[, \ \tau \in T_h,$$

where $u_i(\tau)$ depends on the current state of system i and some exchanged information from other systems.

At time instants $s \in T_c \subseteq T_h \cup t_f$ the team is subject to coupling state constraints

$$\sum\nolimits_{k \in K^l} H_k^l(s) x_k(s) \leq \alpha^l(s), \ l \in L = \{1, \ldots, l^*\}, \tag{2}$$

where $K^l \subseteq I$, $|K^l| \geq 2$; $H_k^l(s) \in \mathbb{R}^{m^l \times n_k}$, $H_k^l(s) \neq 0$ for all $k \in K^l$; $\alpha^l(s) \in \mathbb{R}^{m^l}$.

The control objective is to minimize the worst-case value of a linear terminal penalty

$$\max_{w_k, k \in I} \sum\nolimits_{k \in I} c_k^T x_k(t_f), \tag{3}$$

while satisfying the decoupled input and coupling state constraints (2).

In the following we have to distinguish between the variables used in the optimal control problems for predictions and the real system/plant variables. To this end the latter will be denoted by a superscript $*$. Thus, u_i^* and x_i^* denote the input and the state trajectory which realize in a particular control process, and w_i^* denotes a realized unknown disturbance. It is assumed that at all time instants $\tau \in T_h \cup t_f$ the current state $x_i^*(\tau)$ is completely measured by system i.

3 Centralized Optimal Control

In this section we review some results from [9] on centralized optimal feedback control of dynamical systems subject to bounded disturbances that are needed in the later sections.

When centralized control is implemented, one central controller chooses the inputs for all systems (1) in the team, treating the problem under consideration as a large-scale optimal control problem without taking into account its decoupled dynamics or a specific interconnection structure. The overall system dynamics is then represented in concatenated form

$$\dot{x} = A(t)x + B(t)u + M(t)w, \quad x(t_0) = x_0, \tag{4}$$

where $x(t) \in \mathbb{R}^n$, $u(t) \in \mathbb{R}^r$ and $w(t) \in \mathbb{R}^p$ with $n = \sum_{k \in I} n_k$, $r = \sum_{k \in I} r_k$, $p = \sum_{k \in I} p_k$, denote the state, the input and the disturbance of the overall system at time t, i.e. $x(t) = (x_1(t), \ldots, x_q(t))$, $u(t) = (u_1(t), \ldots, u_q(t))$, $w(t) =$

$(w_1(t),\ldots,w_q(t))$; $A(t) \in \mathbb{R}^{n\times n}$, $B(t) \in \mathbb{R}^{n\times r}$, $M(t) \in \mathbb{R}^{n\times p}$, $t \in [t_0, t_f]$, are the corresponding block diagonal matrices.

In this paper, both in centralized and distributed control schemes, we use only one type of feedback that can be defined for uncertain systems (see e.g. [9,10]), namely the open-loop optimal feedback, which refers to the fact that a feedback strategy is obtained via repetitive (for every time instant $\tau \in T_h$) solution of an open-loop min-max optimal control problem subject to a shrinking control interval $[\tau, t_f]$ and a current overall state $x^*(\tau)$.

The open-loop min-max optimal control problem (centralized) that is solved at time τ is denoted by $\mathcal{P}(\tau)$ and has the form

$$\mathcal{P}(\tau): \qquad J^0(\tau) = \min_u \max_w c^T x(t_f), \tag{5}$$

subject to

$$\dot{x} = A(t)x + B(t)u + M(t)w, \; x(\tau) = x^*(\tau),$$

$$H(s)x(s) \le \alpha(s), \; s \in T_c(\tau) = T_c \cap [\tau, t_f], \;\; u(t) \in U, \; w(t) \in W, \; t \in [\tau, t_f],$$

where $c = (c_1, ..., c_q)$; $H(s) = \begin{pmatrix} H_k^l(s),\, k \in I \\ l \in L \end{pmatrix} \in \mathbb{R}^{m\times n}$, $m = \sum_{l\in L} m^l$, with $H_k^l(s)$ being zero for $k \notin K^l$; $\alpha(s) = (\alpha^l(s), l \in L)$; $U = U_1 \times \ldots \times U_q$, $W = W_1 \times \ldots \times W_q$.

