Abel Symposia 10

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John Erik Fornæss
Marius Irgens
Erlend Fornæss Wold *Editors*

Complex Geometry and Dynamics

The Abel Symposium 2013

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Foreword

The Norwegian government established the Abel Prize in mathematics in 2002, and the first prize was awarded in 2003. In addition to honoring the great Norwegian mathematician Niels Henrik Abel by awarding an international prize for outstanding scientific work in the field of mathematics, the prize shall contribute toward raising the status of mathematics in society and stimulate the interest for science among school children and students. In keeping with this objective, the Niels Henrik Abel Board has decided to finance annual Abel Symposia. The topic of the Symposia may be selected broadly in the area of pure and applied mathematics. The Symposia should be at the highest international level and serve to build bridges between the national and international research communities. The Norwegian Mathematical Society is responsible for the events. It has also been decided that the contributions from these Symposia should be presented in a series of proceedings, and Springer Verlag has enthusiastically agreed to publish the series. The Niels Henrik Abel Board is confident that the series will be a valuable contribution to the mathematical literature.

Helge Holden
Chair of the Niels Henrik Abel Board
Trondheim, Norway

Preface

The theme of the Abel Symposium 2013 was *Complex Geometry*, and it was held at the Norwegian University of Science and Technology, Trondheim, during July 2–5. The event attracted 43 participants and featured presentations by 22 speakers. The scientific agenda primarily focused on geometric problems in Several Complex Variables and Complex Dynamics, including holomorphic laminations/foliations, the $\overline{\partial}$-equation, CR-geometry, pluripotential theory, and function theory. The aim of the Abel Symposium was to present the state of the art on these topics and to discuss future research directions. The speakers and titles were:

1. Bedford, E. *Automorphisms of blowups of projective space*
2. Berndtsson, B.*The openness problem and complex Brunn-Minkowski inequalities*
3. Błocki, Z. *Hörmander's* $\overline{\partial}$*-estimate, some generalizations and new applications*
4. Demailly, J.-P. *On the cohomology of pseudoeffective line bundles*
5. Dihn, T.-C. *Positive closed* (p,p)*-currents and applications in complex dynamics*
6. Ebenfelt, P. *Partial rigidity of degenerate CR-embeddings into spheres*
7. Fornstnerič, F. *Complex analysis and the Calabi-Yau problem*
8. Grushevsky, S. *Meromorphic differentials with real periods and the geometry of the moduli space of Riemann surfaces*
9. Huang, X. *Analyticity of the local hull of holomorphy for a codimension two real-submanifold in* $\mathbb{C}^n$
10. Kohn, J. *Weakly pseudoconvex CR manifolds*
11. McMullen, C. *Entropy and dynamics on complex surfaces*
12. Merker, J. *Siu-Yeung holomorphic sections of* $\mathrm{Sym}^m T_X^*$
13. Mok, N. *On the Zariski closure of an infinite number of totally geodesic subvarietes of* Ω / Γ
14. Nemirovski, S. *Topology and several complex variables*
15. Ohsawa, T. *Levi flats in Hopf surfaces*
16. Sibony, N. *Dynamics of foliations by Riemann surfaces*
17. Stensønes, B. *Real analytic domains and plurisubharmonic functions*
18. Ueda, T. *Semi-parabolic fixed points and their bifurcations in complex dimension 2*
19. Yau, S.-T. *Period integrals, counting curves, and mirror symmetry*

20. Yau, S. *Nonconstant CR morphisms between compact strongly pseudoconvex CR manifolds and etale covering between resolutions of isolated singularities*
21. Yeung, S.-K. *Complex hyperbolicity on the moduli of some higher-dimensional manifolds*
22. Zhou, X. *Some results on L^2-extension problem with optimal estimate*

The scientific committee consisted of John Erik Fornæss (NTNU), Marius Irgens (NTNU), Yum-Tong Siu (Harvard), Erlend F. Wold (Oslo), and Shing-Tung Yau (Harvard).

During the symposium, a dinner was held in honor of Yum-Tong Siu's 70th birthday. We would like to dedicate these proceedings to him.

The participants at the symposium were:

Eric Bedford
Bo Berndtsson
Zbigniew Błocki
Fusheng Deng
Jean-Pierre Demailly
Klas Diederich
Tien-Cuong Dinh
Peter Ebenfelt
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Shing-Tung Yau
Stephen S.T. Yau
Sai-Kee Yeung
Jian-Hua Zheng
Xiang-Yu Zhou
Minxian Zhu

We would like to thank the Norwegian Mathematical Society and the Nils Henrik Abel Memorial Fund for giving us the opportunity to host the Abel Symposium. We would also like to thank the administration at NTNU for their great help with the organizing and Ruth Allewelt at Springer for her help with preparing the proceedings.

Trondheim, Norway — John Erik Fornæss
Trondheim, Norway — Marius Irgens
Oslo, Norway — Erlend Fornæss Wold
May 15, 2015

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Pseudoautomorphisms with Invariant Curves

Eric Bedford, Jeffery Diller, and Kyounghee Kim

Nontrivial automorphisms of complex compact manifolds are typically rare and more typically non-existent. It is interesting to understand which manifolds admit automorphisms, how plentiful they are on any given manifold, and what further special properties distinguish a particular automorphism, or family of automorphisms. These problems have enjoyed much attention in the past fifteen years, motivated largely by work in complex dynamics (e.g. Cantat's thesis [7]). In this introduction, we give a quick account of some of this research, introducing in particular the more general category of pseudoautomorphisms, which occur more frequently in higher dimensions than automorphisms. The final aim of our paper is to present a concrete alternative approach to some recent existence results [22, Theorems 1.1 and 3.1] of Perroni and Zhang for pseudoautomorphisms with invariant curves on rational complex manifolds. Our methods lead to explicit formulas which are especially simple (see Theorems 4.6 and 6.4) when the pseudoautomorphisms correspond to the 'Coxeter element' in an infinite, finitely generated reflection group.

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The topological entropy of an automorphism is a non-negative number that measures the complexity of point orbits. 'Positive entropy' will serve as a precise and reasonable necessary condition for a map to be dynamically interesting. In complex dimension one, i.e. on closed Riemann surfaces, there are no automorphisms of positive entropy. In dimension two, Cantat [7] showed that only three types of complex surfaces can carry automorphisms of positive entropy: tori, $K3$ surfaces (or certain quotients), or rational surfaces. Automorphisms of tori are essentially linear. The cases of $K3$ and rational surfaces are much more interesting. Dynamics of automorphisms of $K3$ surfaces were studied in detail by Cantat [8]. McMullen [17] constructed examples which exhibit rotation domains (two dimensional 'Siegel disks'). The family of all $K3$ surfaces has dimension 20, and the maximum dimension of a continuous family of $K3$ surface automorphisms is even smaller. By contrast, there are continuous families of rational surface automorphisms which have arbitrarily large dimension [5].

It is known [11, 20] that rational complex surfaces X that carry automorphisms of positive entropy are in fact modifications (i.e. compositions of point blowups) $\pi : X \to \mathbf{P}^2$ of the complex projective plane $\mathbf{P}^2$. Thus a rational surface automorphism $F_X : X \to X$ with positive entropy descends via π to a birational 'map' $F : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$ which is locally biholomorphic at generic points but also has a finite union of exceptional curves that are contracted to points and conversely a finite collection $I(F)$ of indeterminate points which are (in a precise sense) each mapped to an algebraic curve. Since the group of all birational maps $F : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$ is quite large, this suggests trying to find automorphisms by looking at a promising family of plane birational maps and identifying those elements whose exceptional/indeterminate behavior can be eliminated by repeated blowup.

The papers [3] and [4] pursued exactly this idea for a well-chosen two parameter family of quadratic birational maps $F : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$ and found a countable set of parameters for which there exists a modification $\pi : X \to \mathbf{P}^2$ lifting F to an automorphism $F_X : X \to X$ with positive entropy. A generic quadratic birational map F on $\mathbf{P}^2$ has three exceptional lines Σ_j, $j = 0, 1, 2$ and three points of indeterminacy e_0, e_1, e_2. The maps F considered in [3] and [4] all share the further property that $F(\Sigma_0) = e_1$ and $F(\Sigma_1) = e_2$. If the parameter is chosen correctly, one can further arrange that $F^n(\Sigma_2) = e_0$ for some (minimal) $n > 0$. In this case, the map F lifts to an automorphism $F_X : X \to X$ of the rational surface $\pi : X \to \mathbf{P}^2$ obtained by blowing up e_1, e_2 and the points $F^j(\Sigma_2)$, $1 \leq j \leq n$. In effect, the exceptional curves and indeterminate points cancel each other out in the blown up space X. It was observed in [3] that F_X has finite order when $n \leq 6$, zero entropy when $n = 7$ and positive entropy for all $n \geq 8$. In fact, it is generally true that one must blow up at least ten points in $\mathbf{P}^2$ to arrive at a rational surface that admits an automorphism with positive entropy.

There is a curious dichotomy, discussed at length in [4], among the automorphisms discovered in [3]. For any fixed $n \geq 8$, there are finitely many maps in the family satisfying $F^n(\Sigma_2) = e_0$. Some, but not all, of these have the additional feature that they preserve a cubic curve $C \subset \mathbf{P}^2$ with a cusp singularity. This curve C, when it exists, contains all points blown up by the modification $\pi : X \to \mathbf{P}^2$, and the proper transform of C by π is a anticanonical curve with one cusp preserved by the automorphism F_X.

By *requiring* the existence of an invariant anticanonical curve, McMullen [18] showed how one can arrive at the examples in [3] synthetically. His approach is to begin with a plausible candidate $F_X^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ for the induced action of the automorphism on the picard group of the surface and then seek an a surface X and an automorphism $F_X : X \to X$ that 'realizes' F_X^*. Here $\mathrm{Pic}(X)$ is equivalent to $\mathbf{Z}^{1+N}$, where N is the number of blowups needed to create X, and if the identification between $\mathrm{Pic}(X)$ and $\mathbf{Z}^{1+N}$ is chosen appropriately, the intersection form on $\mathrm{Pic}(X)$ becomes the standard Lorentz metric on $\mathbf{Z}^{1+N}$. The natural candidates for the action F_X^* are isometries in a certain Coxeter group acting on $\mathbf{Z}^{1+N}$. If one seeks to realize F_X^* with an automorphism F_X that fixes an anticanonical curve C, then the action F_X^* on $\mathrm{Pic}(X)$ must restrict to a corresponding action $(F_X|_C)^*$ on $\mathrm{Pic}(C)$. It is well-known that the component of the identity in $\mathrm{Pic}(C)$ (consisting of divisors of degree 0) identifies naturally with the regular part of C (see e.g. the appendix to [10]). Using this identification and some theory of Coxeter groups, McMullen gave a sufficient condition for realization of F_X^* by an automorphism. In particular, the maps with invariant anticanonical curves discovered in [4] turn out to be realizations of the so-called Coxeter element in the isometry group of $\mathbf{Z}^{1+N}$.

The ideas in [4] and [18] were combined in later work. In particular [10] described all possible rational surface automorphisms with invariant anticanonical curves that are obtained as lifts of *quadratic* birational maps $F : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$. Uehara [24] showed that whether or not one can actually realize a plausible (in McMullen's sense) candidate action F_X^*, one can always construct a rational surface automorphism F_X that is closely related to F_X^* in the sense that the topological entropy of F_X has the correct value (the log of the spectral radius of F_X^*).

Constructing automorphisms on rational k-folds seems to be much more difficult when $k \geq 3$. At present, the only known examples with positive entropy appear in [21]. If one works only with rational k-folds obtained as finite compositions $\pi : X \to \mathbf{P}^k$ of point blowups over projective space, then a recent result of Truong [23] and Bayraktar-Cantat [1] says us that any automorphism of X must have zero entropy. So with this constraint on the manifold X, one must settle for constructing maps which are not quite automorphisms. A birational map $F_X : X \dashrightarrow X$ is a *pseudoautomorphism* [12] if there are sets $S_1, S_2 \subset X$ of codimension ≥ 2 such that $F : X - S_1 \to X - S_2$ is biregular. Equivalently, the image of a hypersurface under both F_X and F_X^{-1} is always a hypersurface and never a subvariety of codimension larger than one.

Having expanded the class of maps we seek, we also modify our criterion for determining which maps are dynamically interesting. Entropy is not an invariant of birational conjugacy (see Guedj [14]), so we employ a related but birationally invariant number, the (first) dynamical degree $\delta_1(F_X)$. For a pseudoautomorphism, $\delta_1(F_X)$ is just the spectral radius of the induced action $F_X^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$. When, as in dimension two, F_X is a genuine automorphism, celebrated results of Gromov [13] and Yomdin [25] imply that $\log \delta_1(F_X)$ is the entropy of F_X. In fact this equality holds generically [9] for birational maps of $\mathbf{P}^k$, but it is not known precisely when it fails. At any rate, the first dynamical degree is much easier to work with for pseudoautomorphisms, so it seems reasonable to substitute $\delta_1(F_X) > 1$ for positive entropy in our criterion for dynamically interesting maps.

Following McMullen's approach, Perroni and Zhang [22] recently showed that one can also construct pseudoautomorphisms $F_X : X \dashrightarrow X$ with $\delta_1(F_X) > 1$ on point blowups X of $\mathbf{P}^k$ (and more generally, on point blowups of products $\mathbf{P}^k \times \cdots \times \mathbf{P}^k$). As in McMullen, they begin with a candidate for the pullback action $F_X^* : \mathrm{Pic}(\mathbf{P}^k) \to \mathrm{Pic}(\mathbf{P}^k)$, chosen from a certain reflection group. They proceed by requiring F_X to preserve a (geometrically) rational curve C with a single cusp (discussed here in Sect. 1) and exploiting the group structure that C_{reg} inherits from its identification with $\mathrm{Pic}_0(C)$; and then they obtain a sufficient criterion for realizing the proposed action F_X^* with a pseudoautomorphism. Their criterion implies in particular that when F_X^* is the 'Coxeter element' in the reflection group, then F_X^* is realizable. One has $\delta_1(F_X^*) > 1$ in this case, so the resulting pseudoautomorphism is, by our standard, dynamically interesting.

Our goal in this article is to follow ideas from [10] in order to make the construction of Perroni and Zhang more explicit, arriving at precise and fairly simple formulas for the maps they discovered. The approach is as follows. We begin with what we call *basic Cremona maps* $F := S \circ J \circ T^{-1}$ on $\mathbf{P}^k$ (discussed at length in Sect. 2). Here $S, T \in PGL(k+1, \mathbf{C})$ are linear automorphisms, and $J : \mathbf{P}^k \dashrightarrow \mathbf{P}^k$ is the Cremona involution $J[x_0 : \cdots : x_k] = [x_0^{-1} : \cdots : x_k^{-1}]$. The exceptional hypersurfaces of J (and hence F) are the coordinate hyperplanes $\Sigma_j := \{x_j = 0\}$. The image $J(\Sigma_j)$ is the point $e_j = [0 : \cdots : 0 : 1 : 0 : \cdots : 0]$ obtained by intersecting all the other coordinate hyperplanes. Conversely, J is indeterminate along the codimension two set consisting of points where two or more coordinate hyperplanes meet.

The effect of the linear maps S and T is to vary the locations of the exceptional hypersurfaces and their images for F and F^{-1}. Specifically, the columns $S(\mathbf{e}_j)$ of S are images of exceptional hypersurfaces for F and the columns of T are the images of exceptional hypersurfaces of F^{-1}. One can use this freedom to try and arrange that there exist integers n_j and a permutation σ of $\{0, \ldots, n\}$ such that

$$F^{n_j-1}(S(\mathbf{e}_j)) = T(e_{\sigma(j)}), \qquad (*)$$

where none of the intermediate points $F^n(\mathbf{e}_j)$, $1 \leq n < n_j - 1$, lie in $I(F)$. Under these conditions it is straightforward to see that when one blows up the intermediate points, then F lifts to pseudoautomorphism $F_X : X \dashrightarrow X$. The data $\{(n_0, \ldots, n_k), \sigma\}$ was called *orbit data* in [2], where it was shown that the orbit data alone are sufficient to determine the dynamical degree $\delta_1(F_X)$. For given orbit data, the condition $(*)$ amounts to a polynomial system of equations satisfied by the entries of the matrices S and T. The simple appearance of this condition is deceptive, however, because it involves equations of many variables and polynomials of very high degree. Moreover, S and T are taken from the noncompact group $\mathrm{Aut}\,(\mathbf{P}^k)$, so one cannot reliably apply intersection theory even to guarantee existence of solutions.

Things become simpler if we require that F preserves the cuspidal curve $C \subset \mathbf{P}^k$ used in [22]. The main result of Sect. 2, and the first step in our construction of pseudoautomorphisms, is Theorem 2.6. It gives an explicit description of those basic Cremona maps that fix the cuspidal curve C in terms of the points $T(\mathbf{e}_j), S(\mathbf{e}_j)$ and the (affine) restriction $F|_C$ of F to C. In particular, preserving C is essentially equivalent to requiring that C_{reg} contains the points $S(\mathbf{e}_j), T(\mathbf{e}_j)$ and therefore also, all intermediate points in $(*)$. The orbits $F^{n_j-1}(S(\mathbf{e}_j))$ are now obtained by iterating an affine map inside a one dimensional set, so the equations imposed by the orbit data are much more tractable.

In Sects. 3 and 4, we therefore use Theorem 2.6 to derive a formula for F in the case where the orbit data corresponds to the action $F_X^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ of the Coxeter element. It turns out that the formulas are much simpler after a linear conjugation, letting $F = L \circ J$, where $L = T^{-1} \circ S$. Lemma 4.1 and Theorem 4.6 combine to give the entries of the matrix L and hence a formula for F.

The connection between rational surface automorphisms and Coxeter groups is more straightforward in dimension two because in that case the intersection product on a surface gives a quadratic form on $\mathrm{Pic}(X)$. The pullback action of any automorphism of X is then an isometry of $\mathrm{Pic}(X)$ that decomposes into a sequence of geometrically natural reflections. In higher dimensions, one needs an auxiliary identification between intersection product of divisors and an actual quadratic form. We describe this identification in Sect. 5. If $\pi : X \to \mathbf{P}^k$ is the blowup of N points $p_1, \ldots, p_N \in \mathbf{P}^k$, then the cohomology group $H^2(X; \mathbf{Z})$ is naturally isomorphic to $\mathrm{Pic}(X)$. A basis for either of these groups is given by the (pullback of the) class of a general hyperplane $E_0 \subset \mathbf{P}^k$, together with the exceptional blowup divisors E_j over p_j. It turns out that there is a unique element $\Phi \in H^{2k-4}(X; \mathbf{Z})$ such that the inner product

$$\langle D, D' \rangle := D \cdot D' \cdot \Phi$$

on classes $D, D' \in \mathrm{Pic}(X)$ is invariant by any pseudoautomorphism $F_X : X \dashrightarrow X$ corresponding to a basic Cremona map. Further, we can decompose the action $F_X^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X)$ into simple reflections as in the surface case. Finally, the maps we arrive at here can be seen to represent the ‘Coxeter element’, which is the isometry of $\mathrm{Pic}(X)$ obtained by composing all of the basic generating reflections.

For simplicity we have confined our attention to pseudoautomorphisms on modifications of $\mathbf{P}^k$ with invariant cuspidal curves, but our methods work more generally. In particular, as in Perroni-Zhang, we can replace $\mathbf{P}^k$ with a product of projective spaces $\mathbf{P}^k \times \cdots \times \mathbf{P}^k$. In Sect. 6 we sketch the main details of our method for *biprojective* spaces $\mathbf{P}^k \times \mathbf{P}^k$. It is worth noting that our methods work when the cuspidal curve, is replaced by various other curves whose regular points admit a group structure. For instance, in Sect. 6 we also say a few words about replacing the cuspidal curve with $k+1$ concurrent lines.

We do not know whether our methods can be used to produce pseudoautomorphisms with invariant curves $C \subset \mathbf{P}^k$ that lack a useful group structure. For instance, a union of $k+1$ mutually disjoint lines in $\mathbf{P}^k$ seems much harder to work with than a set of $k+1$ lines passing through a single point. On the other hand, a related construction of a pseudoautomorphism in $\mathbf{P}^3$ is given in [6]; this construction involves iterated blowups along an invariant curve quite different from the curves treated here.

1 From Cuspidal Curves...

By design, the pseudoautomorphisms we construct in this paper will all have a distinguished invariant curve. In this section we describe the curve and its key properties.

Let $[x_0, \ldots, x_k]$ be homogeneous coordinates on $\mathbf{P}^k$. An irreducible complex curve $C \subset \mathbf{P}^k$ is *rational* if there is a holomorphic parametrization $\psi : \mathbf{P}^1 \to C$. Using affine coordinates on the domain and homogeneous coordinates on the range of ψ, one can write $\psi(t) = [\psi_0(t) : \cdots : \psi_k(t)]$ where $\psi_j(t)$ are polynomials with no common factor. The *degree* of C is the number of intersections, counted with multiplicity, between C and any hyperplane $H \subset \mathbf{P}^k$ that does not contain C. This is a topological invariant, independent of the parametrization ψ. Nevertheless, one sees readily that $\deg C = \max_j \deg \psi_j$.

Let $C = \gamma(\mathbf{P}^1) \subset \mathbf{P}^k$ be the complex curve of degree $k+1$ given by $\gamma(t) = [1 : t : \cdots : t^{k-1} : t^{k+1}]$ for all $t \in \mathbf{C}$ and $\gamma(\infty) = [0 : \cdots : 0 : 1]$. For the sake of brevity we will henceforth refer to C (or any $\mathrm{Aut}(\mathbf{P}^k)$-equivalent curve) somewhat imprecisely as a *cuspidal curve*.

Proposition 1.1 *The curve C has a unique smooth inflection point at $\gamma(0)$, and a unique singularity at $\gamma(\infty)$, which is an ordinary cusp. Moreover,*

- *No hyperplane $H \subset \mathbf{P}^k$ contains C.*
- *No proper linear subspace $L \subset \mathbf{P}^k$ contains more than* $\dim L + 1$ *points of C_{reg}, counted with multiplicity.*
- *Any other degree $k+1$ curve C' that is not contained in a hyperplane and that has a cusp singularity is equal to $T(C)$ for some $T \in$* $\mathrm{Aut}(\mathbf{P}^k)$.

Note that in the second item, if $\dim L < k-1$ then the multiplicity of $L \cap C$ at p is defined to be the minimal (i.e. generic) multiplicity of $H \cap C$ at p among hyperplanes H containing L.

Proof The initial assertions follow from elementary computations. That C is not contained in a hyperplane follows from linear independence of the monomials $\{1, t, \ldots, t^{k-1}, t^{k+1}\}$. The second item now follows for hyperplanes $L = H \subset \mathbf{P}^k$ from the fact that $H \cdot C = k+1$.

Suppose instead that $L \subset \mathbf{P}^k$ is a linear subspace with codimension at least two. Let $S \subset C - L$ be a set of $k - \dim L - 2$ regular points of C. Then there exists a hyperplane H that contains L, S and the cusp of C. Since H does not contain C, and the cusp is a point of multiplicity (at least) two in $H \cap C$, we infer that

$$\#L \cap C_{reg} + \#S + 2 \le k + 1 = H \cdot C,$$

which implies the second item in the proposition.

It remains to establish the third item. If C' is another curve with degree $k+1$ and $p \in C'$ is a singular point, then we can choose a hyperplane $H \ni p$ that meets C' at k distinct points $p_1 = p, p_2, \dots, p_k$. Thus $H \cdot C' \ge (k-1) + \mu$, where μ is the multiplicity of C' at p. Necessarily then $\mu = 2$, and we have equality. In particular, no other point of C' is singular.

To see that C' is isomorphic to C via $\mathrm{Aut}\,(\mathbf{P}^k)$, we show inductively that there exists a flag $L_0 := \{p\} \subset L_1 \subset \cdots \subset L_{k-1} \subset \mathbf{P}^k$ of linear subspaces such that $\dim L_j = j$, and $L_j \cap C = \{p\}$ with multiplicity $j+2$. To this end, suppose we have a partial flag $L_0 \subset \cdots \subset L_{j-1}$ as described. The set of j dimensional subspaces containing L_{j-1} is naturally parametrized by $\mathbf{P}^{k-j+1}$ (i.e. each is determined by L_{j-1} and a choice of normal vector to L_{j-1} at p). Thus we have a map $L : C - \{p\} \to \mathbf{P}^{k-j+1}$ given by $q \mapsto L(q)$ where $L(q)$ is the unique j dimensional subspace containing q and L_{j-1}. Since L is meromorphic on C, and C is one dimensional, the map L extends holomorphically across p. For all $q \in C_{reg}$, we have that $L(q)$ contains p with multiplicity at least $j+1$ and q with multiplicity at least 1. Since $L(q) \to L(p)$ as $q \to p$, it follows that $L_j := L(p)$ contains p with multiplicity μ at least $j+2$. Now if $S \subset C$ is a generic set of $k-j-1$ points, then S and L_j span a hyperplane H, and

$$k + 1 = H \cdot C \ge \#S + \mu = k - j - 1 + \mu.$$

Hence $\mu = j + 2$ exactly as claimed.

Finally, we let $T \in \mathrm{Aut}\,(\mathbf{P}^k)$ be a linear transformation satisfying $T(L_j) = \{x_0 = \cdots = x_{k-j-1} = 0\}$. Let $\psi = (\psi_0, \dots, \psi_k) : \mathbf{P}^1 \to C'$ be a polynomial parametrization satisfying $\psi(0) = [0, \dots, 0, 1] = T(p)$. By hypothesis, $\deg \psi_j \le k+1$ for each j. Since L_j meets C' at p to order $j+2$ and p is a cusp, we have $\psi_{k-j-1}(t) = t^{j+2}\tilde{\psi}_{k-j-1}(t)$ for some polynomial $\tilde{\psi}_{k-j-1}$ such that $\tilde{\psi}_{k-j-1}(0) \neq 0$. Thus we can apply a further 'triangular' transformation $S \in \mathrm{Aut}\,(\mathbf{P}^k)$ so that $S \circ \psi(t) = [t^{k+1}, t^k, \dots, t^2, 1]$. Thus $t^{k+1} S \circ \psi(1/t) = \gamma(t)$; i.e. $S \circ T(C') = C$. □

For any $p \in C_{reg}$, we write $[p] \in \mathrm{Div}(C)$ to indicate the divisor of degree 1 supported at p. The following classical fact about C will be essential in what follows.

Proposition 1.2 (Group Law) *The set* $\mathrm{Pic}_0(C)$ *of linear equivalence classes of degree zero divisors on* C *is isomorphic (as an algebraic group) to* $\mathbf{C}$, *with isomorphism given by* $\mu = \sum n_j[\gamma(t_j)] \mapsto \sum n_j t_j$ *whenever* $\gamma(\infty) \notin \mathrm{supp}\,\mu$. *If, moreover,* $\mu = D|_C$ *is the restriction to* C *of a divisor* $D \subset \mathrm{Div}\,\mathbf{P}^k$, *then* $\sum n_j = \deg D$ *and* $\sum n_j t_j = 0$.

In particular, any divisor $\delta \in \mathrm{Div}(C)$ of degree zero is equivalent to $[\gamma(t)]-[\gamma(0)]$ for some $t \in \mathbf{C}$ and addition in $\mathrm{Pic}_0(C)$ is given by $\sum([\gamma(t_j)]-[\gamma(0)]) = [\gamma(\sum t_j)]-[\gamma(0)]$.

Proof Equivalence between $\mathrm{Pic}_0(C)$ and $\mathbf{C} = C_{reg}$ is classical. The point here is that a divisor $\delta \in \mathrm{Div}(C)$ is principal if and only if $\delta = \mathrm{Div}\, h$ for some rational $h : C \to \mathbf{P}^1$ satisfying (without loss of generality) $h(\gamma(\infty)) = 1$. But because $\gamma'(\infty)$ vanishes to first order, we see that $h \circ \gamma : \mathbf{P}^1 \to \mathbf{P}^1$ is a meromorphic function satisfying $h \circ \gamma(\infty) = 1$ and $(h \circ \gamma)'(\infty) = 0$. Writing $h \circ \gamma = P/Q$, one sees that this is equivalent to $\deg P = \deg Q$ and $\sum_{P(t)=0} t = \sum_{Q(t)=0} t$, where the roots are included with multiplicity in each sum.

Let $H \subset \mathbf{P}^k$ be the hyperplane defined by $x_k = 0$. Note that $H|_C = (k+1)[\gamma(0)]$. And if $D \in \mathrm{Div}(\mathbf{P}^k)$ is another effective divisor with degree d such that $\gamma(0) \notin$ *supp* D, then $D|_C = \sum_{j=1}^{(k+1)d} \gamma(t_j)$ for some $t_j \in C$. Since $D - dH$ is principal in $\mathbf{P}^k$, we have that $[\gamma(\sum t_j)] - [0] \sim (D - dH)|_C \sim 0$ in $\mathrm{Pic}_0(C)$. So $\sum t_j = 0$. □

Proposition 1.3 *Let $T \in \mathrm{Aut}\,(\mathbf{P}^k)$ be a linear transformation satisfying $T(C) = C$. Then $T = T_\lambda$ for some $\lambda \in \mathbf{C}^*$, where $T_\lambda : [x_0, \dots, x_k] \mapsto [x_0, \lambda x_1, \dots \lambda^{k-1} x_{k-1}, \lambda^{k+1} x_k]$*

Proof That $T_\lambda(C) = C$ is easily checked. On the other hand, if $T(C) = C$ for some $T \in \mathrm{Aut}\,(\mathbf{P}^k)$, then the facts that T is a biholomorphism mapping lines to lines and that C has a single cusp $\gamma(\infty)$ and a single inflection point $\gamma(0)$ imply that $T(\gamma(0)) = \gamma(0)$ and $T(\gamma(\infty)) = \gamma(\infty)$. So $T|_{C_{reg}}$ corresponds via γ to the linear map $t \mapsto \lambda t$ for some λ. That is, $T|_C = T_\lambda|_C$. Since C is not contained in a hyperplane, we infer $T = T_\lambda$. □

2 ...To Basic Cremona Maps...

If $X \to \mathbf{P}^k$ is a rational surface obtained by blowing up subvarieties of $\mathbf{P}^k$, and $F_X : X \to X$ is a pseudoautomorphism, then F_X descends to a birational map $F : \mathbf{P}^k \dashrightarrow \mathbf{P}^k$ on projective space. Our method for constructing pseudoautomorphisms reverses this observation. That is, we begin with an appropriate family of birational maps, and then we identify elements F of the family that lift by blowup to pseudoautomorphisms F_X. In this section we describe the family of birational maps we will use.

Let $J : \mathbf{P}^k \dashrightarrow \mathbf{P}^k$, given by $J[x_0, \dots, x_k] = x_0 \dots x_k \cdot [1/x_0 : \cdots : 1/x_k]$, be the standard Cremona involution of degree k on $\mathbf{P}^k$. Since J is a monomial map, the algebraic torus $(\mathbf{C}^*)^k$ is totally invariant by J, and J restricts to a biholomorphism (and group isomorphism) on this set. The complement of $(\mathbf{C}^*)^k$ is the union of all coordinate hyperplanes $\{x_j = 0\}$, and each of these is exceptional, contracted by J to a point. The indeterminacy set of J is the union of all linear subspaces obtained by intersecting two or more coordinate hyperplanes. For any non-empty set of indices $I \subset \{0, \dots, k\}$, we have $J(\bigcap_{i \in I}\{x_i = 0\}) = \bigcap_{i \notin I}\{x_i = 0\}$.

Given $S, T \in \mathrm{Aut}(\mathbf{P}^k)$, we will refer to $F := S \circ J \circ T^{-1}$ as a *basic Cremona map*. The exceptional set of F consists of the images $T(\{x_j = 0\})$ of the coordinate hyperplanes, these are mapped by F to the points $S(\mathbf{e}_j)$. The same is true, with T and S reversed, for F^{-1}. Note that if $\Lambda \in \mathrm{Aut}(\mathbf{P}^k)$ is any 'diagonal' transformation, preserving all coordinate hyperplanes, then $\Lambda \circ J = J \circ \Lambda^{-1}$. Hence replacing S and T by $S \circ \Lambda$ and $T \circ \Lambda$ does not change F. Otherwise, S and T are determined by F.

Note that for *any* birational transformation $F : X \to Y$ one has a natural pullback (or pushforward) map $F^* : \mathrm{Div}(Y) \to \mathrm{Div}(X)$ on divisors, and this descends to at map $F^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(X)$ on linear equivalence classes. On the other hand, we will employ the convention for any hypersurface $H \subset Y$ that $F^{-1}(H) = \overline{F^{-1}(H - I(F^{-1}))}$ is the set-theoretic *proper* transform of H. One has that $F^{-1}(H)$ is irreducible when H is irreducible and that, as a reduced divisor, $F^{-1}(H) = F^*H$ precisely when no hypersurface contracted by F has its image contained in H.

Proposition 2.1 *Suppose that $F = S \circ J \circ T^{-1}$ is a basic Cremona map. Let $H \subset \mathbf{P}^k$ be a hyperplane that does not contain any of the points $T(\mathbf{e}_j)$. Then $F^{-1}(H) = F^*H$ is an irreducible degree k hypersurface, and the multiplicity of F^*H at any point $p \in I(F) \cap F^{-1}(H)$ is $\ell + 1$ where ℓ is the number of exceptional hyperplanes $T(\{x_j = 0\})$ containing p.*

Proof This may be computed directly. □

We will say that F is *centered on C* if all points $T(\mathbf{e}_j)$ lie in C_{reg}.

Proposition 2.2 *Suppose that $F = S \circ J \circ T^{-1}$ is a basic Cremona transformation centered on C.*

- *The cusp $\gamma(\infty)$ of C lies outside the exceptional set of F*
- *For each j, there exists at most one point $p \notin \{T(\mathbf{e}_j) : 0 \le j \le k\}$ where C meets the exceptional hyperplane $T(\{x_j = 0\})$. If there is such a point p, it is not contained in any other exceptional hyperplane; and the intersection between C and $T(\{x_j = 0\})$ is transverse at p.*
- *If there is no such point p, then C is tangent to $T(\{x_j = 0\})$ at some point $T(\mathbf{e}_i)$, $i \neq j$. In this case C meets all other exceptional hyperplanes transversely at $T(\mathbf{e}_i)$.*

In particular $C \cap I(F) = \{T(\mathbf{e}_j) : 0 \le j \le k\}$.

Proof The curve C intersects the exceptional hyperplane $T(\{x_j = 0\})$ at $k+1$ points counting with multiplicity. By hypothesis these include the k points $T(\mathbf{e}_i)$ for all $i \neq j$, and none of these are the cusp of C. Consequently, either C is tangent to $T(\{x_j = 0\})$ at one of the points $T(\mathbf{e}_i)$, or C meets $T(\{x_j = 0\})$ transversely at exactly one other point p.

In particular, if p exists, then it cannot be the cusp of C. Moreover, p cannot lie in another exceptional hyperplane $T(\{x_\ell = 0\})$, $\ell \neq j$, because then the $k-2$-dimensional subspace $T(\{x_\ell = x_j = 0\})$ would contain k points: p and $T(\mathbf{e}_m)$ for all $m \neq \ell, j$, contradicting Proposition 1.1.

The same argument shows that when p does not exist and C is tangent to $T(\{x_j = 0\})$ at $T(\mathbf{e}_i)$, then C cannot be tangent to any other exceptional hyperplane at $T(\mathbf{e}_i)$. □

Proposition 2.2 gives us information about the image of C under F.

Corollary 2.3 *Let F be as in the previous proposition and let $C' = F(C)$. Then*

- *F maps the cusp of C to a cusp of C'; and*
- *$S(\mathbf{e}_j) \in C'$ for all j.*

In particular C' is not contained in any hyperplane.

Proof The first conclusion follows from the fact that F is regular near the cusp $\gamma(\infty)$.

If C meets an exceptional hyperplane $T(\{x_j = 0\})$ at a point p not contained in another coordinate hyperplane, then $F(p) = S(J(\{x_j = 0\})) = S(\mathbf{e}_j) \in C'$. Similarly, one computes readily that if C is tangent to $\{x_j = 0\}$ at $\mathbf{e}_i$, then $(F \circ \gamma)(t) = S(\mathbf{e}_j) \in C'$, where $\gamma(t) = T(\mathbf{e}_j)$.

The final assertion follows from independence of the points $S(\mathbf{e}_j)$. □

Proposition 2.4 *Let F and C' be as in Corollary 2.3. Then* $\deg C' = k + 1$.

Proof We have $\deg C' \geq k$, because otherwise the $k+1$ components of $J \circ T^{-1} \circ \gamma(t)$ would all be polynomials of degree smaller than k and therefore dependent. That is, C' would lie in a hyperplane, contrary to the previous proposition. Moreover, because C' has a cusp, C' is not a rational normal curve. Therefore $\deg C' \geq k + 1$.

For the reverse inequality, let $H \subset \mathbf{P}^k$ be a hyperplane that meets C' transversely at a set K of distinct regular points outside the exceptional hyperplanes for F^{-1}. Then $\deg C' = \#K = \#F^{-1}(K)$. Moreover, all points $F^{-1}(K)$ are regular for C, and none lies in an exceptional hyperplane of F. Thus we may use Proposition 2.1 to compute

$$F^*H|_{T^{-1}(C)} \geq \sum_{p \in F^{-1}(K)} [p] + (k-1) \sum [T(\mathbf{e}_j)],$$

and then infer

$$\deg C' = \#F^{-1}(K) \leq (k+1) \deg F^*H - (k-1)(k+1) = k + 1.$$

□

From Propositions 1.1 and 2.4 and Corollary 2.3, we immediately obtain

Corollary 2.5 *Let F be a basic Cremona map centered on C. Then $F(C)$ is a cuspidal curve, and F^{-1} is centered on $F(C)$.*

Let us say that a basic Cremona map F *properly fixes* C if F is centered on C and $F(C) = C$. Note that then F induces an automorphism on C which corresponds via γ to an affine transformation $t \mapsto \delta t + \tau$ for some $\delta \in \mathbf{C}^*$ and $\tau \in \mathbf{C}$.

We now arrive at the main result of this section. A consequence of the group law (Proposition 1.2), it gives us a good family of birational maps to work with when looking for pseudoautomorphisms.

Theorem 2.6 *Suppose* $\delta \in \mathbf{C}^*$ *and* $t_j^+ \in \mathbf{C}$, $0 \le j \le k$, *are distinct parameters satisfying* $\sum t_j^+ \neq 0$. *Then there exists a unique basic Cremona map* $F = S \circ J \circ T^{-1}$ *and* $\tau \in \mathbf{C}$ *such that*

- *F properly fixes C with $F|_C$ given by* $F(\gamma(t)) = \gamma(\delta t + \tau)$.
- $\gamma(t_j^+) = T(\mathbf{e}_j)$ *for each* $0 \le j \le k$.

Specifically,

- $\tau = \frac{k-1}{k+1}\delta \sum t_j^+$*; and*
- $S(\mathbf{e}_j) = \gamma\left(\delta t_j^+ - \frac{2\tau}{k-1}\right)$.

Note that the points $T(\mathbf{e}_j)$ and $S(\mathbf{e}_j)$ *almost* determine T and S (and therefore F). Below, it will be convenient to use invariance of the cusp to eliminate the remaining ambiguity.

Proof For existence of F, we infer from the condition $\sum t_j^+ \neq 0$ and Proposition 1.2 that the points $\gamma(t_j^+)$ are independent in $\mathbf{P}^k$. Therefore, there exists $T \in \mathrm{Aut}\,(\mathbf{P}^k)$ such that $T(\mathbf{e}_j) = \gamma(t_j^+)$. Then $J \circ T$ is a basic Cremona map centered on C, and $J \circ T(C)$ is an cuspidal curve. So by Proposition 1.1 there exists $S \in \mathrm{Aut}\,(\mathbf{P}^k)$ such that $F = S \circ J \circ T(C) = C$. The restriction $F|_C$ is given by $F(\gamma(t)) = \gamma(\alpha t + \tau)$ for some $\alpha \in \mathbf{C}^*$ and $\tau \in C$. Replacing S with $T_\lambda \circ S$, $\lambda := \delta/\alpha$, we may assume that $\alpha = \delta$. Thus F is the basic Cremona map we seek.

For uniqueness and the remaining assertions about F, suppose we are given $S, T \in \mathrm{Aut}\,(\mathbf{P}^k)$ such that $F = S \circ J \circ T^{-1}$ satisfies the given conditions. If $H \subset \mathbf{P}^k$ is a generic hyperplane, then H meets C in $k+1$ distinct points $\{\gamma(t_0), \dots, \gamma(t_k)\}$ such that $\sum t_j = 0$, and none of these points lies in the exceptional set of F. Hence $F^{-1}(\gamma(t_j)) = \gamma((t_j - \tau)/\delta)$. Also, $F^*H = F^{-1}(H)$ is a hypersurface of degree k that contains with multiplicity $k-1$ all points $T(\mathbf{e}_j) = \gamma(t_j^+)$ that are images of exceptional hyperplanes for $F^{-1}(H)$. This accounts for all $k(k+1)$ points of intersection between C and $F^{-1}(H)$. Using Proposition 1.2 we convert this to the following relationship among parameters

$$0 = \sum \frac{t_j - \tau}{\delta} + k \sum t_j^+ = -\frac{(k+1)\tau}{\delta} + (k-1) \sum t_j^+.$$

That is, $\tau = \frac{k-1}{k+1}\delta \sum t_j^+$.

Now consider an exceptional hyperplane $H = T(\{x_j = 0\})$ for F. We have $H \cap C = \{\gamma(t_i^+) : i \neq j\} \cup \{p_j\}$, where $p_j = \gamma(s_j)$ is as in Proposition 2.2. Thus on the one hand, $F(p_j) = S(\mathbf{e}_j) = \gamma(\delta s_j + \tau)$, and on the other hand Proposition 1.2

gives us that $s_j = -\sum_{i \neq j} t_i^+ = t_j^+ - \frac{k+1}{k-1}\frac{\tau}{\delta}$. So we arrive at

$$S(\mathbf{e}_j) = \gamma\left(\delta t_j^+ - \frac{2\tau}{k-1}\right).$$

To see that the above information completely determines F, let $p \in \mathbf{P}^k$ be a generic point and H be the hyperplane through p and $k-1$ points $\gamma(t_j^+)$. Then $F(H)$ is a hyperplane containing the corresponding set of $k-1$ points $p_j = S(\mathbf{e}_j)$. In addition $H \cap C$ contains two more points $\gamma(s_1), \gamma(s_2) \in C$, so $F(H)$ contains the images $\gamma(\delta s_1 + \tau), \gamma(\delta s_2 + \tau)$. The points p_j and $\gamma(\delta s_j + \tau)$ more than suffice to determine $F(H)$.

By varying the set of $k-1$ points $\gamma(t_j^+)$ used to determine H, we can find a collection of k hyperplanes that intersect uniquely at p and whose images are also hyperplanes intersecting uniquely at $F(p)$. Since the image hyperplanes are completely determined by the data t_j^+, δ, we see that $F(p)$ is uniquely determined by the same data. □

3 ...To Pseudoautomorphisms...

A birational map $F : X \to Y$ is a *pseudoautomorphism* if neither F nor F^{-1} contracts hypersurfaces. Equivalently, $F^{\pm 1}$ have trivial critical divisors. When combined with the following additional observation, Theorem 2.6 allows us to create many pseudoautomorphisms.

Proposition 3.1 *Let $F : \mathbf{P}^k \dashrightarrow \mathbf{P}^k$ be a basic Cremona map properly fixing C. Suppose there is a permutation $\sigma : \{0, 1, \dots, k\} \circlearrowleft$ and, for each point $S(\mathbf{e}_j) \in I(F^{-1})$, an integer $n_j \geq 1$ such that*

- *$F^j(S(\mathbf{e}_j)) \notin I(F)$ for $0 \leq j < n_j - 1$, and*
- *$F^{n_j-1}(S(\mathbf{e}_j)) = T(\mathbf{e}_{\sigma(j)}) \in I(F)$.*

Then F lifts to a pseudoautomorphism $F_X : X \dashrightarrow X$ on the complex manifold obtained by blowing up the points $S(\mathbf{e}_j), \dots, F^{n_j-1}(S(\mathbf{e}_j))$ for all $0 \leq j \leq k$. Moreover, F_X is biholomorphic on a neighborhood of the proper transform C_X of C.

Proof We can write $F = \pi_+ \circ \tilde{F} \circ \pi_-^{-1}$, where $\pi_- : \Gamma_- \to \mathbf{P}^k$ is the blowup of all points $T(\mathbf{e}_j) \in I(F^{-1}) \cap C$, $\pi_+ : \Gamma \to \mathbf{P}^k$ is the blowup of all points $T(\mathbf{e}_j) \in I(F) \cap C$, and $\tilde{F} : \Gamma_- \to \Gamma_+$ is a pseudoautomorphism. More precisely, $I(\tilde{F})$ consists of proper transforms $\pi_-^{-1} S(\{x_i = x_j = 0\})$, $i \neq j$, of intersections between distinct exceptional hyperplanes; $I(\tilde{F}^{-1})$ similarly consists of lifts $\pi_+^{-1} T(\{x_i = x_j = 0\})$ of intersections between F^{-1} exceptional hyperplanes; and $\tilde{F}$ restricts to a biholomorphism

$$\Gamma_- - I(\tilde{F}) \to \Gamma_+ - I(\tilde{F}^{-1}).$$

Proposition 2.2 therefore tells us that $I(\tilde{F}) \cap \pi_-^{-1}(C) = \emptyset = I(\tilde{F}^{-1})$ so that $\tilde{F}$ maps a neighborhood of $\pi_-^{-1}(C)$ biholomorphically onto a neighborhood of $\pi_+^{-1}(C)$.

Now if $\rho : X_1 \to X_0 := \mathbf{P}^k$ is the blowup of the image $S(\mathbf{e}_j)$ of some exceptional hyperplane $T(\{x_j = 0\})$, and $F_1 : X_1 \dashrightarrow X_1$ is the lift of $F_0 := F$ to X_1, then the proper transform of $T(\{x_j = 0\})$ is no longer exceptional for F_1. On the other hand the exceptional hypersurface $E = \rho^{-1}(S(\mathbf{e}_j))$ is either contracted by F_1 or, if $S(\mathbf{e}_j) \in I(F)$, mapped by F_1 onto the proper transform of an exceptional hyperplane $S(\{x_{\sigma(j)} = 0\})$. In any case, the map F_1 again admits a decomposition $F_1 = \pi_+ \circ \tilde{F}_1 \circ \pi_-^{-1}$ where $\pi_\pm$ are compositions of point blowups centered at distinct points in $\rho^{-1}(C)$, and $\tilde{F}_1$ is a pseudoautomorphism mapping a neighborhood of $(\rho \circ \pi_-)^{-1}(C)$ onto a neighborhood of $(\rho \circ \pi_+)^{-1}(C)$.

We may therefore proceed inductively blowing up the points $S(\mathbf{e}_j)$, $F(S(\mathbf{e}_j)), \ldots, F^{n_j-1}(S(\mathbf{e}_j))$ until the map F lifts to a new birational map with one less exceptional hypersurface than F. Moving on to another point $S(\mathbf{e}_j)$ and repeating, the hypothesis of this proposition allow us to finally lift F to a birational map $F_X : X \dashrightarrow X$ with no exceptional hypersurfaces. Clearly in the end, $F_X(C_X) = C_X$ and F_X is biholomorphic near C_X. □

We have a natural restriction map $tr : \mathrm{Div}(X) \to \mathrm{Div}(C_X) \cong \mathrm{Div}(C)$, obtained by intersecting divisors on X with C_X (and then pushing forward by $\pi|_{C_X} : C_X \to C$, which is an isomorphism). The map preserves linear equivalence and so descends to a quotient map $tr : \mathrm{Pic}(X) \to \mathrm{Pic}(C_X) \cong \mathrm{Pic}(C)$. Moreover, the final conclusion of Proposition 3.1 guarantees that pullback commutes with restriction: i.e. $(F|_C)^* \circ tr = tr \circ F_X^*$.

Corollary 3.2 *Let F, X be as in Proposition 3.1. Then the pullback operator $(F|_C)^*$ acts as follows.*

- $tr(K_X) \mapsto tr(K_X)$*;*
- $\mu \mapsto \delta\mu$ *for all* $\mu \in \mathrm{Pic}_0(C)$*, where* $\delta \in \mathbf{C}^*$ *is a root of the characteristic polynomial of* F_X^**.*

In particular, when $K_X \cdot C_X \neq 0$, this information completely characterizes $(F|_C)$.

Proof The first assertion follows from $F_X^* K_X = K_X$, which holds because F_X is a pseudoautomorphism.

For the second assertion, note that $(F|_C)^*$ restricts to the group automorphism on $\mathrm{Pic}_0(C) \cong \mathbf{C}$ given by multiplication by the constant $\delta \in \mathbf{C}^*$ from Theorem 2.6. To see that $P(\delta) = 0$, where P is the characteristic polynomial of F_X^*, let $C^\perp \subset \mathrm{Pic}(X)$ denote the subspace represented by divisors $D \in \mathrm{Div}(X)$ such that $D \cdot C_X = 0$; i.e. $tr(D)$ represents an element of $\mathrm{Pic}_0(C)$. Then $tr(F_X^* D) = \delta\, tr(D)$. Hence, since P has integer coefficients, we may infer that $P(\delta)\, tr(D) = tr(P(F_X^*)D) = 0$. We infer that $P(\delta) = 0$ as long as $tr(C^\perp)$ is non-trivial. But $tr(C^\perp)$ includes all classes of the form $(k+1)[p] - (k+1)[\gamma(0)] = tr((k+1)E - H)$, where $p \in I(F)$ is a point of indeterminacy, $E = \pi^{-1}(p)$ is the corresponding exceptional hypersurface, H is a hyperplane section, and γ is the parametrization of C introduced at the beginning of Sect. 1. Since $I(F)$ consists of $k+1$ distinct points, $k>0$ of these classes are non-trivial in $\mathrm{Pic}_0(C)$.

The final assertion follows from the facts that $\mathrm{Pic}(C)$ is generated by $\mathrm{Pic}_0(C)$ together with one (any) other class of degree one, and that $\deg tr(K_X) = K_X \cdot C_X$. □

Remark 3.3 We have $K_X \cdot C_X \neq 0$ in all the cases we consider. If N is the number of blowups comprising π, then one computes that $K_X \cdot C_X = N(k-1) - (k+1)^2$. Hence $K_X \cdot C_X$ vanishes only when $k=2, N=9$ or $k=3, N=8$ or $k=5, N=9$. The last of these does not occur since $N \geq k+1 = \#I(F)$.

Corollary 3.4 *Let F, X be as in Proposition 3.1. If $K_X \cdot C_X \neq 0$ and $F|_C$ is a translation, then F is linearly conjugate to the standard Cremona map J.*

Proof The condition that $F|_C$ is a translation is equivalent to the condition that $(F|_C)^*$ is the identity operator on $\mathrm{Pic}_0(C)$, i.e. $\delta = 1$ in Corollary 3.2. Since $tr(K_X)$ is also fixed by $(F|_C)^*$ it follows that $(F|_C)^*$ is the identity operator on all of $\mathrm{Pic}(C)$. Therefore $F|_C = \mathrm{id}$.

We infer that $n = n(p) = 1$ for all $p \in I(F^{-1})$ and therefore F^2 is an automorphism of $\mathbf{P}^k$, mapping each exceptional hyperplane for F to an exceptional hyperplane for F^{-1}. Since $F^2(C) = C$, Proposition 1.3 further implies that $F^2 = \mathrm{id}$. That is, F is linearly conjugate to J. □

4 . . . To Formulas

In this section, we derive a formula for a basic Cremona map $F : \mathbf{P}^k \dashrightarrow \mathbf{P}^k$ that fixes a cuspidal curve C and lifts via point blowups along C to a pseudoautomorphism $F_X : X \dashrightarrow X$ as in Proposition 3.1. Specifically, in order to arrive at a formula, we begin with $n > 0$ and suppose that $F = S \circ J \circ T^{-1}$ properly fixes C and satisfies the hypotheses of Proposition 3.1 as follows.

$$\begin{aligned} &S(\mathbf{e}_j) = T(\mathbf{e}_{j+1}) \quad \text{for } j = 0, 1, \dots, k-1, \\ &F^{n-1}(S(\mathbf{e}_k)) = T(\mathbf{e}_0), \\ &F^j(S(\mathbf{e}_k)) \notin I(F) \quad \text{for } 0 \leq j < n-1. \end{aligned} \tag{1}$$

That is, F has 'orbit data' $(n_0, \dots, n_{k-1}, n_k) = (1, \dots, 1, n)$ with cyclic permutation $\sigma : j \mapsto j+1 \mod k+1$.

We assume that $C = \{\gamma(t) = [1, t, \dots, t^{k-1}, t^{k+1}], t \in \mathbf{C} \cup \{\infty\}\}$ is in standard form and that $F|_C$ is not a translation. Note that by Corollary 3.4, the case of a translation can not lead to $\delta > 1$. Thus we may conjugate F by a linear map T_λ to arrange that $\gamma(1)$ is the unique fixed point of $F|_C$ different from the cusp $\gamma(\infty)$, i.e.

$$F|_C : \gamma(t) \mapsto \gamma(\delta(t-1)+1), \quad \text{and} \quad F|_C^{-1} : \gamma(t) \mapsto \gamma\left(\frac{1}{\delta}(t-1)+1\right).$$

Let us also suppose that $\delta \in \mathbf{C}$ is not a root of unity. As in Theorem 2.6, we let t_j^+ denote the parameters for points of indeterminacy of F:

$$\gamma(t_j^+) = T(e_j),\ j = 0, 1, \ldots, k.$$

Lemma 4.1 *The multiplier δ is a root of the polynomial*

$$(\delta^{n+k} - 1)(\delta^2 - 1) - \delta(\delta^{k+1} - 1)(\delta^{n-1} - 1), \tag{2}$$

and the parameters t_j^+ are given by

$$t_j^+ = \delta^j \frac{k+1}{k-1} \cdot \frac{\delta^2 - 1}{\delta(\delta^{k+1} - 1)} - \frac{2}{k-1}.$$

It is convenient that the orbit length n enters into the formula for t_j^+ only through the polynomial defining δ.

Proof Using the formula for $S(\mathbf{e}_j)$ from Theorem 2.6 and $\tau = 1 - \delta$, one rewrites the first condition in (1) as

$$t_j^+ + \frac{2}{k-1} = \delta\left(t_{j-1}^+ + \frac{2}{k-1}\right) = \cdots = \delta^j\left(t_0^+ + \frac{2}{k-1}\right)$$

for all $1 \leq j \leq k$. Similarly, one rewrites the third condition in (1) as

$$\delta^n\left(t_k^+ + \frac{2}{k-1}\right) = t_0^+ + \frac{2}{k-1} + (\delta^{n-1} - 1)\frac{k+1}{k-1}.$$

Comparing this second equation with the case $j = k$ in the first gives

$$t_0^+ + \frac{2}{k-1} = \frac{\delta^{n-1} - 1}{\delta^{n+k} - 1} \cdot \frac{k+1}{k-1}.$$

Hence

$$\sum_{j=0}^{k}\left(t_j^+ + \frac{2}{k-1}\right) = \left(\sum_{j=0}^{k}\delta^j\right)\left(t_0^+ + \frac{2}{k-1}\right) = \frac{\delta^{k+1} - 1}{\delta - 1} \cdot \frac{\delta^{n-1} - 1}{\delta^{n+k} - 1} \cdot \frac{k+1}{k-1}$$

On the other hand, the formula for τ from Theorem 2.6 gives an alternative expression

$$\sum_{j=0}^{k}\left(t_j^+ + \frac{2}{k-1}\right) = \frac{k+1}{k-1}\left(\frac{1}{\delta} + 1\right).$$

Comparing the last two equations, we arrive at

$$\frac{\delta^{n-1}-1}{\delta^{n+k}-1}=\frac{\delta^2-1}{\delta(\delta^{k+1}-1)}.$$

This gives us the defining polynomial for δ and allows us to revise our formula for t_0^+ to the desired equation

$$t_0^+ + \frac{2}{k-1} = \delta^j \frac{k+1}{k-1}\cdot\frac{\delta^2-1}{\delta(\delta^{k+1}-1)}.$$

The formulas for $t_j^+, j>0$ follow immediately. □

Let us consider the following set:

$$\Gamma = \{(2,4),(2,5),(2,6),(2,7),(3,4),(3,5),(4,4),(5,4)\}\cup\{(k,n): n\le 3, 2\le k\}$$

Corollary 4.2 *If $(k,n)\in\Gamma$, then the polynomial for δ given in (2) is a product of cyclotomic factors. If $(k,n)\notin\Gamma$, the polynomial (2) is the product of cyclotomic factors and a Salem polynomial.*

Proof After factoring out $(\delta-1)$, the equation of δ given in (2) can be written as

$$\begin{aligned}\chi(k,n) &:= \delta^n(\delta^{k+2}-\delta^{k+1}-\delta^k+1)+\delta^{k+2}-\delta^2-\delta+1\\ &= \delta^{k+2}(\delta^n-\delta^{n-1}-\delta^{n-2}+1)+\delta^n-\delta^2-\delta+1\end{aligned}$$

Thus we see that if the largest real root is bigger than one, then it is strictly increasing to a Pisot number as $n\to\infty$ and the same is true as $k\to\infty$. The above equation is known as the characteristic polynomial for the Coxeter element of a reflection group $W(2,k+1,n-1)$ with T-shaped Dynkin diagram. (See Sect. 5 for the connection between F and Coxeter elements.) The characteristic polynomial for the Coxeter element of such a reflection group is the product of cyclotomic factors and Salem polynomials (see [16] Proposition 7.1). If $n=1$ and $2\le k$, then F is equivalent to the standard Cremona involution J. For $n=2,3$ we have

$$\chi(k,2)=(\delta-1)(\delta^{k+2}-1),\qquad \chi(k,3)=(\delta-1)^2(\delta+1)(\delta^{k+1}+1)$$

If $(k,n)=(2,8),(3,6),(4,5),(5,5),(6,5)$, then $\chi(k,n)$ has a root bigger than one. We get this Corollary by checking the pairs in Γ directly. □

Thus for a basic Cremona map F satisfying (1), the first dynamical degree $\delta_1(F)$ is strictly bigger than 1 if and only if $(n,k)\notin\Gamma$. Corollary 3.2 and our assumption that the multiplier δ of $F|_C$ is not a root of unity further imply that δ and $\delta_1(F)$ are Galois conjugate.

Corollary 4.3 *Let δ, t_j^+ be as in Lemma 4.1. Then all points $T(\mathbf{e}_j)$, $1 \le j \le k$ and $F^{-i}(T(\mathbf{e}_0))$, $0 \le i \le n-1$ in (1) are distinct.*

Proof It suffices to show that the parameters corresponding to these points are distinct.

From Lemma 4.1 we have for $0 \le i < j \le k$,

$$t_j^+ - t_i^+ = (\delta^j - \delta^i) \cdot \frac{k+1}{k-1} \cdot \frac{\delta^2 - 1}{\delta(\delta^{k+1} - 1)}.$$

Since δ is not a root of unity and $k \ge 2$, it follows that $t_j^+ \ne t_i^+$ for $i \ne j$.

Furthermore, if $0 \le j \le k$ and $1 \le i \le n-1$ are indices such that $t_j^+ = \delta^{-i}(t_0^+ - 1) + 1$, then Lemma 4.1 tells us that

$$\delta(\delta^{k+1} - 1)(\delta^i - 1) - (\delta^{i+j} - 1)(\delta^2 - 1) = 0$$

Notice that we have

$$\begin{aligned} \delta(\delta^{k+1} - 1)(\delta^i - 1) &= (\delta - 1)^2 \delta(\delta^k + \delta^{k-1} + \cdots + 1)(\delta^{i-1} + \delta^{i-2} + \cdots + 1), \\ &= (\delta - 1)^2 (\delta^{k+i} + \sum_{s=2}^{k+j-1} c_s \delta^s + \delta), \end{aligned}$$

where $c_s = \min\{s, k+1, i+1\} \ge 2$. We also have

$$\begin{aligned} (\delta^{i+j} - 1)(\delta^2 - 1) &= (\delta - 1)^2 (\delta^{i+j-1} + \delta^{i+j-2} + \cdots + 1)(\delta + 1) \\ &= (\delta - 1)^2 (\delta^{i+j} + 2 \sum_{s=1}^{i+j-1} \delta^s + 1) \end{aligned}$$

Now δ is a Galois conjugate of the largest real root δ_r of the polynomial given in Lemma 4.1. It follows that the equation above should be divisible by the minimal polynomial of δ_r. In other words, $\delta_r > 1$ must satisfy the above equation. We have three different cases:

- case 1: if $j < k$ then by comparing terms we have

$$\delta(\delta^{k+1} - 1)(\delta^i - 1) - (\delta^{i+j} - 1)(\delta^2 - 1) > 0 \ \text{ for } \ \delta > 1$$

- case 2: if $j = k \ge 2$ and $i \ge 2$, then some of $c_s \ge 3$ and thus

$$\delta(\delta^{k+1} - 1)(\delta^i - 1) - (\delta^{i+j} - 1)(\delta^2 - 1) > 0 \ \text{ for } \ \delta > 1$$

- case 3: if $j = k$ and $i = 1$ then

$$\begin{aligned}&\delta(\delta^{k+1}-1)(\delta^{i}-1)-(\delta^{i+j}-1)(\delta^{2}-1)\\&\quad=(\delta^{k+1}-1)(1-\delta)<0\ \text{ for }\ \delta>1\end{aligned}$$

Thus we have the second part of this Corollary. □

The logic of the computations in Lemma 4.1 is essentially reversible, i.e. if δ and t_j satisfy the conclusions of the Lemma, then (1) holds. From this and Corollary 4.3 it follows that

Theorem 4.4 *Let δ be a root of $(\delta^{n+k}-1)(\delta^2-1)-\delta(\delta^{k+1}-1)(\delta^{n-1}-1)$ that is not also a root of unity. Then there exists a basic Cremona transformation $F:\mathbf{P}^k \dashrightarrow \mathbf{P}^k$ centered on C with multiplier of $F|_C$ equal to δ such that F satisfies the orbit data conditions (1). Up to linear conjugacy, $F = S\circ J\circ T^{-1}$ is uniquely specified by the further conditions*

$$T(\mathbf{e}_j)=\gamma\left(\delta^j\frac{k+1}{k-1}\cdot\frac{\delta^2-1}{\delta(\delta^{k+1}-1)}-\frac{2}{k-1}\right).$$

The matrices S and T defining the basic Cremona map $F = S\circ J\circ T^{-1}$ in this theorem must satisfy

$$\begin{aligned}T&=\left[a_0\gamma(t_0^+)\ a_1\gamma(t_1^+)\cdots\quad\cdots a_k\gamma(t_k^+)\right],\\S&=\left[b_0\gamma(t_1^+)\ b_1\gamma(t_2^+)\cdots b_{k-1}\gamma(t_k^+)\ b_k\gamma((t_0^+-1)/\delta^{n-1}+1)\right],\end{aligned}$$

where t_j^+ are as in Lemma 4.1 and a_i, b_i are non-zero constants. From this it is apparent that the final formulas will be simpler if we conjugate F by T, i.e. if we set $F = L\circ J$, where $L = T^{-1}\circ S$. Letting $L_0,\dots,L_k$ denote the columns of (the matrix of) L, the above information about S and T tells us that $L_j = \frac{b_j}{a_{j+1}}\mathbf{e}_{j+1}$ for $0\le j\le k-1$.

The fact that $\Lambda^{-1}\circ J = J\circ\Lambda$ for any invertible diagonal Λ means that we can further conjugate by Λ (or equivalently, replace S and T with $S\circ\Lambda$ and $T\circ\Lambda$) in order to replace L with $\Lambda^{-1}L\Lambda^{-1}$ to further simplify the columns L_j. It seems convenient to us to make this choice so that $L(1,\dots,1) = (1,\dots,1)$ (i.e. both T and S send the fixed point $(1,\dots,1)$ of J to the cusp $\mathbf{e}_k = \gamma(\infty)$ of C). This results in the following matrix for L

$$\begin{pmatrix}0&0&\dots&0&0&1\\b_0/a_1&0&&0&0&1-b_0/a_1\\0&b_1/a_2&&0&0&1-b_1/a_2\\\vdots&&\ddots&&&\vdots\\0&0&&b_{k-2}/a_{k-1}&0&1-b_{k-2}/a_{k-1}\\0&0&\dots&0&b_{k-1}/a_k&1-b_{k-1}/a_k\end{pmatrix}.$$

We can then evaluate the entries below the main diagonal with the help of the following auxiliary result which we leave to the reader to verify.

Lemma 4.5 *Suppose M is a non-singular $k+1 \times k+1$ matrix whose j-th column is given by $\gamma(t_j)$. If a column vector $v = (v_0, \ldots, v_k)^t$ satisfies $M.v = (0, \ldots, 1)^t$ then for all $0 \leq j \leq k$ we have*

$$v_i = \frac{1}{(\sum t_j) \prod_{j \neq i}(t_j - t_i)}.$$

The condition $T(1, \ldots, 1) = \mathbf{e}_k$ amounts to setting $t_j = t_j^+$ and then taking $(a_1, \ldots, a_k)$ to be the vector $v = (v_1, \ldots, v_k)$ given in the conclusion of the lemma. Hence with the help of Theorem 2.6, we find that

$$a_i = \frac{k+1}{k-1} \cdot \frac{\delta}{1-\delta} \cdot \frac{1}{\prod_{j \neq i}(t_j^+ - t_i^+)}.$$

Likewise, the condition $S(1, \ldots, 1) = \mathbf{e}_k$ amounts to setting $t_j = t_{j+1}^+$ for $0 \leq j \leq k-1$ and $t_k = (t_0^+ - 1)/\delta^{n-1} + 1$ in Lemma 4.5, and then taking $b_i = v_i$ as in the conclusion. So applying Theorem 2.6 with F^{-1} in place of F gives

$$b_{i-1} = \frac{k+1}{k-1} \cdot \frac{1}{\delta - 1} \cdot \frac{1}{\prod_{j \neq i}(t_j^+ - t_i^+)} \cdot \frac{t_0^+ - t_i^+}{(t_0^+ - 1)/\delta^{n-1} + (1 - t_i^+)}.$$

Therefore, the entries of L below the main diagonal are

$$\begin{aligned} b_{i-1}/a_i &= -\frac{1}{\delta} \cdot \frac{t_0^+ - t_i^+}{(t_0^+ - 1)/\delta^{n-1} - (t_i^+ - 1)} = \frac{1}{\delta} \cdot \frac{(\delta^{i+n-1} - \delta^{n-1})\frac{\delta^2 - 1}{\delta(\delta^{k+1} - 1)}}{(1 - \delta^{n+i-1})\frac{\delta^2-1}{\delta(\delta^{k+1}-1)} + (\delta^{n-1} - 1)} \\ &= \frac{1}{\delta} \cdot \frac{(\delta^{i+n-1} - \delta^{n-1})}{(1 - \delta^{n+i-1}) + (\delta^{n+k} - 1)} = \frac{\delta^i - 1}{\delta(\delta^{k+1} - \delta^i)}. \end{aligned}$$

for $1 \leq i \leq k$. The second equality uses the formula for t_i^+ given in Lemma 4.1, and the third uses that δ is a root of the polynomial given in the same lemma. In summary, we have just shown that the map $F := L \circ J$ of Theorem 4.4 has a very convenient expression in terms of the multiplier δ:

Theorem 4.6 *The matrix $L = T^{-1}S$ is given by*

$$L = \begin{pmatrix} 0 & 0 & & & 0 & 1 \\ \beta_1 & 0 & & & 0 & 1-\beta_1 \\ 0 & \beta_2 & 0 & & 0 & 1-\beta_2 \\ & & \ddots & \ddots & & \vdots \\ 0 & & 0 & \beta_{k-1} & 0 & 1-\beta_{k-1} \\ 0 & & & 0 & \beta_k & 1-\beta_k \end{pmatrix}$$

and $\beta_i = (\delta^i - 1)/(\delta(\delta^{k+1} - \delta^i))$ for $i = 1, \ldots, k$.

5 The Connection with Coxeter Groups

Let us consider a basic Cremona map F discussed in the Sect. 3. Let $\rho : X \to \mathbf{P}^k$ be the blowup of $N := \sum n_i$ distinct points $\{F^j(S(e_i)), 0 \le j \le n_i - 1, 0 \le i \le k\}$ as in Proposition 3.1. Let H denote the class of a generic hypersurface in X and let $E_{i,j}$ denote the class of the exceptional divisor over $F^{j-1}(S(e_i))$:

$$E_{i,j} = [\rho^{-1}(F^{n_i - j}(S(e_i)))] \quad \text{for} \;\; 1 \le j \le n_i, \;\; 0 \le i \le k.$$

The Picard group of X is given by

$$\mathrm{Pic}(X) = \langle H, E_{i,j} \; 1 \le j \le n_i, \;\; 0 \le i \le k \rangle.$$

Let us define a symmetric bilinear form $\langle \cdot, \cdot \rangle$ on $\mathrm{Pic}(X)$ as follows:

$$\langle \alpha, \beta \rangle = \alpha \cdot \beta \cdot \Phi \qquad \alpha, \beta \in \mathrm{Pic}(X)$$

where

$$\Phi = (k-1)H^{k-2} + (-1)^k \sum_{i,j} E_{i,j}^{k-2} \in H^{k-2,k-2}(X)$$

and $D^n = D \cdot D \cdots D$ is a n-fold intersection product. Since $H^k = 1, E_{i,j}^k = (-1)^{k-1}$ and everything else is zero, our choice of basis for $\mathrm{Pic}(X)$ gives a geometric basis with respect to the bilinear form:

$$\langle H, H \rangle = k - 1, \quad \langle E_{i,j}, E_{i,j} \rangle = -1$$

and zero otherwise.

Remark 5.1 In case $k = 3$, $-2\Phi = K_X$, where K_X is the canonical class of X.

Remark 5.2 For $k \geq 3$, the anticanonical class $-K_X$ is given by

$$-K_X = (k+1)H - (k-1)\sum_{i,j} E_{i,j}.$$

It follows that the class of $-K_X \cdot \Phi$ in $H^{k-1,k-1}$ is the class of the invariant curve $[C]$. Thus for any hypersurface $D \subset X$, we have

$$\langle D, -K_X \rangle = D \cdot C$$

is the number of intersections between D and C, counted with multiplicity.

Remark 5.3 Since there are N blowups we have

$$\langle K_X, K_X \rangle = (k+1)^2(k-1) - (k-1)^2 N = -(k-1)^2 \left(N - \frac{(k+1)^2}{k-1} \right).$$

and since N is a positive integer, this equation is not equal to zero unless $k-1$ divides 4. In case $k = 2, 3, 5$ one can check this equation vanishes if $N = 9, 8, 9$ respectively. Thus if $(k, N) \neq (2, 9), (3, 8), (5, 9)$, then $\langle K_X, K_X \rangle \neq 0$. Notice that in case $F : \mathbf{P}^k \dashrightarrow \mathbf{P}^k$ has the orbit data $(1, 1, \ldots, n)$ with the cyclic permutation, the total number of blowups is $N = k + n$. Thus if $(k, N) = (2, 9), (3, 8)$, or $(5, 9)$ then $(k, n) = (k, N-k) = (2, 7), (3, 5), (5, 4) \in \Gamma$ where the set Γ is defined in Sect. 4. Thus for these three cases the first dynamical degree of F is equal to 1.

Observe that the N dimensional subspace $K_X^{\perp} \subset \mathrm{Pic}(X)$ may be decomposed into the one dimensional subspace generated by $\alpha_0 := H - \sum_{i,0}^{k} E_{i,1}$ and the complementary subspace generated by the elements $E_{i',j'} - E_{i,j}$. Indeed, if we (re)label the exceptional curves $E_{i,j}$ as $E_0, \ldots, E_{N-1}$, then the elements $\alpha_i := E_i - E_{i-1}$, $1 \leq i \leq N-1$ give a basis for the latter subspace.

In order to see the connection with Coxeter groups, we take in particular $E_i := E_{i,1}$, $0 \leq i \leq k$, $E_{k+1} = E_{k,2}$. The relabeling of the remaining $E_{i,j}$ may be chosen arbitrarily. One then checks easily that $B = \{\alpha_0, \ldots, \alpha_{N-1}\}$ is a basis for $K_X^{\perp}$ satisfying

$$\begin{aligned} \langle \alpha_i, \alpha_i \rangle &= -2 \quad \text{for all } i = 0, \ldots, N-1, \\ \langle \alpha_i, \alpha_{i+1} \rangle &= 1 \quad \text{for all } i = 1, \ldots, N-1, \\ \langle \alpha_0, \alpha_{k+1} \rangle &= 1, \quad \langle \alpha_i, \alpha_j \rangle = 0 \quad \text{otherwise.} \end{aligned}$$

That is, $(\langle \alpha_i, \alpha_j \rangle)_{0 \leq i,j \leq N-1}$ is -2 times the gram matrix of the Coxeter group of Type $T_{2,k+1,N-k-1}$. (See [15, 19].) Let s_i be the reflection through a hyperplane orthogonal to α_i:

$$s_i(D) = D + \langle D, \alpha_i \rangle \, \alpha_i.$$

The Coxeter group $W(2, k+1, N-k-1)$ is the group generated by such reflections. Thus we have identified $W(2, k+1, N-k-1)$ with a subgroup of Aut (Pic(X)) that acts orthogonally relative to the inner product $\langle \cdot, \cdot \rangle$.

Theorem 5.4 *The action F_X^* of the pseudoautomorphism F_X in Proposition 3.1 belongs to $W(2, k+1, N-k-1)$. Hence F_X^* preserves the bilinear form $\langle \cdot, \cdot \rangle$.*

Proof The reflection s_0 corresponds to the action of J^* on Pic(X). The remaining reflections s_i, $i \geq 1$ generate the group of permutations of the exceptional curves $E_{i,j}$ for the modification $X \to \mathbf{P}^k$. Therefore, this Theorem follows from the decomposition (which we leave the reader to check) $F_X^* = s_0 \hat{\sigma} \pi_0 \dots \pi_k$, where π_i cyclicly permutes the curves $E_{i,j}$, $1 \leq j \leq n_j$, and $\hat{\sigma}$ permutes the curves $E_{i,1}$, $0 \leq i \leq k$ according to $\hat{\sigma}(E_{i,1}) = E_{\sigma(i),1}$, where σ is the permutation in the hypothesis of Proposition 3.1. □

6 Pseudoautomorphisms on Multiprojective Spaces

The article [22] considers the more general problem of existence of pseudoautomorphisms $F_X : X \dashrightarrow X$, where $\pi_X : X \to (\mathbf{P}^k)^m$ is a modification of the multiprojective space $(\mathbf{P}^k)^m$, obtained as before by blowing up distinct smooth points along an 'cuspidal curve' C. Our method yields formulas for the pseudoautomorphisms that arise in this case, too. Hence we conclude with a quick sketch of the computations that arise here, laying greatest emphasis on the way things differ from the work presented above. For the sake of simplicity we specialize to the case $m = 2$, i.e. X is a modification of $\mathbf{P}^k \times \mathbf{P}^k$.

Note first of all that Pic$(\mathbf{P}^k \times \mathbf{P}^k) \cong \mathbf{Z}^2$ is generated by 'horizontal' and 'vertical' hyperplanes $H := \mathbf{P}^2 \times L$ and $V := L \times \mathbf{P}^k$, where $L \subset \mathbf{P}^k$ is a generic hyperplane. Let Aut$_0(\mathbf{P}^k \times \mathbf{P}^k)$ denote the connected component of the identity inside Aut $(\mathbf{P}^k \times \mathbf{P}^k)$. The group Aut$_0(\mathbf{P}^k \times \mathbf{P}^k)$ consists of products $T_v \times T_h$ of linear maps $T_v, T_h \in$ Aut $(\mathbf{P}^k)$. For every $a \in \mathbf{C}$, one has an embedding $\gamma_a : C \to X$ given by $t \mapsto (\gamma(t), \gamma(t{-}a))$, but in fact all embeddings of C into X that satisfy $C{\cdot}V = C{\cdot}H = k{+}1$ are equivalent via Aut$_0(\mathbf{P}^k \times \mathbf{P}^k)$ to either γ_0 or γ_1. The latter, which we use here, is in some sense the generic case, distinguished from the former by the condition $(k+1)[\gamma_1(1)] = H|_C \neq V|_C = (k+1)[\gamma_1(0)]$.

The 'standard' Cremona map J is also different in this context. If we write points in $(\mathbf{P}^k)^2$ in bihomogeneous coordinates $(x, y) = ((x_0, \dots, x_k), (y_0, \dots, y_k))$, then J is given by

$$J : (x, y) \mapsto (y/x, 1/x) := ((y_0/x_0, \dots, y_k/x_k), (1/x_0, \dots, 1/x_k)).$$

Note that J contracts the $k+1$ vertical hyperplanes $x_j = 0$ to diagonal points $(\mathbf{e}_j, \mathbf{e}_j)$. Though not an involution, J is reversible with $J^{-1} : (x, y) \mapsto (1/y, x/y)$ conjugate to J via $(x, y) \mapsto (y, x)$. Hence J^{-1} contracts the horizontal hyperplanes $y_j = 0$ to the same diagonal points $(\mathbf{e}_j, \mathbf{e}_j)$.

We recycle the terminology from Sect. 2: a basic Cremona transformation is one of the form $F := S \circ J \circ T^{-1}$ with $S, T \in \mathrm{Aut}(\mathbf{P}^k \times \mathbf{P}^k)$, F is centered on C if $T(\mathbf{e}_j, \mathbf{e}_j) \in C_{reg}$ for all j, and F properly fixes C if, in addition, $F(C) = C$. Proposition 2.3 and Corollary 2.5 apply to the present context with straightforward modifications. The analogue for Theorem 2.6, which we state next, differs in one important way from its predecessor. While there exist automorphisms of $\mathbf{P}^k \times \mathbf{P}^k$ that preserve C, none of these restrict to $C_{reg} \cong \mathrm{Pic}_0(C)$ as group automorphisms (in parametric terms, maps of the form $t \mapsto \delta t$). Therefore, the multiplier δ for $F|_C$ must depend on the choice of parameters t_j^+ for the images of exceptional hyperplanes.

Theorem 6.1 *Let $t_j^+ \in \mathbf{C}$, $0 \le j \le k$, be distinct parameters satisfying $\sum t_j^+ \neq 0$. Then there exists a unique basic Cremona map $F = S \circ J \circ T^{-1} : \mathbf{P}^k \times \mathbf{P}^k \dashrightarrow \mathbf{P}^k \times \mathbf{P}^k$ properly fixing C such that $\gamma_1(t_j^+) = T(\mathbf{e}_j, \mathbf{e}_j)$, $0 \le j \le k$. The restriction $F|_C$ is given by $F \circ \gamma_1(t) = \gamma_1(\delta t + \tau)$, where $\delta \in \mathbf{C}^*$ and $\tau \in \mathbf{C}$ satisfy*

- $\sum t_j^+ = (k+1)(\delta^{-1} + 1)$,
- $\tau = k + (k-1)\delta$, *and*
- $S(\mathbf{e}_j, \mathbf{e}_j) = \gamma_1(\delta(t_j^+ - 2) - 1)$.

We omit most of the proof, deriving only the formulas for τ, δ and $S(\mathbf{e}_j, \mathbf{e}_j)$. First note that regardless of S and T, the induced action F_* on $\mathrm{Pic}(\mathbf{P}^k \times \mathbf{P}^k)$ is given by $F_*V = kH$, $F_*H = V + kH$. Moreover, one computes directly that the images $F(V) = F_*V, F(H) = F_*H$ of generic vertical and horizontal hyperplanes contain the points $S(\mathbf{e}_j, \mathbf{e}_j)$ with multiplicity $k-1$ and k, respectively. Hence assuming that F properly fixes C, we infer that

$$kH|_C = (F_*V)|_C = (F|_C)_*(V|_C) + \sum [(\mathbf{e}_j, \mathbf{e}_j)].$$

In terms of parameters, we may write $S(\mathbf{e}_j, \mathbf{e}_j) = \gamma_1(t_j^-)$ and conclude that

$$k(k+1) = (k+1)\tau + (k-1)\sum t_j^-.$$

Similarly, considering the image of a horizontal hyperplane gives

$$k(k+1) = (k+1)(\delta + \tau) + k\sum t_j^-.$$

Together, the two equations imply $\tau = k + (k-1)\delta$ and $\sum t_j^- = -(k+1)\delta$.

Now consider e.g. the vertical hyperplanes $V_j := T(\{x_j = 0\})$ contracted by F. On the one hand $V_j|_C = [p_j] + \sum_{i \neq j}[T(\mathbf{e}_i, \mathbf{e}_i)]$ for some $p_j \in C$. In terms of parameters, this becomes $s_j = -\sum_{i \neq j} t_i^+$, where $\gamma_1(s_j) = p_j$. On the other hand $\gamma_1(\delta s_j + \tau) = F(p_j) = F(V_j) = S(\mathbf{e}_j, \mathbf{e}_j) = \gamma_1(t_j^-)$. Hence

$$t_j^- = \delta s_j + \tau = \tau - \delta \sum_{i \neq j} t_i^+,$$

So summing the equation over all j gives $\sum t_j^- = (k+1)\tau - k\delta \sum t_j^+$, which implies

$$\delta \sum t_j^+ = (k+1)(\delta+1).$$

Substituting this into the previous display, we arrive at

$$t_j^- = \delta(t_j^+ - 2) - 1,$$

which is the parameter for $S(\mathbf{e}_0, \mathbf{e}_0)$. □

As in Sect. 4, one can choose explicit parameters t_j^+, δ in Theorem 6.1 so that the resulting map $F = S \circ J \circ T^{-1} : \mathbf{P}^k \times \mathbf{P}^k \dashrightarrow \mathbf{P}^k \times \mathbf{P}^k$ has orbit data $(1, \dots, 1, n)$ with cyclic permutation (see (1)). Let us set

$$\Gamma_2 = \{(2,3), (2,4), (3,3), (4,3), (5,3)\} \cup \{(k,1), (k,2),\ k \geq 2\}$$

Lemma 6.2 *The multiplier δ is a root of the polynomial*

$$\chi_{k,n} = \delta^n(\delta^{k+2} - \sum_{j=0}^{k} c_j\delta^j) + \delta^2 \sum_{j=0}^{k} c_j\delta^j - 1$$

where $c_0 = c_k = 1$, and $c_1 = c_2 = \cdots = c_{k-1} = 2$. Furthermore if $(k,n) \in \Gamma_2$ then $\chi_{k,n}$ is a product of cyclotomic polynomials. If $(k,n) \notin \Gamma_2$ then $\chi_{k,n}$ has a Salem polynomial factor and thus the largest real root is bigger than 1.