The optimal open-loop control of $\mathcal{P}(\tau)$ is an input $u^0(t|\tau)$, $t \in [\tau, t_f]$, such that for every realization of the disturbance $w(t) \in W$, $t \in [\tau, t_f]$, the state constraints are satisfied and the worst-case cost is minimized.

Assumption 1. Problem (5) is feasible for $\tau = t_0$.

Under Assumption 1 problem $\mathcal{P}(\tau)$ is feasible for all $\tau \in T_h$ and the centralized optimal feedback control algorithm is specified as follows [9]:

Algorithm 1. (*centralized*)

(1) Set $\tau = t_0$, $x^*(\tau) = x_0$.
(2) Find a solution $u^0(t|\tau)$, $t \in [\tau, t_f]$, to the centralized problem $P(\tau)$.
(3) Apply input $u^*(t) \equiv u^*(\tau) = u^0(\tau|\tau)$, $t \in [\tau, \tau + h[$, to the overall system.
(4) Set $\tau := \tau + h$. If $\tau < t_f$ return to step 2, else stop.

Now we briefly review how the min-max problem $P(\tau)$ is solved. Following [9], problem (5) can be reduced to a deterministic optimal control problem for nominal system (i.e. system (4) without the disturbance term) which constraints are tightened to ensure robust feasibility of the inputs in (5).

Denote by $F(t) \in \mathbb{R}^{n\times n}$, $t \in [t_0, t_f]$, the fundamental matrix of the overall system (4): $\dot{F}(t) = A(t)F(t)$, $F(t_0) = I^n$, where $I^n \in \mathbb{R}^{n\times n}$ is an identity matrix.

For a given input $u(t)$, $t \in [\tau, t_f]$, and disturbance $w(t)$, $t \in [\tau, t_f]$, the overall output $y(s) = H(s)x(s)$ at time instant $s \in T_c(\tau)$, can be found as

$$y(s) = H(s)F(s)F^{-1}(\tau)x^*(\tau) + \int_\tau^s H(s)F(s)F^{-1}(t)[B(t)u(t) + M(t)w(t)]dt.$$

Introduce matrix functions $\Phi(s,t) \in R^{m\times n}$, $t \in [t_0, s]$, $k \in I$, such that $\Phi(s,t) = H(s)F(s)F^{-1}(t)$. Obviously, $\partial\Phi(s,t)/\partial t = -\Phi(s,t)A(t)$, $\Phi(s,s) = H(s)$. Then

$$y(s) = \Phi(s,\tau)x^*(\tau) + \int_\tau^s \Phi(s,t)[B(t)u(t) + M(t)w(t)]dt.$$

The input $u(t)$, $t \in [\tau, t_f]$, is feasible in (5) for every possible realization of the disturbance w, if and only if $y(s) \le \alpha(s)$ for all $s \in T_c(\tau)$ which translate into the inequalities

$$\Phi(s,\tau)x^*(\tau) + \int_\tau^s \Phi(s,t)B(t)u(t)dt + \gamma(s|\tau) \le \alpha(s),\ s \in T_c(\tau).$$

Here the first two terms are the output of the overall nominal system (4) and the term $\gamma(s|\tau) \in \mathbb{R}^m$, $s \in T_c(\tau)$, corresponds to the worst-case realization of the disturbances: $\gamma(s|\tau) = (\gamma_j(s|\tau), j = \overline{1,m})$,

$$\gamma_j(s|\tau) = \int_\tau^s \max_{w\in W} \phi_j(s,t)^T M(t)wdt,$$

where $\phi_j(s,t)^T$ is the j-th row of the matrix $\Phi(s,t)$.