Starting from Theorem 6.1 in place of Theorem 2.6, the proof of the first part of this lemma is essentially identical to the proof of Lemma 4.1. The polynomial $\chi_{k,n}$ is the characteristic polynomial of the generalized Coxeter group $W(3, k+1, n-1)$, so it follows that $\chi_{k,n}$ is a product of cyclotomic polynomials and at most one Salem polynomial. As in Corollary 4.2, we see that the largest root of $\chi_{k,n}$ increases to a root of $\delta^{k+2} - \delta^k - 2\sum_{j=1}^{k-1} \delta^j - 1$ as $n \to \infty$ and to a root of $\delta^{n+2} - \delta^n - 2\sum_{j=1}^{n-1} \delta^j - 1$ as $k \to \infty$. For each $k \geq 2$, we have $\chi_{k,1} = (x^{k+1} - 1)(x^2 + x + 1)$ and $\chi_{k,2} = x^{k+4} - 1$. Checking the other five elements in Γ_2 directly we see that $\chi_{k,n}$ is a product of cyclotomic polynomial if $(k,n) \in \Gamma_2$. The largest roots of $\chi_{2,5}$, $\chi_{3,4}$, and $\chi_{6,3}$ are 1.40127, 1.40127, and 1.17628 respectively. Thus, by monotonicity, we get the final assertion in the lemma.

Lemma 6.3 *The parameters t_j^+ are given by*

$$t_j^+ = \frac{\delta^{j-1}}{\delta^{k+1} - 1}\left[k(k+1) - \delta(\delta+1)\right] - \frac{k - 2\delta^2 - \delta + 1}{\delta(\delta - 1)}.$$

Furthermore all points $T(\mathbf{e}_j, \mathbf{e}_j)$, $1 \leq j \leq k$ and $F^{-i}(T(\mathbf{e}_0, \mathbf{e}_0))$, $0 \leq i \leq n-1$ are distinct.

Proof The given orbit data and permutation, together with Theorem 2.6, give $t_i^- = \delta^i t_0^- + (k-2\delta)(\delta^i - 1)/(\delta - 1)$ for $j = 1, \dots, k$, and also $\sum t_j^- = -(k+1)\delta$ and $t_j^- = \delta(t_j^+ - 2) - 1$. The formula for t_j^+ follows from these equations.

Now since

$$t_j^+ - t_i^- = (\delta^j - \delta^i)(k(k+1) - \delta(\delta+1))/(\delta(\delta^{k+1} - 1))$$

it follows that $t_j^+ \neq t_i^+$ for $j \neq i$. Applying F^{-i} to $T(\mathbf{e}_0, \mathbf{e}_0)$, we see that if there are i and j such that $T(\mathbf{e}_j, \mathbf{e}_j) = F^{-i}(T(\mathbf{e}_0, \mathbf{e}_0))$, then

$$k - \delta^2 - \delta - \delta(\delta^i - 1)(\delta^k + \delta^{k-1} + \cdots + 1) = 0.$$

Since δ is a Galois conjugate of a Salem number, above equation should be divisible by a Salem polynomial. However if $\delta > 1$ we see that the left hand side of the equation is strictly negative. □

Finally, we can imitate the argument for Theorem 4.6 to get a formula for $T^{-1}S = L_1 \times L_2 \in \mathrm{Aut}_0(\mathbf{P}^k \times \mathbf{P}^k)$. We let both T and S send the fixed point $((1,\dots,1),(1,\dots,1))$ of J to the cusp of C and thus L fixes the point $((1,\dots,1),(1,\dots,1))$.

Theorem 6.4 *The matrix for $L_i \in \mathrm{Aut}(\mathbf{P}^k)$, $i = 1, 2$ is given by*

$$\begin{pmatrix} 0 & 0 & & & 0 & s_i \\ \beta_1 & 0 & & & 0 & s_i - \beta_1 \\ 0 & \beta_2 & 0 & & 0 & s_i - \beta_2 \\ & & \ddots & \ddots & & \vdots \\ 0 & & 0 & \beta_{k-1} & 0 & s_i - \beta_{k-1} \\ 0 & & & 0 & \beta_k & s_i - \beta_k \end{pmatrix}$$

where $s_1 = 1, s_2 = (\delta^2 + \delta + 1)/\delta$ and $\beta_j = (\delta^j - 1)(\delta + 1)/(\delta^2(\delta^{k+1} - \delta^j))$ for $j = 1, \dots, k$.

Concluding remarks. So far, we have described a construction of pseudoautomorphisms which is achieved by blowing up points on the cuspidal curve. The same procedure works with other invariant curves. Two of these that occur in all dimensions are: (i) the rational normal curve and a tangent line, (ii) $k+1$ concurrent lines in general position. We will make a few comments on case (ii). First we work in $\mathbf{P}^k$, and then we consider multi-projective space.

In the case of concurrent lines, we let L_j, $0 \le j \le k$, denote the line passing through $[1 : \cdots : 1]$ and e_j. For the parametrizations, we may use: $\psi_c^{(j)} : \mathbf{C} \to \mathbf{P}^k$ with $\psi_c^{(0)}(t) = [-t : 1 : \cdots : 1]$, and for $1 \le j \le k$, $\psi_c^{(j)}(t) = [t : 0 : \cdots : 1 : \cdots : 0]$, where there is one '1', and this appears in the jth slot. If we wish to work on multiprojective spaces, we use the parametrized curve $\Psi : \mathbf{C} \to (\mathbf{P}^k)^m = \mathbf{P}^k \times \cdots \times \mathbf{P}^k$ given by $\Psi(t) = (\psi(t-\tau_0), \psi(t-\tau_1), \ldots, \psi(t-\tau_{m-1}))$.

Now let us consider multi-projective spaces $(\mathbf{P}^k)^m = \mathbf{P}^k \times \cdots \times \mathbf{P}^k$. We write a point as $(x, y) = (x, y^{(1)}, \ldots, y^{(m-1)}) \in (\mathbf{P}^k)^m$. As a basic Cremona map, we start with

$$J(x, y) : (x, y^{(1)}, \ldots, y^{(m-1)}) \mapsto (y^{(1)}/x, \ldots, y^{(m-1)}/x, 1/x),$$

where as before $y^{(s)}/x = [y_0^{(s)}/x_0 : \cdots : y_k^{(s)}/x_k]$. The exceptional hypersurfaces are given, as in the case $m = 2$, by $J : \{x_j = 0\} \mapsto (\mathbf{e}_j, \ldots, \mathbf{e}_j)$.

With the curve Ψ, it is possible to carry through the same principle of construction as in the preceding sections. We consider the case (ii) of concurrent lines and give the map $L = L_0 \times \cdots \times L_{m-1} \in \mathrm{Aut}_0(\mathbf{P}^k \times \cdots \times \mathbf{P}^k)$ so that the map $f := L \circ J$ will have orbit data $\{(1, \ldots, 1, n(k+1)), \sigma\}$, where σ is a cyclic permutation. The orbit length is divisible by $k + 1$ because the orbit of Σ_k moves cyclically through each of the $k + 1$ lines. With this orbit data, the resulting pseudoautomorphism will represent the Coxeter element of a T-shaped diagram [19]. We let δ be any Galois conjugate of the dynamical degree δ_1 for this orbit data, and the desired matrices are given by

$$L_j = \begin{pmatrix} 0 & 0 & 0 & 0 & s_j \\ v & 0 & 0 & 0 & s_j - v \\ 0 & v & 0 & 0 & s_j - v \\ 0 & 0 & \ddots & 0 & s_j - v \\ 0 & 0 & 0 & v & s_j - v \end{pmatrix}, \; v = -\delta \frac{\delta^m - 1}{\delta - 1}, \; s_j = \frac{(\delta^m - 1)(\delta^{j+1} - 1)}{\delta^j(\delta - 1)(\delta^{m-j} - 1)}$$

for $j = 0, \ldots, m - 1$.

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The Openness Conjecture and Complex Brunn-Minkowski Inequalities

Bo Berndtsson

Dedicated to Bradley Manning and Edward Snowden in recognition of their work for openness

1 Introduction

Let u be a plurisubharmonic function defined in the unit ball, B, of $\mathbb{C}^n$. The *openness conjecture* of Demailly and Kollár [10] states that the interval of numbers $p > 0$ such that

$$\int_{rB} e^{-pu} < \infty, \tag{1}$$

for some $r > 0$, is open. This is quite easy to see in one variable, since the singularities of u are then given explicitly by the Green potential of Δu, but the higher dimensional case is much more subtle. The openness conjecture was first proved in dimension 2 by Favre and Jonsson (see [11]), and then for all dimensions in [2]. After that, simpler proofs and generalizations to the so called *strong openness conjecture* (see below), have been given by Guan-Zhou [13] and [14]; see also [16] and [19] for variants of the proof and simplifications.

The proof of the openness conjecture from [2] was based on positivity properties of certain vector bundles from [4], that in some ways can be seen as a complex variables generalization of the Brunn-Minkowski theorem. The aim of this survey article is to describe these 'complex Brunn-Minkowski inequalities' and explain how they can be applied in this context. In this we will basically restrict ourselves to the simplest cases and refer to the original articles for complete statements and proofs. It should be stressed that the recent proofs of Guan-Zhou, Hiep and Lempert are actually simpler than the method to prove the openness conjecture

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presented here, but we hope that the original proof still has some interest and that the exposition here also may serve as an introduction to how our 'complex Brunn-Minkowski inequalities' can be applied. The method here also gives for free a bound on the exponents p, that can also be obtained in the setting of Guan-Zhou [15], but with more work.

In the last section we also give a very brief account of the strong openness conjecture and sketch a conjectural picture how the strong openness theorem might fit into our method.

2 The Brunn-Minkowski Theorem

The classical Brunn-Minkowski theorem for convex bodies (see e.g. [12] for a nice account, including history and applications) can be formulated in the following way.

Theorem 2.1 *Let A_0 and A_1 be two convex bodies in $\mathbb{R}^n$, and denote by*

$$A_t := \{a = ta_1 + (1-t)a_0; a_0 \in A_0 \text{ and } a_1 \in A_1\}, \tag{2}$$

for $0 \leq t \leq 1$. Let $|A|$ be the Lebesgue measure of a set A. Then the function

$$t \to |A_t|^{1/n}$$

is concave.

An equivalent statement that relies less on the additive structure on $\mathbb{R}^n$, and is more suitable for the complex variants that we will describe later is the following.

Theorem 2.2 *Let $\mathcal{A}$ be a convex body in $\mathbb{R}^{n+1}$ and denote by*

$$A_t = \{a \in \mathbb{R}^n; (t, a) \in \mathcal{A}\}. \tag{3}$$

Then the function $t \to |A_t|^{1/n}$ is concave.

The equivalence of these two statements is not hard to see. In order to deduce the first statement from the second we let $\mathcal{A}$ be the convex hull in $\mathbb{R}^{n+1}$ of $\{0\} \times A_0 \cup \{1\} \times A_1$, and for the converse it suffices to observe that if A_t is defined by (3), then

$$tA_1 + (1-t)A_0 \subset A_t,$$

if $\mathcal{A}$ is convex.

There is yet another version of the theorem that will be useful for us. To prove Theorem 2.1 it actually suffices to prove the seemingly weaker inequality

$$|A_t| \geq \min(|A_0|, |A_1|),$$

which of course trivially follows from the concavity. This is because we can rescale the sets and use the homogenity of Lebesgue measure. Therefore we see that Theorems 2.1 and 2.2 are also equivalent to saying that

$$t \to \log |A_t|$$

is concave, since this also implies the min-statement. This is sometimes called the multiplicative form of the Brunn-Minkowski theorem.

One reason why the multiplicative form is often more useful is that it allows a functional version, known as Prékopa's theorem [21].

Theorem 2.3 *Let $\phi(t,x)$ be a convex function on $\mathbb{R}^{n+1} = \mathbb{R}_t \times \mathbb{R}^n_x$. Then*

$$\tilde{\phi}(t) := \log \int_{R^n} e^{-\phi(t,x)} dx \tag{4}$$

is concave.

To get the multiplicative version of Theorem 2.2 from Prékopa's theorem one lets ϕ be the convex function that equals 0 on $\mathcal{A}$ and ∞ on the complement of $\mathcal{A}$. (The reader that does not like functions that take infinite values can instead use an appropriate limit of finite functions.) Then $\tilde{\phi}(t) = \log |A_t|$ so $\log |A_t|$ is concave.

Both the Brunn-Minkowski and Prékopa theorems have a variety of proofs, often pointing in different directions of generalizations. The proof of Prékopa's theorem that is most relevant for us is the one by Brascamp and Lieb [7]. Their proof is based on a weighted Poincaré inequality, with weight function $e^{-\phi}$. This weighted Poincaré inequality is actually a real variable version of Hörmander's [17] L^2-estimate for the $\bar{\partial}$-equation, cf. [8]. It is therefore natural to ask which, if any, inequalities in the complex domain correspond to Prékopa's inequality. This question was discussed in [3, 4] and [5], and in the next section we will give a very brief account of this.

3 A Complex Variant of the Brunn-Minkowski Theorem

We shall now describe the simplest version of the results in [4]. Let D be a pseudoconvex domain in $\mathbb{C}^n_z$ and let Ω be a domain in $\mathbb{C}_t$. If $\phi(t,z)$ is a plurisubharmonic function in $\mathcal{D} := \Omega \times D$ we consider for each fixed t in Ω the Bergman space

$$A^2_t := \{h \in H(D); \int_D |h(z)|^2 e^{-\phi(t,z)} d\lambda(z) < \infty\},$$

equipped with the Bergman norm

$$\|h\|^2_t = \int_D |h(z)|^2 e^{-\phi(t,z)} d\lambda(z).$$

If we put an appropriate restriction on the growth of ϕ, like

$$|\phi(t, z) - \psi(z)| \leq C(t),$$

for some fixed function ψ and function $C(t)$, then all the Bergman spaces $A^2_t = A^2$ are the same as sets, but their norms vary with t. They therefore together make up a trivial vector bundle

$$E := \Omega \times A^2,$$

with an hermitian norm $\| \cdot \|_t$. This is thus an hermitian holomorphic vector bundle, and although it has in general infinite rank it has a Chern connection and a curvature, Θ^E, as in the finite rank case, see [4].

Theorem 3.1 *The curvature of E is positive (in the nonstrict sense).*

To understand the meaning of this statement it is not absolutely necessary to resort to the technical definition of curvature and its extension to bundles of infinite rank – although there are explicit formulas for the curvature that are sometimes of interest, see [5]. Recall that for vector bundles of finite rank, the curvature is negative if and only if the logarithm of the norm of any holomorphic section is (pluri)subharmonic. This can be taken as the definition of negative curvature also for bundles of infinite rank, and then we say that a bundle has positive curvature if its dual has negative curvature. For the bundle E above we can e.g. construct holomorphic sections of its dual in the following way. Let

$$t \to f(t)$$

be a holomorphic map from Ω to D. For each t we then let $\xi(t) \in E^*_t$ be defined as evaluation at $f(t)$, so that

$$\langle \xi(t), h \rangle = h(f(t))$$

if h is in A^2. Clearly this defines a holomorphic section of E^*. The squared norm of the evaluation functional at a point z for the norm on A^2_t is by definition $B_t(z)$, the (diagonal) Bergman kernel for A^2_t at z, so

$$\|\xi(t)\|^2_t = B_t(f(t)).$$

Hence Theorem 3.1 implies in particular that $\log B_t(f(t))$ is subharmonic for any holomorphic map f, so

$$\log B_t(z),$$

is plurisubharmonic on $\Omega \times D$ (see [3] for a more general statement).

For line bundles it is also true that the logarithm of a holomorphic nonvanishing section is (pluri)superharmonic if the curvature is positive, but we stress that this does not hold for bundles of higher rank. Thus we have no direct statements about functions like

$$t \to \log \int_B |h(z)|^2 e^{-\phi(t,z)} d\lambda(z)$$

(with h holomorphic), except in special cases when the rank of the bundle is one. This actually happens in some cases. If $D = \mathbb{C}^n$ and $\phi(t,z)$ grows like $(n+1)\log(1+|z|^2)$ at infinity, then A^2 contains only constants, and we can conclude that

$$\tilde{\phi} := -\log \int_B e^{-\phi(t,z)} d\lambda(z)$$

is subharmonic. This is of course in close analogy with Theorem 2.3. On the other hand, if ϕ does not satisfy such a bound, $\tilde{\phi}$ is not necessarily subharmonic, as can be see from the simple example $\phi(t,z) = |z-\bar{t}|^2 - |t|^2 = |z|^2 - 2\mathrm{Re}\, tz$, cf. [18]. Then $\tilde{\phi} = c_n - |t|^2$ is not subharmonic.

Theorem 3.1 is the simplest version of what we here call ‘complex Brunn-Minkowski’ theorems. There are many variants of the setting and the result, the most general involving fibrations of Kähler manifolds, holomorphic n-forms and holomorphic line bundles with positively curved metrics. For our purposes here Theorem 3.1 is enough and we refer to [4] and [5] for proofs and generalizations. Let us just mention here, in order to make a first contact with Sect. 2, that Theorem 3.1 can be proved by computing

$$i\partial\bar{\partial}_t \|h\|_t^2$$

and applying Hörmander’s theorem in a way quite similar to how Brascamp and Lieb proved Prékopa’s theorem.

On a formal level, the analogy between Theorems 3.1 and 2.3 is that we have replaced the convex function in Theorem 2.3 by a plurisubharmonic function, and the constant function 1 in Prékopa’s theorem by a holomorphic function h. Although the statement of Theorem 3.1 has nothing to do with volumes of sets, it turns out that it can be seen as a stronger version of Theorem 2.3 and implies Theorem 2.3 as a special case. To see this we shall apply Theorem 3.1 when D and ϕ have some symmetry properties.

We first look at the case when D and $\phi(t,\cdot)$ are invariant under the natural S^1-action on $\mathbb{C}^n$

$$(z_1, \dots z_n) \to (e^{i\theta} z_1, \dots e^{i\theta} z_n) := s_\theta z.$$

Thus we assume that D is invariant under s_θ for all real θ and that for each t, $\phi(t, s_\theta z) = \phi(t,z)$. First we moreover assume that D is even closed under the maps

$z \to \lambda z$, if $|\lambda| \leq 1$. Then D can be written as

$$D = \{z; \psi(z) < 0\},$$

for some ψ, plurisubharmonic in all of $\mathbb{C}^n$, that is logarithmically homogenous, i.e. $\psi(\lambda z) = \psi(z) + \log|\lambda|$ if λ is a nonzero complex number. In particular, D is Runge, so the polynomials are dense in A^2.

Hence our Bergman space A^2 splits as a direct sum

$$A^2 = \bigoplus_0^\infty H_m$$

where H_m is the space of polynomials, and these spaces are orthogonal for all the norms $\|\cdot\|_t$. Therefore we also get an orthogonal splitting

$$E = \bigoplus_0^\infty E_m$$

of the hermitian vector bundle E. Since E has positive curvature it follows that all the subbundles E_m are also positively curved (see e.g. [23] for more on this). In particular E_0 is positively curved. The fibers of E_0 consist of polynomials of degree zero, i.e. constants, so E_0 is a line bundle with the constant function 1 as trivializing section. Since

$$\|1\|_t^2 = \int_D e^{-\phi(t,z)} d\lambda(z),$$

the positivity of E_0 means that

$$t \to \log \int_D e^{-\phi(t,z)} d\lambda(z)$$

is superharmonic, which is a(nother) complex version of Prékopa's theorem. It is only the positivity at the level $m = 0$ that gives Prékopa-like statements, higher degrees give corresponding statements for matrices $M = (M_{\alpha,\beta})(t)$ where

$$M_{\alpha,\beta}(t) = \int_D z^\alpha \bar{z}^\beta e^{-\phi(t,z)} d\lambda(z)$$

where $|\alpha| = |\beta| = m$. More precisely, we see that

$$\Theta_m := i\bar{\partial}(M^{-1}\partial M) \geq 0$$

as a curvature operator.

In a similar way, the usual Prékopa theorem follows from Theorem 3.1 when we assume full toric symmetry. We only sketch this and refer to an article by Raufi [23] where this is explained and used to get a matrixvalued Prékopa theorem. We then assume that both D and ϕ are invariant under the full torus action

$$z \to (e^{i\theta_1} z_1, \dots e^{i\theta_n} z_n).$$

Then ϕ depends only on $r_j = |z_j|$ and is a convex function of $\log r_j$, and $D = T^n \times D_{\mathbb{R}}$ where $D_{\mathbb{R}}$ is logarithmically convex. The orthogonal splitting

$$A^2 = \bigoplus H_\alpha,$$

where the sum is now over all multiindices α, then gives Prékopa's theorem after a logarithmic change of variables. Here all multiindices α give the same information; the change in α just means that the weight function changes by a linear term.

Even in this case of full toric symmetry, Theorem 3.1 is a bit more general than Prékopa's theorem, since we do not need to assume any symmetry in t:

Theorem 3.2 *Let $\Phi(t, x_1, \dots x_n) = \Phi(t, z_1, \dots z_n)$ be plurisubharmonic in $\Omega \times V_{\mathbb{R}} \times i\mathbb{R}^n$ and independent of the imaginary part of z. Then*

$$\tilde{\Phi}(t) := -\log \int_{V_{\mathbb{R}}} e^{-\Phi(t, x_1, \dots x_n)} dx$$

is subharmonic in Ω. In particular, if Φ is also independent of $\mathrm{Im}\, t$*, then $\tilde{\Phi}$ is convex.*

In this sense one can perhaps say that the classical Brunn-Minkowski theorem is the special case of the complex theorem when we have maximal symmetry. In another direction Theorem 3.2 can be seen as a generalization of a well known result of Kiselman [18].

Theorem 3.3 (Kiselman) *Under the same assumptions as in Theorem 3.2, let*

$$\hat{\Phi}(t) := \inf_{x \in V_{\mathbb{R}}} \Phi(t, x).$$

Then $\hat{\Phi}$ is subharmonic.

(This follows from Theorem 3.2 since $\hat{\Phi} = \lim_{p \to \infty} \widetilde{(p\Phi)}/p$.)

4 The Openness Problem

We now return to the openness problem. We have given a plurisubharmonic function u in the ball, which we assume to be negative and such that

$$\int_B e^{-u} < \infty,$$

where B is the unit ball. For any $s > 0$ we let $u_s = \max(u+s, 0) = \max(u, -s) + s$. We also extend this to when s is complex with $\mathrm{Re}\, s > 0$ by putting $u_s = u_{\mathrm{Re}\, s}$. Then $u(s,z) = u_s(z)$ is plurisubharmonic on $\Omega \times B$, with Ω being the halfplane. Obviously $0 \le u_s \le s$, so u_s stays uniformly bounded for s bounded. Let, for h holomorphic and square integrable in the unit ball

$$\|h\|_s^2 := \int_B |h|^2 e^{-2u_s}$$

(note the factor 2 in the exponent!). Then $\|h\|_0$ is the standard unweighted L^2-norm and for s in a bounded set $\|h\|_s$ is equivalent to $\|h\|_0$. The next proposition says in particular that we can express the norm

$$\int_B |h|^2 e^{-u}$$

in terms of $\|h\|_s$.

Proposition 4.1 *Assume $u < 0$ and $0 < p < 2$. Then for h square integrable in B*

$$\int_B |h|^2 e^{-pu} = a_p \int_0^\infty e^{ps} \|h\|_s^2 ds + b_p \|h\|_0^2$$

for a_p and b_p suitable positive constants.

Proof First note that if $x < 0$

$$\int_0^\infty e^{ps} e^{-2\max(x+s,0)} ds = \int_0^{-x} e^{ps} ds + \int_{-x}^\infty e^{-2x} e^{(p-2)s} ds = C_p e^{-px} - 1/p.$$

Applying this with $x = u$ we find that

$$C_p \int_B |h|^2 e^{-pu} = \int_0^\infty e^{ps} \|h\|_s^2 ds + (1/p) \int_B |h|^2,$$

which proves the proposition. □

The moral of Proposition 4.1 is that we have translated questions about the norm of a scalar valued (holomorphic) function h over an n-dimensional domain B, to questions about the norm of a vector (A^2) valued (constant) function over the interval $(0, \infty)$. These norms $\|h\|_s$ depend only on $\mathrm{Re}\, s$ and enjoy a certain convexity property by Theorem 3.1, and we shall see how this reduces the openness problem to a problem about integrability of convex functions. The next very simple proposition illustrates the idea.

Proposition 4.2 *Let $k(s)$ be a convex function on $[0, \infty)$. Then*

$$\int_0^\infty e^{-k(s)} ds < \infty$$

if and only if

$$\lim_{s\to\infty} k(s)/s > 0.$$

Proof We may of course assume that $k(0) = 0$. Then $k(s)/s$ is increasing so its limit at infinity exists. If the limit is smaller than or equal to zero, then $k(s) \leq 0$ for all s, so the integral cannot converge. The other direction is obvious. □

It follows trivially from Proposition 4.2 that if

$$\int_0^\infty e^{s-k(s)} ds < \infty,$$

then for some $p > 1$

$$\int_0^\infty e^{ps-k(s)} ds < \infty.$$

In view of Proposition 4.1 this is a version of the openness statement for one dimensional spaces. The next theorem is a vector valued analog of Proposition 4.2.

Theorem 4.3 *Let H_0 be a (separable) Hilbert space equipped with a family of equivalent Hilbert norms $\|\cdot\|_s$ for $\operatorname{Re} s \geq 0$. Assume these norms depend only on the real part of s and define a hermitian metric on the trivial bundle $\Omega \times H_0$, where Ω is the right half plane, of positive curvature. Let H be the subspace of H_0 of elements h such that*

$$\|h\|^2 := \int_0^\infty e^s \|h\|_s^2 ds < \infty.$$

Then, for any h in H, $\epsilon > 0$ and $s > 1/\epsilon$ there is an element h_s in H_0 such that

$$\|h - h_s\|_0^2 \leq 2\epsilon \|h\|^2, \tag{5}$$

and

$$\|h_s\|_s^2 \leq e^{-(1+\epsilon)s} \|h\|_0^2. \tag{6}$$

Proof Take $\epsilon > 0$ and fix $s > 1/\epsilon$. By assumption there is a self adjoint bounded linear operator T_s on H_0 such that

$$\langle u, v\rangle_s = \langle T_s u, v\rangle_0.$$

By the spectral theorem (see [24]) we can (for s fixed!) realize our Hilbert space H_0 as an L^2-space over a measure space X, with respect to some positive measure $d\mu$, in such a way that

$$\|h\|_0^2 = \int_X |h|^2 d\mu(x)$$

and

$$\|h\|_s^2 = \int_X |h|^2 e^{-s\lambda(x)} d\mu(x).$$

We define h_s by $h_s = \chi(x)h$, where χ is the characteristic function of the set $\lambda > (1+\epsilon)$. Let $r_s = h - h_s$. Clearly

$$\|h_s\|_s^2 = \int_{\lambda > 1+\epsilon} |h|^2 e^{-s\lambda(x)} d\mu(x) \leq e^{-(1+\epsilon)s} \int_X |h|^2 d\mu = e^{-(1+\epsilon)s} \|h\|_0^2.$$

Hence (6) is satisfied. For (5) we will use a comparison with a flat family of metrics, which we define for $0 \leq \operatorname{Re} t \leq s$ by

$$|h|_t^2 = \int_X |h|^2 e^{-\operatorname{Re} t\lambda(x)} d\mu(x). \tag{7}$$

This is a flat metric in the sense that it is isometric to $\|\cdot\|_0 = |\cdot|_0$ via a map T_ζ, linear on H_0 and holomorphic in ζ. Indeed, if

$$h \to T_\zeta(h) := h_\zeta := h e^{\zeta\lambda/2},$$

then

$$|h_\zeta|_{\operatorname{Re}\zeta} = \|h\|_0.$$

Since $|h|_t$ coincides with $\|h\|_t^2$ for $t = 0$ and $t = s$ it follows that

$$\|h\|_t^2 \geq |h|_t^2$$

for t between 0 and s. This is a consequence of a minimum principle for positively curved metrics that we will return to shortly. Accepting this for the moment the argument continues as follows.

Since r_s and h_s are orthogonal for the scalar product defined by $|\cdot|_t$,

$$\int_0^s e^t \|h\|_t^2 dt \geq \int_0^s e^t |h|_t^2 dt \geq \int_0^s e^t |r_s|_t^2 dt.$$

By the definition of r_s

$$|r_s|_t^2 \geq e^{-t(1+\epsilon)} \|r_s\|_0^2.$$

Hence

$$\int_0^s e^t |r_s|_t^2 dt \geq \int_0^s e^{-\epsilon t} dt \|r_s\|_0^2 \geq 1/(2\epsilon)\|r_s\|_0^2,$$

since $s > 1/\epsilon$. All in all

$$\|r_s\|_0^2 \leq 2\epsilon \|h\|^2$$

so we have proved (5).

Let us now finally return to the minimum principle used above. For bundles of finite rank, this is a consequence of a well known theorem, see [1], Lemma 8.11, and the references there. In our case, when one of the bundles is flat, the proof is actually easier, as pointed out to us by László Lempert (Private communication), see also [20]. Let $\|\cdot\|_{-t}$ and $|\cdot|_{-t}$ be the dual norms of $\|\cdot\|_t$ and $|\cdot|_t$ respectively, with respect to the pairing $\langle\cdot,\cdot\rangle_0$. By this we mean that

$$\|h\|_{-t} = \sup_g |\langle g, h\rangle_0|,$$

with the supremum taken over all g in H_0 with $\|g\|_t \leq 1$, and $|\cdot|_{-t}$ is defined in an analogous way. Then clearly

$$|h|_{-t}^2 = \int_X |h|^2 e^{\mathrm{Re}\, t\lambda(x)} d\mu(x),$$

so $|\cdot|_{-t}$ is also a flat metric on the vector bundle $\Omega \times H_0$.

It suffices to prove that the negatively curved metric $\|\cdot\|_{-\tau}$ is smaller than the flat metric $|\cdot|_{-\tau}$ for $\mathrm{Re}\,\tau$ between 0 and s. Take $0 < t_0 < s$. Since $|\cdot|_{-\tau}$ is flat we can, as explained above find for any h in H_0 a holomorphic h_τ which equals h for $\tau = t_0$ and has $|h_\tau|_{-\tau} = \|h\|_0$ constant. Then

$$\psi(\tau) := \log \|h_\tau\|_{-\tau} - \log |h_\tau|_{-\tau}$$

is subharmonic and equal to zero when $\mathrm{Re}\,\tau$ equals zero or s. By the maximum principle $\psi(t_0) \leq 0$ so $\|h\|_{-t_0} \leq |h|_{-t_0}$ as we wanted. □

We are now ready to prove the openness theorem.

Theorem 4.4 *Let u be a negative plurisubharmonic function in the unit ball B. Assume that*

$$\int_B e^{-u} = A < \infty.$$

Then for $p < 1 + c_n/A$, where c_n is a constant depending only on the dimension,

$$\int_{B/2} e^{-pu} < \infty.$$

Proof We apply Theorem 4.3 to

$$H_0 = \{h \in H(B); \int_B |h|^2 < \infty\}$$

and $h = 1$. Note first that there is a constant δ_n such that if g is holomorphic in the ball and

$$\int_B |g|^2 \leq \delta_n$$

then $\sup_{B/2} |g| \leq 1/2$. By Theorem 4.3, we can for any $\epsilon > 0$ and $s > 1/\epsilon$ find a holomorphic function h_s in the ball such that

$$\|h_s\|_s^2 \leq |B| e^{-(1+\epsilon)s}$$

and

$$\|h - h_s\|_0^2 \leq 2\epsilon A.$$

If $\epsilon \leq \delta_n/(2A)$ it follows that $\sup_{B/2} |h - h_s| < 1/2$. Since h is identically 1, this implies that $|h_s| > 1/2$ on $B/2$. Hence

$$\int_{B/2} e^{-2u_s} \leq 4 \int_B |h_s|^2 e^{-2u_s} = 4\|h_s\|_s^2 \leq 4|B| e^{-(1+\epsilon)s}.$$

Hence, if $p < (1 + \epsilon/2)$

$$e^{ps} \int_{B/2} e^{-2u_s} \leq C e^{-s\epsilon/2}.$$

Integrating from $1/\epsilon$ to infinity we find by applying Proposition 4.1 again (with B replaced by $B/2$) that

$$\int_{B/2} e^{-pu} < \infty.$$

□

5 A Conjectural Picture for Strong Openness

Let us first state the 'strong openness conjecture' from Demailly [9]. It says that if as before u is plurisubharmonic in the ball and h is holomorphic in the ball, then the set of $p > 0$ such that

$$|h|^2 e^{-pu}$$

is integrable in some neighbourhood of the origin, is open. The original openness conjecture is thus the case of strong openness when $h = 1$. The strong openness conjecture was proved by Guan-Zhou in [13], see also Hiep [16] and Lempert [19]. Here we will discuss how this problem might be related to the methods described above, in the simpler case when we look at integrability over a compact manifold instead of some neighbourhood of the origin.

First of all, to motivate our discussion, let us say a few words about the openness problem in one variable. Then the subharmonic function u can be written locally as the sum of a harmonic part – which does not affect the integrability – and a potential

$$P(z) = \int \log|z - \zeta|^2 d\mu(\zeta),$$

where $\mu = \Delta u$ is a positive measure. It is very easy to see that

$$e^{-pu}$$

is integrable in some neighbourhood of the origin if and only if $\mu(\{0\}) < 1/p$. In the same way

$$|h|^2 e^{-pu}$$

is integrable if and only if $\mu(\{0\}) < (1 + k)/p$, where k is the order of the zero of h at the origin.

We now elaborate a little bit on the 'moral' of Proposition 4.1 as described in Sect. 4. To simplify matters somewhat we consider a variant of the setting in Sect. 4, where instead of a space of holomorphic functions in the ball we look at the space of sections of a line bundle over a compact manifold, so that we are dealing with finite dimensional spaces.

Let $L \to X$ be an ample line bundle over a compact manifold. We will consider the space $H^0(X, K_X + kL)$ of holomorphic sections of the adjoint bundles $K_X + kL$. On L we consider two metrics, ϕ and ϕ_0, where ϕ_0 is positively curved and smooth whereas ϕ is a singular metric with $i\partial\bar\partial\phi \geq 0$. We are interested in when integrals

$$\int_X |h|^2 e^{-(\phi+(k-1)\phi_0)},$$

with h in $H^0(X, K_X + kL)$, are finite. Let $u = (\phi - \phi_0)$. Then

$$\int_X |h|^2 e^{-(\phi+(k-1)\phi_0)} = \int_X |h|^2 e^{-(u+k\phi_0)},$$

and u is ω-plurisubharmonic for $\omega = i\partial\bar{\partial}\phi_0$, i.e. $i\partial\bar{\partial}\phi + \omega \geq 0$. As in the previous section we put $u_s = \max(u + \operatorname{Re} s, 0)$ for $\operatorname{Re} s \geq 0$. Then u_s is ω-plurisubharmonic on the product of the halfplane, Ω, with X. We let

$$\|h\|_s^2 := \int_X |h|^2 e^{-(2u_s+k\phi_0)}.$$

If $k \geq 2$ then $i\partial\bar{\partial}_{s,X}(2u_s + k\phi_0) \geq 0$. Hence we can apply the manifold version of Theorem 3.1 from [3] and conclude that the trivial vector bundle $\Omega \times H^0(X, K_x + kL)$ is positively curved when equipped with the metric $\| \cdot \|_s$. As before we get that if h is in $H^0(X, K_X + kL)$ then

$$\int_X |h|^2 e^{-(pu+k\phi_0)} = a_p \int_0^\infty e^{ps} \|h\|_s^2 ds + b_p \|h\|_0. \tag{8}$$

Since the metric on E depends only on the real part of s we can make a change of variables $s = -\log \zeta$, where ζ is in the punctured unit disk. Abusing notation slightly we let $\|h\|_\zeta = \|h\|_s$ if $s = -\log \zeta$. Then

$$\int_0^\infty e^{ps} \|h\|_s^2 ds = (2\pi)^{-1} \int_\Delta e^{-(p+2)\log|\zeta|} \|h\|_\zeta^2 d\lambda(\zeta). \tag{9}$$

We can now extend (the trivial) bundle E as a vector bundle over the entire disk, including the origin, and consider

$$e^{-(p+2)\log|\zeta|} \|h\|_\zeta^2$$

as a singular metric defined over the whole disk, see [6, 22]. The only serious singularities of the metric are of course at the origin. Ideally, we would now have that this singular metric has a curvature Θ which has a smooth (or almost smooth) part outside the origin, plus a singular part Θ_{sing} supported at the origin. With respect to a holomorphic frame, Θ_{sing} would then be represented by a matrix of Dirac masses at the origin which could be diagonalized in a suitable frame. The convergence of (9) should then be governed by the sizes of these Dirac masses, so that

$$\int_X |h|^2 e^{-\phi+(k-1)\phi_0},$$

is finite precisely when h lies in the union of the eigenspaces corresponding to Dirac masses strictly smaller than 1. In fact, our discussion of the one dimensional openness conjecture above says precisely that this holds when the rank of the bundle is one.

There are obstacles to making this picture rigorous. First, Raufi [22], has given an example of a positively curved singular vector bundle metric, with only an isolated singularity, whose curvature does not have measure coefficients, but contains derivatives of Dirac measures. Still, it might be true that this cannot occur if the metric is S^1-invariant as it is in our case. If this turns out to be so it seems likely that the rest of the argument would go through so that the study of multiplier ideals, at least over compact manifolds, could be reduced to a vector valued problem over the disk.

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Estimates for $\bar{\partial}$ and Optimal Constants

Zbigniew Błocki

Dedicated to Professor Yum-Tong Siu on the occasion of his 70th birthday

1 Introduction

The fundamental extension result of Ohsawa-Takegoshi [12] says that if Ω is a pseudoconvex domain and H is an affine complex subspace of $\mathbb{C}^n$ then for any plurisubharmonic φ in Ω ($\varphi \equiv 0$ is an especially interesting case) and any holomorphic f in $\Omega' := \Omega \cap H$ there exists a holomorphic extension F to Ω satisfying the estimate

$$\int_\Omega |F|^2 e^{-\varphi} d\lambda \leq C\pi \int_{\Omega'} |f|^2 e^{-\varphi} d\lambda', \tag{1}$$

where C is a constant depending only on n and the diameter of Ω.

The original proof of this result used $\bar{\partial}$-theory on complete Kähler manifolds and complicated commutator identities. This approach was simplified by Siu [13] who used only Hörmander's formalism in $\mathbb{C}^n$ and proved in addition that the constant C depends only on the distance of Ω from H: he showed that if $\Omega \subset \{|z_n| < 1\}$ and $H = \{z_n = 0\}$ then one can take $C = 64/9\sqrt{1 + 1/4e} = 6.80506\ldots$ in (1). This was improved to $C = 4$ in [1] and $C = 1.95388\ldots$ in [10]. The optimal constant here, $C = 1$, was recently obtained in [6]. A slightly more general result was shown: if $\Omega \subset \mathbb{C}^{n-1} \times D$ and $0 \in D$ then (1) holds with $C = c_D(0)^{-2}$, where $c_D(0)$ is the logarithmic capacity of $\mathbb{C} \setminus D$ with respect to 0. This gave in particular

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a one-dimensional estimate

$$c_D(z)^2 \le \pi K_D(z,z),$$

where K_D is the Bergman kernel, and settled a conjecture of Suita [14].

The main tool in proving the optimal version of (1) was a new L^2–estimate for $\bar\partial$. On one hand, this new result, using some ideas of Berndtsson [2] and B.-Y. Chen [8], easily follows from the classical Hörmander estimate [11]. On the other hand, it also implies some other $\bar\partial$-estimates due to Donnelly-Fefferman and Berndtsson, even with optimal constants as will turn out. Important contribution here is due to B.-Y. Chen [8] who showed that the Ohsawa-Takegoshi theorem, unlike in [12, 13] or [1], can be deduced directly from Hörmander's estimate.

2 Estimates for $\bar\partial$

Let Ω be a pseudoconvex domain in $\mathbb{C}^n$. For

$$\alpha = \sum_j \alpha_j d\bar z_j \in L^2_{loc,(0,1)}(\Omega)$$

we look for $u \in L^2_{loc}(\Omega)$ solving the equation

$$\bar\partial u = \alpha. \tag{2}$$

Such u always exists and we are interested in weighted L^2-estimates for solutions of (2).

The classical one is due to Hörmander [11]: for smooth, strongly plurisubharmonic φ in Ω one can find a solution of (2) satisfying

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \le \int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi} e^{-\varphi} d\lambda, \tag{3}$$

where

$$|\alpha|^2_{i\partial\bar\partial\varphi} = \sum_{j,k} \varphi^{jk} \bar\alpha_j \alpha_k$$

is the length of α with respect to the Kähler metric with potential φ. (Here (φ^{jk}) is the inverse transposed of the complex Hessian $(\partial^2\varphi/\partial z_j \partial\bar z_k)$.) It was observed in [4] that the Hörmander estimate (3) also holds for arbitrary plurisubharmonic φ but one should replace $|\alpha|^2_{i\partial\bar\partial\varphi}$ with any nonnegative $H \in L^\infty_{loc}(\Omega)$ satisfying

$$i\bar\alpha \wedge \alpha \le H\, i\partial\bar\partial\varphi.$$

Another very useful estimate (see e.g. [7]) for (2) is due to Donnelly-Feffermann [9]: if ψ is another plurisubharmonic function in Ω such that

$$i\partial\psi \wedge \bar\partial\psi \le i\partial\bar\partial\psi$$

(that is $|\bar\partial\psi|^2_{i\partial\bar\partial\psi} \le 1$) then there exists a solution of (2) with

$$\int_\Omega |u|^2 e^{-\varphi} d\lambda \le C \int_\Omega |\alpha|^2_{i\partial\bar\partial\psi} e^{-\varphi} d\lambda, \tag{4}$$

where C is an absolute constant. We will show that $C = 4$ is optimal here.

The Donnelly-Feffermann estimate (4) was generalized by Berndtsson [1]: if $0 \le \delta < 1$ then we can find appropriate u with

$$\int_\Omega |u|^2 e^{\delta\psi-\varphi} d\lambda \le \frac{4}{(1-\delta)^2} \int_\Omega |\alpha|^2_{i\partial\bar\partial\psi} e^{\delta\psi-\varphi} d\lambda. \tag{5}$$

This particular constant was obtained in [3] (originally in [1] it was $\frac{4}{\delta(1-\delta)^2}$) and we will prove in Sect. 3 that it is the best possible.

Berndtsson's estimate (5) is closely related to the Ohsawa-Takegoshi extension theorem [12] but the latter cannot be deduced from it directly (it could be if (5) were true for $\delta = 1$). The following version from [5] makes up for this disadvantage: if in addition $|\bar\partial\psi|^2_{i\partial\bar\partial\psi} \le \delta < 1$ on $\operatorname{supp}\alpha$ then we can find a solution of (2) with

$$\int_\Omega |u|^2 (1 - |\bar\partial\psi|^2_{i\partial\bar\partial\psi}) e^{\psi-\varphi} d\lambda \le \frac{1}{(1-\sqrt{\delta})^2} \int_\Omega |\alpha|^2_{i\partial\bar\partial\psi} e^{\psi-\varphi} d\lambda. \tag{6}$$

The best constant in the Ohsawa-Takegoshi theorem that one can get from (6) is $1.95388\ldots$ (see [5]), originally obtained in [10].

To get the optimal constant 1 in the Ohsawa-Takegoshi theorem the following estimate for $\bar\partial$ was obtained in [6]:

Theorem 1 *Assume that $\alpha \in L^2_{loc,(0,1)}(\Omega)$ is $\bar\partial$-closed form in a pseudoconvex domain Ω in $\mathbb{C}^n$. Let φ be plurisubharmonic in Ω and $\psi \in W^{1,2}_{loc}(\Omega)$, locally bounded from above, satisfy $|\bar\partial\psi|^2_{i\partial\bar\partial\varphi} \le 1$ in Ω and $|\bar\partial\psi|^2_{i\partial\bar\partial\varphi} \le \delta$ on $\operatorname{supp}\alpha$. Then there exists $u \in L^2_{loc}(\Omega)$ solving (2) and such that*

$$\int_\Omega |u|^2 (1 - |\bar\partial\psi|^2_{i\partial\bar\partial\varphi}) e^{2\psi-\varphi} d\lambda \le \frac{1+\sqrt{\delta}}{1-\sqrt{\delta}} \int_\Omega |\alpha|^2_{i\partial\bar\partial\varphi} e^{2\psi-\varphi} d\lambda. \tag{7}$$

Theorem 1 can be quite easily deduced from the Hörmander estimate (3) using some ideas of Berndtsson [2] and Chen [8], see [6] for details. On the other hand, note that we can recover (3) from Theorem 1 if we take $\psi \equiv 0$. We can also easily get (5): take $\tilde{\varphi} = \varphi + \psi$ and $\tilde{\psi} = \frac{1+\delta}{2}\psi$. Then $2\tilde{\psi} - \tilde{\varphi} = \delta\psi - \varphi$ and

$$|\bar{\partial}\tilde{\psi}|^2_{i\partial\bar{\partial}\tilde{\varphi}} \leq \frac{(1+\delta)^2}{4} =: \tilde{\delta}$$

(since $|\bar{\partial}\psi|^2_{i\partial\bar{\partial}\psi} \leq 1$). From (7) we obtain (5) with the constant

$$\frac{1+\sqrt{\tilde{\delta}}}{(1-\sqrt{\tilde{\delta}})(1-\tilde{\delta})} = \frac{4}{(1-\delta)^2}.$$

3 Optimal Constants

We will show that the constant in (5) is optimal for every δ. For $\delta = 0$ this gives $C = 4$ in the Donnelly-Fefferman estimate (4). We consider $\Omega = \Delta$, the unit disc, $\varphi \equiv 0$ and $\psi(z) = -\log(-\log|z|)$, so that

$$\psi_{z\bar{z}} = |\psi_z|^2 = \frac{1}{4|z|^2\log^2|z|}.$$

We also take functions of the form

$$v(z) = \frac{\eta(-\log|z|)}{z} \tag{8}$$

for $\eta \in C_0^1([0,\infty))$, and set

$$\alpha := \bar{\partial} v = -\frac{\eta'(-\log|z|)}{2|z|^2}\, d\bar{z}. \tag{9}$$

The crucial observation is that v is the minimal solution to $\bar{\partial}u = \alpha$ in $L^2(\Delta, e^{\delta\psi})$. Indeed, using polar coordinates we can easily show that $\{z^n\}_{n\geq 0}$ is an orthogonal system in $L^2(\Delta, e^{\delta\psi}) \cap \ker\bar{\partial}$ and that

$$\langle v, z^n\rangle_{L^2(\Delta, e^{\delta\psi})} = 0, \quad n = 0, 1, \ldots$$

Berndtsson's estimate (5) now gives the following version of the Hardy-Poincaré inequality

$$\int_0^\infty \eta^2 t^{-\delta} dt \leq \frac{4}{(1-\delta)^2}\int_0^\infty (\eta')^2 t^{2-\delta} dt \tag{10}$$

if $0 \leq \delta < 1$ and $\eta \in C_0^1([0,\infty))$.

We are thus reduced to proving that this constant is optimal:

Proposition 2 *The constant* $4/(1-\delta)^2$ *in* (10) *cannot be improved.*

Proof Set

$$\eta(t) = \begin{cases} t^{-a}, & 0 < t \le 1 \\ t^{-b}, & t \ge 1. \end{cases}$$

Then both left and right-hand sides of (10) are finite iff $a < (1-\delta)/2$ and $b > (1-\delta)/2$. Assuming this, and since $\eta(t)$ is monotone and converges to 0 as $t \to \infty$, we can find an appropriate approximating sequence in $C_0^1([0,\infty))$. Thus (10) holds also for this η. We compute

$$\int_0^\infty \eta^2 t^{-\delta} dt = \frac{1}{1-\delta-2a} + \frac{1}{\delta-1+2b}$$

and

$$\int_0^\infty (\eta')^2 t^{2-\delta} dt = \frac{a^2}{1-\delta-2a} + \frac{b^2}{\delta-1+2b}.$$

The ratio between these quantities is equal to

$$\frac{2}{(1-\delta)(a+b)-2ab}$$

and it tends to $4/(1-\delta)^2$ as both a and b tend to $(1-\delta)/2$. □

Finally, since the same argument would work for any radially symmetric weights in Δ or an annulus $\{r < |z| < 1\}$ where $0 \le r < 1$, from (5) with α given by (9) and φ, ψ of the form $\varphi = g(-\log|z|)$, $\psi = h(-\log|z|)$ we can get the following weighted Poincaré inequalities:

Theorem 3 *Let* g, h *be convex, decreasing functions on* $(0,\infty)$. *Assume in addition that* h *is* C^2 *smooth,* $h'' > 0$ *and* $(h')^2 \le h''$. *Then, if* $0 \le \delta < 1$, *for* $\eta \in C_0^1([0,\infty))$ *one has*

$$\int_0^\infty \eta^2 e^{\delta h - g} dt \le \frac{4}{(1-\delta)^2} \int_0^\infty \frac{(\eta')^2}{h''} e^{\delta h - g} dt.$$

□

Theorem 4 *Let* g, h *be convex functions on* $(0,T)$, *where* $0 < T \le \infty$. *Assume that* h *is* C^2 *smooth,* $h'' > 0$ *and* $(h')^2 \le h''$. *If* $0 \le \delta < 1$ *it follows that for any* $\eta \in W^{1,2}_{loc}((0,T))$ *with*

$$\int_0^T \eta e^{\delta h - g} dt = 0 \tag{11}$$

we have

$$\int_0^T \eta^2 e^{\delta h - g} dt \le \frac{4}{(1-\delta)^2} \int_0^T \frac{(\eta')^2}{h''} e^{\delta h - g} dt$$

provided that both integrals exist. □

The condition (11) is necessary to ensure that in the case of an annulus the solution given by (8) is minimal in the $L^2(\{r < |z| < 1\}, e^{\delta\psi - \varphi})$-norm: it is enough to check that it is perpendicular to every element of the orthogonal system $\{z^k\}_{k\in\mathbb{Z}}$ in $\ker\bar\partial$. For $k \neq -1$ it is sufficient to use the fact that the weight is radially symmetric and for $k = -1$ one has to use (11).

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On the Cohomology of Pseudoeffective Line Bundles

Jean-Pierre Demailly

Dedicated to Professor Yum-Tong Siu on the occasion of his 70th birthday

1 Introduction and Statement of the Main Results

Let X be a compact Kähler n-dimensional manifold, equipped with a Kähler metric, i.e. a positive definite Hermitian $(1,1)$-form $\omega = \mathrm{i} \sum_{1\le j,k\le n} \omega_{jk}(z)\, dz_j \wedge d\overline{z}_k$ such that $d\omega = 0$. By definition a holomorphic line bundle L on X is said to be *pseudo-effective* if there exists a singular hermitian metric h on L, given by $h(z) = e^{-\varphi(z)}$ with respect to a local trivialization $L_{|U} \simeq U \times \mathbb{C}$, such that the curvature form

$$\mathrm{i}\,\Theta_{L,h} := \mathrm{i}\,\partial\overline{\partial}\varphi \tag{1}$$

is (semi) positive in the sense of currents, i.e. φ is locally integrable and $i\Theta_{L,h} \ge 0$: in other words, the weight function φ is plurisubharmonic (psh) on the corresponding trivializing open set U. A basic concept is the notion of *multiplier ideal sheaf*, introduced in [50].

Definition 1 To any psh function φ on an open subset U of a complex manifold X, one associates the "multiplier ideal sheaf" $\mathscr{I}(\varphi) \subset \mathscr{O}_{X|U}$ of germs of holomorphic functions $f \in \mathscr{O}_{X,x}$, $x \in U$, such that $|f|^2 e^{-\varphi}$ is integrable with respect to the Lebesgue measure in some local coordinates near x. We also define the global multiplier ideal sheaf $\mathscr{I}(h) \subset \mathscr{O}_X$ of a hermitian metric h on $L \in \mathrm{Pic}(X)$ to be equal to $\mathscr{I}(\varphi)$ on any open subset U where $L_{|U}$ is trivial and $h = e^{-\varphi}$. In such a

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definition, we may in fact assume $\mathrm{i}\,\Theta_{L,h} \geq -C\omega$, i.e. locally $\varphi = \mathrm{psh} + C^\infty$, we say in that case that φ is *quasi-psh*.

Let us observe that a multiplier ideal sheaf $\mathscr{I}(\varphi)$ is left unmodified by adding a smooth function to φ; for such purposes, the additional C^∞ terms are irrelevant in quasi-psh functions. A crucial and well-known fact is that the ideal sheaves $\mathscr{I}(\varphi) \subset \mathscr{O}_{X|U}$ and $\mathscr{I}(h) \subset \mathscr{O}_X$ are always *coherent analytic sheaves*; when $U \subset X$ is a coordinate open ball, this can be shown by observing that $\mathscr{I}(\varphi)$ coincides with the locally stationary limit $\mathscr{J} = \lim\uparrow_{N\to+\infty}\mathscr{J}_N$ of the increasing sequence of coherent ideals $\mathscr{J}_N = (g_j)_{0\leq j<N}$ associated with a Hilbert basis $(g_j)_{j\in\mathbb{N}}$ of the Hilbert space of holomorphic functions $f \in \mathscr{O}_X(U)$ such that $\int_U |f|^2 e^{-\varphi} dV_\omega < +\infty$. The proof is a consequence of Hörmander's L^2 estimates applied to weights of the form

$$\psi(z) = \varphi(z) + (n+k)\log|z-x|^2.$$

This easily shows that $\mathscr{I}(\varphi)_x + \mathfrak{m}_x^k = \mathscr{J}_x + \mathfrak{m}_x^k$, and one then concludes that $\mathscr{I}(\varphi)_x = \mathscr{J}_x$ by the Krull lemma. When X is *projective algebraic*, Serre's GAGA theorem implies that $\mathscr{I}(h)$ is in fact a *coherent algebraic sheaf*, in spite of the fact that φ may have very "wild" analytic singularities – e.g. they might be everywhere dense in X in the Euclidean topology. Therefore, in some sense, the multiplier ideal sheaf is a powerful tool to extract algebraic (or at least analytic) data from arbitrary singularities of psh functions. In this context, assuming strict positivity of the curvature, one has the following well-known fundamental vanishing theorem.

Theorem 1 (Nadel Vanishing Theorem, [22, 50]) *Let (X, ω) be a compact Kähler n-dimensional manifold, and let L be a holomorphic line bundle over X equipped with a singular Hermitian metric h. Assume that $\mathrm{i}\,\Theta_{L,h} \geq \varepsilon\omega$ for some $\varepsilon > 0$ on X. Then*

$$H^q\big(X, \mathscr{O}(K_X \otimes L) \otimes \mathscr{I}(h)\big) = 0 \qquad \text{for all } q \geq 1,$$

where $K_X = \Omega_X^n = \Lambda^n T_X^*$ denotes the canonical line bundle.

The proof follows from an application of Hörmander's L^2 estimates with singular weights, themselves derived from the Bochner-Kodaira identity (see [19, 21, 40]). One should observe that the strict positivity assumption implies L to be big, hence X must be projective, since every compact manifold that is Kähler and Moishezon is also projective (cf. [48, 52, 53]). However, when relaxing the strict positivity assumption, one can enter the world of general compact Kähler manifolds, and their study is one of our main goals.

In many cases, one has to assume that the psh functions involved have milder singularities. We say that a psh or quasi-psh function φ has *analytic singularities* if locally on the domain of definition U of φ one can write

$$\varphi(z) = c \log \sum_{j=1}^{N} |g_j|^2 + O(1) \tag{2}$$

where the g_j's are holomorphic functions, $c \in \mathbb{R}_+$ and $O(1)$ means a locally bounded remainder term. Assumption (2) implies that the set of poles $Z = \varphi^{-1}(-\infty)$ is an analytic set, locally defined as $Z = \bigcap g_j^{-1}(0)$, and that φ is locally bounded on $U \smallsetminus Z$. We also refer to this situation by saying that φ has *logarithmic poles*. In general, one introduces the following comparison relations for psh or quasi-psh functions φ and hermitian metrics $h = e^{-\varphi}$; a more flexible comparison relation will be introduced in Sect. 5.

Definition 2 Let φ_1, φ_2 be psh functions on an open subset U of a complex manifold X. We say that

(a) φ_1 has less singularities than φ_2, and write $\varphi_1 \preccurlyeq \varphi_2$, if for every point $x \in U$, there exists a neighborhood V of x and a constant $C \geq 0$ such that $\varphi_1 \geq \varphi_2 - C$ on V.
(b) φ_1 and φ_2 have equivalent singularities, and write $\varphi_1 \sim \varphi_2$, if locally near any point of U we have $\varphi_1 - C \leq \varphi_2 \leq \varphi_1 + C$.

Similarly, given a pair of hermitian metrics h_1, h_2 on a line bundle $L \to X$,

(a') we say that h_1 is less singular than h_2, and write $h_1 \preccurlyeq h_2$, if locally there exists a constant $C > 0$ such that $h_1 \leq Ch_2$.
(b') we say that h_1, h_2 have equivalent singularities, and write $h_1 \sim h_2$, if locally there exists a constant $C > 0$ such that $C^{-1}h_2 \leq h_1 \leq Ch_2$.

(of course when h_1 and h_2 are defined on a compact manifold X, the constant C can be taken global on X in (a') and (b')).

Important features of psh singularities are the semi-continuity theorem (see [27]) and the strong openness property recently proved by Guan and Zhou [36–38]. Let U be an open set in a complex manifold X and φ a psh function on U. Following [27], we define the log canonical threshold of φ at a point $z_0 \in U$ by

$$c_{z_0}(\varphi) = \sup\left\{c > 0 : \ e^{-2c\varphi} \text{ is } L^1 \text{ on a neighborhood of } z_0\right\} \in \,]0, +\infty] \tag{3}$$

(Here L^1 integrability refers to the Lebesgue measure with respect to local coordinates). It is an important invariant of the singularity of φ at z_0. We refer to [25–27, 29, 45, 50, 55, 61] for further information about properties of the log canonical threshold. In this setting, the semi-continuity theorem can be stated as follows.

Theorem 2 (cf. [27]) *For any given $z_0 \in U$, the map* $\mathrm{PSH}(U) \to \,]0, +\infty]$, $\varphi \mapsto c_{z_0}(\varphi)$ *is upper semi-continuous with respect to the topology of weak convergence on the space of psh functions* (*the latter topology being actually the same as the topology of L^1_{loc} convergence*).

The original proof of [27] was rather involved and depended on uniform polynomial approximation, combined with a reduction to a semi-continuity theorem for algebraic singularities; the Ohsawa-Takegoshi L^2 extension theorem [51] was used in a crucial way. We will give here a simpler and more powerful derivation due to Hiep [54], still depending on the Ohsawa-Takegoshi theorem, that simultaneously yields effective versions of Berndtsson's result [3] on the openness conjecture, as well as Guan and Zhou's proof of the strong openness conjecture for multiplier ideal sheaves.

Theorem 3 ([36–38]) *Let φ, ψ_j, $j \in \mathbb{N}$, be psh functions on an open set U in a complex manifold X. Assume that $\psi_j \le \varphi$ and that ψ_j converges to φ in L^1_{loc} topology as $j \to +\infty$. Then for every relatively compact subset $U' \Subset U$, the multiplier ideal sheaves $\mathscr{I}(\psi_j)$ coincide with $\mathscr{I}(\varphi)$ on U' for $j \ge j_0(U') \gg 1$.*

Before going further, notice that the family of multiplier ideals $\lambda \mapsto \mathscr{I}(\lambda\varphi)$ associated with a psh function φ is nonincreasing in $\lambda \in \mathbb{R}_+$. By the Noetherian property of ideal sheaves, they can jump only for a locally finite set of values λ in $[0, +\infty[$, and in particular, there exists a real value $\lambda_0 > 1$ such that

$$\mathscr{I}_+(\varphi) := \lim_{\varepsilon \to 0+} \mathscr{I}((1+\varepsilon)\varphi) = \mathscr{I}(\lambda\varphi), \quad \forall \lambda \in \,]1, \lambda_0]. \tag{4}$$

We will say that $\mathscr{I}_+(\varphi)$ is the upper semicontinuous regularization of the multiplier ideal sheaf. Berndtsson's result [3] states that the equality $\mathscr{I}(\varphi) = \mathscr{O}_X$ implies $\mathscr{I}_+(\varphi) = \mathscr{O}_X$. If we take $\psi_j = (1 + 1/j)\varphi$ and assume (without loss of generality) that $\varphi \le 0$, Theorem 3 implies in fact

Corollary 1 *For every psh function φ, the upper semicontinuous regularization coincides with the multiplier ideal sheaf, i.e. $\mathscr{I}_+(\varphi) = \mathscr{I}(\varphi)$.*

Now, if L is a pseudoeffective line bundle, it was observed in [23] that there always exists a unique equivalence class $h_{\min}$ of singular hermitian metrics with minimal singularities, such that $\mathrm{i}\,\Theta_{L,h_{\min}} \ge 0$ (by this we mean that $h_{\min}$ is unique up to equivalence of singularities). In fact, if h_∞ is a smooth metric on L, one can define the corresponding weight $\varphi_{\min}$ of $h_{\min}$ as an upper envelope

$$\varphi_{\min}(z) = \sup\big\{\varphi(z)\,;\ \mathrm{i}\,\Theta_{L,h_\infty} + \mathrm{i}\,\partial\overline{\partial}\varphi \ge 0,\ \varphi \le 0 \text{ on } X\big\}, \tag{5}$$

and put $h_{\min} = h_\infty e^{-\varphi_{\min}}$. In general, $h_{\min}$ need not have analytic singularities.

An important fact is that one can approximate arbitrary psh functions by psh functions with analytic singularities. The appropriate technique consists of using an asymptotic Bergman kernel procedure (cf. [21] and Sect. 2). If φ is a holomorphic function on a ball $B \subset \mathbb{C}^n$, one puts

$$\varphi_m(z) = \frac{1}{2m} \log \sum_{\ell \in \mathbb{N}} |g_{m,\ell}(z)|^2$$

where $(g_{m,\ell})_{\ell \in \mathbb{N}}$ is a Hilbert basis of the space $\mathscr{H}(B, m\varphi)$ of L^2 holomorphic functions on B such that $\int_B |f|^2 e^{-2m\varphi} dV < +\infty$. When $T = \alpha + dd^c\varphi$ is a closed $(1,1)$-current on X in the same cohomology class as a smooth $(1,1)$-form α and φ is a quasi-psh potential on X, a sequence of global approximations T_m can be produced by taking a finite covering of X by coordinate balls (B_j). A partition of unity argument allows to glue the local approximations $\varphi_{m,j}$ of φ on B_j into a global potential φ_m, and one sets $T_m = \alpha + dd^c\varphi_m$. These currents T_m converge weakly to T, are smooth in the complement $X \smallsetminus Z_m$ of an increasing family of analytic subsets $Z_m \subset X$, and their singularities approach those of T. More precisely, the Lelong numbers $\nu(T_m, z)$ converge uniformly to those of T, and whenever $T \geq 0$, it is possible to produce a current T_m that only suffers a small loss of positivity, namely $T_m \geq -\varepsilon_m \omega$ where $\lim_{m \to +\infty} \varepsilon_m = 0$. These considerations lead in a natural way to the concept of *numerical dimension* of a closed positive $(1,1)$-current T. We define

$$\mathrm{nd}(T) = \max \Big\{p = 0, 1, \ldots, n\,;\ \limsup_{m \to +\infty} \int_{X \smallsetminus Z_m} (T_m + \varepsilon_m \omega)^p \wedge \omega^{n-p} > 0\Big\}. \tag{6}$$

One can easily show (see Sect. 5) that the right hand side of (6) does not depend on the sequence (T_m), provided that the singularities approach those of T (we call this an “asymptotically equisingular approximation”).

These concepts are very useful to study cohomology groups with values in pseudoeffective line bundles (L, h). Without assuming any strict positivity of the curvature, one can obtain at least a hard Lefschetz theorem with coefficients in L. The technique is based on a use of harmonic forms with respect to suitable “equisingular approximations” φ_m of the weight φ of h (in that case we demand that $\mathscr{I}(\varphi_m) = \mathscr{I}(\varphi)$ for all m); the main idea is to work with complete Kähler metrics in the open complements $X \smallsetminus Z_m$ where φ_m is smooth, and to apply a variant of the Bochner formula on these sets. More details can be found in Sect. 4 and in [31].

Theorem 4 ([31]) *Let (L, h) be a pseudo-effective line bundle on a compact Kähler manifold (X, ω) of dimension n, let $\Theta_{L,h} \geq 0$ be its curvature current and $\mathscr{I}(h)$ the associated multiplier ideal sheaf. Then, the wedge multiplication operator $\omega^q \wedge \bullet$ induces a surjective morphism*

$$\Phi^q_{\omega,h} : H^0(X, \Omega_X^{n-q} \otimes L \otimes \mathscr{I}(h)) \longrightarrow H^q(X, \Omega_X^n \otimes L \otimes \mathscr{I}(h)).$$

The special case when L is nef is due to Takegoshi [62]. An even more special case is when L is semipositive, i.e. possesses a smooth metric with semipositive curvature. In that case the multiple ideal sheaf $\mathscr{I}(h)$ coincides with $\mathscr{O}_X$ and we get the following consequence already observed by Enoki [33] and Mourougane [49].

Corollary 2 *Let (L, h) be a semipositive line bundle on a compact Kähler manifold (X, ω) of dimension n. Then, the wedge multiplication operator $\omega^q \wedge \bullet$ induces a surjective morphism*

$$\Phi^q_\omega : H^0(X, \Omega^{n-q}_X \otimes L) \longrightarrow H^q(X, \Omega^n_X \otimes L).$$

It should be observed that although all objects involved in Theorem 4 are algebraic when X is a projective manifold, there is no known algebraic proof of the statement; it is not even clear how to define algebraically $\mathscr{I}(h)$ for the case when $h = h_{min}$ is a metric with minimal singularity. However, even in the special circumstance when L is nef, the multiplier ideal sheaf is crucially needed.

The next statement is taken from the PhD thesis of Junyan Cao [13]. The proof is a combination of our Bergman regularization techniques, together with an argument of Ch. Mourougane [49] relying on a use of the Calabi-Yau theorem for Monge-Ampère equations.

Theorem 5 ([13, 14]) *Let (L, h) be a pseudoeffective line bundle on a compact Kähler n-dimensional manifold X. Then*

$$H^q(X, K_X \otimes L \otimes \mathscr{I}(h)) = 0 \qquad \text{for every } \ q \geq n - \mathrm{nd}(L, h) + 1,$$

where $\mathrm{nd}(L, h) := \mathrm{nd}(\mathrm{i}\,\Theta_{L,h})$.

Cao's technique of proof actually yields the result for the upper semicontinuous regularization

$$\mathscr{I}_+(h) = \lim_{\varepsilon \to 0} \mathscr{I}(h^{1+\varepsilon}) \tag{7}$$

instead of $\mathscr{I}(h)$, but we can apply Guan-Zhou's Theorem 3 to see that the equality $\mathscr{I}_+(h) = \mathscr{I}(h)$ always holds. As a final geometric application of this circle of ideas, we present the following result which was obtained in [16].

Theorem 6 ([16]) *Let X be a compact Kähler threefold that is "strongly simple" in the sense that it has no nontrivial analytic subvariety. Then the Albanese morphism $\alpha : X \to \mathrm{Alb}(X)$ is a biholomorphism, and therefore X is biholomorphic to a 3-dimensional complex torus $\mathbb{C}^3/\Lambda$.*

I would like to thank the referee wholeheartedly for numerous suggestions that led to substantial improvements of the exposition.

2 Approximation of psh Functions and of Closed (1,1)-Currents

We first recall here the basic result on the approximation of psh functions by psh functions with analytic singularities. The main idea is taken from [Dem92] and relies on the Ohsawa-Takegoshi extension theorem, For other applications to algebraic geometry, see [Dem93b] and Demailly-Kollár [DK01]. Let φ be a psh function on an open set $\Omega \subset \mathbb{C}^n$. Recall that the Lelong number of φ at a point $x_0 \in \Omega$ is defined to be

$$\nu(\varphi, x_0) = \liminf_{z \to x_0} \frac{\varphi(z)}{\log|z - x_0|} = \lim_{r \to 0_+} \frac{\sup_{B(x_0,r)} \varphi}{\log r}. \tag{8}$$

In particular, if $\varphi = \log|f|$ with $f \in \mathcal{O}(\Omega)$, then $\nu(\varphi, x_0)$ is equal to the vanishing order

$$\mathrm{ord}_{x_0}(f) = \sup\{k \in \mathbb{N}\,; D^\alpha f(x_0) = 0,\ \forall |\alpha| < k\}.$$

Theorem 7 *Let φ be a plurisubharmonic function on a bounded pseudoconvex open set $\Omega \subset \mathbb{C}^n$. For every $m > 0$, let $\mathcal{H}_\Omega(m\varphi)$ be the Hilbert space of holomorphic functions f on Ω such that $\int_\Omega |f|^2 e^{-2m\varphi} dV_{2n} < +\infty$ and let $\varphi_m = \frac{1}{2m} \log \sum |g_{m,\ell}|^2$ where $(g_{m,\ell})$ is an orthonormal basis of $\mathcal{H}_\Omega(m\varphi)$. Then there are constants $C_1, C_2 > 0$ independent of m such that*

(a) $\varphi(z) - \frac{C_1}{m} \le \varphi_m(z) \le \sup_{|\zeta - z| < r} \varphi(\zeta) + \frac{1}{m} \log \frac{C_2}{r^n}$ *for every $z \in \Omega$ and $r < d(z, \partial\Omega)$. In particular, φ_m converges to φ pointwise and in L^1_{loc} topology on Ω when $m \to +\infty$ and*

(b) $\nu(\varphi, z) - \frac{n}{m} \le \nu(\varphi_m, z) \le \nu(\varphi, z)$ *for every $z \in \Omega$.*

Proof

(a) Note that $\sum |g_{m,\ell}(z)|^2$ is the square of the norm of the evaluation linear form $\mathrm{ev}_z : f \mapsto f(z)$ on $\mathcal{H}_\Omega(m\varphi)$, since $g_{m,\ell}(z) = \mathrm{ev}_z(g_{m,\ell})$ is the ℓ-th coordinate of ev_z in the orthonormal basis $(g_{m,\ell})$. In other words, we have

$$\sum |g_{m,\ell}(z)|^2 = \sup_{f \in B(1)} |f(z)|^2$$

where $B(1)$ is the unit ball of $\mathcal{H}_\Omega(m\varphi)$ (The sum is called the *Bergman kernel* associated with $\mathcal{H}_\Omega(m\varphi)$). As φ is locally bounded from above, the L^2 topology is actually stronger than the topology of uniform convergence on compact subsets of Ω. It follows that the series $\sum |g_{m,\ell}|^2$ converges uniformly on Ω and that its sum is real analytic. Moreover, by what we just explained, we have

$$\varphi_m(z) = \sup_{f\in B(1)} \frac{1}{2m} \log |f(z)|^2 = \sup_{f\in B(1)} \frac{1}{m} \log |f(z)|.$$

For $z_0 \in \Omega$ and $r < d(z_0, \partial\Omega)$, the mean value inequality applied to the psh function $|f|^2$ implies

$$|f(z_0)|^2 \le \frac{1}{\pi^n r^{2n}/n!} \int_{|z-z_0|<r} |f(z)|^2 dV_{2n}(z)$$
$$\le \frac{1}{\pi^n r^{2n}/n!} \exp\Big(2m \sup_{|z-z_0|<r} \varphi(z)\Big) \int_\Omega |f|^2 e^{-2m\varphi} dV_{2n}.$$

If we take the supremum over all $f \in B(1)$ we get

$$\varphi_m(z_0) \le \sup_{|z-z_0|<r} \varphi(z) + \frac{1}{2m} \log \frac{1}{\pi^n r^{2n}/n!}$$

and the second inequality in (a) is proved – as we see, this is an easy consequence of the mean value inequality. Conversely, the Ohsawa-Takegoshi extension theorem ([51]) applied to the 0-dimensional subvariety $\{z_0\} \subset \Omega$ shows that for any $a \in \mathbb{C}$ there is a holomorphic function f on Ω such that $f(z_0) = a$ and

$$\int_\Omega |f|^2 e^{-2m\varphi} dV_{2n} \le C_3 |a|^2 e^{-2m\varphi(z_0)},$$

where C_3 only depends on n and diam Ω. We fix a such that the right hand side is 1. Then $\|f\| \le 1$ and so we get

$$\varphi_m(z_0) \ge \frac{1}{m} \log |f(z_0)| = \frac{1}{m} \log |a| = \varphi(z) - \log \frac{C_3}{m}.$$

The inequalities given in (a) are thus proved. Taking $r = 1/m$, we find that $\lim_{m\to+\infty} \sup_{|\zeta-z|<1/m} \varphi(\zeta) = \varphi(z)$ by the upper semicontinuity of φ, and therefore $\lim \varphi_m(z) = \varphi(z)$, since $\lim \frac{1}{m} \log(C_2 m^n) = 0$.

(b) The above estimates imply

$$\sup_{|z-z_0|<r} \varphi(z) - \frac{C_1}{m} \le \sup_{|z-z_0|<r} \varphi_m(z) \le \sup_{|z-z_0|<2r} \varphi(z) + \frac{1}{m} \log \frac{C_2}{r^n}.$$

After dividing by $\log r < 0$ when $r \to 0$, we infer

$$\frac{\sup_{|z-z_0|<2r} \varphi(z) + \frac{1}{m}\log\frac{C_2}{r^n}}{\log r} \le \frac{\sup_{|z-z_0|<r} \varphi_m(z)}{\log r} \le \frac{\sup_{|z-z_0|<r} \varphi(z) - \frac{C_1}{m}}{\log r},$$

and from this and definition (8), it follows immediately that

$$\nu(\varphi, x) - \frac{n}{m} \le \nu(\varphi_m, z) \le \nu(\varphi, z).$$

□

Theorem 7 implies in a straightforward manner the deep result of [57] on the analyticity of the Lelong number upperlevel sets.

Corollary 3 ([57]) *Let φ be a plurisubharmonic function on a complex manifold X. Then, for every $c > 0$, the Lelong number upperlevel set*

$$E_c(\varphi) = \big\{z \in X\,;\ \nu(\varphi, z) \ge c\big\}$$

is an analytic subset of X.

Proof Since analyticity is a local property, it is enough to consider the case of a psh function φ on a pseudoconvex open set $\Omega \subset \mathbb{C}^n$. The inequalities obtained in Theorem 7 (b) imply that

$$E_c(\varphi) = \bigcap_{m \ge m_0} E_{c-n/m}(\varphi_m).$$

Now, it is clear that $E_c(\varphi_m)$ is the analytic set defined by the equations $g^{(\alpha)}_{m,\ell}(z) = 0$ for all multi-indices α such that $|\alpha| < mc$. Thus $E_c(\varphi)$ is analytic as a (countable) intersection of analytic sets. □

Remark 1 It has been observed by Dano Kim [44] that the functions φ_m produced by Theorem 7 do not in general satisfy $\varphi_{m+1} \succcurlyeq \varphi_m$, in other words their singularities may not always increase monotonically to those of φ. Thanks to the subbadditivity result of [18], this is however the case for any subsequence φ_{m_k} such that m_k divides m_{k+1}, e.g. $m_k = 2^k$ or $m_k = k!$ (we will refer to such a sequence below as being a "*multiplicative sequence*"). In that case, a use of the Ohsawa-Takegoshi theorem on the diagonal of $\Omega \times \Omega$ shows that one can obtain $\varphi_{m_{k+1}} \le \varphi_{m_k}$ (after possibly replacing φ_{m_k} by $\varphi_{m_k} + C/m_k$ with C large enough), see [18] and [31].

Our next goal is to study the regularization process more globally, i.e. on a compact complex manifold X. For this, we have to take care of cohomology class. It is convenient to introduce $d^c = \frac{\mathrm{i}}{4\pi}(\overline{\partial} - \partial)$, so that $dd^c = \frac{\mathrm{i}}{2\pi}\partial\overline{\partial}$. Let T be a closed $(1,1)$-current on X. We assume that T is *quasi-positive*, i.e. that there exists a $(1,1)$-form γ with continuous coefficients such that $T \ge \gamma$; observe that a function φ is quasi-psh iff its complex Hessian is bounded below by a $(1,1)$-form with continuous or locally bounded coefficients, that is, if $dd^c\varphi$ is quasi-positive. The case of positive currents ($\gamma = 0$) is of course the most important.

Lemma 1 *There exists a smooth closed* $(1,1)$-*form* α *representing the same* $\partial\overline{\partial}$-*cohomology class as* T *and an* quasi-psh *function* φ *on* X *such that* $T = \alpha + dd^c\varphi$.

Proof Select an open covering (B_j) of X by coordinate balls such that $T = dd^c\varphi_j$ over B_j, and construct a global function $\varphi = \sum \theta_j\varphi_j$ by means of a partition of unity $\{\theta_j\}$ subordinate to B_j. Now, we observe that $\varphi - \varphi_k$ is smooth on B_k because all differences $\varphi_j - \varphi_k$ are smooth in the intersections $B_j \cap B_k$ and we can write $\varphi - \varphi_k = \sum \theta_j(\varphi_j - \varphi_k)$. Therefore $\alpha := T - dd^c\varphi$ is smooth. □

Thanks to Lemma 1, the problem of approximating a quasi-positive closed $(1,1)$-current is reduced to approximating a quasi-psh function. In this way, we get

Theorem 8 *Let* $T = \alpha + dd^c\varphi$ *be a quasi-positive closed* $(1,1)$-*current on a compact Hermitian manifold* (X,ω) *such that* $T \geq \gamma$ *for some continuous* $(1,1)$-*form* γ. *Then there exists a sequence of quasi-positive currents* $T_m = \alpha + dd^c\varphi_m$ *whose local potentials have the same singularities as* $1/2m$ *times a logarithm of a sum of squares of holomorphic functions and a decreasing sequence* $\varepsilon_m > 0$ *converging to* 0, *such that*

(a) T_m *converges weakly to* T,
(b) $\nu(T,x) - \dfrac{n}{m} \leq \nu(T_m,x) \leq \nu(T,x)$ *for every* $x \in X$;
(c) $T_m \geq \gamma - \varepsilon_m\omega$.

We say that our currents T_m *are approximations of* T *with analytic singularities* (*possessing logarithmic poles*). *Moreover, for any multiplicative subsequence* m_k, *one can arrange that* $T_{m_k} = \alpha + dd^c\varphi_{m_k}$ *where* (φ_{m_k}) *is a non-increasing sequence of potentials.*

Proof We just briefly sketch the idea – essentially a partition of unity argument – and refer to [21] for the details. Let us write $T = \alpha + dd^c\varphi$ with α smooth, according to Lemma 1. After replacing T with $T - \alpha$ and γ with $\gamma - \alpha$, we can assume without loss of generality that $\{T\} = 0$, i.e. that $T = dd^c\varphi$ with a quasi-psh function φ on X such that $dd^c\varphi \geq \gamma$. Now, for $\varepsilon > 0$ small, we select a finite covering $(B_j)_{1\leq j\leq N(\varepsilon)}$ of X by coordinate balls on which there exists an ε-approximation of γ as

$$\sum_{1\leq\ell\leq n} \lambda_{j,\ell}\, \mathrm{i}\, dz^j_\ell \wedge d\overline{z}^j_\ell \leq \gamma_{|B_j} \leq \sum_{1\leq\ell\leq n} (\lambda_{j,\ell} + \varepsilon)\, \mathrm{i}\, dz^j_\ell \wedge d\overline{z}^j_\ell$$

in terms of holomorphic coordinates $(z^j_\ell)_{1\leq\ell\leq n}$ on B_j (for this we just diagonalize $\gamma(a_j)$ at the center a_j of B_j, and take the radius of B_j small enough). By construction $\psi_{j,\varepsilon}(z) = \varphi(z) - \sum_{1\leq\ell\leq n} \lambda_{j,\ell}|z^j_\ell|^2$ is psh on B_ℓ, and we can thus obtain approximations $\psi_{j,\varepsilon,m}$ of ψ_j by the Bergman kernel process applied on each ball B_j. The idea is to define a global approximation of φ by putting

$$\varphi_{\varepsilon,m}(x) = \frac{1}{m}\log\Bigg(\sum_{1\le j\le N(\varepsilon)} \theta_{j,\varepsilon}(x)\,\exp\Big(m\Big(\psi_{j,\varepsilon,m}(x) + \sum_{1\le\ell\le n}(\lambda_{j,\ell}-\varepsilon)|z^j_\ell|^2\Big)\Big)\Bigg)$$

where $(\theta_{j,\varepsilon})_{1\le j\le N(\varepsilon)}$ is a partition of unity subordinate to the B_j's. If we take $\varepsilon = \varepsilon_m$ and $\varphi_m = \varphi_{\varepsilon_m,m}$ where ε_m decays very slowly, then it is not hard to check that $T_m = dd^c\varphi_m$ satisfies the required estimates; it is essentially enough to observe that the derivatives of $\theta_{j,\varepsilon}$ are "killed" by the factor $\frac{1}{m}$ when $m \gg \frac{1}{\varepsilon}$. □

We need a variant of Theorem 8 providing more "equisingularity" in the sense that the multiplier ideal sheaves are preserved. If one adds the requirement to obtain a non-increasing sequence of approximations of the potential, one can do this only at the expense of accepting "transcendental" singularities, which can no longer be guaranteed to be logarithmic poles.

Theorem 9 *Let $T = \alpha + dd^c\varphi$ be a closed $(1,1)$-current on a compact Hermitian manifold (X,ω), where α is a smooth closed $(1,1)$-form and φ a quasi-psh function. Let γ be a continuous real $(1,1)$-form such that $T \ge \gamma$. Then one can write $\varphi = \lim_{m\to+\infty}\tilde\varphi_m$ where*

(a) *$\tilde\varphi_m$ is smooth in the complement $X \smallsetminus Z_m$ of an analytic set $Z_m \subset X$;*
(b) *$\{\tilde\varphi_m\}$ is a non-increasing sequence, and $Z_m \subset Z_{m+1}$ for all m;*
(c) *$\int_X (e^{-\varphi} - e^{-\tilde\varphi_m})dV_\omega$ is finite for every m and converges to 0 as $m \to +\infty$;*
(d) *("equisingularity") $\mathscr{I}(\tilde\varphi_m) = \mathscr{I}(\varphi)$ for all m ;*
(e) *$T_m = \alpha + dd^c\tilde\varphi_m$ satisfies $T_m \ge \gamma - \varepsilon_m\omega$, where $\lim_{m\to+\infty}\varepsilon_m = 0$.*

Proof (A substantial simplification of the original proof in [31].) As in the previous proof, we may assume that $\alpha = 0$ and $T = dd^c\varphi$, and after subtracting a constant to φ we can also achieve that $\varphi \le -1$ everywhere on X. For every germ $f \in \mathscr{O}_{X,x}$, (c) implies $\int_U |f|^2(e^{-\varphi} - e^{-\tilde\varphi_m})dV_\omega < +\infty$ on some neighborhood U of x, hence the integrals $\int_U |f|^2e^{-\varphi}dV_\omega$ and $\int_U |f|^2e^{-\tilde\varphi_m}dV_\omega$ are simultaneously convergent or divergent, and (d) follows trivially. We define

$$\tilde\varphi_m(x) = \sup_{k\ge m}(1+2^{-k})\varphi_{p_k}$$

where (p_k) is a multiplicative sequence that grows fast enough, with $\varphi_{p_{k+1}} \le \varphi_{p_k} \le 0$ for all k. Clearly $\tilde\varphi_m$ is a non-increasing sequence, and

$$\lim_{m\to+\infty}\tilde\varphi_m(x) = \lim_{k\to+\infty}\varphi_{p_k}(x) = \varphi(x)$$

at every point $x \in X$. If Z_m is the set of poles of φ_{p_m}, it is easy to see that

$$\tilde{\varphi}_m(x) = \lim_{\ell \to +\infty} \sup_{k \in [m,\ell]} (1 + 2^{-k})\varphi_{p_k}$$

converges uniformly on every compact subset of $X \smallsetminus Z_m$, since any new term $(1 + 2^{-\ell})\varphi_{p_\ell}$ may contribute to the sup only in case

$$\varphi_{p_\ell} \geq \frac{1 + 2^{-p_m}}{1 + 2^{-p_\ell}} \varphi_{p_m} \quad (\geq 2\varphi_{p_m}),$$

and the difference of that term with respect to the previous term $(1+2^{-(\ell-1)})\varphi_{p_{\ell-1}} \geq (1 + 2^{-(\ell-1)})\varphi_{p_\ell}$ is less than $2^{-\ell}|\varphi_{p_\ell}| \leq 2^{1-\ell}|\varphi_{p_m}|$. Therefore $\tilde{\varphi}_m$ is continuous on $X \smallsetminus Z_m$, and getting it to be smooth is only a matter of applying Richberg's approximation technique ([24, 56]). The only serious thing to prove is property (c). To achieve this, we observe that $\{\varphi < \tilde{\varphi}_m\}$ is contained in the union $\bigcup_{k \geq m} \{\varphi < (1 + 2^{-k})\varphi_{p_k}\}$, therefore

$$\int_X \big(e^{-\varphi} - e^{-\tilde{\varphi}_m}\big) dV_\omega \leq \sum_{k=m}^{+\infty} \int_X \mathbf{1}_{\varphi < (1+2^{-k})\varphi_{p_k}} e^{-\varphi} dV_\omega \tag{9}$$

and

$$\begin{aligned}
\int_X \mathbf{1}_{\varphi < (1+2^{-k})\varphi_{p_k}} e^{-\varphi} dV_\omega &= \int_X \mathbf{1}_{\varphi < (1+2^{-k})\varphi_{p_k}} \exp\big(2^k\varphi - (2^k + 1)\varphi\big) dV_\omega \\
&\leq \int_X \mathbf{1}_{\varphi < (1+2^{-k})\varphi_{p_k}} \exp\big((2^k + 1)(\varphi_{p_k} - \varphi)\big) dV_\omega \\
&\leq \int_X \mathbf{1}_{\varphi < (1+2^{-k})\varphi_{p_k}} \exp\big(2p_k(\varphi_{p_k} - \varphi)\big) dV_\omega
\end{aligned} \tag{10}$$

if we take $p_k > 2^{k-1}$ (notice that $\varphi_{p_k} - \varphi \geq 0$). Now, by Lemma 2 below, our integral (10) is finite. By Lebesgue's monotone convergence theorem, we have for k fixed

$$\lim_{p \to +\infty} \int_X \mathbf{1}_{\varphi < (1+2^{-k})\varphi_p} e^{-\varphi} dV_\omega = 0$$

as a decreasing limit, and we can take p_k so large that $\int_{\varphi < (1+2^{-k})\varphi_{p_k}} e^{-\varphi} dV_\omega \leq 2^{-k}$. This ensures that property (c) holds true by (9). □

Lemma 2 *On a compact complex manifold, for any quasi-psh potential φ, the Bergman kernel procedure leads to quasi-psh potentials φ_m with analytic singularities such that*

$$\int_X e^{2m(\varphi_m - \varphi)} dV_\omega < +\infty.$$

Proof By definition of φ_m in Theorem 7, $\exp(2m(\varphi_m))$ is (up to the irrelevant partition of unity procedure) equal to the Bergman kernel $\sum_{\ell\in\mathbb{N}}|g_{m,\ell}|^2$. By local uniform convergence and the Noetherian property, it has the same local vanishing behavior as a finite sum $\sum_{\ell\le N(m)}|g_{m,\ell}|^2$ with $N(m)$ sufficiently large. Since all terms $g_{m,\ell}$ have L^2 norm equal to 1 with respect to the weight $e^{-2m\varphi}$, our contention follows. □

Remark 2 A very slight variation of the proof would yield the improved condition

(c') $\forall\lambda\in\mathbb{R}_+,\quad \int_X(e^{-\lambda\varphi}-e^{-\lambda\tilde\varphi_m})dV_\omega\le 2^{-m}$ for $m\ge m_0(\lambda)$,

and thus an equality $\mathscr{I}(\lambda\tilde\varphi_m)=\mathscr{I}(\lambda\varphi)$ for $m\ge m_0(\lambda)$. We just need to replace estimate (9) by

$$\int_X\big(e^{-m\varphi}-e^{-m\tilde\varphi_m}\big)dV_\omega\le\sum_{k=m}^{+\infty}\int_X\mathbf{1}_{\varphi<(1+2^{-k})\varphi_{p_k}}e^{-k\varphi}dV_\omega$$

and take p_k so large that $2p_k\ge k(2^k+1)$ and $\int_{\varphi<(1+2^{-k})\varphi_{p_k}}e^{-k\varphi}dV_\omega\le 2^{-k-1}$. □

We also quote the following very simple consequence of Lemma 2, which will be needed a bit later. Since φ_m is less singular than φ, we have of course an inclusion $\mathscr{I}(\lambda\varphi)\subset\mathscr{I}(\lambda\varphi_m)$ for all $\lambda\in\mathbb{R}_+$. Conversely :

Corollary 4 *For every pair of positive real numbers $\lambda'>\lambda>0$, we have an inclusion of multiplier ideals*

$$\mathscr{I}(\lambda'\varphi_m)\subset\mathscr{I}(\lambda\varphi)\qquad\textit{as soon as}\ \ m\ge\left\lceil\frac{1}{2}\frac{\lambda\lambda'}{\lambda'-\lambda}\right\rceil.$$

Proof If $f\in\mathscr{O}_{X,x}$ and U is a sufficiently small neighborhood of x, the Hölder inequality for conjugate exponents $p,q>1$ yields

$$\int_U|f|^2e^{-\lambda\varphi}dV_\omega\le\Big(\int_U|f|^2e^{-\lambda'\varphi_m}dV_\omega\Big)^{1/p}\Big(\int_U|f|^2e^{\frac{q}{p}\lambda'\varphi_m-q\lambda\varphi}dV_\omega\Big)^{1/q}.$$

Therefore, if $f\in\mathscr{I}(\lambda'\varphi_m)_x$, we infer that $f\in\mathscr{I}(\lambda\varphi)_x$ as soon as the integral $\int_X e^{\frac{q}{p}\lambda'\varphi_m-q\lambda\varphi}dV_\omega$ is convergent. If we select $p\in\,]1,\lambda'/\lambda]$, this is implied by the condition $\int_X e^{q\lambda(\varphi_m-\varphi)}dV_\omega<+\infty$. If we further take $q\lambda=2m_0$ to be an even integer so large that

$$p=\frac{q}{q-1}=\frac{2m_0/\lambda}{2m_0/\lambda-1}\le\frac{\lambda'}{\lambda},\qquad\text{e.g. }\ m_0=m_0(\lambda,\lambda')=\left\lceil\frac{1}{2}\frac{\lambda\lambda'}{\lambda'-\lambda}\right\rceil,$$

then we indeed have $\int_X e^{2m_0(\varphi_m-\varphi)}dV_\omega\le\int_X e^{2m(\varphi_m-\varphi)}dV_\omega<+\infty$ for $m\ge m_0(\lambda,\lambda')$, thanks to Lemma 2. □

Remark 3 Without the monotonicity requirement (b) for the sequence $(\tilde\varphi_m)$ in Theorem 9, the strong openness conjecture proved in the next section would directly provide an equisingular sequence, simply by taking

$$\widehat{\varphi}_m = \left(1 + \frac{1}{m}\right)\varphi_m$$

where φ_m is the Bergman approximation sequence. In fact all $\widehat{\varphi}_m$ have analytic have analytic singularities and Corollary 4 applied with $\lambda = 1$ and $\lambda' = 1 + 1/m$ shows that $\mathscr{I}(\widehat{\varphi}_m) \subset \mathscr{I}(\varphi)$. Since $\widehat{\varphi}_m \geq (1 + \frac{1}{m})\varphi$, the equality $\mathscr{I}(\widehat{\varphi}_m) = \mathscr{I}(\varphi)$ holds for m large by strong openness, and properties (a), (c), (d), (e) can be seen to hold. However, the sequence $(\widehat{\varphi}_m)$ is not monotone.

3 Semi-continuity of psh Singularities and Proof of the Strong Openness Conjecture

In this section, we present a proof of the strong openness conjecture for multiplier ideal sheaves. Let Ω be a domain in $\mathbb{C}^n$, $f \in \mathscr{O}(\Omega)$ a holomorphic function, and $\varphi \in \mathrm{PSH}(\Omega)$ a psh function on Ω. For every holomorphic function f on Ω, we introduce the *weighted log canonical threshold* of φ with weight f at z_0

$$c_{f,z_0}(\varphi) = \sup\left\{c > 0 :\ |f|^2 e^{-2c\varphi} \text{ is } L^1 \text{ on a neighborhood of } z_0\right\} \in\ ]0, +\infty].$$

The special case $f = 1$ yields the usual log canonical threshold $c_{z_0}(\varphi)$ that was defined in the introduction. The openness conjectures can be stated as follows.

Conjectures

(a) (openness conjecture, [27])
The set $\{c > 0 :\ e^{-2c\varphi}$ is L^1 on a neighborhood of $z_0\}$ *equals the open interval* $]0, c_{z_0}(\varphi)[$.

(b) (strong openness conjecture, [23])
The set $\{c > 0 :\ |f|^2 e^{-2c\varphi}$ is L^1 on a neighborhood of $z_0\}$ *equals the open interval* $]0, c_{f,z_0}(\varphi)[$.

The openness conjecture (a) was first established by Favre and Jonsson [34] in dimension 2 (see also [42, 43]), and 8 years later by Berndtsson [3] in arbitrary dimension. The strong form (b), which is equivalent to Corollary 1, was settled very recently by Guan and Zhou [36]. Their proof uses a sophisticated version of the L^2-extension theorem of Ohsawa and Takegoshi in combination with the curve selection lemma. They have also obtained related semi-continuity statements in [37] and "effective versions" in [38]. A simplified proof along the same lines has been given by Lempert in [47].

Here, we follow Pham Hoang Hiep's approach [54], which is more straightforward and avoids the curve selection lemma. It is based on the original version [51] of the L^2-extension theorem, applied to members of a standard basis for a multiplier ideal sheaf of holomorphic functions associated with a plurisubharmonic function φ. In this way, by means of a simple induction on dimension, one can obtain the strong openness conjecture, and give simultaneously an effective version of the semicontinuity theorem for weighted log canonical thresholds. The main results are contained in the following theorem.

Theorem 10 ([54]) *Let f be a holomorphic function on an open set Ω in $\mathbb{C}^n$ and let φ be a psh function on Ω.*

(i) ("Semicontinuity theorem") *Assume that $\int_{\Omega'} e^{-2c\varphi} dV_{2n} < +\infty$ on some open subset $\Omega' \subset \Omega$ and let $z_0 \in \Omega'$. Then there exists $\delta = \delta(c, \varphi, \Omega', z_0) > 0$ such that for every $\psi \in \mathrm{PSH}(\Omega')$, $\|\psi - \varphi\|_{L^1(\Omega')} \le \delta$ implies $c_{z_0}(\psi) > c$. Moreover, as ψ converges to φ in $L^1(\Omega')$, the function $e^{-2c\psi}$ converges to $e^{-2c\varphi}$ in L^1 on every relatively compact open subset $\Omega'' \Subset \Omega'$.*

(ii) ("Strong effective openness") *Assume that $\int_{\Omega'} |f|^2 e^{-2c\varphi} dV_{2n} < +\infty$ on some open subset $\Omega' \subset \Omega$. When $\psi \in \mathrm{PSH}(\Omega')$ converges to φ in $L^1(\Omega')$ with $\psi \le \varphi$, the function $|f|^2 e^{-2c\psi}$ converges to $|f|^2 e^{-2c\varphi}$ in L^1 norm on every relatively compact open subset $\Omega'' \Subset \Omega'$.*

Corollary 5 ("Strong openness") *For any plurisubharmonic function φ on a neighborhood of a point $z_0 \in \mathbb{C}^n$, the set $\{c > 0 : |f|^2 e^{-2c\varphi}$ is L^1 on a neighborhood of $z_0\}$ is an open interval $(0, c_{f,z_0}(\varphi))$.*

Corollary 6 ("Convergence from below") *If $\psi \le \varphi$ converges to φ in a neighborhood of $z_0 \in \mathbb{C}^n$, then $c_{f,z_0}(\psi) \le c_{f,z_0}(\varphi)$ converges to $c_{f,z_0}(\varphi)$.*

In fact, after subtracting a large constant to φ, we can assume $\varphi \le 0$ in both corollaries. Then Corollary 5 is a consequence of assertion (ii) of the main theorem when we take Ω' small enough and $\psi = (1 + \delta)\varphi$ with $\delta \searrow 0$. In Corollary 6, we have by definition $c_{f,z_0}(\psi) \le c_{f,z_0}(\varphi)$ for $\psi \le \varphi$, but again (ii) shows that $c_{f,z_0}(\psi)$ becomes $\ge c$ for any given value $c \in (0, c_{f,z_0}(\varphi))$, whenever $\|\psi - \varphi\|_{L^1(\Omega')}$ is sufficiently small.

Remark 4 One cannot remove condition $\psi \le \varphi$ in assertion (ii) of the main theorem. Indeed, choose $f(z) = z_1$, $\varphi(z) = \log|z_1|$ and $\varphi_j(z) = \log|z_1 + \frac{z_2}{j}|$, for $j \ge 1$. One has $\varphi_j \to \varphi$ in $L^1_{\mathrm{loc}}(\mathbb{C}^n)$, however $c_{f,0}(\varphi_j) = 1 < c_{f,0}(\varphi) = 2$ for all $j \ge 1$. On the other hand, condition (i) of Theorem 10 does not require any given inequality between φ and ψ. Modulo Berndtsson's solution of the openness conjecture, (i) follows from the effective semicontinuity result of [27], but (like Guan and Zhou) Hiep's technique will reprove both by a direct and easier method.

A few preliminaries According to standard techniques in the theory of Gröbner bases, one equips the ring $\mathcal{O}_{\mathbb{C}^n,0}$ of germs of holomorphic functions at 0 with the homogeneous lexicographic order of monomials $z^\alpha = z_1^{\alpha_1} \ldots z_n^{\alpha_n}$, that is, $z_1^{\alpha_1} \ldots z_n^{\alpha_n} < z_1^{\beta_1} \ldots z_n^{\beta_n}$ if and only if $|\alpha| = \alpha_1 + \ldots + \alpha_n < |\beta| = \beta_1 + \ldots + \beta_n$

or $|\alpha| = |\beta|$ and $\alpha_i < \beta_i$ for the first index i with $\alpha_i \neq \beta_i$. For each $f(z) = a_{\alpha^1} z^{\alpha^1} + a_{\alpha^2} z^{\alpha^2} + \ldots$ with $a_{\alpha^j} \neq 0$, $j \geq 1$ and $z^{\alpha^1} < z^{\alpha^2} < \ldots$, we define the *initial coefficient*, *initial monomial* and *initial term* of f to be respectively $\mathrm{IC}(f) = a_{\alpha^1}$, $\mathrm{IM}(f) = z^{\alpha^1}$, $\mathrm{IT}(f) = a_{\alpha^1} z^{\alpha^1}$, and the support of f to be $\mathrm{SUPP}(f) = \{z^{\alpha^1}, z^{\alpha^2}, \ldots\}$. For any ideal $\mathscr{I}$ of $\mathscr{O}_{\mathbb{C}^n,0}$, we define $\mathrm{IM}(\mathscr{I})$ to be the ideal generated by $\{\mathrm{IM}(f)\}_{\{f \in \mathscr{I}\}}$. First, we recall the division theorem of Hironaka and the concept of standard basis of an ideal.

Theorem 11 (Division theorem of Hironaka, [1, 5, 6, 32, 35]) *Let $f, g_1, \ldots, g_k \in \mathscr{O}_{\mathbb{C}^n,0}$. Then there exist $h_1, \ldots, h_k, s \in \mathscr{O}_{\mathbb{C}^n,0}$ such that*

$$f = h_1 g_1 + \ldots + h_k g_k + s,$$

and $\mathrm{SUPP}(s) \cap \langle \mathrm{IM}(g_1), \ldots, \mathrm{IM}(g_k)\rangle = \emptyset$, *where* $\langle \mathrm{IM}(g_1), \ldots, \mathrm{IM}(g_k)\rangle$ *denotes the ideal generated by the family* $(\mathrm{IM}(g_1), \ldots, \mathrm{IM}(g_k))$.

Standard basis of an ideal Let $\mathscr{I}$ be an ideal of $\mathscr{O}_{\mathbb{C}^n,0}$ and let $g_1, \ldots, g_k \in \mathscr{I}$ be such that $\mathrm{IM}(\mathscr{I}) = \langle \mathrm{IM}(g_1), \ldots, \mathrm{IM}(g_k)\rangle$. Take $f \in \mathscr{I}$. By the division theorem of Hironaka, there exist $h_1, \ldots, h_k, s \in \mathscr{O}_{\mathbb{C}^n,0}$ such that

$$f = h_1 g_1 + \ldots + h_k g_k + s,$$

and $\mathrm{SUPP}(s) \cap \mathrm{IM}(\mathscr{I}) = \emptyset$. On the other hand, since $s = f - h_1 g_1 + \ldots + h_k g_k \in \mathscr{I}$, we have $\mathrm{IM}(s) \in \mathrm{IM}(\mathscr{I})$. Therefore $s = 0$ and the g_j's are generators of $\mathscr{I}$. By permuting the g_j's and performing ad hoc subtractions, we can always arrange that $\mathrm{IM}(g_1) < \mathrm{IM}(g_2) < \ldots < \mathrm{IM}(g_k)$, and we then say that $(g_1, \ldots, g_k)$ is a standard basis of $\mathscr{I}$.
Theorem 10 will be proved by induction on dimension n. All statements are trivial for $n = 0$. Assume that the theorem holds for dimension $n - 1$. Thanks to the L^2-extension theorem of Ohsawa and Takegoshi ([51]), one obtains the following key lemma.

Lemma 3 *Let $\varphi \leq 0$ be a plurisubharmonic function and f be a holomorphic function on the polydisc Δ_R^n of center 0 and (poly)radius $R > 0$ in $\mathbb{C}^n$, such that for some $c > 0$*

$$\int_{\Delta_R^n} |f(z)|^2 e^{-2c\,\varphi(z)} dV_{2n}(z) < +\infty.$$

Let $\psi_j \leq 0$, $j \geq 1$, be a nequence of plurisubharmonic functions on Δ_R^n with $\psi_j \to \varphi$ in $L^1_{\mathrm{loc}}(\Delta_R^n)$, and assume that either $f = 1$ identically or $\psi_j \leq \varphi$ for all $j \geq 1$. Then for every $r < R$ and $\varepsilon \in (0, \frac{1}{2}r]$, there exist a value $w_n \in \Delta_\varepsilon \smallsetminus \{0\}$, an index j_0, a constant $\tilde{c} > c$ and a sequence of holomorphic functions F_j on Δ_r^n, $j \geq j_0$, such *that* $\mathrm{IM}(F_j) \le \mathrm{IM}(f)$, $F_j(z) = f(z) + (z_n - w_n)\sum a_{j,\alpha} z^\alpha$ *with* $|w_n||a_{j,\alpha}| \le r^{-|\alpha|}\varepsilon$ *for all* $\alpha \in \mathbb{N}^n$, *and*

$$\int_{\Delta_r^n} |F_j(z)|^2 e^{-2\tilde{c}\,\psi_j(z)} dV_{2n}(z) \le \frac{\varepsilon^2}{|w_n|^2} < +\infty, \quad \forall j \ge j_0.$$

Moreover, one can choose w_n *in a set of positive measure in the punctured disc* $\Delta_\varepsilon \smallsetminus \{0\}$ (*the index* $j_0 = j_0(w_n)$ *and the constant* $\tilde{c} = \tilde{c}(w_n)$ *may then possibly depend on* w_n).

Proof By Fubini's theorem we have

$$\int_{\Delta_R} \left[\int_{\Delta_R^{n-1}} |f(z', z_n)|^2 e^{-2c\,\varphi(z', z_n)} dV_{2n-2}(z') \right] dV_2(z_n) < +\infty.$$

Since the integral extended to a small disc $z_n \in \Delta_\eta$ tends to 0 as $\eta \to 0$, it will become smaller than any preassigned value, say $\varepsilon_0^2 > 0$, for $\eta \le \eta_0$ small enough. Therefore we can choose a set of positive measure of values $w_n \in \Delta_\eta \smallsetminus \{0\}$ such that

$$\int_{\Delta_R^{n-1}} |f(z', w_n)|^2 e^{-2c\,\varphi(z', w_n)} dV_{2n-2}(z') \le \frac{\varepsilon_0^2}{\pi \eta^2} < \frac{\varepsilon_0^2}{|w_n|^2}.$$

Since the main theorem is assumed to hold for $n-1$, for any $\rho < R$ there exist $j_0 = j_0(w_n)$ and $\tilde{c} = \tilde{c}(w_n) > c$ such that

$$\int_{\Delta_\rho^{n-1}} |f(z', w_n)|^2 e^{-2\tilde{c}\,\psi_j(z', w_n)} dV_{2n-2}(z') < \frac{\varepsilon_0^2}{|w_n|^2}, \quad \forall j \ge j_0.$$

(For this, one applies part (i) in case $f = 1$, and part (ii) in case $\psi_j \le \varphi$, using the fact that $\psi = \frac{\tilde{c}}{c}\psi_j$ converges to φ as $\tilde{c} \to c$ and $j \to +\infty$). Now, by the L^2-extension theorem of Ohsawa and Takegoshi (see [51]), there exists a holomorphic function F_j on $\Delta_\rho^{n-1} \times \Delta_R$ such that $F_j(z', w_n) = f(z', w_n)$ for all $z' \in \Delta_\rho^{n-1}$, and

$$\begin{aligned}\int_{\Delta_\rho^{n-1} \times \Delta_R} |F_j(z)|^2 e^{-2\tilde{c}\,\psi_j(z)} dV_{2n}(z) &\le C_n R^2 \int_{\Delta_\rho^{n-1}} |f(z', w_n)|^2 e^{-2\tilde{c}\,\psi_j(z', w_n)} dV_{2n-2}(z') \\ &\le \frac{C_n R^2 \varepsilon_0^2}{|w_n|^2},\end{aligned}$$

where C_n is a constant which only depends on n (the constant is universal for $R = 1$ and is rescaled by R^2 otherwise). By the mean value inequality for the plurisubharmonic function $|F_j|^2$, we get

$$|F_j(z)|^2 \le \frac{1}{\pi^n(\rho-|z_1|)^2\dots(\rho-|z_n|)^2}\int_{\Delta_{\rho-|z_1|}(z_1)\times\ldots\times\Delta_{\rho-|z_n|}(z_n)}|F_j|^2dV_{2n}$$
$$\le \frac{C_nR^2\varepsilon_0^2}{\pi^n(\rho-|z_1|)^2\dots(\rho-|z_n|)^2|w_n|^2},$$

where $\Delta_\rho(z)$ is the disc of center z and radius ρ. Hence, for any $r<R$, by taking $\rho=\frac{1}{2}(r+R)$ we infer

$$\|F_j\|_{L^\infty(\Delta_r^n)} \le \frac{2^nC_n^{\frac{1}{2}}R\varepsilon_0}{\pi^{\frac{n}{2}}(R-r)^n|w_n|}. \tag{11}$$

Since $F_j(z',w_n)-f(z',w_n)=0$, $\forall z'\in\Delta_r^{n-1}$, we can write $F_j(z)=f(z)+(z_n-w_n)g_j(z)$ for some function $g_j(z)=\sum_{\alpha\in\mathbb{N}^n}a_{j,\alpha}z^\alpha$ on $\Delta_r^{n-1}\times\Delta_R$. By (11), we get

$$\|g_j\|_{\Delta_r^n}=\|g_j\|_{\Delta_r^{n-1}\times\partial\Delta_r}\le\frac{1}{r-|w_n|}\Big(\|F_j\|_{L^\infty(\Delta_r^n)}+\|f\|_{L^\infty(\Delta_r^n)}\Big)$$
$$\le\frac{1}{r-|w_n|}\Big(\frac{2^nC_n^{\frac{1}{2}}R\varepsilon_0}{\pi^{\frac{n}{2}}(R-r)^n|w_n|}+\|f\|_{L^\infty(\Delta_r^n)}\Big).$$

Thanks to the Cauchy integral formula, we find

$$|a_{j,\alpha}|\le\frac{\|g_j\|_{\Delta_r^n}}{r^{|\alpha|}}\le\frac{1}{(r-|w_n|)r^{|\alpha|}}\Big(\frac{2^nC_n^{\frac{1}{2}}R\varepsilon_0}{\pi^{\frac{n}{2}}(R-r)^n|w_n|}+\|f\|_{L^\infty(\Delta_r^n)}\Big).$$

We take in any case $\eta\le\varepsilon_0\le\varepsilon\le\frac{1}{2}r$. As $|w_n|<\eta\le\frac{1}{2}r$, this implies

$$|w_n||a_{j,\alpha}|\,r^{|\alpha|}\le\frac{2}{r}\Big(\frac{2^nC_n^{\frac{1}{2}}R\varepsilon_0}{\pi^{\frac{n}{2}}(R-r)^n}+\|f\|_{L^\infty(\Delta_r^n)}|w_n|\Big)\le C'\varepsilon_0,$$

for some constant C' depending only on n, r, R and f. This yields the estimates of Lemma 3 for $\varepsilon_0:=C''\varepsilon$ with C'' sufficiently small. Finally, we prove that $\mathrm{IM}(F_j)\le\mathrm{IM}(f)$. Indeed, if $\mathrm{IM}(g_j)\ge\mathrm{IM}(f)$, since $|w_n\|a_{j,\alpha}|\le r^{-|\alpha|}\varepsilon$, we can choose ε small enough such that $\mathrm{IM}(F_j)=\mathrm{IM}(f)$ and $\left|\frac{\mathrm{IC}(F_j)}{\mathrm{IC}(f)}\right|\in(\frac{1}{2},2)$. Otherwise, if $\mathrm{IM}(g_j)<\mathrm{IM}(f)$, we have $\mathrm{IM}(F_j)=\mathrm{IM}(g_j)<\mathrm{IM}(f)$. □

Proof of Theorem 10 By well-known properties of (pluri)potential theory, the L^1 convergence of ψ to φ implies that $\psi\to\varphi$ almost everywhere, and the assumptions guarantee that φ and ψ are uniformly bounded on every relatively compact subset of Ω'. In particular, after shrinking Ω' and subtracting constants, we can assume that

$\varphi \le 0$ on Ω. Also, since the L^1 topology is metrizable, it is enough to work with a sequence $(\psi_j)_{j\ge1}$ converging to φ in $L^1(\Omega')$. Again, we can assume that $\psi_j \le 0$ and that $\psi_j \to \varphi$ almost everywhere on Ω'. By a trivial compactness argument, it is enough to show (i) and (ii) for some neighborhood Ω'' of a given point $z_0 \in \Omega'$. We assume here $z_0 = 0$ for simplicity of notation, and fix a polydisc Δ_R^n of center 0 with R so small that $\Delta_R^n \subset \Omega'$. Then $\psi_j(\bullet, z_n) \to \varphi(\bullet, z_n)$ in the topology of $L^1(\Delta_R^{n-1})$ for almost every $z_n \in \Delta_R$.

Proof of statement (i) *in Theorem 10* We have here $\int_{\Delta_R^n} e^{-2c\varphi} dV_{2n} < +\infty$ for $R > 0$ small enough. By Lemma 3 with $f = 1$, for every $r < R$ and $\varepsilon > 0$, there exist $w_n \in \Delta_\varepsilon \smallsetminus \{0\}$, an index j_0, a number $\tilde c > c$ and a sequence of holomorphic functions F_j on Δ_r^n, $j \ge j_0$, such that $F_j(z) = 1 + (z_n - w_n)\sum a_{j,\alpha} z^\alpha$, $|w_n||a_{j,\alpha}|\, r^{-|\alpha|} \le \varepsilon$ and

$$\int_{\Delta_r^n} |F_j(z)|^2 e^{-2\tilde c\,\psi_j(z)} dV_{2n}(z) \le \frac{\varepsilon^2}{|w_n|^2}, \quad \forall j \ge j_0.$$

For $\varepsilon \le \frac12$, we conclude that $|F_j(0)| = |1 - w_n a_{j,0}| \ge \frac12$ hence $c_0(\psi_j) \ge \tilde c > c$ and the first part of (i) is proved. In fact, after fixing such ε and w_n, we even obtain the existence of a neighborhood Ω'' of 0 on which $|F_j| \ge \frac14$, and thus get a uniform bound $\int_{\Omega''} e^{-2\tilde c\,\psi_j(z)} dV_{2n}(z) \le M < +\infty$. The second assertion of (i) then follows from the estimate

$$\begin{aligned}\int_{\Omega''} \left|e^{-2c\,\psi_j(z)} - e^{-2c\,\varphi(z)}\right| dV_{2n}(z) \le &\int_{\Omega''\cap\{|\psi_j|\le A\}} \left|e^{-2c\,\psi_j(z)} - e^{-2c\,\varphi(z)}\right| dV_{2n}(z)\\ &+ \int_{\Omega''\cap\{\psi_j<-A\}} e^{-2c\,\varphi(z)} dV_{2n}(z)\\ &+ e^{-2(\tilde c - c)A} \int_{\Omega''\cap\{\psi_j<-A\}} e^{-2\tilde c\,\psi_j(z)} dV_{2n}(z).\end{aligned}$$

In fact the last two terms converge to 0 as $A \to +\infty$, and, for A fixed, the first integral in the right hand side converges to 0 by Lebesgue's bounded convergence theorem, since $\psi_j \to \varphi$ almost everywhere on Ω''.

Proof of statement (ii) *in Theorem 10* Take $f_1, \dots, f_k \in \mathscr{O}_{\mathbb{C}^n,0}$ such that $(f_1, \dots, f_k)$ is a standard basis of $\mathscr{I}(c\varphi)_0$ with $\mathrm{IM}(f_1) < \dots < \mathrm{IM}(f_k)$, and Δ_R^n a polydisc so small that

$$\int_{\Delta_R^n} |f_l(z)|^2 e^{-2c\,\varphi(z)} dV_{2n}(z) < +\infty, \quad l = 1, \dots, k.$$

Since the germ of f at 0 belongs to the ideal $(f_1, \dots, f_k)$, we can essentially argue with the f_l's instead of f. By Lemma 3, for every $r < R$ and $\varepsilon_l > 0$, there exist $w_{n,l} \in \Delta_{\varepsilon_l} \smallsetminus \{0\}$, an index $j_0 = j_0(w_{n,l})$, a number $\tilde c = \tilde c(w_{n,l}) > c$ and a sequence of holomorphic functions $F_{j,l}$ on Δ_r^n, $j \ge j_0$, such that $F_{j,l}(z) = 1 + (z_n - w_{n,l})\sum a_{j,l,\alpha} z^\alpha$,

$|w_{n,l}||a_{j,l,\alpha}|\, r^{-|\alpha|} \le \varepsilon_l$ and

$$\int_{\Delta_r^n} |F_{j,l}(z)|^2 e^{-2\tilde{c}\,\psi_j(z)} dV_{2n}(z) \le \frac{\varepsilon_l^2}{|w_{n,l}|^2}, \quad \forall l = 1, \dots, k, \ \ \forall j \ge j_0. \tag{12}$$

Since $\psi_j \le \varphi$ and $\tilde{c} > c$, we get $F_{j,l} \in \mathscr{I}(\tilde{c}\,\psi_j)_0 \subset \mathscr{I}(c\,\varphi)_0$. The next step of the proof consists in modifying $(F_{j,l})_{1 \le l \le k}$ in order to obtain a standard basis of $\mathscr{I}(c\,\varphi)_0$. For this, we proceed by selecting successively $\varepsilon_1 \gg \varepsilon_2 \gg \dots \gg \varepsilon_k$ (and suitable $w_{n,l} \in \Delta_{\varepsilon_l} \smallsetminus \{0\}$). We have $\mathrm{IM}(F_{j,1}), \dots, \mathrm{IM}(F_{j,k}) \in \mathrm{IM}(\mathscr{I}(c\,\varphi)_0$, in particular $\mathrm{IM}(F_{j,1})$ is divisible by $\mathrm{IM}(f_l)$ for some $l = 1, \dots, k$. Since $\mathrm{IM}(F_{j,1}) \le \mathrm{IM}(f_1) < \dots < \mathrm{IM}(f_k)$, we must have $\mathrm{IM}(F_{j,1}) = \mathrm{IM}(f_1)$ and thus $\mathrm{IM}(g_{j,1}) \ge \mathrm{IM}(f_1)$. As $|w_{n,1}||a_{j,1,\alpha}| \le \varepsilon_1$, we will have $\left|\frac{\mathrm{IC}(F_{j,1})}{\mathrm{IC}(f_1)}\right| \in (\frac{1}{2}, 2)$ for ε_1 small enough. Now, possibly after changing ε_2 to a smaller value, we show that there exists a polynomial $P_{j,2,1}$ such that the degree and coefficients of $P_{j,2,1}$ are uniformly bounded, with $\mathrm{IM}(F_{j,2} - P_{j,2,1}F_{j,1}) = \mathrm{IM}(f_2)$ and $\frac{|\,\mathrm{IC}(F_{j,2} - P_{j,2,1}F_{j,1})|}{|\,\mathrm{IC}(f_2)|} \in (\frac{1}{2}, 2)$. We consider two cases:

Case 1: If $\mathrm{IM}(g_{j,2}) \ge \mathrm{IM}(f_2)$, since $|w_{n,2}||a_{j,2,\alpha}| \le r^{-|\alpha|}\varepsilon_2$, we can choose ε_2 so small that $\mathrm{IM}(F_{j,2}) = \mathrm{IM}(f_2)$ and $\frac{|\,\mathrm{IC}(F_{j,2})|}{|\,\mathrm{IC}(f_2)|} \in (\frac{1}{2}, 2)$. We then take $P_{j,2,1} = 0$.

Case 2: If $\mathrm{IM}(g_{j,2}) < \mathrm{IM}(f_2)$, we have $\mathrm{IM}(g_{j,2}) = \mathrm{IM}(F_{j,2}) \in \mathrm{IM}(\mathscr{I}(c\,\varphi)_0)$. Hence $\mathrm{IM}(g_{j,2})$ is divisible by $\mathrm{IM}(f_l)$ for some $l = 1, \dots, k$. However, since $\mathrm{IM}(g_{j,2}) < \mathrm{IM}(f_2) < \dots < \mathrm{IM}(f_k)$, the only possibility is that $\mathrm{IM}(g_{j,2})$ be divisible by $\mathrm{IM}(f_1)$. Take $b \in \mathbb{C}$ and $\beta, \gamma \in \mathbb{N}^n$ such that $\mathrm{IT}(g_{j,2}) := a_{j,2,\gamma} z^\gamma = b z^\beta\ \mathrm{IT}(F_{j,1})$. We have $z^\beta \le z^\gamma = \mathrm{IM}(g_{j,2}) < \mathrm{IM}(f_2)$ and

$$|w_{n,2}||b| = |w_{n,2}| \frac{|\,\mathrm{IC}(g_{j,2})|}{|\,\mathrm{IC}(F_{j,1})|} \le \frac{2|w_{n,2}||a_{j,2,\gamma}|}{|\,\mathrm{IC}(f_1)|} \le \frac{2r^{-|\gamma|}\varepsilon_2}{|\,\mathrm{IC}(f_1)|}$$

can be taken arbitrarily small. Set $\tilde{g}_{j,2}(z) = g_{j,2}(z) - b z^\beta F_{j,1}(z) = \sum \tilde{a}_{j,2,\alpha} z^\alpha$ and

$$\widetilde{F}_{j,2}(z) = f_2(z) + (z_n - w_{n,2})\tilde{g}_{j,2}(z) = F_{j,2}(z) - b(z_n - w_{n,2}) z^\beta F_{j,1}(z).$$

We have $\mathrm{IM}(\tilde{g}_{j,2}) > \mathrm{IM}(g_{j,2})$. Since $|w_{n,2}||b| = O(\varepsilon_2)$ and $|w_{n,2}||a_{j,2,\alpha}| = O(\varepsilon_2)$, we get $|w_{n,2}||\tilde{a}_{j,2,\alpha}| = O(\varepsilon_2)$ as well. Now, we consider two further cases. If $\mathrm{IM}(\tilde{g}_{j,2}) \ge \mathrm{IM}(f_2)$, we can again change ε_2 for a smaller value so that $\mathrm{IM}(\widetilde{F}_{j,2}) = \mathrm{IM}(f_2)$ and $\frac{|\,\mathrm{IC}(\widetilde{F}_{j,2})|}{|\,\mathrm{IC}(f_2)|} \in (\frac{1}{2}, 2)$. Otherwise, if $\mathrm{IM}(\tilde{g}_{j,2}) < \mathrm{IM}(f_2)$, we have $\mathrm{IM}(F_{j,2}) = \mathrm{IM}(g_{j,2}) < \mathrm{IM}(\widetilde{F}_{j,2}) = \mathrm{IM}(\tilde{g}_{j,2}) < \mathrm{IM}(f_2)$. Notice that $\{z^\gamma : z^\gamma < \mathrm{IM}(f_2)\}$ is a finite set. By using similar arguments a finite number of times, we find ε_2 so small that $\mathrm{IM}(F_{j,2} - P_{j,2,1}F_{j,1}) = \mathrm{IM}(f_2)$ and $\frac{|\,\mathrm{IC}(F_{j,2} - P_{j,2,1}F_{j,1})|}{|\,\mathrm{IC}(f_2)|} \in (\frac{1}{2}, 2)$ for some polynomial $P_{j,2,1}$. Repeating the same arguments for $F_{j,3}, \dots, F_{j,k}$, we select inductively ε_l, $l = 1, \dots, k$, and construct linear combinations

$$F'_{j,l} = F_{j,l} - \sum_{1 \le m \le l-1} P_{j,l,m} F'_{j,m}$$

with polynomials $P_{j,l,m}$, $1 \le m < l \le k$, possessing uniformly bounded coefficients and degrees, such that $\mathrm{IM}(F'_{j,l}) = \mathrm{IM}(f_l)$ and $\dfrac{|\,\mathrm{IC}(F'_{j,l})|}{|\,\mathrm{IC}(f_l)|} \in (\frac{1}{2}, 2)$ for all $l = 1, \dots, k$ and $j \ge j_0$. This implies that $(F'_{j,1}, \dots, F'_{j,k})$ is also a standard basis of $\mathscr{I}(c\varphi)_0$. By Theorem 1.2.2 in [35], we can find ρ, $K > 0$ so small that there exist holomorphic functions $h_{j,1}, \dots, h_{j,k}$ on Δ^n_ρ with $\rho < r$, such that

$$f = h_{j,1}F'_{j,1} + h_{j,2}F'_{j,2} + \dots + h_{j,k}F'_{j,k} \quad \text{on } \Delta^n_\rho$$

and $\|h_{j,l}\|_{L^\infty(\Delta^n_\rho)} \le K\|f\|_{L^\infty(\Delta^n_r)}$, for all $l = 1, \dots, k$ (ρ and K only depend on $f_1, \dots, f_k$). By (12), this implies a uniform bound

$$\int_{\Delta^n_\rho} |f(z)|^2 e^{-2\tilde{c}\,\psi_j(z)} dV_{2n}(z) \le M < +\infty$$

for some $\tilde{c} > c$ and all $j \ge j_0$. Take $\Omega'' = \Delta^n_\rho$. We obtain the L^1 convergence of $|f|^2 e^{-2c\,\psi_j}$ to $|f|^2 e^{-2c\,\varphi}$ almost exactly as we argued for the second assertion of part (i), by using the estimate

$$\begin{aligned}
\int_{\Omega''} |f|^2 \left| e^{-2c\,\psi_j(z)} - e^{-2c\,\varphi(z)} \right| dV_{2n}(z) \qquad\qquad & \\
\le \int_{\Omega'' \cap \{|\psi_j| \le A\}} & |f|^2 \left| e^{-2c\,\psi_j(z)} - e^{-2c\,\varphi(z)} \right| dV_{2n}(z) \\
& + \int_{\Omega'' \cap \{\psi_j < -A\}} |f|^2 e^{-2c\,\varphi(z)} dV_{2n}(z) \\
& + e^{-2(\tilde{c}-c)A} \int_{\Omega'' \cap \{\psi_j < -A\}} |f|^2 e^{-2\tilde{c}\,\psi_j(z)} dV_{2n}(z).
\end{aligned}$$

4 Hard Lefschetz Theorem for Pseudoeffective Line Bundles

4.1 A Variant of the Bochner Formula

We first recall a variation of the Bochner formula that is required in the proof of the Hard Lefschetz Theorem with values in a positively curved (and therefore non flat) line bundle (L, h). Here the base manifold is a Kähler (non necessarily compact) manifold (Y, ω). We denote by $|\ \ | = |\ \ |_{\omega,h}$ the pointwise Hermitian norm on

$\Lambda^{p,q}T_Y^* \otimes L$ associated with ω and h, and by $\| \ \| = \| \ \|_{\omega,h}$ the global L^2 norm

$$\|u\|^2 = \int_Y |u|^2 dV_\omega \qquad \text{where} \quad dV_\omega = \frac{\omega^n}{n!}$$

We consider the $\overline{\partial}$ operator acting on (p,q)-forms with values in L, its adjoint $\overline{\partial}_h^*$ with respect to h and the complex Laplace-Beltrami operator $\Delta''_h = \overline{\partial}\overline{\partial}_h^* + \overline{\partial}_h^*\overline{\partial}$. Let v be a smooth $(n-q,0)$-form with compact support in Y. Then $u = \omega^q \wedge v$ satisfies

$$\|\overline{\partial}u\|^2 + \|\overline{\partial}_h^* u\|^2 = \|\overline{\partial}v\|^2 + \int_Y \sum_{I,J} \Big(\sum_{j\in J} \lambda_j\Big) |u_{IJ}|^2 \tag{13}$$

where $\lambda_1 \le \cdots \le \lambda_n$ are the curvature eigenvalues of $\Theta_{L,h}$ expressed in an orthonormal frame $(\partial/\partial z_1, \ldots, \partial/\partial z_n)$ (at some fixed point $x_0 \in Y$), in such a way that

$$\omega_{x_0} = \mathrm{i} \sum_{1\le j\le n} dz_j \wedge d\overline{z}_j, \qquad (\Theta_{L,h})_{x_0} = dd^c\varphi_{x_0} = \mathrm{i} \sum_{1\le j\le n} \lambda_j dz_j \wedge d\overline{z}_j.$$

Formula (13) follows from the more or less straightforward identity

$$(\overline{\partial}_\varphi^* \overline{\partial} + \overline{\partial}\,\overline{\partial}_\varphi^*)(v \wedge \omega^q) - (\overline{\partial}_\varphi^* \overline{\partial} v) \wedge \omega^q = q\,\mathrm{i}\,\partial\overline{\partial}\varphi \wedge \omega^{q-1} \wedge v,$$

by taking the inner product with $u = \omega^q \wedge v$ and integrating by parts in the left hand side (we leave the easy details to the reader). Our formula is thus established when v is smooth and compactly supported. In general, we have:

Proposition 1 *Let* (Y,ω) *be a* complete *Kähler manifold and* (L,h) *a smooth Hermitian line bundle such that the curvature possesses a uniform lower bound* $\Theta_{L,h} \ge -C\omega$. *For every measurable* $(n-q,0)$*-form* v *with* L^2 *coefficients and values in L such that* $u = \omega^q \wedge v$ *has differentials* $\overline{\partial}u$, $\overline{\partial}^* u$ *also in* L^2, *we have*

$$\|\overline{\partial}u\|^2 + \|\overline{\partial}_h^* u\|^2 = \|\overline{\partial}v\|^2 + \int_Y \sum_{I,J} \Big(\sum_{j\in J} \lambda_j\Big) |u_{IJ}|^2$$

(*here, all differentials are computed in the sense of distributions*).

Proof Since (Y,ω) is assumed to be complete, there exists a sequence of smooth forms v_ν with compact support in Y (obtained by truncating v and taking the convolution with a regularizing kernel) such that $v_\nu \to v$ in L^2 and such that $u_\nu = \omega^q \wedge v_\nu$ satisfies $u_\nu \to u$, $\overline{\partial}u_\nu \to \overline{\partial}u$, $\overline{\partial}^* u_\nu \to \overline{\partial}^* u$ in L^2. By the curvature assumption, the final integral in the right hand side of (13) must be under control (i.e. the integrand becomes nonnegative if we add a term $C\|u\|^2$ on both sides, $C \gg 0$).

We thus get the equality by passing to the limit and using Lebesgue's monotone convergence theorem. □

4.2 *Proof of Theorem 4*

Here X denotes a compact Kähler manifold equipped with a Kähler metric ω, and (L, h) is a pseudoeffective line bundle on X. To fix the ideas, we first indicate the proof in the much simpler case when (L, h) has a smooth metric h (so that $\mathscr{I}(h) = \mathscr{O}_X$), and then treat the general case.

4.2.1 Special Case: (L, h) *is Hermitian Semipositive (with a Smooth Metric)*

Let $\{\beta\} \in H^q(X, \Omega^n_X \otimes L)$ be an arbitrary cohomology class. By standard L^2 Hodge theory, $\{\beta\}$ can be represented by a smooth harmonic $(0, q)$-form β with values in $\Omega^n_X \otimes L$. We can also view β as a (n, q)-form with values in L. The pointwise Lefschetz isomorphism produces a unique $(n-q, 0)$-form α such that $\beta = \omega^q \wedge \alpha$. Proposition 1 then yields

$$\|\overline{\partial}\alpha\|^2 + \int_Y \sum_{I,J} \Big(\sum_{j\in J} \lambda_j\Big) |\alpha_{IJ}|^2 = \|\overline{\partial}\beta\|^2 + \|\overline{\partial}^*_h \beta\|^2 = 0,$$

and the curvature eigenvalues λ_j are nonnegative by our assumption. Hence $\overline{\partial}\alpha = 0$ and $\{\alpha\} \in H^0(X, \Omega^{n-q}_X \otimes L)$ is mapped to $\{\beta\}$ by $\Phi^q_{\omega,h} = \omega^q \wedge \bullet$.

4.2.2 General Case

There are several difficulties. The first difficulty is that the metric h is no longer smooth and we cannot directly represent cohomology classes by harmonic forms. We circumvent this problem by smoothing the metric on an (analytic) Zariski open subset and by avoiding the remaining poles on the complement. However, some careful estimates have to be made in order to take the error terms into account.

Fix $\varepsilon = \varepsilon_\nu$ and let $h_\varepsilon = h_{\varepsilon_\nu}$ be an approximation of h, such that h_ε is smooth on $X \smallsetminus Z_\varepsilon$ (Z_ε being an analytic subset of X), $\Theta_{L,h_\varepsilon} \geq -\varepsilon\omega$, $h_\varepsilon \leq h$ and $\mathscr{I}(h_\varepsilon) = \mathscr{I}(h)$. This is possible by Theorem 9. Now, we can find a family

$$\omega_{\varepsilon,\delta} = \omega + \delta(\mathrm{i}\,\partial\overline{\partial}\psi_\varepsilon + \omega), \qquad \delta > 0$$

of *complete Kähler* metrics on $X \smallsetminus Z_\varepsilon$, where ψ_ε is a quasi-psh function on X with $\psi_\varepsilon = -\infty$ on Z_ε, ψ_ε smooth on $X \smallsetminus Z_\varepsilon$ and $\mathrm{i}\,\partial\overline{\partial}\psi_\varepsilon + \omega \geq 0$ (see e.g. [19], Théorème 1.5). By construction, $\omega_{\varepsilon,\delta} \geq \omega$ and $\lim_{\delta\to 0} \omega_{\varepsilon,\delta} = \omega$. We look at the L^2

Dolbeault complex $K^\bullet_{\varepsilon,\delta}$ of $(n,\bullet)$-forms on $X \smallsetminus Z_\varepsilon$, where the L^2 norms are induced by $\omega_{\varepsilon,\delta}$ on differential forms and by h_ε on elements in L. Specifically

$$K^q_{\varepsilon,\delta} = \Big\{u{:}X \smallsetminus Z_\varepsilon \to \Lambda^{n,q}T^*_X \otimes L;\ \int_{X\smallsetminus Z_\varepsilon}(|u|^2_{\Lambda^{n,q}\omega_{\varepsilon,\delta}\otimes h_\varepsilon} + |\overline{\partial}u|^2_{\Lambda^{n,q+1}\omega_{\varepsilon,\delta}\otimes h_\varepsilon})dV_{\omega_{\varepsilon,\delta}} < \infty\Big\}.$$

Let $\mathscr{K}^q_{\varepsilon,\delta}$ be the corresponding sheaf of germs of locally L^2 sections on X (the local L^2 condition should hold on X, not only on $X \smallsetminus Z_\varepsilon$!). Then, for all $\varepsilon > 0$ and $\delta \geq 0$, $(\mathscr{K}^q_{\varepsilon,\delta}, \overline{\partial})$ is a resolution of the sheaf $\Omega^n_X \otimes L \otimes \mathscr{I}(h_\varepsilon) = \Omega^n_X \otimes L \otimes \mathscr{I}(h)$. This is because L^2 estimates hold locally on small Stein open sets, and the L^2 condition on $X \smallsetminus Z_\varepsilon$ forces holomorphic sections to extend across Z_ε ([19], Lemma 6.9).

Let $\{\beta\} \in H^q(X, \Omega^n_X \otimes L \otimes \mathscr{I}(h))$ be a cohomology class represented by a smooth form with values in $\Omega^n_X \otimes L \otimes \mathscr{I}(h)$ (one can use a Čech cocycle and convert it to an element in the $\mathscr{C}^\infty$ Dolbeault complex by means of a partition of unity, thanks to the usual De Rham-Weil isomorphism, see also the final proof in Sect. 6 for more details). Then

$$\|\beta\|^2_{\varepsilon,\delta} \leq \|\beta\|^2 = \int_X |\beta|^2_{\Lambda^{n,q}\omega\otimes h} dV_\omega < +\infty.$$

The reason is that $|\beta|^2_{\Lambda^{n,q}\omega\otimes h} dV_\omega$ decreases as ω increases. This is just an easy calculation, shown by comparing two metrics ω, ω' which are expressed in diagonal form in suitable coordinates; the norm $|\beta|^2_{\Lambda^{n,q}\omega\otimes h}$ turns out to decrease faster than the volume dV_ω increases; see e.g. [19], Lemma 3.2; a special case is $q = 0$, then $|\beta|^2_{\Lambda^{n,q}\omega\otimes h} dV_\omega = i^{n^2}\beta \wedge \overline{\beta}$ with the identification $L \otimes \overline{L} \simeq \mathbb{C}$ given by the metric h, hence the integrand is even independent of ω in that case.

By the proof of the De Rham-Weil isomorphism, the map $\alpha \mapsto \{\alpha\}$ from the cocycle space $Z^q(\mathscr{K}^\bullet_{\varepsilon,\delta})$ equipped with its L^2 topology, into $H^q(X, \Omega^n_X \otimes L \otimes \mathscr{I}(h))$ equipped with its finite vector space topology, is continuous. Also, Banach's open mapping theorem implies that the coboundary space $B^q(\mathscr{K}^\bullet_{\varepsilon,\delta})$ is closed in $Z^q(\mathscr{K}^\bullet_{\varepsilon,\delta})$. This is true for all $\delta \geq 0$ (the limit case $\delta = 0$ yields the strongest L^2 topology in bidegree (n,q)). Now, β is a $\overline{\partial}$-closed form in the Hilbert space defined by $\omega_{\varepsilon,\delta}$ on $X \smallsetminus Z_\varepsilon$, so there is a $\omega_{\varepsilon,\delta}$-harmonic form $u_{\varepsilon,\delta}$ in the same cohomology class as β, such that

$$\|u_{\varepsilon,\delta}\|_{\varepsilon,\delta} \leq \|\beta\|_{\varepsilon,\delta}. \tag{14}$$

Let $v_{\varepsilon,\delta}$ be the unique $(n-q,0)$-form such that $u_{\varepsilon,\delta} = v_{\varepsilon,\delta} \wedge \omega^q_{\varepsilon,\delta}$ ($v_{\varepsilon,\delta}$ exists by the pointwise Lefschetz isomorphism). Then

$$\|v_{\varepsilon,\delta}\|_{\varepsilon,\delta} = \|u_{\varepsilon,\delta}\|_{\varepsilon,\delta} \leq \|\beta\|_{\varepsilon,\delta} \leq \|\beta\|.$$

As $\sum_{j\in J}\lambda_j \geq -q\varepsilon$ by the assumption on Θ_{L,h_ε}, the Bochner formula yields

$$\|\overline{\partial}v_{\varepsilon,\delta}\|^2_{\varepsilon,\delta} \leq q\varepsilon\|u_{\varepsilon,\delta}\|^2_{\varepsilon,\delta} \leq q\varepsilon\|\beta\|^2.$$

These uniform bounds imply that there are subsequences $u_{\varepsilon,\delta_\nu}$ and $v_{\varepsilon,\delta_\nu}$ with $\delta_\nu \to 0$, possessing weak-L^2 limits $u_\varepsilon = \lim_{\nu\to+\infty} u_{\varepsilon,\delta_\nu}$ and $v_\varepsilon = \lim_{\nu\to+\infty} v_{\varepsilon,\delta_\nu}$. The limit $v_\varepsilon = \lim_{\nu\to+\infty} v_{\varepsilon,\delta_\nu}$ is with respect to $L^2(\omega) = L^2(\omega_{\varepsilon,0})$. To check this, notice that in bidegree $(n-q,0)$, the space $L^2(\omega)$ has the weakest topology of all spaces $L^2(\omega_{\varepsilon,\delta})$; indeed, an easy calculation made in ([19], Lemma 3.2) yields

$$|f|^2_{\Lambda^{n-q,0}\omega\otimes h}dV_\omega \leq |f|^2_{\Lambda^{n-q,0}\omega_{\varepsilon,\delta}\otimes h}dV_{\omega_{\varepsilon,\delta}} \qquad \text{if } f \text{ is of type } (n-q,0).$$

On the other hand, the limit $u_\varepsilon = \lim_{\nu\to+\infty} u_{\varepsilon,\delta_\nu}$ takes place in all spaces $L^2(\omega_{\varepsilon,\delta})$, $\delta > 0$, since the topology gets stronger and stronger as $\delta \downarrow 0$ [possibly not in $L^2(\omega)$, though, because in bidegree (n,q) the topology of $L^2(\omega)$ might be strictly stronger than that of all spaces $L^2(\omega_{\varepsilon,\delta})$]. The above estimates yield

$$\|v_\varepsilon\|^2_{\varepsilon,0} = \int_X |v_\varepsilon|^2_{\Lambda^{n-q,0}\omega\otimes h_\varepsilon}dV_\omega \leq \|\beta\|^2,$$
$$\|\overline{\partial}v_\varepsilon\|^2_{\varepsilon,0} \leq q\varepsilon\|\beta\|^2_{\varepsilon,0},$$
$$u_\varepsilon = \omega^q \wedge v_\varepsilon \equiv \beta \qquad \text{in } H^q(X,\Omega^n_X\otimes L\otimes\mathscr{I}(h_\varepsilon)).$$

Again, by arguing in a given Hilbert space $L^2(h_{\varepsilon_0})$, we find L^2 convergent subsequences $u_\varepsilon \to u$, $v_\varepsilon \to v$ as $\varepsilon \to 0$, and in this way get $\overline{\partial}v = 0$ and

$$\|v\|^2 \leq \|\beta\|^2,$$
$$u = \omega^q \wedge v \equiv \beta \qquad \text{in } H^q(X,\Omega^n_X\otimes L\otimes\mathscr{I}(h)).$$

Theorem 4 is proved. Notice that the equisingularity property $\mathscr{I}(h_\varepsilon) = \mathscr{I}(h)$ is crucial in the above proof, otherwise we could not infer that $u \equiv \beta$ from the fact that $u_\varepsilon \equiv \beta$. This is true only because all cohomology classes $\{u_\varepsilon\}$ lie in the same fixed cohomology group $H^q(X,\Omega^n_X\otimes L\otimes\mathscr{I}(h))$, whose topology is induced by the topology of $L^2(\omega)$ on $\overline{\partial}$-closed forms (e.g. through the De Rham-Weil isomorphism). □

Remark 5 In (14), the existence of a harmonic representative holds true only for $\omega_{\varepsilon,\delta}$, $\delta > 0$, because we need to have a complete Kähler metric on $X \smallsetminus Z_\varepsilon$. The trick of employing $\omega_{\varepsilon,\delta}$ instead of a fixed metric ω, however, is not needed when Z_ε is (or can be taken to be) empty. This is the case if (L,h) is such that $\mathscr{I}(h) = \mathscr{O}_X$ and L is nef. Indeed, by definition, L is nef iff there exists a sequence of smooth metrics h_ν such that $\mathrm{i}\,\Theta_{L,h_\nu} \geq -\varepsilon_\nu\omega$, so we can take the φ_ν's to be everywhere smooth in Theorem 9. However, multiplier ideal sheaves are needed in the surjectivity statement even in case L is nef, as it may happen that $\mathscr{I}(h_{\min}) \neq \mathscr{O}_X$

even then, and $h := \lim h_\nu$ is anyway always more singular than $h_{\min}$. Let us recall a standard example (see [30, 31]). Let B be an elliptic curve and let V be the rank 2 vector bundle over B which is defined as the (unique) non split extension

$$0 \to \mathscr{O}_B \to V \to \mathscr{O}_B \to 0.$$

In particular, the bundle V is numerically flat, i.e. $c_1(V) = 0$, $c_2(V) = 0$. We consider the ruled surface $X = \mathbb{P}(V)$. On that surface there is a unique section $C = \mathbb{P}(\mathscr{O}_B) \subset X$ with $C^2 = 0$ and

$$\mathscr{O}_X(C) = \mathscr{O}_{\mathbb{P}(V)}(1)$$

is a nef line bundle. One can check that $L = \mathscr{O}_{\mathbb{P}(V)}(3)$ leads to a *zero* Lefschetz map

$$\omega \wedge \bullet : \quad H^0(X, \Omega_X^1 \otimes L) \longrightarrow H^1(X, K_X \otimes L) \simeq \mathbb{C},$$

so this is a counterexample to Corollary 2 in the nef case. Incidentally, this also shows (in a somewhat sophisticated way) that $\mathscr{O}_{\mathbb{P}(V)}(1)$ is nef but not semipositive, a fact that was first observed in [30].

5 Numerical Dimension of Currents

A large part of this section borrows ideas from S. Boucksom's [7, 8] and Junyan Cao's [14] PhD theses. We try however to give here a slightly more formal exposition. The main difference with S. Boucksom's approach is that we insist on keeping track of singularities of currents and leaving them unchanged, instead of trying to minimize them in each cohomology class.

5.1 Monotone Asymptotically Equisingular Approximations

Let X be a compact complex n-dimensional manifold. We consider the closed convex cone of *pseudoeffective classes*, namely the set $\mathscr{E}(X)$ of cohomology classes $\{\alpha\} \in H^{1,1}(X, \mathbb{R})$ containing a closed positive $(1,1)$-current $T = \alpha + dd^c\varphi$ (in the non Kähler case one should use Bott-Chern cohomology groups here, but we will be mostly concerned with the Kähler case in the sequel). We also introduce the set $\mathscr{S}(X)$ of singularity equivalence classes of closed positive $(1,1)$-currents $T = \alpha + dd^c\varphi$ (i.e., α being fixed, up to equivalence of singularities of the potentials φ, cf. Definition 2). Clearly, there is a fibration

$$\pi : \mathscr{S}(X) \to \mathscr{E}(X), \qquad T \mapsto \{\alpha\} \in \mathscr{E}(X) \subset H^{1,1}(X, \mathbb{R}). \tag{15}$$

We will denote by $\mathscr{S}_\alpha(X)$ the fiber $\pi^{-1}(\{\alpha\})$ of $\mathscr{S}(X)$ over a given cohomology class $\{\alpha\} \in \mathscr{E}(X)$. Observe that the base $\mathscr{E}(X)$ is a closed convex cone in a finite dimensional vector space, but in general the fiber $\mathscr{S}_\alpha(X)$ must be viewed as a very complicated infinite dimensional space : if we take e.g. $\{\alpha_1\} \in H^{1,1}(\mathbb{P}^n, \mathbb{R})$ to be the unit class $c_1(\mathscr{O}(1))$, then any current $T = \frac{1}{d}[H]$ where H_d is an irreducible hypersurface of degree d defines a point in $\mathscr{S}_{\alpha_1}(\mathbb{P}^n)$, and these points are all distinct. The set $\mathscr{S}(X)$ is nevertheless equipped in a natural way with an addition law $\mathscr{S}(X) \times \mathscr{S}(X) \to \mathscr{S}(X)$ that maps $\mathscr{S}_\alpha(X) + \mathscr{S}_\beta(X)$ into $\mathscr{S}_{\alpha+\beta}(X)$, a scalar multiplication $\mathbb{R}_+ \times \mathscr{S}(X) \to \mathscr{S}(X)$ that takes $\lambda \cdot \mathscr{S}_\alpha(X)$ to the fiber $\mathscr{S}_{\lambda\alpha}(X)$. In this way, $\mathscr{S}(X)$ should be viewed as some sort of infinite dimensional convex cone. The fibers $\mathscr{S}_\alpha(X)$ also possess a partial ordering $\preccurlyeq$ (cf. Definition 2) such that $\forall j,\ S_j \preccurlyeq T_j \Rightarrow \sum S_j \preccurlyeq \sum T_j$, and a fiberwise "min" operation

$$\begin{aligned} &\min : \mathscr{S}_\alpha(X) \times \mathscr{S}_\alpha(X) \longrightarrow \mathscr{S}_\alpha(X), \\ &(T_1, T_2) = (\alpha + dd^c\varphi_1, \alpha + dd^c\varphi_2) \longmapsto T = \alpha + dd^c \max(\varphi_1, \varphi_2), \end{aligned} \tag{16}$$

with respect to which the addition is distributive, i.e.

$$\min(T_1 + S, T_2 + S) = \min(T_1, T_2) + S.$$

Notice that when $T_1 = \frac{1}{d}[H_1]$, $T_2 = \frac{1}{d}[H_2]$ are effective $\mathbb{Q}$-divisors, all these operations $+, \cdot$, $\min(\bullet)$ and the ordering $\preccurlyeq$ coincide with the usual ones known for divisors. Following Junyan Cao [14] (with slightly more restrictive requirements that do not produce much change in practice), we introduce

Definition 3 Let $T = \alpha + dd^c\varphi$ be a closed positive $(1,1)$-current on X, where α is a smooth closed $(1,1)$-form and φ is a quasi-psh function on X. We say that the sequence of currents $T_k = \alpha + dd^c\psi_k$, $k \in \mathbb{N}$, is a "monotone asymptotically equisingular approximation of T by currents with analytic singularities" if the sequence of potentials (ψ_k) satisfies the following properties:

(a) *(monotonicity)* The sequence (ψ_k) is non-increasing and converges to φ at every point of X.
(b) The functions ψ_k have analytic singularities (and $\psi_k \preccurlyeq \psi_{k+1}$ by (a)).
(c) *(lower bound of positivity)*

$$\alpha + dd^c\psi_k \geq -\varepsilon_k \cdot \omega \qquad \text{with} \quad \lim_{k\to+\infty} \varepsilon_k = 0$$

for any given smooth positive hermitian $(1,1)$-form ω on X.
(d) *(asymptotic equisingularity)* For every pair of positive numbers $\lambda' > \lambda > 0$, there exists an integer $k_0(\lambda, \lambda') \in \mathbb{N}$ such that

$$\mathscr{I}(\lambda'\psi_k) \subset \mathscr{I}(\lambda\varphi) \qquad \text{for } k \geq k_0(\lambda, \lambda').$$

Remark 6 Without loss of generality, one can always assume that the quasi-psh potentials $\varphi_k = c_k \log |g_k|^2 + O(1)$ have rational coefficients $c_k \in \mathbb{Q}_+$; here again, g_k is a tuple of locally defined holomorphic functions. In fact, after subtracting constants, one can achieve that $\varphi \le 0$ and $\psi_k \le 0$ for all k. If the c_k are arbitrary nonnegative real numbers, one can always replace ψ_k by $\psi'_k = (1-\delta_k)\psi_k$ with a decreasing sequence $\delta_k \in]0,1[$ such that $\lim \delta_k = 0$ and $(1-\delta_k)c_k \in \mathbb{Q}_+$. Then (a), (b), (d) are still valid, and (c) holds with $\varepsilon'_k = (1-\delta_k)\varepsilon_k + C\delta_k$ and C a constant such that $\alpha \ge -C\omega$. □

The fundamental observation is:

Theorem 12 *If $\psi_k := \varphi_{m_k}$ is the sequence of potentials obtained by the Bergman kernel approximation of $T = \alpha + dd^c\varphi$ given in the proof of Theorem 8 and (m_k) is a multiplicative sequence, then the ψ_k can be arranged to satisfy the positivity, monotonicity and asymptotic equisingularity properties of Definition 3. Moreover, if we start with currents $T \preccurlyeq T'$ in the same cohomology class $\{\alpha\}$, we obtain corresponding approximations that satisfy $\psi_k \preccurlyeq \psi'_k$.*

Proof By Corollary 4, the asymptotic equisingularity property (d) in Definition 3 is satisfied for $m_k \ge \lceil \frac{1}{2}\frac{\lambda\lambda'}{\lambda'-\lambda}\rceil$. The other properties are already known or obvious, especially the coefficients $c_k = \frac{1}{m_k}$ are just inverses of integers in that case. □

The following proposition provides a precise comparison of analytic singularities of potentials when their multiplier ideal sheaves satisfy inclusion relations.

Proposition 2 *Let φ, ψ be quasi-psh functions with analytic singularities, let $c > 0$ be the constant such that φ can be expressed as $c\log\sum|g_j|^2 + O(1)$ with holomorphic functions g_j, and let $\lambda \in \mathbb{R}_+$. Denoting $t_+ := \max(t,0)$, we have the implications*

(a) $\forall f \in \mathscr{O}_{X,x}, \quad \displaystyle\int_{B_x \ni x} |f|^2 e^{-\lambda\varphi} dV < +\infty \quad \Rightarrow \quad \log|f|^2 \succcurlyeq \tfrac{1}{c}\big(\lambda c - n\big)_+\varphi,$

(b) $\mathscr{I}(\psi) \subset \mathscr{I}(\lambda\varphi) \quad \Rightarrow \quad \displaystyle\int e^{\psi-\lambda\varphi} dV < +\infty$ *and* $\psi \succcurlyeq \tfrac{1}{c}\big(\lambda c - n\big)_+\varphi$ *(locally).*

Proof Since everything is local, we may assume that φ, ψ are psh functions on a small ball $B \subset \mathbb{C}^n$, and $\varphi(z) = c\log|g|^2 = c\log\sum_{1\le j\le N}|g_j(z)|^2$.

(a) The convergence of the integral on a small ball B_x of center x implies

$$\int_{B_x} |f|^2|g|^{-2\lambda c} dV \le \mathrm{Const}\int_{B_x} |f|^2 e^{-\lambda\varphi} dV < +\infty$$

By the openness of convergence exponents, one gets

$$\int_{B_x} |f|^2|g|^{-2\lambda+\varepsilon} dV < +\infty$$

for $\varepsilon > 0$ small enough (this can be seen e.g. by using a log resolution of the ideal sheaf (f, g_j)). Now, if $\lambda c \geq n$, Skoda's division theorem [60] implies that each f can be written $f = \sum h_j g_j$ where h_j satisfies a similar estimate where the exponent of $|g|^{-2}$ is decreased by 1. An iteration of the Skoda division theorem for the h_j yields $f \in (g_j)^k$ where $k = (\lfloor \lambda c \rfloor - (n-1))_+ \geq (\lambda c - n)_+$. Hence

$$\log |f|^2 \leq k \log |g|^2 + C \leq \frac{k}{c}\varphi + C'$$

and (a) is proved.

(b) If $(f_\ell)_{\ell \in \mathbb{N}}$ is a Hilbert basis of the space of L^2 holomorphic functions f with $\int_B |f|^2 e^{-\psi} dV < +\infty$, the proof of Theorem 7 yields $\psi \leq C + \log \sum |f_\ell|^2$ (and locally the singularity is achieved by a finite sum of f_ℓ's by the Noetherian property). After possibly shrinking B, the relations $f_\ell \in \mathscr{I}(\psi) \subset \mathscr{I}(\lambda\varphi)$ imply

$$\int_B |f_\ell|^2 e^{-\lambda\varphi} dV < +\infty,$$

hence $\int e^{\psi - \lambda\varphi} dV < +\infty$ locally by taking the sum over ℓ. The inequality proved in (a) for each $f = f_\ell$ also yields

$$\psi \leq \log \sum |f_\ell|^2 + C \leq \frac{1}{c}\big(\lambda c - n\big)_+ \varphi + C',$$

and our singularity comparison relation follows.

□

Corollary 7 *If* $T = \alpha + dd^c\varphi$ *is a closed positive* $(1,1)$*-current and* (ψ_k)*,* (ψ'_k) *are two monotone asymptotically equisingular approximations of* φ *with analytic singularities, then for every* k *and every* $\varepsilon > 0$*, there exists* ℓ *such that* $(1-\varepsilon)\psi_k \preccurlyeq \psi'_\ell$ *(and vice versa by exchanging the roles of* (ψ_k) *and* (ψ'_k)*).*

Proof Let $c > 0$ be the constant occurring in the logarithmic poles of ψ_k (k being fixed). By condition (d) in Definition 3, for $\lambda' > \lambda \gg 1$ we have $\mathscr{I}(\lambda'\psi'_\ell) \subset \mathscr{I}(\lambda\varphi) \subset \mathscr{I}(\lambda\psi_k)$ for $\ell \geq \ell_0(\lambda, \lambda')$ large enough. Proposition 2 (b) implies the singularity estimate $\psi'_\ell \succcurlyeq \frac{1}{c\lambda'}(c\lambda - n)_+ \psi_k$, and the final constant in front of ψ_k can be taken arbitrary close to 1. □

Our next observation is that the min(•) procedure defined above for currents is well behaved in terms of asymptotic equisingular approximations.

Proposition 3 *Let* $T = \alpha + dd^c\varphi$ *and* $T' = \alpha + dd^c\varphi'$ *be closed positive* $(1,1)$*-currents in the same cohomology class* $\{\alpha\}$*. Let* (ψ_k) *and* (ψ'_k) *be respective monotone asymptotically equisingular approximations with analytic singularities and rational coefficients. Then* $\max(\psi_k, \psi'_k)$ *provides a monotone asymptotically equisingular approximation of* $\min(T, T') = \alpha + dd^c \max(\varphi, \varphi')$ *with analytic singularities and rational coefficients.*

Proof If $\psi_k = c_k \log|g_k|^2 + O(1)$ and $\psi'_k = c'_k \log|g'_k|^2 + O(1)$, we can write $c_k = p_k/q_k$, $c'_k = p'_k/q'_k$ and

$$\max(\psi_k, \psi'_k) = \frac{1}{q_k q'_k} \log\left(|g_k|^{2p_k} + |g'_k|^{2p'_k}\right) + O(1),$$

hence $\max(\psi_k, \psi'_k)$ also has analytic singularities with rational coefficients (this would not be true with our definitions when the ratio c'_k/c_k is irrational, but of course we could just extend a little bit the definition of what we call analytic singularities, e.g. by allowing arbitrary positive real exponents, in order to avoid this extremely minor annoyance). It is well known that

$$\alpha + dd^c\psi_k \geq -\varepsilon_k\omega, \quad \alpha + dd^c\psi'_k \geq -\varepsilon'_k\omega$$
$$\Rightarrow \quad \alpha + dd^c \max(\psi_k, \psi'_k) \geq -\max(\varepsilon_k, \varepsilon'_k)\omega.$$

Finally, if $\psi_{B,k}$ (resp. $\psi'_{B,k}$ and $\tilde{\psi}_{B,k}$)) comes from the Bergman approximation of φ (resp. of φ' and $\tilde{\varphi} := \max(\varphi, \varphi')$), we have

$$\tilde{\varphi} \geq \varphi \;\;\Rightarrow\;\; \tilde{\psi}_{B,k} \geq \psi_{B,k}, \qquad \tilde{\varphi} \geq \varphi' \;\;\Rightarrow\;\; \tilde{\psi}_{B,k} \geq \psi'_{B,k}$$

hence $\tilde{\psi}_{B,k} \geq \max(\psi_{B,k}, \psi'_{B,k})$ and so $\tilde{\psi}_{B,k} \preccurlyeq \max(\psi_{B,k}, \psi'_{B,k})$. However, for every $\varepsilon > 0$, one has $(1-\varepsilon)\psi_{B_k} \preccurlyeq \psi_\ell$ and $(1-\varepsilon)\psi'_{B_k} \preccurlyeq \psi'_\ell$ for $\ell \geq \ell_0(k, \varepsilon)$ large, therefore $(1-\varepsilon)\tilde{\psi}_{B,k} \preccurlyeq \max(\psi_\ell, \psi'_\ell)$. This shows that $\max(\psi_\ell, \psi'_\ell)$ has enough singularities (the "opposite" inequality $\max(\psi_\ell, \psi'_\ell) \geq \tilde{\varphi} = \max(\varphi, \varphi')$, i.e. $\max(\psi_\ell, \psi'_\ell) \preccurlyeq \tilde{\varphi}$, holds trivially). □

Following Junyan Cao [15], we now investigate the additivity properties of the Bergman approximation procedure.

Theorem 13 *Let $T = \alpha + dd^c\varphi$ and $T' = \beta + dd^c\varphi'$ be closed $(1,1)$-currents in cohomology classes $\{\alpha\}$, $\{\beta\} \in \mathscr{E}(X)$. Then for every multiplicative sequence (m_k), the sum $\varphi_{m_k} + \varphi'_{m_k}$ of the Bergman approximations of φ, φ' gives a monotone asymptotically equisingular approximation of $\varphi + \varphi'$ and $T + T'$.*

Proof Let $\tilde{\varphi}_m$ be the Bergman kernel approximations of $\tilde{\varphi} = \varphi + \varphi'$. By the subadditivity property of ideal sheaves $\mathscr{I}(m\varphi + m\varphi') \subset \mathscr{I}(m\varphi)\mathscr{I}(m\varphi')$ ([18, Th. 2.6]), hence we have $\varphi_m + \varphi'_m \preccurlyeq \tilde{\varphi}_m$. By Definition 3 (d), Theorem 12 and Corollary 7, to prove Theorem 13, it is sufficient to prove that for every $m \in \mathbb{N}$ fixed, there exists a positive sequence $\lim_{p\to+\infty} \varepsilon_p = 0$ such that

$$(1-\varepsilon_p)\tilde{\varphi}_m \preccurlyeq \varphi_p + \varphi'_p \qquad \text{for every } p \gg 1. \tag{17}$$

For every $m \in \mathbb{N}$ fixed, there exists a bimeromorphic map $\pi : \widetilde{X} \to X$, such that

$$\tilde{\varphi}_m \circ \pi = \sum_i c_i \ln |s_i| + C^\infty \qquad \text{for some } c_i > 0, \tag{18}$$

and the effective divisor $\sum_i \mathrm{Div}(s_i)$ is normal crossing. By the construction of $\tilde{\varphi}_m$, we have $\tilde{\varphi}_m \preccurlyeq \varphi + \varphi'$. Therefore

$$\tilde{\varphi}_m \circ \pi \preccurlyeq (\varphi + \varphi') \circ \pi. \tag{19}$$

By Siu's decomposition formula for closed positive currents applied to $dd^c(\varphi \circ \pi)$, $dd^c(\varphi' \circ \pi)$ respectively, the divisorial parts add up to produce a divisor that is at least equal to the divisorial part in $dd^c(\tilde{\varphi}_m \circ \pi)$, thus (19) and (18) imply the existence of numbers $a_i,\ b_i \geq 0$ satisfying

(i) $a_i + b_i = c_i$ for every i,
(ii) $\sum_i a_i \ln |s_i| \preccurlyeq \varphi \circ \pi$ and $\sum_i b_i \ln |s_i| \preccurlyeq \varphi' \circ \pi$.

Let $p \in \mathbb{N}$ be an integer, J be the Jacobian of π, $f \in \mathscr{I}(p\varphi)_x$ and $g \in \mathscr{I}(p\varphi')_x$ for some $x \in X$. The inequalities in (ii) and a change of variables $w = \pi(z)$ in the L^2 integrals yield

$$\int_{\pi^{-1}(U_x)} \frac{|f \circ \pi|^2 |J|^2}{\prod_i |s_i|^{2pa_i}} < +\infty \quad \text{and} \quad \int_{\pi^{-1}(U_x)} \frac{|g \circ \pi|^2 |J|^2}{\prod_i |s_i|^{2pb_i}} < +\infty \tag{20}$$

for some small open neighborhood U_x of x. Since $\sum_i \mathrm{Div}(s_i)$ is normal crossing, (20) implies that

$$\sum_i (pa_i - 1) \ln |s_i| \preccurlyeq \ln(|f \circ \pi|) + \ln |J| \text{ and } \sum_i (pb_i - 1) \ln |s_i| \preccurlyeq \ln(|g \circ \pi|) + \ln |J|.$$

Combining this with (i), we get

$$\sum_i (pc_i - 2) \ln |s_i| \preccurlyeq \ln(|(f \cdot g) \circ \pi|) + 2 \ln |J|. \tag{21}$$

Note that J is independent of p, and $c_i > 0$. (21) implies thus that, when $p \to +\infty$, we can find a sequence $\varepsilon_p \to 0^+$, such that

$$\sum_i pc_i (1 - \varepsilon_p) \ln |s_i| \preccurlyeq \ln |(f \cdot g) \circ \pi|. \tag{22}$$

Since f (respectively g) is an arbitrary element in $\mathscr{I}(p\varphi)$ (respectively $\mathscr{I}(p\varphi')$), by the construction of φ_p and φ'_p, (22) implies that

$$\sum_i c_i(1-\varepsilon_p)\ln|s_i| \preccurlyeq (\varphi_p+\varphi'_p)\circ\pi.$$

Combining this with the fact that $(1-\varepsilon_p)\tilde\varphi_m\circ\pi \sim \sum_i c_i(1-\varepsilon_p)\ln|s_i|$, we get

$$(1-\varepsilon_p)\tilde\varphi_m\circ\pi \preccurlyeq (\varphi_p+\varphi'_p)\circ\pi.$$

Therefore $(1-\varepsilon_p)\tilde\varphi_m \preccurlyeq \varphi_p+\varphi'_p$ and (17) is proved. □

This motivates the following formal definition.

Definition 4 For each class $\{\alpha\}\in\mathscr{E}(X)$, we define $\widehat{\mathscr{S}}_\alpha(X)$ as a set of equivalence classes of sequences of quasi-positive currents $T_k=\alpha+dd^c\psi_k$ such that

(a) $T_k=\alpha+dd^c\psi_k\ge-\varepsilon_k\cdot\omega$ with $\lim_{k\to+\infty}\varepsilon_k=0$,
(b) the functions ψ_k have analytic singularities and $\psi_k\preccurlyeq\psi_{k+1}$ for all k.

We say that (T_k) is weakly less singular than (T'_k) in $\widehat{\mathscr{S}}_\alpha(X)$, and write $(T_k)\preccurlyeq_W(T'_k)$, if for every $\varepsilon>0$ and k, there exists ℓ such that $(1-\varepsilon)T_k\preccurlyeq T'_\ell$. Finally, we write $(T_k)\sim_W(T'_k)$ when we have $(T_k)\preccurlyeq_W(T'_k)$ and $(T'_k)\preccurlyeq_W(T_k)$, and define $\widehat{\mathscr{S}}_\alpha(X)$ to be the quotient space by this equivalence relation.