Concluding, the optimal open-loop control $u^0(t|\tau)$, $t \in [\tau, t_f]$, of problem $\mathcal{P}(\tau)$ is obtained by the solution of the deterministic optimal control problem

$$\min_u c^T x(t_f), \tag{6}$$

subject to

$$\dot{x} = A(t)x + B(t)u,\ x(\tau) = x^*(\tau),$$
$$H(s)x(s) \le \alpha(s) - \gamma(s|\tau),\ s \in T_c(\tau),\quad u(t) \in U,\ t \in [\tau, t_f].$$

The resulting cost of problem $\mathcal{P}(\tau)$ is given by

$$J^0(\tau) = \gamma^0(\tau) + c^T x^0(t_f|\tau) = \gamma^0(\tau) + \phi^0(\tau)^T x^*(\tau) + \int_\tau^{t_f} \phi^0(t)^T B(t)u^0(t|\tau)dt,$$

where $x^0(t|\tau)$, $t \in [\tau, t_f]$, is the optimal overall trajectory of (6) and $\gamma^0(\tau) = \int_\tau^{t_f} \max_{w\in W} \phi^0(t)^T M(t)wdt$, $\phi^0(t)^T = c^T F(t_f)F^{-1}(t)$, $t \in [t_0, t_f]$.

4 Distributed Optimal Control

In this section an algorithm for distributed optimal feedback control of a team of systems (1) is developed. Each system predicts its future control inputs on the base of its own current state $x_i^*(\tau)$ and some information received from neighboring systems, where neighbors are defined by the coupling constraints. It is assumed that there is a communication delay equal to the sampling time h. A centralized controller described in Sect. 3 is employed offline at initialization stage and is not available for any online computations.

4.1 Local Optimal Control Problem

To achieve control of systems (1) in a distributed fashion we associate an optimal control problem $\mathcal{P}_i(\tau)$ with each system i, minimizing over only local inputs u_i subject to local and coupling constraints. To formulate such a local optimal control problem we first define the interconnection topology for the multi-agent system under consideration.

Systems i and j are coupled by the constraints and are called neighbors if they enter the same constraint in (2). Denote by L_i all indices of the constraints (2) containing a term for system i, i.e. $L_i = \{l \in L : i \in K^l\}$. Then $N_i = \cup_{l \in L_i} K^l \setminus \{i\}$ is a set of indices of all neighbors of system i. Note that the interconnection topology here is time-invariant. It is assumed that system i can communicate only to its neighbors $k \in N_i$. The information that is exchanged over the communication network will be specified in Sect. 4.2.

In the following the $u_i^d(\cdot|\tau) = (u_i^d(t|\tau),\ t \in [\tau, t_f])$ denotes the distributed input predicted by system i at time τ, i.e. the optimal open-loop control of local problem $\mathcal{P}_i(\tau)$. Concatenated distributed input $u^d(\cdot|\tau) = (u_k^d(\cdot|\tau), k \in I)$ will be also referred to as the optimal distributed open-loop control. The corresponding state trajectory of the nominal system (1) with the initial state $x_i(\tau) = x_i^*(\tau)$ is denoted by $x_i^d(\cdot|\tau) = (x_i^d(t|\tau), t \in [\tau, t_f])$. Furthermore, $y_i^l(s|\tau) = H_i^l(s)x_i^d(s|\tau)$, $s \in T_c(\tau)$, $l \in L_i$, denote the outputs of system i predicted at time τ. The overall distributed output corresponding to the l-th constraint (2) at time instant $s \in T_c$ is $y^l(s|\tau) = \sum_{k \in K^l} y_k^l(s|\tau)$.

Following [8], define the open-loop min-max optimal control problem $\mathcal{P}_i(\tau)$ for system i at time instant $\tau \in T_h \setminus t_0$:

$$\mathcal{P}_i(\tau): \qquad J_i(\tau) = \min_{u_i} \max_{w_i} \sum_{k \in I} c_k^T x_k(t_f),$$

subject to

$$\begin{gathered}
\dot{x}_i = A_i(t)x_i + B_i(t)u_i + M_i(t)w_i,\ x_i(\tau) = x_i^*(\tau), \\
\dot{x}_k = A_k(t)x_k + B_k(t)u_k^d(t|\tau - h),\ x_k(\tau) = x_k^d(\tau|\tau - h),\ k \in N_i, \\
\sum_{k \in K^l} H_k^l(s)x_k(s) \le \alpha_i^l(s|\tau),\ s \in T_c(\tau), l \in L_i, \\
u_i(t) \in U_i,\ w_i(t) \in W_i,\ t \in [\tau, t_f].
\end{gathered} \tag{7}$$