The set

$$\widehat{\mathscr{S}}(X)=\bigcup_{\{\alpha\}\in\mathscr{E}(X)}\widehat{\mathscr{S}}_\alpha(X) \tag{23}$$

is by construction a fiber space $\hat\pi:\widehat{\mathscr{S}}(X)\to\mathscr{E}(X)$, and, by fixing a multiplicative sequence such as $m_k=2^k$, we find a natural "Bergman approximation functional"

$$\mathbf{B}:\mathscr{S}(X)\to\widehat{\mathscr{S}}(X),\qquad T=\alpha+dd^c\varphi\longmapsto(T_{B,k}),\ \ T_k=\alpha+dd^c\psi_{B,k} \tag{24}$$

where $\psi_{B,k}:=\varphi_{m_k}$ is the corresponding subsequence of the sequence of Bergman approximations (φ_m).

The set $\widehat{\mathscr{S}}(X)$ is equipped with a natural addition $(T_k)+(T'_k)=(T_k+T'_k)$, with a scalar multiplication $\lambda\cdot(T_k)=(\lambda T_k)$ for $\lambda\in\mathbb{R}_+$, as well as with the min(•) operation $\min((T_k),(T'_k))=(\min(T_k,T'_k))$ obtained by taking $\max(\psi_k,\psi'_k)$ of the corresponding potentials. By Proposition 3, **B** is a morphism for the min(•) operation, and by Theorem 13, **B** is also a morphism for addition. Accordingly, it is natural to define a weak equivalence of singularities for closed positive currents by

$$T\preccurlyeq_W T' \iff_{\text{def}} (T_{B,k})\preccurlyeq_W(T'_{B,k}), \tag{25}$$

$$T\sim_W T' \iff T\preccurlyeq_W T' \text{ and } T'\preccurlyeq_W T. \tag{26}$$

Related ideas are discussed in [4] (especially § 5), using the theory of valuations. One can summarize the above results in the following statement.

Theorem 14 *The Bergman approximation functional*

$$\mathbf{B} : \mathscr{S}(X) \to \widehat{\mathscr{S}}(X), \qquad T = \alpha + dd^c\varphi \longmapsto (T_{B,k})$$

is a morphism for addition and for the $\min(\bullet)$ *operation on currents. Moreover* $\mathbf{B}$ *induces an injection* $\mathscr{S}(X)/\!\sim_W \to \widehat{\mathscr{S}}(X)$.

Remark 7 It is easy to see that the induced map $\mathscr{S}(X)/\!\sim_W \to \widehat{\mathscr{S}}(X)$ is an isomorphism when $\dim X = 1$. However, this map is not always surjective when $\dim X \geq 2$. In fact, [30, Example 1.7] exhibits a ruled surface over an elliptic curve Γ and a nef line bundle L over X, such that $\alpha = c_1(L)$ contains a unique closed positive current $T = [C]$, for some curve $C \subset X$ that is a section of $X \to \Gamma$. Then the Bergman approximation is (up to equivalence of singularities) the constant sequence $T_{B,k} = T$, while $\widehat{\mathscr{S}}_\alpha(X)$ also contains a sequence of smooth currents $T_k \geq -\varepsilon_k\omega$. This implies that $\mathscr{S}(X) \to \widehat{\mathscr{S}}(X)$ is not surjective in this situation. The following proposition shows however that the "formal elements" (T_k) from $\widehat{\mathscr{S}}(X)$ do not carry larger singularities than the closed positive current classes in $\mathscr{S}(X)$ (the latter being constrained by the singularities of the "limiting currents" T representing the class).

Proposition 4 *Let* $T_k = \alpha + dd^c\psi_k$ *be a sequence of closed* $(1,1)$*-currents representing an element in* $\widehat{\mathscr{S}}_\alpha(X)$*. Then there exists a closed positive current* $T \in \alpha$ *such that* $(T_k) \preccurlyeq_W (T_{B,k})$.

Proof We have $T_k \geq -\varepsilon_k\omega$ and $\psi_k \preccurlyeq \psi_{k+1}$ for some decreasing sequence $\varepsilon_k \downarrow 0$. We replace ψ_k by setting

$$\tilde{\psi}_k(x) = \sup\Big\{\tau(x)\,;\ \sup_X \tau \leq 0,\ \alpha + dd^c\tau \geq -\varepsilon_k\omega,\ \text{and}\ \exists C > 0,\ \tau \leq \psi_k + C\Big\}.$$

Then $(\tilde{\psi}_k)$ is a decreasing sequence for the usual order relation $\leq$ and $\tilde{\psi}_k \sim \psi_k$ (the argument to prove the equivalence of singularities is similar to the one already used in the proof of Theorem 13, clearly $\tilde{\psi}_k \geq \psi_k - M_k$ where $M_k = \sup_X \psi_k$, and the converse inequality $\tilde{\psi}_k \leq \psi_k + C_k$ is seen by using a blow-up to make the singularities of ψ_k divisorial). We take

$$\varphi = \lim_{k\to+\infty} \tilde{\psi}_k \quad \text{and} \quad T = \alpha + dd^c\varphi.$$

Since $\alpha + dd^c\tilde{\psi}_k \geq -\varepsilon_k\omega$, we get in the limit $T = \alpha + dd^c\varphi \geq 0$. Let (φ_m) be the Bergman approximation sequence of φ. Since $\varphi \leq \tilde{\psi}_\ell \leq \psi_\ell + C_\ell$, Proposition 2 (a) applied with $\lambda = 2m$ shows that $\varphi_m \succcurlyeq \frac{1}{2mc_\ell}(2mc_\ell - n)_+\psi_\ell$ where $c_\ell > 0$ is the coefficient of the log singularity of ψ_ℓ. Therefore, if we take $T_{B,k} = \alpha + dd^c\varphi_{m_k}$, we get in the limit $(T_{B,k}) \succcurlyeq_W (T_\ell)$. □

Remark 8 When X is projective algebraic and $\{\alpha\}$ belongs to the Neron-Severi space

$$\mathrm{NS}_{\mathbb{R}}(X) = (H^{1,1}(X,\mathbb{C}) \cap H^2(X,\mathbb{Z})/\mathrm{torsion}) \otimes_{\mathbb{Z}} \mathbb{R},$$

the fiber $\widehat{\mathscr{S}}_\alpha(X)$ is essentially an algebraic object. In fact, we could define $\widehat{\mathscr{S}}_\alpha(X)$ as the set of suitable equivalence classes of "formal limits" $\lim_{c_1(D)\to\{\alpha\}} \lim_{k\to+\infty} \frac{1}{k}\mathfrak{a}_k$ associated with sequences of graded ideals $\mathfrak{a}_k \subset H^0(X, \mathscr{O}_X(kD))$ satisfying the subadditive property $\mathfrak{a}_{k+\ell} \subset \mathfrak{a}_k\mathfrak{a}_\ell$, where D are big $\mathbb{Q}$-divisors whose first Chern classes $c_1(D)$ approximate $\{\alpha\} \in \mathrm{NS}_{\mathbb{R}}(X)$. Many related questions are discussed in the algebraic setting in Lazarfeld's book [46]. It is nevertheless an interesting point, even in the projective case, that one can "extrapolate" these concepts to all transcendental classes, and get in this way a global space $\widehat{\mathscr{S}}(X)$ which looks well behaved, e.g. semicontinuous, under variation of the complex structure of X.

5.2 Intersection Theory on $\mathscr{S}(X)$ and $\hat{\mathscr{S}}(X)$

Let X be a compact Kähler n-dimensional manifold equipped with a Kähler metric ω. We consider closed positive $(1,1)$-currents $T_j = \alpha_j + dd^c\varphi_j$, $1 \le j \le p$. Let us first assume that the functions φ_j have analytic singularities, and let $Z \subset X$ be an analytic set such that the φ_j's are locally bounded on $X \smallsetminus Z$. The (p,p)-current

$$\Theta = \mathbf{1}_{X\smallsetminus Z} T_1 \wedge \ldots \wedge T_k$$

is well defined on $X \smallsetminus Z$, thanks to Bedford and Taylor [10], and it is a closed positive current there. By [10] such a current does not carry mass on any analytic set, so we can enlarge Z without changing the total mass of Θ. In fact, Θ extends as a closed positive current on the whole of X. To see this, let us take a simultaneous *log resolution* of the T_j's, i.e. a modification

$$\mu : \widehat{X} \to X$$

such that if $\varphi_j = c_j \log \sum_\ell |g_{j,\ell}|^2 + O(1)$, then the pull-back of the ideals $(g_{j,\ell})_\ell$, namely $\mu^*(g_{j,\ell})_\ell = (g_{j,\ell} \circ \mu)_\ell$ is a purely divisorial ideal sheaf $\mathscr{O}_{\widehat{X}}(-D_j)$ on $\widehat{X}$. Let $u_j = 0$ be a local holomorphic equation of the divisor D_j on $\widehat{X}$. Since $\log \sum_\ell |g_{j,\ell}|^2 = \log |u_j|^2 + \log \sum_\ell |g_{j,\ell}/uj|^2 = \log |u_j|^2 + v_j$, where $v_j \in C^\infty$ and $dd^c \log |u_j|^2 = [D_j]$ by the Lelong-Poincaré equation, we find

$$\mu^* T_j = \mu^*\alpha_j + dd^c(\varphi_j \circ \mu) = c_j[D_j] + \widehat{T}_j, \quad \text{where} \quad \widehat{T}_j = \mu^*\alpha_j + dd^c\widehat{\varphi}_j \tag{27}$$

and $\widehat{\varphi}_j$ is a locally bounded potential on $\widehat{X}$ such that $\widehat{T}_j \geq 0$. Now, if $E = \mu^{-1}(Z)$, we get

$$\mathbf{1}_{X\smallsetminus Z}T_1 \wedge \ldots \wedge T_p = \mu_*(\mathbf{1}_{\widehat{X}\smallsetminus E}\widehat{T}_1 \wedge \ldots \wedge \widehat{T}_p) = \mu_*(\widehat{T}_1 \wedge \ldots \wedge \widehat{T}_p). \tag{28}$$

Hence the right-hand side defines the desired extension of $\mathbf{1}_{X\smallsetminus Z}T_1 \wedge \ldots \wedge T_p$ to X as the direct image of a closed positive current on $\widehat{X}$ carrying no mass on E. An essential point is the following monotonicity lemma – the reader will find a more general version for non-pluripolar products in [2, Theorem 1.16].

Lemma 4 *Assume that we have closed positive* $(1,1)$*-currents with analytic singularities* T_j, $T_j' \in \{\alpha_j\}$ *with* $T_j \preccurlyeq T_j'$, $1 \leq j \leq p$, *and let* $\gamma \geq 0$ *be a closed positive smooth* $(n-p, n-p)$*-form on* X*. If* Z *is an analytic set containing the poles of all* T_j *and* T_j'*, we have*

$$\int_X \mathbf{1}_{X\smallsetminus Z}T_1 \wedge \ldots \wedge T_p \wedge \gamma \geq \int_X \mathbf{1}_{X\smallsetminus Z}T_1' \wedge \ldots \wedge T_p' \wedge \gamma.$$

Proof We take a log-resolution $\mu : \widehat{X} \to X$ that works for all T_j and T_j' simultaneously. By (27) and (28), we have $\mu^*T_j = c_j[D_j] + \widehat{T}_j$ where $\widehat{T}_j \geq 0$ has a locally bounded potential on $\widehat{X}$, and

$$\int_X \mathbf{1}_{X\smallsetminus Z}T_1 \wedge \ldots \wedge T_p \wedge \gamma = \int_{\widehat{X}} \widehat{T}_1 \wedge \ldots \wedge \widehat{T}_p \wedge \mu^*\gamma.$$

There are of course similar formulas $\mu^*T_j' = c_j[D_j'] + \widehat{T}_j'$ for the T_j''s, and our assumption $T_j \preccurlyeq T_j'$ means that the corresponding divisors satisfy $c_jD_j \leq c_j'D_j'$, hence $\Delta_j := c_j'D_j' - c_jD_j \geq 0$. In terms of cohomology, we have

$$\mu^*\{\alpha_j\} = \{\mu^*T_j\} = \{\widehat{T}_j\} + \{c_jD_j\} = \{\mu^*T_j'\} = \{\widehat{T}_j'\} + \{c_j'D_j'\},$$

hence $\{\widehat{T}_j\} = \{\widehat{T}_j'\} + \{\Delta_j\}$ in $H^2(\widehat{X}, \mathbb{R})$. By Stokes' theorem, we conclude that

$$\begin{aligned}\int_{\widehat{X}} \widehat{T}_1 \wedge \widehat{T}_2 \wedge \ldots \wedge \widehat{T}_p \wedge \mu^*\gamma &= \int_{\widehat{X}} (\widehat{T}_1' + \{\Delta_1\}) \wedge \widehat{T}_2 \wedge \ldots \wedge \widehat{T}_p \wedge \mu^*\gamma \\ &\geq \int_{\widehat{X}} \widehat{T}_1' \wedge \widehat{T}_2 \wedge \ldots \wedge \widehat{T}_p \wedge \mu^*\gamma\end{aligned}$$

thanks to the positivity of our currents $\widehat{T}_j$, $\widehat{T}_j'$ and the fact that the product of such currents with bounded potentials by the current of integration $[\Delta_j]$ is well defined and positive ([10]). By replacing successively all terms $\{\widehat{T}_j\}$ by $\{\widehat{T}'_j\} + \{\Delta_j\}$ we infer

$$\int_{\widehat{X}} \widehat{T}_1 \wedge \ldots \wedge \widehat{T}_p \wedge \mu^*\gamma \geq \int_{\widehat{X}} \widehat{T}'_1 \wedge \ldots \wedge \widehat{T}'_p \wedge \mu^*\gamma. \qquad \square$$

Now, assume that we have arbitrary closed positive $(1,1)$-currents $T_1, \ldots, T_p$. For each of them, we take a sequence $T_{j,k} = \alpha_j + i\partial\overline{\partial}\psi_{j,k}$ of monotone asymptotically equisingular approximations by currents with analytic singularities, $T_{j,k} \geq -\varepsilon_{j,k}\omega$, $\lim_{k\to+\infty} \varepsilon_{j,k} = 0$. We have $T_{j,k} \preccurlyeq T_{j,k+1}$, and we may also assume without loss of generality that $\varepsilon_{j,k} \geq \varepsilon_{j,k+1} > 0$ for all j, k. Let Z_k be an analytic containing all poles of the $T_{j,k}$, $1 \leq j \leq p$. It follows immediately from the above discussion and especially from Lemma 4 that the integrals

$$\int_X \mathbf{1}_{X\smallsetminus Z_k}(T_{1,k} + \varepsilon_{1,k}\omega) \wedge \ldots \wedge (T_{p,k} + \varepsilon_{p,k}\omega) \wedge \gamma \geq 0$$

are well defined and nonincreasing in k (the fact that $\varepsilon_{j,k}$ is non increasing even helps here). From this, we conclude

Theorem 15 *For every $p = 1, 2, \ldots, n$, there is a well defined p-fold intersection product*

$$\widehat{\mathscr{S}}(X) \times \cdots \times \widehat{\mathscr{S}}(X) \longrightarrow H^{p,p}_+(X, \mathbb{R})$$

which assigns to any p-tuple of equivalence classes of monotone sequences $(T_{j,k})$ in $\widehat{\mathscr{S}}(X)$, $1 \leq j \leq p$, the limit cohomology class

$$\lim_{k\to+\infty} \left\{\mathbf{1}_{X\smallsetminus Z_k}(T_{1,k} + \varepsilon_{1,k}\omega) \wedge \ldots \wedge (T_{p,k} + \varepsilon_{p,k}\omega)\right\} \in H^{p,p}_+(X, \mathbb{R})$$

where $H^{p,p}_+(X, \mathbb{R}) \subset H^{p,p}(X, \mathbb{R})$ denotes the cone of cohomology classes of closed positive (p,p)-currents. This product is additive and homogeneous in each argument in the space $\widehat{\mathscr{S}}(X)$.

Corollary 8 *By combining the above formal intersection product with the Bergman approximation operator $\mathbf{B} : \mathscr{S}(X) \to \widehat{\mathscr{S}}(X)$, we get an intersection product*

$$\mathscr{S}(X) \times \cdots \times \mathscr{S}(X) \longrightarrow H^{p,p}_+(X, \mathbb{R}) \quad \textit{denoted} \quad (T_1, \ldots, T_p) \longmapsto \langle T_1, \ldots, T_p\rangle^+,$$

which is homogeneous and additive in each argument.

Proof (of Theorem 15) The existence of a limit in cohomology is seen by fixing a dual basis $(\{\gamma_j\})$ of $H^{n-p,n-p}(X)$, using the Serre duality pairing

$$H^{p,p}(X, \mathbb{R}) \times H^{n-p,n-p}(X) \to \mathbb{R}, \qquad (\beta, \gamma) \mapsto \int_X \beta \wedge \gamma.$$

Since X is Kähler, we can take $\gamma_1 = \omega^{n-p}$ and replace if necessary γ_j by $\gamma_j + C\omega^{n-p}$, $C \gg 1$, to get $\gamma_j \geq 0$ for all $j \geq 2$. Then the integrals

$$\int_X \mathbf{1}_{X \smallsetminus Z_k}(T_{1,k} + \varepsilon_{1,k}\omega) \wedge \ldots \wedge (T_{p,k} + \varepsilon_{p,k}\omega) \wedge \gamma_j \geq 0$$

are nonincreasing in k, and the limit must therefore exist by monotonicity. □

Remark 9 It is natural to ask how the above intersection product compares with the (cohomology class of the) "non-pluripolar product" $\langle T_1, \ldots, T_p \rangle$ defined in [2, § 1]. In fact, the above product only neglects analytic parts of the currents involved. The simple example of a probability measure T without atoms supported on a polar set of a compact Riemann surface X yields e.g. $\langle T \rangle^+ = 1$, while the non-pluripolar part $\langle T \rangle$ vanishes.

5.3 Kähler Definition of the Numerical Dimension

Using the intersection product defined in Theorem 15, we can give a precise definition of the numerical dimension.

Definition 5 Let (X, ω) be a compact Kähler n-dimensional manifold. We define the numerical dimension $\mathrm{nd}(T)$ of a closed positive $(1, 1)$-current T on X to be the largest integer $p = 0, 1, \ldots, n$ such that $\langle T^p \rangle^+ \neq 0$, i.e. $\int_X \langle T^p \rangle^+ \wedge \omega^{n-p} > 0$.

Accordingly, if (L, h) be a pseudoeffective line bundle on X, we define its numerical dimension to be

$$\mathrm{nd}(L, h) = \mathrm{nd}(\,\mathrm{i}\,\Theta_{L,h}). \tag{29}$$

By the results of the preceding subsection, $\mathrm{nd}(L, h)$ depends only on the weak equivalence class of singularities of the metric h.

Remark 10 H. Tsuji [63] has defined a notion of numerical dimension by a more algebraic method:

Definition 6 Let X be a projective variety and (L, h) a pseudo-effective line bundle. When V runs over all irreducible algebraic suvarieties of X, one defines

$$\nu_{\mathrm{num}}(L, h) = \sup\left\{p = \dim V\,;\ \limsup_{m\to\infty} \frac{h^0\big(\widetilde{V}, \mu^*(L^{\otimes m}) \otimes \mathscr{I}(\mu^* h^m)\big)}{m^p} > 0\right\}$$

where $\mu : \widetilde{V} \to V \subset X$ is an embedded desingularization of V in X.

Junyan Cao [14] has shown that $\nu_{\mathrm{num}}(L, h)$ coincides with $\mathrm{nd}(L, h)$ as defined in (29). The idea is to make a reduction to the "big" case $\mathrm{nd}(L, h) = \dim X$ and to use holomorphic Morse inequalities [20] in combination with a regularization procedure. We omit the rather technical details here.

Remark 11 If L is pseudo-effective there is also a natural concept of numerical dimension $\mathrm{nd}(L)$ that does not depend on the choice of a metric h on L. One can set e.g.

$$\mathrm{nd}(L) = \max \Big\{ p \in [0,n]\,;\ \exists c > 0,\ \forall \varepsilon > 0,\ \exists h_\varepsilon,\ \ \Theta_{L,h_\varepsilon} \geq -\varepsilon\omega, \ \text{ such that} \\ \int_{X \smallsetminus Z_\varepsilon} (\mathrm{i}\,\Theta_{L,h_\varepsilon} + \varepsilon\omega)^p \wedge \omega^{n-p} \geq c \Big\},$$

where h_ε runs over all metrics with analytic singularities on L. It may happen in general that $\mathrm{nd}(L, h_{\min}) < \mathrm{nd}(L)$, even when L is nef; in that case the h_ε can be taken to be smooth in the definition of $\mathrm{nd}(L)$, and therefore $\mathrm{nd}(L)$ is the largest integer p such that $c_1(L)^p \neq 0$. In fact, for the line bundle L already mentioned in Remark 5, it is shown in [30] that there is unique positive current $T \in c_1(L)$, namely the current of integration $T = [C]$ on the negative curve $C \subset X$, hence $\mathrm{nd}(L, h_{\min}) = \mathrm{nd}([C]) = 0$, although we have $\mathrm{nd}(L) = 1$ here.

6 Proof of Junyan Cao's Vanishing Theorem

This section is a brief account and a simplified exposition of Junyan Cao's proof, as detailed in his PhD thesis [13]. The key curvature and singularity estimates are contained in the following technical statement, which depends in a crucial way on Bergman regularization and on Yau's theorem [64] for solutions of Monge-Ampère equations.

Proposition 5 *Let (L, h) be a pseudoeffective line bundle on a compact Kähler manifold (X, ω). Let us write $T = \frac{\mathrm{i}}{2\pi}\Theta_{L,h} = \alpha + dd^c\varphi$ where α is smooth and φ is a quasi-psh potential. Let $p = \mathrm{nd}(L, h)$ be the numerical dimension of (L, h). Then, for every $\gamma \in \,]0, 1]$ and $\delta \in \,]0, 1]$, there exists a quasi-psh potential $\Phi_{\gamma,\delta}$ on X satisfying the following properties:*

(a) *$\Phi_{\gamma,\delta}$ is smooth in the complement $X \smallsetminus Z_\delta$ of an analytic set $Z_\delta \subset X$.*
(b) *$\alpha + \delta\omega + dd^c\Phi_{\gamma,\delta} \geq \frac{\delta}{2}(1-\gamma)\omega$ on X.*
(c) *$(\alpha + \delta\omega + dd^c\Phi_{\gamma,\delta})^n \geq a\,\gamma^n\delta^{n-p}\omega^n$ on $X \smallsetminus Z_\delta$.*
(d) *$\Phi_{\gamma,\delta} \leq (1 + b\delta)\psi_{B,k} + C_{\gamma,\delta}$ where $\psi_{B,k} \geq \varphi$ is a Bergman approximation of φ of sufficiently high index $k = k_0(\delta)$.*
(e) *$\sup_X \Phi_{1,\delta} = 0$, and for all $\gamma \in \,]0, 1]$ there are estimates $\Phi_{\gamma,\delta} \leq A$ and*

$$\exp\big(-\Phi_{\gamma,\delta}\big) \leq e^{-(1+b\delta)\varphi} \exp\big(A - \gamma\Phi_{1,\delta}\big)$$

(f) *For* $\gamma_0,\ \delta_0 > 0$ *small,* $\gamma \in\ ]0,\gamma_0]$, $\delta \in\ ]0,\delta_0]$ *and* $k = k_0(\delta)$ *large enough, we have*

$$\mathscr{I}(\Phi_{\gamma,\delta}) = \mathscr{I}_+(\varphi) = \mathscr{I}(\varphi).$$

Here $a,\ b,\ A,\ \gamma_0,\ \delta_0,\ C_{\gamma,\delta} > 0$ *are suitable constants* ($C_{\gamma,\delta}$ *being the only one that depends on* γ*,* δ*).*

Proof Denote by $\psi_{B,k}$ the nonincreasing sequence of Bergman approximations of φ (obtained with denominators $m_k = 2^k$, say). We have $\psi_{B,k} \geq \varphi$ for all k, the $\psi_{B,k}$ have analytic singularities and $\alpha + dd^c\psi_{B,k} \geq -\varepsilon_k\omega$ with $\varepsilon_k \downarrow 0$. Then $\varepsilon_k \leq \frac{\delta}{4}$ for $k \geq k_0(\delta)$ large enough, and so

$$\begin{aligned}\alpha + \delta\omega + dd^c\big((1+b\delta)\psi_{B,k}\big) &\geq \alpha + \delta\omega - (1+b\delta)(\alpha+\varepsilon_k\omega)\\ &\geq \delta\omega - (1+b\delta)\varepsilon_k\omega - b\delta\alpha \geq \tfrac{\delta}{2}\omega\end{aligned}$$

for $b > 0$ small enough (independent of δ and k). Let $\mu : \widehat{X} \to X$ be a log-resolution of $\psi_{B,k}$, so that

$$\mu^*\big(\alpha + \delta\omega + dd^c((1+b\delta)\psi_{B,k})\big) = c_k[D_k] + \beta_k$$

where $\beta_k \geq \frac{\delta}{2}\mu^*\omega \geq 0$ is a smooth closed $(1,1)$-form on $\widehat{X}$ that is > 0 in the complement $\widehat{X} \smallsetminus E$ of the exceptional divisor, $c_k = \frac{1+b\delta}{m_k} > 0$, and D_k is a divisor that includes all components E_ℓ of E. The map μ can be obtained by Hironaka [39] as a composition of a sequence of blow-ups with smooth centers, and we can even achieve that D_k and E are normal crossing divisors. In this circumstance, it is well known that there exist arbitrary small numbers $\eta_\ell > 0$ such that $\beta_k - \sum\eta_\ell[E_\ell]$ is a Kähler class on $\widehat{X}$. Hence we can find a quasi-psh potential $\widehat{\theta}_k$ on $\widehat{X}$ such that $\widehat{\beta}_k := \beta_k - \sum\eta_\ell[E_\ell] + dd^c\widehat{\theta}_k$ is a Kähler metric on $\widehat{X}$, and by taking the η_ℓ small enough, we may assume that $\int_{\widehat{X}}(\widehat{\beta}_k)^n \geq \frac{1}{2}\int_{\widehat{X}}\beta_k^n$. Now, we write

$$\begin{aligned}\alpha + \delta\omega + dd^c\big((1+b\delta)\psi_{B,k}\big) &\geq \alpha + \varepsilon_k\omega + dd^c\psi_{B,k} + (\delta-\varepsilon_k)\omega - b\delta(\alpha+\varepsilon_k\omega)\\ &\geq (\alpha + \varepsilon_k\omega + dd^c\psi_{B,k}) + \tfrac{\delta}{2}\omega\end{aligned}$$

for $k \geq k_0(\delta)$ and $b > 0$ small (independent of δ and k). The assumption on the numerical dimension of $\frac{i}{2\pi}\Theta_{L,h} = \alpha + dd^c\varphi$ implies the existence of a constant $c > 0$ such that, with $Z = \mu(E) \subset X$, we have

$$\begin{aligned}\int_{\widehat{X}}\beta_k^n &= \int_X \mathbf{1}_{X\smallsetminus Z}\big(\alpha + \delta\omega + dd^c((1+b\delta)\psi_{B,k})\big)^n\\ &\geq \binom{n}{p}\Big(\frac{\delta}{2}\Big)^{n-p}\int_{X\smallsetminus Z}\big(\alpha + \varepsilon_k\omega + dd^c\psi_{B,k}\big)^p \wedge \omega^{n-p} \geq c\,\delta^{n-p}\int_X \omega^n\end{aligned}$$

for all $k \geq k_0(\delta)$. Therefore, we may assume

$$\int_{\widehat{X}} (\widehat{\beta}_k)^n \geq \frac{c}{2}\,\delta^{n-p} \int_X \omega^n.$$

By Yau's theorem [64], there exists a quasi-psh potential $\widehat{\tau}_k$ on $\widehat{X}$ such that $\widehat{\beta}_k + dd^c\widehat{\tau}_k$ is a Kähler metric on $\widehat{X}$ with a prescribed volume form $\hat{f} > 0$ such that $\int_{\widehat{X}} f = \int_{\widehat{X}} \widehat{\beta}_k^n$. By the above discussion, we can take here $\hat{f} > \frac{c}{3}\delta^{n-p}\mu^*\omega^n$ everywhere on $\widehat{X}$.

Now, we consider $\theta_k = \mu_*\widehat{\theta}_k$ and $\tau_k = \mu_*\widehat{\tau}_k \in L^1_{\mathrm{loc}}(X)$. Since $\widehat{\theta}_k$ was defined in such a way that $dd^c\widehat{\theta}_k = \widehat{\beta}_k - \beta_k + \sum_\ell \eta_\ell[E_\ell]$, we get

$$\mu^*\big(\alpha + \delta\omega + dd^c((1 + b\delta)\psi_{B,k} + \gamma(\theta_k + \tau_k))\big)$$
$$= c_k[D_k] + (1-\gamma)\beta_k + \gamma\Big(\sum_\ell \eta_\ell[E_\ell] + \widehat{\beta}_k + dd^c\widehat{\tau}_k\Big) \geq 0.$$

This implies in particular that $\Phi_{\gamma,\delta} := (1 + b\delta)\psi_{B,k} + \gamma(\theta_k + \tau_k)$ is a quasi-psh potential on X and that

$$\mu^*\big(\alpha + \delta\omega + dd^c\Phi_{\gamma,\delta}\big) \geq (1-\gamma)\beta_k \geq \frac{\delta}{2}(1-\gamma)\,\mu^*\omega,$$

thus condition (b) is satisfied. Putting $Z_\delta = \mu(|D_k|) \supset \mu(E) = Z$, we also have

$$\mu^*\mathbf{1}_{X\smallsetminus Z_\delta}\big(\alpha + \delta\omega + dd^c\Phi_{\gamma,\delta}\big)^n \geq \gamma^n\,\widehat{\beta}_k^n \geq \frac{c}{3}\,\gamma^n\delta^{n-p}\mu^*\omega^n,$$

therefore condition (c) is satisfied as well with $a = c/3$. Property (a) is clear, and (d) holds since the quasi-psh function $\widehat{\theta}_k + \widehat{\tau}_k$ must be bounded from above on $\widehat{X}$. We will actually adjust constants in $\widehat{\theta}_k + \widehat{\tau}_k$ (as we may), so that $\sup_X \Phi_{1,\delta} = 0$. Since $\varphi \leq \psi_{B,k} \leq \psi_{B,0} \leq A_0 := \sup_X \psi_{B,0}$ and

$$\Phi_{\gamma,\delta} = (1 + b\delta)\psi_{B,k} + \gamma\big(\Phi_{1,\delta} - \psi_{B,k}\big) = (1 - \gamma + b\delta)\psi_{B,k} + \gamma\Phi_{1,\delta},$$

we have

$$(1 + b\delta)\varphi - \gamma(A_0 - \psi_{B,k}) \leq \Phi_{\gamma,\delta} \leq (1 - \gamma + b\delta)A_0$$

and the estimates in (e) follow with $A = (1 + b)A_0$. The only remaining property to be proved is (f). Condition (d) actually implies $\mathscr{I}(\Phi_{\gamma,\delta}) \subset \mathscr{I}((1 + b\delta)\psi_{B,k})$, and Corollary 4 also gives $\mathscr{I}((1 + b\delta)\psi_{B,k}) \subset \mathscr{I}((1 + b\delta/2)\varphi)$ if we take $k \geq k_0(\delta)$ large enough, hence $\mathscr{I}(\Phi_{\gamma,\delta}) \subset \mathscr{I}_+(\varphi)$ for $\delta \leq \delta_0$ small. In the opposite direction, we observe that $\Phi_{1,\gamma}$ satisfies $\alpha + \omega + dd^c\Phi_{1,\delta} \geq 0$ and $\sup_X \Phi_{1,\delta} = 0$, hence $\Phi_{1,\delta}$ belongs to a compact family of quasi-psh functions. A standard result of potential theory then shows the existence of a uniform small constant $c_0 > 0$

such that $\int_X \exp(-c_0\Phi_{1,\delta})dV_\omega < +\infty$ for all $\delta \in\,]0,1]$. If $f \in \mathscr{O}_{X,x}$ is a germ of holomorphic function and U a small neighborhood of x, the Hölder inequality combined with estimate (e) implies

$$\int_U |f|^2 \exp(-\Phi_{\gamma,\delta})dV_\omega \le e^A\Big(\int_U |f|^2 e^{-p(1+b\delta)\varphi}dV_\omega\Big)^{\frac{1}{p}}\Big(\int_U |f|^2 e^{-q\gamma\Phi_{1,\delta}}dV_\omega\Big)^{\frac{1}{q}}.$$

We fix $\lambda_0 > 1$ so that $\mathscr{I}(\lambda_0\varphi) = \mathscr{I}_+(\varphi)$, $p \in\,]1,\lambda_0[$ (say $p = 1+\lambda_0)/2$), and take

$$\gamma \le \gamma_0 := \frac{c_0}{q} = c_0\frac{\lambda_0 - 1}{\lambda_0 + 1} \quad \text{and} \quad \delta \le \delta_0 \in\,]0,1] \text{ so small that } p(1+b\delta_0) \le \lambda_0.$$

Then clearly $f \in \mathscr{I}(\lambda_0\varphi)$ implies $f \in \mathscr{I}(\Phi_{\gamma,\delta})$, and (f) is proved. □

The rest of the arguments proceeds along the lines of [19, 49] and [28]. Let (L,h) be a pseuffective line bundle and $p = \mathrm{nd}(L,h) = \mathrm{nd}(\mathrm{i}\,\Theta_{L,h})$. We equip L be the hermitian metric h_δ defined by the quasi-psh weight $\Phi_\delta = \Phi_{\gamma_0,\delta}$ obtained in Proposition 5, with $\delta \in\,]0,\delta_0]$. Since Φ_δ is smooth on $X \smallsetminus Z_\delta$, the well-known Bochner-Kodaira identity shows that for every smooth (n,q)-form u with values in $K_X \otimes L$ that is compactly supported on $X \smallsetminus Z_\delta$, one has

$$\|\overline{\partial}u\|^2_\delta + \|\overline{\partial}^* u\|^2_\delta \ge 2\pi\int_X (\lambda_{1,\delta} + \ldots + \lambda_{q,\delta} - q\delta)|u|^2 e^{-\Phi_\delta}dV_\omega,$$

where $\|u\|^2_\delta := \int_X |u|^2_{\omega,h_\delta}dV_\omega = \int_X |u|^2 e^{-\Phi_\delta}dV_\omega$ and

$$0 < \lambda_{1,\delta}(x) \le \ldots \le \lambda_{n,\delta}(x)$$

are, at each point $x \in X$, the eigenvalues of $\alpha + \delta\omega + dd^c\Phi_\delta$ with respect to the base Kähler metric ω. Notice that the $\lambda_{j,\delta}(x) - \delta$ are the actual eigenvalues of $\frac{\mathrm{i}}{2\pi}\Theta_{L,h_\delta} = \alpha + dd^c\Phi_\delta$ with respect to ω and that the inequality $\lambda_{j,\delta}(x) \ge \frac{\delta}{2}(1-\gamma) > 0$ is guaranteed by Proposition 5 (b). After dividing by $2\pi q$ (and neglecting that constant in the left hand side), we get

$$\|\overline{\partial}u\|^2_\delta + \|\overline{\partial}^* u\|^2_\delta + \delta\|u\|^2_\delta \ge \int_X (\lambda_{1,\delta} + \ldots + \lambda_{q,\delta})|u|^2 e^{-\Phi_\delta}dV_\omega. \tag{30}$$

A standard Hahn-Banach argument in the L^2-theory of the $\overline{\partial}$-operator then yields the following conclusion.

Proposition 6 *For every L^2 section of $\Lambda^{n,q}T^*_X \otimes L$ such that $\|f\|_\delta < +\infty$ and $\overline{\partial}f = 0$ in the sense of distributions, there exists a L^2 section $v = v_\delta$ of $\Lambda^{n,q-1}T^*_X \otimes L$ and a*

L^2 section $w = w_\delta$ of $\Lambda^{n,q}T_X^* \otimes L$ such that $f = \overline{\partial} v + w$ with

$$\|v\|_\delta^2 + \frac{1}{\delta}\|w\|_\delta^2 \le \int_X \frac{1}{\lambda_{1,\delta} + \ldots + \lambda_{q,\delta}} |f|^2 e^{-\Phi_\delta} dV_\omega.$$

Because of the singularities of the weight on Z_δ, one should in fact argue first on $X \smallsetminus Z_\delta$ and approximate the base Kähler metric ω by a metric $\widehat{\omega}_{\delta,\varepsilon} = \omega + \varepsilon\widehat{\omega}_\delta$ that is complete on $X \smallsetminus Z_\delta$, exactly as explained in [19]; we omit the (by now standard) details here. A consequence of Proposition 6 is that the "error term" w satisfies the L^2 bound

$$\int_X |w|^2 e^{-\Phi_\delta} dV_\omega \le \int_X \frac{\delta}{\lambda_{1,\delta} + \ldots + \lambda_{q,\delta}} |f|^2 e^{-\Phi_\delta} dV_\omega. \tag{31}$$

The idea for the next estimate is taken from Mourougane's PhD thesis [49].

Lemma 5 *The ratio $\rho_\delta(x) := \delta/(\lambda_{1,\delta}(x) + \ldots + \lambda_{q,\delta}(x))$ is uniformly bounded on X (independently of δ), and, as soon as $q \ge n - \mathrm{nd}(L,h) + 1$, there exists a subsequence $(\rho_{\delta\ell})$, $\delta_\ell \to 0$, that tends almost everywhere to 0 on X.*

Proof By estimates (b,c) in Proposition 5, we have $\lambda_{j,\delta}(x) \ge \frac{\delta}{2}(1 - \gamma_0)$ and

$$\lambda_{1,\delta}(x) \ldots \lambda_{n,\delta}(x) \ge a\gamma_0^n \delta^{n-p} \quad \text{where } \ p = \mathrm{nd}(L,h). \tag{32}$$

Therefore we already find $\rho_\delta(x) \le 2/q(1 - \gamma_0)$. Now, we have

$$\int_{X \smallsetminus Z_\delta} \lambda_{n,\delta}(x) dV_\omega \le \int_X (\alpha + \delta\omega + dd^c\Phi_\delta) \wedge \omega^{n-1} = \int_X (\alpha + \delta\omega) \wedge \omega^{n-1} \le \text{Const},$$

therefore the "bad set" $S_\varepsilon \subset X \smallsetminus Z_\delta$ of points x where $\lambda_{n,\delta}(x) > \delta^{-\varepsilon}$ has a volume $\mathrm{Vol}(S_\varepsilon) \le C\delta^\varepsilon$ converging to 0 as $\delta \to 0$ (with a slightly more elaborate argument we could similarly control any elementary symmetric function in the $\lambda_{j,\delta}$'s, but this is not needed here). Outside of S_ε, the inequality (32) yields

$$\lambda_{q,\delta}(x)^q \delta^{-\varepsilon(n-q)} \ge \lambda_{q,\delta}(x)^q \lambda_{n,\delta}(x)^{n-q} \ge a\gamma_0^n \delta^{n-p}$$

hence

$$\lambda_{q,\delta}(x) \ge c\delta^{\frac{n-p+(n-q)\varepsilon}{q}} \quad \text{and} \quad \rho_\delta(x) \le C\delta^{1-\frac{n-p+(n-q)\varepsilon}{q}}.$$

If we take $q \ge n - p + 1$ and $\varepsilon > 0$ small enough, the exponent of δ in the final estimate is positive, and Lemma 5 follows. □

Proof (of Junyan Cao's Theorem 5) Let $\{f\}$ be a cohomology class in the group $H^q(X, K_X \otimes L \otimes \mathscr{I}_+(h))$, $q \ge n - \mathrm{nd}(L,h) + 1$. Consider a finite Stein open covering $\mathscr{U} = (U_\alpha)_{\alpha=1,\ldots,N}$ by coordinate balls U_α. There is an isomorphism between Čech cohomology $\check{H}^q(\mathscr{U}, \mathscr{F})$ with values in the sheaf $\mathscr{F} = \mathscr{O}(K_X \otimes L) \otimes \mathscr{I}_+(h)$ and the cohomology of the complex $(K_\delta^\bullet, \overline{\partial})$ of (n,q)-forms u such that both u and $\overline{\partial}u$ are L^2 with respect to the weight Φ_δ, i.e. $\int_X |u|^2 \exp(-\Phi_\delta) dV_\omega < +\infty$ and $\int_X |\overline{\partial}u|^2 \exp(-\Phi_\delta) dV_\omega < +\infty$. The isomorphism comes from Leray's theorem and from the fact that the sheafified complex $(\mathscr{K}_\delta^\bullet, \overline{\partial})$ is a complex of $\mathscr{C}^\infty$-modules that provides a resolution of the sheaf $\mathscr{F}$: the main point here is that $\mathscr{I}(\Phi_\delta) = \mathscr{I}_+(\varphi) = \mathscr{I}_+(h)$, as asserted by Proposition 5 (f), and that we can locally solve $\overline{\partial}$-equations by means of Hörmander's estimates [40].

Let (ψ_α) be a partition of unity subordinate to $\mathscr{U}$. The explicit isomorphism between Čech cohomology and L^2 cohomology yields a smooth L^2 representative $f = \sum_{|I|=q} f_I(z) dz_1 \wedge \ldots \wedge dz_n \wedge d\overline{z}_I$ which is a combination

$$f = \sum_{\alpha_0} \psi_{\alpha_0} c_{\alpha_0\alpha_1\ldots\alpha_q} \overline{\partial}\omega_{\alpha_1} \wedge \ldots \wedge \overline{\partial}\psi_{\alpha_q}$$

of the components of the corresponding Čech cocycle

$$c_{\alpha_0\alpha_1\ldots\alpha_q} \in \Gamma\big(U_{\alpha_0} \cap U_{\alpha_1} \cap \ldots \cap U_{\alpha_q}, \mathscr{O}(\mathscr{F})\big).$$

Estimate (e) in Proposition 5 implies the Hölder inequality

$$\int_X \rho_\delta |f|^2 \exp(-\Phi_\delta) dV_\omega \le e^A \Big(\int_X \rho_\delta^p |f|^2 e^{-p(1+b\delta)\varphi} dV_\omega \Big)^{\frac{1}{p}} \Big(\int_X |f|^2 e^{-q\gamma_0 \Phi_{1,\delta}} dV_\omega \Big)^{\frac{1}{q}}.$$

Our choice of $\delta \le \delta_0$, γ_0 and p, q shows that the integrals in the right hand side are convergent, and especially $\int_X |f|^2 e^{-p(1+b\delta)\varphi} dV_\omega < +\infty$. Lebesgue's dominated convergence theorem combined with Lemma 5 implies that the L^p-part goes to 0 as $\delta = \delta_\ell \to 0$, hence the "error term" w converges to 0 in L^2 norm by estimate (31). If we express the corresponding class $\{w\}$ in Čech cohomology and use Hörmander's estimates on the intersections $U_\alpha = \bigcap U_{\alpha_j}$, we see that $\{w\}$ will be given by a Čech cocycle $(\tilde{w}_\alpha)$ such that $\int_{U_\alpha} |\tilde{w}_\alpha|^2 e^{-\Phi_\delta} dV_\omega \to 0$ as $\delta = \delta_\ell \to 0$ (we may lose here some fixed constants since Φ_δ is just quasi-psh on our balls, but this is irrelevant thanks to the uniform lower bounds for the Hessian). The inequality $\Phi_\delta \le A$ in Proposition 5 (e) shows that we have as well an unweighted L^2 estimate $\int_{U_\alpha} |\tilde{w}_\alpha|^2 dV \to 0$. However it is well-known that when one takes unweighted L^2 norms on spaces of Čech cocyles (or uniform convergence on compact subsets, for that purpose), the resulting topology on the finite dimensional space $\check{H}^q(\mathscr{U}, \mathscr{F})$ is Hausdorff, so the subspace of coboundaries is closed in the space of cocycles. Hence we conclude from the above that f is a coboundary, as desired. □

Remark 12 In this proof, it is remarkable that one can control the error term w, but a priori completely loses control on the element v such that $\overline{\partial}v \approx f$ when $\delta \to 0$!

7 Compact Kähler Threefolds Without Nontrivial Subvarieties

The bimeromorphic classification of compact Kähler manifolds leads to considering those, termed as "simple" by Campana, that have as little internal structure as possible, and are somehow the elementary bricks needed to reconstruct all others through meromorphic fibrations (cf. [11, 12]).

Definition 7 A compact Kähler manifold X is said to be simple if there does not exist any irreducible analytic subvariety Z with $0 < \dim Z < \dim X$ through a very general point $x \in X$, namely a point x in the complement $X \smallsetminus \bigcup S_j$ of a countable union of analytic sets $S_j \subsetneq X$.

Of course, every one dimensional manifold X is simple, but in higher dimensions $n > 1$, one can show that a very general torus $X = \mathbb{C}^n/\Lambda$ has no nontrivial analytic subvariety Z at all (i.e. none beyond finite sets and X itself), in any dimension n. In even dimension, a very general Hyperkähler manifold can be shown to be simple as well. It has been known since Kodaira that there are no other simple Kähler surfaces (namely only very general 2-dimensional tori and K3 surfaces). Therefore, the next dimension to be investigated is dimension 3. In this case, Campana, Höring and Peternell have shown in [17] that X is bimeromorphically a quotient of a torus by a finite group (see Theorem 18 at the end). Following [16], we give here a short self-contained proof for "strongly simple" Kähler threefolds, namely threefolds that do not possess any proper analytic subvariety.

The simplicity assumption implies that the algebraic dimension is $a(X) = 0$, in particular X cannot be projective, and cannot either be uniruled (i.e. covered by rational curves). By the Kodaira embedding theorem, we also infer that $H^0(X, \Omega_X^2) \neq 0$, otherwise X would be projective. One of the most crucial arguments is the following strong and difficult theorem of Brunella [9].

Theorem 16 ([9]) *Let X be a compact Kähler manifold with a 1-dimensional holomorphic foliation F given by a nonzero morphism of vector bundle $L \to T_X$, where L is a line bundle on X, and T_X is its holomorphic tangent bundle. If L^{-1} is not pseudoeffective, the closures of the leaves of F are rational curves, and X is thus uniruled.*

We use this result in the form of the following corollary, which has been observed in [41], Proposition 4.2.

Corollary 9 *If X is a non uniruled n-dimensional compact Kähler manifold with $H^0(X, \Omega_X^{n-1}) \neq 0$, then K_X is pseudoeffective.*

Proof Ω_X^{n-1} is canonically isomorphic to $K_X \otimes T_X$. Any nonzero section of Ω_X^{n-1} thus provides a nonzero map $K_X^{-1} \to T_X$, and an associated foliation. □

It follows from the above that the canonical line bundle K_X of our simple threefold X must be pseudoeffective. We then use the following simple observation.

Proposition 7 *Assume that X is a strongly simple compact complex manifold. Then every pseudoeffective line bundle (L, h) is nef, and all multiplier sheaves $\mathscr{I}(h^m)$ are trivial, i.e. $\mathscr{I}(h^m) = \mathscr{O}_X$. Moreover, we have $c_1(L)^n = 0$.*

Proof Since there are not positive dimensional analytic subvarieties, the zero varieties of the ideal sheaves $\mathscr{I}(h^m)$ must be finite sets of points, hence, by Skoda [60], the Lelong numbers $\nu(\mathrm{i}\,\Theta_{L,h}, x)$ are zero except on a countable set $S \subset X$. By [21], this implies that L is nef and $c_1(L)^n \geq \sum_{x\in S} \nu(\mathrm{i}\,\Theta_{L,h}, x)^n$. However, by the Grauert-Riemenschneider conjecture solved in [58, 59] and [20], the positivity of $c_1(L)^n$ would imply that $a(X) = n$ (i.e. X Moishezon, a contradiction). Therefore $c_1(L)^n = 0$ and $S = \emptyset$. □

Proposition 8 *Let X be a compact Kähler manifold of dimension $n > 1$ without any non-trivial subvariety, and with K_X pseudoeffective. Then*

$$h^j(X, K_X^{\otimes m}) \leq h^0(X, \Omega_X^j \otimes K_X^{\otimes m}) \leq \binom{n}{j} \quad \textit{for every } j \geq 0,$$

and the Hilbert polynomial $P(m) := \chi(X, K_X^{\otimes m})$ is constant, equal to $\chi(X, \mathscr{O}_X)$.

Proof The inequality $h^j(X, K_X^{\otimes m}) \leq h^0(X, \Omega_X^j \otimes K_X^{\otimes m})$ follows from the Hard Lefschetz Theorem 4 applied with $L = K_X$ and the corresponding trivial multiplier ideal sheaf. Also, for any holomorphic vector bundle E on X, we have $h^0(X, E) \leq \operatorname{rank}(E)$, otherwise, some ratios of determinants of sections would produce a nonconstant meromorphic function, and thus $a(X) > 0$, contradiction; here we take $E = \Omega_X^j \otimes K_X^{\otimes m}$ and get $\operatorname{rank} E = \binom{n}{j}$. The final claim is clear because a polynomial function $P(m)$ which remains bounded as $m \to +\infty$ is necessarily constant. □

Corollary 10 *Let X be a strongly simple Kähler threefold. Let $h^{i,j} = \dim H^{i,j}(X, \mathbb{C})$ be the Hodge numbers. We have*

$$c_1(X)^3 = c_1(X) \cdot c_2(X) = 0, \quad \chi(X, \mathscr{O}_X) = 0 \quad \textit{and} \quad q := h^{1,0} > 0.$$

Proof The intersection number $K_X^3 = -c_1(X)^3$ vanishes because it is the leading term of $P(m)$, up to the factor 3!. The Riemann-Roch formula then gives

$$P(m) = \frac{(1 - 12\,m)}{24}\, c_1(X) \cdot c_2(X).$$

The boundedness of $P(m)$ implies $\chi(X, \mathscr{O}_X) = \frac{1}{24} c_1(X) \cdot c_2(X) = 0$. Now, we write

$$0 = \chi(X, \mathscr{O}_X) = 1 - h^{1,0} + h^{2,0} - h^{3,0}.$$

By Kodaira's theorem, $h^{2,0} > 0$ since X is not projective, and $h^{3,0} \le 1$ since $a(X) = 0$. Thus $0 = 1 - h^{1,0} + h^{2,0} - h^{3,0} \ge 1 - q + 1 - 1 = 1 - q$, and $q > 0$. □

Everything is now in place for the final conclusion.

Theorem 17 *Let X be a strongly simple Kähler threefold. Then the Albanese map $\alpha : X \to \mathrm{Alb}(X)$ is a biholomorphism of 3-dimensional tori.*

Proof Since $q = h^{1,0} > 0$, the Albanese map α is non constant. By simplicity, X cannot possess any fibration with positive dimensional fibers, so we must have $\dim \alpha(X) = \dim X = 3$, and as $q = h^{1,0} = h^0(X, \Omega^1_X) \le 3$ (Proposition 8 with $j = 1, m = 0$) the Albanese map α must be surjective. The function $\det(d\alpha)$ cannot vanish, otherwise we would get a non trivial divisor, so α is étale. Therefore X is a 3-dimensional torus, as a finite étale cover of the 3-dimensional torus $\mathrm{Alb}(X)$, and α must be an isomorphism. □

In [17], the following stronger result is established as a consequence of the existence of good minimal models for Kähler threefolds:

Theorem 18 *Let X be smooth compact Kähler threefold. If X is simple, there exists a bimeromorphic morphism $X \to T/G$ where T is a torus and G a finite group acting on T.*

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Null Holomorphic Curves in $\mathbb{C}^3$ and Applications to the Conformal Calabi-Yau Problem

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1 On Null Curves in $\mathbb{C}^3$ and Minimal Surfaces in $\mathbb{R}^3$

An open connected Riemann surface M is said to be a *bordered Riemann surface* if it is the interior of a compact one dimensional complex manifold $\overline{M}$ with smooth boundary $bM \neq \emptyset$ consisting of finitely many closed Jordan curves. The closure $\overline{M} = M \cup bM$ of such M is a *compact bordered Riemann surface*. By classical results every compact bordered Riemann surface is conformally equivalent to a smoothly bounded domain in an open (or compact) Riemann surface R by a map smoothly extending to the boundary.

Let M be an open Riemann surface. A *holomorphic* immersion $F = (F_1, F_2, F_3): M \to \mathbb{C}^3$ is said to be a *null curve* if it is directed by the conical quadric subvariety

$$\mathfrak{A} = \{(z_1, z_2, z_3) \in \mathbb{C}^3 : z_1^2 + z_2^2 + z_3^2 = 0\}, \tag{1}$$

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in the sense that the derivative $F' = (F'_1, F'_2, F'_3): M \to \mathbb{C}^3$ with respect to any local holomorphic coordinate on M takes values in $\mathfrak{A} \setminus \{0\}$. If M is a bordered Riemann surface, then the same definition applies to smooth maps $\overline{M} \to \mathbb{C}^3$ which are holomorphic in M.

The real and the imaginary part of a null curve $M \to \mathbb{C}^3$ are *conformal* (angle preserving) *minimal immersions* $M \to \mathbb{R}^3$; that is, having mean curvature identically zero. Conversely, every conformal minimal immersion $M \to \mathbb{R}^3$ is locally (on any simply-connected domain in M) the real part of a null curve; this fails on non-simply connected Riemann surfaces due to the periods of the harmonic conjugate. This connection enables the use of complex analytic tools in minimal surface theory, a major topic of differential geometry since the times of Euler and Lagrange. One of the quintessential examples of this statement is the use of Runge's approximation theorem for holomorphic functions on open Riemann surfaces in the study of *the Calabi-Yau problem for surfaces*.

In 1965, Calabi [24] conjectured the nonexistence of *complete* minimal surfaces in $\mathbb{R}^3$ with bounded projection into a straight line; this would imply in particular the nonexistence of bounded complete minimal surfaces in $\mathbb{R}^3$. Recall that an immersion $F: M \to \mathbb{R}^n$, $n \in \mathbb{N}$, is said to be complete if the pullback F^*ds^2 by F of the Riemannian metric on $\mathbb{R}^n$ is a complete metric on M. Calabi's conjecture turned out to be false by the groundbreaking counterexample by Jorge and Xavier [59] in 1980 who constructed a complete conformal minimal immersion $F = (F_1, F_2, F_3): \mathbb{D} \to \mathbb{R}^3$ of the disc $\mathbb{D} = \{z = x + \imath y \in \mathbb{C}: |z| < 1\}$ with $F_3(z) = \Re(z) = x$. (Here $\imath = \sqrt{-1}$.) Their method consists of applying the classical Runge theorem on a labyrinth of compact sets in $\mathbb{D}$ in order to construct a suitable harmonic function $(F_1, F_2): \mathbb{D} \to \mathbb{R}^2$. Refinements of Jorge and Xavier's technique have given rise to a number of examples of complete minimal surfaces in $\mathbb{R}^3$ with a bounded coordinate function (see [65, 66, 83]); in particular it was recently proven by Alarcón, Fernández, and López [1–3] that every open Riemann surface carrying non-constant bounded harmonic functions is the underlying conformal structure of such a surface.

As pointed out by S.-T. Yau in his 1982 problem list [98, Problem 91], the question of whether there exist complete bounded minimal surfaces in $\mathbb{R}^3$ (which became known in the literature as *the Calabi-Yau problem*) remained open. (See also Yau's *Millenium Lecture 2000* [99].) It was Nadirashvili [74] who in 1996 answered this question in the affirmative by constructing a conformal complete bounded minimal immersion of the disc $\mathbb{D}$ into $\mathbb{R}^3$. Nadirashvili's method relies on a recursive use of Runge's approximation theorem on labyrinths of compact subsets of $\mathbb{D}$, and it has been the seed of several construction techniques, leading to a variety of examples. In particular, in 2012 Ferrer et al. [38] constructed complete properly immersed minimal surfaces with *arbitrary topology* in any given either convex or bounded and smooth domain of $\mathbb{R}^3$; see also Alarcón et al. [4] for the case of finite topology.

A question that appeared early in the history of the Calabi-Yau problem was whether there exist complete bounded minimal surfaces in $\mathbb{R}^3$ whose conjugate surfaces also exist and are bounded; that is, whether there exist complete bounded null curves in $\mathbb{C}^3$. The first such examples were provided only very recently by Alarcón and López [9] who constructed complete null curves with arbitrary topology properly immersed in any given convex domain of $\mathbb{C}^3$; this answers a question by Martín, Umehara, and Yamada [68, Problem 1]. Their method, which is different from Nadirashvili's one, relies on a Runge-Mergelyan type theorem for null curves in $\mathbb{C}^3$ [8], a new and powerful tool that gave rise to a number of constructions of both minimal surfaces in $\mathbb{R}^3$ and null curves in $\mathbb{C}^3$ (see [2, 8–10, 12]). Very recently, Ferrer, Martín, Umehara, and Yamada [39] showed that Nadirashvili's method can be adapted to null curves, giving an alternative proof of the existence of complete bounded null discs in $\mathbb{C}^3$.

In the opposite direction, Colding and Minicozzi [28] proved in 2005 that every complete *embedded* minimal surface of finite topology in $\mathbb{R}^3$ is proper in $\mathbb{R}^3$, hence unbounded. This result was extended by Meeks, Pérez, and Ros [71] to surfaces with finite genus and countably many ends. It follows that the original Calabi's conjecture is true for embedded surfaces of finite topology. Although being embedded is a strong constraint for a complete minimal surface in $\mathbb{R}^3$, it is no constraint for null curves in $\mathbb{C}^3$ as the following result shows.

Theorem 1.1 ([6, Corollary 6.2]) *There exist complete null curves with arbitrary topology properly embedded in any given convex domain of* $\mathbb{C}^3$.

This answers a question by Martín, Umehara, and Yamada [68, Problem 2].

The key to the proof is that *the general position of null curves in* $\mathbb{C}^3$ *is embedded*. In fact, Theorem 1.1 easily follows from the existence of complete properly immersed null curves in convex domains of $\mathbb{C}^3$ [9] and the following desingularization result from [6].

Theorem 1.2 ([6, Theorem 2.4 and Theorem 2.5]) *Let M be a bordered Riemann surface. Then every null immersion $\overline{M} \to \mathbb{C}^3$ can be approximated in the $\mathscr{C}^1$-topology on $\overline{M}$ by null embeddings $\overline{M} \hookrightarrow \mathbb{C}^3$. Similarly, if M is any open Riemann surface then any null immersion $M \to \mathbb{C}^3$ can be approximated uniformly on compacts by null embeddings $M \hookrightarrow \mathbb{C}^3$.*

Since complex submanifolds of complex Euclidean spaces are area minimizing [37], the Calabi-Yau problem is closely related to a question, posed by Yang [96, 97] in 1977, whether there exist complete bounded complex submanifolds of $\mathbb{C}^n$ for $n > 1$. The first result in this subject was obtained by Jones [58] who constructed holomorphic immersions $\mathbb{D} \to \mathbb{C}^2$ and embeddings $\mathbb{D} \hookrightarrow \mathbb{C}^3$ with bounded image and complete induced metric. His method is based on the BMO duality theorem. In 2009, Martín, Umehara, and Yamada [69] extended Jones' result to complete bounded complex curves in $\mathbb{C}^2$ with arbitrary finite genus. Finally, Alarcón and López [9] constructed complete complex curves with arbitrary topology properly immersed in any given convex domain of $\mathbb{C}^2$, as well as the first example of a complete bounded *embedded* complex curve in $\mathbb{C}^2$ [13].

The use of Runge's theorem in the recursive procedure of Nadirashvili's method for constructing complete bounded minimal surfaces does not enable one to control the placement in $\mathbb{R}^3$ of the entire surface at each step. Therefore one is forced to cut away some small pieces of the surface in order to keep it suitably bounded, so it is impossible to control the conformal structure on the surface when applying this technique to non-simply connected Riemann surfaces (the simply-connected ones are of course conformally equivalent to the disc $\mathbb{D}$). This phenomenon has been present in every construction of complete bounded minimal surfaces in $\mathbb{R}^3$ and null curves in $\mathbb{C}^3$ (see [9] and references therein), and also in every construction of complete bounded complex curves in $\mathbb{C}^2$ with non-trivial topology up to this point (cf. [9, 13, 69]).

This constraint has recently been overcome by the authors [5] in the case of bordered Riemann surfaces in $\mathbb{C}^n$, $n \geq 2$.

Theorem 1.3 ([5, Theorem 1]) *Every bordered Riemann surface carries a complete proper holomorphic immersion to the unit ball of $\mathbb{C}^n$, $n \geq 2$, which can be chosen an embedding for $n \geq 3$.*

The construction in the proof of Theorem 1.3 is inspired by that of Alarcón and López [9], but it requires additional complex analytic tools. The key point is to replace Runge's theorem by approximate solutions of certain *Riemann-Hilbert boundary value problems* in $\mathbb{C}^n$ (see Sect. 3 below for an exposition of this subject). This gives sufficient control of the placement of the whole curve in the space to avoid shrinking, thereby enabling one to control its conformal structure. Another important tool is the method of Forstnerič and Wold [45] for *exposing boundary points* of a complex curve. This method, together with a local version of the Mergelyan theorem, enables one to incorporate suitable arcs to a compact bordered Riemann surface in $\mathbb{C}^n$ without modifying its conformal structure.

However, the case of minimal surfaces in $\mathbb{R}^3$ and null curves in $\mathbb{C}^3$ is still much harder and requires more refined complex analytic tools. The following version of the *Riemann-Hilbert problem for null curves* has been obtained recently in [7].

Theorem 1.4 ([7, Theorem 3.4]) *Let M be a bordered Riemann surface, and let I be a compact subarc of bM which is not a connected component of bM. Choose a small annular neighborhood $A \subset \overline{M}$ of bM and a smooth retraction $\rho\colon A \to bM$. Let $F\colon \overline{M} \to \mathbb{C}^3$ be a null holomorphic immersion, let $\vartheta \in \mathfrak{A} \setminus \{0\}$ be a null vector, let $\mu\colon bM \to \mathbb{R}_+$ be a continuous function supported on I, and consider the continuous map*

$$\varkappa\colon bM \times \overline{\mathbb{D}} \to \mathbb{C}^3, \qquad \varkappa(x,\xi) = F(x) + \mu(x)\,\xi\,\vartheta.$$

Then for any number $\epsilon > 0$ there exist an arbitrarily small open neighborhood Ω of I in A and a null holomorphic immersion $G\colon \overline{M} \to \mathbb{C}^3$ satisfying the following properties:

(a) $\mathrm{dist}(G(x), \varkappa(x, b\mathbb{D})) < \epsilon$ *for all $x \in bM$.*
(b) $\mathrm{dist}(G(x), \varkappa(\rho(x), \overline{\mathbb{D}})) < \epsilon$ *for all $x \in \Omega$.*
(c) *G is ϵ-close to F in the $\mathscr{C}^1$ topology on $\overline{M} \setminus \Omega$.*

The authors gave a direct proof by explicit calculation in the special case when M is the disc $\mathbb{D}$, using the so called spinor representation of the null quadric (1). This can be used locally on small discs abutting the boundary of M. The proof of the general case depends on the technique of *gluing holomorphic sprays* (which amounts to a nonlinear version of the $\overline{\partial}$-problem in complex analysis, see [42, Chapter 5]) applied to the derivatives of null curves. To use the method of gluing sprays in the present setting, one must control the periods of some maps in the amalgamated spray in order to get well defined null curves by integration; in addition, delicate estimates are needed to ensure that the resulting null curves have the desired properties. A more precise outline of the proof is given in Sect. 4.

The following result, whose proof uses the techniques described above, is the authors' main contribution to the Calabi-Yau problem.

Theorem 1.5 ([7, Theorem 1.1]) *Every bordered Riemann surface carries a complete proper null embedding into the unit ball of* $\mathbb{C}^3$.

If $F\colon M \to \mathbb{C}^3$ is a null curve, then the Riemannian metric F^*ds^2 induced by the Euclidean metric of $\mathbb{C}^3$ via F is twice the one induced by the Euclidean metric of $\mathbb{R}^3$ via the real part $\Re F$ (cf. [75]). Indeed, write $F = (F^1, F^2, F^3)\colon M \to \mathbb{C}^3$ and let $\zeta = x + \imath y$ be a local holomorphic coordinate on M. Then

$$0 = \sum_{j=1}^{3} (F^j_\zeta)^2 = \sum_{j=1}^{3} (F^j_x)^2 = \sum_{j=1}^{3} \big((\Re F^j)_x + \imath(\Im F^j)_x\big)^2$$
$$= \sum_{j=1}^{3} \big(((\Re F^j)_x)^2 - ((\Im F^j)_x)^2\big) + 2\imath \sum_{j=1}^{3} (\Re F^j)_x(\Im F^j)_x.$$

Since $(\Im F)_y = -(\Re F)_x$ by the Cauchy-Riemann equations, the above is equivalent to $|(\Re F)_x| = |(\Im F)_y|$ and $\langle(\Re F)_x, (\Im F)_y\rangle = 0$. It follows that the minimal immersion $\Re F\colon M \to \mathbb{R}^3$ is conformal, harmonic, and

$$F^*ds^2 = |F_x|^2(dx^2 + dy^2) = 2|(\Re F)_x|^2(dx^2 + dy^2) = 2(\Re F)^*ds^2.$$

In particular, the real part of a complete null curve in $\mathbb{C}^3$ is a complete conformal minimal immersion in $\mathbb{R}^3$. In view of Theorem 1.5, we obtain the following result on the so-called *conformal Calabi-Yau problem* for surfaces.

Theorem 1.6 ([7, Corollary 1.2]) *Every bordered Riemann surface M carries a conformal complete minimal immersion $M \to \mathbb{R}^3$ with bounded image.*

We emphasize that, in Theorems 1.5 and 1.6, *the conformal structure on the source Riemann surface is not changed*.

Notice that we do not control the asymptotic behavior of the surfaces in Theorem 1.6; in particular we do not know whether there exist conformal complete *proper* minimal immersions from any bordered Riemann surface into a ball of $\mathbb{R}^3$.

At this point we wish to draw a certain analogy with the old problem whether every open Riemann surface admits a proper holomorphic embedding into $\mathbb{C}^2$ (see Bell and Narasimhan [17, Conjecture 3.7, p. 20]). It is classical that every open Riemann surface properly holomorphically embeds in $\mathbb{C}^3$ and immerses in $\mathbb{C}^2$ [42, §8.2]. By introducing the technique of exposing boundary points alluded to above, combined with the Andersén-Lempert theory of holomorphic automorphisms of $\mathbb{C}^n$ for $n > 1$, Forstnerič and Wold proved in 2009 that, if a compact bordered Riemann surface $\overline{M}$ admits a (nonproper) holomorphic embedding in $\mathbb{C}^2$ then its interior M admits a *proper* holomorphic embedding in $\mathbb{C}^2$ [45]. Further applications of their technique can be found in [67] and [46]. For example, *every circular domain in the Riemann sphere admits a proper holomorphic embedding in* $\mathbb{C}^2$ [46]. (The case of finitely connected plane domains was established by Globevnik and Stensønes in 1995 [52].) The main novelty in [45, 46] is that, unlike in the earlier constructions, no cutting of the surface is needed and hence the conformal structure is preserved. It is considerably easier to show that every open oriented real surface M admits a complex structure J such that the open Riemann surface (M, J) admits a proper holomorphic embedding into $\mathbb{C}^2$ (Alarcón and López [11]; for the case of finite topology see [27]).

Another good example of how Runge's theorem has been exploited in minimal surface theory is the construction of proper minimal surfaces in $\mathbb{R}^3$ with hyperbolic conformal structure. (An open Riemann surface is said to be *hyperbolic* if it carries negative non-constant subharmonic functions; otherwise it is called *parabolic*.) An old conjecture of Sullivan [70] asserted that every properly immersed minimal surface in $\mathbb{R}^3$ with finite topology must have parabolic conformal structure. The first counterexample was given by Morales in [72] who in 2003 constructed a proper *conformal* minimal immersion $\mathbb{D} \to \mathbb{R}^3$.

In the same line, Schoen and Yau [84] asked in 1985 whether a minimal surface in $\mathbb{R}^3$ properly projecting into a plane must be parabolic. The question was answered by Alarcón and López [8] who showed that in fact every open Riemann surface M carries a conformal minimal immersion $X = (X_1, X_2, X_3)\colon M \to \mathbb{R}^3$ such that $(X_1, X_2)\colon M \to \mathbb{R}^2$ is a proper map, and a null curve $F = (F_1, F_2, F_3)\colon M \to \mathbb{C}^3$ such that $(F_1, F_2)\colon M \to \mathbb{C}^2$ is proper. (See also [12].) Taking into account Theorem 1.2 we obtained the following extension of the previous results.

Theorem 1.7 ([6, Theorem 8.1]) *Every open Riemann surface M carries a proper null* embedding $F = (F_1, F_2, F_3)\colon M \hookrightarrow \mathbb{C}^3$ *such that* $(F_1, F_2)\colon M \to \mathbb{C}^2$ *is a proper map.*

The constructions in [8] and [6] involved particular versions of Runge's theorem for minimal surfaces in $\mathbb{R}^3$ and null curves in $\mathbb{C}^3$ that do not give any control on the third coordinate. However, the newly developed complex analytic methods involved in the proof of Theorem 1.5 above enable us to overcome this constraint in the case of bordered Riemann surfaces as null curves in $\mathbb{C}^3$. The following is our main result in this line.

Theorem 1.8 ([7, Theorem 1.4]) *Every bordered Riemann surface M carries a null holomorphic embedding $F = (F_1, F_2, F_3): M \hookrightarrow \mathbb{C}^3$ such that $(F_1, F_2): M \to \mathbb{C}^2$ is a proper map and the function $F_3: M \to \mathbb{C}$ is bounded on M.*

Joining together the methods in the proof of the above result and the Mergelyan theorem for null curves [8] (see also [6]), we also get the following result.

Theorem 1.9 ([7, Theorem 1.8]) *Every orientable noncompact smooth real surface M without boundary admits a complex structure J such that the Riemann surface (M, J) carries a proper holomorphic null embedding $(F_1, F_2, F_3): (M, J) \to \mathbb{C}^3$ such that F_3 is a bounded function on M.*

Recall that the only properly immersed minimal surfaces in $\mathbb{R}^3$ with a bounded coordinate function are planes by the Half-Space theorem of Hoffman and Meeks [57]. Therefore, if $F = (F_1, F_2, F_3): M \hookrightarrow \mathbb{C}^3$ is a proper null curve as those in Theorem 1.8, then $\Re(e^{\imath t} F): M \to \mathbb{R}^3$ is a conformal complete *non-proper* minimal immersion for all $t \in \mathbb{R}$.

2 On Null Curves in $SL_2(\mathbb{C})$ and Bryant Surfaces in $\mathbb{H}^3$

A *holomorphic* immersion $F: M \to SL_2(\mathbb{C})$ from an open Riemann surface into the special linear group

$$SL_2(\mathbb{C}) = \left\{ z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \in \mathbb{C}^4 : \det z = z_{11}z_{22} - z_{12}z_{21} = 1 \right\}$$

is said to be a *null curve* if it is directed by the quadric variety

$$\mathfrak{E} = \left\{ z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} : \det z = z_{11}z_{22} - z_{12}z_{21} = 0 \right\} \subset \mathbb{C}^4, \tag{2}$$

meaning that the derivative $F': M \to \mathbb{C}^4$ with respect to any local holomorphic coordinate on M belongs to $\mathfrak{E} \setminus \{0\}$. If M is a bordered Riemann surface, then the same definition applies for smooth maps $\overline{M} \to SL_2(\mathbb{C})$ being holomorphic in M.

In 1987, Bryant [22] discovered that, if $F: M \to SL_2(\mathbb{C})$ is a null curve, then $F \cdot \overline{F}^T$ takes values in the hyperbolic space $\mathbb{H}^3 = SL_2(\mathbb{C})/SU(2)$ and is a *conformal* immersion with constant mean curvature one; conversely, every simply-connected *Bryant surface* (i.e., with constant mean curvature one in $\mathbb{H}^3$) is the projection of a null curve in $SL_2(\mathbb{C})$. As is the case of minimal surfaces in $\mathbb{R}^3$, the above connection enables the use of complex analytic tools in Bryant surface theory which made this subject a fashionable research topic in the last decade (see e.g. [29, 82, 93] for the background). Moreover, the *Lawson correspondence* [63] implies that every simply connected Bryant surface is isometric to a minimal surface in $\mathbb{R}^3$ and vice versa;

hence problems on minimal surface theory are automatically natural also for Bryant surfaces.

Some of the results described in the previous section hold not only for null curves in $\mathbb{C}^3$, but also for immersions with derivative in an arbitrary conical subvariety A of $\mathbb{C}^n$ ($n \geq 3$) which is smooth away from the origin; i.e., *A-immersions*.

Theorem 2.1 ([6, Theorem 2.5 and Corollary 2.7]) *Let M be an open Riemann surface, and let A be a closed conical subvariety of $\mathbb{C}^n$, $n \geq 3$, which is not contained in any hyperplane and such that $A \setminus \{0\}$ is a smooth Oka manifold. Then:*

- *(Desingularization theorem) Every A-immersion $M \to \mathbb{C}^n$ can be approximated uniformly on compacts by A-embeddings $M \to \mathbb{C}^n$.*
- *(Runge theorem) Every A-immersion $U \to \mathbb{C}^n$ on an open neighborhood U of a compact Runge set $K \subset M$ can be approximated uniformly on K by A-immersions $M \to \mathbb{C}^n$.*

Recall that a complex manifold Y is said to be an *Oka manifold* if every holomorphic map from an open neighborhood of a compact convex set $K \subset \mathbb{C}^N$ to Y can be approximated, uniformly on K, by entire maps $\mathbb{C}^N \to Y$; see [42] for a reference on Oka theory.

Although the variety $\mathfrak{E}$ (2) controlling null curves in $SL_2(\mathbb{C})$ meets the requirements of Theorem 2.1, the results do not apply directly since the $\mathfrak{E}$-immersions $M \to \mathbb{C}^4$ furnished by the theorem need not lie in $SL_2(\mathbb{C})$. In order to force the image to lie in $SL_2(\mathbb{C})$, one must add another equation expressing the condition that the tangent vector to the curve is also tangent to $SL_2(\mathbb{C})$ (as a submanifold of $\mathbb{C}^4$). Unfortunately the resulting system of equations is no longer autonomous (the second equation depends on the point in space), and hence the methods of [6] do not apply.

However, Martín, Umehara, and Yamada [68] discovered in 2009 that the biholomorphic map $\mathcal{T}\colon \mathbb{C}^3 \setminus \{z_3 = 0\} \to SL_2(\mathbb{C}) \setminus \{z_{11} = 0\}$, given by

$$\mathcal{T}(z_1, z_2, z_3) = \frac{1}{z_3}\begin{pmatrix} 1 & z_1 + \imath z_2 \\ z_1 - \imath z_2 & z_1^2 + z_2^2 + z_3^2 \end{pmatrix}, \tag{3}$$

carries null curves into null curves. This transformation allows us to obtain a succulent list of corollaries to the results in the previous section.

In view of Theorem 1.2 we get the following result concerning the general position of null curves in $SL_2(\mathbb{C}) \setminus \{z_{11} = 0\}$.

Theorem 2.2 ([6, Corollary 2.8]) *Let M be a bordered Riemann surface. Every immersed null curve $\overline{M} \to SL_2(\mathbb{C}) \setminus \{z_{11} = 0\}$ can be approximated in the $\mathscr{C}^1$ topology on $\overline{M}$ by embedded null curves $\overline{M} \to SL_2(\mathbb{C}) \setminus \{z_{11} = 0\}$.*

Observe that for any constant $c > 0$ the biholomorphism $\mathcal{T}$ (3) maps complete bounded null curves in $\mathbb{C}^3 \setminus \{|z_3| > c\}$ into complete bounded null curves in $SL_2(\mathbb{C})$, which in turn project to complete bounded Bryant surfaces in $\mathbb{H}^3$ [68].

From Theorems 1.1 and 1.5, we get the following results regarding Calabi-Yau type questions.

Theorem 2.3 ([6], [7, Corollary 1.9])

- *There exist complete bounded embedded null curves in* $SL_2(\mathbb{C})$ *and immersed Bryant surfaces in* $\mathbb{H}^3$ *with arbitrary topology.*
- *Every bordered Riemann surface* M *admits a complete null holomorphic embedding* $M \to SL_2(\mathbb{C})$ *with bounded image, and it is conformally equivalent to a complete bounded immersed Bryant surface in* $\mathbb{H}^3$.

The latter part of the former item in the above theorem was already proven in [8] where also complete bounded *immersed* null curves in $SL_2(\mathbb{C})$ with arbitrary topology were given. Complete bounded immersed simply connected null holomorphic curves in $SL_2(\mathbb{C})$, hence complete bounded simply-connected Bryant surfaces in $\mathbb{H}^3$, were provided in [39, 68].

Finally, observe that applying $\mathcal{T}$ to a proper null curve $F = (F_1, F_2, F_3): M \to \mathbb{C}^3$ such that $0 < c_1 < |F_3| < c_2$ on M one gets a proper null curve in $SL_2(\mathbb{C})$, which in turn projects to a proper Bryant surface in $\mathbb{H}^3$. Therefore, Theorems 1.8 and 1.9 provide the following examples of proper null curves in $SL_2(\mathbb{C})$ and Bryant surfaces in $\mathbb{H}^3$.

Theorem 2.4 ([7, Corollaries 1.5 and 1.6 and Theorem 1.8])

- *There exist properly embedded null curves in* $SL_2(\mathbb{C})$ *and properly immersed Bryant surfaces in* $\mathbb{H}^3$ *with arbitrary topology.*
- *Every bordered Riemann surface* M *admits a proper null holomorphic embedding* $M \to SL_2(\mathbb{C})$*, and it is conformally equivalent to a properly immersed Bryant surface in* $\mathbb{H}^3$.

Connecting to Sullivan's conjecture for minimal surfaces [70], the ones in the latter item of Theorem 2.4 are the first examples of proper null curves in $SL_2(\mathbb{C})$ and proper Bryant surfaces in $\mathbb{H}^3$ with finite topology and hyperbolic conformal structure.

3 The Riemann-Hilbert Problem and Proper Holomorphic Maps of Bordered Riemann Surfaces

In this and the following section we explain how a certain version of the classical *Riemann-Hilbert boundary value problem* is used in the construction of holomorphic curves satisfying various additional properties (for example, proper and/or complete and bounded).

The linear *Riemann boundary value problem* on the disc $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ asks for holomorphic functions $f(z) = u(z) + \imath v(z)$ on $\mathbb{D}$, continuous on the closed disc $\overline{\mathbb{D}}$ and satisfying the following condition on the circle $\mathbb{T} = b\,\mathbb{D} = \{|z| = 1\}$:

$$a(z)\, u(z) - b(z)\, v(z) = c(z), \qquad z \in \mathbb{T}, \tag{4}$$

where a, b, and c are given real-valued continuous functions on $\mathbb{T}$. This classical problem considered by Riemann in his dissertation [81], and its extension to non-simply connected domains and bordered Riemann surfaces, is one of the main boundary value problems of analytic function theory. Geometrically speaking, (4) demands that the boundary value of f at any point $z \in \mathbb{T}$ lies in a certain affine real line $l_z \subset \mathbb{C}$ depending on z.

Writing $f(z) = f_+(z)$ and introducing the conjugate function on $\mathbb{C} \setminus \mathbb{D}$ by

$$f_-(z) = \overline{f_+\,(\bar{z}^{-1})}, \qquad |z| \geq 1,$$

we have $f_-(z) = \overline{f_+(z)}$ on the circle $\mathbb{T}$, and (4) can be written as

$$\alpha(z)f_+(z) + \beta(z)f_-(z) = c(z), \qquad |z| = 1, \tag{5}$$

where $\alpha(z) = (a(z) + \imath b(z))/2$ and $\beta(z) = (a(z) - \imath b(z))/2$. Hilbert's generalization [55] asks for functions f_+ and f_-, holomorphic on $\mathbb{D}$ and $\mathbb{C} \setminus \overline{\mathbb{D}}$, respectively, continuous up to the circle and satisfying (5) where α, β, and c are arbitrary complex-valued functions on $\mathbb{T}$. One may consider the same problem for vector-valued or matrix-valued holomorphic functions. For example, the *Birkhoff factorization problem* [19] amounts to finding solutions of the equation $f_+(z) = G(z)f_-(z)$ on $|z| = 1$, where the matrix-valued functions $f_\pm$ are holomorphic inside and outside of the disc $\mathbb{D}$, respectively.

Riemann's boundary value problem was motivated by the problem of finding a linear differential equation of Fuchsian type with given singular points and a given monodromy group; this is *Hilbert's 21st problem* on the famous list of 23 problems from his 1900 ICM lecture. In 1905, Hilbert [55] obtained some progress on the general Riemann-Hilbert problem (5) by reducing it to an integral equation. The homogeneous vector-valued Riemann-Hilbert problem was solved in 1908 by Plemelj [76, 77] by using his *jump formula* for Cauchy integrals. This also gave a solution of Hilbert's 21st problem, except for a gap in some cases (see Bolibrukh [21] and Kostov [61]). In 1909 Birkhoff [19] obtained a factorization theorem on matrix-valued functions which implies that any holomorphic vector bundle over the Riemann sphere is a direct sum of holomorphic line bundles. The non-homogeneous Riemann-Hilbert problem was considered in 1934 by Privalov [80]. The Riemann-Hilbert problem has a rich variety of applications in many fields; see in particular the monographs by Gakhov [47], Muskhelishvili [73], Vekua [94], and Wegert [95].

We now return to Riemann's boundary value problem (4), but replacing the affine lines by closed Jordan curves $l_z \subset \mathbb{C}$ containing the origin in the bounded component of $\mathbb{C} \setminus l_z$ for every $z \in \mathbb{T}$. This *nonlinear Riemann-Hilbert problem* was solved by Forstnerič in 1988 [41] who showed that the graphs of solutions fill the polynomial hull of the torus $T = \{(z, w) \in \mathbb{C}^2 : z \in \mathbb{T},\ w \in l_z\}$. (See also the papers [14, 40, 86].) An extension to more general sets was given by Slodkowski [85]. Berndtsson and Ransford [18] used these ideas to give another proof of Carleson's Corona Theorem for the disc. Černe [26] found some solutions of the nonlinear Riemann-Hilbert problem on bordered Riemann surfaces.

A closely related direction, which offers a more geometric point of view on Riemann's boundary value problem, is the existence and perturbation theory of analytic discs (and of bordered Riemann surfaces) with boundaries attached to a certain real submanifold in a complex manifold. This point of view was pioneered by Bishop [20] who studied local envelopes of holomorphy of real surfaces in $\mathbb{C}^2$ and, more generally, of real submanifolds near complex singularities, by introducing and solving the so-called *Bishop equation*. His work was continued and developed in several directions by Kenig and Webster [60], Lempert [64], Bedford and Gaveau [15], Forstnerič [40], Trépreau [89], Tumanov [90–92], Bedford and Klingenberg [16], Globevnik [50], Černe [25] and many others. This subject also received considerable attention on almost complex manifolds starting with Gromov's seminal work in 1985 [54] on the use of pseudoholomorphic curves in symplectic geometry. For these developments see e.g. the survey by Eliashberg [36] on the technique of filling by holomorphic discs, the collection by Audin and Lafontaine [62], and the papers [56, 87, 88], among others. This line of work also plays an important role in modern topology, in particular in Floer homology. We are unable to present a comprehensive survey of this large body of results in a short space, so we apologize to the authors whose contributions are not mentioned in the above summary.

We wish to emphasize that these are very difficult analytic problems whose exact solutions are typically extremely difficult or impossible to find. However, in many applications one only needs an *approximate solution*, and this is usually a much easier problem. The following version of the approximate Riemann-Hilbert problem appears in many different constructions: of proper holomorphic maps, of bounded complete holomorphic curves, in formulas for extremal functions in pluripotential theory, etc.

Let X be a complex manifold (or a complex space). We are given a holomorphic map $f\colon \overline{\mathbb{D}} \to X$, also called an *analytic disc* in X, and for each point $z \in \mathbb{T}$ a holomorphic map $g_z\colon \overline{\mathbb{D}} \to X$ such that $g_z(0) = f(z)$ and the discs g_z depend continuously on $z \in \mathbb{T}$. Set $T_z = g_z(\mathbb{T}) \subset X$ and $S_z = g_z(\overline{\mathbb{D}}) \subset X$ for $z \in \mathbb{T}$. Fix a distance function dist on X. Given numbers $0 < r < 1$ and $\epsilon > 0$, the *approximate Riemann-Hilbert problem* asks for a holomorphic map $F\colon \overline{\mathbb{D}} \to X$ satisfying the following properties for some $r' \in [r, 1)$:

(a) $\mathrm{dist}(F(z), T_z) < \epsilon$ for $z \in \mathbb{T}$,
(b) $\mathrm{dist}(F(\rho z), S_z) < \epsilon$ for $z \in \mathbb{T}$ and $r' \le \rho \le 1$, and
(c) $\mathrm{dist}(F(z), f(z)) < \epsilon$ for $|z| \le r'$.

These conditions can be adapted to any bordered Riemann surface $\overline{M}$ instead of the disc $\overline{\mathbb{D}}$ as the domain of the maps f and F. (Compare with the statement of Theorem 1.4. The domain of the maps g_z is always the closed disc $\overline{\mathbb{D}}$.)

When $X = \mathbb{C}^n$, this approximate Riemann-Hilbert problem is easily solved as follows (see [44] or [34] for the details). Consider the map

$$\mathbb{T} \times \overline{\mathbb{D}} \ni (z, w) \mapsto g_z(w) - f(z) \in \mathbb{C}^n$$

which is continuous in z and holomorphic in w. Note that this vanishes at $w = 0$ for any $z \in \mathbb{T}$ since $g_z(0) = f(z)$. We can approximate it arbitrarily closely by a rational map

$$G(z, w) = z^{-m} \sum_{j=1}^{N} A_j(z) w^j \in \mathbb{C}^n,$$

where the A_j's are $\mathbb{C}^n$-valued holomorphic polynomials and $m \in \mathbb{N}$. Pick $k \in \mathbb{N}$ and set

$$F(z) = f(z) + G(z, z^k) = f(z) + z^{k-m} \sum_{j=1}^{N} A_j(z) z^{k(j-1)}, \quad z \in \overline{\mathbb{D}}. \tag{6}$$

The pole at $z = 0$ cancels if $k > m$, and one easily verifies that F satisfies the properties (a)–(c) if the integer k is chosen big enough.

It is not clear how to solve this problem in an arbitrary complex manifold X and with the disc $\mathbb{D}$ (as the domain of f) replaced by a bordered Riemann surface M. However, in most applications (in particular, in those related to the present survey) it suffices to solve the problem on a small disc $\overline{D} \subset \overline{M}$ which intersects the boundary bM in a compact arc $I \subset M$ around a given point $p \in bM$. Furthermore, replacing the disc $g_p \colon \overline{\mathbb{D}} \to X$ by its graph in $\overline{\mathbb{D}} \times X$ which has an open Stein neighborhood in $\mathbb{C} \times X$, we can reduce the local approximation problem over $\overline{D}$ to the standard case of discs in $\mathbb{C}^n$.

To enable the gluing of a local solution (on $\overline{D}$) with the given map f (to get a new map on $\overline{M}$) we actually solve the following modified Riemann-Hilbert problem. Pick a pair of smaller arcs $I_0, I_1 \subset bM$ such that $p \in I_0 \subset I_1 \subset I$ and a cut-off function $\chi \colon bM \to [0, 1]$ such that $\chi = 1$ on I_0 and $\chi = 0$ on $bM \setminus I_1$. Set $\tilde{g}_z(w) = g_z(\chi(z)\, w)$ for $z \in bM$ and $w \in \overline{\mathbb{D}}$. Note that $\tilde{g}_z$ agrees with g_z for $z \in I_1$ and is the constant disc $w \to f(z)$ for any point $z \in bM \setminus I_1$. We define $\tilde{g}_z$ as the constant disc $f(z)$ for points $z \in bD \setminus I_1$. Let $\tilde{F} : \overline{D} \to X$ be an approximate solution of the Riemann-Hilbert problem with the data $f|_{\overline{D}}$ and $\tilde{g}_z$, $z \in bD$. By choosing the integer k in (6) big enough, $\tilde{F}$ satisfies condition (a) for $z \in I_0$, it satisfies condition (b) for $z \in bD$, and it is uniformly close to f on $\overline{D} \setminus U$ where $U \subset \overline{M}$ is any given neighborhood of the arc I_1. In particular, we can write $\overline{M} = A \cup B$ where $A, B \subset \overline{M}$ are closed smoothly bounded domains such that A is the complement of a small neighborhood of the arc I_1, $B \subset \overline{D}$ contains a small neighborhood of I_1, we have the separation property $\overline{A \setminus B} \cap \overline{B \setminus A} = \emptyset$ (any such pair (A, B) is called a *Cartan pair*), and $\tilde{F}$ is uniformly close to f on the domain $C = A \cap B$.

At this point we wish to glue f and $\tilde{F}$. If $X = \mathbb{C}^n$, this is a *Cousin-I problem with bounds*: the difference $c = \tilde{F} - f$ is holomorphic and small on $C = A \cap B$, and by solving the $\overline{\partial}$-equation with bounds on $\overline{M}$ it can be split as the difference $c = b - a$ where a, b are holomorphic and small on A and B, respectively. Then $\tilde{F} - b = f - a$ on

$C = A \cap B$ and hence the two sides amalgamate into a holomorphic map $F: \overline{M} \to X$ satisfying the appropriate analogues of the conditions (a)–(c).

This simple method does not work in the absence of a linear structure on the manifold X. However, we can still reduce the problem to the linear case by the method of *gluing holomorphic sprays*. We give an outline and refer for the details (in this precise setting) to [34]. For the general method of gluing sprays see [42, Chapter 5].

We begin by embedding f as the core map $f = f_0$ in a family of holomorphic maps $f_t: \overline{M} \to X$, depending holomorphically on a parameter $t \in U \subset \mathbb{C}^N$ in a neighborhood of $0 \in \mathbb{C}^N$ for some $N \in \mathbb{N}$, such that the partial differential

$$\left.\frac{\partial f_t(z)}{\partial t}\right|_{t=0} : T_0\mathbb{C}^N \cong \mathbb{C}^N \to T_{f_0(z)}X$$

is surjective for every point $z \in \overline{M}$. Such a family $\{f_t\}$ is called a *dominating (holomorphic) spray of maps*. Next we shrink U around $0 \in \mathbb{C}^N$ and solve the Riemann-Hilbert problem over $\overline{D}$ to get a holomorphic family $\tilde{F}_t: \overline{D} \to X$ $(t \in U)$ approximating f_t on $C = A \cap B$. If the approximation is close enough, there exists a holomorphic map $C \times U \ni (x, t) \mapsto \gamma(x, t) \in \mathbb{C}^N$ close to the map $(x, t) \mapsto t$ such that

$$f_t(x) = \tilde{F}_{\gamma(x,t)}(x), \quad x \in C,\ t \in U.$$

(The set U shrinks again around 0.) Now the difficult part is to split γ in the form

$$\gamma(x, \alpha(x, t)) = \beta(x, t), \quad x \in C,\ t \in W,$$

where $W \subset U$ is a neighborhood of $0 \in \mathbb{C}^N$ and $\alpha: A \times W \to \mathbb{C}^N$, $\beta: B \times W \to \mathbb{C}^N$ are holomorphic maps close to the map $(x, t) \mapsto t$. This is achieved by solving the Cousin-I problem with bounds and using the implicit function theorem in Banach spaces. Then

$$f_{\alpha(t,x)} = \tilde{F}_{\beta(x,t)}(x), \quad x \in C,\ t \in W,$$

so the two sides define a spray of maps $\overline{M} \to X$. By taking $t = 0$ we get a map $F: \overline{M} \to X$ satisfying the desired properties provided that the approximations were sufficiently close.

Having explained the problem and the method of solving it, we now give a brief survey of applications of this technique to the construction of proper holomorphic maps. The use of this method in the construction of complete bounded holomorphic immersions of bordered Riemann surfaces is discussed in the following section.

Suppose that M is a bordered Riemann surface and $f: \overline{M} \to \mathbb{C}^n$ is a holomorphic map where $n > 1$. Assume that $0 \notin f(bM)$, a condition which holds for a generic f. In order to push the boundary $f(bM)$ further towards infinity, we choose for every

$z \in bM$ a unit vector $V(z) \in \mathbb{C}^n$ orthogonal to $f(z)$ and consider the linear disc $g_z\colon \overline{\mathbb{D}} \to \mathbb{C}^n$ defined by $g_z(w) = f(z) + wV(z)$. Furthermore, to localize the problem as explained above, we introduce a cut-off function $\chi\colon bM \to [0, 1]$ with support on a small arc $I \subset bM$ and consider instead the maps

$$g_z(w) = f(z) + w\delta\chi(z)V(z), \quad z \in bM,\ w \in \overline{\mathbb{D}};$$

here $\delta > 0$ is a constant. If the arc I is short enough (so that the image $f(z)$ for $z \in I$ does not vary very much), we can use a fixed vector V orthogonal to the point $f(p)$ for some $p \in I$. A solution $F\colon \overline{M} \to \mathbb{C}^n$ of the Riemann-Hilbert problem with this data then approximates f on most of $\overline{M}$, but on the arc $I_0 \subset I$ where $\chi = 1$ we have $F(z) \approx f(z) + \delta V(z)$, so $|F(z)| \geq |f(z)| + c\delta^2$ for some constant $c \in (0, 1)$ close to 1. We have thus increased the distance from the origin by a fixed amount on the arc $I_0 \subset bM$. By systematically repeating this construction on finitely many arcs which cover bM we increase the distance by a fixed amount on all of bM, while at the same time approximating the map as closely as desired on a given compact subset of M. We can perform this operation inductively so that the resulting sequence $F_k\colon \overline{M} \to \mathbb{C}^n$ converges uniformly on compacta in M to a proper holomorphic map $F = \lim_{k\to\infty} F_k\colon M \to \mathbb{C}^n$.

This method easily adapts to the case when $\mathbb{C}^n$ is replaced by an arbitrary complex manifold X of dimension $n > 1$ which admits a smooth exhaustion function $\rho\colon X \to \mathbb{R}$ whose Levi form (which equals the complex Hessian in any system of local holomorphic coordinates on X) has at least two positive eigenvalues at every point of X. (Such a manifold is said to be $(n-1)$-complete; it is $(n-1)$-convex if the condition holds outside some compact subset of X. See Grauert [53] for the theory of q-convexity.) In this case one uses small holomorphic discs g_z in X ($z \in bM$) lying in the zero locus of the Levi polynomial of the function ρ at the point $f(z) \in X$ in the direction of a positive eigenvalue. Along this zero set the quadratic holomorphic term in the Taylor expansion of ρ at $f(z)$ vanishes, so ρ equals the Levi form plus terms of higher order, and therefore it increases quadratically along the image of g_z. (Taking the exhaustion function $\rho = |z_1|^2 + \ldots + |z_n|^2$ on $\mathbb{C}^n$ gives linear discs orthogonal to the given point of $\mathbb{C}^n$.)

For applications of this method to the construction of proper holomorphic mappings from open Riemann surfaces see the papers [31–33, 43, 44, 48, 49, 51], listed chronologically. The paper [33] contains the most general results in this direction and also includes a survey of the previous work. This list does not include papers on *embedding* Riemann surfaces in $\mathbb{C}^2$ where different techniques are used; see e.g. [45, 46, 52, 67]. Some work has also been done on almost complex Stein manifolds of real dimension 4 [30].

A similar technique is employed in the *Poletsky theory of discs* to obtain formulas expressing the envelopes of various disc functionals as pointwise minima over a family of analytic discs through a given point. This gives fairly explicit formulas for several extremal functions in pluripotential theory. The initial point was the remarkable discovery of Poletsky [78, 79] and Bu and Schachermayer [23] of the formula for computing the biggest plurisubharmonic minorant of an upper semicontinuous function as the envelope of the Poisson functional. This also gives Poletsky's characterization of polynomially convex hulls by sequences of analytic discs. For recent developments on this subject see the papers [34, 35] and the references therein.

4 The Riemann-Hilbert Problem for Null Holomorphic Curves

Approximate solutions of Riemann-Hilbert boundary value problems, described in the previous section, have recently been used by the authors [5] in the construction of proper *complete* holomorphic immersions of any bordered Riemann surface M into the ball of $\mathbb{C}^2$ (see Theorem 1.3 above). Furthermore, in [7] the Riemann-Hilbert technique was applied for the first time to the construction of complete bounded null curves (see Theorem 1.5) and of proper null curves in $\mathbb{C}^3$ with a bounded coordinate function (Theorem 1.9). We now describe the main ideas behind these developments.

Let $f\colon \overline{M} \to \mathbb{C}^n$ be a holomorphic map such that $0 \notin f(bM)$. Fix a point $p \in bM$. By pushing in the direction orthogonal to $f(p) \in \mathbb{C}^n$ for the amount $\delta > 0$ (using the Riemann-Hilbert method) we increase the length of curves in M ending near p by approximately δ, while the outer radius only increases by the order of δ^2. Performing this construction recursively with a sequence $\delta_k > 0$ such that $\sum_k \delta_k = \infty$ while $\sum_k \delta_k^2 < \infty$ one therefore expects to get a bounded complete immersion (proper in a ball if one controls the procedure sufficiently carefully). However, there is a difficulty in controlling the distance estimate for divergent curves in M on long boundary segments of M; undesired shortcuts may appear as is seen in the *sliding curtain model*. (Imagine that a curtain is held fixed at the lower end and is pulled horizontally at the upper end; this stretches each thread of the curtain, but the vertical distance between the edges remains the same.)

To eliminate this problem, the authors in [5] adjusted to this purpose the method of *exposing boundary points*, originally developed by Forstnerič and Wold [45] in the construction of proper holomorphic embeddings of bordered Riemann surfaces into $\mathbb{C}^2$. One divides each of the boundary curves of M into finitely many adjacent arcs $I_1, \ldots, I_m$ which are short enough so that the distance estimates for divergent curves in M terminating on any of these arcs (and approaching the arc sufficiently 'radially') can be controlled. Let $p_1, \ldots, p_m$ be the endpoints of this collection of arcs. Fix a small number $\delta > 0$. In order to eliminate any shortcuts (after the deformation) which could be formed by curves in M wandering almost horizontally in the vicinity of bM, we first attach to $f(\overline{M})$ at the point $f(p_k)$ an embedded arc $\lambda_k \subset \mathbb{C}^n$ of length $> \delta$ which stays very close to $f(p_k)$. It is now possible to deform the image $f(M)$ so that the deformation is arbitrarily small away from the points p_k, while at $f(p_k)$ the image of $\overline{M}$ is stretched within a thin tube around λ_k so that $f(p_k)$ goes to the other endpoint of λ_k. The effect of this deformation is that curves in M which come sufficiently close to one of the points p_k get elongated by at least δ (this happens in particular to curves ending on bM near p_k), while the outer radius of the image almost does not increase. Now we complete the picture by applying the Riemann-Hilbert method on each of the arcs I_k without destroying the effect of the first step. The cumulative effect of both deformations is that the length of any divergent curve in M increases by at least $\delta > 0$ while the outer radius only increases by $O(\delta^2)$. The proof is finished by a recursive procedure. The details are considerable (cf. [5]).

A similar construction can be done for null curves, except that the estimates become even more subtle (cf. [7, §3]). We now give a brief outline of this method.

Sketch of proof of Theorem 1.4 The special case $M = \mathbb{D}$ is done by an explicit calculation (cf. [7, Lemma 3.1]), using the so called *spinor representation* of the null quadric $\mathfrak{A}$:

$$\pi\colon \mathbb{C}^2 \to \mathfrak{A}, \quad \pi(u,v) = \big(u^2 - v^2, \imath(u^2 + v^2), 2uv\big).$$

The restriction $\pi\colon \mathbb{C}^2 \setminus \{0\} \to \mathfrak{A}^* = \mathfrak{A} \setminus \{0\}$ is an unbranched two-sheeted covering map, so the derivative $F'\colon \overline{\mathbb{D}} \to \mathfrak{A}^*$ of any null disc $F\colon \overline{\mathbb{D}} \to \mathbb{C}^3$ lifts to a map $(u,v) : \overline{\mathbb{D}} \to \mathbb{C}^2 \setminus \{0\}$. One applies the Riemann-Hilbert problem to the map (u,v) with a suitably chosen boundary data, projects the result by π to the null quadric $\mathfrak{A}^*$, and integrates to get a null disc $G\colon \overline{\mathbb{D}} \to \mathbb{C}^3$ satisfying properties (a), (b), and (c) in Theorem 1.4, and also the following:

(d) G is ϵ-close to F in the $\mathscr{C}^1$ topology on $\overline{\mathbb{D}} \setminus U$, where $U \subset \overline{\mathbb{D}}$ is an arbitrarily small neighborhood of the arc $I \subset \mathbb{T}$.

To prove Theorem 1.4 in the general case we proceed as follows. Choose closed oriented Jordan curves $C_1, \ldots, C_m \subset M$ which determine a basis of the 1st homology group $H_1(M;\mathbb{Z})$. Pick a nowhere vanishing holomorphic 1-form θ on $\overline{M}$ (such exists by the Oka-Grauert principle). Given a map $f\colon \overline{M} \to \mathbb{C}^3$, we denote by $\mathcal{P}_j(f) \in \mathbb{C}^3$ for $j = 1, \ldots, m$ the period vector $\int_{C_j} f\theta$, and by $\mathcal{P}(f) \in \mathbb{C}^{3m}$ the period matrix with columns $\mathcal{P}_j(f) \in \mathbb{C}^3$. Observe that we have a bijective correspondence (up to constants)

$$\{F\colon \overline{M} \to \mathbb{C}^3 \text{ null curve}\} \longleftrightarrow \{f\colon \overline{M} \to \mathfrak{A}^* \text{ holomorphic, } f\theta \text{ exact}\}$$

$$F(x) = F(p) + \int_p^x f\theta, \qquad dF = f\theta.$$

Let $F\colon \overline{M} \to \mathbb{C}^3$ be a holomorphic null curve. Set $f = dF/\theta : \overline{M} \to \mathfrak{A}^*$ and embed it as the core map $f = f_0$ in a dominating holomorphic spray of maps $f_t\colon \overline{M} \to \mathfrak{A}^*$, where the parameter t belongs to an open ball $B \subset \mathbb{C}^N$ around the origin. Furthermore, we ensure that the associated period map $B \ni t \mapsto \mathcal{P}(f_t) \in \mathbb{C}^{3m}$ is submersive at $t = 0$.

Choose a small disc $D \subset M$ whose closure $\overline{D} \subset \overline{M}$ intersects bM along a compact arc $J \subset bM$ which contains the given arc I (containing the support of the function μ in Theorem 1.4) in its relative interior. Fix a point $p \in D$ and apply the special case of Theorem 1.4 to the family of maps $F_t : \overline{D} \to \mathbb{C}^3$ given by $F_t(z) = F(p) + \int_p^z f_t \theta$ $(t \in B)$ and the Riemann-Hilbert boundary data from Theorem 1.4. (The integral is calculated along any path in $\overline{D}$ from p to z; note that $F_0 = F|_{\overline{D}}$.) This gives a new spray of null discs $\tilde{G}_t : \overline{D} \to \mathbb{C}^3$ satisfying the properties (a)–(c) in Theorem 1.4 and also property (d) above (with G replaced by $\tilde{G}_t$ and F replaced by F_t). The spray of maps $\tilde{g}_t = d\tilde{G}_t/\theta : \overline{M} \to \mathfrak{A}^*$ then approximates the initial spray $f_t : \overline{D} \to \mathfrak{A}^*$ in the $\mathscr{C}^1$ topology on $\overline{D} \setminus U$. If the approximations are close enough, we can glue these two sprays by the method described in Sect. 3 to get a new holomorphic spray $g_t : \overline{M} \to \mathfrak{A}^*$ for t in a smaller ball $0 \in B' \subset \mathbb{C}^N$. The spray g_t approximates f_t on $\overline{M} \setminus U$. Since we can choose the loops generating $H_1(M; \mathbb{Z})$ in this set, the period map $t \mapsto \mathcal{P}(g_t)$ approximates the map $t \mapsto \mathcal{P}(f_t)$. Since the latter map was chosen submersive at $t = 0$, there exists a $t_0 \in B'$ close to the origin such that the map $g = g_{t_0} : \overline{M} \to \mathfrak{A}^*$ has vanishing periods. Hence g integrates to a null curve $G(z) = F(p) + \int_p^z g\theta$ $(z \in \overline{M})$. It can be verified that G satisfies Theorem 1.4 provided that all approximations were close enough.

The approximate solutions of the Riemann-Hilbert problem for null curves, established in [7, §3], have an additional feature which is the basis of the proof of Theorem 1.8, and hence of the results in Sect. 2. When deforming a null disc $F : \overline{\mathbb{D}} \to \mathbb{C}^3$ in the direction of a null vector $V \in \mathfrak{A}^*$ near a certain boundary point $p \in \mathbb{T}$ (see Theorem 1.4), the component in the direction orthogonal to the 2-plane spanned by the vectors $F'(p) \in \mathfrak{A}^*$ and V changes arbitrarily little in the $\mathscr{C}^0(\mathbb{D})$ sense. To see how this is used, let us give

Sketch of proof of Theorem 1.8 We outline the main idea in the simplest case when M is the disc $\mathbb{D}$. Choose a couple of orthogonal null vectors in $\mathbb{C}^2 \times \{0\} \subset \mathbb{C}^3$; for example, $V_1 = (1, \imath, 0)$ and $V_2 = (1, -\imath, 0)$. We begin with the linear null embedding $\overline{\mathbb{D}} \ni z \mapsto zV_1$. Using Theorem 1.4 with $M = \mathbb{D}$ we deform this embedding near the boundary in the direction of the vector V_2. Ignoring the nullity condition, one could simply take $\overline{\mathbb{D}} \ni z \mapsto zV_1 + z^N V_2$ for a big integer N. To get a null curve, the third component must be involved, but it can be chosen arbitrarily $\mathscr{C}^0$ small if N is big enough. Next we deform the map from the previous step again in the direction of the vector V_1, changing the third component only slightly in the $\mathscr{C}^0(\mathbb{D})$ norm. Repeating this alternating procedure gives a sequence of holomorphic null maps which converges to a proper null curve in $\mathbb{C}^3$ with a bounded third component.

For a general bordered Riemann surface M this construction is performed locally on small discs abutting the boundary bM, using also the method of exposed arcs in order to prevent any shorts. The local modifications are assembled together by the method of gluing sprays as described in the above sketch of proof of Theorem 1.4. □

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Real-Normalized Differentials and the Elliptic Calogero-Moser System

Samuel Grushevsky and Igor Krichever

1 Introduction

This paper is part of a series studying the geometry of the moduli space of curves using meromorphic differentials with real periods (so-called real-normalized differentials, see below). These differentials were introduced in [12] in full generality by the second author, while the idea for some special case goes back at least to Maxwell. More recently, in [4] we have used these differentials to give a direct proof of Diaz' theorem [3] on the dimension of complete subvarieties of the moduli space of curves. Further, in [5] we have described the local infinitesimal structure of the foliation on the moduli space defined using the periods of a real-normalized differential. In [7] together with Chaya Norton we will study in detail the behavior of real-normalized differentials under degeneration. In the current paper we first review the main framework of this setup and then present part of the motivation for the constructions using two real-normalized differentials simultaneously. Namely, we show that if one takes the two foliations on the moduli space corresponding to two different real-normalized differentials, then some intersections of their leaves

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are in fact algebraic subvarieties, which are equal to the suitable loci of spectral curves of the elliptic Calogero-Moser completely integrable system (see Theorem 6 for a precise statement). This result provides motivation for some of the conjectures that we make in [5]. In [6] we will further use this motivation and the degeneration techniques of [7] to obtain results on cusps of plane curves and on the cohomology of the moduli space of curves.

2 Real-Normalized Differentials and Foliations by Absolute Periods

Definition 1 Let $\mathscr{M} := \mathscr{M}_{g,1}(1)$ denote the moduli space of genus g Riemann surfaces C with one marked point $p \in C$ and a 1-jet of a local coordinate at p: that is to say a local coordinate z on a small analytic neighborhood of p in C, where we identify z and z' iff $z' = z + O(z^2)$. Algebro-geometrically, $\mathscr{M}$ is the total space of the relative tangent bundle at p over $\mathscr{M}_{g,1}$ with the zero section removed (as the local coordinate must be non-degenerate). We denote a point $(C, p, z) \in \mathscr{M}$ by X.

Definition 2 A meromorphic differential η on a Riemann surface C is called *real-normalized* if all its periods are real, i.e. if for any closed loop $\gamma \in H_1(C, \mathbb{Z})$ we have $\int_\gamma \eta \in \mathbb{R}$.

We notice in particular that the residue of a real-normalized differential at any point must be purely imaginary, so that its integral over a small loop around that point is real. From the positive-definiteness of the imaginary part of the period matrix of a Riemann surface it follows that the only *holomorphic* real-normalized differential is identically zero. Since any $\mathbb{R}$-linear combination of real-normalized differentials is real-normalized, it follows that the singularities of a real-normalized differential determine it uniquely, and in particular we have

Proposition 1 *For any $X \in \mathscr{M}$ there exists a unique meromorphic real-normalized differential $\Psi = \Psi_X$ on C whose only singularity is a double pole at p, where its singular part is equal to dz/z^2 in the chosen jet.*

We refer to [4] for a detailed proof and more comments and references on the history of real-normalized differentials. For further use, we will also denote $\Psi' = \Psi'_X$ the real-normalized differential with the unique singularity being a double pole at p of the form idz/z^2.

More generally, a unique real-normalized meromorphic differential exists for any prescribed collection of singular parts with zero residues. If such a real-normalized differential η is exact, $\eta = df$, then $f : C \to \mathbb{P}^1$ is a meromorphic function with prescribed poles at the singularities of η. This is exactly to say that f exhibits C as a cover of $\mathbb{P}^1$ with prescribed ramification, and the space of such f is precisely the Hurwitz space. The Lyashko-Looijenga coordinates on the Hurwitz space are then defined to be the critical values of f, that is the values of f at its critical points, the points where $\eta = df = 0$. It is known that Lyashko-Looijenga coordinates give local coordinates on the Hurwitz space (see [17] for an exposition of the theory), and similarly one can define local coordinates using real-normalized differentials in general. This construction is also a special (real-normalized) case of the generalization to meromorphic differentials of the well-known construction for holomorphic differentials, see [19] for a survey of these ideas.

For any $X \in \mathcal{M}$ we denote by $q_1, \dots, q_{2g}$ the zeroes of Ψ_X, written with multiplicity.

Definition 3 If $X \in \mathcal{M}$ is such that all q_i are distinct, we choose a sufficiently small analytic neighborhood of X where the zeroes remain distinct, and moreover over which we can choose a continuously varying symplectic basis $A_1, \dots, A_g, B_1, \dots, B_g$ of $H_1(C, \mathbb{Z})$ (that is to say, $A_i \cdot B_j = \delta_{ij}$, while $A_i \cdot A_j = B_i \cdot B_j = 0$). The *absolute periods* of Ψ are the integrals $\alpha_i := \int_{A_i} \Psi \in \mathbb{R}$ and $\beta_i := \int_{B_i} \Psi \in \mathbb{R}$. These are real-analytic functions on such a neighborhood of X. The *relative periods* of Ψ are the integrals $\int_*^{q_1} \Psi, \dots, \int_*^{q_{2g}} \Psi$, where the paths of integration are chosen not to intersect A_i or B_i, and the basepoint $*$ is chosen so that the sum of all relative periods is equal to zero. We note that the full collection of absolute and relative periods is in fact the pairing of Ψ with a basis of the relative homology of the punctured surface $H_1(C \setminus \{p\}, \{q_1, \dots, q_{2g}\})$, see [8].

It turns out that these periods give local coordinates:

Proposition 2 ([16], see also [4]) *The collection of absolute and relative periods gives real-analytic local coordinates, with values in* $\mathbb{R}^{2g} \times \mathbb{C}^{2g-1}$*, near any* $X \in \mathcal{M}$ *where all zeroes of* Ψ *are distinct, which we call the* period coordinates.

This statement should be viewed as a generalization of the fact that we have Lyashko-Looijenga coordinates on the Hurwitz space. Indeed, the proposition above holds in full generality for any prescribed singular parts, and in that case if we restrict to the locus where all the absolute periods are zero—so that Ψ is exact—the relative periods are precisely the Lyashko-Looijenga coordinates on the corresponding Hurwitz space.

Remark 4 The above definition of the period coordinates can be generalized to define local coordinates near a point of $\mathcal{M}$ where Ψ has multiple zeroes—in that case locally a zero of Ψ of multiplicity k may be perturbed to become at most k zeroes. In a neighborhood, we then choose paths to these at most k zeroes that coincide outside a small neighborhood of zero (basically going to where the k-multiple zero was, and then perturbing the ends of the paths to go to the individual zeroes), and then instead of individual relative periods consider the symmetric function of the corresponding up to k relative periods, counted with multiplicity (so that at the original point we have symmetric functions of a k-tuple of equal values). This is discussed in more detail [4], and provides a viewpoint on the coordinates on various strata of holomorphic differentials with prescribed configurations of zeroes studied in Teichmüller dynamics—but will not play in a role in the current paper.

A crucial observation that made the results of [4] possible is that while the values of absolute periods are not well-defined on $\mathcal{M}$, as they depend on a choice of a symplectic homology basis, the condition that they are locally constant is well-defined.

Definition 5 For any $X \in \mathcal{M}$ we define a *leaf of a (big) foliation* $\mathcal{L}_X$ passing through X to locally be the locus of points in $\mathcal{M}$ where the absolute periods are constant and equal to those on X. As noted above, this condition is independent of the choice of the symplectic homology basis, and we thus define a foliation $\mathcal{L}$ of $\mathcal{M}$ to consist of these leaves (the analogous foliation for holomorphic differentials is sometimes called the REL foliation or the absolute period foliation in Teichmüller dynamics.) We note that relative periods give local holomorphic coordinates on each leaf, and we thus have a tangentially complex foliation on $\mathcal{M}$: the tangent space to $\mathcal{L}_X$ at any X is complex. In particular note that all leaves of $\mathcal{L}$ are smooth immersed complex submanifolds of $\mathcal{M}$ of complex codimension g.

Note that the leaves of $\mathcal{L}$ are only defined locally, and their global geometry in general can be extremely complicated, with the closure in the Deligne-Mumford compactification being especially badly behaved, see [18] for a study of a similar situation for holomorphic differentials, and [7] for a study of degenerations and the behavior of local coordinates in our setting. However, note that (for more general singularities) a leaf of $\mathcal{L}$ corresponding to all absolute periods being zero is the suitable Hurwitz space, and is algebraic. More generally, if say all absolute periods α_i, β_i on a leaf $\mathcal{L}$ are rational (or, still more generally, generate an additive subgroup of $\mathbb{R}$ in which 0 is an isolated point), then $\mathcal{L}$ is an embedded (as opposed to only immersed) submanifold of $\mathcal{M}$. Indeed, if two points X_1, X_2 that are close in $\mathcal{M}$ both lie in $\mathcal{L}$, in a neighborhood of X_1 choose a continuously varying basis for $H_1(X, \mathbb{Z})$. Then the periods of Ψ over this basis must lie in the same discrete subgroup of $\mathbb{R}$ for both X_1 and X_2, and thus all the periods of Ψ over this basis of cycles must be the same for X_1 and X_2. This means that X_1 and X_2 lie on the same connected component of the intersection of $\mathcal{L}$ with a small neighborhood of X_1, which is to say that the intersection of $\mathcal{L}$ with a small neighborhood in $\mathcal{M}$ of any its point is connected, and thus $\mathcal{L} \subset \mathcal{M}$ is then embedded.

All of the above constructions were completely general, and developed in [4] in full generality for arbitrary prescribed singularities. We now take into account the specifics of our situation, where we naturally have *two* real-normalized differentials Ψ and Ψ'. We first have the easy observation:

Lemma 3 *For any $X = (C, p, z) \in \mathcal{M}$ any real-normalized differential on C with a double pole at p is an $\mathbb{R}$-linear combination of Ψ_X and Ψ'_X.*

Proof Indeed, suppose such a real-normalized differential η has a singular part $(a + bi)dz/z^2$, for $a, b \in \mathbb{R}$ (notice that η cannot have a residue at its unique pole on a compact Riemann surface). Then the differential $\eta - a\Psi - b\Psi'$ is a holomorphic real-normalized differential on C, and thus must be zero.

It thus turns out that some properties of the pair (Ψ, Ψ') are independent of the choice of the local coordinate, and thus make sense on $\mathcal{M}_{g,1}$—this will be used in [6] to study the homology classes of the loci in $\mathcal{M}_{g,1}$ where Ψ and Ψ' have a prescribed number of common zeroes. We thus overlap the above structures defined by Ψ and Ψ'.

Definition 6 For any $X \in \mathcal{M}$, let $\mathcal{L}_X$ be the leaf of the foliation defined above. Let $\mathcal{L}'_X$ be the leaf of the analogous foliation on $\mathcal{M}$ defined using the period coordinates associated to Ψ'. We then let $\hat{\mathcal{S}}_X = \mathcal{L}_X \cap \mathcal{L}'_X \subset \mathcal{M}$ be the intersection of these two leaves. From the lemma above we see that the condition of local constancy of absolute periods of *both* Ψ and Ψ' implies the constancy of periods of the real-normalized differential with any fixed double pole. That is to say, $\hat{\mathcal{S}}_X$ does not depend on the choice of a local coordinate, and is a preimage of some locus $\mathcal{S}_X \subset \mathcal{M}_{g,1}$.

By a slight abuse of language, we will call $\mathcal{S}$ the "small" foliation on $\mathcal{M}$, and call $\mathcal{S}_X$ its leaves. Here we use the term foliation loosely: indeed, one of the main conjectures of [5] is precisely the statement that all the leaves $\mathcal{S}_X$ are smooth (and of the same dimension). Thus what we mean by a foliations is only that there exists a subvariety $\mathcal{S}_X$ through every point of $\mathcal{M}_{g,1}$, and distinct such sets do not intersect. However, the main result of the current paper is precisely the statement that there exists an everywhere dense collection of leaves $\mathcal{S}_X$ that are in fact smooth algebraic subvarieties of $\mathcal{M}$. To prove this, we identify these leaves as the loci of suitable spectral curves of the elliptic Calogero-Moser integrable systems, and thus *any* leaf $\mathcal{S}_X$ can be interpreted as a suitable perturbation of the spectral curves of the elliptic Calogero-Moser system, which motivates our conjectures in [5].

3 The Calogero-Moser Locus via Real-Normalized Differentials

Definition 7 We define the Calogero-Moser locus $\mathcal{K}_g \subset \mathcal{M}_{g,1}$ to be the locus of all (C, p) for which there exist two $\mathbb{R}$-linearly independent differentials of the second kind $\Phi_1, \Phi_2 \in H^0(C, K_C + 2p)$ with all periods *integer*.

We note that in particular Φ_1 and Φ_2 are real-normalized, and since by Lemma 3 the space of real-normalized differentials with a unique double pole is two-dimensional over $\mathbb{R}$, any real-normalized differential with a unique double pole at p is equal to $r_1\Phi_1 + r_2\Phi_2$ for some $r_1, r_2 \in \mathbb{R}$. We thus have

Proposition 4 *The Calogero-Moser locus $\mathcal{K}_g \subset \mathcal{M}_{g,1}$ is the union of all leaves of the small foliation $\mathcal{S}$ for which there exist $r_1, r_2, r'_1, r'_2 \in \mathbb{R}$ such that all (absolute) periods of Ψ lie in $r_1\mathbb{Z} + r_2\mathbb{Z}$ and all periods of Ψ' lie in $r'_1\mathbb{Z} + r'_2\mathbb{Z}$.*

Remark 8 We note that the locus $\mathscr{K}_g$ is easily seen to be dense in $\mathscr{M}_{g,1}$, as for example it includes the set of all curves for which all absolute periods of Ψ and Ψ' are rational (there are finitely many periods, so we can just choose $r_1 = r_1'$ to be the inverse of the largest denominator). Of course the locus $\mathscr{K}_g$ consists of infinitely many leaves of the foliation $\mathscr{S}$ corresponding to different r's, and has infinitely many connected components—distinguished at least by the least common multiple of the denominators of periods. Instead of working with all of $\mathscr{K}_g$, we will thus first represent it as a countable union of loci corresponding to different degrees of the covers of the elliptic curve that is inherent in the picture.

Indeed, note that if Φ_1 and Φ_2 have integer periods, then so does any their $\mathbb{Z}$-linear combination. Thus for $(C,p) \in \mathscr{K}_g$ we choose Φ_1, Φ_2 generating the lattice in $H^0(C, K_C + 2p)$ for which all periods are integer, and let

$$\tau := \frac{\Phi_2}{\Phi_1}(p) \tag{1}$$

(which means taking the ratio of the singular parts of Φ_2 and Φ_1 at p). Since Φ_i are $\mathbb{R}$-linearly independent, $\operatorname{Im}\tau \neq 0$, and by swapping Φ_1 and Φ_2 if necessary we may assume that $\operatorname{Im}\tau > 0$. While τ depends on the choice of generators of the lattice, and is only well-defined up to an action of $\mathrm{SL}(2,\mathbb{Z})$, note that the differential $\Phi_2 - \tau\Phi_1$ is *holomorphic* on C, and all its periods lie in $\mathbb{Z} + \tau\mathbb{Z}$. Thus integrating this differential gives a holomorphic map

$$z : C \to E := \mathbb{C}/\mathbb{Z} + \tau\mathbb{Z}; \qquad q \mapsto z(q) := \int_p^q (\Phi_2 - \tau\Phi_1) \tag{2}$$

We note that the isomorphism class of the elliptic curve E and the map $z : C \to E$ do not depend on the choice of Φ_1, Φ_2.

Definition 9 For $N \in \mathbb{Z}$ we denote $\mathscr{K}_{g,N} \subset \mathscr{K}_g$ the locus of Calogero-Moser curves for which the degree of the map $z : C \to E$ is equal to N.

We will see that $\mathscr{K}_{g,N}$ is empty if $N < g$ as will follow from our explicit parametrization of these loci; it can also be shown that for any $N \geq g$ the locus $\mathscr{K}_{g,N}$ is in fact non-empty, as can be seen by studying suitable degenerations and then perturbing—but we will not need this result here.

Since the degree is an integer that depends continuously on the Calogero-Moser curve, it is locally constant on $\mathscr{K}_g$, and thus each $\mathscr{K}_{g,N}$ is a union of some collection of connected components of $\mathscr{K}_g$. Analytically, N can be computed as

$$N = \sum_{k=1}^{g} \left(\int_{A_k} \Phi_2 \int_{B_k} \Phi_1 - \int_{B_k} \Phi_2 \int_{S_k} \Phi_1 \right) = 2\pi i \operatorname{res}_p (F_1 \Phi_2) . \tag{3}$$

In what follows we will always fix the generators Φ_1, Φ_2, or equivalently fix a basis of $H_1(E,\mathbb{Z}) \equiv \mathbb{Z}^2$. For any $\tau \in \mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$ we then denote $\mathscr{K}_{g,N}^{\tau} \subset \mathscr{K}_{g,N} \subset \mathscr{K}_g \subset$

$\mathcal{M}_{g,1}$ the subset where $E = E_\tau$. Since the above constructions and definition only depend on the absolute periods of Ψ_1 and Ψ_2, we have the following

Proposition 5 *The locus $\mathcal{K}_{g,N}^\tau$ is a union of leaves of the small foliation $\mathcal{S} \subset \mathcal{M}_{g,1}$; that is to say, if any leaf of $\mathcal{S}$ intersects $\mathcal{K}_{g,N}^\tau$ (for $N \in \mathbb{Z}, \tau \in \mathbb{H}/\mathrm{SL}(2,\mathbb{Z})$ fixed), then this leaf is contained in $\mathcal{K}_{g,N}^\tau$.*

The main result of this paper is justifying the name "Calogero-Moser" for this locus, i.e. the identification of $\mathcal{K}_g$ as the locus of spectral curves of the elliptic Calogero-Moser system. Our main result is the following

Theorem 6 *The locus $\mathcal{K}_{g,N}$ is the locus of curves that are normalizations $\tilde{C}_{cm}$ of spectral curves C_{cm} of the N-particle elliptic Calogero-Moser system (i.e. of curves given by* (8) *below).*

Note that in particular this theorem implies that the loci $\mathcal{K}_{g,N}^\tau$ and $\mathcal{K}_{g,N}$, are all algebraic, as they arise from the algebro-geometric constructions associated to the elliptic Calogero-Moser system.

4 The Elliptic Calogero-Moser System

Definition 10 The *elliptic Calogero-Moser (CM) system* introduced in [1] is a system of N particles on an elliptic curve E with pairwise interactions. The phase space of this system is

$$\begin{aligned}\mathcal{P}_N :&= (\mathbb{C} \times E)^{\times N} \setminus \{\text{diagonals in } E\} \\ &= \left\{q_1, \dots, q_N \in \mathbb{C}, x_1, \dots, x_N \in E, x_i \neq x_j\right\},\end{aligned}$$

where we think of the variables x_i as the positions of the particles, and of q_i as their momenta, lying in the cotangent space to E, which is trivial and identified with $\mathbb{C}$. The evolution of this system is a trajectory of a set of particles in the phase space.

The *elliptic Calogero-Moser Hamiltonian* is the meromorphic function $H_2 : \mathcal{P}_N \to \mathbb{C}$ given by

$$H_2 := \frac{1}{2}\sum_{i=1}^{N} q_i^2 - 2\sum_{i\neq j} \wp(x_i - x_j), \tag{4}$$

where $\wp$ denotes the Weierstrass $\wp$-function on E.

The Hamiltonian equations of motions are then

$$\dot{x}_i = -\frac{\partial H_2}{\partial q_i}; \quad \dot{q}_i = \frac{\partial H_2}{\partial x_i};$$

where from now on the dot denotes the partial derivative $\partial/\partial t$ with respect to time. These equations of motion determine the evolution of the system of particles completely starting from the given initial conditions.

In [11] the second author showed that the equations of motion of the elliptic CM system admit a Lax representation with "elliptic spectral parameter z". This is to say that the Hamiltonian equations of motion above for the Hamiltonian H_2 are equivalent to the matrix-valued differential equation $\dot{L} = [L, M]$, where $L = L(z)$ and $M = M(z)$ are $N \times N$ matrices depending on the point $z \in E$, given explicitly by

$$L_{ii}(z) = \frac{1}{2}\, q_i; \quad L_{ij}(z) = F(x_i - x_j, z) \quad \text{for } i \neq j, \tag{5}$$

and

$$M_{ii}(z) = \wp(z) - 2\sum_{k\neq i} \wp(x_i - x_k); \quad M_{ij}(z) = -2F'(x_i - x_j, z), \tag{6}$$

with the function F defined by

$$F(x, z) := \frac{\sigma(z - x)}{\sigma(z)\sigma(x)} e^{\zeta(z)x}, \tag{7}$$

for ζ and σ the standard Weierstrass elliptic functions, and where F' denotes the derivative of F with respect to x.

Definition 11 The *spectral curve* C_{cm} of the elliptic CM system is the normalization at the point $(k, z) = (\infty, 0)$ of the closure in $\mathbb{P}^1 \times E$ of the affine curve $C^o_{cm} \subset \mathbb{C} \times (E \setminus \{0\})$ given by the equation

$$R(k, z) := \det(k \cdot I + L(z)) = 0, \tag{8}$$

where I is the $N \times N$ identity matrix. For further use, we expand this determinant as a polynomial in powers of k, denoting the coefficients $r_i(z)$, so that $R(k, z) = \sum_{i=0}^{N} r_i(z)k^{N-i}$, in particular with $r_0(z) = 1$. We note that C_{cm} is singular at all singularities of C^o_{cm}, and denote $\tilde{C}_{cm}$ the normalization of C_{cm}.

Remark 12 It is easy to see that the Lax equation $\dot{L} = [M, L]$ directly implies that the characteristic equation satisfied by the differential operator L does not depend on t, i.e. the spectral curve can be regarded as "integrals of motion" (is time-invariant). The general algebro-geometric integration scheme of soliton systems based on a concept of the Baker-Akhiezer functions in fact establishes the one-to-one correspondence of the open sets of the phase space of the system and the Jacobian bundle over the family of the corresponding spectral curves. Under this correspondence the equations of motion of the system become the equations of the linear flow on the Jacobian.

From the Riemann-Hurwitz formula it follows that the arithmetic genus of C_{cm} is equal to N; thus the genus of its normalization $\tilde{C}_{cm}$ is strictly less than N if and only if C_{cm} is singular.

From the explicit formula (5) for $L(z)$ one sees that each $r_i(z)$ is a meromorphic function of $z \in E$ with a pole of order i at $z = 0$. As shown in [11], near $z = 0$ the polynomial $R(k, z)$ admits a factorization of the form

$$R(k, z) = \prod_{i=1}^{N} (k + a_i z^{-1} + h_i + O(z)), \tag{9}$$

with $a_1 = 1 - N$ and $a_i = 1$ for $i > 1$, for some $h_i \in \mathbb{C}$. This implies that the closure $\overline{C^o_{cm}} \subset \mathbb{P}^1 \times E$ of C^o_{cm} is obtained by adding one point $(\infty, 0)$, at which $N-1$ branches of $\overline{C^o_{cm}}$ are tangent to each other (corresponding to $a_2 = \ldots = a_N = 1$), and one branch is transverse to them. Thus if we blow up the point $(\infty, 0) \in \mathbb{P}^1 \times E$, on the strict transform of $\overline{C^o_{cm}}$ under this blowup we would have a smooth point p corresponding to the first branch, and a point p' contained in the $N - 1$ branches. Thus generically the partial normalization C_{cm} of $\overline{C^o_{cm}}$ at $(\infty, 0)$ is obtained by doing the second blowup at p', and irrespective of this we have $\#\{C_{cm} \setminus C^o_{cm}\} = N$.

Proposition 7 ([2]) *For a fixed elliptic curve E and a fixed integer N the space of Calogero-Moser spectral curves C_{cm} is equal to $\mathbb{C}^N$, i.e. is parameterized by N free complex parameters.*

These N free complex parameters were found in [2] by observing that a polynomial $R(k, z)$ has a unique representation of the form

$$R(k, z) := f(k - \zeta(z), z), \tag{10}$$

where

$$f(\phi, z) = \frac{1}{\sigma(z)}\, \sigma\left(z + \frac{\partial}{\partial \phi}\right) H(\phi) = \frac{1}{\sigma(z)} \sum_{n=0}^{N} \frac{1}{n!}\, \partial_z^n \sigma(z) \frac{\partial^n H}{\partial \phi^n}. \tag{11}$$

and H is the monic degree N polynomial

$$H(\phi) = \phi^N + \sum_{i=0}^{N-1} I_i \phi^i$$

whose coefficients $I_0, \ldots, I_{n-1} \in \mathbb{C}$ give parameters for the space of Calogero-Moser spectral curves.

We are now ready to prove one direction of our main result, that the spectral curves of the elliptic Calogero-Moser system in fact lie in the locus $\mathcal{K}_g$ defined using differentials with real periods:

Proposition 8 *For a fixed elliptic curve $E = E_\tau$ and a fixed integer N, the normalization $\tilde{C}_{cm}$ of the spectral curve C_{cm} of the Calogero-Moser system lies in the Calogero-Moser locus $\mathcal{K}^\tau_{g,N} \subset \mathcal{M}_{g,1}$.*

Proof Indeed, to prove this we need to construct two differentials Φ_1 and Φ_2 on C_{cm} with all periods integer (and then also verify that the resulting N is correct). To construct these differentials, we will think of the curve $C^o_{cm} \subset \mathbb{C} \times (E \setminus \{0\})$, so that we can pull back differentials from both factors, and try to look for Φ_i of the form $f_i(k)dk + g_i(z)dz$. Note that since all periods of differentials on $\mathbb{C}$ are zero, the first summand contributes nothing to the periods of Φ_i, while the periods of the second summand are the periods of that differential on the image in E of a cycle in C, and thus lie in $\mathbb{Z} + \tau\mathbb{Z}$. Thus we will take g_i so that the two periods of the meromorphic differential $g_i(z)dz$ on E are integer, and choose f_i to ensure that the singularities are as required.

Indeed, when we compactify, the differential dk has a double pole at $(\infty, 0) \in \mathbb{P}^1 \times E$. As discussed above, this point has N preimages on C_{cm} corresponding to the N local branches near $(\infty, 0)$, with local expressions given by (9), and the preimage of $(\infty, 0)$ under the first blowup consists of two point p, with one branch through it, and p' lying on $N-1$ branches. Since dk has a double pole at $\infty \in \mathbb{P}^1$, at all the $N-1$ branches where the coefficients $a_i = 1$ in (9), so that the branch is locally given by $k + z^{-1} + h_i = O(z)$, we can cancel the double pole of dk near $k = \infty$ by taking locally $-dz/z^2$; since we are working on the elliptic curve, this means we should globally take $-\wp(z)dz$, which precisely this double pole at $z = 0$. Thus we are looking for $\Phi_i = a_i(dk - \wp(z)dz) + c_i dz$ for some constants $a_i, c_i \in \mathbb{C}$, where the constant c_i is determined to ensure that the periods are integers. Solving we thus get

$$\Phi_1 := \frac{1}{2\pi i}(dk - \wp(z)dz) + c_1 dz, \qquad \Phi_2 := \frac{\tau}{2\pi i}(dk - \wp(z)dz) + c_2 dz. \tag{12}$$

where

$$c_1 := \frac{1}{2\pi i}\int_0^1 \wp(z)dz \quad \text{and} \quad c_2 := \frac{1}{2\pi i}\int_0^\tau \wp(z)dz.$$

We have thus shown that the curve C_{cm} indeed lies in $\mathcal{K}_g$, and by construction the value of N is as required.

The other direction of the main result, Theorem 6, is the statement that any curve $(C, p) \in \mathcal{K}^\tau_{g,N}$, arises as the spectral curve C_{cm} of the elliptic Calogero-Moser system. This uses the methods of integrable systems: in [9, 10] the second author gave a general construction to obtain an algebro-geometric solution of the KP equation starting from such a curve. The statement that we need to prove is essentially that in the case when Φ_1 and Φ_2 both have integral periods the solution of KP equation obtained in this way is elliptic.

To explain how this argument works, we recall the definition of the Baker-Akhiezer function and related constructions.

Definition 13 For $(C, p, z) \in \mathscr{M}$, for a fixed generic set of g points $\gamma_1, \ldots, \gamma_g \in C$ (that is, forming an effective divisor $Z_0 := \gamma_1 + \ldots + \gamma_g$ of degree g on C with $h^0(C, D) = 1$), and for fixed $x, t \in \mathbb{C}$ the Baker-Akhiezer function $\psi(x, t, p)$ is the unique function on C, which is meromorphic on $C \setminus \{p\}$, with only singularities being the simple poles at γ_i, and such that in a neighborhood of p it has an essential singularity that admits an expression of the form

$$\psi(x, t, z) = e^{xz^{-1}+tz^{-2}} \left(1 + \sum_{s=1}^{\infty} \xi_s(x, t) z^{-s}\right) \tag{13}$$

where each ξ_s is some holomorphic function of x, t.

The uniqueness of the Baker-Akhiezer function follows easily from observing that the ratio of two such functions would be holomorphic on all of C, since the simple poles cancel, with value 1 at p, where the essential singularities cancel. To see the existence of the Baker-Akhiezer function, note that an explicit expression for it was obtained in [9]:

$$\psi(x, t, q) = \frac{\theta(A(q) + Ux + Vt + Z_0)\,\theta(A(p) + Z_0)}{\theta(A(p) + Ux + Vt + Z_0)\,\theta(A(q) + Z_0)}\, e^{x\int^q \Omega_2 + t\int^q \Omega_3}, \tag{14}$$

where $A : C \hookrightarrow J(C)$ is the Abel-Jacobi embedding of the curve into its Jacobian, and U and V are the vectors of B-periods of the normalized (i.e. with all A-periods zero) differentials Ω_2 and Ω_3, with poles at p of second and third order, respectively, and holomorphic elsewhere.

The construction of CM curves was crucial for the identification of the theory of the CM system and the theory of the elliptic solutions of the Kadomtsev-Petviashvili (KP) equation established in [11]. This identification is based on the following result:

Lemma 9 *The equation*

$$\left(\partial_t - \partial_x^2 + u(x, t)\right) \psi(x, t) = 0 \tag{15}$$

with elliptic potential (i.e. u(x,t) is an elliptic function of the variable x) has a meromorphic in x solution ψ if and only if u is of the form

$$u = 2\sum_{i=1}^{N} \wp(x - x_i(t)) \tag{16}$$

with poles $x_i(t)$ satisfying the equations of motion of the CM system.

Remark 14 In [11] a slightly weaker form of the lemma was proven. Namely, its assertion was proved under the assumption that Eq. (16) has a family of *double-Bloch* solutions (i.e. meromorphic solutions with monodromy $\psi(x + \omega_\alpha, t) =$

$w_\alpha \psi(x,t)$, where ω_a are periods of the elliptic curve and w_a are constants.) This weaker version is sufficient for our further purposes, but for completeness we included above the strongest form of the lemma, proven in [14] (see [15] for details).

As shown in [9], the Baker-Akhiezer function satisfies partial differential equation (15) with the potential $u(x,t)$ given explicitly as

$$u(x,t) = 2\partial_x^2 \ln \theta(Ux + Vt + Z_0). \tag{17}$$

We will now use the Baker-Akhiezer function to obtain our main result.

Proof (Proof of main Theorem 6) Starting from any curve (C, p, z) (and a collection of g points on C in a general position) we can construct uniquely a Baker-Akhiezer function, given explicitly by (14). We first show that the curves $(C,p) \in \mathscr{K}_g$ are characterized within $\mathscr{M}_{g,1}$ by the property that the vector U in (14) and (17) spans an elliptic curve in the Jacobian of C (i.e. $\mathbb{C}U \subset J(C)$ is closed).

Indeed, recall that U is the vector of B-periods of the meromorphic differential Ω_2, which has a double pole at p, and all of which A-periods are zero. For $(C,p) \in \mathscr{K}_g$ we then have two holomorphic differentials $\Omega_2 - \Phi_1$ and $\tau\Omega_2 - \Phi_2$. Since all the A-periods of Ω_2 are zero, the A-periods of these two differentials are all integer, and thus both $\Omega_2 - \Phi_1$ and $\tau\Omega_2 - \Phi_2$ must be linear combinations of a basis $\omega_1 \ldots \omega_g$ of holomorphic differentials on C dual to the A-cycles, with *integer* coefficients. Thus we have

$$\Omega_2 = \Phi_1 + w_1 = (\Phi_2 + w_2)/\tau$$

where holomorphic differentials w_1, w_2 are integral linear combinations of ω_i. From the first equality it follows that the vector U of B-periods of Ω_2 is equal to the sum of B-periods of Φ_1, which are integers, and the B-periods of w_1, which are integral linear combinations of the periods of ω_i. This means that we have $U \in \mathbb{Z}^g + \tau_C\mathbb{Z}^g$, where τ_C is the period matrix of C (the matrix of B-periods of ω_i). Similarly from the second expression for Ω_2 it follows that also $\tau U \in \mathbb{Z}^g + \tau_C\mathbb{Z}^g$. Finally since $\mathbb{C}U = \mathbb{R}U + \mathbb{R}\tau U$ is then a complex line containing two non-proportional vectors in the lattice $\mathbb{Z}^g + \tau_C\mathbb{Z}^g$, its image in the Jacobian $J(C) := \mathbb{C}^g/\mathbb{Z}^g + \tau_C\mathbb{Z}^g$ is compact, and thus U spans an elliptic curve in $J(C)$.

Let now N be the degree of the restriction of the theta function from $J(C)$ to the elliptic curve E generated by U. Then the restriction of the theta function of $J(C)$ to E can be written as a product in terms of its zeroes using the elliptic σ function:

$$\theta(\tau_C, Ux + Vt + Z_0) = f(t, Z_0) \prod_{i=1}^{N} \sigma(x - x_i(t)) \tag{18}$$

for t and Z_0 fixed, where f is non-zero (and of course depends on t and Z_0 holomorphically), and $x_1(t), \ldots x_N(t)$ are the zeroes of the restriction of the theta function of $J(C)$ to the corresponding translate of E. Substituting this expression into (17) implies that u is of the form (16). The Baker-Akhiezer function is a meromorphic function of x. From Lemma 9 it then follows that $q_i(t)$ in (18) satisfy the equations of motion of the CM system.

Remark 15 For the last statement a weaker version of Lemma 9 suffices, because from the definition of the Baker-Akhiezer function it follows that it has monodromy given by

$$\psi(x+1,t,p) = e^{2\pi i F_1(p)}\psi(x,t,p), \qquad \psi(x+\tau,t,p) = e^{2\pi i F_2(p)}\psi(x,t,p). \tag{19}$$

Such monodromy was called in [13] the double-Bloch property. Geometrically, it means that as a function of x for t and p fixed, ψ is a (meromorphic) section of a certain bundle on the elliptic curve $E = \mathbb{C}/\mathbb{Z}+\tau\mathbb{Z}$, where this bundle depends on p.

Thus, starting from $(C,p) \in \mathcal{K}_{g,N}$, we have used the Baker-Akhiezer function (which is a solution of the KP system) to then construct a solution of the elliptic Calogero-Moser system. This elliptic Calogero-Moser system has a spectral curve given explicitly by Eq. (8). It thus remains to show that the $\tilde{C}_{cm}$ of the spectral curve of this CM system indeed coincides with the original curve C. To this end it is enough to check that

$$\psi(x,t,p) = \sum_{i=1}^{N} c_i(t,p)F(x-x_i(t),z)e^{kx+k^2t} \tag{20}$$

satisfies all the defining properties of the Baker-Akhiezer function on $\tilde{C}_{cm}$—and thus by uniqueness is the Baker-Akhiezer function. Here the functions c_i are coordinates of the vector $C = (c_1,\dots,c_N)$ satisfying

$$(L(t,z)+k)\ C = 0, \quad \dot{C} = M(t,z)C\,. \tag{21}$$

This verification is straightforward, and the proof is thus complete.

Since we have constructed all curves in $\mathcal{K}_g$ as normalizations of spectral curves of the Calogero-Moser system, and normalizing can only reduce the arithmetic genus, we get in particular

Corollary 10 *The locus $\mathcal{K}^{\tau}_{g,N}$ is empty if $g > N$.*

Remark 16 We note that $\mathcal{K}_{g,g}$ by construction is the locus of spectral curves that are smooth. Indeed, for $C_{cm} \in \mathcal{K}_{g,g}$, the differentials Φ_1 and Φ_2 cannot have common zeroes. If for some point $p \in C_{cm}$ we had $\Phi_1(p) = \Phi_2(p) = 0$, then also $dk(p) = dz(p) = 0$, as these two differentials are linear combinations of Φ_1 and Φ_2. However, if both dk and dz vanish at a point of $\overline{C^o_{cm}}$, this point is singular, while we assumed the curve to be smooth. Following this line of thought, one would expect the common zeroes of Φ_1 and Φ_2 on Calogero-Moser curves to be closely related to the singularities of the curve. For the two simplest possible classes of singularities—nodes and simple cusps—the situation is as follows: the differentials Ψ_1 and Ψ_2 (or equivalently dk and dz) do not have a zero at a point of $\tilde{C}_{cm}$ that is a preimage of a node on C_{cm}, and have a simple common zero at a preimage of a cusp (and a multiple common zero at a preimage of any more complicated singularity). It can be shown that a Zariski open subset of $\mathscr{K}_{g,N}$ corresponds to singular CM curves having $N-g$ nodes.

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On the CR Transversality of Holomorphic Maps into Hyperquadrics

Dedicated to Professor Yum-Tong Siu on the occasion of his 70th birthday

Xiaojun Huang and Yuan Zhang

1 Introduction and the Main Theorems

Let M_1 and M_2 be two connected smooth CR hypersurfaces in $\mathbf{C}^n$ and $\mathbf{C}^N$, respectively, with $3 \leq n \leq N$. Let F be a holomorphic map from some small neighborhood $U \subset \mathbf{C}^n$ of M_1 into $\mathbf{C}^N$ with $F(M_1) \subset M_2$. Given a point $p \in M_1$, denote by $T_p^{(1,0)}M_1$ and $T_{F(p)}^{(1,0)}M_2$ the holomorphic tangent vector spaces of M_1 at p and M_2 at $F(p)$, respectively. Assume F does not send a neighborhood of p in $\mathbf{C}^n$ into M_2. An important question in the study of the geometric structure of F is to understand the geometric conditions for the manifolds in which F is CR transversal to M_1 at p. Recall that F is said to be CR transversal at p if

$$T_{F(p)}^{(1,0)}M_2 + dF(T_p^{(1,0)}\mathbf{C}^n) = T_{F(p)}^{(1,0)}\mathbf{C}^N.$$

Roughly speaking, the CR transversality property can be interpreted as an non-vanishing property of the normal derivative of the normal components for the map.

The problem has been extensively investigated in the literature. When both the target and the source manifolds are strongly pseudoconvex, CR transversality always holds due to the classical Hopf lemma. In the equal dimensional case ($n = N$), work has been done by Pinchuk [18], Fornaess [11], Baouendi-Rothschild

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[2], Baouendi-Huang-Rothschild [3], Ebenfelt-Rothschild [8], Huang [12] and the references therein. See also Baouendi-Ebenfelt-Rothschild [4] and D'Angelo [7] for more general related topics. The study of the higher codimensional case starts with the work of Baouendi-Huang in [1] where it is proved that the CR transversality always holds when the manifolds are hyperquadrics of the same signature. Baouendi-Ebenfelt-Rothschild [5] proved, under rather general setting, that the CR transversality holds in an open dense subset. See also a recent paper of Ebenfelt-Son [9] and the references therein.

While there exist examples where CR transversality fails on certain thin sets (see, for instance [5]), as mentioned above, the rigidity theorem due to Baouendi-Huang [1] indicates that the CR transversality holds everywhere when both M_1 and M_2 are hyperquadrics of the same signature ℓ. Enlightened by this result, the following conjecture concerning the CR transversality was asked by Baouendi and the first author in the year of 2005:

Conjectures (Baouendi-Huang [1]) *Let $M_1 \subset \mathbf{C}^n$ and $M_2 \subset \mathbf{C}^N$ be two (connected) Levi non-degenerate real analytic hypersurfaces with the same signature $\ell > 0$. Here $3 \le n < N$. Let F be a holomorphic map defined in a neighborhood U of M_1, sending M_1 into M_2. Then either F is a local CR embedding from M_1 into M_2 or F is totally degenerate in the sense that it maps a neighborhood U of M_1 in $\mathbf{C}^n$ into M_2.*

We point out that, for the M_1 and M_2 given in the conjecture, the fact that F is CR transversal at p is equivalent to the fact that F is a CR embedding from a neighborhood of p in M_1 into M_2. Along these lines, in a recent paper of the authors [15], by developing a new technique, we showed the CR transversality holds when $M_2 = H_\ell^{n+1}$ and the point under study is not CR umbilical in the sense of Chern-Moser. Recall that a hyperquadric H_ℓ^n of signature ℓ in $\mathbf{C}^n$ is defined by

$$H_\ell^n := \left\{(z,w) \in \mathbf{C}^{n-1} \times \mathbf{C} : \Im w = |z|_\ell^2\right\},$$

where for any n-tuples a and b, $\langle a, \bar{b}\rangle_\ell := -\sum_{j=1}^{\ell} a_j \bar{b}_j + \sum_{j=\ell+1}^{n} a_j \bar{b}_j$ and $|a|_\ell^2 = \langle a, \bar{a}\rangle_\ell$.

In this paper, combining a quantitative version of a very useful lemma due to the first author with the tools developed in [15], we are able to drop the geometric assumption of the umbilicality and relax the codimension-one restriction in [15]. The generalization of the above mentioned lemma in [13] will be addressed in detail in Sect. 3.

We next state our main theorems:

Theorem 1 *Let M_ℓ be a smooth Levi non-degenerate hypersurface of signature ℓ in $\mathbf{C}^n$ with $n \ge 3$ and $0 \in M_\ell$. Suppose that F is a holomorphic map in a small neighborhood U of $0 \in \mathbf{C}^n$ such that*

$$F(M_\ell \cap U) \subset H_\ell^N$$

with $N - n < \frac{n-1}{2}$. *If* $F(U) \not\subset H_\ell^N$, *then* F *is CR transversal to* M_ℓ *at* 0, *or equivalently,* F *is a CR embedding from a small neighborhood of* $0 \in M_\ell$ *into* H_ℓ^N.

Theorem 2 *Let* M_ℓ *be a germ of a smooth Levi non-degenerate hypersurface at 0 of signature* ℓ *in* $\mathbf{C}^n$, $n \geq 3$. *Suppose that there exists a holomorphic map* F *in a neighborhood* U *of* 0 *in* $\mathbf{C}^n$ *sending* M_ℓ *into* H_ℓ^N *but* $F(U) \not\subset H_\ell^N$, $N < 2n - 1$. *Then* M_ℓ *is CR embeddable into* H_ℓ^N *near* 0. *Equivalently, there exists a holomorphic map* $\tilde{F} : M_\ell \to H_\ell^N$ *near* 0, *which is CR transversal to* M_ℓ *at* 0.

The idea of the proof is based on a re-scaling technique that was initially introduced in [15]. With the aid of a quantitative lemma of the first author in [13], we generate a formal CR transversal map which, by a result of Meylan-Mir-Zaitsev proved in [16], is necessarily convergent. Finally, using a rigidity result in [10], F differs from the CR transversal map only by an automorphisms of the target and hence it is CR transversal as well.

The outline of the paper is as follows. In Sect. 2, the notations and a normalization procedure of Baouendi-Huang is revisited. A modified lemma in [13] is discussed and proved in Sect. 3. Section 4 is devoted to the proof of the main theorem.

2 Notations and a Normalization Procedure

Let M_ℓ be a germ at 0 of a smooth Levi non-degenerate hypersurface of signature ℓ in $\mathbf{C}^n$. After a holomorphic change of coordinates, M_ℓ near the origin can be expressed as follows.

$$M_\ell = \left\{(z, w) \in \mathbf{C}^{n-1} \times \mathbf{C} : \Im w = |z|_\ell^2 - \frac{1}{4}\mathscr{S}(z) + o(4)\right\}. \tag{1}$$

Here $\mathscr{S}(z) := \sum_{1 \leq \alpha,\beta,\gamma,\delta \leq n} s_{\alpha\bar{\beta}\gamma\bar{\delta}} z_\alpha \bar{z}_\beta z_\gamma \bar{z}_\delta$ is a homogeneous polynomial of bi-degree (2,2), called the Chern-Moser-Weyl curvature function of M_ℓ at 0. See [6] for more details. Without loss of generality, we always assume that $\ell \leq (n-1)/2$ so ℓ becomes an invariant.

As in [6], assign the weighted degree 1 to variable z and 2 to variable w. Given a holomorphic function h, denote by $h^{(k)}$ the terms of weighted degree k, and by $h^{(\mu,\nu)}$ the terms of degree μ in z variable and of degree ν in w variable in the power series expansion of h at 0. For each integer $k \geq 0$, we write $o(k)$ for terms of degree larger than k, and $o_{wt}(k)$ for terms of weighted degree larger than k. To simplify our notation, we also preassign the coefficient of h with negative degrees to be 0.

Let $\tilde{M}_\ell$ be a germ at 0 of another smooth Levi-nondegenerate hypersurface in $\mathbf{C}^N$ of signature ℓ given by

$$\tilde{M}_\ell = \left\{(\tilde{z}, \tilde{w}) \in \mathbf{C}^{N-1} \times \mathbf{C} : \Im \tilde{w} = |\tilde{z}|_\ell^2 - \frac{1}{4}\tilde{\mathscr{S}}(\tilde{z}) + o(4)\right\}. \tag{2}$$

Here $\tilde{\mathscr{S}}$ is the corresponding Chern-Moser curvature tensor function of $\tilde{M}_\ell$ at 0.

Let F be a smooth CR map sending $(M_\ell, 0)$ into $(\tilde{M}_\ell, 0)$. Write

$$F := (\tilde{f}, g) = (f, \phi, g) \tag{3}$$

with $f = (f_1, \ldots, f_{n-1})$ and $\phi = (\phi_1, \ldots, \phi_{N-n})$ being components of F. Assume that F is CR transversal at 0. Then, following a normalization procedure as in [1, §2], we have

$$\begin{aligned} \tilde{z} = (f_1(z,w), \ldots, f_{n-1}(z,w), \phi_1(z,w), \ldots, \phi_{N-n}(z,w)) = \lambda z U + \mathbf{a} w + O(|(z,w)|^2) \\ \tilde{w} = g(z,w) = \sigma\lambda^2 w + O(|(z,w)|^2). \end{aligned} \tag{4}$$

Here U can be extended to an $(N-1)\times(N-1)$ matrix $\tilde{U} \in SU(N-1,\ell)$ (namely $\langle X\tilde{U}, Y\overline{\tilde{U}}\rangle_\ell = \langle X, Y\rangle_\ell$ for any $X, Y \in \mathbf{C}^{N-1}$), $\mathbf{a} \in \mathbf{C}^{N-1}$ and $\lambda > 0, \sigma = \pm 1$ with $\sigma = 1$ for $\ell < \frac{n-1}{2}$. When $\sigma = -1$, by considering $F \circ \tau_{n-1/2}$ instead of F, where $\tau_{\frac{n-1}{2}}(z_1, \ldots, z_{\frac{n-1}{2}}, z_{\frac{n-1}{2}+1}, \ldots, z_{n-1}, w) = (z_{\frac{n-1}{2}+1}, \ldots, z_{n-1}, z_1, \ldots, z_{\frac{n-1}{2}}, -w)$, we can make $\sigma = 1$. Hence, we will assume in what follows that $\sigma = 1$. Moreover, as in [14], F can be normalized as follows:

Proposition 1 ([14]) *Let M_ℓ and $\tilde{M}_\ell$ be defined by (1) and (2), respectively, and let F be a smooth CR map sending M_ℓ into $\tilde{M}_\ell$ given by (3) and (4) with $\sigma = 1$. Then after composing F from the left by some automorphism $T \in Aut_0(H_\ell^N)$ preserving the origin, the following holds:*

$$F^\sharp = (f^\sharp, \phi^\sharp, g^\sharp) := T \circ F,$$

with

$$\begin{aligned} f^\sharp(z,w) &= z + \tfrac{i}{2} a^{(1,0)}(z) w + o_{wt}(3), \\ \phi^\sharp(z,w) &= \phi^{(2,0)}(z) + o_{wt}(2), \\ g^\sharp(z,w) &= w + o_{wt}(4), \end{aligned}$$

and

$$\langle a^{(1,0)}(z), \bar{z}\rangle_\ell |z|_\ell^2 = |\phi^{(2,0)}(z)|^2 + \frac{1}{4}(\mathscr{S}(z) - \lambda^{-2}\tilde{\mathscr{S}}(\lambda(z,0)\tilde{U})).$$

In particular, the automorphism T is given by

$$T(\tilde{z}, \tilde{w}) = \frac{(\lambda^{-1}(\tilde{z} - \lambda^{-2}\mathbf{a}\tilde{w})\tilde{U}^{-1}, \lambda^{-2}\tilde{w})}{q(\tilde{z}, \tilde{w})}$$

with $r_0 = \frac{1}{2}\Re\{g''_{ww}(0)\}$, $q(\tilde{z}, \tilde{w}) = 1 + 2i\langle \tilde{z}, \lambda^{-2}\overline{\mathbf{a}}\rangle_\ell + \lambda^{-4}(r_0 - i|\mathbf{a}|_\ell^2)\tilde{w}$. *Moreover,* $F^\sharp$ *sends* M_ℓ *into* $\tilde{M}^\sharp := T(\tilde{M}_\ell)$ *given by*

$$\tilde{M}^\sharp = \{(\tilde{z}^\sharp, \tilde{w}^\sharp) \in \mathbf{C}^{N+1} : \Im\tilde{w}^\sharp = |\tilde{z}^\sharp|_\ell^2 + \frac{1}{4}\tilde{\mathscr{S}}^\sharp(\tilde{z}^\sharp) + o(4)\}$$

with $\tilde{\mathscr{S}}^\sharp(z^\sharp) = \lambda^{-2}\tilde{\mathscr{S}}(\lambda z^\sharp \tilde{U})$.

3 A Quantitative Version of a Basic Lemma

In this section, some simple preparation facts will be given without proof at first. In the second part of the section, we will discuss a quantitative version of a lemma obtained in [13], which played crucial role for us to get the convergence in our rescaling argument.

Given a polynomial ϕ, define $\|\phi\|$ to be the maximum modulus of all the coefficients in ϕ. For a given vector-valued polynomial $\phi = (\phi_1, \ldots, \phi_s)$, $\|\phi\| := \max_{1\le j\le s}\|\phi_j\|$. We first refer to a lemma in [15] without proof.

Lemma 1 ([15])

(1) Let $X(z, z)$ *and* $Y(z, z)$ *be two polynomials such that* $X(z, z) = Y(z, z)|z|_\ell^2$. *Then* $\|Y\|$ *is bounded by a constant depending only on* $\|X\|$ *and the degree of* X.

(2) Let $h(z)$ *be a homogeneous holomorphic polynomial of degree* d *in* $z \in \mathbf{C}^n$. *If* $|h(z)| \le c|z|^d$ *on* $\{|z|_\ell^2 = 0\}$, *then* $\|h\| \le C$ *for some* C *depending only on* c *and* d.

In various rigidity problems concerning CR immersions, the following lemma in [13] plays an essential role in deriving key identities to eventually conclude uniqueness:

Lemma 2 ([13]) *Let* $\{\phi_j\}_{j=1}^{n-1}$ *and* $\{\psi_j\}_{j=1}^{n-1}$ *be two families of holomorphic functions in* $\mathbf{C}^n$. *Let* $B(z, \xi)$ *be a real-analytic function in* (z, ξ). *Suppose that*

$$\sum_{j=1}^{n-1}\phi_j(z)\psi_j(\xi) = B(z, \xi)\langle z, \xi\rangle_\ell.$$

Then $B(z, \xi) = \sum_{j=1}^{n-1}\phi_j(z)\psi_j(\xi) = 0$.

We find a quantitative version of the above lemma serves our purpose under this context perfectly well.

Lemma 3 *Let $\{\phi_j\}_{j=1}^{n-1}$ and $\{\psi_j\}_{j=1}^{n-1}$ be two families of holomorphic polynomials of degree k and m in $\mathbf{C}^n$, respectively. Let $H(z,\xi), B(z,\xi)$ be two polynomials in (z,ξ). Suppose that*

$$\sum_{j=1}^{n-1} \phi_j(z)\psi_j(\xi) = H(z,\xi) + B(z,\xi)\langle z,\xi\rangle_\ell$$

and $\|H\| \le C$. Then $\|B\| \le \tilde{C}$ and $\|\sum_{j=1}^{n-1} \phi_j(z)\psi_j(\xi)\| \le \tilde{C}$ with $\tilde{C}$ dependent only on (C,k,m,n).

The proof of the lemma is based on the following algorithm together with Lemma 2. First, let us formulate the algorithm procedure so as to re-adjust two families $\{\phi_j\}_{j=1}^{n-1}$ and $\{\psi_j\}_{j=1}^{n-1}$ in Lemma 3.

Lemma 4 *Let $\phi := \{\phi_j\}_{j=1}^{s}$ and $\psi := \{\psi_j\}_{j=1}^{s}$ be two families of holomorphic polynomials of degree k and m in $\mathbf{C}^n$, respectively. There exist two families $\tilde{\phi} := \{\tilde{\phi}_j\}_{j=1}^{s}$ and $\tilde{\psi} := \{\tilde{\psi}_j\}_{j=1}^{s}$ of holomorphic polynomials of degree k and m in $\mathbf{C}^n$, respectively, such that*

$$\sum_{j=1}^{s} \phi_j(z)\psi_j(\xi) = \sum_{j=1}^{s} \tilde{\phi}_j(z)\tilde{\psi}_j(\xi) \tag{5}$$

and

$$\|\tilde{\phi}\| \le 1, \ \ C\|\tilde{\psi}\| \le \|\sum_{j=1}^{s} \tilde{\phi}_j(z)\tilde{\psi}_j(\xi)\| \le s\|\tilde{\psi}\| \tag{6}$$

for some positive constant C dependent only on s.

Proof of Lemma 4 Without loss of generality, assume $\|\phi_j\| \ne 0$ for all $1 \le j \le s$ and $\{\phi_j\}_{j=1}^{s}$ are linearly independent. Moreover, by replacing ϕ_j and ψ_j by $\frac{\phi_j}{\|\phi_j\|}$ and $\|\phi_j\|\psi_j$, respectively, one can assume that $\|\phi_j\| = 1$ for all $1 \le j \le s$. Denote by $\{e_l\}_{l=1}^{d(k)}$ a basis of unit monomials to span the polynomial spaces of degree k and write $\phi_j = \sum_{1\le l\le d(k)} D_j^l e_l$, $1 \le j \le s$. Here $d(k)$ is the dimension of polynomial spaces of degree k. Hence $\|\phi_j\| = \max_{1\le l\le d(k)} D_j^l$ for each $1 \le j \le s$. Arranging the order of $\{e_l\}$ if necessary, we can make $D_1^1 = 1$ and $|D_1^l| \le 1$.