Here the input u_i of the i-th system is the optimization variable, and the inputs u_k of systems $k \in I_i$ are held as fixed parameters equal to their distributed inputs $u_k^d(\cdot|\tau - h)$ predicted at the previous time $\tau - h$. Thus, system i assumes that its neighbors keep controls predicted at time $\tau - h$ also for the current time τ and besides they follow nominal trajectories, i.e. $w_k(t) \equiv 0$, $t \in [\tau - h, t_f]$, $k \in N_i$. Then their predicted states $x_k^d(\tau|\tau - h)$ are used in $\mathcal{P}_i(\tau)$ as initial states at time instant τ. The initial state of system i is its current state $x_i^*(\tau)$.

In $\mathcal{P}_i(\tau)$ the coupling state constraints have a modified right hand side

$$\alpha_i^l(s|\tau) = y^l(s|\tau - h) + \Omega_i^l(s|\tau)[\alpha^l - y^l(s|\tau - h)],\ l \in L_i, \tag{8}$$

where $\Omega_i^l(\tau) \in \mathbb{R}^{m^l \times m^l}$ is a diagonal matrix of weight parameters for system i in constraint l, $\sum_{k \in K^l} \Omega_k^l(s|\tau) = I^{m^l}$ for all $s \in T_c(\tau)$, $l \in L$.

The idea behind constraints modifications (8) is to guarantee feasibility of the optimal distributed open-loop control $u^d(\cdot|\tau)$ with respect to the overall system. This feasibility result is proved in Sect. 4.3.

In can be seen from (7) and (8) that in order to construct problem $\mathcal{P}_i(\tau)$ system i needs to know the dynamics of the neighboring systems, their predicted states $x_k^d(\tau|\tau-h)$, whole input trajectories $u_k^d(\cdot|\tau-h)$, and the outputs $y_k^l(s|\tau-h)$, $s \in T_c(\tau)$, $k \in N_i$. However, in contrast to [8], where systems with coupled dynamics are studied, some information here is abundant. In the next section we derive an equivalent formulation of problem $\mathcal{P}_i(\tau)$ that compared to (7) has a reduced dimension and requires less data from other systems.

4.2 An Equivalent Formulation of $\mathcal{P}_i(\tau)$ and the Algorithm

Since in problem $\mathcal{P}_i(\tau)$ dynamics of systems $k \in N_i$ is deterministic and doesn't depend on input u_i, their trajectories $x_k^d(\cdot|\tau-h)$ are known parameters. They can be excluded from the dynamics (7) and embedded into the modified state constraints. The latter take the form

$$H_i^l(s)x_i(s) + \sum_{k \in K^l \setminus i} H_k^l(s)x_k^d(s|\tau-h) =$$

$$= H_i^l(s)x_i(s) + \sum_{k \in K^l \setminus i} y_k^l(s|\tau-h) \leq \alpha_i^l(\tau),\ s \in T_c(\tau),\ l \in L_i.$$

Denote $\bar{\alpha}_i^l(\tau) = \alpha_i^l(\tau) - \sum_{k \in K^l \setminus i} y_k^l(s|\tau-h) = y_i^l(s|\tau-h) + \Omega_i^l(s|\tau)(\alpha^l(s) - y^l(s|\tau-h))$ to obtain the new state constraints

$$H_i^l(s)x_i(s) \leq \bar{\alpha}_i^l(s|\tau),\ s \in T_c(\tau),\ l \in L_i.$$

The resulting local optimal control problem for system i at time $\tau \in T_h \setminus t_0$ is

$$\mathcal{P}_i^d(\tau): \qquad J_i^d(\tau) = \min_{u_i} \max_{w_i} c_i^T x_i(t_f),$$

subject to

$$\dot{x}_i = A_i(t)x_i + B_i(t)u_i + M_i(t)w_i,\ x_i(\tau) = x_i^*(\tau),$$
$$H_i^l(s)x_i(s) \leq \bar{\alpha}_i^l(s|\tau),\ s \in T_c(\tau),\ l \in L_i,$$
$$u_i(t) \in U_i,\ w_i(t) \in W_i,\ t \in [\tau, t_f].$$

Similarly to Sect. 3, problem $\mathcal{P}_i^d(\tau)$ can be reduced to a deterministic problem for the nominal system (1) with the tightened constraints:

$$\min_{u_i} c_i^T x_i(t_f),$$

subject to

$$\dot{x}_i = A_i(t)x_i + B_i(t)u_i,\ x_i(\tau) = x_i^*(\tau),$$

$$H_i^l(s)x_i(s) \le \bar{\alpha}_i^l(s|\tau) - \gamma_i^l(s|\tau),\ s \in T_c(\tau),\ l \in L_i,\ \ u_i(t) \in U_i,\ t \in [\tau, t_f].$$

where $\gamma_i^l(s|\tau) = (\gamma_{i,j}^l(s|\tau), j = \overline{1, m^l})$: $\gamma_{i,j}^l(s|\tau) = \int_\tau^s \max_{w_i \in W_i} \phi_{i,j}^l(s,t)^T M_i(t) w_i dt$, and $\phi_{i,j}^l(s,t)^T$ is the j-th row of the matrix $\Phi_i^l(s,t) = H_i^l(s)F_i(s)F_i^{-1}(t)$, and $F_i(t) \in \mathbb{R}^{n_i \times n_i}$, $t \in [t_0, t_f]$, denotes the fundamental matrix of system (1): $\dot{F}_i(t) = A_i(t)F_i(t)$, $F_i(t_0) = I^{n_i}$.

The cost of problem $\mathcal{P}_i^d(\tau)$ is equal to

$$J_i^d(\tau) = \gamma_i^0(\tau) + \phi_i^0(\tau)^T x_i^*(\tau) + \int_\tau^{t_f} \phi_i^0(t)^T B_i(t) u_i^d(t|\tau) dt, \tag{9}$$

where $\gamma_i^0(\tau) = \int_\tau^{t_f} \max_{w_i \in W_i} \phi_i^0(t)^T M_i(t) w_i dt$, $\phi_i^0(t)^T = c_i^T F_i(t_f) F_i^{-1}(t)$.

Analyzing problem $\mathcal{P}_i^d(\tau)$ we conclude that system i needs the following information at time $\tau \in T_h \setminus t_0$:

(1) its complete current state $x_i^*(\tau)$;
(2) from all neighboring systems $k \in N_i$ delayed by h the outputs $y_k^l(s|\tau - h) = H_k^l(s)x_k^d(s|\tau - h)$, $l \in L_i$, corresponding to the distributed input $u_k^d(\cdot|\tau - h)$ predicted at time $\tau - h$.

The distributed optimal feedback control algorithm is specified as follows:

Algorithm 2. (*distributed*)

(1) Set $\tau = t_0$, $x^*(\tau) = x_0$.
(2) Find a solution $u^0(\cdot|t_0)$ to the centralized problem $P(t_0)$ and set $u_i^d(t|t_0) = u_i^0(t|t_0)$, $t \in [t_0, t_f]$, $i \in I$.

For each system $i \in I$ (in parallel):

(3) Apply input $u_i^*(t) \equiv u_i^*(\tau) = u_i^d(\tau|\tau)$, $t \in [\tau, \tau + h[$.
(4) Communicate the outputs $y_i^l(\tau)$ to neighbors $k \in K^l \setminus i$, $l \in L_i$.
(5) Set $\tau := \tau + h$. If $\tau = t_f$ stop.
(6) Solve problem $\mathcal{P}_i^d(\tau)$ to find $u_i^d(t|\tau)$, $t \in [\tau, t_f]$. Return to step 3.

4.3 Properties of Distributed Control

The important properties that a distributed control scheme should possess are feasibility of distributed inputs with respect to coupling constraints (2), suboptimality of the distributed input with respect to a centralized one, and a recursive feasibility of the optimal control problems $\mathcal{P}_i^d(\tau)$ solved at each sampling time $\tau \in T_h \setminus t_0$. In this section we prove these properties for Algorithm 2.