Step 1: Let ${}^1\phi_1 := \phi_1, {}^1\phi_j := \phi_j - D_j^1 \cdot \phi_1, 2 \le j \le s$. Then in terms of the basis representation ${}^1\phi_j :={}^1 D_j^l \cdot e_l$, one has

$${}^1D_1^1 = 1, \ |{}^1D_1^l| \le 1, \ 2 \le l \le d(k);$$

$$ {}^1D_j^1 = 0,\ |{}^1D_j^l| \le 2,\ 2 \le j \le s,\ \ 2 \le l \le d(k). $$

Moreover, letting ${}^1\psi_1 := \psi_1 + \sum_{j=2}^{s} D_j^1 \cdot \psi_j, {}^1\psi_j := \psi_j, 2 \le j \le s$, then

$$ \sum_{j=1}^{s} \phi_j(z)\psi_j(\xi) = \sum_{j=1}^{s} {}^1\phi_j(z) \cdot {}^1\psi_j(\xi). \tag{7} $$

Step 2: Normalize ${}^1\phi_j, 2 \le j \le s$ by replacing ${}^1\phi_j, {}^1\psi_j$ by $\frac{{}^1\phi_j}{\|{}^1\phi_j\|}$ and $\|{}^1\phi_j\| \cdot {}^1\psi_j$, respectively. By abuse of notation, we still denote them by ${}^1\phi_j, {}^1\psi_j$ and the representation matrix under the basis $\{e_l\}$ by $\{{}^1D_j^l\}$. Moreover, since $\{\phi_j\}_{j=1}^s$ are linearly independent, by rearranging the order of $\{e_l\}_{l=2}^{d(k)}$ if necessary, we have (7) holds with

$$ \begin{aligned} {}^1D_1^1 &= 1,\ |{}^1D_1^l| \le 1,\ 2 \le l \le d(k); \\ {}^1D_2^1 &= 0,\ {}^1D_2^2 = 1,\ |{}^1D_2^l| \le 1,\ 3 \le l \le d(k); \\ {}^1D_j^1 &= 0,\ |{}^1D_j^l| \le 1,\ 3 \le j \le s,\ \ 2 \le l \le d(k) \end{aligned} $$

and for each $1 \le j \le s$,

$$ \max_{1 \le l \le d(k)} {}^1D_j^l = 1. $$

Step 3: Let ${}^2\phi_2 = {}^1\phi_2, {}^2\phi_j := {}^1\phi_j - {}^1D_j^2 \cdot {}^1\phi_2$ for $1 \le j \le s, j \ne 2$. Then in terms of the basis representation ${}^2\phi_j := {}^2D_j^l \cdot e_l$, we deduce

$$ \begin{aligned} {}^2D_1^1 &= 1,\ {}^2D_1^2 = 0,\ |{}^2D_1^l| \le 2,\ 3 \le l \le d(k); \\ {}^2D_2^1 &= 0,\ {}^2D_2^2 = 1,\ |{}^2D_2^l| \le 1,\ 3 \le l \le d(k); \\ {}^2D_j^1 &= 0,\ {}^2D_j^2 = 0,\ |{}^2D_j^l| \le 2,\ 3 \le j \le s,\ \ 3 \le l \le d(k). \end{aligned} $$

Moreover, letting ${}^2\psi_2 := {}^1\psi_2 + \sum_{j \ne 2} {}^1D_j^2 \cdot {}^1\psi_j, {}^2\psi_j := {}^1\psi_j, 1 \le j \le s$ with $j \ne 2$, then

$$ \sum_{j=1}^{s} \phi_j(z)\psi_j(\xi) = \sum_{j=1}^{s} {}^2\phi_j(z) \cdot {}^2\psi_j(\xi). \tag{8} $$

Step 4: Normalize ${}^2\phi_j, 1 \le j \le s, j \ne 2$ by replacing ${}^2\phi_j, {}^2\psi_j$ by $\frac{{}^2\phi_j}{\|{}^2\phi_j\|}$ and $\|{}^2\phi_j\| \cdot {}^2\psi_j$, respectively. As before, we still denote them by ${}^2\phi_j, {}^2\psi_j$ and the representation matrix under the basis $\{e_l\}$ by $\{{}^2D_j^l\}$. Furthermore, (8) holds with

$$1 \geq {}^2D_1^1 \geq \frac{1}{2},\ {}^2D_1^2 = 0,\ |{}^2D_1^l| \leq 1,\ 3 \leq l \leq d(k);$$
$${}^2D_2^1 = 0,\ {}^2D_2^2 = 1,\ |{}^2D_2^l| \leq 1,\ 3 \leq l \leq d(k);$$
$${}^2D_j^1 = 0,\ {}^2D_j^2 = 0,\ |{}^2D_j^l| \leq 1,\ 3 \leq j \leq s,\ \ 3 \leq l \leq d(k)$$

and for each $1 \leq j \leq s$,

$$\max_{1 \leq l \leq d(k)} {}^2D_j^l = 1.$$

Step 5: Continue the above process until we get new families $\{{}^s\phi_j\}_{j=1}^s, \{{}^s\psi_j\}_{j=1}^s$ such that under the basis representation, ${}^s\phi_j := {}^sD_j^l \cdot e_l$ with

$$
{}^sD = \begin{bmatrix}
{}^sD_1^1 & 0 & 0 & \cdots & 0 & {}^sD_1^{s+1} & \cdots & {}^sD_1^{d(k)} \\
0 & {}^sD_2^2 & 0 & \cdots & 0 & {}^sD_2^{s+1} & \cdots & {}^sD_2^{d(k)} \\
0 & 0 & {}^sD_3^3 & \cdots & 0 & {}^sD_2^{s+1} & \cdots & {}^sD_3^{d(k)} \\
 & & & & \cdots & & & \\
 & & & & \cdots & & & \\
 & & & & \cdots & & & \\
0 & 0 & 0 & \cdots & {}^sD_s^s & {}^sD_s^{s+1} & \cdots & {}^sD_s^{d(k)}
\end{bmatrix},
$$

where

$$1 \geq {}^sD_j^j \geq \frac{1}{2^{s-j}},\ 1 \leq j \leq s-1;\ \ {}^sD_s^s = 1;$$

and for each $1 \leq j \leq s$,

$$\max_{1 \leq l \leq d(k)} {}^sD_j^l = 1.$$

Moreover,

$$\sum_{j=1}^{s} \phi_j(z)\psi_j(\xi) = \sum_{j=1}^{s} {}^s\phi_j(z) \cdot {}^s\psi_j(\xi).$$

Let $\tilde{\phi}_j := {}^s\phi_j, \tilde{\psi}_j := {}^s\psi_j, 1 \leq j \leq s$. Then from the construction, for $1 \leq j \leq s$, $\|\tilde{\phi}_j\| = 1$ with $\sum_{j=1}^{s} \phi_j(z)\psi_j(\xi) = \sum_{j=1}^{s} \tilde{\phi}_j(z)\tilde{\psi}_j(\xi)$. Hence

$$\|\sum_{j=1}^{s} \phi_j(z)\psi_j(\xi)\| \leq \sum_{j=1}^{s} \|\tilde{\phi}_j\|\|\tilde{\psi}_j\| \leq s\|\tilde{\psi}\|.$$

Furthermore, since ${}^sD_j^j \geq \frac{1}{2^{s-j}}$ when $1 \leq j \leq s$,

$$\|\sum_{j=1}^{s} \phi_j(z)\psi_j(\xi)\| \geq \max_{1\leq j\leq s} {}^sD_j^j \cdot \|\tilde{\psi}_j\| \geq \frac{1}{2^{s-1}}\|\tilde{\psi}\|.$$

The proof of Lemma 4 is therefore complete. □

Proof of Lemma 3 Assume by contradiction that there exist families of $\{\phi^\lambda\}$ and $\{\psi^\lambda\}$, such that

$$\sum_{j=1}^{n-1} \phi_j^\lambda(z)\psi_j^\lambda(\xi) = H^\lambda(z,\xi) + B^\lambda(z,\xi)\langle z,\xi\rangle_\ell \tag{9}$$

with $\|H^\lambda\| \leq C$ while $\|\sum_{j=1}^{n-1} \phi_j^\lambda(z)\psi_j^\lambda(\xi)\| = \lambda \to \infty$. Applying Lemma 4 to ϕ^λ and ψ^λ if necessary, we can further assume that ϕ^λ and ψ^λ satisfy

$$\|\phi^\lambda\| \leq 1, \quad C\|\psi^\lambda\| \leq \|\sum_{j=1}^{n-1} \phi_j^\lambda(z)\psi_j^\lambda(\xi)\| = \lambda \leq (n-1)\|\psi^\lambda\|.$$

In special, for each $1 \leq j \leq n-1$,

$$\|\phi_j^\lambda\| \leq 1, \quad \frac{1}{n-1} \leq \|\frac{\psi_j^\lambda}{\lambda}\| \leq \frac{1}{C}. \tag{10}$$

Dividing both sides of (9) by λ, then one obtains for some polynomial $\tilde{B}^\lambda$ that

$$\sum_{j=1}^{n-1} \phi_j^\lambda(z)\frac{\psi_j^\lambda(\xi)}{\lambda} = \frac{H^\lambda(z,\xi)}{\lambda} + \tilde{B}^\lambda(z,\xi)\langle z,\xi\rangle_\ell. \tag{11}$$

Since ϕ^λ and ψ^λ satisfy (10), we deduce after passing to a subsequence that ϕ^λ and $\frac{\psi^\lambda}{\lambda}$ converges, say, to polynomials ϕ^∞ and ψ^∞. Moreover, the same inequalities in (10) pass onto ϕ^∞ and ψ^∞ without change, i.e., $\|\phi^\infty\| \leq 1$, $\frac{1}{n-1} \leq \|\psi^\infty\| \leq C$ and $C\|\psi^\infty\| \leq \|\sum_{j=1}^{n-1} \phi_j^\infty(z)\psi_j^\infty(\xi)\| \leq (n-1)\|\psi^\infty\|$.

On the other hand, from (11) and Lemma 1, after passing $\lambda \to \infty$, there exists some polynomial B^∞ such that

$$\sum_{j=1}^{n-1} \phi_j^\infty(z)\psi_j^\infty(\xi) = B^\infty(z,\xi)\langle z,\xi\rangle_\ell.$$

According to Lemma 2, it immediately gives that

$$\sum_{j=1}^{n-1} \phi_j^{\infty}(z)\psi_j^{\infty}(\xi) = 0.$$

This however contradicts with the fact that $\|\sum_{j=1}^{n-1} \phi_j^{\infty}(z)\psi_j^{\infty}(\xi)\| \geq C\|\psi^{\infty}\| \geq \frac{C}{n-1}$. Therefore, there exists some $\tilde{C}$ dependent only on (C, k, m, n) such that $\|\sum_{j=1}^{n-1} \phi_j(z)\psi_j(\xi)\| \leq \tilde{C}$ and hence $\|B\| \leq \tilde{C}$ because of Lemma 1. □

With a routine induction process, Lemmas 1 and 3 combined together can be used to show the following:

Lemma 5 *Let $\{\phi_{jr}\}_{j=1}^{n-1}$ and $\{\psi_{jr}\}_{j=1}^{n-1}$ be two families of holomorphic polynomials in $\mathbf{C}^n$, $1 \leq r \leq m$. Let $H(z,\xi), B(z,\xi)$ be two polynomials in (z,ξ). Suppose that*

$$\sum_{r=1}^{m}\left(\sum_{j=1}^{n-1} \phi_{jr}(z)\psi_{jr}(\xi)\right)\langle z,\xi\rangle_\ell^r = H(z,\xi) + B(z,\xi)\langle z,\xi\rangle_\ell^{m+1}$$

and $\|H\| \leq C$. Then $\|B\| \leq \tilde{C}$ and $\|\sum_{j=1}^{n-1} \phi_{jr}(z)\psi_{jr}(\xi)\| \leq \tilde{C}$ for all $1 \leq r \leq m$ with $\tilde{C}$ dependent only on (C, n, m) and the degrees of ϕ_{jr}, ψ_{jr} for all $1 \leq r \leq m$.

4 Proof of the Main Theorems

The proof of the main theorems is motivated by the ideas in [15] and [10]. Assume F is not CR transversal to M_ℓ at 0 and $F(U) \not\subset H_\ell^N$. Assume also $N - n < n - 1$ for the moment.

By a result of [5], the set of points where the CR transversality holds for such an F forms an open dense subset in M_ℓ. Choose a sequence $\{p_j\} \in M_\ell$ such that $p_j \to 0$ and F is CR transversal at each p_j with $j \geq 1$. Write $q_j := F(p_j)$. Now for each j, applying the normalization process to F at p_j as in Sect. 2, we obtain $F_{p_j}^{\sharp}$ in the following form:

$$F_{p_j}^{\sharp} = (f_{p_j}^{\sharp}, \phi_{p_j}^{\sharp}, g_{p_j}^{\sharp}) = (f_1{}_{p_j}^{\sharp}, \ldots, f_n{}_{p_j}^{\sharp}, \phi_{p_j}^{\sharp}, g_{p_j}^{\sharp}) := T_{p_j} \circ \tau_{F(p_j)} \circ F \circ \sigma_{p_j}, \quad (12)$$

where

$$f^{\sharp}_{p_j}(z,w) = z + \tfrac{i}{2}a^{(1,0)}_{p_j}(z)w + o_{wt}(3),$$
$$\phi^{\sharp}_{p_j}(z,w) = \phi^{(2,0)}_{p_j}(z) + o_{wt}(2),$$
$$g^{\sharp}_{p_j}(z,w) = w + o_{wt}(4),$$

with the following CR Gauss-Codazzi equation

$$\langle a^{(1,0)}_{p_j}(z), \bar{z}\rangle_\ell |z|^2_\ell = |\phi^{(2,0)}_{p_j}(z)|^2 + \frac{1}{4}\mathscr{S}_{p_j}(z). \tag{13}$$

Here $\tau_{F(p_j)}$ is the translation map of H^N_ℓ sending $F(p_j)$ to 0, σ_{p_j} is a biholomorphic map sending 0 to p_j such that $\sigma^{-1}_{p_j}(M_\ell)$ is normalized up to the 4th order, and $\mathscr{S}_{p_j}$ is the resulting Chern-Moser-Weyl curvature function of M_ℓ at p_j. Note σ_{p_j} depends smoothly on p_j. Since F is not CR transversal at 0, $\lim_{j\to\infty}\lambda_{p_j} = 0$ with λ_{p_j} defined in (4) for the map $\tau_{F(p_j)}\circ F\circ\sigma_{p_j}$. By construction, at each point p_j, $F^{\sharp}_{p_j}$ sends $\sigma^{-1}_{p_j}(M_\ell)$ into H^N_ℓ. We then have for $(z,u)\approx 0$,

$$-\Im g^{\sharp}_{p_j}(z, u + i(|z|^2_\ell + o_{wt}(3))) + |f^{\sharp}_{p_j}(z, u + i(|z|^2_\ell + o_{wt}(3)))|^2_\ell + \\ + |\phi^{\sharp}_{p_j}(z, u + i(|z|^2_\ell + o_{wt}(3)))|^2 = 0, \tag{14}$$

Here $(z, u + i(|z|^2_\ell + o_{wt}(3)))$ is a local parametrization of $\sigma^{-1}_{p_j}(M_\ell)$ near 0. Due to the smooth dependence of σ_{p_j} with respect to p_j, the error term $o_{wt}(3)$ depends smoothly on p_j. With an abuse of notation, we shall suppress $\sharp$ and the subindex j of p for the map in (14).

Given any positive integer $k \geq 2$, collect terms of weighted degree k in the power series expansion of (14). We have:

$$\Im g^{(k)}_p(z,w) - 2\Re\langle f^{(k-1)}_p(z,w), \bar{z}\rangle_\ell = (|\phi_p(z,w)|^2)^{(k)} \\ + \quad H(g^{(r)}_p|_{0\leq r\leq k-1}, f^{(r)}_p|_{0\leq r\leq k-2}, (|\phi_p|^2)^{(r)}|_{0\leq r\leq k-1}) \tag{15}$$

on $w = u + i|z|^2_\ell$. Here H is a certain bounded polynomial on all its variables. From now on and in what follows, we use C in general to represent constants independent of p, and use $H(\cdot,\cdot)$ in general to represent polynomials whose norm is bounded by C. C and H may be different in different contexts.

Lemma 6 *Assume that $N - n < n - 1$. For F_p constructed as above and for each k, $\|F^{(k)}_p\| \leq C$ with C independent of p.*

Proof of Lemma 6 According to the normalization procedure conducted in Sect. 2, $\|g^{(k)}_p\| \leq C$, $\|f^{(k-1)}_p\| \leq C$, $\|(|\phi_p|^2)^{(k)}\| \leq C$ automatically hold when $k \leq 4$. Indeed,

$\|g_p^{(k)}\| \leq 1, \|f_p^{(k-2)}\| \leq 1, \|(|\phi_p|^2)^{(k-1)}\| = 0, k \leq 4$ by (12). Moreover, since $\|\mathscr{S}_p\| \leq C$, applying Lemma 3 to (13), one has $\|f_p^{(3)}\| \leq C, \|(|\phi_p|^2)^{(4)}\| \leq C$.

Assuming by induction that $(\|g_p^{(j)}\|, \|f_p^{(j-1)}\|, \|(|\phi_p|^2)^{(j)}\|)$ are all uniformly bounded by some constant independent of p for $j \leq k$, we shall show the unform boundedness of $(\|g_p^{(k+1)}\|, \|f_p^{(k)}\|, \|(|\phi_p|^2)^{(k+1)}\|)$. Complexifying (15) at level $k+1$, we obtain

$$\begin{aligned} g_p^{(k+1)}(z,w) - \bar{g}_p^{(k+1)}(\xi,\eta) - 2i\langle f_p^{(k)}(z,w), \xi\rangle_\ell - 2i\langle \bar{f}_p^{(k)}(\xi,\eta), z\rangle_\ell \\ = 2i\langle \phi_p(z,w), \bar{\phi}_p(\xi,\eta)\rangle^{(k+1)} + H(z,\xi,w,\eta) \end{aligned} \tag{16}$$

which holds on $w - \eta = 2i\langle z,\xi\rangle_\ell$.

Let $L_j = \frac{\partial}{\partial z_j} + 2i\delta_j\xi_j\frac{\partial}{\partial w}, 1 \leq j \leq n-1$ with $\delta_j = -1$ when $j \leq \ell$ and $\delta_j = 1$ with $j \geq \ell + 1$. Then L_j is a holomorphic tangent vector field on $w - \eta = 2i\langle z,\xi\rangle_\ell$ for each j. Applying L_j onto (16), we get

$$\begin{aligned} L_j g_p^{(k+1)}(z,w) - 2i\langle L_j f_p^{(k)}(z,w), \xi\rangle_\ell - 2i\bar{f}_{p,j}^{(k)}(\xi,\eta) \\ = 2iL_j\langle \phi_p(z,w), \bar{\phi}_p(\xi,\eta)\rangle^{(k+1)} + H(z,\xi,w,\eta) \end{aligned} \tag{17}$$

on $w - \eta = 2i\langle z,\xi\rangle_\ell$.

Now we expand $g_p^{(k+1)}, f_p^{(k)}, \langle \phi_p(z,w), \bar{\phi}_p(\xi,\eta)\rangle^{(k+1)}$ in the following manner:

$$g_p^{(k+1)}(z,w) = \sum_{\mu+2\nu=k+1} (g_p)_{\mu\nu}(z)w^\nu;$$

$$f_p^{(k)}(z,w) = \sum_{\mu+2\nu=k} (f_p)_{\mu\nu}(z)w^\nu;$$

$$\langle \phi_p(z,w), \bar{\phi}_p(\xi,\eta)\rangle^{(k+1)} = \sum_{\mu+\gamma+2(\nu+\delta)=k+1} (A_p)_{\mu\gamma\nu\delta}(z,\xi)w^\nu\eta^\delta.$$

Here $(g_p)_{\mu\nu}$ and $(f_p)_{\mu\nu}$ are homogeneous polynomials of degree μ in z, $(A_p)_{\mu\gamma\nu\delta}$ is a homogeneous polynomial of bi-degree (μ,γ) in (z,ξ).

Let $w = 0, \eta = -2i\langle z,\xi\rangle_\ell$ in (16). Then we have

$$\begin{aligned} (g_p)_{(k+1)0}(z) - \sum_{\mu+2\nu=k+1} (\bar{g}_p)_{\mu\nu}(\xi)\eta^\nu - 2i\langle (f_p)_{k0}(z), \xi\rangle_\ell \\ -2i\langle \sum_{\mu+2\nu=k} (\bar{f}_p)_{\mu\nu}(\xi)\eta^\nu, z\rangle_\ell = 2i \sum_{\mu+\gamma+2\delta=k+1} (A_p)_{\mu\gamma 0\delta}(z,\xi)\eta^\delta + H(z,\xi,\eta) \end{aligned} \tag{18}$$

on $\eta = -2i\langle z,\xi\rangle_\ell$.

Collect terms in (z,ξ) of bi-degree $(k+1,0)$ and $(k,1)$ in (18). By the fact that $\phi_p(0) = \frac{\partial\phi_p}{\partial z}(0) = \frac{\partial\phi_p}{\partial w}(0) = 0$ and the definition of $(A_p)_{\mu\gamma\nu\delta}$,

$$(A_p)_{k+1,0,0,0} = (A_p)_{k,1,0,0} = (A_p)_{k-1,0,0,1} = 0. \tag{19}$$

Then we have that

$$\|(f_p)_{k0}\| \leq C, \quad \|(g_p)_{(k+1)0}\| \leq C.$$

Hence for each $1 \leq j \leq n-1$,

$$L_j f_p^{(k)}(z,0) = 2i\delta_j \xi_j \sum_{\mu=k-2} (f_p)_{\mu 1}(z) + H(z);$$

$$L_j g_p^{(k+1)}(z,0) = 2i\delta_j \xi_j \sum_{\mu=k-1} (g_p)_{\mu 1}(z) + H(z). \tag{20}$$

Collecting terms in (z,ξ) of bi-degree $(\alpha,\beta), \beta \geq 2$ with $\alpha+\beta = k+1$ in (18) gives

$$-(\bar{g}_p)_{\beta-\alpha,\alpha}(\xi)\eta^\alpha - 2i\langle z, (\bar{f}_p)_{\beta-\alpha+1,\alpha-1}(\xi)\eta^{\alpha-1}\rangle_\ell$$
$$= 2i\sum_{\theta=0}^{\alpha-2} (A_p)_{\alpha-\theta,\beta-\theta,0,\theta}(z,\xi)\eta^\theta + H(z,\xi,\eta) \tag{21}$$

with $\eta = -2i\langle z,\xi\rangle_\ell$. Here once again we used the fact that $\phi_p(0) = \frac{\partial\phi_p}{\partial z}(0) = 0$ which implies $(A_p)_{1,(\beta-\alpha-1),0,(\alpha-1)} = (A_p)_{0,\beta-\alpha,0,\alpha} = 0$, so the summation on the right hand side runs only till $\theta = \alpha - 2$. Recall from the definition of A_p, $(A_p)_{\mu\gamma\nu\delta}(z,\xi) = \sum_{j=1}^{N-n} \phi_{p,j}^{(\mu,\nu)}(z,1)\bar{\phi}_{p,j}^{(\gamma,\delta)}(\xi,1)$. Since $N-n < n-1$ by assumption, we immediately have, by applying Lemma 5 to (21) and by using (19), that

$$\|(\bar{g}_p)_{\beta-\alpha,\alpha}(\xi)\langle z,\xi\rangle_\ell - \langle z, (\bar{f}_p)_{\beta-\alpha+1,\alpha-1}(\xi)\rangle_\ell\| \leq C \tag{22}$$

with $\beta \geq 2$, and

$$\|(A_p)_{\mu\gamma 0\delta}\| \leq C.$$

Hence from the above inequality,

$$L_j(A_p)(z,\xi,0,\eta) = 2i\delta_j\xi_j \sum_{\mu+\gamma+2\delta=k-1} (A_p)_{\mu\gamma 1\delta}(z,\xi)\eta^\delta + H(z,\xi,\eta). \tag{23}$$

Letting $w = 0, \eta = -2i\langle z,\xi\rangle_\ell$ and then substituting (20) and (23) in (17), we have for each $1 \leq j \leq n-1$,

$$2i\delta_j\xi_j \sum_{\mu=k-1} (g_p)_{\mu 1}(z) - 2i\langle 2i\delta_j\xi_j \sum_{\mu=k-2} (f_p)_{\mu 1}(z), \xi\rangle_\ell - 2i \sum_{\mu+2\nu=k+1} (\bar{f}_{p,j})_{\mu\nu}(\xi)\eta^\nu$$
$$= 2i\delta_j\xi_j \sum_{\mu+\gamma+2\delta=k-1} (A_p)_{\mu\gamma 1\delta}(z,\xi)\eta^\delta + H(z,\xi,\eta) \tag{24}$$

on $\eta = -2i\langle z, \xi\rangle_\ell$. Collect terms in (z, ξ) of bi-degree $(k-1, 1)$ and $(k-2, 2)$ in (24). Since $(A_p)_{k-1,0,1,0} = (A_p)_{k-3,0,1,1} = 0$, one obtains that

$$\|(g_p)_{(k-1)1}\| \le C,$$
$$\|2i\delta_j\langle \xi_j (f_p)_{(k-2)1}(z), \xi\rangle_\ell + (\bar{f}_{p,j})_{(4-k)(k-2)}(\xi)(-2i\langle z, \xi\rangle_\ell)^{k-2}\| \le C. \tag{25}$$

Here we have used the convention that $h_\mu = 0$ if μ is negative.

Moreover, collecting terms of bi-degree (α, β) in (z, ξ) with $\beta \ge 3$ and $\alpha+\beta = k$ in (24), one gets for each $1 \le j \le n$,

$$(\bar{f}_{p,j})_{(\beta-\alpha)\alpha}(\xi)(-2i\langle z, \xi\rangle_\ell)^\alpha$$
$$= -\delta_j\xi_j \sum_{\theta=0}^{\alpha-1} (A_p)_{(\alpha-\theta)(\beta-\theta-1)1\theta}(z, \xi)(-2i\langle z, \xi\rangle_\ell)^\theta + H(z, \xi).$$

Here we use the fact that $(A_p)_{0,\beta-\alpha-1,1,\alpha} = 0$, so the summation on the right hand sides runs only till $\alpha - 1$. Applying Lemma 5 onto the above identity as before, we obtain $\|(f_p)_{\mu\nu}\| \le C$ for $\mu + 2\nu = k$ with $\mu + \nu \ge 3$. When $\mu + 2\nu = k \ge 4$ with $\mu + \nu \le 2$, or equivalently, when $\mu = 0, \nu = 2$, one substitutes the fact that $\|(f_p)_{21}\| \le C$ into (25) and gets $\|(f_p)_{02}\| \le C$ and hence

$$\|(f_p)_{\mu\nu}\| \le C \tag{26}$$

for $\mu + 2\nu = k$. Substitute (26) into (22), then

$$\|(g_p)_{\mu\nu}\| \le C \tag{27}$$

for $\mu + 2\nu = k + 1$ (with $\mu + \nu \ge 3$, which is always fulfilled when $\mu + 2\nu = k + 1 \ge 5$).

Using Eq. (16), we then have from (26) and (27),

$$\langle \phi_p(z, w), \bar{\phi}_p(\xi, \eta)\rangle^{(k+1)} = H(z, \xi, w, \eta) \tag{28}$$

on $w - \eta = 2i\langle z, \xi\rangle_\ell$.

We claim that, for arbitrary (z, w, ξ, η), we have

$$\langle \phi_p(z, w), \bar{\phi}_p(\xi, \eta)\rangle^{(k+1)} = H(z, \xi, w, \eta).$$

Indeed, by (28), we have

$$\sum_{\mu+\gamma+2(\nu+\delta)=k+1} (A_p)_{\mu\gamma\nu\delta}(z, \xi)\left(\eta + 2i\langle z, \xi\rangle_\ell\right)^\nu \eta^\delta = H(z, \xi, \eta) \tag{29}$$

near 0. If $\|(A_p)_{\mu\gamma\nu\delta}\| \le C$ does not hold uniformly in p, then there exists a smallest integer δ_0 such that $\|(A_p)_{\mu\gamma\nu\delta_0}\| \to \infty$ as $p \to 0$ after passing to a subsequence if necessary. Moving the terms with $\delta < \delta_0$ to the right, we obtain

$$\sum_{\mu+\gamma+2(\nu+\delta_0)=k+1} (A_p)_{\mu\gamma\nu\delta_0}(z,\xi)\big(2i\langle z,\xi\rangle_\ell\big)^\nu = H(z,\xi).$$

Collecting terms in (z,ξ) of bi-degree (α,β) with $\alpha+\beta = k+1-2\delta_0$ in the above expression, we get

$$\sum_{\theta=0}^{\alpha}(A_p)_{(\alpha-\theta)(\beta-\theta)\theta\delta_0}(z,\xi)\big(2i\langle z,\xi\rangle_\ell\big)^\theta = H(z,\xi).$$

Recall $(A_p)_{\mu\gamma\nu\delta} = \sum_{j=1}^{N-n}\phi_{p,j}^{(\mu,\nu)}(z,1)\bar{\phi}_{p,j}^{(\gamma,\delta)}(\xi,1)$ by definition and $N-n < n-1$. Applying Lemma 5 to the above identity, one deduces $\|(A_p)_{\mu\gamma\nu\delta_0}\| \le C$ for $\mu+\gamma+2(\nu+\delta_0) = k+1$. This is a contradiction! Hence the claim holds.

The induction is thus complete. Consequently, for each k, $\|\phi_p^{(k)}\| \le C$ with C independent of p. We have shown for each fixed k, $\{\|F_{p_j}^{(k)}\|\}_{j=1}^\infty$ is bounded by some constant independent of j. □

We are now in a position to prove Theorems 1 and 2, making use of the result in [16, 17].

Proof of Theorem 2 If F is CR transversal to M_ℓ at 0, then we are done. Assume F is not CR transversal at 0. Then there exists $p_j \to 0$ such that F_{p_j} as constructed at the beginning of the section satisfies (12). Moreover, for each k, $\|F_{p_j}^{(k)}\| \le C$ with C independent of j by Lemma 6. Following the same trick as in [15], for each k, $\{F_{p_j}^{(k)}\}_{j=1}^\infty$ converges as $j \to \infty$ after passing to subsequences, to a certain $F^{*(k)}$. By the way these maps were constructed, the nontrivial formal map $F^*(= (f^*,\phi^*,g^*)) := \sum_k F^{*(k)}$ sends M_ℓ into H_ℓ^N satisfying the following normalization:

$$f^*(z,w) = z + o_{wt}(2),$$
$$\phi^*(z,w) = o_{wt}(1),$$
$$g^*(z,w) = w + o_{wt}(4).$$

According to a result of Meylan-Mir-Zaitsev [16], the formal map F^* is convergent. Hence, F^* is a CR immersion at 0 sending M_ℓ into H_ℓ^N. □

Proof of Theorem 1 Assume by contradiction that F neither is CR transversal to M_ℓ at 0 nor sends U into H_ℓ^N. Then there exists a CR immersion F^* sending M_ℓ into H_ℓ^N by Theorem 2. On the other hand, since any two CR transversal maps between a Levi-nondegenerate hypersurface and a hyperquadric of the same signature differ only by an automorphism of the hyperquadric (see [10]) when the codimension is less than $\frac{n-1}{2}$, there exists an automorphism T of H_ℓ^N such that near $p_j \approx 0$, and hence at all points in M_ℓ near the origin,

$$F = T \circ F^*.$$

Since T extends to an automorphism of the projective space $\mathbf{P}^N$ and $T(0) = 0$, F must be CR transversal at 0. This is a contradiction. □

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Rationality in Differential Algebraic Geometry

Joël Merker

1 Rationality

The natural integer numbers:

$$1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10,\ 11,\ 12,\ 13,\ \ldots,\ 2013,\ \ldots\ldots$$

necessarily hint at some '*invention*' of the Zero. While for the Greeks, the actual '∞' and the actual '0' did not '*exist*', the Babylonians used the symbol '0' in numeration. In India (cf. e.g. Brahmagupta), the zero comes from self-subtraction:

$$0 \overset{\text{def}}{:=} \mathbf{a} - \mathbf{a}.$$

In rational numbers:

$$\frac{p}{q} \qquad\qquad (q \neq 0),$$

division by zero is and must be excluded.

The present paper aims at showing that *higher abstract conceptions in advanced mathematics depend upon archetypical rational computational phenomena.*

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Several instances of deeper rationality facts will hence be surveyed:

- ☐ In Cartan's theory of exterior differential systems;
- ☐ In Complex Algebraic Geometry.

2 Equivalences of 5-Dimensional CR Manifolds

> Despite their importance, until now, the invariants of strictly pseudoconvex domains have been fully computed, to our knowledge, only in the case of the unit ball $\mathbb{B}^{n+1}$, where they all vanish!
> Sidney Webster [73].

Real analytic ($\mathscr{C}^\omega$) *CR-generic* submanifolds $M \subset \mathbb{C}^{n+c}$ of codimension $c \geqslant 0$ are those satisfying $TM + J(TM) = T\mathbb{C}^{n+c}\big|_M$, where $J \colon T\mathbb{C}^{n+c} \longrightarrow T\mathbb{C}^{n+c}$ denotes the standard complex structure and then $TM \cap J(TM)$ has constant real dimension $2n =: 2\,\mathsf{CRdim}\,M$, while $\mathsf{dim}_{\mathbb{R}} M = 2n + c$; general $\mathscr{C}^\omega$ CR submanifolds $M \subset \mathbb{C}^\nu$, i.e. those for which $\mathsf{dim}\,\big(T_pM \cap J(T_pM)\big)$ is constant for $p \in M$, are always locally CR-generic in some complex submanifold [50], hence CR-genericity is not a restriction.

Problem 1 *Classify local $\mathscr{C}^\omega$ CR-generic submanifolds $M^{2n+c} \subset \mathbb{C}^{n+c}$ under local biholomorphisms of $\mathbb{C}^{n+c}$ up to dimension $2n + c \leqslant \mathbf{5}$.*

If $c = 0$, then $M \cong \mathbb{C}^n$, where '$\cong$' means '*locally biholomorphic*'; if $n = 0$, then $M \cong \mathbb{R}^c$. Assume therefore $c \geqslant 1$ and $n \geqslant 1$. The possible CR dimensions and real codimensions are:

$$2n + c = \mathbf{3} \quad \Longrightarrow \quad \big\{ n = \mathbf{1}, \quad c = \mathbf{1},$$

$$2n + c = \mathbf{4} \quad \Longrightarrow \quad \big\{ n = \mathbf{1}, \quad c = \mathbf{2},$$

$$2n + c = \mathbf{5} \quad \Longrightarrow \quad \begin{cases} n = \mathbf{1}, & c = \mathbf{3}, \\ n = \mathbf{2}, & c = \mathbf{1}. \end{cases}$$

In local coordinates $\big(z_1, \dots, z_n, w_1, \dots, w_c\big) = \big(x_1 + \sqrt{-1}\,y_1, \dots, x_n + \sqrt{-1}\,y_n,\ u_1 + \sqrt{-1}\,v_1, \dots, u_c + \sqrt{-1}\,v_c\big)$, represent with graphing $\mathscr{C}^\omega$ functions $\varphi_\bullet$:

$$M^3 \subset \mathbb{C}^2\colon \qquad \Big[\ v = \varphi(x, y, u),$$

$$M^4 \subset \mathbb{C}^3\colon \qquad \left[\begin{aligned} v_1 &= \varphi_1(x, y, u_1, u_2), \\ v_2 &= \varphi_2(x, y, u_1, u_2), \end{aligned}\right.$$

$$M^5 \subset \mathbb{C}^4\colon \quad \left[\begin{aligned} v_1 &= \varphi_1(x, y, u_1, u_2, u_3), \\ v_2 &= \varphi_2(x, y, u_1, u_2, u_3), \\ v_3 &= \varphi_3(x, y, u_1, u_2, u_3), \end{aligned} \right.$$

$$M^5 \subset \mathbb{C}^3\colon \quad \big[\; v = \varphi(x_1, y_1, x_2, y_2, u).$$

Before proceeding further, answer (partly) Webster's quote.

2.1 Explicit Characterization of Sphericity

Consider for instance a hypersurface $M^3 \subset \mathbb{C}^2$. As its graphing function φ is real analytic, the equation can be solved explicitly for w [42, 50]:

$$w = \Theta\big(z, \bar{z}, \overline{w}\big).$$

Letting the 'round' unit 3-sphere $S^3 \subset \mathbb{C}^2$ be:

$$1 = z\bar{z} + w\overline{w} = x^2 + y^2 + u^2 + v^2,$$

a Cayley transform [42] maps $S^3 \backslash \{p_\infty\}$ with $p_\infty := (0, -1)$ biholomorphically onto the *Heisenberg sphere*:

$$w = \overline{w} + 2\sqrt{-1}\, z\bar{z}.$$

An intrinsic local generator for the fundamental subbundle:

$$T^{1,0}M := \big\{X - \sqrt{-1}\, J(X)\colon X \in TM \cap J(TM)\big\}$$

of $TM \otimes_{\mathbb{R}} \mathbb{C}$ is:

$$L = \frac{\partial}{\partial z}.$$

Also, an intrinsic generator for $T^{0,1}M := \overline{T^{1,0}M}$ is:

$$\overline{L} := \frac{\partial}{\partial \bar{z}} - \frac{\Theta_{\bar{z}}(z, \bar{z}, \overline{w})}{\Theta_{\overline{w}}(z, \bar{z}, \overline{w})}\, \frac{\partial}{\partial \overline{w}}.$$

In the Lie bracket:

$$\big[L, \overline{L}\big] = \left[\frac{\partial}{\partial z}, \frac{\partial}{\partial \bar{z}} - \frac{\Theta_{\bar{z}}}{\Theta_{\overline{w}}}\, \frac{\partial}{\partial \overline{w}}\right] = \left(\frac{-\,\Theta_{\overline{w}}\, \Theta_{z\bar{z}} + \Theta_{\bar{z}}\, \Theta_{z\overline{w}}}{\Theta_{\overline{w}}\, \Theta_{\overline{w}}}\right) \frac{\partial}{\partial \overline{w}},$$

an explicit *Levi factor in coordinates* appears:

$$\frac{-\boldsymbol{\Theta}_{\overline{w}}\,\boldsymbol{\Theta}_{z\overline{z}}+\boldsymbol{\Theta}_{\overline{z}}\,\boldsymbol{\Theta}_{z\overline{w}}}{\boldsymbol{\Theta}_{\overline{w}}\,\boldsymbol{\Theta}_{\overline{w}}}.$$

The assumption that $M^3 \subset \mathbb{C}^2$ is smooth reads:

$$0 \neq \boldsymbol{\Theta}_{\overline{w}} \quad \text{vanishes nowhere.}$$

The assumption that M is Levi nondegenerate reads:

$$0 \neq -\boldsymbol{\Theta}_{\overline{w}}\,\boldsymbol{\Theta}_{z\overline{z}}+\boldsymbol{\Theta}_{\overline{z}}\,\boldsymbol{\Theta}_{z\overline{w}} \quad \text{also vanishes nowhere.}$$

General principle *Various geometric assumptions enter into the computations in the denominators.*

Here is a first illustration.

Theorem 1 *An arbitrary real analytic hypersurface $M^3 \subset \mathbb{C}^2$ which is Levi nondegenerate:*

$$w = \Theta\big(z,\,\overline{z},\,\overline{w}\big),$$

is locally biholomorphically equivalent to the Heisenberg sphere if and only if:

$$0 \equiv \left(\frac{-\,\Theta_{\overline{w}}}{\Theta_{\overline{z}}\Theta_{z\overline{w}}-\Theta_{\overline{w}}\Theta_{z\overline{z}}}\,\frac{\partial}{\partial\overline{z}}+\frac{\Theta_{\overline{z}}}{\Theta_{\overline{z}}\Theta_{z\overline{w}}-\Theta_{\overline{w}}\Theta_{z\overline{z}}}\,\frac{\partial}{\partial\overline{w}}\right)^2\big[\mathsf{AJ}^4(\Theta)\big]$$

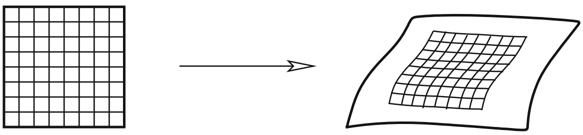
identically in $\mathbb{C}\{z,\overline{z},\overline{w}\}$, where:

$$\begin{aligned}
\mathsf{AJ}^4(\Theta) := \frac{1}{[\boldsymbol{\Theta}_{\overline{z}}\boldsymbol{\Theta}_{z\overline{w}}-\boldsymbol{\Theta}_{\overline{w}}\boldsymbol{\Theta}_{z\overline{z}}]^3}&\Bigg\{\Theta_{zz\overline{z}\overline{z}}\left(\Theta_{\overline{w}}\Theta_{\overline{w}}\begin{vmatrix}\Theta_{\overline{z}} & \Theta_{\overline{w}}\\ \Theta_{z\overline{z}} & \Theta_{z\overline{w}}\end{vmatrix}\right)-\\
&-2\Theta_{zz\overline{z}\overline{w}}\left(\Theta_{\overline{z}}\Theta_{\overline{w}}\begin{vmatrix}\Theta_{\overline{z}} & \Theta_{\overline{w}}\\ \Theta_{z\overline{z}} & \Theta_{z\overline{w}}\end{vmatrix}\right)+\Theta_{zz\overline{w}\overline{w}}\left(\Theta_{\overline{z}}\Theta_{\overline{z}}\begin{vmatrix}\Theta_{\overline{z}} & \Theta_{\overline{w}}\\ \Theta_{z\overline{z}} & \Theta_{z\overline{w}}\end{vmatrix}\right)+\\
&+\Theta_{zz\overline{z}}\left(\Theta_{\overline{z}}\Theta_{\overline{z}}\begin{vmatrix}\Theta_{\overline{w}} & \Theta_{\overline{w}\overline{w}}\\ \Theta_{z\overline{w}} & \Theta_{z\overline{w}\overline{w}}\end{vmatrix}-2\Theta_{\overline{z}}\Theta_{\overline{w}}\begin{vmatrix}\Theta_{\overline{w}} & \Theta_{\overline{z}\overline{w}}\\ \Theta_{z\overline{w}} & \Theta_{z\overline{z}\overline{w}}\end{vmatrix}+\Theta_{\overline{w}}\Theta_{\overline{w}}\begin{vmatrix}\Theta_{\overline{w}} & \Theta_{\overline{z}\overline{z}}\\ \Theta_{z\overline{w}} & \Theta_{z\overline{z}\overline{z}}\end{vmatrix}\right)+\\
&+\Theta_{zz\overline{w}}\left(-\Theta_{\overline{z}}\Theta_{\overline{z}}\begin{vmatrix}\Theta_{\overline{z}} & \Theta_{\overline{w}\overline{w}}\\ \Theta_{z\overline{z}} & \Theta_{z\overline{w}\overline{w}}\end{vmatrix}+2\Theta_{\overline{z}}\Theta_{\overline{w}}\begin{vmatrix}\Theta_{\overline{z}} & \Theta_{\overline{z}\overline{w}}\\ \Theta_{z\overline{z}} & \Theta_{z\overline{z}\overline{w}}\end{vmatrix}-\Theta_{\overline{w}}\Theta_{\overline{w}}\begin{vmatrix}\Theta_{\overline{z}} & \Theta_{\overline{z}\overline{z}}\\ \Theta_{z\overline{z}} & \Theta_{z\overline{z}\overline{z}}\end{vmatrix}\right)\Bigg\}.
\end{aligned}$$

In fact, $\boldsymbol{\Theta}_{\overline{w}}$ *also* enters into the denominator, but erases in the equation '$= 0$'.

2.2 Theorema Egregium of Gauss

Briefly, here is a second illustration. On an embedded surface $S^2 \subset \mathbb{R}^3 \ni (x, y, z)$, consider local curvilinear bidimensional coordinates (u, v):



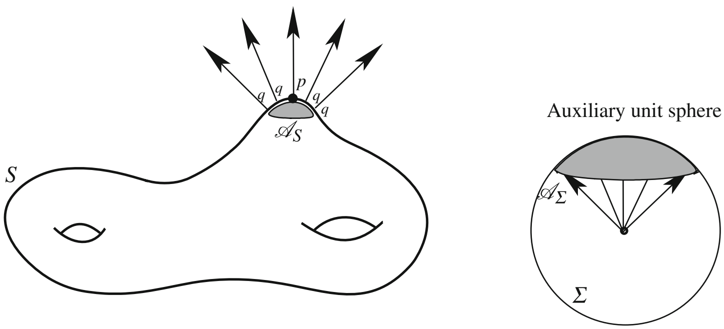
through parametric equations

$$x = x(u, v), \qquad y = y(u, v), \qquad z = z(u, v).$$

The flat Pythagorean metric $dx^2 + dy^2 + dz^2$ on $\mathbb{R}^3$ induces on S^2:

$$ds^2 = \big|(du, dv)\big|^2 = E\,du^2 + 2F\,dudv + G\,dv^2,$$

$$\text{with:} \quad E := x_u^2 + y_u^2 + z_u^2, \quad F := x_u x_v + y_u y_v + z_u z_v, \quad G := x_v^2 + y_v^2 + z_v^2.$$



The *Gaussian curvature* of S at one of its points p is:

$$\mathsf{Curvature}(p) := \lim_{\mathscr{A}_S \to p} \frac{\text{area of the region } \mathscr{A}_\Sigma \text{ on the auxiliary unit sphere}}{\text{area of the region } \mathscr{A}_S \text{ on the surface}}.$$

When S is graphed as $z = \varphi(x, y)$, a first formula is:

$$\mathsf{Curvature} = \frac{\varphi_{xx}\,\varphi_{yy} - \varphi_{xy}\,\varphi_{xy}}{1 + \varphi_x^2 + \varphi_y^2}.$$

A splendid computation by Gauss established its intrinsic meaning:

$$\begin{aligned}\text{Curvature} = \frac{1}{(EG-F^2)^2}\bigg\{ & E\left[\frac{\partial E}{\partial v}\cdot\frac{\partial G}{\partial v} - 2\,\frac{\partial F}{\partial u}\cdot\frac{\partial G}{\partial v} + \left(\frac{\partial G}{\partial u}\right)^2\right] + \\ & + F\left[\frac{\partial E}{\partial u}\cdot\frac{\partial G}{\partial v} - \frac{\partial E}{\partial v}\cdot\frac{\partial G}{\partial u} - 2\,\frac{\partial E}{\partial v}\cdot\frac{\partial F}{\partial v} + 4\,\frac{\partial F}{\partial u}\cdot\frac{\partial F}{\partial v} - 2\,\frac{\partial F}{\partial u}\cdot\frac{\partial G}{\partial u}\right] + \\ & + G\left[\frac{\partial E}{\partial u}\cdot\frac{\partial G}{\partial u} - 2\,\frac{\partial E}{\partial u}\cdot\frac{\partial F}{\partial v} + \left(\frac{\partial E}{\partial v}\right)^2\right] - \\ & - 2\,\big(EG-F^2\big)\left[\frac{\partial^2 E}{\partial v^2} - 2\,\frac{\partial^2 F}{\partial u\partial v} + \frac{\partial^2 G}{\partial u^2}\right]\bigg\},\end{aligned}$$

and, in the denominator appears $E\,G - F^2$ which is > 0 in any metric.

2.3 Propagation of Sphericity Across Levi Degenerate Points

Here is one application of *explicit rational expressions* as above. Developing Pinchuk's techniques of extension along Segre varieties [61], Kossovskiy-Shafikov [33] showed that local biholomorphic equivalence to a model $(k, n-k)$*-pseudo-sphere:*

$$w = \overline{w} + 2i\big(-z_1\overline{z}_1 - \cdots - z_k\overline{z}_k + z_{k+1}\overline{z}_{k+1} + \cdots + z_n\overline{z}_n\big),$$

propagates on any connected real analytic hypersurface $M \subset \mathbb{C}^{n+1}$ which is Levi nondegenerate outside some n-dimensional complex hypersurface $\Sigma \subset M$. A more general statement, not known with Segre varieties techniques, is:

Theorem 2 ([45]) *If a connected $\mathscr{C}^\omega$ hypersurface $M \subset \mathbb{C}^{n+1}$ is locally biholomorphic, in a neighborhood of one of its points p, to some $(k, n-k)$-pseudo-sphere, then locally at every other Levi nondegenerate point $q \in M\backslash\Sigma_{\mathsf{LD}}$, this hypersurface M is also locally biholomorphic to some Heisenberg $(l, n-l)$-pseudo-sphere, with [33], possibly $l \neq k$.*

The proof, suggested by Beloshapka, consists first for $n = 1$ in observing that after expansion, Theorem 1 characterizes sphericity as:

$$0 \equiv \frac{\mathsf{polynomial}\big(\big(\Theta_{z_j\overline{z}_k\overline{w}_l}\big)_{1\leqslant j+k+l\leqslant 6}\big)}{\big[\Theta_{\overline{z}}\,\Theta_{z\overline{w}} - \Theta_{\overline{w}}\,\Theta_{z\overline{z}}\big]^7},$$

at every Levi nondegenerate point $(z_p, w_p) \in M$ at which the denominator is $\neq 0$. But this means that the numerator is $\equiv 0$ near $(z_p, \overline{z}_p, \overline{w}_p)$, and at every other Levi nondegenerate point $(z_q, w_q) \in M$ close to (z_p, w_p), the numerator is also locally $\equiv 0$

by analytic continuation. Small translations of coordinates are needed; the complete arguments appear in [45].

In dimensions $n \geqslant 2$, the explicit characterization of $(k, n-k)$-pseudo-sphericity is also *rational*. Indeed, in local holomorphic coordinates:

$$t = (z, w) \in \mathbb{C}^n \times \mathbb{C},$$

represent similarly a $\mathscr{C}^\omega$ hypersurface $M^{2n+1} \subset \mathbb{C}^{n+1}$ as:

$$w = \Theta\big(z, \bar{z}, \overline{w}\big) = \Theta\big(z, \bar{t}\big).$$

Introduce the Levi form Jacobian-like determinant:

$$\Delta := \begin{vmatrix} \Theta_{\bar{z}_1} & \cdots & \Theta_{\bar{z}_n} & \Theta_{\overline{w}} \\ \Theta_{z_1\bar{z}_1} & \cdots & \Theta_{z_1\bar{z}_n} & \Theta_{z_1\overline{w}} \\ \cdot\cdot & \cdots & \cdot\cdot & \cdot\cdot \\ \Theta_{z_n\bar{z}_1} & \cdots & \Theta_{z_n\bar{z}_n} & \Theta_{z_n\overline{w}} \end{vmatrix} = \begin{vmatrix} \Theta_{\bar{t}_1} & \cdots & \Theta_{\bar{t}_n} & \Theta_{\bar{t}_{n+1}} \\ \Theta_{z_1\bar{t}_1} & \cdots & \Theta_{z_1\bar{t}_n} & \Theta_{z_1\bar{t}_{n+1}} \\ \cdot\cdot & \cdots & \cdot\cdot & \cdot\cdot \\ \Theta_{z_n\bar{t}_1} & \cdots & \Theta_{z_n\bar{t}_n} & \Theta_{z_n\bar{t}_{n+1}} \end{vmatrix}.$$

It is nonzero at a point $p = (z_p, \bar{t}_p)$ if and only if M is Levi nondegenerate at p. For any index $\mu \in \{1, \ldots, n, n+1\}$ and for any index $\ell \in \{1, \ldots, n\}$, let also $\Delta^{\mu}_{[0_{1+\ell}]}$ denote the same determinant, but with its μ-th column replaced by the transpose of the line $(0 \cdots 1 \cdots 0)$ with 1 at the $(1+\ell)$-th place, and 0 elsewhere, its other columns being untouched. Similarly, for any indices $\mu, \nu, \tau \in \{1, \ldots, n, n+1\}$, denote by $\Delta^{\tau}_{[\bar{t}_\mu \bar{t}_\nu]}$ the same determinant as Δ, but with only its τ-th column replaced by the transpose of the line:

$$\big(\Theta_{\bar{t}_\mu \bar{t}_\nu} \quad \Theta_{z_1\bar{t}_\mu\bar{t}_\nu} \quad \cdots \quad \Theta_{z_n\bar{t}_\mu\bar{t}_\nu}\big),$$

other columns being again untouched. All these determinants Δ, $\Delta^{\mu}_{[0_{1+\ell}]}$, $\Delta^{\tau}_{[\bar{t}_\mu\bar{t}_\nu]}$ depend upon the third-order jet $J^3_{z,\bar{z},\overline{w}}\Theta$.

Theorem 3 ([40, 45]) *A $\mathscr{C}^\omega$ hypersurface $M \subset \mathbb{C}^{n+1}$ with $n \geqslant 2$ which is Levi nondegenerate at some point $p = (z_p, \bar{z}_p, \overline{w}_p)$ is $(k, n-k)$-pseudo-spherical at p for some k if and only if, identically for $(z, \bar{z}, \overline{w})$ near $(z_p, \bar{z}_p, \overline{w}_p)$:*

$$\begin{aligned} 0 \equiv \frac{1}{\Delta^3} \Bigg[\sum_{\mu=1}^{n+1} \sum_{\nu=1}^{n+1} \Bigg[\Delta^{\mu}_{[0_{1+\ell_1}]} \cdot \Delta^{\nu}_{[0_{1+\ell_2}]} \Bigg\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{k_1} \partial z_{k_2} \partial \bar{t}_\mu \partial \bar{t}_\nu} - \sum_{\tau=1}^{n+1} \Delta^{\tau}_{[\bar{t}_\mu \bar{t}_\nu]} \cdot \frac{\partial^3 \Theta}{\partial z_{k_1} \partial z_{k_2} \partial \bar{t}_\tau} \Bigg\} - \\ - \tfrac{\delta_{k_1, \ell_1}}{n+2} \sum_{\ell'=1}^{n} \Delta^{\mu}_{[0_{1+\ell'}]} \cdot \Delta^{\nu}_{[0_{1+\ell_2}]} \Bigg\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{\ell'} \partial z_{k_2} \partial \bar{t}_\mu \partial \bar{t}_\nu} - \sum_{\tau=1}^{n+1} \Delta^{\tau}_{[\bar{t}_\mu \bar{t}_\nu]} \cdot \frac{\partial^3 \Theta}{\partial z_{\ell'} \partial z_{k_2} \partial \bar{t}_\tau} \Bigg\} - \\ - \tfrac{\delta_{k_1, \ell_2}}{n+2} \sum_{\ell'=1}^{n} \Delta^{\mu}_{[0_{1+\ell_1}]} \cdot \Delta^{\nu}_{[0_{1+\ell'}]} \Bigg\{ \Delta \cdot \frac{\partial^4 \Theta}{\partial z_{\ell'} \partial z_{k_2} \partial \bar{t}_\mu \partial \bar{t}_\nu} - \sum_{\tau=1}^{n+1} \Delta^{\tau}_{[\bar{t}_\mu \bar{t}_\nu]} \cdot \frac{\partial^3 \Theta}{\partial z_{\ell'} \partial z_{k_2} \partial \bar{t}_\tau} \Bigg\} - \end{aligned}$$

$$
\begin{aligned}
&-\tfrac{\delta_{k_2,\ell_1}}{n+2}\sum_{\ell'=1}^{n}\Delta^{\mu}_{[0_{1+\ell'}]}\cdot\Delta^{\nu}_{[0_{1+\ell_2}]}\bigg\{\Delta\cdot\frac{\partial^4\Theta}{\partial z_{k_1}\partial z_{\ell'}\partial\bar t_\mu\partial\bar t_\nu}-\sum_{\tau=1}^{n+1}\Delta^{\tau}_{[\bar t_\mu\bar t_\nu]}\cdot\frac{\partial^3\Theta}{\partial z_{k_1}\partial z_{\ell'}\partial\bar t_\tau}\bigg\}-\\
&-\tfrac{\delta_{k_2,\ell_2}}{n+2}\sum_{\ell'=1}^{n}\Delta^{\mu}_{[0_{1+\ell_1}]}\cdot\Delta^{\nu}_{[0_{1+\ell'}]}\bigg\{\Delta\cdot\frac{\partial^4\Theta}{\partial z_{k_1}\partial z_{\ell'}\partial\bar t_\mu\partial\bar t_\nu}-\sum_{\tau=1}^{n+1}\Delta^{\tau}_{[\bar t_\mu\bar t_\nu]}\cdot\frac{\partial^3\Theta}{\partial z_{k_1}\partial z_{\ell'}\partial\bar t_\tau}\bigg\}+\\
&\quad+\tfrac{1}{(n+1)(n+2)}\cdot\big[\delta_{k_1,\ell_1}\delta_{k_2,\ell_2}+\delta_{k_2,\ell_1}\delta_{k_1,\ell_2}\big]\cdot\\
&\quad\cdot\sum_{\ell'=1}^{n}\sum_{\ell''=1}^{n}\Delta^{\mu}_{[0_{1+\ell'}]}\cdot\Delta^{\nu}_{[0_{1+\ell''}]}\bigg\{\Delta\cdot\frac{\partial^4\Theta}{\partial z_{\ell'}\partial z_{\ell''}\partial\bar t_\mu\partial\bar t_\nu}-\sum_{\tau=1}^{n+1}\Delta^{\tau}_{[\bar t_\mu\bar t_\nu]}\cdot\frac{\partial^3\Theta}{\partial z_{\ell'}\partial z_{\ell''}\partial\bar t_\tau}\bigg],\\
&\qquad\qquad\qquad\qquad(1\leqslant k_1,\,k_2\leqslant n;\ \ 1\leqslant\ell_1,\,\ell_2\leqslant n).
\end{aligned}
$$

Then as in the case $n = 1$, propagation of pseudo-sphericity 'jumps' across Levi degenerate points, because above, the denominator Δ^3 locates Levi nondegenerate points. This explicit expression is a translation of Hachtroudi's characterization [29] of equivalence to $w'_{z'_{k_1}z'_{k_2}}(z') = 0$ of completely integrable PDE systems:

$$
w_{z_{k_1}z_{k_2}}(z)=\Phi_{k_1,k_2}\big(z,\,w(z),\,w_{z_1}(z),\ldots,w_{z_n}(z)\big)\qquad(1\leqslant k_1,\,k_2\leqslant n).
$$

Question still open *Compute explicitly the Chern-Moser-Webster* 1*-forms and curvatures [10, 72] in terms of a local graphing function for a Levi nondegenerate* $M^{2n+1}\subset\mathbb{C}^{n+1}$ *(rigid and tube cases are treated in [30]).*

This would, in particular, provide an alternative proof of Theorem 3.

2.4 Zariski-Generic $\mathscr{C}^\omega$ CR Manifolds of Dimension $\leqslant 5$

Coming back to $M^{2n+c}\subset\mathbb{C}^{n+c}$ of dimension $2n+c\leqslant 5$, and calling *Zariski-open* any complement $M\backslash\Sigma$ of some *proper* real analytic subset $\Sigma\subsetneq M$, treat at first the:

Problem 2 (Accessible subquestion of Problem 1) *Set up all possible initial geometries of connected $\mathscr{C}^\omega$ CR-generic submanifolds $M^{2n+c}\subset\mathbb{C}^{n+c}$ at Zariski-generic points.*

For general $M^{2n+c}\subset\mathbb{C}^{n+c}$, recall that the fundamental invariant bundle is:

$$
T^{1,0}M:=\big\{X-\sqrt{-1}\,J(X)\colon X\in TM\cap J(TM)\big\}.
$$

Lemma 1 ([46]) *If a CR-generic $M^{2n+c}\subset\mathbb{C}^{n+c}$ is locally graphed as:*

$$
v_j=\varphi_j\big(x_1,\ldots,x_n,y_1,\ldots,y_n,u_1,\ldots,u_c\big)\qquad(1\leqslant j\leqslant c),
$$

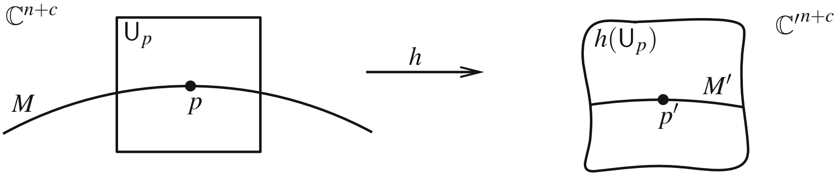
a local frame $\{\mathscr{L}_1, \dots, \mathscr{L}_n\}$ *for* $T^{1,0}M$ *consists of the n vector fields:*

$$\mathscr{L}_i = \frac{\partial}{\partial z_k} + A_i^1(x_\bullet, y_\bullet, u_\bullet)\,\frac{\partial}{\partial u_1} + \cdots\cdots + A_i^c(x_\bullet, y_\bullet, u_\bullet)\,\frac{\partial}{\partial u_c} \qquad (1 \leqslant i \leqslant n),$$

having rational *coefficient-functions:*

$$A_i^1 = \frac{\begin{vmatrix} -\varphi_{1,z_i} & \varphi_{1,u_2} & \cdots & \varphi_{1,u_c} \\ -\varphi_{2,z_i} & \sqrt{-1}+\varphi_{2,u_2} & \cdots & \varphi_{2,u_c} \\ \vdots & \vdots & \ddots & \vdots \\ -\varphi_{c,z_i} & \varphi_{c,u_2} & \cdots & \sqrt{-1}+\varphi_{c,u_c} \end{vmatrix}}{\begin{vmatrix} \sqrt{-1}+\varphi_{1,u_1} & \cdots & \varphi_{1,u_c} \\ \varphi_{2,u_1} & \cdots & \varphi_{2,u_c} \\ \vdots & \ddots & \vdots \\ \varphi_{c,u_1} & \cdots & \sqrt{-1}+\varphi_{c,u_c} \end{vmatrix}}, \quad \dots\dots\dots, \quad A_i^c = \frac{\begin{vmatrix} \sqrt{-1}+\varphi_{1,u_1} & \cdots & -\varphi_{1,z_i} \\ \varphi_{2,u_1} & \cdots & -\varphi_{2,z_i} \\ \vdots & \ddots & \vdots \\ \varphi_{c,u_1} & \cdots & -\varphi_{c,z_i} \end{vmatrix}}{\begin{vmatrix} \sqrt{-1}+\varphi_{1,u_1} & \cdots & \varphi_{1,u_c} \\ \varphi_{2,u_1} & \cdots & \varphi_{2,u_c} \\ \vdots & \ddots & \vdots \\ \varphi_{c,u_1} & \cdots & \sqrt{-1}+\varphi_{c,u_c} \end{vmatrix}}.$$

Nonvanishing of the denominator is equivalent to CR-genericity of M.
Here is how M transfers through local biholomorphisms.



Lemma 2 ([46]) *Given a connected* $\mathscr{C}^\omega$ *CR-generic submanifold* $M^{2n+c} \subset \mathbb{C}^{n+c}$ *and a local biholomorphism between open subsets:*

$$h\colon \quad \mathsf{U}_p \xrightarrow{\sim} h(\mathsf{U}_p) = \mathsf{U}'_{p'} \subset \mathbb{C}'^{n+c},$$

with $p \in M$, $p' = h(p)$, *setting:*

$$M' := h(M) \subset \mathbb{C}'^{n+c} \qquad (c = \mathsf{codim}\, M',\ n = \mathsf{CRdim}\, M'),$$

then for any two local frames:

$$\{\mathscr{L}_1, \dots, \mathscr{L}_n\} \quad \textit{for}\ \ T^{1,0}M \qquad \textit{and} \qquad \{\mathscr{L}'_1, \dots, \mathscr{L}'_n\} \quad \textit{for}\ \ T^{1,0}M',$$

there exist uniquely defined $\mathscr{C}^\omega$ *local coefficient-functions:*

$$a'_{i_1 i_2}\colon \quad M' \longrightarrow \mathbb{C} \qquad (1 \leqslant i_1, i_2 \leqslant n),$$

satisfying:

$$h_*(\mathscr{L}_1) = a'_{11}\,\mathscr{L}'_1 + \cdots + a'_{n1}\,\mathscr{L}'_n,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$h_*(\mathscr{L}_n) = a'_{1n}\,\mathscr{L}'_1 + \cdots + a'_{nn}\,\mathscr{L}'_n.$$

Definition 1 Taking any local 1-form $\rho_0 \colon TM \to \mathbb{R}$ whose extension to $\mathbb{C} \otimes_{\mathbb{R}} TM$ satisfies:

$$T^{1,0}M \oplus T^{0,1}M = \{\rho_0 = 0\},$$

the Hermitian matrix of the *Levi form* of M at various points $p \in M$ is:

$$\begin{pmatrix} \rho_0\big(\sqrt{-1}\,[\mathscr{L}_1, \overline{\mathscr{L}}_1]\big) & \cdots & \rho_0\big(\sqrt{-1}\,[\mathscr{L}_n, \overline{\mathscr{L}}_1]\big) \\ \vdots & \ddots & \vdots \\ \rho_0\big(\sqrt{-1}\,[\mathscr{L}_1, \overline{\mathscr{L}}_n]\big) & \cdots & \rho_0\big(\sqrt{-1}\,[\mathscr{L}_n, \overline{\mathscr{L}}_n]\big) \end{pmatrix}(p),$$

the extra factor $\sqrt{-1}$ being present in order to counterbalance the change of sign:

$$\overline{[\mathscr{L}_j, \overline{\mathscr{L}}_k]} = -[\mathscr{L}_k, \overline{\mathscr{L}}_j].$$

As an application, show the invariance of Levi nondegeneracy. For $M^3 \subset \mathbb{C}^2$ equivalent to $M'^3 \subset \mathbb{C}'^2$, whence $n = n' = 1$, introduce local vector field generators:

$$\mathscr{L} \text{ for } T^{1,0}M \qquad \text{and} \qquad \mathscr{L}' \text{ for } T^{1,0}M'.$$

Lemma 3 At every point $q \in M$ near p:

$$\mathbf{3} = \mathsf{rank}_{\mathbb{C}}\Big(\mathscr{L}\big|_q,\ \overline{\mathscr{L}}\big|_q,\ [\mathscr{L}, \overline{\mathscr{L}}]\big|_q\Big)$$
$$\Updownarrow$$
$$\mathbf{3} = \mathsf{rank}_{\mathbb{C}}\Big(\mathscr{L}'\big|_{h(q)},\ \overline{\mathscr{L}}'\big|_{h(q)},\ [\mathscr{L}', \overline{\mathscr{L}}']\big|_{h(q)}\Big).$$

Proof By what precedes, there exists a function $a' \colon M' \longrightarrow \mathbb{C}\backslash\{0\}$ with:

$$h_*(\mathscr{L}) = a'\,\mathscr{L}' \qquad \text{and} \qquad h_*(\overline{\mathscr{L}}) = \overline{a}'\,\overline{\mathscr{L}}'.$$

Consequently:

$$\begin{aligned} h_*\big([\mathscr{L},\overline{\mathscr{L}}]\big) &= \big[h_*(\mathscr{L}),\, h_*(\overline{\mathscr{L}})\big] \\ &= \big[a'\mathscr{L}',\, \overline{a}'\overline{\mathscr{L}}'\big] \\ &= a'\overline{a}'\,\big[\mathscr{L}',\,\overline{\mathscr{L}}'\big] + a'\,\mathscr{L}'(\overline{a}')\cdot\overline{\mathscr{L}}' - \overline{a}'\,\overline{\mathscr{L}}'(a')\cdot\mathscr{L}'. \end{aligned}$$

Dropping the mention of h_*, because the *change of frame* matrix:

$$\begin{pmatrix} \mathscr{L} \\ \overline{\mathscr{L}} \\ [\mathscr{L},\overline{\mathscr{L}}] \end{pmatrix} = \begin{pmatrix} a' & 0 & 0 \\ 0 & \overline{a}' & 0 \\ * & * & a'\overline{a}' \end{pmatrix} \begin{pmatrix} \mathscr{L}' \\ \overline{\mathscr{L}}' \\ [\mathscr{L}',\overline{\mathscr{L}}'] \end{pmatrix}$$

is visibly of rank 3, the result follows. □

Since $T^{1,0}M$ and $T^{0,1}M$ are Frobenius-involutive [46], only iterated Lie brackets between $T^{1,0}M$ and $T^{0,1}M$ are nontrivial. For a $\mathscr{C}^\omega$ connected $M^{2n+c} \subset \mathbb{C}^{n+c}$ with $T^{1,0}M$ having local generators $\mathscr{L}_1, \dots, \mathscr{L}_n$, set:

$$\mathbb{L}^1_{\mathscr{L},\overline{\mathscr{L}}} := \mathscr{C}^\omega\text{-linear combinations of } \mathscr{L}_1,\dots,\mathscr{L}_n,\ \overline{\mathscr{L}}_1,\dots,\overline{\mathscr{L}}_n,$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$\mathbb{L}^{\nu+1}_{\mathscr{L},\overline{\mathscr{L}}} := \mathscr{C}^\omega\text{-linear combinations of vector fields } \mathscr{M}^\nu \in \mathbb{L}^\nu_{\mathscr{L},\overline{\mathscr{L}}}$$
$$\text{and of brackets } \big[\mathscr{L}_k, \mathscr{M}^\nu\big],\ \big[\overline{\mathscr{L}}_k, \mathscr{M}^\nu\big].$$

Set:

$$\mathbb{L}^{\mathsf{Lie}}_{\mathscr{L},\overline{\mathscr{L}}} := \bigcup_{\nu\geqslant 1} \mathbb{L}^\nu_{\mathscr{L},\overline{\mathscr{L}}}.$$

By real analyticity [50], there exists an integer c_M with $0 \leqslant c_M \leqslant c$ and a proper real analytic subset $\Sigma \subsetneqq M$ such that at every point $q \in M\backslash\Sigma$:

$$\mathsf{dim}_{\mathbb{C}}\Big(\mathbb{L}^{\mathsf{Lie}}_{\mathscr{L},\overline{\mathscr{L}}}(q)\Big) = 2n + c_M.$$

Theorem 4 (Known, [38, 50]) *Every point $q \in M\backslash\Sigma$ has a small open neighborhood $\mathsf{U}_q \subset \mathbb{C}^{n+c}$ in which:*

$$M^{2n+c} \cong \underline{M}^{2n+c}$$

biholomorphically, with a CR and $\mathscr{C}^\omega$:

$$\underline{M}^{2n+c} \subset \mathbb{C}^{n+c_M} \times \mathbb{R}^{c-c_M}.$$

The case $c_M \leqslant c-1$ must hence be considered as *degeneration*, excluded in classification of initial geometries at Zariski-generic points.

A well known fact is that, at a Zariski-generic point, a connected $\mathscr{C}^\omega$ hypersurface $M^3 \subset \mathbb{C}^2$ is either $\cong \mathbb{C} \times \mathbb{R}$ or is Levi nondegenerate:

$$3 = \mathsf{rank}_{\mathbb{C}}\big(T^{1,0}M,\ T^{0,1}M,\ \big[T^{1,0}M,\, T^{0,1}M\big]\big).$$

Theorem 5 ([46]) *Excluding degenerate CR manifolds, there are precisely* ***six*** *general classes of nondegenerate connected $M^{2n+c} \subset \mathbb{C}^{n+c}$ having dimension:*

$$2n + c \leqslant \mathbf{5},$$

hence having CR dimension $n = 1$ or $n = 2$, namely if:

$$\{\mathscr{L}\} \qquad \textit{or} \qquad \{\mathscr{L}_1, \mathscr{L}_2\},$$

denotes any local frame for $T^{1,0}M$:

- **General Class** I*: Hypersurfaces $M^3 \subset \mathbb{C}^2$ with $\big\{\mathscr{L},\, \overline{\mathscr{L}},\, \big[\mathscr{L}, \overline{\mathscr{L}}\big]\big\}$ constituting a frame for $\mathbb{C} \otimes_{\mathbb{R}} TM$, with:*

$$\text{Model I:} \qquad v = z\bar{z},$$

- **General Class** II*: CR-generic $M^4 \subset \mathbb{C}^3$ with $\big\{\mathscr{L},\, \overline{\mathscr{L}},\, \big[\mathscr{L}, \overline{\mathscr{L}}\big],\, \big[\mathscr{L},\, \big[\mathscr{L}, \overline{\mathscr{L}}\big]\big]\big\}$ constituting a frame for $\mathbb{C} \otimes_{\mathbb{R}} TM$, with:*

$$\text{Model II:} \qquad v_1 = z\bar{z}, \qquad v_2 = z^2\bar{z} + z\bar{z}^2,$$

- **General Class** III_1*: CR-generic $M^5 \subset \mathbb{C}^4$ with $\big\{\mathscr{L}, \overline{\mathscr{L}},\, \big[\mathscr{L}, \overline{\mathscr{L}}\big],\, \big[\mathscr{L},\, \big[\mathscr{L}, \overline{\mathscr{L}}\big]\big],\, \big[\overline{\mathscr{L}},\, \big[\mathscr{L}, \overline{\mathscr{L}}\big]\big]\big\}$ constituting a frame for $\mathbb{C} \otimes_{\mathbb{R}} TM$, with:*

$$\text{Model III}_1\text{:} \qquad v_1 = z\bar{z}, \qquad v_2 = z^2\bar{z} + z\bar{z}^2, \qquad v_3 = \sqrt{-1}\,\big(z^2\bar{z} - z\bar{z}^2\big),$$

- **General Class** III_2*: CR generic $M^5 \subset \mathbb{C}^4$ with $\big\{\mathscr{L}, \overline{\mathscr{L}},\, \big[\mathscr{L}, \overline{\mathscr{L}}\big],\, \big[\mathscr{L},\, \big[\mathscr{L}, \overline{\mathscr{L}}\big]\big],\, \big[\mathscr{L},\, \big[\mathscr{L},\, \big[\mathscr{L}, \overline{\mathscr{L}}\big]\big]\big]\big\}$ constituting a frame for $\mathbb{C} \otimes_{\mathbb{R}} TM$, while $\mathbf{4} = \mathsf{rank}_{\mathbb{C}}\Big(\mathscr{L}, \overline{\mathscr{L}},\, \big[\mathscr{L}, \overline{\mathscr{L}}\big], \big[\mathscr{L},\, \big[\mathscr{L}, \overline{\mathscr{L}}\big]\big],\, \big[\overline{\mathscr{L}},\, \big[\mathscr{L}, \overline{\mathscr{L}}\big]\big]\Big)$, with:*

$$\text{Model III}_2\text{:} \qquad v_1 = z\bar{z}, \qquad v_2 = z^2\bar{z} + z\bar{z}^2, \qquad v_3 = 2\,z^3\bar{z} + 2\,z\bar{z}^3 + 3\,z^2\bar{z}^2,$$

- **General Class** IV_1*: Hypersurfaces $M^5 \subset \mathbb{C}^3$ with $\big\{\mathscr{L}_1,\, \mathscr{L}_2,\, \overline{\mathscr{L}}_1,\, \overline{\mathscr{L}}_2,\, \big[\mathscr{L}_1, \overline{\mathscr{L}}_1\big]\big\}$ constituting a frame for $\mathbb{C} \otimes_{\mathbb{R}} TM$, and with the Levi-Form of M being of rank* 2 *at every point $p \in M$, with:*

$$\text{Model(s) IV}_1\text{:} \qquad v = z_1\bar{z}_1 \pm z_2\bar{z}_2,$$

- **General Class IV_2:** *Hypersurfaces $M^5 \subset \mathbb{C}^3$ with $\{\mathscr{L}_1, \mathscr{L}_2, \overline{\mathscr{L}}_1, \overline{\mathscr{L}}_2, [\mathscr{L}_1, \overline{\mathscr{L}}_1]\}$ constituting a frame for $\mathbb{C} \otimes_{\mathbb{R}} TM$, with the Levi-Form being of rank 1 at every point $p \in M$ while the Freeman-Form (defined below) is nondegenerate at every point, with:*

$$\text{Model } \mathsf{IV}_2: \qquad v = \frac{z_1\bar{z}_1 + \frac{1}{2}\,z_1z_1\bar{z}_2 + \frac{1}{2}\,z_2\bar{z}_1\bar{z}_1}{1 - z_2\bar{z}_2}.$$

The models II, III_1 appear in the works of Beloshapka [2, 3] which exhibit a wealth of higher dimensional models widening the biholomorphic equivalence problem. Before proceeding, explain (only) how the General Classes IV_1 and IV_2 occur.

2.5 Concept of Freeman Form

If $M^5 \subset \mathbb{C}^3$ is connected $\mathscr{C}^\omega$ and belongs to Class IV_2, so that:

$$1 = \mathrm{rank}_{\mathbb{C}}\big(\text{Levi-Form}^M(p)\big) \qquad (\forall\, p \in M),$$

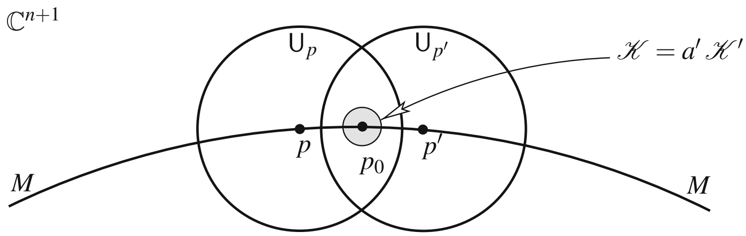
then there exists a unique rank 1 complex vector subbundle:

$$K^{1,0}M \subset T^{1,0}M$$

such that, at every point $p \in M$:

$$K_p^{1,0}M \ni \mathscr{K}_p \iff \mathscr{K}_p \in \mathrm{Kernel}\big(\text{Levi-Form}^M(p)\big).$$

Local trivializations of $K^{1,0}M$ match up on intersections of balls [46].



Furthermore, three known involutiveness conditions hold [46]:

$$\begin{aligned}
\big[K^{1,0}M,\, K^{1,0}M\big] &\subset K^{1,0}M, \\
\big[K^{0,1}M,\, K^{0,1}M\big] &\subset K^{0,1}M, \\
\big[K^{1,0}M,\, K^{0,1}M\big] &\subset K^{1,0}M \oplus K^{0,1}M.
\end{aligned} \tag{1}$$

In local coordinates, $M^5 \subset \mathbb{C}^3$ is graphed as:

$$v = \varphi\big(x_1, x_2, y_1, y_2, u\big),$$

and two local generators of $T^{1,0}M$ with their conjugates are:

$$\begin{aligned}
\mathscr{L}_1 &= \frac{\partial}{\partial z_1} + A_1 \frac{\partial}{\partial u}, & \overline{\mathscr{L}}_1 &= \frac{\partial}{\partial \bar{z}_1} + \overline{A_1} \frac{\partial}{\partial u}, \\
\mathscr{L}_2 &= \frac{\partial}{\partial z_2} + A_2 \frac{\partial}{\partial u}, & \overline{\mathscr{L}}_2 &= \frac{\partial}{\partial \bar{z}_2} + \overline{A_2} \frac{\partial}{\partial u},
\end{aligned}$$

with:

$$\begin{aligned}
A_1 &:= -\frac{\varphi_{z_1}}{\sqrt{-1} + \varphi_u}, \\
A_2 &:= -\frac{\varphi_{z_2}}{\sqrt{-1} + \varphi_u}.
\end{aligned}$$

Taking as a 1-form ρ_0 with $\{\rho_0 = 0\} = T^{1,0}M \oplus T^{0,1}M$:

$$\rho_0 := -A_1\, dz_1 - A_2\, dz_2 - \overline{A_1}\, d\bar{z}_1 - \overline{A_2}\, d\bar{z}_2 + du,$$

the top-left entry of the:

$$\begin{aligned}
\textsf{Levi-Matrix} &= \begin{pmatrix} \rho_0\big(\sqrt{-1}\big[\mathscr{L}_1, \overline{\mathscr{L}}_1\big]\big) & \rho_0\big(\sqrt{-1}\big[\mathscr{L}_2, \overline{\mathscr{L}}_1\big]\big) \\ \rho_0\big(\sqrt{-1}\big[\mathscr{L}_1, \overline{\mathscr{L}}_2\big]\big) & \rho_0\big(\sqrt{-1}\big[\mathscr{L}_2, \overline{\mathscr{L}}_2\big]\big) \end{pmatrix} \\
&= \begin{pmatrix} \sqrt{-1}\big(\mathscr{L}_1\big(\overline{A_1}\big) - \overline{\mathscr{L}}_1\big(A_1\big)\big) & \sqrt{-1}\big(\mathscr{L}_2\big(\overline{A_1}\big) - \overline{\mathscr{L}}_1\big(A_2\big)\big) \\ \sqrt{-1}\big(\mathscr{L}_1\big(\overline{A_2}\big) - \overline{\mathscr{L}}_2\big(A_1\big)\big) & \sqrt{-1}\big(\mathscr{L}_2\big(\overline{A_2}\big) - \overline{\mathscr{L}}_2\big(A_2\big)\big) \end{pmatrix},
\end{aligned} \tag{2}$$

expresses as:

$$\sqrt{-1}\,\big(\mathscr{L}_1(\overline{A_1}) - \overline{\mathscr{L}}_1(A_1)\big) = \frac{1}{(\sqrt{-1}+\varphi_u)^2(-\sqrt{-1}+\varphi_u)^2}\Big\{2\,\varphi_{z_1\overline{z}_1} + 2\,\varphi_{z_1\overline{z}_1}\varphi_u\varphi_u - 2\sqrt{-1}\,\varphi_{\overline{z}_1}\varphi_{z_1u} - 2\,\varphi_{\overline{z}_1}\varphi_{z_1u}\varphi_u + 2\sqrt{-1}\,\varphi_{z_1}\varphi_{\overline{z}_1u} + 2\,\varphi_{z_1}\varphi_{\overline{z}_1}\varphi_{uu} - 2\,\varphi_{z_1}\varphi_{\overline{z}_1u}\varphi_u\Big\},$$

with quite similar expressions for the remaining three entries. As M belongs to Class IV_2:

$$0 \equiv \det\begin{pmatrix} \sqrt{-1}\big(\overline{A_1}_{z_1} + A_1\,\overline{A_1}_u - A_{1\overline{z_1}} - \overline{A_1}\,A_{1u}\big) & \sqrt{-1}\big(\overline{A_1}_{z_2} + A_2\,\overline{A_1}_u - A_{2\overline{z_1}} - \overline{A_1}\,A_{2u}\big) \\ \sqrt{-1}\big(\overline{A_2}_{z_1} + A_1\,\overline{A_2}_u - A_{1\overline{z_2}} - \overline{A_2}\,A_{1u}\big) & \sqrt{-1}\big(\overline{A_2}_{z_2} + A_2\,\overline{A_2}_u - A_{2\overline{z_2}} - \overline{A_2}\,A_{2u}\big) \end{pmatrix},$$

that is to say in terms of φ:

$$\begin{aligned} 0 \equiv \frac{4}{(\sqrt{-1}+\varphi_u)^3(-\sqrt{-1}+\varphi_u)^3}\Big\{&\varphi_{z_2\overline{z}_2}\varphi_{z_1\overline{z}_1} - \varphi_{z_2\overline{z}_1}\varphi_{z_1\overline{z}_2} + \\ &+ \varphi_{z_2\overline{z}_1}\varphi_{\overline{z}_2}\varphi_{z_1u}\varphi_u - \varphi_{z_2\overline{z}_1}\varphi_{\overline{z}_2}\varphi_{z_1}\varphi_{uu} - \varphi_{\overline{z}_1}\varphi_{z_2u}\varphi_{z_1}\varphi_{\overline{z}_2u} + \varphi_{\overline{z}_1}\varphi_{z_2u}\varphi_u\varphi_{z_1\overline{z}_2} - \\ &- \varphi_{z_2}\varphi_{\overline{z}_1u}\varphi_{\overline{z}_2}\varphi_{z_1u} - \varphi_{z_2}\varphi_{\overline{z}_1}\varphi_{uu}\varphi_{z_1\overline{z}_2} + \varphi_{z_2}\varphi_{\overline{z}_1u}\varphi_u\varphi_{z_1\overline{z}_2} - \varphi_{z_2\overline{z}_2}\varphi_{\overline{z}_1}\varphi_{z_1u}\varphi_u + \\ &+ \varphi_{z_2\overline{z}_2}\varphi_{z_1}\varphi_{\overline{z}_1}\varphi_{uu} - \varphi_{z_2\overline{z}_2}\varphi_{z_1}\varphi_{\overline{z}_1u}\varphi_u + \varphi_{z_2\overline{z}_1}\varphi_{z_1}\varphi_{\overline{z}_2u}\varphi_u + \varphi_{z_2}\varphi_{\overline{z}_2u}\varphi_{\overline{z}_1}\varphi_{z_1u} - \\ &- \varphi_{z_2}\varphi_{\overline{z}_2u}\varphi_{z_1\overline{z}_1}\varphi_u + \varphi_{\overline{z}_2}\varphi_{z_2u}\varphi_{z_1}\varphi_{\overline{z}_1u} - \varphi_{\overline{z}_2}\varphi_{z_2u}\varphi_u\varphi_{z_1\overline{z}_1} + \varphi_{\overline{z}_2}\varphi_{z_2}\varphi_{uu}\varphi_{z_1\overline{z}_1} + \\ &+ \sqrt{-1}\big(\varphi_{z_2\overline{z}_2}\varphi_{z_1}\varphi_{\overline{z}_1u} + \varphi_{\overline{z}_1}\varphi_{z_2u}\varphi_{z_1\overline{z}_2} + \varphi_{z_2\overline{z}_1}\varphi_{\overline{z}_2}\varphi_{z_1u} + \varphi_{z_2}\varphi_{\overline{z}_2u}\varphi_{z_1\overline{z}_1}\big) - \\ &- \sqrt{-1}\big(\varphi_{\overline{z}_2}\varphi_{z_2u}\varphi_{z_1\overline{z}_1} + \varphi_{z_2\overline{z}_1}\varphi_{z_1}\varphi_{\overline{z}_2u} + \varphi_{z_2}\varphi_{\overline{z}_1u}\varphi_{z_1\overline{z}_2} + \varphi_{z_2\overline{z}_2}\varphi_{\overline{z}_1}\varphi_{z_1u}\big) - \\ &- \varphi_{z_2\overline{z}_1}\varphi_{z_1\overline{z}_2}\varphi_u\varphi_u + \varphi_{z_2\overline{z}_2}\varphi_{z_1\overline{z}_1}\varphi_u\varphi_u\Big\}. \end{aligned} \tag{3}$$

Since the rank of the Levi Matrix (2) equals 1 everywhere, after permutation, its top-left entry vanishes nowhere. Hence a local generator for $K^{1,0}M$ is:

$$\mathscr{K} = k\,\mathscr{L}_1 + \mathscr{L}_2,$$

with:

$$k = -\frac{\mathscr{L}_2\big(\overline{A_1}\big) - \overline{\mathscr{L}}_1\big(A_2\big)}{\mathscr{L}_1\big(\overline{A_1}\big) - \overline{\mathscr{L}}_1\big(A_1\big)},$$

namely:

$$k=\frac{\varphi_{z_2\overline{z}_1}+\varphi_{z_2\overline{z}_1}\varphi_u\varphi_u-\sqrt{-1}\varphi_{\overline{z}_1}\varphi_{z_2u}-\varphi_{\overline{z}_1}\varphi_{z_2u}\varphi_u+\sqrt{-1}\varphi_{z_2}\varphi_{\overline{z}_1u}+\varphi_{z_2}\varphi_{\overline{z}_1}\varphi_{uu}-\varphi_{z_2}\varphi_{\overline{z}_1u}\varphi_u}{-\varphi_{z_1\overline{z}_1}-\varphi_{z_1\overline{z}_1}\varphi_u\varphi_u+\sqrt{-1}\varphi_{\overline{z}_1}\varphi_{z_1u}+\varphi_{\overline{z}_1}\varphi_{z_1u}\varphi_u-\sqrt{-1}\varphi_{z_1}\varphi_{\overline{z}_1u}-\varphi_{z_1}\varphi_{\overline{z}_1}\varphi_{uu}+\varphi_{z_1}\varphi_{\overline{z}_1u}\varphi_u},$$

and there is a surprising computational fact that this function k happens to be also equal to the other two quotients ([46], II, p. 82):

$$k=-\frac{\mathscr{L}_2\big(\overline{A_1}\big)}{\mathscr{L}_1\big(\overline{A_1}\big)}=-\frac{-\overline{\mathscr{L}}_1\big(A_2\big)}{-\overline{\mathscr{L}}_1\big(A_1\big)}. \tag{4}$$

Heuristically, this fact becomes transparent when $\varphi=\varphi(x_1,x_2,y_1,y_2)$ is independent of u, whence the Levi matrix becomes:

$$\begin{pmatrix}2\,\varphi_{z_1\overline{z}_1} & 2\,\varphi_{z_2\overline{z}_1}\\ 2\,\varphi_{z_1\overline{z}_2} & 2\,\varphi_{z_2\overline{z}_2}\end{pmatrix},$$

and clearly:

$$k=-\frac{2\,\varphi_{z_2\overline{z}_1}}{2\,\varphi_{z_1\overline{z}_1}}=-\frac{\varphi_{z_2\overline{z}_1}}{\varphi_{z_1\overline{z}_1}}=-\frac{\mathscr{L}_2(\varphi_{\overline{z}_1})}{\mathscr{L}_1(\varphi_{\overline{z}_1})}=-\frac{-\overline{\mathscr{L}}_1(\varphi_{z_2})}{-\overline{\mathscr{L}}_1(\varphi_{z_1})}.$$

Proposition 1 ([46]) *In any system of holomorphic coordinates, for any choice of Levi-kernel adapted local $T^{1,0}M$-frame $\{\mathscr{L}_1,\mathscr{K}\}$ satisfying:*

$$K^{1,0}M=\mathbb{C}\,\mathscr{K},$$

and for any choice of differential 1-forms $\{\rho_0,\kappa_0,\zeta_0\}$ satisfying:

$$\begin{aligned}\{0=\rho_0\}&=T^{1,0}M\oplus T^{0,1}M,\\ \{0=\rho_0=\kappa_0=\overline{\kappa}_0=\overline{\zeta}_0\}&=K^{1,0}M,\end{aligned}$$

the quantity:

$$\kappa_0\big(\big[\mathscr{K},\overline{\mathscr{L}}_1\big]\big),$$

is, at one fixed point $p\in M$, either zero or nonzero, independently of any choice.

Definition 2 The *Freeman form* at a point $p\in M$ is the value of $\kappa_0\big(\big[\mathscr{K},\overline{\mathscr{L}}_1\big]\big)(p)$, and it depends only on $\kappa_0(p)$, $\mathscr{K}\big|_p$, $\overline{\mathscr{L}}_1\big|_p$.

With a $T^{1,0}M$-frame $\{\mathscr{K}, \mathscr{L}_1\}$ satisfying $K^{1,0}M = \mathbb{C}\mathscr{K}$, define quite equivalently:

$$\mathsf{Freeman\text{-}Form}^M(p)\colon \left[\begin{array}{l} K_p^{1,0}M \times \big(T_p^{1,0}M \mod K_p^{1,0}M\big) \longrightarrow \mathbb{C} \\ \qquad\qquad\qquad\qquad \big(\mathscr{K}_p, \mathscr{L}_{1p}\big) \longmapsto \big[\mathscr{K}, \overline{\mathscr{L}}_1\big](p) \\ \qquad\qquad\qquad\qquad\qquad\qquad \mod \big(K^{1,0}M \oplus T^{0,1}M\big), \end{array}\right.$$

the result being independent of vector field extensions $\mathscr{K}\big|_p = \mathscr{K}_p$ and $\mathscr{L}_1\big|_p = \mathscr{L}_{1p}$.

General Classes IV_1, IV_2. For a connected $\mathscr{C}^\omega$ hypersurface $M^5 \subset \mathbb{C}^3$, if the Levi form is of rank 2 at one point, it is of rank 2 at every Zariski-generic point. Excluding Levi degenerate points, this brings IV_1.

If the Levi form is identically zero, then as is known $M \cong \mathbb{C}^2 \times \mathbb{R}$.

If the Levi form is of rank 1, the Freeman form creates bifurcation:

Proposition 2 ([46]) *A $\mathscr{C}^\omega$ hypersurface $M^5 \subset \mathbb{C}^3$ having at every point p:*

$$\mathsf{rank}_{\mathbb{C}}\big(\mathsf{Levi\text{-}Form}^M(p)\big) = \mathbf{1}$$

has an identically vanishing:

$$\mathsf{Freeman\text{-}Form}^M(p) \equiv \mathbf{0},$$

if and only if it is locally biholomorphic to a product:

$$M^5 \cong M^3 \times \mathbb{C}$$

with a $\mathscr{C}^\omega$ hypersurface $M^3 \subset \mathbb{C}^2$.

Proof With:

$$\mathscr{K} = k\,\mathscr{L}_1 + \mathscr{L}_2 = k\,\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \big(k A_1 + A_2\big)\,\frac{\partial}{\partial u},$$

the involutiveness (1):

$$\big[\mathscr{K}, \overline{\mathscr{K}}\big] = \text{function} \cdot \mathscr{K} + \text{function} \cdot \overline{\mathscr{K}},$$

and the fact that this bracket does not contain either $\partial\big/\partial z_2$ or $\partial\big/\partial \overline{z}_2$:

$$\begin{aligned} \big[\mathscr{K}, \overline{\mathscr{K}}\big] &= \left[k\,\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \big(k A_1 + A_2\big)\,\frac{\partial}{\partial u},\ \overline{k}\,\frac{\partial}{\partial \overline{z}_1} + \frac{\partial}{\partial \overline{z}_2} + \big(\overline{k}\,\overline{A}_1 + \overline{A}_2\big)\,\frac{\partial}{\partial u}\right] \\ &= \mathscr{K}\big(\overline{k}\big)\,\frac{\partial}{\partial \overline{z}_1} - \overline{\mathscr{K}}\big(k\big)\,\frac{\partial}{\partial z_1} + \Big(\mathscr{K}\big(\overline{k}\,\overline{A}_1 + \overline{A}_2\big) - \overline{\mathscr{K}}\big(k A_1 + A_2\big)\Big)\,\frac{\partial}{\partial u}, \end{aligned}$$

entail:

$$0 \equiv \overline{\mathscr{K}}(k) \equiv \mathscr{K}\big(\overline{k}\big).$$

Next, identical vanishing of the Freeman form:

$$\big[\mathscr{K}, \overline{\mathscr{L}}_1\big] \equiv 0 \qquad \mathrm{mod} \quad \big(\mathscr{K}, \overline{\mathscr{K}}, \overline{\mathscr{L}}_1\big),$$

with:

$$\begin{aligned}\big[\mathscr{K}, \overline{\mathscr{L}}_1\big] &= \bigg[k\,\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \big(kA_1 + A_2\big)\,\frac{\partial}{\partial u},\ \ \frac{\partial}{\partial \overline{z}_1} + \overline{A}_1\,\frac{\partial}{\partial u}\bigg] \\ &= \overline{\mathscr{L}}_1(k)\,\frac{\partial}{\partial z_1} + \text{something}\,\frac{\partial}{\partial u},\end{aligned}$$

reads as:

$$0 \equiv \overline{\mathscr{L}}_1(k),$$

and since $\big\{\overline{\mathscr{K}}, \overline{\mathscr{L}}_1\big\}$ is a $T^{0,1}M$-frame,

The $\mathscr{C}^\omega$ slanting function k is a CR function!

Moreover, the last coefficient-function of:

$$\mathscr{K} = k\,\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} + \big(kA_1 + A_2\big)\,\frac{\partial}{\partial u}$$

is *also* a CR function, namely it is annihilated by $\overline{\mathscr{L}}_1$, $\overline{\mathscr{L}}_2$, because firstly:

$$0 \overset{?}{=} \overline{\mathscr{L}}_1\big(kA_1 + A_2\big) = k\overline{\mathscr{L}}_1(A_1) + \overline{\mathscr{L}}_1(A_2) \overset{\text{ok}}{=} 0,$$

thanks to (4), and secondly because a direct computation gives:

$$\begin{aligned}0 &\overset{?}{=} \overline{\mathscr{L}}_2\big(kA_1 + A_2\big) \\ &= k\overline{\mathscr{L}}_2(A_1) + \overline{\mathscr{L}}_2(A_2) \\ &= \frac{-\,\text{numerator of the Levi determinant}}{(\sqrt{-1} + \varphi_u)\,[\varphi_{z_1\overline{z}_1} + \varphi_{z_1\overline{z}_1}\varphi_u\varphi_u - \sqrt{-1}\,\varphi_{\overline{z}_1}\varphi_{z_1u} - \varphi_{\overline{z}_1}\varphi_{z_1u}\varphi_u + \sqrt{-1}\,\varphi_{z_1}\varphi_{\overline{z}_1u}\varphi_u + \varphi_{z_1}\varphi_{\overline{z}_1}\varphi_{uu}]} \\ &\equiv 0.\end{aligned}$$

In conclusion, the $(1,0)$ field $\mathscr{K}$ has $\mathscr{C}^\omega$ CR coefficient-functions, hence $\mathscr{K}$ is locally extendable to a neighborhood of M in $\mathbb{C}^3$ as a $(1,0)$ field having holomorphic coefficients, and a straightening $\mathscr{K} = \partial/\partial z_2$ yields $M^5 \cong M^3 \times \mathbb{C}$. □

Excluding therefore such a degeneration, and focusing attention on a Zariski-generic initial classification, it therefore remains only the General Class IV_2.

2.6 Existence of the Six General Classes I, II, III_1, III_2, IV_1, IV_2

Graphing functions are essentially free and arbitrary.

Proposition 3 ([46]) *Every CR-generic submanifold belonging to the four classes* I, II, III_1, IV_1 *may be represented in suitable local holomorphic coordinates as:*

$$\underline{(\mathsf{I})}: \quad \Big[v = z\bar{z} + z\bar{z}\,\mathrm{O}_1\big(z,\bar{z}\big) + z\bar{z}\,\mathrm{O}_1(u),$$

$$\underline{(\mathsf{II})}: \quad \begin{bmatrix} v_1 = z\bar{z} \qquad\qquad + z\bar{z}\,\mathrm{O}_2\big(z,\bar{z}\big) + z\bar{z}\,\mathrm{O}_1(u_1) + z\bar{z}\,\mathrm{O}_1(u_2), \\ v_2 = z^2\bar{z} + z\bar{z}^2 + z\bar{z}\,\mathrm{O}_2\big(z,\bar{z}\big) + z\bar{z}\,\mathrm{O}_1(u_1) + z\bar{z}\,\mathrm{O}_1(u_2), \end{bmatrix}$$

$$\underline{(\mathsf{III})_1}: \quad \begin{bmatrix} v_1 = z\bar{z} \qquad\qquad\qquad + z\bar{z}\,\mathrm{O}_2\big(z,\bar{z}\big) + z\bar{z}\,\mathrm{O}_1(u_1) + z\bar{z}\,\mathrm{O}_1(u_2) + z\bar{z}\,\mathrm{O}_1(u_3), \\ v_2 = z^2\bar{z} + z\bar{z}^2 \qquad\quad + z\bar{z}\,\mathrm{O}_2\big(z,\bar{z}\big) + z\bar{z}\,\mathrm{O}_1(u_1) + z\bar{z}\,\mathrm{O}_1(u_2) + z\bar{z}\,\mathrm{O}_1(u_3), \\ v_3 = \sqrt{-1}\,\big(z^2\bar{z} - z\bar{z}^2\big) + z\bar{z}\,\mathrm{O}_2\big(z,\bar{z}\big) + z\bar{z}\,\mathrm{O}_1(u_1) + z\bar{z}\,\mathrm{O}_1(u_2) + z\bar{z}\,\mathrm{O}_1(u_3), \end{bmatrix}$$

$$\underline{(\mathsf{IV})_1}: \quad \Big[v = z_1\bar{z}_1 \pm z_2\bar{z}_2 + \mathrm{O}_3\big(z_1, z_2, \bar{z}_1, \bar{z}_2, u\big),$$

with arbitrary remainder functions. For class:

$$\underline{(\mathrm{III})_2}: \quad \begin{aligned} v_1 &= z\bar{z} + c_1\, z^2\bar{z}^2 + z\bar{z}\,\mathrm{O}_3\big(z,\bar{z}\big) + z\bar{z}\,u_1\,\mathrm{O}_1\big(z,\bar{z},u_1\big) + \\ &\qquad + z\bar{z}\,u_2\,\mathrm{O}_1\big(z,\bar{z},u_1,u_2\big) + z\bar{z}\,u_3\,\mathrm{O}_1\big(z,\bar{z},u_1,u_2,u_3\big), \\ v_2 &= z^2\bar{z} + z\bar{z}^2 + z\bar{z}\,\mathrm{O}_3\big(z,\bar{z}\big) + z\bar{z}\,u_1\,\mathrm{O}_1\big(z,\bar{z},u_1\big) + \\ &\qquad + z\bar{z}\,u_2\,\mathrm{O}_1\big(z,\bar{z},u_1,u_2\big) + z\bar{z}\,u_3\,\mathrm{O}_1\big(z,\bar{z},u_1,u_2,u_3\big), \\ v_3 &= 2\,z^3\bar{z} + 2\,z\bar{z}^3 + 3\,z^2\bar{z}^2 + z\bar{z}\,\mathrm{O}_3\big(z,\bar{z}\big) + z\bar{z}\,u_1\,\mathrm{O}_1\big(z,\bar{z},u_1\big) + \\ &\qquad + z\bar{z}\,u_2\,\mathrm{O}_1\big(z,\bar{z},u_1,u_2\big) + z\bar{z}\,u_3\,\mathrm{O}_1\big(z,\bar{z},u_1,u_2,u_3\big), \end{aligned}$$

the 3 *graphing functions* φ_1, φ_2, φ_3 *are subjected to the identical vanishing:*

$$0 \equiv \begin{vmatrix} \mathscr{L}(\overline{A}_1)-\overline{\mathscr{L}}(A_1) & \mathscr{L}(\overline{A}_2)-\overline{\mathscr{L}}(A_2) & \mathscr{L}(\overline{A}_3)-\overline{\mathscr{L}}(A_3) \\ \begin{array}{c}\mathscr{L}(\mathscr{L}(\overline{A}_1))-2\mathscr{L}(\overline{\mathscr{L}}(A_1))+ \\ +\overline{\mathscr{L}}(\mathscr{L}(A_1))\end{array} & \begin{array}{c}\mathscr{L}(\mathscr{L}(\overline{A}_2))-2\mathscr{L}(\overline{\mathscr{L}}(A_2))+ \\ +\overline{\mathscr{L}}(\mathscr{L}(A_2))\end{array} & \begin{array}{c}\mathscr{L}(\mathscr{L}(\overline{A}_3))-2\mathscr{L}(\overline{\mathscr{L}}(A_3))+ \\ +\overline{\mathscr{L}}(\mathscr{L}(A_3))\end{array} \\ \begin{array}{c}-\overline{\mathscr{L}}(\overline{\mathscr{L}}(A_1))+2\overline{\mathscr{L}}(\mathscr{L}(\overline{A}_1))- \\ -\mathscr{L}(\overline{\mathscr{L}}(\overline{A}_1))\end{array} & \begin{array}{c}-\overline{\mathscr{L}}(\overline{\mathscr{L}}(A_2))+2\overline{\mathscr{L}}(\mathscr{L}(\overline{A}_2))- \\ -\mathscr{L}(\overline{\mathscr{L}}(\overline{A}_2))\end{array} & \begin{array}{c}-\overline{\mathscr{L}}(\overline{\mathscr{L}}(A_3))+2\overline{\mathscr{L}}(\mathscr{L}(\overline{A}_3))- \\ -\mathscr{L}(\overline{\mathscr{L}}(\overline{A}_3))\end{array} \end{vmatrix}.$$

Lastly, for class:

$\underline{(\mathrm{IV})_2}$: $\quad \big[\, v = z_1\overline{z}_1 + \frac{1}{2}\, z_1 z_1 \overline{z}_2 + \frac{1}{2}\, z_2 \overline{z}_1 \overline{z}_1 + \mathrm{O}_4\big(z_1, z_2, \overline{z}_1, \overline{z}_2\big) + u\, \mathrm{O}_2\big(z_1, z_2, \overline{z}_1, \overline{z}_2, u\big),$

the graphing function φ *is subjected to the identical vanishing of the Levi determinant* (3).

2.7 Cartan Equivalences and Curvatures

Problem 3 (Subproblem of Problem 1) *Perform Cartan's local equivalence procedure for these six general classes* I, II, III_1, III_2, IV_1, IV_2 *of nondegenerate CR manifolds up to dimension* **5**.

Class I equivalences. Consider firstly Class I hypersurfaces $M^3 \subset \mathbb{C}^2$ graphed as $v = \varphi(x, y, u)$ with:

$$\begin{cases} \mathscr{L} = \dfrac{\partial}{\partial z} - \dfrac{\varphi_z}{\sqrt{-1} + \varphi_u}\, \dfrac{\partial}{\partial u}, \qquad \overline{\mathscr{L}} = \dfrac{\partial}{\partial \overline{z}} - \dfrac{\varphi_{\overline{z}}}{-\sqrt{-1} + \varphi_u}\, \dfrac{\partial}{\partial u}, \\ \mathscr{T} := \sqrt{-1}\,\big[\mathscr{L}, \overline{\mathscr{L}}\big] = \ell\, \dfrac{\partial}{\partial u} \end{cases}$$

making a frame for $\mathbb{C} \otimes_{\mathbb{R}} TM$, where the Levi-factor (nonvanishing) function:

$$\begin{aligned} \ell &:= \sqrt{-1}\,\Big(\mathscr{L}\big(\overline{A}\big) - \overline{\mathscr{L}}(A)\Big) \\ &= \frac{1}{(\sqrt{-1} + \varphi_u)^2\,(-\sqrt{-1} + \varphi_u)^2} \Big\{ 2\,\varphi_{z\overline{z}} + 2\,\varphi_{z\overline{z}}\,\varphi_u\,\varphi_u - 2\,\sqrt{-1}\,\varphi_{\overline{z}}\,\varphi_{zu} - 2\,\varphi_{\overline{z}}\,\varphi_{zu}\,\varphi_u + \\ &\qquad\qquad + 2\,\sqrt{-1}\,\varphi_z\,\varphi_{\overline{z}u} + 2\,\varphi_z\,\varphi_{\overline{z}}\,\varphi_{uu} - 2\,\varphi_z\,\varphi_{\overline{z}u}\,\varphi_u \Big\} \end{aligned} \tag{5}$$

will, notably, enter computations in denominator place.

Introduce also the dual coframe for $\mathbb{C} \otimes_{\mathbb{R}} T^*M$:

$$\{\rho_0,\, \overline{\zeta}_0,\, \zeta_0\},$$

satisfying:

$$\begin{array}{lll} \rho_0(\mathscr{T}) = 1 & \rho_0(\overline{\mathscr{L}}) = 0 & \rho_0(\mathscr{L}) = 0, \\ \overline{\zeta}_0(\mathscr{T}) = 0 & \overline{\zeta}_0(\overline{\mathscr{L}}) = 1 & \overline{\zeta}_0(\mathscr{L}) = 0, \\ \zeta_0(\mathscr{T}) = 0 & \zeta_0(\overline{\mathscr{L}}) = 0 & \zeta_0(\mathscr{L}) = 1. \end{array}$$

Since:

$$[\mathscr{L},\, \mathscr{T}] = \left[\frac{\partial}{\partial z} + A\,\frac{\partial}{\partial u},\ \ell\,\frac{\partial}{\partial u}\right] = \Big(\ell_z + A\,\ell_u - \ell\,A_u\Big)\,\frac{\partial}{\partial u} = \overbrace{\frac{\ell_z + A\,\ell_u - \ell\,A_u}{\ell}}^{=:P}\,\mathscr{T},$$

the *initial Darboux structure* reads dually as:

$$\begin{aligned} d\rho_0 &= P\,\rho_0 \wedge \zeta_0 + \overline{P}\,\rho_0 \wedge \overline{\zeta}_0 + \sqrt{-1}\,\zeta_0 \wedge \overline{\zeta}_0, \\ d\overline{\zeta}_0 &= 0, \\ d\zeta_0 &= 0, \end{aligned}$$

with a *single* fundamental function P. Élie Cartan in 1932 performed his equivalence procedure (well presented in [58]) for such $M^3 \subset \mathbb{C}^2$, but the completely explicit aspects must be endeavoured once again for systematic treatment of Problem 3. Within Tanaka's theory, Ezhov-McLaughlin-Schmalz [23] already constructed a Cartan connection on a certain principal bundle $N^8 \to M^3$, whose effective aspects have been explored further in [51, 52].

Indeed, as already observed in Lemma 3 (*see* also [46, 52]), the initial *G-structure* for (local) biholomorphic equivalences of such hypersurfaces is:

$$\mathsf{G}_{\mathsf{IV}_2}^{\mathsf{initial}} := \left\{ \begin{pmatrix} \mathsf{a}\overline{\mathsf{a}} & 0 & 0 \\ \overline{\mathsf{b}} & \overline{\mathsf{a}} & 0 \\ \mathsf{b} & 0 & \mathsf{a} \end{pmatrix} \in \mathsf{GL}_3(\mathbb{C}):\ \ \mathsf{a} \in \mathbb{C}\backslash\{0\},\ \mathsf{b} \in \mathbb{C} \right\}.$$

Cartan's method is to deal with the so-called *lifted coframe:*

$$\begin{pmatrix} \rho \\ \overline{\zeta} \\ \zeta \end{pmatrix} := \begin{pmatrix} \mathsf{a}\overline{\mathsf{a}} & 0 & 0 \\ \overline{\mathsf{b}} & \overline{\mathsf{a}} & 0 \\ \mathsf{b} & 0 & \mathsf{a} \end{pmatrix} \begin{pmatrix} \rho_0 \\ \overline{\zeta}_0 \\ \zeta_0 \end{pmatrix},$$

in the space of $\big(x, y, u, \mathsf{a}, \overline{\mathsf{a}}, \mathsf{b}, \overline{\mathsf{b}}\big)$. After two absorbtions-normalizations and after one prolongation:

Theorem 6 (M.-Sabzevari, [52]) *The biholomorphic equivalence problem for $M^3 \subset \mathbb{C}^2$ lifts as some explicit eight-dimensional coframe:*

$$\{\rho, \overline{\zeta}, \zeta, \alpha, \beta, \overline{\alpha}, \overline{\beta}, \delta\},$$

on a certain manifold $N^8 \longrightarrow M^3$ having $\{e\}$-structure equations:

$$\begin{aligned}
d\rho &= \alpha \wedge \rho + \overline{\alpha} \wedge \rho + \sqrt{-1}\, \zeta \wedge \overline{\zeta}, \\
d\overline{\zeta} &= \overline{\beta} \wedge \rho + \overline{\alpha} \wedge \overline{\zeta}, \\
d\zeta &= \beta \wedge \rho + \alpha \wedge \zeta, \\
d\alpha &= \delta \wedge \rho + 2\sqrt{-1}\, \zeta \wedge \overline{\beta} + \sqrt{-1}\, \overline{\zeta} \wedge \beta, \\
d\beta &= \delta \wedge \zeta + \beta \wedge \overline{\alpha} + \mathfrak{I}\, \overline{\zeta} \wedge \rho, \\
d\overline{\alpha} &= \delta \wedge \rho - 2\sqrt{-1}\, \overline{\zeta} \wedge \beta - \sqrt{-1}\, \zeta \wedge \overline{\beta}, \\
d\overline{\beta} &= \delta \wedge \overline{\zeta} + \overline{\beta} \wedge \alpha + \overline{\mathfrak{I}}\, \zeta \wedge \rho, \\
d\delta &= \delta \wedge \alpha + \delta \wedge \overline{\alpha} + \sqrt{-1}\, \beta \wedge \overline{\beta} + \mathfrak{T}\, \rho \wedge \zeta + \overline{\mathfrak{T}}\, \rho \wedge \overline{\zeta},
\end{aligned}$$

with the single primary complex invariant:

$$\begin{aligned}
\mathfrak{I} := \frac{1}{6}\, \frac{1}{\mathsf{a}\overline{\mathsf{a}}^3} \Big(- 2\, \overline{\mathscr{L}}\big(\mathscr{L}\big(\overline{\mathscr{L}}\big(\overline{P}\big)\big)\big) + 3\, \overline{\mathscr{L}}\big(\overline{\mathscr{L}}\big(\mathscr{L}\big(\overline{P}\big)\big)\big) - 7\, \overline{P}\, \overline{\mathscr{L}}\big(\mathscr{L}\big(\overline{P}\big)\big) + \\
+ 4\, \overline{P}\, \mathscr{L}\big(\overline{\mathscr{L}}\big(\overline{P}\big)\big) - \mathscr{L}\big(\overline{P}\big)\, \overline{\mathscr{L}}\big(\overline{P}\big) + 2\, \overline{P}\, \overline{P}\, \mathscr{L}\big(\overline{P}\big) \Big),
\end{aligned}$$

and with one secondary invariant:

$$\mathfrak{T} = \frac{1}{\overline{\mathsf{a}}} \left(\overline{\mathscr{L}}(\overline{\mathfrak{I}}) - \overline{P}\, \overline{\mathfrak{I}} \right) - \sqrt{-1}\, \frac{\mathsf{b}}{\mathsf{a}\overline{\mathsf{a}}}\, \overline{\mathfrak{I}}.$$

Explicitness obstacle. In terms of the function P, the formulas for $\mathfrak{I}$, for $\mathfrak{T}$ and for the 1-forms constituting the $\{e\}$-structure are writable, but when expressing everything in terms of the graphing function φ, because P involves the Levi factor ℓ in denominator place, formulas 'explode'.

Indeed, the real and imaginary parts Δ_1 and Δ_2 in:

$$\mathfrak{I} = \frac{4}{\mathsf{a}\overline{\mathsf{a}}^3} \big(\boldsymbol{\Delta}_1 + \sqrt{-1}\, \boldsymbol{\Delta}_4 \big)$$

have numerators containing respectively [52]:

1 553 198 and **1 634 457**

monomials in the differential ring in $\binom{6+3}{3} - 1 = 83$ variables:

$$\mathbb{Z}\big[\varphi_x, \varphi_y, \varphi_{x^2}, \varphi_{y^2}, \varphi_{u^2}, \varphi_{xy}, \varphi_{xu}, \varphi_{yu}, \ldots\ldots, \varphi_{x^6}, \varphi_{y^6}, \varphi_{u^6}, \ldots\big].$$

Strikingly, though, in the so-called *rigid case* (often useful as a case of study-exploration) where $\varphi = \varphi(x, y)$ is independent of u so that:

$$P = \frac{\varphi_{zz\overline{z}}}{\varphi_{z\overline{z}}}, \qquad \text{Levi factor at denominator} = \ell = \varphi_{z\overline{z}},$$

$\mathfrak{I}$ is easily writable:

$$\mathfrak{I}\Big|_{\substack{\text{rigid}\\\text{case}}} = \frac{1}{6}\,\frac{1}{\mathsf{a}\overline{\mathsf{a}}^3}\bigg(\frac{\varphi_{z^2\overline{z}^4}}{\varphi_{z\overline{z}}} - 6\,\frac{\varphi_{z^2\overline{z}^3}\,\varphi_{z\overline{z}^2}}{(\varphi_{z\overline{z}})^2} - \frac{\varphi_{z\overline{z}^4}\,\varphi_{z^2\overline{z}}}{(\varphi_{z\overline{z}})^2} - 4\,\frac{\varphi_{z\overline{z}^3}\,\varphi_{z^2\overline{z}^2}}{(\varphi_{z\overline{z}})^2} + \\ + 10\,\frac{\varphi_{z\overline{z}^3}\,\varphi_{z^2\overline{z}}\,\varphi_{z\overline{z}^2}}{(\varphi_{z\overline{z}})^3} + 15\,\frac{(\varphi_{z\overline{z}^2})^2\,\varphi_{z^2\overline{z}^2}}{(\varphi_{z\overline{z}})^3} - 15\,\frac{(\varphi_{z\overline{z}^2})^3\,\varphi_{z^2\overline{z}}}{(\varphi_{z\overline{z}})^4}\bigg),$$

and this therefore shows that *there is a spectacular contrast of computational complexity when passing from the rigid case to the general case*. The reason of this contrast mainly comes from the size of the Levi factor ℓ in (5) *appearing at denominator place in subsequent differentiations*.

Class IV_2 equivalences. Consider secondly a Class IV_2 hypersurface $M^5 \subset \mathbb{C}^3$, and let a Levi-kernel adapted frame for $\mathbb{C} \otimes_{\mathbb{R}} TM$ be:

$$\big\{\mathscr{T},\, \overline{\mathscr{L}}_1,\, \overline{\mathscr{K}},\, \mathscr{L}_1,\, \mathscr{K}\big\}, \qquad \overline{\mathscr{T}} = \mathscr{T},$$
$$\mathscr{T} := \sqrt{-1}\,\big[\mathscr{L}_1, \overline{\mathscr{L}}_1\big] = \ell\,\frac{\partial}{\partial u}, \qquad \ell := \sqrt{-1}\,\big(\mathscr{L}_1\big(\overline{A}_1\big) - \overline{\mathscr{L}}_1(A_1)\big).$$

Lemma 4 ([46, 62]) *The initial Lie structure of this frame consists of* $10 = \binom{5}{2}$ *brackets:*

$$\begin{aligned}
\big[\mathscr{T}, \overline{\mathscr{L}}_1\big] &= -\overline{P}\cdot\mathscr{T},\\
\big[\mathscr{T}, \overline{\mathscr{K}}\big] &= \overline{\mathscr{L}}_1\big(\overline{k}\big)\cdot\mathscr{T} + \mathscr{T}\big(\overline{k}\big)\cdot\overline{\mathscr{L}}_1,\\
\big[\mathscr{T}, \mathscr{L}_1\big] &= -P\cdot\mathscr{T},\\
\big[\mathscr{T}, \mathscr{K}\big] &= \mathscr{L}_1(k)\cdot\mathscr{T} + \mathscr{T}(k)\cdot\mathscr{L}_1,\\
\big[\overline{\mathscr{L}}_1, \overline{\mathscr{K}}\big] &= \overline{\mathscr{L}}_1\big(\overline{k}\big)\cdot\overline{\mathscr{L}}_1,\\
\big[\overline{\mathscr{L}}_1, \mathscr{L}_1\big] &= \sqrt{-1}\,\mathscr{T},\\
\big[\overline{\mathscr{L}}_1, \mathscr{K}\big] &= \overline{\mathscr{L}}_1(k)\cdot\mathscr{L}_1,\\
\big[\overline{\mathscr{K}}, \mathscr{L}_1\big] &= -\,\mathscr{L}_1\big(\overline{k}\big)\cdot\overline{\mathscr{L}}_1,\\
\big[\overline{\mathscr{K}}, \mathscr{K}\big] &= 0,\\
\big[\mathscr{L}_1, \mathscr{K}\big] &= \mathscr{L}_1(k)\cdot\mathscr{L}_1,
\end{aligned}$$

in terms of the 2 *fundamental functions:*

$$k := -\frac{\mathscr{L}_2\big(\overline{A_1}\big) - \overline{\mathscr{L}}_1\big(A_2\big)}{\mathscr{L}_1\big(\overline{A_1}\big) - \overline{\mathscr{L}}_1\big(A_1\big)}, \qquad\qquad P := \frac{\ell_{z_1} + A_1\,\ell_u - \ell\,A_{1,u}}{\sqrt{-1}\,\big(\mathscr{L}_1\big(\overline{A}_1\big) - \overline{\mathscr{L}}_1\big(A_1\big)\big)}$$

(in [62], M^5 is graphed as $u = F(x_1, y_1, x_2, y_2, v)$ instead, hence P changes).

Introduce then the coframe:

$$\big\{\rho_0,\, \overline{\kappa}_0,\, \overline{\zeta}_0,\, \kappa_0,\, \zeta_0\big\}$$

which is dual to the frame:

$$\big\{\mathscr{T},\, \overline{\mathscr{L}}_1,\, \overline{\mathscr{K}},\, \mathscr{L}_1,\, \mathscr{K}\big\},$$

the notations being the same as in Proposition 1, that is to say:

$$\begin{aligned}
\rho_0 &= \frac{du - A_1 dz_1 - A_2 dz_2 - \overline{A}_1 d\overline{z}_1 - \overline{A}_2 d\overline{z}_2}{\ell}, \\
\kappa_0 &= dz_1 - k\, dz_2, \\
\zeta_0 &= dz_2.
\end{aligned}$$

The initial Darboux structure is:

$$\begin{aligned}
d\rho_0 &= \overline{P}\cdot \rho_0 \wedge \overline{\kappa}_0 - \overline{\mathscr{L}}_1\big(\overline{k}\big)\cdot \rho_0 \wedge \overline{\zeta}_0 + P\cdot \rho_0 \wedge \kappa_0 - \mathscr{L}_1(k)\cdot \rho_0 \wedge \zeta_0 + \sqrt{-1}\,\kappa_0 \wedge \overline{\kappa}_0, \\
d\overline{\kappa}_0 &= -\,\mathscr{T}\big(\overline{k}\big)\cdot \rho_0 \wedge \overline{\zeta}_0 - \overline{\mathscr{L}}_1\big(\overline{k}\big)\cdot \overline{\kappa}_0 \wedge \overline{\zeta}_0 + \mathscr{L}_1\big(\overline{k}\big)\cdot \overline{\zeta}_0 \wedge \kappa_0, \\
d\overline{\zeta}_0 &= 0, \\
d\kappa_0 &= -\,\mathscr{T}(k)\cdot \rho_0 \wedge \zeta_0 - \overline{\mathscr{L}}_1(k)\cdot \overline{\kappa}_0 \wedge \zeta_0 - \mathscr{L}_1(k)\cdot \kappa_0 \wedge \zeta_0, \\
d\zeta_0 &= 0.
\end{aligned}$$

The initial associated G-structure is:

$$\mathsf{G}_{\mathsf{IV}_2}^{\mathsf{initial}} := \left\{ \begin{pmatrix} \mathsf{c} & 0 & 0 & 0 & 0 \\ \mathsf{b} & \mathsf{a} & 0 & 0 & 0 \\ 0 & 0 & \overline{\mathsf{c}} & 0 & 0 \\ 0 & 0 & \overline{\mathsf{b}} & \overline{\mathsf{a}} & 0 \\ \mathsf{e} & \mathsf{d} & \overline{\mathsf{e}} & \overline{\mathsf{d}} & \mathsf{a}\overline{\mathsf{a}} \end{pmatrix} \in \mathscr{M}_{5\times 5}(\mathbb{C}) :\ \mathsf{a},\, \mathsf{c} \in \mathbb{C}\backslash\{0\},\ \mathsf{b},\, \mathsf{d},\, \mathsf{e} \in \mathbb{C} \right\}.$$

Theorem 7 (Pocchiola, [62]) *Two fundamental explicit invariants both having denominators related to the nondegeneracy of the Freeman form:*

$$W := \frac{2}{3}\,\frac{\mathscr{L}_1\big(\overline{\mathscr{L}}_1(k)\big)}{\overline{\mathscr{L}}_1(k)} + \frac{2}{3}\,\frac{\mathscr{L}_1\big(\mathscr{L}_1(\bar{k})\big)}{\mathscr{L}_1(\bar{k})} + \\ + \frac{1}{3}\,\frac{\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(k)\big)\,\mathscr{K}\big(\overline{\mathscr{L}}_1(k)\big)}{\overline{\mathscr{L}}_1(k)^3} - \frac{1}{3}\,\frac{\mathscr{K}\big(\overline{\mathscr{L}}_1\big(\overline{\mathscr{L}}_1(k)\big)\big)}{\overline{\mathscr{L}}_1(k)^2} + \frac{i}{3}\,\frac{\mathscr{T}(k)}{\overline{\mathscr{L}}_1(k)},$$

and:

$$J := \frac{5}{18}\,\frac{\mathscr{L}_1\big(\mathscr{L}_1(\bar{k})\big)^2}{\mathscr{L}_1(\bar{k})^2} + \frac{1}{3}\,P\,\mathscr{L}_1(P) - \frac{1}{9}\,P^2\,\frac{\mathscr{L}_1\big(\mathscr{L}_1(\bar{k})\big)}{\mathscr{L}_1(\bar{k})} + \frac{20}{27}\,\frac{\mathscr{L}_1\big(\mathscr{L}_1(\bar{k})\big)^3}{\mathscr{L}_1(\bar{k})^3} - \\ - \frac{5}{6}\,\frac{\mathscr{L}_1\big(\mathscr{L}_1(\bar{k})\big)\,\mathscr{L}_1\big(\mathscr{L}_1\big(\mathscr{L}_1(\bar{k})\big)\big)}{\mathscr{L}_1(\bar{k})^2} + \frac{1}{6}\,\frac{\mathscr{L}_1\big(\mathscr{L}_1(\bar{k})\big)\,\mathscr{L}_1(P)}{\mathscr{L}_1(\bar{k})} - \\ -\frac{1}{6}\,P\,\frac{\mathscr{L}_1\big(\mathscr{L}_1\big(\mathscr{L}_1(\bar{k})\big)\big)}{\mathscr{L}_1(\bar{k})} - \frac{2}{27}\,P^3 - \frac{1}{6}\,\mathscr{L}_1\big(\mathscr{L}_1(P)\big) + \frac{1}{6}\,\frac{\mathscr{L}_1\big(\mathscr{L}_1\big(\mathscr{L}_1\big(\mathscr{L}_1(\bar{k})\big)\big)\big)}{\mathscr{L}_1(\bar{k})},$$

occur in the local biholomorphic equivalence problem for Class IV_2 *real analytic hypersurfaces* $M^5 \subset \mathbb{C}^3$*. Such a* M^5 *is locally biholomorphically equivalent to the light cone model:*

$$u = \frac{z_1\bar{z}_1 + \frac{1}{2}\,z_1z_1\bar{z}_2 + \frac{1}{2}\,z_2\bar{z}_1\bar{z}_1}{1 - z_2\bar{z}_2} \underset{\substack{\text{locally}\\ \text{by [24]}}}{\cong} (\mathsf{Re}\,z_1')^2 - (\mathsf{Re}\,z_2')^2 - (\mathsf{Re}\,z_3')^2, \qquad (\mathsf{LC})$$

having 10*-dimensional symmetry group* $\mathsf{Aut}_{CR}(\mathsf{LC}) \cong \mathsf{Sp}(4,\mathbb{R})$*, if and only if:*

$$0 \equiv W \equiv J.$$

Moreover, if either $W \not\equiv 0$ *or* $J \not\equiv 0$*, an absolute parallelism is constructed on* M *(after relocalization), and in this case, the local Lie group of* $\mathscr{C}^\omega$ *CR automorphisms of* M *always has:*

$$\dim \mathsf{Aut}_{CR}(M) \leqslant 5.$$

The latter dimension drop was obtained by Fels-Kaup [25] under the assumption that $\mathsf{Aut}_{CR}(M)$ is locally transitive, while Cartan's method embraces *all* Class IV_2 hypersurfaces $M^5 \subset \mathbb{C}^3$.

Reduction to an absolute parallelism on a 10-dimensional bundle $N^{10} \to M$ has been obtained previously by Isaev-Zaitsev [31] and Medori-Spiro [37]. The explicitness of W and J, the equivalence bifurcation $W \not\equiv 0$ or $J \not\equiv 0$ and the dimension drop $10 \to 5$ provide a complementary aspect. Furthermore, here is an application of the explicit rational expressions of J and W in the spirit of Theorem 2.

Corollary 1 *Let $M^5 \subset \mathbb{C}^3$ be a connected $\mathscr{C}^\omega$ hypersurface whose Levi form is of rank* 1 *at Zariski-generic points, possibly of rank* 0 *somewhere, and whose Freeman form is also nondegenerate at Zariski-generic points. If M is locally biholomorphic to the light cone model* (LC) *at some Freeman nondegenerate point, then M is also locally biholomorphic to* (LC) *at every other Freeman nondegenerate point.*

Equivalences of remaining Classes II, III_1, III_2, IV_1. Class II has been treated optimally by Beloshapka-Ezhov-Schmalz [4] who directly constructed a Cartan connection on a principal bundle $N^5 \to M^4$ with fiber $\cong \mathbb{R}$. Recently, Pocchiola [63] provided an alternative construction the elements of which are computed deeper.

Class IV_1 is reduced to an absolute parallelism with a Cartan connection by Chern-Moser [10] inspired by Hachtroudi [29], though not explicity in terms of a local graphing function (question still open).

Recently, jointly with Sabzevari, Class III_1 has been recently settled. Beloshapka (*see* also [1]), proved that the Lie algebra $\mathfrak{aut}_{CR} = 2\,\mathrm{Re}\,\mathfrak{hol}$ of Aut_{CR} of the cubic:

$$\text{Model } \mathsf{III}_1\text{:} \qquad v_1 = z\bar{z}, \qquad v_2 = z^2\bar{z} + z\bar{z}^2, \qquad v_3 = \sqrt{-1}\left(z^2\bar{z} - z\bar{z}^2\right),$$

is 7-dimensional generated by:

$$\begin{aligned}
T &:= \partial_{w_1}, \\
S_1 &:= \partial_{w_2}, \\
S_2 &:= \partial_{w_3}, \\
L_1 &:= \partial_z + (2\sqrt{-1}\,z)\,\partial_{w_1} + (2\sqrt{-1}\,z^2 + 4w_1)\,\partial_{w_2} + 2z^2\,\partial_{w_3}, \\
L_2 &:= \sqrt{-1}\,\partial_z + (2\,z)\,\partial_{w_1} + (2z^2)\,\partial_{w_2} - (2\sqrt{-1}\,z^2 - 4w_1)\,\partial_{w_3}, \\
D &:= z\,\partial_z + 2w_1\,\partial_{w_1} + 3w_2\,\partial_{w_2} + 3w_3\,\partial_{w_3}, \\
R &:= \sqrt{-1}\,z\,\partial_z - w_3\,\partial_{w_2} + w_2\,\partial_{w_3}.
\end{aligned}$$

Hence for a CR-generic $M^5 \subset \mathbb{C}^4$ belonging to Class III_1 graphed as:

$$v_1 = \varphi_1(x, y, u_1, u_2, u_3), \qquad v_2 = \varphi_2(x, y, u_1, u_2, u_3), \qquad v_3 = \varphi_3(x, y, u_1, u_2, u_3),$$

reduction to an absolute parallelism on a certain bundle $N^7 \to M^5$ can be expected, and in fact, similarly as in Theorem 7, finer equivalence bifurcations will occur.

Lemma 1 showed that a generator for $T^{1,0}M$ is:

$$\mathscr{L} = \frac{\partial}{\partial z} + \frac{\Lambda_1}{\Delta}\frac{\partial}{\partial u_1} + \frac{\Lambda_2}{\Delta}\frac{\partial}{\partial u_2} + \frac{\Lambda_3}{\Delta}\frac{\partial}{\partial u_3},$$

where:

$$\Delta := \begin{vmatrix} \sqrt{-1} + \varphi_{1,u_1} & \varphi_{1,u_2} & \varphi_{1,u_3} \\ \varphi_{2,u_1} & \sqrt{-1} + \varphi_{2,u_2} & \varphi_{2,u_3} \\ \varphi_{3,u_1} & \varphi_{3,u_2} & \sqrt{-1} + \varphi_{3,u_3} \end{vmatrix},$$

and where:

$$\Lambda_1 := \begin{vmatrix} -\varphi_{1,z} & \varphi_{1,u_2} & \varphi_{1,u_3} \\ -\varphi_{2,z} & \sqrt{-1} + \varphi_{2,u_2} & \varphi_{2,u_3} \\ -\varphi_{3,z} & \varphi_{3,u_2} & \sqrt{-1} + \varphi_{3,u_3} \end{vmatrix},$$

with similar Λ_2, Λ_3. By definition, on a Class III_1 CR-generic $M^5 \subset \mathbb{C}^4$ the fields:

$$\big\{\overline{\mathscr{S}},\, \mathscr{S},\, \mathscr{T},\, \overline{\mathscr{L}},\, \mathscr{L}\big\},$$

where:

$$\mathscr{T} := \sqrt{-1}\,\big[\mathscr{L}, \overline{\mathscr{L}}\big], \qquad \mathscr{S} := \big[\mathscr{L},\, \mathscr{T}\big], \qquad \overline{\mathscr{S}} := \big[\overline{\mathscr{L}},\, \mathscr{T}\big],$$

make up a frame for $\mathbb{C} \otimes_{\mathbb{R}} TM^5$. Computing these (iterated) Lie brackets, there are certain coefficient-polynomials:

$$\Upsilon_i = \Upsilon_i\Big(\varphi_{1,x^jy^ku_1^{l_1}u_2^{l_2}u_3^{l_3}},\ \varphi_{2,x^jy^ku_1^{l_1}u_2^{l_2}u_3^{l_3}},\ \varphi_{3,x^jy^ku_1^{l_1}u_2^{l_2}u_3^{l_3}}\Big)_{1\leqslant j+k+l_1+l_2+l_3\leqslant 2} \qquad (i=1,2,3),$$

$$\Pi_i = \Pi_i\Big(\varphi_{1,x^jy^ku_1^{l_1}u_2^{l_2}u_3^{l_3}},\ \varphi_{2,x^jy^ku_1^{l_1}u_2^{l_2}u_3^{l_3}},\ \varphi_{3,x^jy^ku_1^{l_1}u_2^{l_2}u_3^{l_3}}\Big)_{1\leqslant j+k+l_1+l_2+l_3\leqslant 3} \qquad (i=1,2,3),$$

so that (mind exponents in denominators):

$$\begin{aligned}
\mathscr{T} &= \frac{\Upsilon_1}{\Delta^2\,\overline{\Delta}^2}\,\frac{\partial}{\partial u_1} + \frac{\Upsilon_2}{\Delta^2\,\overline{\Delta}^2}\,\frac{\partial}{\partial u_2} + \frac{\Upsilon_3}{\Delta^2\,\overline{\Delta}^2}\,\frac{\partial}{\partial u_3}, \\
\mathscr{S} &= \frac{\Pi_1}{\Delta^4\,\overline{\Delta}^3}\,\frac{\partial}{\partial u_1} + \frac{\Pi_2}{\Delta^4\,\overline{\Delta}^3}\,\frac{\partial}{\partial u_2} + \frac{\Pi_3}{\Delta^4\,\overline{\Delta}^3}\,\frac{\partial}{\partial u_3}, \\
\overline{\mathscr{S}} &= \frac{\overline{\Pi}_1}{\Delta^3\,\overline{\Delta}^4}\,\frac{\partial}{\partial u_1} + \frac{\overline{\Pi}_2}{\Delta^3\,\overline{\Delta}^4}\,\frac{\partial}{\partial u_2} + \frac{\overline{\Pi}_3}{\Delta^3\,\overline{\Delta}^4}\,\frac{\partial}{\partial u_3}.
\end{aligned}$$

Explicitness obstacle. The expansions of Υ_1, Υ_2, Υ_3 as polynomials in their $3 \cdot 20$ variables incorporate 41 964 monomials while those of Π_1, Π_2, Π_3 as polynomials in their $3 \cdot 55$ variables would incorporate more than (no computer succeeded) 100 000 000 terms.

Hence one never can achieve complete explicitness.

Between the 5 fields $\{\overline{\mathscr{S}}, \mathscr{S}, \mathscr{T}, \overline{\mathscr{L}}, \mathscr{L}\}$, there are $10 = \binom{5}{2}$ Lie brackets. Assign therefore formal names to the uncomputable appearing coefficient-functions.

Lemma 5 ([53]) *In terms of 5 fundamental coefficient-functions:*

$$P, \quad Q, \quad R, \quad A, \quad B,$$

the 10 Lie bracket relations write as:

$$\begin{aligned}
[\overline{\mathscr{S}}, \mathscr{S}] &= \overline{K}_{\mathsf{rpl}} \cdot \overline{\mathscr{S}} - K_{\mathsf{rpl}} \cdot \mathscr{S} - \sqrt{-1}\, J_{\mathsf{rpl}} \cdot \mathscr{T}, \\
[\overline{\mathscr{S}}, \mathscr{T}] &= -\overline{F}_{\mathsf{rpl}} \cdot \overline{\mathscr{S}} - \overline{G}_{\mathsf{rpl}} \cdot \mathscr{S} - \overline{E}_{\mathsf{rpl}} \cdot \mathscr{T}, \\
[\overline{\mathscr{S}}, \overline{\mathscr{L}}] &= -\overline{Q} \cdot \overline{\mathscr{S}} - \overline{R} \cdot \mathscr{S} - \overline{P} \cdot \mathscr{T}, \\
[\overline{\mathscr{S}}, \mathscr{L}] &= -\overline{B} \cdot \overline{\mathscr{S}} - B \cdot \mathscr{S} - A \cdot \mathscr{T}, \\
[\mathscr{S}, \mathscr{T}] &= -G_{\mathsf{rpl}} \cdot \overline{\mathscr{S}} - F_{\mathsf{rpl}} \cdot \mathscr{S} - E_{\mathsf{rpl}} \cdot \mathscr{T}, \\
[\mathscr{S}, \overline{\mathscr{L}}] &= -\overline{B} \cdot \overline{\mathscr{S}} - B \cdot \mathscr{S} - A \cdot \mathscr{T}, \\
[\mathscr{S}, \mathscr{L}] &= -R \cdot \overline{\mathscr{S}} - Q \cdot \mathscr{S} - P \cdot \mathscr{T}, \\
[\mathscr{T}, \overline{\mathscr{L}}] &= -\overline{\mathscr{S}}, \\
[\mathscr{T}, \mathscr{L}] &= -\mathscr{S}, \\
[\overline{\mathscr{L}}, \mathscr{L}] &= \sqrt{-1}\, \mathscr{T},
\end{aligned}$$

the coefficient-functions E_{rpl}, G_{rpl}, H_{rpl}, J_{rpl}, K_{rpl} *being secondary:*

$$\begin{aligned}
E_{\mathsf{rpl}} &= \sqrt{-1}\left(\mathscr{L}(A) - \overline{\mathscr{L}}(P) + A\overline{B} + BP - AQ - \overline{P}R\right), \\
F_{\mathsf{rpl}} &= \sqrt{-1}\left(\mathscr{L}(B) - \overline{\mathscr{L}}(Q) + A + B\overline{B} - R\overline{R}\right), \\
G_{\mathsf{rpl}} &= \sqrt{-1}\left(\mathscr{L}(\overline{B}) - \overline{\mathscr{L}}(R) + \overline{BB} + BR - P - \overline{B}Q - R\overline{Q}\right),
\end{aligned}$$

with similar, longer expressions for J_{rpl}, K_{rpl}.

Introduce then the coframe:

$$\{\overline{\sigma_0},\, \sigma_0,\, \rho_0,\, \overline{\zeta_0},\, \zeta_0\},$$

which is dual to the frame:

$$\{\overline{\mathscr{S}},\, \mathscr{S},\, \mathscr{T},\, \overline{\mathscr{L}},\, \mathscr{L}\}.$$

Organize the ten Lie brackets as a convenient auxiliary array:

$$
\begin{array}{lccccccccccc}
 & \overline{\mathscr{S}} & & \mathscr{S} & & \mathscr{T} & & \overline{\mathscr{L}} & & \mathscr{L} & \\
 & \boxed{d\overline{\sigma_0}} & & \boxed{d\sigma_0} & & \boxed{d\rho_0} & & \boxed{d\overline{\zeta_0}} & & \boxed{d\zeta_0} & \\
[\overline{\mathscr{S}}, \mathscr{S}] = & \overline{K}_{\mathsf{rpl}} \cdot \overline{\mathscr{S}} & + & -K_{\mathsf{rpl}} \cdot \mathscr{S} & + & -\sqrt{-1}\, J_{\mathsf{rpl}} \cdot \mathscr{T} & + & 0 & + & 0 & \boxed{\overline{\sigma_0} \wedge \sigma_0} \\
[\overline{\mathscr{S}}, \mathscr{T}] = & -\overline{F}_{\mathsf{rpl}} \cdot \overline{\mathscr{S}} & + & -\overline{G}_{\mathsf{rpl}} \cdot \mathscr{S} & + & -\overline{E}_{\mathsf{rpl}} \cdot \mathscr{T} & + & 0 & + & 0 & \boxed{\overline{\sigma_0} \wedge \rho_0} \\
[\overline{\mathscr{S}}, \overline{\mathscr{L}}] = & -\overline{Q} \cdot \overline{\mathscr{S}} & + & -\overline{R} \cdot \mathscr{S} & + & -\overline{P} \cdot \mathscr{T} & + & 0 & + & 0 & \boxed{\overline{\sigma_0} \wedge \overline{\zeta_0}} \\
[\overline{\mathscr{S}}, \mathscr{L}] = & -\overline{B} \cdot \overline{\mathscr{S}} & + & -B \cdot \mathscr{S} & + & -A \cdot \mathscr{T} & + & 0 & + & 0 & \boxed{\overline{\sigma_0} \wedge \zeta_0} \\
[\mathscr{S}, \mathscr{T}] = & -G_{\mathsf{rpl}} \cdot \overline{\mathscr{S}} & + & -F_{\mathsf{rpl}} \cdot \mathscr{S} & + & -E_{\mathsf{rpl}} \cdot \mathscr{T} & + & 0 & + & 0 & \boxed{\sigma_0 \wedge \rho_0} \\
[\mathscr{S}, \overline{\mathscr{L}}] = & -\overline{B} \cdot \overline{\mathscr{S}} & + & -B \cdot \mathscr{S} & + & -A \cdot \mathscr{T} & + & 0 & + & 0 & \boxed{\sigma_0 \wedge \overline{\zeta_0}} \\
[\mathscr{S}, \mathscr{L}] = & -R \cdot \overline{\mathscr{S}} & + & -Q \cdot \mathscr{S} & + & -P \cdot \mathscr{T} & + & 0 & + & 0 & \boxed{\sigma_0 \wedge \zeta_0} \\
[\mathscr{T}, \overline{\mathscr{L}}] = & -\overline{\mathscr{S}} & + & 0 & + & 0 & + & 0 & + & 0 & \boxed{\rho_0 \wedge \overline{\zeta_0}} \\
[\mathscr{T}, \mathscr{L}] = & 0 & + & -\mathscr{S} & + & 0 & + & 0 & + & 0 & \boxed{\rho_0 \wedge \zeta_0} \\
[\overline{\mathscr{L}}, \mathscr{L}] = & 0 & + & 0 & + & \sqrt{-1}\, \mathscr{T} & + & 0 & + & 0 & \boxed{\overline{\zeta_0} \wedge \zeta_0}
\end{array}
$$

Read *vertically* and put an overall minus sign to get the *initial Darboux structure*:

$$
\begin{aligned}
d\overline{\sigma_0} &= -\overline{K}_{\mathsf{rpl}} \cdot \overline{\sigma_0} \wedge \sigma_0 + \overline{F}_{\mathsf{rpl}} \cdot \overline{\sigma_0} \wedge \rho_0 + \overline{Q} \cdot \overline{\sigma_0} \wedge \overline{\zeta}_0 + \overline{B} \cdot \overline{\sigma_0} \wedge \zeta_0 + \\
&\qquad + G_{\mathsf{rpl}} \cdot \sigma_0 \wedge \rho_0 + \overline{B} \cdot \sigma_0 \wedge \overline{\zeta}_0 + R \cdot \sigma_0 \wedge \zeta_0 + \rho_0 \wedge \overline{\zeta}_0, \\
d\sigma_0 &= K_{\mathsf{rpl}} \cdot \overline{\sigma_0} \wedge \sigma_0 + \overline{G}_{\mathsf{rpl}} \cdot \overline{\sigma_0} \wedge \rho_0 + \overline{R} \cdot \overline{\sigma_0} \wedge \overline{\zeta}_0 + B \cdot \overline{\sigma_0} \wedge \zeta_0 + \\
&\qquad + F_{\mathsf{rpl}} \cdot \sigma_0 \wedge \rho_0 + B \cdot \sigma_0 \wedge \overline{\zeta}_0 + Q \cdot \sigma_0 \wedge \zeta_0 + \rho_0 \wedge \zeta_0, \\
d\rho_0 &= \sqrt{-1}\, J_{\mathsf{rpl}} \cdot \overline{\sigma_0} \wedge \sigma_0 + \overline{E}_{\mathsf{rpl}} \cdot \overline{\sigma_0} \wedge \rho_0 + \overline{P} \cdot \overline{\sigma_0} \wedge \overline{\zeta}_0 + A \cdot \overline{\sigma_0} \wedge \zeta_0 + \\
&\qquad + E_{\mathsf{rpl}} \cdot \sigma_0 \wedge \rho_0 + A \cdot \sigma_0 \wedge \overline{\zeta}_0 + P \cdot \sigma_0 \wedge \zeta_0 - \sqrt{-1}\, \overline{\zeta}_0 \wedge \zeta_0, \\
d\overline{\zeta}_0 &= 0, \\
d\zeta_0 &= 0.
\end{aligned}
$$

The initial G-structure is:

$$
\mathsf{G}^{\mathsf{initial}}_{\mathsf{III}_1} := \left\{ \begin{pmatrix} \mathsf{a} & 0 & 0 & 0 & 0 \\ 0 & \overline{\mathsf{a}} & 0 & 0 & 0 \\ \mathsf{b} & \overline{\mathsf{b}} & \mathsf{a}\overline{\mathsf{a}} & 0 & 0 \\ \mathsf{e} & \mathsf{d} & \mathsf{c} & \mathsf{a}\mathsf{a}\overline{\mathsf{a}} & 0 \\ \overline{\mathsf{d}} & \overline{\mathsf{e}} & \overline{\mathsf{c}} & 0 & \mathsf{a}\overline{\mathsf{a}\mathsf{a}} \end{pmatrix} \in \mathscr{M}_{5\times 5}(\mathbb{C}) \colon \ \mathsf{a} \in \mathbb{C}\backslash\{0\}, \ \mathsf{b}, \mathsf{c}, \mathsf{d}, \mathsf{e} \in \mathbb{C} \right\}.
$$

The lifted coframe is:

$$\begin{pmatrix} \overline{\sigma} \\ \sigma \\ \rho \\ \overline{\zeta} \\ \zeta \end{pmatrix} := \begin{pmatrix} \mathsf{a}\overline{\mathsf{a}}\overline{\mathsf{a}} & 0 & 0 & 0 & 0 \\ 0 & \mathsf{a}\mathsf{a}\overline{\mathsf{a}} & 0 & 0 & 0 \\ \overline{\mathsf{c}} & \mathsf{c} & \mathsf{a}\overline{\mathsf{a}} & 0 & 0 \\ \overline{\mathsf{e}} & \mathsf{d} & \overline{\mathsf{b}} & \overline{\mathsf{a}} & 0 \\ \overline{\mathsf{d}} & \mathsf{e} & \mathsf{b} & 0 & \mathsf{a} \end{pmatrix} \begin{pmatrix} \overline{\sigma_0} \\ \sigma_0 \\ \rho_0 \\ \overline{\zeta_0} \\ \zeta_0 \end{pmatrix}.$$

Performing absorption of torsion and normalization of group variables thanks to remaining essential torsion [53, 58], the coefficient-function R is an invariant which creates bifurcation. Even in terms of P, Q, R, A, B, the expressions of some of the curvatures happen to be large and the study of their mutual independencies requires to take account of iterated Jacobi identities, an aspect of the subject which remains invisible in non-parametric Cartan method.

Theorem 8 (M.-Sabzevari, [53]) *Within the branch $R = 0$, the biholomorphic equivalence problem for $M^5 \subset \mathbb{C}^4$ in Class III_1 reduces to various absolute parallelisms namely to $\{e\}$-structures on certain manifolds of dimension 6, or directly on the 5-dimensional basis M, unless all existing essential curvatures vanish identically, in which case M is (locally) biholomorphic to the cubic model with a characterization of such a condition being explicit in terms of the five fundamental functions P, Q, R, A, B.*

Within the branch $R \not\equiv 0$, reduction to an absolute parallelism on the 5-dimensional basis M always takes place, whence:

$$\dim \mathsf{Aut}_{CR}(M) \leqslant 5.$$

Class III_2 was recently settled also.

Theorem 9 (Pocchiola, [64]) *If $M^5 \subset \mathbb{C}^4$ is a local $\mathscr{C}^\omega$ CR-generic submanifold belonging to Class III_1, then there exists a 6-dimensional principal bundle $P^6 = M^5 \times \mathbb{R}^*$ and there exists a coframe on P^6:*

$$\varpi := \big(\lambda,\, \tau,\, \sigma,\, \rho,\, \zeta,\, \overline{\zeta}\big)$$

such that any local $\mathscr{C}^\omega$ CR-diffeomorphism $\mathsf{H}_M \colon M \to M$ lifts as a bundle isomorphism $\hat{\mathsf{H}}_M \colon P \to P$ which satisfies $\mathsf{H}^*(\varpi) = \varpi$. Moreover, the structure equations of ϖ on P are of the form:

$$d\tau = 4\,\lambda \wedge \tau + I_1\,\tau \wedge \zeta - I_1\,\tau \wedge \overline{\zeta} + 3\,I_1\,\sigma \wedge \rho + \sigma \wedge \zeta + \sigma \wedge \overline{\zeta},$$

$$d\sigma = 3\,\lambda \wedge \sigma + I_2\,\tau \wedge \rho + I_3\,\tau \wedge \zeta + \overline{I}_3\,\tau \wedge \overline{\zeta} + I_4\,\sigma \wedge \rho -$$
$$- \tfrac{1}{2}\,I_1\,\sigma \wedge \zeta + \tfrac{1}{2}\,I_1\,\sigma \wedge \overline{\zeta} + \rho \wedge \zeta + \rho \wedge \overline{\zeta},$$

$$d\rho = 2\,\lambda \wedge \rho + I_5\,\tau \wedge \sigma + I_6\,\tau \wedge \rho + I_7\,\tau \wedge \zeta + \overline{I}_7\,\tau \wedge \overline{\zeta} + I_8\,\sigma \wedge \rho + \\ + I_9\,\sigma \wedge \zeta + \overline{I}_9\,\sigma \wedge \overline{\zeta} - \tfrac{1}{2}\,I_1\,\rho \wedge \zeta + \tfrac{1}{2}\,I_1\,\rho \wedge \overline{\zeta} + \sqrt{-1}\,\zeta \wedge \overline{\zeta},$$

$$d\zeta = \lambda \wedge \zeta + I_{10}\,\tau \wedge \sigma + I_{11}\,\tau \wedge \rho + I_{12}\,\tau \wedge \zeta + I_{13}\,\tau \wedge \overline{\zeta} + I_{14}\,\sigma \wedge \rho + I_{15}\,\sigma \wedge \zeta,$$

for function $I_\bullet$, $J_{\bullet\bullet}$ on P together with:

$$d\lambda = \sum_{\nu,\mu} J_{\nu\mu}\,\nu \wedge \mu \qquad \big(\mu, \nu = \tau, \sigma, \rho, \zeta, \overline{\zeta}\big).$$

For both Classes III_1 and III_2, there also exist canonical Cartan connections naturally related to the final $\{e\}$-structures [54, 64].

These works complete the program of performing parametric *Cartan equivalences for all CR manifolds up to dimension* 5.

In dimension 6, Ezhov-Isaev-Schmalz [22] treated elliptic and hyperbolic $M^6 \subset \mathbb{C}^4$. A wealth of higher dimensional biholomorphic equivalence problems exists, e.g. [34] for CR-generic $M^{2+c} \subset \mathbb{C}^{1+c}$ in relation to classification of nilpotent Lie algebras [26].

3 Kobayashi Hyperbolicity

The dominant theme is the *interplay* between the extrinsic projective geometry of algebraic subvarieties of $\mathbb{P}^n(\mathbb{C})$ and their intrinsic geometric features. Phillip Griffiths.

In local Cartan theory, as seen in what precedes, denominators therefore play a central role in the differential ring generated by derivatives of the fundamental graphing functions φ_j. Similarly, in arithmetics of rational numbers p/q, like e.g. in multizeta calculus [44] involving a wealth of nested Cramer-type determinants, a growing complexity, potentially infinite, exists, and in fact, the complexity of rational numbers *also enters* high order covariant derivatives of Cartan curvatures, as an expression of the unity of mathematics. It is now time to show how *explicit rationality* also concerns the core of global algebraic geometry.

Let X be a compact complex n-dimensional projective manifold that is of *general type*, namely whose canonical bundle $K_X = \Lambda^n T_X^*$ is *big* in the sense that $\dim H^0\big(X, (K_X)^{\otimes m}\big) \geqslant c\, m^n$ when $m \to \infty$ for some constant $c > 0$. It is known that smooth hypersurfaces $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ are so if and only if their degree d is $\geqslant n+3$, since the *adjunction formula* shows that $K_X \cong \mathscr{O}_X(d-n-2)$ (the related rationality aspects will be discussed later).

A conjecture of Green-Griffiths-Lang expects that all nonconstant entire holomorphic curves $f\colon \mathbb{C} \to X$ should in fact land (be 'canalized') inside a certain proper subvariety $Y \subsetneqq X$. The current state of the art is still quite (very) far from reaching such a statement in this optimality. Furthermore, a companion Picard-type conjecture dating back to Kobayashi 1970 expects that all entire curves valued in Zariski-generic hypersurfaces $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ of degree $d \geqslant 2n + 1$ should necessarily be *constant* (see [32, 35, 67]), and the state of the art is also still quite far from understanding properly when this occurs, even in dimension 2, because of the lack of an appropriate explicit rational theory.

The current general method towards these conjectures consists as a first step in setting up a great number of nonzero differential equations $P\big(f, f', f'', \dots, f^{(\kappa)}\big) = 0$ of some order κ (necessarily $\geqslant n$) satisfied by all nonconstant $f\colon \mathbb{C} \to X$, and then as a second step, in trying to *eliminate* from such numerous differential equations the true derivatives $f', f'', \dots, f^{(\kappa)}$ in order to get certain purely algebraic nonzero equations $Q(f) = 0$ involving no derivatives anymore.

In 1979, by computing the Euler-Poincaré characteristic of a natural vector bundle nowadays called the *Green-Griffiths jet bundle* $\mathscr{E}^{\rm GG}_{\kappa,m} T_X^* \to X$, and by relying upon a H^2-cohomology vanishing theorem due to Bogomolov, Green and Griffiths [27] showed the existence of differential equations satisfied by entire curves valued in smooth *surfaces* $X^2 \subset \mathbb{P}^3$ of degree $d \geqslant 5$. In 1996, a breakthrough article by Siu-Yeung [71] showed Kobayashi-hyperbolicity of complements $\mathbb{P}^2 \backslash X^1$ of generic curves of degree $\geqslant 10^{13}$ (rounding off). Around 2000, McQuillan [36], by importing ideas from (multi)foliation theory considered entire maps valued in compact surfaces of general type having Chern numbers $\mathsf{c}_1^2 - \mathsf{c}_2 > 0$, which, for the case of $X^2 \subset \mathbb{P}^3$, improved very substantially the degree bound to $d \geqslant 36$, and this was followed in a work of Demailly and El Goul (*see* [16]) by the improvement $d \geqslant 21$. Later, using the Demailly-Semple bundle of jets that are invariant under reparametrization of the source $\mathbb{C}$, Rousseau was the first to treat in great details threefolds $X^3 \subset \mathbb{P}^4$ and he established algebraic degeneracy of entire curves in degree $d \geqslant 593$ [65]. Previously, in two conference proceedings of the first 2000 years (ICM [68] and Abel Symposium [69]), Siu showed the existence of differential equations on hypersurfaces $X^n \subset \mathbb{P}^{n+1}$; the recent publication [70] of his extended preprint of that time confirmed the validity and the strength of his approach, which will be pursued *infra*.

Invariant jets used in [16, 21, 39, 65] are in fact deeply connected to rationality.

Indeed, another instance of the key role of denominators appears in computational invariant theory. Starting with an ideal $\mathscr{I} \subset \mathbb{C}[X_1, \dots, X_n]$ and with a nonzero $f \in \mathbb{C}[X_1, \dots, X_n]$, the *$f$-saturation* of $\mathscr{I}$ is:

$$\mathscr{I}^{\mathsf{sat}} \equiv \frac{\mathscr{I}}{f^\infty} \overset{\text{def}}{=} \big\{g \in \mathbb{C}[X]\colon f^m\, g \in \mathscr{I} \text{ for some } m \in \mathbb{N}\big\},$$

with increasing union stabilizing by noetherianity:

$$\frac{\mathscr{I}}{f} \subset \frac{\mathscr{I}}{f^2} \subset \frac{\mathscr{I}}{f^3} \subset \cdots \subset \frac{\mathscr{I}}{f^m} = \frac{\mathscr{I}}{f^{m+1}} = \cdots .$$

The *Kernel algorithm*, discovered in the nineteenth Century, consists in:

$$\begin{aligned} \langle \mathsf{g}_0, \ldots, \mathsf{g}_{n_0} \rangle &= \text{Initial ideal}, \\ \langle \mathsf{g}_0, \ldots, \mathsf{g}_{n_0}, \ldots, \mathsf{g}_{n_1} \rangle &= \text{Saturation}_f \langle \mathsf{g}_0, \ldots, \mathsf{g}_{n_0} \rangle, \\ \langle \mathsf{g}_0, \ldots, \mathsf{g}_{n_0}, \ldots, \mathsf{g}_{n_1}, \ldots, \mathsf{g}_{n_2} \rangle &= \text{Saturation}_f \langle \mathsf{g}_0, \ldots, \mathsf{g}_{n_0}, \ldots, \mathsf{g}_{n_1} \rangle, \qquad \textit{etc.}, \end{aligned}$$

and it has been applied in [39] to set up an algorithm which generates all polynomials in the κ-jet of a local holomorphic map $\mathbb{D} \to \mathbb{C}^n$, $\zeta \longmapsto (f_1(\zeta), \ldots, f_n(\zeta))$ from the unit disc $\mathbb{D} = \{|\zeta| < 1\}$ that are invariant under all biholomorphic reparametrizations of $\mathbb{D}$ with saturation with respect to the first derivative f_1'. The explicit generators for $n = 4 = \kappa$ and for $n = 2$, $\kappa = 5$ given in [39] show well that saturation (division) by f_1' generates some unpredictable complexity, a well known phenomenon in invariant theory.

Later, Berczi and Kirwan [7], by developing concepts and tools from reductive geometric invariant theory, showed that the concerned algebra is always finitely generated. A challenging still open question is to get information about the number of generators and about the structure of relations they share. In any case, the prohibitive complexity of these algebras still prevents to hope for reaching arbitrary dimension $n \geqslant 2$ and jet order $\kappa \geqslant n$ for Green-Griffiths and Kobayashi conjectures with invariant jets.

Around the same time, under the direction of Demailly and using an algebraic version of holomorphic Morse inequalities delineated by Trapani, Diverio studied a certain *subbundle* of the bundle of invariant jets, already introduced before by Demailly in [16]. This, for the first time after Siu, opened the door to arbitrary dimension $n \geqslant 2$, though this was clearly not sufficient to reach the first step towards the Green-Griffiths-Lang conjecture. In fact, an inspection of [21] shows that on hypersurfaces $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ of degree d approximately $\geqslant 2^{n^5}$ (rounding off), many differential equations exist with jet order $\kappa = n$ equal to the dimension, but when increasing the jet order $\kappa = n+1, n+2, \ldots$, an unpleasant stabilization of the degree gain occurs, so that there is absolutely no hope to reach the optimal $d \geqslant n + 3$ for the first step towards the Green-Griffiths-Lang conjecture (as did Green-Griffiths in 1979 in dimension $n = 2$) with Diverio's technique (even) for hypersurfaces $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$.

For all of these reasons, it became undoubtedly clear that the whole theory had to come back to the bundle of (plain) Green-Griffiths jets.

On an n-dimensional complex manifold X^n, for a jet order $\kappa \geqslant 1$ and for an homogeneous order $m \geqslant 1$, the *Green-Griffiths bundle* $\mathscr{E}_{\kappa,m}^{\mathsf{GG}} T_X^* \longrightarrow X$ in a local chart $(z_1, \ldots, z_n)\colon \mathsf{U} \longrightarrow \mathbb{C}^n$ with $\mathsf{U} \subset X$ open, has general local holomorphic sections which are polynomials in the derivatives $z', z'', \ldots, z^{(\kappa)}$ of the z_i (considered as functions of a single variable $\zeta \in \mathbb{D}$) of the form:

$$\sum_{|\alpha_1|+2\,|\alpha_2|+\cdots+\kappa\,|\alpha_\kappa|=m} P_{\alpha_1,\ldots,\alpha_\kappa}(z)\,\big(z'\big)^{\alpha_1}\,\big(z''\big)^{\alpha_2}\,\cdots\,\big(z^{(\kappa)}\big)^{\alpha_\kappa},$$

the $P_{\alpha_1,\dots,\alpha_\kappa}$ being holomorphic in U (this local definition ignores rationality features of the $P_{\alpha,\dots,\alpha_\kappa}$ which will be explored *infra*).

The memoir [43] established that on a hypersurface $X = X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ of degree:

$$d \geqslant n + 3,$$

if $\mathscr{A} \to X$ is any ample line bundle—take e.g. simply $\mathscr{A} := \mathscr{O}_X(1)$—, then:

$$h^0\big(X,\, \mathscr{E}_{\kappa,m}^{GG} T_X^* \otimes \mathscr{A}^{-1}\big) \geqslant \frac{m^{(\kappa+1)n-1}}{(\kappa!)^n\,((\kappa+1)n-1)!}\Big\{\tfrac{(\log\kappa)^n}{n!} d(d-n-2)^n - \mathsf{Constant}_{n,d}\cdot(\log\kappa)^{n-1}\Big\} - \mathsf{Constant}_{n,d,\kappa}\cdot m^{(\kappa+1)n-2},$$

a formula in which the right-hand side minorant visibly tends to ∞, as soon as both $\kappa \geqslant \kappa_{n,d}^0$ and $m \geqslant m_{n,d,\kappa}^0$ do (no explicit expressions of the constants was provided there). This, then, generalized to dimension $n \geqslant 2$ the Green and Griffiths surface theorem, by estimating the asymptotic quantitative behavior of weighted Young diagrams and by applying partial (good enough) results of Brückmann [9] concerning the cohomology of Schur bundles $\mathscr{S}^{(\ell_1,\dots,\ell_n)}T_X^*$.

Also coming back to plain Green-Griffiths jets, but developing completely different elaborate negative jet curvature estimates which go back to an article of Cowen and Griffiths [11] and which had been 'in the air' for some time, though 'blocked for deep reasons' by the untractable algebraic complexity of invariant jets, Demailly [17] realized the next significant advance towards the conjecture by establishing, under the sole assumption that X be of general type (not necessarily a hypersurface), that nonconstant entire holomorphic curves $f\colon \mathbb{C} \longrightarrow X$ always satisfy (many) nonzero differential equations. The Bourbaki Seminar 1061 by Paŭn [60] is a useful guide to enter the main concepts and techniques of the topic.

However, according to [20], it is impossible to reach the Green-Griffiths conjecture for *all* general type compact complex manifolds by applying the jet differential technique. This justifies to restrict attention to hypersurfaces or to complete intersections in the projective space, and in this case, as recently highlighted once again by Siu [70], the only convincing strategy towards a first solution to Kobayashi's conjecture in arbitrary dimension $n \geqslant 1$ is to develop a new systematic theory of explicit *rational* holomorphic sections of jet bundles.

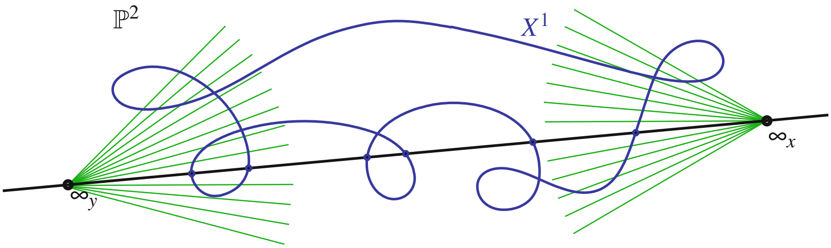
3.1 Holomorphic Jet Differentials

Let $[X_0 \colon X_1 \colon \cdots \colon X_n] \in \mathbb{P}^n(\mathbb{C})$ be homogeneous coordinates. Recall that for $t \in \mathbb{N}$, holomorphic sections of $\mathscr{O}_{\mathbb{P}^n}(t)$ are represented on $\mathsf{U}_i = \{X_i \neq 0\}$ as quotients:

$$\ell_i([X]) := \frac{P(X_0 \colon X_1 \colon \cdots \colon X_n)}{(X_i)^t},$$

for some polynomial $P \in \mathbb{C}[X]$ homogeneous of degree t, while, for $t \in \mathbb{Z}\backslash\mathbb{N}$, meromorphic sections are:

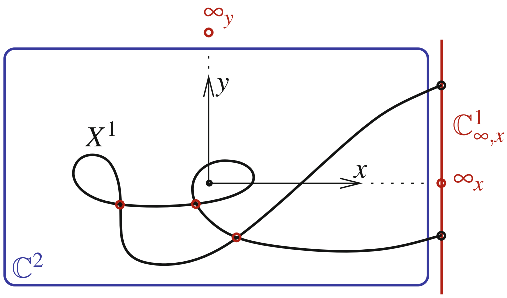
$$\ell_i([X]) := \frac{(X_i)^t}{P(X_0 \colon X_1 \colon \cdots \colon X_n)}.$$



To review another global rationality phenomenon, consider a complex algebraic curve X^1 smooth or with simple normal crossings in $\mathbb{P}^2(\mathbb{C})$ of degree $d \geqslant 1$, choose two points $\infty_x, \infty_y \notin X$ so that the line $\overline{\infty_x \infty_y}$ intersects X^1 transversally in d distinct points, and adapt homogeneous coordinates $[T \colon X \colon Y] \in \mathbb{P}^2$ with affine $x := \frac{X}{T}$ and $y := \frac{Y}{T}$ so that $\overline{\infty_x \infty_y} = \mathbb{P}^\infty_1 = \{[0 \colon X \colon Y]\}$, $\infty_x = [0 \colon 1 \colon 0]$, $\infty_y = [0 \colon 0 \colon 1]$, whence:

$$X^1 \cap \mathbb{C}^2_{(x,y)} = \big\{(x,y) \in \mathbb{C}^2 \colon R(x,y) = 0\big\},$$

for some polynomial $R \in \mathbb{C}[x,y]$ of degree d.



Within the intrinsic theory, in terms of the ambient line bundles $\mathscr{O}_{\mathbb{P}^2}(t) \longrightarrow \mathbb{P}^2$ ($t \in \mathbb{Z}$), the adjunction formula tells:

$$T_X^* \;\cong\; \mathscr{O}_X(d-3) \;\overset{\text{def}}{=}\; \mathscr{O}_{\mathbb{P}^2}(d-3)\big|_X.$$

The genus formula is of great importance, because it exposes the relationship between the '*intrinsic*' topological invariant g of the curve X^1 and the '*extrinsic*' quantity d. Phillip Griffiths.

Theorem 10 (Inspirational) *On a smooth degree d algebraic curve $X^1 \subset \mathbb{P}^2$:*

$$\tfrac{(d-1)(d-2)}{2} \;=\; \mathsf{dim}\, H^0\big(X, T_X^*\big) \;=\; \mathsf{genus}(X) \;=\; g.$$

But the extrinsic theory [28] tells more. Differentiating once $0 = R(x,y)$:

$$0 = R_x\, dx + R_y\, dy \qquad\Longleftrightarrow\qquad \frac{dy}{R_x} \;=\; -\,\frac{dx}{R_y},$$

denominators must appear. If X^1 is smooth, $X^1 \cap \mathbb{C}^2 = \{R_x \neq 0\} \cup \{R_y \neq 0\}$, and global holomorphic sections of T_X^* are represented by multiplications:

$$G(x,y)\bigg(\frac{dy}{R_x} \;=\; -\,\frac{dx}{R_y}\bigg),$$

with $G \in \mathbb{C}[x,y]$ having degree $\leqslant d-3$, the space of such G being of dimension $\frac{(d-3+2)(d-3+1)}{2}$, since changing affine chart in order to capture $\mathbb{P}^1_\infty \backslash \{\infty_x\}$:

$$(x,y) \;\longmapsto\; \bigg(\frac{x}{y},\, \frac{1}{y}\bigg) =: \big(x_2, y_2\big), \qquad\qquad R_2(x_2,y_2) \;:=\; (y_2)^d\, R\bigg(\frac{x_2}{y_2},\, \frac{1}{y_2}\bigg),$$

knowing $dy = -\,dy_2/(y_2)^2$, the left side $G\, dy/R_x$ transfers to:

$$G(x,y)\, \frac{dy}{R_x} \;=\; G\bigg(\frac{x_2}{y_2},\, \frac{1}{y_2}\bigg)\, (y_2)^{d-3}\, \frac{-\,dy_2}{R_{2,x_2}(x_2,y_2)},$$

the denominator $R_{2,x_2}(x_2, y_2)$ being nonzero on X at every point of $\{y_2 = 0\} = \mathbb{C}^1_{\infty,y}$, while $(y_2)^{d-3}$ compensates the poles of $G\big(\frac{x_2}{y_2}, \frac{1}{y_2}\big)$ as soon as $\deg G \leqslant d-3$.