Theorem 1. *For any $\tau \in T_h \setminus t_0$ the optimal distributed open-loop control $u^d(\cdot|\tau) = (u_k(\cdot|\tau), k \in I)$ is a feasible input in the centralized optimal control problem $\mathcal{P}(\tau)$.*

Proof. Let instant τ be fixed. We have to prove that the overall distributed trajectory $x^d(\cdot|\tau)$, corresponding to the input $u^d(\cdot|\tau)$ satisfies the inequalities

$$H(s)x^d(s|\tau) \le \alpha - \gamma(s|\tau), s \in T_c(\tau), \tag{10}$$

as defined by problem (6) equivalent to $\mathcal{P}(\tau)$.

Consider problem $\mathcal{P}_k^d(\tau)$. Its optimal open-loop control $u_k^d(\cdot|\tau)$ is feasible, therefore the corresponding optimal trajectory $x_k^d(\cdot|\tau)$ satisfies the inequalities

$$H_k^l(s)x_k^d(s|\tau) \leq \bar{\alpha}_k^l(s|\tau) - \gamma_k^l(s|\tau),\ s \in T_c(\tau),\ l \in L_i. \tag{11}$$

Summing (11) over all $k \in K^l$ for a fixed $l \in L$ and taking into account that

$$\sum_{k\in K^l} \bar{\alpha}_k^l(s|\tau) = \sum_{k\in K^l} \left[y_k^l(s|\tau-h) + \Omega_k^l(s|\tau)(\alpha^l(s) - y^l(s|\tau-h))\right] = \alpha^l(s),$$

$$\gamma(s|\tau) = (\gamma^l(s|\tau), l \in L),\ \ \gamma^l(s|\tau) = \sum_{i\in K^l} \gamma_i^l(s|\tau),\ l \in L,$$

obtain

$$\sum_{k\in K^l} H_k^l(s)x_k^d(s|\tau) \leq \alpha^l(s) - \gamma^l(s|\tau),\ l \in L.$$

The latter in concatenated form is given by (10), therefore $u^d(\cdot|\tau)$ is feasible in the centralized problem $\mathcal{P}(\tau)$. □

To guarantee recursive feasibility of the distributed algorithm, i.e. existence of solution of problem $\mathcal{P}_i^d(\tau)$ for all $\tau \in T_h \setminus t_0$, we assume:

Assumption 2. $\Omega_i^l(s|\tau)$ is such that $\gamma_i^l(s|\tau-h) = \Omega_i^l(s|\tau)\gamma^l(s|\tau-h)$.

The following theorem implies that if a centralized solution for $\tau = t_0$ exists and the weights are properly chosen, then Algorithm 2 can indeed be implemented for distributed feedback control.

Theorem 2. *Under Assumptions 1,2 problem $\mathcal{P}_i^d(\tau)$ is feasible for all $\tau \in T_h \setminus t_0$.*

Proof. To prove the assertion it is sufficient to show that the distributed input $u_i^d(\cdot|\tau-h)$, predicted by system i at time $\tau-h$, is feasible in problem $\mathcal{P}_i^d(\tau)$. Then according to optimal control existence theorems there exists a solution $u_i^d(\cdot|\tau)$ to problem $\mathcal{P}_i^d(\tau)$.

For the trajectory $x_i(\cdot|\tau)$ of nominal system (1) corresponding to the control $u_i^d(\cdot|\tau-h)$ the following is true

$$H_i^l(s)x_i(s|\tau) = \Phi_i^l(s,\tau)x_i^*(\tau) + \int_\tau^s \Phi_i^l(s,t)B_i(t)u_i^d(t|\tau-h)dt = y_i^l(s|\tau-h) +$$

$$+\Phi_i^l(s,\tau)(x_i^*(\tau) - x_i^d(\tau|\tau-h)) = y_i^l(s|\tau-h) + \int_{\tau-h}^\tau \Phi_i^l(s,t)M_i(t)w_i^*(t)dt,$$

where, due to Assumption 2,

$$\int_{\tau-h}^\tau \Phi_i^l(s,t)M_i(t)w_i^*(t)dt \leq \gamma_i^l(s|\tau-h) - \gamma_i^l(s|\tau) = \Omega_i^l(s|\tau)\gamma^l(s|\tau-h) - \gamma_i^l(s|\tau).$$

Theorem 1 implies that

$$y^l(s|\tau-h) = \sum_{k\in K^l} H_k^l(s)x_k^d(s|\tau-h) \leq \alpha^l(s) - \gamma^l(s|\tau-h),$$

therefore $\gamma^l(s|\tau-h) \leq \alpha^l(s) - y^l(s|\tau-h)$.

Concluding,

$$H_i^l(s)x_i(s|\tau) \le y_i^l(s|\tau-h) + \Omega_i^l(s|\tau)\gamma^l(s|\tau-h) - \gamma_i^l(s|\tau) \le$$

$$\le y_i^l(s|\tau-h) + \Omega_i^l(s|\tau)[\alpha^l(s) - y^l(s|\tau-h)] - \gamma_i(s|\tau) = \bar{\alpha}_i^l(s|\tau) - \gamma_i^l(s|\tau).$$

Thus, $u_i^d(\cdot|\tau-h)$ satisfies the constraints of $\mathcal{P}_i^d(\tau)$ and the latter is feasible. □

Note that the feasibility results do not use optimality of distributed inputs and therefore hold for any type of performance index, not only for (3). Its linearity is, however, important for deriving the suboptimality properties of the distributed scheme.

Since according to Theorem 1 the optimal distributed open-loop control $u^d(\cdot|\tau)$ is a feasible input in problem $\mathcal{P}(\tau)$, one can calculate its resulting worst-case cost (3) as $\sum_{k\in I} J_k^d(\tau)$, where $J_i^d(\tau)$ is the optimal value of the performance index of problem $\mathcal{P}_i^d(\tau)$ as defined by (9).

The following theorem asserts that the cost, corresponding to the optimal distributed open-loop control $u^d(\cdot|\tau)$, is a nonincreasing function of τ.

Theorem 3. *Under Assumptions 1,2 the inequalities hold*

$$J^0(\tau) \le \sum\nolimits_{k\in I} J_k^d(\tau) \le \sum\nolimits_{k\in I} J_k^d(\tau-h) \le J^0(t_0).$$

Proof. The first inequality is a consequence of the fact, that $u^d(\cdot|\tau)$ is feasible, but not necessarily optimal in $\mathcal{P}(\tau)$.

The second inequality is obtained via the following arguments. Since, according to Theorem 2, $u_k^d(\cdot|\tau-h)$ is feasible in problem $\mathcal{P}_k^d(\tau)$, its resulting cost is not less that the optimal value $J_k^d(\tau)$:

$$J_k^d(\tau) \le \phi_k^0(\tau)^T x_k^*(\tau) + \int_\tau^{t_f} \phi_k^0(t)^T B_k(t) u_k^d(t|\tau-h) + \gamma_k^0(\tau).$$

Summing over all $k \in I$, obtain

$$\sum_{k\in I} J_k^d(\tau) \le \phi^0(\tau)^T x^*(\tau) + \int_\tau^{t_f} \phi^0(t)^T B(t) u^d(t|\tau-h) + \gamma^0(\tau)$$

$$= \phi^0(\tau)^T[x^*(\tau) - x^d(\tau|\tau-h)] + \sum_{k\in I} J_k^d(\tau-h) - \gamma^0(\tau-h) + \gamma^0(\tau) \le \sum_{k\in I} J_k^d(\tau-h).$$

The third inequality results from initialization of the distributed algorithm with the centralized optimal open-loop control $u^0(\cdot|t_0)$. □

An important consequence of Theorem 3 is that the performance of the overall system under the distributed feedback tends to centralized closed-loop performance under open-loop optimal feedback when $\max_{w\in W} ||w|| \to 0$ for all $i \in I$ and, as a result, $J^0(\tau) \to J^0(t_0)$. Therefore, for small disturbances the distributed scheme produces a suboptimal feedback.

5 Example

As an illustrative example consider an optimal control of five identical point masses moving in the plane under disturbances and coupled only by the constraints. The dynamics of the systems is given by the equations

$$m\ddot{x}_{i,k} + k_1\dot{x}_{i,k} = u_{i,k} + w_{i,k},\ \ k = 1, 2,$$

where $i \in I = \{1, \ldots, 5\}$ and all systems have same parameters: $m = 1$, $k_1 = 1$, $k_2 = 2$. For all $i \in I$ and $t \geq 0$ the input $u_i = (u_{i,1}, u_{i,2})$ has to satisfy a local constraint of the form $||u_i(t)||_1 \leq 1$ and the disturbance $w_i = (w_{i,1}, w_{i,2})$ is bounded: $||w_i(t)||_\infty \leq w^*$, where $w^* = 0.05$.

The interconnection topology is given by a circle, i.e. every system $i = 2, 3, 4$ has systems $i + 1$ and $i - 1$ as their neighbors ($N_i = \{i - 1, i + 1\}$) and the first and the last systems are coupled to their closest neighbor and each other: $N_1 = \{2, 5\}$, $N_5 = \{4, 1\}$.

The control objective is to drive in finite time $t_f = 10$ all systems closer to their neighbors

$$|x_{i,k}(t_f) - x_{j,k}(t_f)| \leq 1, k = 1, 2,\ \ j \in N_i, i \in I,$$

while maximizing the worst-case terminal velocity along the vertical axes:

$$\max_{u_i} \min_{w_i}\ \dot{x}_{i,2}(t_f),\ \ i \in I.$$

Figure 1 shows simulation results when applying the distributed Algorithm 2 in comparison with centralized optimal control by Algorithm 1. In the simulation presented at time $t = 0$ all masses were stationary at different positions and for $t \in [0, t_f]$ subjected to the following constant disturbances: $w_1(t) = w^*(-0.5, -1)^T$, $w_2(t) = -w_1(t)$, $w_3(t) = w^*(0.5, 0.5)^T$, $w_4(t) = w^*(0, 1)^T$, $w_5(t) = w^*(0.5, -1)^T$. The weights in (8) were all equal to 1/2.

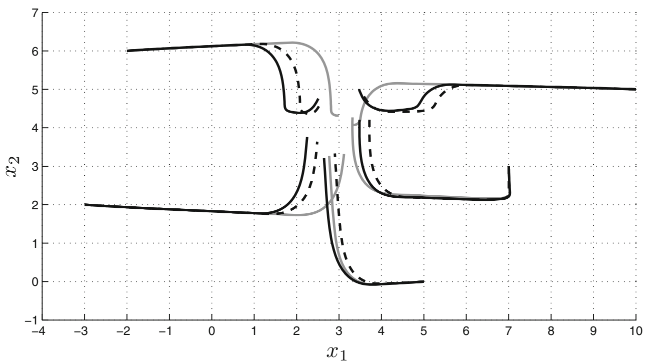

Fig. 1. State trajectories under centralized (solid), distributed (dash) and open-loop (grey) inputs

It can be seen from Fig. 1 that distributed control recovers the behavior of the centralized controller. The performance index of the centralized open-loop optimal feedback was 2.39263225 while the one of the distributed feedback was 2.36410018, which constitutes a loss in performance of only 1.1925 per cent. In other simulations for dynamically coupled [8] and decoupled systems the percentage error was not over five per cent.

For the reference the disturbed optimal open-loop trajectories calculated at the initial time $t = 0$ and used for initialization of the distributed algorithm are also presented in Fig. 1. It can be seen that the distributed trajectories can deviate quite far from the initial centralized plan.

6 Conclusions

This paper presents a robust distributed control scheme for optimal control of linear dynamically decoupled systems with coupling constraints. The key advantages of the algorithm are (1) parallel solutions of local optimization problems without interactions during iterations, (2) small amount of communication data, (3) robust constraint satisfaction, (4) less conservatism due to an assumption of the nominal and not worst-case performance of the neighbor systems. Future research will focus on obtaining suboptimality estimates for robust distributed feedbacks and development of the scheme for weakly interconnected systems.

Acknowledgments. This work is partially supported by the Belarusian Republican Foundation for Fundamental Research (grant Φ14MC-005).

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