Next, the *intrinsic* Riemann-Roch theorem states that, given any divisor D on a compact, abstract, Riemann surface S, if $\mathscr{O}_D$ denotes the sheaf of meromorphic functions $f \in \Gamma(\mathscr{M}_S)$ with $\operatorname{div} f \geqslant -D$, then:

$$\dim H^0\big(S, \mathscr{O}_D\big) - \dim H^1\big(S, \mathscr{O}_D\big) = \deg D - \operatorname{genus}(S) + 1.$$

For compact Riemann surfaces S, there exists a satisfactory correspondence between intrinsic features and extrinsic embeddings: all S admit a representation as a curve $X^1 \subset \mathbb{P}^2$, smooth or having normal crossings singularities.

Using Brill-Noether duality, the Riemann-Roch theorem can be proved [28] for such $X^1 \subset \mathbb{P}^2$ by means of two inequalities:

$$\begin{aligned}\deg D - \mathsf{g}(S) + 1 &\leqslant \dim \big\{f \in \mathscr{M}(S)\colon \operatorname{div}(f) \geqslant -D\big\} = \dim H^0\big(S, \mathscr{O}_D\big),\\ -\deg D + \mathsf{g}(S) - 1 &\leqslant \dim \big\{\omega \in \mathscr{M}T^*_X(S)\colon \operatorname{div}(\omega) \geqslant +D\big\} = \dim H^1\big(S, \mathscr{O}_D\big).\end{aligned}$$

For instance, the second inequality is proved by means of *extrinsic* meromorphic differential forms:

$$\frac{G}{H}\left(\frac{dy}{R_x} = -\frac{dx}{R_y}\right),$$

with $G, H \in \mathbb{C}[x,y]$ subjected to appropriate conditions with respect to D.

Jets of order 2. Next, consider second order jets of holomorphic maps $\mathbb{D} \to X^1 \subset \mathbb{P}^2$, use x', y' instead of dx, dy, and x'', y''. Differentiate $0 \equiv R\big(x(\zeta), y(\zeta)\big)$ twice:

$$0 = x' R_x + y' R_y, \qquad\qquad 0 = x'' R_x + y'' R_y + (x')^2 R_{xx} + 2\,x'y' R_{xy} + (y')^2 R_{yy},$$

divide by $R_x R_y$, solve for y'', replace y' on the right:

$$\frac{y'}{R_x} = -\frac{x'}{R_y}, \qquad\qquad \frac{y''}{R_x} = -\frac{x''}{R_y} - \frac{(x')^2}{R_y}\left[\frac{R_{xx}}{\boxed{R_x}} - 2\,\frac{R_{xy}}{R_y} + \frac{R_x}{R_y}\frac{R_{yy}}{R_y}\right].$$

To erase the division by R_x, multiply the first equation by $x'\,\frac{R_{xx}}{R_x}$ and subtract:

$$\frac{y''}{R_x} - \frac{y'x'}{R_x}\frac{R_{xx}}{R_x} = -\frac{x''}{R_y} - \frac{(x')^2}{R_y}\left[-2\,\frac{R_{xy}}{R_y} + \frac{R_x}{R_y}\frac{R_{yy}}{R_y}\right].$$

Little further, this expression can be symmetrized [48]:

$$\frac{y''}{R_x}+\frac{(y')^2}{R_x}\bigg[-\frac{R_{xy}}{R_x}+\frac{R_y}{R_x}\frac{R_{xx}}{R_x}\bigg]=-\frac{x''}{R_y}-\frac{(x')^2}{R_y}\bigg[-\frac{R_{xy}}{R_y}+\frac{R_x}{R_y}\frac{R_{yy}}{R_y}\bigg],$$

and this provides *second order holomorphic jet differentials* on $X^1 \subset \mathbb{P}^2$ when $d \geqslant 4$, after checking holomorphicity on the $\mathbb{P}^1_\infty$. For jets of order 3 [48]:

$$\begin{aligned}
&\frac{y'''}{R_x}+\frac{y''y'}{R_x}\bigg[-3\frac{R_{xy}}{R_x}+3\bigg(\frac{R_y}{R_x}\bigg)\frac{R_{xx}}{R_x}\bigg]+\\
&\quad+\frac{(y')^3}{R_x}\bigg[-6\bigg(\frac{R_y}{R_x}\bigg)\frac{R_{xy}}{R_x}\frac{R_{xx}}{R_x}+3\bigg(\frac{R_y}{R_x}\bigg)^2\frac{R_{xx}}{R_x}\frac{R_{xx}}{R_x}+3\bigg(\frac{R_y}{R_x}\bigg)\frac{R_{xxy}}{R_x}-\bigg(\frac{R_y}{R_x}\bigg)^2\frac{R_{xxx}}{R_x}\bigg]=\\
&=-\frac{x'''}{R_y}-\frac{x''x'}{R_y}\bigg[-3\frac{R_{xy}}{R_y}+3\bigg(\frac{R_x}{R_y}\bigg)\frac{R_{yy}}{R_y}\bigg]-\\
&\quad-\frac{(x')^3}{R_y}\bigg[-6\bigg(\frac{R_x}{R_y}\bigg)\frac{R_{xy}}{R_y}\frac{R_{yy}}{R_y}+3\bigg(\frac{R_x}{R_y}\bigg)^2\frac{R_{yy}}{R_y}\frac{R_{yy}}{R_y}+3\bigg(\frac{R_x}{R_y}\bigg)\frac{R_{xyy}}{R_y}-\bigg(\frac{R_x}{R_y}\bigg)^2\frac{R_{yyy}}{R_y}\bigg].
\end{aligned}$$

Strikingly, and quite interestingly, there appear explicit rational expressions belonging to $\mathbb{Z}\big[\frac{R_{\cdot\cdot}}{R_x}\big]$ on the left, and to $\mathbb{Z}\big[\frac{R_{\cdot\cdot}}{R_y}\big]$ on the right.

Jets of arbitrary order. The *Green-Griffiths* bundle $\mathscr{E}^{\mathrm{GG}}_{\kappa,m}T^*_{X^1} \longrightarrow X^1 \subset \mathbb{P}^2$ consists, for a jet order $\kappa \geqslant 1$, in the m-homogeneous polynomialization of the bundle $J^\kappa\big(\mathbb{D}, X^1\big)$, and is a *vector* bundle of:

$$\mathsf{rank}\,\big(\mathscr{E}^{\mathrm{GG}}_{\kappa,m}T^*_{X^1}\big) \;=\; \mathsf{Card}\Big\{\big(m_1, m_2, \dots, m_\kappa\big)\in\mathbb{N}^\kappa\colon m_1+2\,m_2+\cdots+\kappa\,m_\kappa=m\Big\}.$$

It admits a natural filtration whose associated graded vector bundle is [43]:

$$\mathsf{Gr}^\bullet\mathscr{E}^{\mathrm{GG}}_{\kappa,m}T^*_{X^1}\;\cong\;\bigoplus_{\substack{m_1+\cdots+\kappa m_\kappa=m\\ m_1\geqslant 0,\,\dots,\,m_\kappa\geqslant 0}}\Big(\mathsf{Sym}^{m_1}T^*_X\otimes\cdots\otimes\mathsf{Sym}^{m_\kappa}T^*_X\Big),$$

whence:

$$\mathsf{Gr}^\bullet\mathscr{E}^{\mathrm{GG}}_{\kappa,m}T^*_{X^1}\;\cong\;\bigoplus_{\substack{m_1+\cdots+\kappa m_\kappa=m\\ m_1\geqslant 0,\,\dots,\,m_\kappa\geqslant 0}}\mathscr{O}_X\Big(\big(m_1+\cdots+m_\kappa\big)\,(d-3)\Big).$$

Knowing that for $t \geqslant d$:

$$\mathsf{dim}\,H^0\big(X,\mathscr{O}_X(t)\big)\;=\;\binom{t+2}{2}-\binom{t-d+2}{2},$$

and knowing $H^1\big(X, \mathscr{O}_X(t)\big) = 0$, it follows:

$$\dim H^0\big(X,\, \mathscr{E}_{\kappa,m}^{\rm GG} T_{X^1}^*\big) \,=\, \sum_{m_1+\cdots+\kappa m_\kappa = m} \bigg\{ \binom{(m_1+\cdots+m_\kappa)(d-3)+2}{2} - \\ - \binom{(m_1+\cdots+m_\kappa)(d-3)-d+2}{2} \bigg\},$$

which, asymptotically, becomes:

$$\dim H^0\big(X,\, \mathscr{E}_{\kappa,m}^{\rm GG} T_X^*\big) \,\geqslant\, \frac{m^\kappa}{\kappa!\,\kappa!} \Big[d^2 \,\log \kappa + d^2\, {\rm O}(1) + {\rm O}(d) \Big] + {\rm O}\big(m^{\kappa-1}\big).$$

Theorem 11 ([48]) *Given an arbitrary jet order $\kappa \geqslant 1$, for every $1 \leqslant \lambda \leqslant \kappa$, if $\deg R \geqslant \kappa + 3$, there exist perfectly symmetric expressions:*

$$\mathsf{J}_R^\lambda := \begin{cases} \dfrac{y^{(\lambda)}}{R_x} + \displaystyle\sum_{\mu_1+\cdots+(\lambda-1)\mu_{\lambda-1}=\lambda} \frac{\big(y'\big)^{\mu_1} \cdots \big(y^{(\lambda-1)}\big)^{\mu_{\lambda-1}}}{R_x} \, \mathscr{J}^\lambda_{\mu_1,\ldots,\mu_{\lambda-1}} \bigg(\frac{R_y}{R_x}, \Big(\frac{R_{x^iy^j}}{R_x} \Big)_{\substack{2 \leqslant i+j \leqslant \\ \leqslant -1+\mu_1+\cdots+\mu_{\lambda-1}}} \bigg), \\ -\dfrac{x^{(\lambda)}}{R_y} - \displaystyle\sum_{\mu_1+\cdots+(\lambda-1)\mu_{\lambda-1}=\lambda} \frac{\big(x'\big)^{\mu_1} \cdots \big(x^{(\lambda-1)}\big)^{\mu_{\lambda-1}}}{R_y} \, \mathscr{J}^\lambda_{\mu_1,\ldots,\mu_{\lambda-1}} \bigg(\frac{R_x}{R_y}, \Big(\frac{R_{y^ix^j}}{R_y} \Big)_{\substack{2 \leqslant i+j \leqslant \\ \leqslant -1+\mu_1+\cdots+\mu_{\lambda-1}}} \bigg), \\ 0 \qquad on \;\; X^1 \cap \mathbb{P}^1_\infty, \end{cases}$$

which define generating *holomorphic jet differentials on the smooth curve $X^1 \subset \mathbb{P}^2$, notably on the two open subsets:*

$$\begin{aligned} &\big\{R_x \neq 0\big\} \;\; \textit{where the bundle } J^\kappa(\mathbb{D}, X^1) \textit{ has intrinsic coordinates:} \\ &\qquad\qquad \big(y; y', y'', \ldots, y^{(\kappa)}\big), \\ &\big\{R_y \neq 0\big\} \;\; \textit{where the bundle } J^\kappa(\mathbb{D}, X^1) \textit{ has intrinsic coordinates:} \\ &\qquad\qquad \big(x; x', x'', \ldots, x^{(\kappa)}\big). \end{aligned}$$

All J_R^λ vanish on the ample divisor $X^1 \cap \mathbb{P}^1_\infty$, and involve universal polynomials:

$$\mathscr{J}^\lambda_{\mu_1,\ldots,\mu_{\lambda-1}} = \mathscr{J}^\lambda_{\mu_1,\ldots,\mu_{\lambda-1}} \Big(\mathsf{R}_{0,1},\, \big(\mathsf{R}_{i,j}\big)_{2 \leqslant i+j \leqslant -1+\mu_1+\cdots+\mu_{\lambda-1}} \Big)$$

with coefficients in $\mathbb{Z}$, and in terms of these generating jet differentials, holomorphic sections of $\mathscr{E}_{\kappa,m}^{\rm GG} T_{X^1}^*$ are generally represented as:

$$\sum_{m_1+2m_2+\cdots+\kappa m_\kappa = m} \big(\mathsf{J}_R^1\big)^{m_1} \big(\mathsf{J}_R^2\big)^{m_2} \cdots \big(\mathsf{J}_R^\kappa\big)^{m_\kappa} \cdot \mathsf{G}_{m_1,m_2,\ldots,m_\kappa}(x,y),$$

with polynomials:

$$\mathsf{G}_{m_1,m_2,\dots,m_\kappa} = \mathsf{G}_{m_1,m_2,\dots,m_\kappa}(x,y)$$

of degrees:

$$\deg\, \mathsf{G}_{m_1,m_2,\dots,m_\kappa} \leqslant \underbrace{m_1(d-3)+m_2(d-4)+\cdots+m_\kappa\big(d-\kappa-2\big)}_{=:\,\delta},$$

which belong to the quotient spaces:

$$\mathbb{C}_\delta[x,y]\Big/R\cdot\mathbb{C}_{\delta-d}[x,y],$$

the total number of such sections being equal to:

$$\sum_{m_1+\cdots+\kappa m_\kappa=m} \left\{\binom{m_1(d-3)+\cdots+m_\kappa(d-\kappa-2)+2}{2}-\binom{m_1(d-3)+\cdots+m_\kappa(d-\kappa-2)-d+2}{2}\right\},$$

that is to say, also asymptotically equal to:

$$\frac{m^\kappa}{\kappa!\,\kappa!}\Big[d^2\,\log\kappa + d^2\,\mathrm{O}(1)+\mathrm{O}(d)\Big]+\mathrm{O}\big(m^{\kappa-1}\big).$$

This generalizes directly to the case of complete intersection *curves* $X^1 \subset \mathbb{P}^{1+c}(\mathbb{C})$:

$$0 = R^1\big(z_1,\dots,z_c,z_{c+1}\big),\quad \cdots\cdots\cdots,\quad 0 = R^c\big(z_1,\dots,z_c,z_{c+1}\big),$$

minors of the Jacobian matrix naturally occupying denominator places:

$$\frac{z_1'}{\begin{vmatrix} R^1_{z_2} & \cdots & R^1_{z_{c+1}} \\ \cdot\cdot & \cdots & \cdot\cdot \\ R^c_{z_2} & \cdots & R^c_{z_{c+1}} \end{vmatrix}} = \cdots\cdots = (-1)^c\,\frac{z_{c+1}'}{\begin{vmatrix} R^1_{z_1} & \cdots & R^1_{z_c} \\ \cdot\cdot & \cdots & \cdot\cdot \\ R^c_{z_1} & \cdots & R^c_{z_c} \end{vmatrix}}.$$

Holomorphic sections of the canonical bundle. For $X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$ a smooth hypersurface:

$$0 = R\big(z_1,\dots,z_n,z_{n+1}\big),$$

affinely represented as the zero-set of a degree $d \geqslant 1$ polynomial, whence:

$$0 = R_{z_1}\, dz_1 + \cdots + R_{z_n}\, dz_n + R_{z_{n+1}}\, dz_{n+1},$$

holomorphic sections of the *canonical bundle* $K_X := \Lambda^n T_X^*$ (which generalizes the cotangent T_X^* of $X^1 \subset \mathbb{P}^2$), are represented by the equalities:

$$\begin{aligned}\frac{dz_1 \wedge \cdots \wedge dz_n}{R_{z_{n+1}}} &= -\frac{dz_1 \wedge \cdots \wedge dz_{n-1} \wedge dz_{n+1}}{R_{z_n}} = \cdots\cdots\cdots \\ &= (-1)^n \frac{dz_2 \wedge \cdots \wedge dz_{n+1}}{R_{z_1}},\end{aligned}$$

that are always holomorphic on the $\mathbb{P}^n_\infty$ as soon as $d \geqslant n+3$.

Question. *For $X^n \subset \mathbb{P}^{n+c}$ a complete intersection $\{0 = R^1 = \cdots = R^c\}$ of codimension c, are there explicit rational holomorphic sections of $\mathscr{E}^{\rm GG}_{\kappa,m} T_X^*$ having as denominators the appropriate minors of the Jacobian matrix $\big(R^j_{z_k}\big)$ and numerators in $\mathbb{Z}\big[R^j_{z_1^{\alpha_1}\cdots z_{n+c}^{\alpha_{n+c}}}\big]$?*

Surfaces $X^2 \subset \mathbb{P}^3$. Let $X^2 \subset \mathbb{P}^3$ be a smooth surface represented in affine coordinates $(x, y, z) \in \mathbb{C}^3 \subset \mathbb{P}^3$ as:

$$0 = R(x, y, z),$$

for some polynomial $R \in \mathbb{C}[x, y, z]$ of degree $d \geqslant 1$. Differentiate this once:

$$0 = x'\, R_x + y'\, R_y + z'\, R_z.$$

Three natural open sets $\{R_x \neq 0\}$, $\{R_y \neq 0\}$, $\{R_z \neq 0\}$ cover X^2, by smoothness. On $X^2 \cap \{R_x \neq 0\}$, coordinates are (y, z), cotangent (fiber) coordinates are (y', z'). The change of trivialization for $T_X^* \cong J^1(\mathbb{D}, X)$ from above $X^2 \cap \{R_x \neq 0\}$ to above $X^2 \cap \{R_y \neq 0\}$:

$$\big(y, z, y', z'\big) \longmapsto \big(x, z, x', z'\big)$$

amounts to just solving:

$$y' = -x' \frac{R_x}{R_y} - z' \frac{R_z}{R_y}.$$

Inspired by what precedes for curves $X^1 \subset \mathbb{P}^2$, seek global holomorphic sections of:

$$\mathscr{E}^{\rm GG}_{1,m} T_X^* \cong \mathsf{Sym}^m\, T_X^*$$

(symmetric differentials) under the form:

$$\sum_{j+k=m} \mathsf{coeff}_{j,k} \cdot (y')^j \, (z')^k \quad \underset{\text{trivialization}}{\overset{\text{change of}}{\longmapsto}} \quad \sum_{j+k=m} \mathsf{coeff}_{j,k}^{\sim} \cdot (x')^j \, (z')^k,$$

with all coefficient-functions $\mathsf{coeff}_{j,k}(y,z)$ being holomorphic on $X^2 \cap \{R_x \neq 0\}$ and all $\mathsf{coeff}_{j,k}^{\sim}(x,z)$ being holomorphic on $X^2 \cap \{R_y \neq 0\}$. A question arises about the nature of these coefficients, which should be related to the defining polynomial R. A proposal of answer, inspired by $X^1 \subset \mathbb{P}^2$, is that they belong to:

$$\frac{1}{R_x}\,\mathbb{Z}\bigg[\frac{R_y}{R_x},\, \frac{R_z}{R_x}\bigg] \qquad \text{and to:} \qquad \frac{1}{R_y}\,\mathbb{Z}\bigg[\frac{R_x}{R_y},\, \frac{R_z}{R_y}\bigg],$$

because then, such jet differentials would vanish on the $\mathbb{P}^2_\infty$, as soon as $\mathsf{deg}\, R \geqslant 2m+2$. Of course, in this special case, the intrinsic theory [8, 19] already knows that whenever $\kappa < \frac{n}{c}$, on a complete intersection $X^n \subset \mathbb{P}^{n+c}$:

$$0 = H^0\big(X,\, \mathscr{E}_{\kappa,m}^{\mathrm{GG}} T_X^*\big).$$

Here with $n=2$, $c=1$, $\kappa=1$, a confirmation is:

Proposition 4 *For every $m \geqslant 1$, if polynomials $\Pi_{j,k} \in \mathbb{Z}[\mathsf{U},\mathsf{V}]$ are such that the rational differential expression having only $\frac{1}{R_x}$-denominators:*

$$\begin{aligned}
\sum_{j+k=m} \frac{(y')^j\,(z')^k}{R_x}\, \Pi_{j,k}\bigg(\frac{R_y}{R_x},\, \frac{R_z}{R_x}\bigg) &= \sum_{j+k=m} \frac{\big(-x'\,\frac{R_x}{R_y} - z'\,\frac{R_z}{R_y}\big)^j\,(z')^k}{R_x}\, \Pi_{j,k}\bigg(\frac{R_y}{R_x},\, \frac{R_z}{R_x}\bigg)\\
&= \sum_{j+k=m} \frac{(x')^j\,(z')^k}{R_y}\, \Pi_{j,k}^{\sim}\bigg(\frac{R_x}{R_y},\, \frac{R_z}{R_y}\bigg)
\end{aligned}$$

becomes, after changing trivialization, an expression having only $\frac{1}{R_y}$-denominators, then all $\Pi_{j,k} \equiv 0$.

However, the intrinsic theory knows already [8, 16, 19, 21, 65, 68, 69, 71] that for $X^n \subset \mathbb{P}^{n+c}$, global holomorphic sections of $\mathscr{E}_{\kappa,m}^{\mathrm{GG}} T_X^*$ exist when $\kappa \geqslant \frac{n}{c}$.

Hence for $n=2$, $c=1$, $\kappa=2$, differentiate *twice:*

$$\begin{aligned}
0 &= x'\,R_x + y'\,R_y + z'\,R_z,\\
0 &= x''\,R_x + y''\,R_y + z''\,R_z + (x')^2\,R_{xx} + (y')^2\,R_{yy} + (z')^2\,R_{zz} + 2x'y'\,R_{xy}\\
&\quad + 2x'z'\,R_{xz} + 2y'z'\,R_{yz}.
\end{aligned}$$

The interesting question (to which no answer is known not up to date) is whether there exist holomorphic jet differentials of the rational form:

$$\sum_{j_1+k_1+j_2+k_2=m} \frac{(y')^{j_1}\,(z')^{k_1}\,(y'')^{j_2}\,(z'')^{k_2}}{R_x} \cdot \Pi_{j_1k_1j_2k_2} \left(\frac{R_y}{R_x}, \frac{R_z}{R_x}, \frac{R_{xx}}{R_x}, \frac{R_{yy}}{R_x}, \frac{R_{zz}}{R_x}, \frac{R_{xy}}{R_x}, \frac{R_{xz}}{R_x}, \frac{R_{yz}}{R_x}\right),$$

having the property that, after replacement of:

$$\begin{aligned} y' &= -x'\,\frac{R_x}{R_y} - z'\,\frac{R_z}{R_y},\\ y'' &= -x''\,\frac{R_x}{R_y} - z''\,\frac{R_z}{R_y} - (x')^2\,\frac{R_{xx}}{R_y} - (y')^2\,\frac{R_{yy}}{R_y} - (z')^2\,\frac{R_{zz}}{R_y} - 2x'y'\,\frac{R_{xy}}{R_y} - 2x'z'\,\frac{R_{xz}}{R_y}\\ &\quad - 2y'z'\,\frac{R_{yz}}{R_y}, \end{aligned}$$

after expansion and after reorganization, a similar jet-rational expression is got:

$$\sum_{j_1+k_1+j_2+k_2=m} \frac{(x')^{j_1}\,(z')^{k_1}\,(y'')^{j_2}\,(z'')^{k_2}}{R_y} \cdot \Pi^{\sim}_{j_1k_1j_2k_2} \left(\frac{R_x}{R_y}, \frac{R_z}{R_y}, \frac{R_{xx}}{R_y}, \frac{R_{yy}}{R_y}, \frac{R_{zz}}{R_y}, \frac{R_{xy}}{R_y}, \frac{R_{xz}}{R_y}, \frac{R_{yz}}{R_y}\right)$$

which involves division by only R_y. The number of variables becomes 8 (large):

$$\begin{aligned} \Pi_{j_1k_1j_2k_2} &\in \frac{1}{R_x} \cdot \mathbb{Z}\left[\frac{R_y}{R_x}, \frac{R_z}{R_x}, \frac{R_{xx}}{R_x}, \frac{R_{yy}}{R_x}, \frac{R_{zz}}{R_x}, \frac{R_{xy}}{R_x}, \frac{R_{xz}}{R_x}, \frac{R_{yz}}{R_x}\right],\\ \Pi^{\sim}_{j_1k_1j_2k_2} &\in \frac{1}{R_y} \cdot \mathbb{Z}\left[\frac{R_x}{R_y}, \frac{R_z}{R_y}, \frac{R_{xx}}{R_y}, \frac{R_{yy}}{R_y}, \frac{R_{zz}}{R_y}, \frac{R_{xy}}{R_y}, \frac{R_{xz}}{R_y}, \frac{R_{yz}}{R_y}\right]. \end{aligned}$$

By anticipation, for $X^n \subset \mathbb{P}^{n+1}$ and for jets of order $\kappa = n$:

$$\#\Big(\text{partial derivatives}\, R_{z_1^{\alpha_1}\cdots z_n^{\alpha_n} z_{n+1}^{\alpha_{n+1}}}\Big) = \binom{n+1+n}{n} \sim 2^{2n+1}\,\frac{1}{\sqrt{\pi n}},$$

hence something is *intimately exponential* in the subject. For $X^2 \subset \mathbb{P}^3$ of degree $d \gg 1$, a patient cohomology sequences chasing shows that there exist nonzero second-order holomorphic jet differentials in $H^0\big(X, \mathscr{E}_{2,m}^{\rm GG} T_X^*\big)$ only when:

$$m \geqslant 14$$

(similarly, by [8], for $X^2 \subset \mathbb{P}^4$ of bidegrees $d_1, d_2 \gg 1$, it is necessary that $m \geqslant 10$). Hence combinatorially, there is a complexity obstacle, and moreover, an inspection of what holds true for curves $X^1 \subset \mathbb{P}^2$ shows that it is quite probable that the degrees of the $\Pi_{j_1k_1j_2k_2}$ are about to be approximately equal to $m \geqslant 14$, whence the total number of monomials they involve:

$$\binom{14+8}{8} = 319\,770,$$

would be already rather large to determine in a really effective way whether they exist.

It happens to be a bit easier to work with the Wronskians:

$$\square := \begin{vmatrix} y' & z' \\ y'' & z'' \end{vmatrix} = y'z'' - z'y'', \qquad \Delta := \begin{vmatrix} z' & x' \\ z'' & x'' \end{vmatrix} = z'x'' - x'z''.$$

Two fundamental transition formulas are:

$$\begin{aligned}
\frac{y'}{R_x} &= -\frac{x'}{R_y} - \frac{z'}{R_y}\frac{R_z}{R_x}, \\
\frac{\begin{vmatrix} y' & z' \\ y'' & z'' \end{vmatrix}}{R_x} &= \frac{\begin{vmatrix} z' & x' \\ z'' & x'' \end{vmatrix}}{R_y} - \frac{(x')^2 z'}{R_y}\left[\left(\frac{R_x}{R_y}\right)\frac{R_{yy}}{R_y} - 2\,\frac{R_{xy}}{R_y} + \frac{R_{xx}}{R_x}\right] - \\
&\quad - 2\,\frac{x'(z')^2}{R_y}\left[\left(\frac{R_z}{R_y}\right)\frac{R_{yy}}{R_y} - \frac{R_{yz}}{R_y} + \frac{R_{xz}}{R_x} - \left(\frac{R_z}{R_y}\right)\frac{R_{xy}}{R_x}\right] - \\
&\quad - \frac{(z')^3}{R_y}\left[\left(\frac{R_z}{R_y}\right)^2\frac{R_{yy}}{R_x} - 2\left(\frac{R_z}{R_y}\right)\frac{R_{yz}}{R_x} + \frac{R_{zz}}{R_x}\right].
\end{aligned}$$

Set as abbreviated new notations:

$$\begin{aligned}
\frac{R_x}{R_y} &=: r_x, & \frac{R_z}{R_y} &=: r_z, \\
\frac{R_{xx}}{R_y} &=: r_{yy}, & \frac{R_{xy}}{R_y} &=: r_{xy}, \\
\frac{R_{yy}}{R_y} &=: r_{zz}, & \frac{R_{xz}}{R_y} &=: r_{xz}, \\
\frac{R_{zz}}{R_y} &=: r_{xx}, & \frac{R_{yz}}{R_y} &=: r_{yz}.
\end{aligned}$$

Rewrite:

$$\begin{aligned} y' &= -x'\,r_x - z'\,r_z, \\ \square &= \Delta\, r_x - (x')^2 z' \Big[r_x\, r_x\, r_{yy} - 2\, r_x\, r_{xy} + r_{xx} \Big] - \\ &\qquad - 2\, x' (z')^2 \Big[r_x\, r_z\, r_{yy} - r_x\, r_{yz} + r_{xz} - r_z\, r_{xy} \Big] - \\ &\qquad - (z')^3 \Big[r_z\, r_z\, r_{yy} - 2\, r_z\, r_{yz} + r_{zz} \Big]. \end{aligned}$$

Divide both sides by:

$$R_x = r_x\, R_y,$$

and obtain:

$$\begin{aligned} \frac{y'}{R_x} &= -\frac{x'}{R_y} - \frac{z'}{R_y}\,\frac{r_z}{r_x}, \\ \frac{\square}{R_x} &= \frac{\Delta}{R_y} - \frac{(x')^2 z'}{R_y} \Big[r_x\, r_{yy} - 2\, r_{xy} + \frac{r_{xx}}{r_x} \Big] - \\ &\qquad - 2\,\frac{x'(z')^2}{R_y} \Big[r_z\, r_{yy} - r_{yz} + \frac{r_{xz}}{r_x} - \frac{r_z\, r_{xy}}{r_x} \Big] - \\ &\qquad \frac{(z')^3}{R_y} \Big[\frac{r_z\, r_z\, r_{yy}}{r_x} - 2\,\frac{r_z\, r_{yz}}{r_x} + \frac{r_{zz}}{r_x} \Big]. \end{aligned}$$

In the case $n = 2 = \kappa$, $c = 1$, the question formulated above becomes:

Question. *Do there exist nontrivial linear combinations of:*

$$\frac{\big(-x' r_x - z' r_z\big)^j (z')^k \left(\begin{array}{l} \Delta r_x - (x')^2 z' \big[r_x r_x r_{yy} - 2 r_x r_{xy} + r_{xx} \big] \\ \qquad - 2x'(z')^2 \big[r_x r_z r_{yy} - r_x r_{yz} + r_{xz} - r_z r_{xy} \big] - \\ \qquad - (z')^3 \big[r r_z r_z r_{yy} - 2 r_z r_{yz} + r_{zz} \big] \end{array} \right)^l}{R_x\, r_x} \times$$

$$\times \left(\frac{1}{r_x}\right)^a \left(\frac{r_z}{r_x}\right)^b \left(\frac{r_{xx}}{r_x}\right)^c \left(\frac{r_{yy}}{r_x}\right)^d \left(\frac{r_{zz}}{r_x}\right)^e \left(\frac{r_{xy}}{r_x}\right)^f \left(\frac{r_{xz}}{r_x}\right)^g \left(\frac{r_{yz}}{r_x}\right)^h,$$

for some nonnegative integers:

$$j,\ k,\ l, \qquad a,\ b,\ c,\ d,\ e,\ f,\ g,\ h,$$

in which any $\frac{1}{r_x}$ would have disappeared?

3.2 Slanted Vector Fields

To construct holomorphic jet differentials on a hypersurface $X^n \subset \mathbb{P}^{n+1}$ defined as:

$$0 = R(z_1, \ldots, z_n, z_{n+1}) = \sum_{\alpha_1+\cdots+\alpha_n+\alpha_{n+1}\leqslant d} a_{\alpha_1\ldots\alpha_n\alpha_{n+1}}\, z_1^{\alpha_1} \cdots z_n^{\alpha_n} z_{n+1}^{\alpha_{n+1}},$$

two strategies exist, the first one (still open) being to work (only) in the ring (of fractions) of all partial derivatives of R:

$$\mathbb{Z}\Big[\Big(R_{z_1^{\beta_1}\cdots z_{n+1}^{\beta_{n+1}}}\Big)_{\beta_1+\cdots+\beta_{n+1}\leqslant\kappa}\Big],$$

and the second one (currently active) being to work in the ring of coefficients:

$$\mathbb{Z}\Big[\Big(a_{\alpha_1\ldots\alpha_{n+1}}\Big)_{\alpha_1+\cdots+\alpha_{n+1}\leqslant d}\Big].$$

For instance, differentiate $0 = \sum_\alpha a_\alpha\, z^\alpha$ up to order, say, 4:

$$0 = \sum_\alpha a_\alpha\, z^\alpha$$

$$0 = \sum_\alpha a_\alpha \bigg(\sum_{j_1} \frac{\partial(z^\alpha)}{\partial z_{j_1}}\, z'_{j_1}\bigg)$$

$$0 = \sum_\alpha a_\alpha \bigg(\sum_{j_1} \frac{\partial(z^\alpha)}{\partial z_{j_1}}\, z''_{j_1} + \sum_{j_1,j_2} \frac{\partial^2(z^\alpha)}{\partial z_{j_1}\partial z_{j_2}}\, z'_{j_1} z'_{j_2}\bigg)$$

$$0 = \sum_\alpha a_\alpha \bigg(\sum_{j_1} \frac{\partial(z^\alpha)}{\partial z_{j_1}}\, z'''_{j_1} + \sum_{j_1,j_2} \frac{\partial^2(z^\alpha)}{\partial z_{j_1}\partial z_{j_2}}\, 3\, z'_{j_1} z''_{j_2} + \sum_{j_1,j_2,j_3} \frac{\partial^3(z^\alpha)}{\partial z_{j_1}\partial z_{j_2}\partial z_{j_3}}\, z'_{j_1} z'_{j_2} z'_{j_3}\bigg)$$

$$\begin{aligned} 0 = \sum_\alpha a_\alpha \bigg(&\sum_{j_1} \frac{\partial(z^\alpha)}{\partial z_{j_1}}\, z''''_{j_1} + \sum_{j_1,j_2} \frac{\partial^2(z^\alpha)}{\partial z_{j_1}\partial z_{j_2}} \big(4\, z'_{j_1} z'''_{j_2} + 3\, z''_{j_1} z''_{j_2}\big) + \\ &+ \sum_{j_1,j_2,j_3} \frac{\partial^3(z^\alpha)}{\partial z_{j_1}\partial z_{j_2}\partial z_{j_3}}\, 6\, z'_{j_1} z'_{j_2} z''_{j_3} + \sum_{j_1,j_2,j_3,j_4} \frac{\partial^4(z^\alpha)}{\partial z_{j_1}\partial z_{j_2}\partial z_{j_3}\partial z_{j_4}}\, z'_{j_1} z'_{j_2} z'_{j_3} z'_{j_4}\bigg). \end{aligned}$$

Lemma 6 ([41]) *The equation obtained by differentiating the condition $R\big(f(\zeta)\big) \equiv 0$ up to an arbitrary order $\kappa \geqslant 1$ reads in closed form as follows:*

$$\begin{aligned} 0 = \sum_{\alpha\in\mathbb{N}^{n+1}} a_\alpha \sum_{e=1}^{\kappa} \sum_{1\leqslant\lambda_1<\cdots<\lambda_e\leqslant\kappa} \sum_{\mu_1\geqslant1,\ldots,\mu_e\geqslant1} \sum_{\mu_1\lambda_1+\cdots+\mu_e\lambda_e=\kappa} \frac{\kappa!}{(\lambda_1!)^{\mu_1}\mu_1!\cdots(\lambda_e!)^{\mu_e}\mu_e!} \\ \sum_{j_1^1,\ldots,j_{\mu_1}^1=1}^{n+1} \cdots \sum_{j_1^e,\ldots,j_{\mu_e}^e=1}^{n+1} \frac{\partial^{\mu_1+\cdots+\mu_e}\big(z^\alpha\big)}{\partial z_{j_1^1}\cdots\partial z_{j_{\mu_1}^1}\cdots\partial z_{j_1^e}\cdots\partial z_{j_{\mu_e}^e}}\, z_{j_1^1}^{(\lambda_1)} \cdots z_{j_{\mu_1}^1}^{(\lambda_1)} \cdots z_{j_1^e}^{(\lambda_e)} \cdots z_{j_{\mu_e}^e}^{(\lambda_e)}. \end{aligned}$$

These equations for $\kappa = 0, 1, \dots, \kappa$ define a certain (projectivizable) subvariety:

$$J^{\kappa}_{\text{vert}} \subset \mathbb{C}^{n+1}_{(z_k)} \times \mathbb{C}^{\frac{(n+1+d)!}{(n+1)!\,d!}}_{(a_\alpha)},$$

complete intersection of codimension $\kappa + 1$ outside $\{z'_1 = \cdots = z'_{n+1} = 0\}$. Vector fields tangent to J^{κ}_{vert} write under the general form:

$$\mathsf{T} = \sum_{i=1}^{n+1} \mathsf{Z}_i \frac{\partial}{\partial z_i} + \sum_{\substack{\alpha \in \mathbb{N}^{n+1} \\ |\alpha| \leqslant d}} \mathsf{A}_\alpha \frac{\partial}{\partial a_\alpha} + \sum_{k=1}^{n+1} \mathsf{Z}'_k \frac{\partial}{\partial z'_k} + \sum_{k=1}^{n+1} \mathsf{Z}''_k \frac{\partial}{\partial z''_k} + \cdots + \sum_{k=1}^{n+1} \mathsf{Z}^{(\kappa)}_k \frac{\partial}{\partial z^{(\kappa)}_k}.$$

Notably, the next theorem works with the quotient ring of $\mathbb{Z}[a_\alpha]$, not of $\mathbb{Z}[R_{z^\beta}]$.

Theorem 12 ([41, 55, 69, 70]) *With $\kappa \leqslant d$, at every point of $J^{\kappa}_{\text{vert}} \backslash \{z'_i = 0\}$, there exist $j^d_{n,\kappa} := \dim J^{\kappa}_{\text{vert}}$ global holomorphic sections $\mathsf{T}_1, \dots, \mathsf{T}_{j^d_{n,\kappa}}$ of the twisted tangent bundle:*

$$T_{J^{\kappa}_{\text{vert}}} \otimes \mathscr{O}_{\mathbb{P}^{n+1}}\big(\kappa^2 + 2\,\kappa\big) \otimes \mathscr{O}_{\mathbb{P}^{\frac{(n+1+d)!}{(n+1)!\,d!}-1}}(1),$$

which generate the tangent space:

$$\mathbb{C}\mathsf{T}_1\big|_p \oplus \cdots \oplus \mathbb{C}\mathsf{T}_{j^d_{n,\kappa}}\big|_p = T_{J^n_{\text{vert}},\,p}.$$

According to Siu [69, 70], these fields can be used to show that, for X generic, entire curves $f\colon \mathbb{C} \to X$ land in the base locus of *all* global algebraic jet differentials belonging to the space:

$$H^0\big(X, E^{\text{GG}}_{n,m} T^*_X \otimes K_X^{-\delta m}\big) \neq 0, \tag{7}$$

which is shown in [21] to be nonzero for small enough $\delta \in \mathbb{Q}_{>0}$, for $\kappa = n$, for $m \gg 1$, provided:

$$d = \deg X \geqslant 2^{n^5}.$$

More precisely, by an abstract argument, extend locally any such jet differential:

$$\mathsf{P}(z, a) = \sum_{|\gamma_1| + \cdots + n|\gamma_n| = m} \mathsf{p}_\gamma(z, a)\, (z')^{\gamma_1} \cdots (z^{(n)})^{\gamma_n},$$

for a generic. Use the vector fields of Theorem 12 to eliminate $(z')^{\gamma_1} \dots (z^{(n)})^{\gamma_n}$, and get the:

Proposition 5 *Nonconstant entire curves algebraically degenerate inside the proper algebraic subset of X:*

$$Y := \big\{z \in X\colon \underbrace{\mathsf{p}_\gamma\big(z, a\big) = 0, \quad \forall\, |\gamma_1| + \cdots + n\,|\gamma_n| = m}_{\text{all coefficients, very numerous}}\big\}.$$

Here, the total number of algebraic equations $p_\gamma(z, a) = 0$ is exponentially large $\approx m^n \gg (2^{n^5})^n$. Naturally, the common zero-set should conjecturally be empty, whence Kobayashi's conjecture—not in optimal degree—seems to be almost established. However, all *intrinsic* techniques which provide global holomorphic sections like (7) above, namely either a decomposition of jet bundles in Schur bundles, or asymptotic Morse inequalities, or else probabilistic curvature estimates, are up to now unable to provide a partial explicit expression of even a single algebraic coefficient $\mathsf{p}_\gamma(z, a)$.

This is why a refoundation towards rational effectiveness is necessary.

At least before refounding the construction of holomorphic jet differentials, such intrinsic approaches may be pushed further to improve the degree bound $d \geqslant 2^{n^5}$, and to treat new geometric situations.

Brotbek [8] produced holomorphic jet differentials on general complete intersections $X^n \subset \mathbb{P}^{n+c}$ of multidegrees $d_1, \ldots, d_c \gg 1$. Mourougane [55] showed that for general moving enough families of high enough degree hypersurfaces in $\mathbb{P}^{n+1}$, there is a proper algebraic subset of the total space that contains the image of all sections.

Yet getting information about 'high enough' degrees represents a substantial computational work.

The most substantial recent progress concerning degree bounds is mainly due to Berczi [5, 6], in the case of $f\colon \mathbb{C} \to X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$, with $d \geqslant n^{8n}$, instead of $d \geqslant 2^{n^5}$. Logarithmic jet bundles introduced by Noguchi [56, 57] improve this bound. Using the above vector fields and probabilistic curvature estimates for Green-Griffiths jets, Demailly obtained in [18], still in the case $f\colon \mathbb{C} \to X^n \subset \mathbb{P}^{n+1}(\mathbb{C})$:

$$d \geqslant \frac{n^4}{3}\,\big(n \log\big(n\,\big(\log(24\,n)\big)\big)^n.$$

Theorem 13 (Darondeau, [12–14, 66]) *Let $X^{n-1} \subset \mathbb{P}^n(\mathbb{C})$ be a smooth complex projective algebraic hypersurface of degree:*

$$d \;\geqslant\; (5\,n)^2\, n^n.$$

If X^{n-1} is Zariski-generic, then there exists a proper algebraic subvariety $Y \subsetneqq \mathbb{P}^n$ of codimension $\geqslant 2$ such that every nonconstant entire holomorphic curve $f\colon \mathbb{C} \to \mathbb{P}^n \backslash X$ actually lands in Y, namely $f(\mathbb{C}) \subset Y$.

Consider again the *universal family* of degree d hypersurfaces of $\mathbb{P}^n$:

$$\mathscr{H} := \Big\{\big([Z], [A]\big) \in \mathbb{P}^n \times \mathbb{P}^{\binom{n+d}{d}-1}\colon \sum A_\alpha\, Z^\alpha = 0\Big\}.$$

With an additional variable $W \in \mathbb{C}$, introduce the family of hypersurfaces of $\mathbb{P}^{n+1}$:

$$W^d = \sum_\alpha A_\alpha Z^\alpha.$$

The space $J^\kappa_{\text{vert}}(-\log)$ of *vertical logarithmic κ-jets* is associated to jets of local holomorphic maps $f : \mathbb{D} \to \mathbb{P}^n \backslash \mathscr{H}_A$ valued in the complement of a hypersurface $\mathscr{H}_A$ corresponding to a fixed A and having a certain determined behavior near $\{W = 0\}$. The counterpart of Theorem 12 useful *infra* is:

Theorem 14 (Darondeau, [14, 41, 59]) *With $\kappa \leqslant d$, the twisted tangent bundle to the space of logarithmic κ-jets:*

$$T_{J^\kappa_{\text{vert}}(-\log)} \otimes \mathscr{O}_{\mathbb{P}^n}\big(\kappa^2 + 2\kappa\big) \otimes \mathscr{O}_{\mathbb{P}^{\frac{(n+d)!}{n!\,d!}}}(1)$$

is generated by its global holomorphic sections at every point not in $\{W = 0\} \cup \{Z_i' = 0\}$.

3.3 Prescribing the Base Locus of Siu-Yeung Jet Differentials

In [49, 71], it is shown that a surface $X^2 \subset \mathbb{P}^3$ having affine equation: $z^d = R(x, y)$, where $R \in \mathbb{C}[x, y]$ is a generic polynomial of high enough degree $d \gg 1$, the following holds. For every collection of polynomials $A_{j,k,p,q} \in \mathbb{C}[x, y]$ having degrees $\deg A_{j,k,p,q} \leqslant d - 3\,m - 1$, the meromorphic jet differential:

$$\frac{\mathsf{J}(x, y, x', y', x'', y'')}{R_y \cdot z^{m(d-1)}} = \frac{1}{R_y \cdot z^{m(d-1)}} \sum_{j+k+p+3q=m} A_{j,k,p,q}\, (x')^j\, (y')^k\, (R')^p \begin{vmatrix} x' & R' \\ x'' & R'' \end{vmatrix}^q (R)^{m-p-q},$$

where:

$$R' := R_x\, x' + R_y\, y', \qquad R'' := R_x\, x'' + R_y\, y'' + R_{xx}\, (x')^2 + 2\, R_{xy}\, x'\, y' + R_{yy}\, (y')^2,$$

possesses a restriction to X^2 which is a *holomorphic* section of the bundle of the Green-Griffiths jet bundle $\mathscr{E}^{\mathrm{GG}}_{2,m} T^*_X$, provided only that the polynomial numerator:

$$\mathsf{J}(x, y, x', y', x'', y'') \;\equiv\; R_y(x, y)\, \tilde{\mathsf{J}}(x, y, x', y', x'', y'')$$

is divisible by R_y, which happens to be satisfiable for $m = 81$ and $d = 729$, and more generally whenever $m \geqslant 81$ and $d \geqslant 3\,m$.

An expansion yields:

$$\mathsf{J} = \sum_{\alpha+\beta+3\gamma=m} \Lambda_{\alpha,\beta,\gamma}\big(A_\bullet, J^2_{x,y} R\big)\, (x')^\alpha\, (y')^\beta \begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}^\gamma,$$

in terms of some $\Lambda_\bullet$ that are linear in the $A_\bullet$ and polynomial in the 2-jet $J^2_{x,y} R$.

Since the vector fields of Theorem 14 have a maximal pole order 8 here, lowering $\deg A_{j,k,p,q} \leqslant d - 11\,m - 1$ enables to conclude, as in Proposition 5, that for a generic curve $\{R = 0\} \subset \mathbb{P}^2$, nonconstant entire holomorphic maps $f\colon \mathbb{C} \to \mathbb{P}^2 \backslash \{R = 0\}$ land inside the common zero set of all the $\Lambda_\bullet$, for $d \geqslant 2916$.

Open Problem *Control or prescribe the base locus of coordinate jet differentials.*

This problem is related to Debarre's ampleness conjecture [15, 47], known to hold for complete intersections X^n in $\mathbb{P}^{n+c}(\mathbb{C})$ with $c \geqslant n$ in degrees $\geqslant n^{n^3}$, but ampleness of second order jet bundles is more delicate. A conjecturally accessible strategy is as follows, of course extendable to arbitrary dimensions. For convenience, replace $m \mapsto 3\,m$. Decompose $\mathsf{J} = \mathsf{J}^{\mathsf{top}} + \mathsf{J}^{\mathsf{cor}}_{\mathsf{sub}}$, where:

$$\mathsf{J}^{\mathsf{top}} := 1 \cdot \begin{vmatrix} x' & R' \\ x'' & R'' \end{vmatrix}^m (R)^{2m} = \left(\begin{vmatrix} x' & y' \\ x'' & y'' \end{vmatrix}^m R_y + (x')^3\, R_{xx} + 2\,(x')^2 y'\, R_{xy} + x'(y')^2\, R_{yy} \right)^m (R)^{2m},$$

$$\mathsf{J}^{\mathsf{cor}}_{\mathsf{sub}} := \sum_{\substack{j+k+p+3q=3m \\ q \leqslant m-1}} A_{j,k,p,q}(x,y)\,(x')^j\,(y')^k\,(R')^p \begin{vmatrix} x' & R' \\ x'' & R'' \end{vmatrix}^q (R)^{m-p-q}.$$

Since $\big((x'y'' - y'x'')\,R_y\big)^m$ in $\mathsf{J}^{\mathsf{top}}$ is divisible by R_y, Proposition 5 would show that entire curves land in $\{(R_y)^{m-1} = 0\}$, and exchanging $x \leftrightarrow y$, in $\{(R_x)^{m-1} = 0\}$, hence are constant because $\emptyset = \{0 = R = R_x = R_y\}$ by smoothness of $\{R = 0\}$.

However, *all* $\Lambda_{\alpha,\beta,\gamma}$, not just $\Lambda_{0,0,m}$, should be divisible by R_y in order that the restriction to the projectivization of $\{z^d = R(x,y)\}$ of $\mathsf{J}\big/\big(R_y\,z^{3m(d-1)}\big)$ be a holomorphic jet differential, because modulo R_y:

$$\mathsf{J}^{\mathsf{top}} \equiv \Big((x')^3\, R_{xx} + 2\,(x')^2 y'\, R_{xy} + x'(y')^2\, R_{yy}\Big)^m (R)^{2m}$$

is nonzero. The strategy is to use $\mathsf{J}^{\mathsf{cor}}_{\mathsf{sub}}$ in order to *correct* this remainder. Conjecturally, the linear map which, to the $A_\bullet$ of $\mathsf{J}^{\mathsf{cor}}_{\mathsf{sub}}$, associates the coefficients of a basis $x^h y^i$ of $\mathbb{C}[x,y]/\langle R_y \rangle$ in all the monomials $(x')^\alpha (y')^\beta$ with $\alpha + \beta = 3m$ is submersive, also in arbitrary dimension, which would hence terminate.

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A Survey on Levi Flat Hypersurfaces

Takeo Ohsawa

Dedicated to Professor Yum-Tong Siu on the occasion of his 70th birthday

1 Introduction

In the theory of several complex variables, the notion of pseudoconvexity is of basic importance since its discovery by Hartogs and the solution of the Levi problem by Oka: Unramified domains over $\mathbb{C}^n$ are pseudoconvex if and only if they are domains of holomorphy. The situation becomes subtler for the domains over complex manifolds. Grauert [19] first noted that general pseudoconvex domains in complex manifolds are not necessarily holomorphically convex. Narasimhan [36] showed that a generically chosen complex torus of dimension ≥ 2 contains a pseudoconvex domain which contains a real hypersurface foliated by dense complex leaves. Accordingly they do not admit nonconstant holomorphic functions by the maximum principle. Grauert [20] showed that a tubular neighborhood of the zero section of a generically chosen line bundle over a non-rational Riemann surface also serves as such an example. In spite of these counterexamples, it was noticed in [39] and [12] that generic disc bundles over non-rational compact Riemann surfaces are Stein manifolds although their boundaries are foliated by complex leaves. These examples naturally raised a question of classifying compact real hypersurfaces in complex manifolds that are foliated by complex submanifolds of codimension one. Such hypersurfaces are said to be Levi flat. The principal purpose of the present article is to give an account of the studies towards a classification of Levi flat hypersurfaces. After giving some preliminaries in Sect. 1 and a glance at some species of Levi flat hypersurfaces in Sect. 2, we shall review the works

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on the classification of Levi flat hypersurfaces in compact complex manifolds. An emphasis will be put on the existence and nonexistence results in $\mathbb{CP}^n$, complex tori and Hopf surfaces (cf. [3, 5, 28, 29, 34, 43, 45, 48, 49, 55]). The author thanks the referee for useful comments.

2 Examples of Levi Flat Hypersurfaces

Let M be a connected complex manifold of dimension n and let $X \subset M$ be a closed and smooth real hypersurface of class $\mathcal{C}^2$. X is said to be Levi flat if X is locally pseudoconvex from the both sides, i.e., for any point $x \in X$ one can find a neighborhood U of x such that $U \setminus X$ is Stein. If X is Levi flat, the Levi form of X vanishes everywhere, which implies that X admits a foliation whose leaves are of dimension $n - 1$ and everywhere tangent to the holomorphic tangent vectors of X. Since such a foliation is unique on X, we shall call it the Levi foliation of X. Trivial examples of Levi flat hypersurfaces are simple closed curves in Riemann surfaces. More general but still trivial ones are the preimages of such simple closed curves by proper and smooth holomorphic maps. As for the background of complex spaces and the Levi problem, the reader may consult [21, 53] and [10]. (For the motivation from foliation theory, see [4] and [7].)

The following is a collection of Levi flat hypersurfaces from the literature.

2.1 *Levi Flat Boundaries of Non-Stein Domains*

Let M be a compact complex manifold of dimension ≥ 1 and let $E \to M$ be a holomorphic vector bundle whose transition matrices are all unitary. Then the zero section of E admits a pseudoconvex neighborhood system consisting of fiber vectors of length less than constants. None of these neighborhoods are Stein and their boundaries are Levi flat if and only if the rank of E is one. The Levi foliations have dense leaves if and only if no nonzero tensor power of E is trivial. This observation was first due to Grauert [20].

2.2 *Stein Domains and Non-Stein Domains with Product Structure*

Let $\hat{\mathbb{C}}$ $(= \mathbb{CP}^1)$ be the Riemann sphere with inhomogeneous coordinate ζ and let $Y = (\mathbb{C} \setminus \{0\})/\mathbb{Z}$, where the action of the infinite cyclic group $\mathbb{Z}$ on $\mathbb{C} \setminus \{0\}$ is generated by $z \mapsto 2z$. Let $X = \{(\zeta, [z]); \mathrm{Im}(\zeta z) = 0\}$ and $\Omega = \{\mathrm{Im}(\zeta z) > 0\}$. Then X is Levi flat, $\partial\Omega = X$, and Ω is equivalent to the product of $\mathbb{C} \setminus \{0\}$ and an annulus by the map $(\zeta, [z]) \mapsto (\zeta, [\zeta z])$. This example is taken from [39]. Note that the fibers of the projection from Ω to Y are equivalent to the upper half plane $H = \{\mathrm{Im}\zeta > 0\}$ and $\Omega \approx H \times (\mathbb{C} \setminus \{0\}) / \Gamma$, where Γ is the subgroup of $\mathrm{Aut}(H \times (\mathbb{C} \setminus \{0\}))$ generated by $(\zeta, z) \mapsto (2\zeta, 2z)$. That Ω is equivalent to the product of an annulus and $\mathbb{C} \setminus \{0\}$ can be seen from this, too. Generalizing this, Diederich and Fornaess [11] gave an answer to Grauert's question whether or not every smoothly bounded pseudoconvex domain in a compact complex manifold admits a plurisubharmonic exhaustion function. Namely, they considered the manifold $\Omega = (H \times (\mathbb{C}^n \setminus \{0\}))/\Gamma, n \geq 2$, where Γ is generated by $(\zeta, z_1, \ldots, z_n) \mapsto (2\zeta, 2z_1, \ldots, 2z_n)$. Then Ω is a fibre bundle over a Hopf manifold and bounded by a Levi flat hypersurface in the associated $\hat{\mathbb{C}}$-bundle. Since $\Omega \approx \{\exp(-2\pi^2/\log 2) < |\zeta| < 1\} \times (\mathbb{C}^n \setminus \{0\})$, we have that Ω is not holomorphically convex, so that it does not admit any plurisubharmonic exhaustion function.

2.3 Disc Bundles

The above examples of pseudoconvex domains with Levi flat boundaries are all disc bundles over compact manifolds since $H \approx \mathbb{D} := \{\zeta; |\zeta| < 1\}$. A project started in [12] to classify Stein manifolds arising in this way. A basic result obtained there is that $\mathbb{D}$-bundles over compact Kähler manifolds admit plurisubharmonic exhaustion functions. For the proof, one observes at first that $\mathbb{D}$-bundles with locally constant transition functions over compact Riemannian manifolds admit either harmonic sections with respect to the Poincaré metric on the fibers, or locally constant sections on the boundary, i.e., locally constant sections of the associated $\hat{\mathbb{C}}$-bundles whose images are contained in the boundaries of the $\mathbb{D}$-bundles. This follows from the energy decreasing property of the solutions of certain heat equations on the base manifold which has been established by Eells and Sampson [13]. Besides the application of this existence result, a crucial step is to exploit Siu's variant of the Bochner trick in [54] which shows the pluriharmonicity of harmonic maps from compact Kähler manifolds to locally symmetric spaces of negative curvature (see also [6]). As a result, the harmonic sections of $\mathbb{D}$-bundles over compact Kähler manifolds turn out to be pluriharmonic, so that plurisubharmonic exhaustion functions are obtained either as the logarithm of the fiberwise Bergman kernel functions with respect to the fibre coordinates centered at the images of harmonic sections, or as the squared length of the fiber vectors of the associated pluriharmonic vector bundles. In the latter part, the classical Hodge theory is applied by mimicking the method of Ueda [60], where the convexity properties of neighborhoods of complex curves has been studied in detail. As for the function theoretic property of $\mathbb{D}$-bundles over compact Riemann surfaces (= compact and nonsingular complex curves), the conclusion is summarized as follows.

Theorem 1 *A $\mathbb{D}$-bundle over a compact Riemann surface is Stein if and only if it does not admit any holomorphic section.*

This has some similarity to Hartshorn's question asking whether or not the complement of a compact complex curve $\mathcal{C}$ in a projective algebraic surface over $\mathbb{C}$ is Stein if $\mathcal{C}$ intersects with all the other curves. Note that the set of equivalence classes of $\mathbb{D}$-bundles is naturally identified with a subset of the equivalence classes of Aut$\mathbb{D}$ representations of the fundamental groups of the base manifolds, so that it carries a natural topology. Teichmüller spaces of compact Riemann surfaces are connected components of this space. By the way, the existence of pluriharmonic sections is applied also to study the fundamental groups of compact Kähler manifolds (cf. [59]). This direction is closely related to a conjecture by Shafarevich asserting that the universal covering space of every compact Kähler manifold is holomorphically convex. A recent successful work of Eyssidieux, Katzarkov, Pantev and Ramachandran [15] including a solution of the Shafarevich conjecture for the residually finite fundamental group has a very wide scope.

2.4 Levi Flat Hypersurfaces in Torus Bundles

The projection from the domain Ω in Sect. 2.2 to the first factor $\hat{\mathbb{C}}$ induces another fiber structure on Ω; a bundle over $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ whose fibers are annuli. By generalizing this situation, Nemirovski [37] has presented a construction of certain Levi flat hypersurfaces bounding Stein domains in a torus bundle over any projective manifold $\mathcal{S}$. Such a torus bundle arises, for instance, as the quotient of a $\mathbb{C}^*$-bundle, say $B \to S$, by the action of $\mathbb{Z}$ generated by $(w, z) \mapsto (w, 2z)$ in terms of the local coordinate w of S and the fibre coordinate z. Then, for any meromorphic section s of the $\hat{\mathbb{C}}$-bundle associated to B such that its zeros and poles are mutually disjoint and of order one, a Levi flat hypersurface X in $B/\mathbb{Z}$ is defined as the closure of the union of $(\mathbb{C}^* \cap \mathbb{R}) \cdot s(x)/\mathbb{Z}$, where x runs through the complement of $s^{-1}(0) \cup s^{-1}(\infty)$. If $S \setminus (s^{-1}(0) \cup s^{-1}(\infty))$ is Stein, X bounds a Stein domain because holomorphic fibre bundles with Stein fibers and Stein bases are Stein if the dimension of the fibers are one (cf. [35]). This construction of Levi flat hypersurfaces can be generalized for other torus bundles (cf. [41, 45]), and for the quotients of B by a more general action of $\mathbb{Z}$ (cf. [48]). Nowadays real analytic Levi flat hypersurfaces in Hopf surfaces are completely classified from this viewpoint (see Sect. 5).

3 Tools from Cohomology Theory

For the proof of certain nonexistence theorems for Levi flat hypersurfaces, Hartogs type existence theorems are useful. The idea is that the Levi flatness is inconsistent with the strict pseudoconvexity inside. Among several approaches to Hartogs type extension theorems, one based on the cohomology vanishing theorem is effective here. The L^2-method for the $\bar{\partial}$-operator is available to show such a vanishing theorem. We shall recall it below for the reader's convenience. Let (Ω, g) be a (not necessarily connected) Hermitian manifold of dimension n and let (E, h) be a Hermitian holomorphic vector bundle over Ω. We denote by $C^{p,q}(\Omega, E)$ the space of E-valued $\mathcal{C}^\infty$-smooth (p, q)-forms on Ω. Recall that $C^{p,q}(\Omega, E)$ is identified with the set of $\mathcal{C}^\infty$ sections of the bundle $\bigwedge^p(T^{1,0}\Omega)^* \otimes \bigwedge^q(T^{0,1}\Omega)^* \otimes E$, where $T^{1,0}\Omega$ (resp. $T^{0,1}\Omega$) denotes the holomorphic (resp. antiholomorphic) tangent bundle of Ω. We let $C_0^{p,q}(\Omega, E)$ denote the subset of $C^{p,q}(\Omega, E)$ consisting of compactly supported forms. Let $\overline{\partial}$ (resp. ∂) denote the complex derivative of type $(0, 1)$ (resp. $(1, 0)$) on Ω. By an abuse of notation the operator $\partial\overline{\partial}$ will often be identified with the complex Hessian. The $\overline{\partial}$-cohomology groups $H^{p,q}(\Omega, E)$ and $H_0^{p,q}(\Omega, E)$ are defined by

$$H^{p,q}(\Omega, E) := \mathrm{Ker}\overline{\partial} \cap C^{p,q}(\Omega, E)/\mathrm{Im}\overline{\partial} \cap C^{p,q}(\Omega, E), \tag{1}$$

and

$$H_0^{p,q}(\Omega, E) := \mathrm{Ker}\overline{\partial} \cap C_0^{p,q}(\Omega, E)/\mathrm{Im}\overline{\partial} \cap C_0^{p,q}(\Omega, E), \tag{2}$$

respectively.

In order to analyse the $\overline{\partial}$-cohomology groups, the metric structure (g, h) is useful. The pointwise length of $u \in C^{p,q}(\Omega, E)$ with respect to g and h is denoted by $|u|$. Then the L^2-norm of u, denoted by $\|u\|$, is defined as the square root of the integral of $|u|^2$ on Ω whenever $u \in C_0^{p,q}(\Omega, E)$.

Let $L_{(2)}^{p,q}(\Omega, E)$ be the completion of the pre-Hilbert space $C_0^{p,q}(\Omega, E)$ with respect to the L^2-norm. The operator $\overline{\partial}$ will also stand for a densely defined closed linear operator on $L_{(2)}^{p,q}(\Omega, E)$ whose domain of definition, denoted by $\mathrm{Dom}\overline{\partial}$, is $\{f \in L^{p,q}(\Omega, E); \overline{\partial}f \in L^{p,q+1}(\Omega, E)\}$, where $\overline{\partial}f$ is defined in the sense of distribution. We define the L^2 $\overline{\partial}$-cohomology groups $H_2^{p,q}(\Omega, E)$ by

$$H_{(2)}^{p,q}(\Omega, E) := \mathrm{Ker}\overline{\partial} \cap L_{(2)}^{p,q}(\Omega, E)/\mathrm{Im}\overline{\partial} \cap L_{(2)}^{p,q}(\Omega, E). \tag{3}$$

The bundle E will not be referred to if E is the trivial line bundle $\Omega \times \mathbb{C}$.

The adjoint of $\overline{\partial}$ will be denoted by $\overline{\partial}^*$. A basic fact is that

$$H_{(2)}^{p,q}(\Omega, E) \approx \mathrm{Ker}\overline{\partial} \cap \mathrm{Ker}\overline{\partial}^* \cap L_{(2)}^{p,q}(\Omega, E) \tag{4}$$

if $\mathrm{Im}\overline{\partial} \cap L_{(2)}^{p,q}(\Omega, E)$ is closed (cf. [24]).

Commutator relations of the operators including $\partial_h = h^{-1} \circ \partial \circ h$, where h is regarded as a smooth section of $\mathrm{Hom}(E, \overline{E}^*)$ and maps $C^{p,q}(\Omega, E)$ to $C^{p,q}(\Omega, \overline{E}^*)$, are used to obtain the L^2-estimates of the Poincaré type for the operator $\overline{\partial}^*$. Here E^* stands for the dual of E and $\overline{E}^*$ its complex conjugate.

Let Θ_h be the curvature form of h. Recall that $\Theta_h \in C^{1,1}(\Omega, \mathrm{Hom}(E, E))$, $\Theta_h = \partial_h\overline{\partial} + \overline{\partial}\partial_h$ as an operator, and that there are positivity notions of Θ_h as a quadratic form on $E \otimes T^{1,0}\Omega$.

The positivity of Θ_h is that of $\partial\bar{\partial}\varphi$ if rank$E = 1$ and $h = e^{-\varphi}$ locally. (E, h) is said to be flat if $\Theta_h = 0$. Vanishing of the L^2-$\bar{\partial}$-cohomology holds under some positivity condition on Θ_h. The following is one of the most basic theorems in this theory.

Theorem 2 (cf. [2] and [1]) $H^{p,q}_{(2)}(\Omega, E) = 0$ *for* $p+q > n$ *if* g *is a complete Kähler metric on* Ω *and* $g = \Theta_h$.

For any Hermitian form Θ along the fibers of $T^{1,0}\Omega$, we shall denote by $\lambda_p(\Theta), p = 1, \ldots, n$, the eigenvalues of Θ with respect to g ordered in such a way that $\lambda_1(\Theta) \geq \lambda_2(\Theta) \geq \cdots \geq \lambda_n(\Theta)$. The following is a variant of Grauert-Riemenschneider's vanishing theorem in [22]and essentially contained in [57] up to the Serre duality.

Theorem 3 *Let* (Ω, g) *be a connected complete Kähler manifold of dimension n and let* (L, h) *be a Hermitian holomorphic line bundle over* Ω. *Suppose that there exists a positive integer k such that* $\sum_{p=k}^{n} \lambda_p(\Theta_h)$ *is everywhere nonnegative and greater than some positive number outside a proper compact subset of* Ω. *Then* $H^{n,n-k+1}_{(2)}(\Omega, L) = \{0\}$ *and* $H^{0,k-1}_{(2)}(\Omega, L^*) = \{0\}$. *Here* L^* *denotes the dual bundle of* L *equipped with the dual fiber metric.*

Outline of the proof By the curvature condition, there exist $c > 0$ and a compact set $K \subset \Omega$ such that

$$\int_{\Omega\setminus K} |u|^2 dV \leq c \int (\|\bar{\partial}u\|^2 + \|\bar{\partial}^* u\|^2), \tag{5}$$

holds for any $u \in \mathrm{Dom}\bar{\partial} \cap \mathrm{Dom}\bar{\partial}^* \cap L^{n,n-k+1}_{(2)}(\Omega, L)$, where dV denotes the volume form.

From this estimate, it follows by Rellich's lemma and Hörmander's theorem that $\mathrm{Im}\bar{\partial} \cap L^{n,n-k+1}_{(2)}(\Omega, L)$ is closed, and $\dim H^{n,n-k+1}_{(2)}(\Omega, L) < \infty$ (cf. Theorem 1.1.3 in [23]). Hence $H^{n,n-k+1}_{(2)}(\Omega, E) \approx \mathrm{Ker}\bar{\partial} \cap \mathrm{Ker}\bar{\partial}^* \cap L^{n,n-k+1}_{(2)}(\Omega, E)$. By (5) it then follows that every element of $\mathrm{Ker}\bar{\partial} \cap \mathrm{Ker}\bar{\partial}^* \cap L^{n,n-k+1}_{(2)}(\Omega, E)$ is zero outside some compact subset of Ω, and so it vanishes identically by Aronszajn's unique continuation theorem. Hence $H^{n,n-k+1}_{(2)}(\Omega, L) = \{0\}$. Similarly one obtains that $H^{0,k-1}_{(2)}(\Omega, L^*) = \{0\}$. □

Corollary 4 *Let* (Ω, g) *be as above and let* $\varphi : \Omega \to [0, \infty)$ *be a* $\mathcal{C}^\infty$ *exhaustion function on* Ω *such that* $\sum_{j=k}^{n} \lambda_j(\partial\bar{\partial}\varphi)$ *is everywhere nonnegative and greater than some positive constant outside a proper compact subset of* Ω. *Then* $H^{n,p}(\Omega) = \{0\}$ *(resp.* $H^{0,p}_0(\Omega) = \{0\}$*) hold for* $p \geq n - k + 1$ *(resp.* $p \leq k - 1$*).*

Proof Let $u \in \mathrm{Ker}\bar{\partial} \cap C^{n,n-k+1}(\Omega)$. Then, one can find a convex increasing function ν such that $u \in L^{n,n-k+1}_{(2)}(\Omega)$ with respect to the fiber metric $e^{-\nu(\varphi)}$. Hence, by Theorem 3 and the strong ellipticity of the Laplacian we have that

$u \in \mathrm{Im}(\bar{\partial}|C^{n,n-k}(\Omega))$. Hence $H^{n,n-k+1}(\Omega) = \{0\}$, Similarly one has $H^{n,p}(\Omega) = \{0\}$ for $p \geq n-k+1$. Hence, by the Serre duality $H_0^{0,p}(\Omega) = \{0\}$ for $p \leq k-1$. □

In particular, the following Hartogs type extension theorem holds.

Theorem 5 *Let Ω be a connected complex manifold of dimension n admitting a complete Kähler metric and a $\mathcal{C}^\infty$ exhaustion function φ such that the sum of any $n-2$ eigenvalues of $\partial\bar{\partial}\varphi$ is everywhere nonnegative and bigger than some positive constant outside a proper compact subset of Ω. Then the natural restriction map $H^{0,1}(\Omega) \to \varinjlim H^{0,1}(\Omega \setminus K)$ is surjective. Here $\varinjlim$ denotes the inductive limit and K runs through the compact subsets of Ω.*

The following is proved similarly as above.

Theorem 6 (cf. [9, 38, 51]) *Let (Ω, g) be a complete Kähler manifold and let (E, h) be a flat Hermitian vector bundle over Ω. Assume that there exists a $\mathcal{C}^\infty$-smooth plurisubharmonic function φ on Ω such that $\partial\bar{\partial}\varphi$ is of rank $\geq n-k+1$ outside a compact subset of Ω. Then the natural inclusion homomorphism from $H_0^{p,q}(\Omega, E)$ to $H^{p,q}(\Omega, E)$ is injective for $p+q \leq n-k$.*

Corollary 7 *In the above situation, the restriction maps $H^{p,q}(\Omega, E) \to \varinjlim H^{p,q}(\Omega \setminus K, E)$ are surjective for $p+q \leq n-k-1$.*

Let's consider the case $\Omega = M \setminus X$ where M is compact and X is Levi flat. For simplicity we shall assume moreover that X is of class C^∞. The following basic observation is essentially contained in [18].

Proposition 8 *If M admits a Kähler metric, then $M \setminus X$ admits a complete Kähler metric.*

Proof Let $\delta(z)$ denote the distance from a point z of M to X with respect to a Kähler metric g^0 on M. Then there exists a neighborhood U of X such that $g^0 - \partial\bar{\partial}\log\delta$ is a complete Kähler metric on $U \setminus X$. Then one takes a $\mathcal{C}^\infty$-smooth function ρ : $M \to [0, 1]$ which is identically 1 on a neighborhood of X and supp $\rho \subset U$. Clearly, $Ag^0 - \partial\bar{\partial}\log\delta$ becomes a complete Kähler metric on $M \setminus X$ for sufficiently large positive number A. □

As for the global pseudoconvexity of $M \setminus X$, an open question is whether or not $M \setminus X$ admits a plurisubharmonic exhaustion function in the situation of Proposition 8. It is known that $-\log\delta$ is plurisubharmonic (resp. strictly plurisubharmonic) near X if the holomorphic bisectional curvature of M is semipositive (resp. positive) near X (cf. [14, 56, 58]. See also [40] and [50]).

Remark 9 It is not difficult to extend the result to X of class $\mathcal{C}^2$ because of an approximation theorem for plurisubharmonic functions by smooth "nearly plurisubhamonic" ones on Kähler manifolds (cf. [8]). Similarly, by the method of [51], the smoothness condition on φ can be weakened to $\mathcal{C}^2$ in Theorem 6 and Corollary 8.

4 Nonexistence Theorems in $\mathbb{CP}^n$ ($n \geq 3$) and Generalizations

We shall focus here on the nonexistence question for the Levi flat hypersurfaces in $\mathbb{CP}^n$ for $n \geq 3$. Positive results (on the nonexistence) and their generalizations will be reviewed. The problem arose as a part of a conjecture that, for any holomorphic foliation of codimension one (with singularities) on $\mathbb{CP}^n$, any leaf accumulates to a singular point of the foliation (cf. [4] and [7]).

Definition 10 A holomorphic foliation of codimension k on a complex manifold M is a coherent analytic subsheaf, say $\mathcal{F}$, of the sheaf $\mathcal{O}(T^{1,0}M)$ of the germs of holomorphic sections of $T^{1,0}M$, such that there exists a Zariski dense open subset $U \subset M$ and a bijectively embedded submanifold $\iota : F \hookrightarrow U$ of codimension k for which $\iota^*\mathcal{F} = \mathcal{O}(T^{1,0}F)$ holds. The subbundle $E \subset T^{1,0}M|_U$ satisfying $T^{1,0}F = \iota^*E$ is called the tangent bundle of $\mathcal{F}$. The quotient of $T^{1,0}M|_U$ by E is called the normal bundle of $\mathcal{F}$.

We denote by Sing$\mathcal{F}$ the set of points around which $\mathcal{F}$ is not locally free. Clearly $U \subset M\setminus$Sing$\mathcal{F}$. Connected components of F are called the leaves of F. A nonempty closed subset of M is called a stable set of $\mathcal{F}$ if it is a union of leaves of $\mathcal{F}$. It is an open question whether or not the closure of every leaf of $\mathcal{F}$ intersects with Sing$\mathcal{F}$ if $M = \mathbb{CP}^n$ ($n \geq 2$) and $\mathcal{F}$ is of codimension one (cf. [7]). LinsNeto [29] proved the following.

Theorem 11 *On* $\mathbb{CP}^n$ ($n \geq 3$)*, holomorphic foliations of codimension one do not have stable sets.*

The proof in [29] is based on a study of holomorphic foliations on $\mathbb{CP}^2$ in [4]. Theorem 11 implies the following nonexistence theorem for the real analytic Levi flat hypersurfaces.

Theorem 12 *There is no real analytic Levi flat hypersurfaces in* $\mathbb{CP}^n$ ($n \geq 3$).

Proof If there existed one, say X, then $\mathbb{CP}^n \setminus X$ is Stein by a theorem of Fujita (cf. [16]). On the other hand, by the real analyticity of X, there exists a holomorphic foliation say $\mathcal{G}$ on a neighborhood of X whose leaves intersecting with X are those of the Levi foliation. Since $\mathbb{CP}^n \setminus X$ is Stein and of dimension ≥ 3, we have that $\mathcal{G}$ extends to a coherent analytic sheaf say $\tilde{\mathcal{G}}$ on $\mathbb{CP}^n$ (cf. [52]). $\tilde{\mathcal{G}}$ is obviously a holomorphic foliation of codimension one on $\mathbb{CP}^n$, so that a contradiction with Theorem 11 arises. □

We will now give an alternative proof based on the idea in [55]. It is more straightforward and independent of Theorem 11.

Another proof of Theorem 12 Suppose the contrary. Then one has X and $\mathcal{G}$ as above. The normal bundle of $\mathcal{G}$, say $\mathcal{N}$, is positive on a neighborhood $W \supset X$ because it is a quotient of $T^{1,0}\mathbb{CP}^n$. Let Θ be the curvature form of $\mathcal{N}$ such that $\Theta|_W > 0$. By shrinking W if necessary, one can find a smooth 1-form θ on W satisfying $d\theta =$

Θ, since $\mathcal{N}|_X$ is topologically trivial. Let $\theta^{0,1}$ be the $(0,1)$ component of θ. Then $\overline{\partial}\theta^{0,1} = 0$ since Θ is of type $(1,1)$. Since $\mathbb{CP}^n \setminus X$ is Stein and $n \geq 3$, we have that $\theta^{0,1}$ extends as a smooth $\overline{\partial}$-closed form on $\mathbb{CP}^n$. Since $H^{0,1}(\mathbb{CP}^n) = 0$, there exists a real valued function φ on $\mathbb{CP}^n$ such that $\overline{\partial}\varphi = \theta$. Hence $\Theta = \partial\overline{\partial}\varphi$ on W so that φ is strictly plurisubharmonic along any leaf of the Levi foliation. Since X is compact, this contradicts the maximum principle for plurisubharmonic functions.

From this point, Theorem 12 has been generalised in two different directions. First, the method of the above alternate proof of Theorem 12 works also to show the nonexistence of X under a weaker regularity assumption. Indeed, Siu combined this approach with a technique of solving the $\overline{\partial}$-equation with boundary regularity to obtain the following.

Theorem 13 *There exist no $\mathcal{C}^{12}$-smooth Levi flat hypersurfaces in $\mathbb{CP}^3$.*

Combining Theorem 13 with the argument of taking a generic hyperplane section, one can deduce that the same is true for $\mathbb{CP}^n$ $(n \geq 3)$. Cao and Shaw [5] further generalised the result to the hypersurfaces of Lipschitz class, where the Levi flatness is to be understood as the two-sided pseudoconvexity, too. Another generalization of Theorem 11 was done in [43] to compact Kähler manifolds.

Theorem 14 (cf. [43]) *Let M be a compact Kähler manifold and let X be a real analytic Levi flat hypersurface in M. Then $M \setminus X$ does not admit a plurisubharmonic exhaustion function of class $\mathcal{C}^2$ whose Levi form has at least 3 positive eigenvalues outside a compact subset of $M \setminus X$.*

Proof Suppose that there were a plurisubharmonic exhaustion function φ on $M \setminus X$ such that $\partial\overline{\partial}\varphi$ is of rank ≥ 3 outside a compact subset of $M \setminus X$. As in the above proof of Theorem 11, we extend the Levi foliation of X to a neighborhood as a holomorphic foliation say $\mathcal{G}$, and let $\mathcal{N}$ be the normal bundle of $\mathcal{G}$. Since $\mathcal{N}|_X$ is topologically trivial, $\mathcal{N}$ is defined on a neighborhood of X by a multiplicative 1-cocycle which is the image of some additive 1-cocycle by the exponential map. In virtue of the Dolbeault isomorphism, any additive holomorphic 1-cocycle extends to M in view of Corollary 5 and Proposition 8 (see also the remark after them). Therefore there would exist a holomorphic line bundle $\tilde{\mathcal{N}}$ over M such that $\tilde{\mathcal{N}}$ is topologically trivial and extends $\mathcal{N}$. Since M is Kähler, $\tilde{\mathcal{N}}$ admits a flat Hermitian structure. On the other hand, since $\mathcal{G}$ is a holomorphic foliation of codimension one, there exists a system of locally defined closed holomorphic 1-forms ω_α $(\alpha \in \mathbb{N})$, such that $\cup_\alpha \mathrm{Ker}\omega_\alpha$ is the tangent bundle of $\mathcal{G}$. Then ω_α is naturally indentified with an N^*-valued holomorphic 1-from. Since $\tilde{\mathcal{N}}^*$ is a flat extension of $\mathcal{N}^*$ to M, Corollary 5 implies that there exists an $\tilde{\mathcal{N}}^*$-valued holomorphic 1-form $\tilde{\omega}$ extending $\{\omega_\alpha\}$. Since $\tilde{\mathcal{N}}^*$ is unitary flat and M is Kähler, $d\omega = 0$ must hold. This is a natural extension of Kodaira's observation (cf. [25], Lemma 1.1.2).

Consequently, one has a system of closed holomorphic 1-forms, say $\tilde{\omega}_\alpha$, defining $\mathcal{G}$ satisfying the transition relations $\tilde{\omega}_\alpha = e^{i\theta_{\alpha\beta}}\tilde{\omega}_\beta$ with $\theta_{\alpha\beta} \in \mathbb{R}$. In other words, the leaves of $\mathcal{G}$ locally consists of the level sets of holomorphic functions say f_α satisfying the transition relations $f_\alpha = e^{i\theta_{\alpha\beta}} f_\beta + c_{\alpha\beta}$ $(c_{\alpha\beta} \in \mathbb{C})$, from which it follows that there exist a neighborhood $U \supset X$ and a neighborhood V of the diagonal of $U \times U$ such that $|f(z) - f(w)|$ is a well defined continuous function on V, say $\mathrm{d}(z, w)$. Then we put

$$\delta(z) = \inf\{\mathrm{d}(z, w) : (z, w) \in V \cap (U \times X)\}, \tag{6}$$

and choose a sufficiently small positive number ϵ so that $\delta^{-1}(\epsilon)$ is compact. Then $\varphi|_{\delta^{-1}(\epsilon)}$ obviously violates the maximum principle. □

Combining the above method with a technique of algebraic geometry, a similar phenomenon was observed for the complement of analytic sets in compact Kähler manifolds.

Theorem 15 (cf. [44]) *Let M be a compact Kähler manifold and let $A \subset M$ be a complex analytic subset of codimension one. Assume that there exists an effective divisor D supported on A such that its associated line bundle is topologically trivial on A. Then $M \setminus A$ admits no $\mathcal{C}^\infty$-smooth plurisubharmonic exhaustion function whose Levi form is of rank ≥ 3 outside a compact subset of $M \setminus A$.*

As for recent results related to Theorem 15, see [46] and [61, 62]. Coming back to Levi flat hypersurfaces, more in the spirit of Siu, Brunella [3] proved the following.

Theorem 16 *Let M be a compact Kähler manifold of dimension ≥ 3 and let X be a Levi flat hypersurface of class $\mathcal{C}^{2,\alpha}$ $(\alpha > 0)$. If there exist a neighborhood $U \supset X$ and a holomorphic foliation $\mathcal{F}$ of codimension one on U for which X is stable, then the normal bundle of $\mathcal{F}$ is not positive.*

It turned out later that the curvature condition on the normal bundle can be weakened to "semipositive of rank ≥ 3" alike Theorem 14 (cf. [47]).

5 Classification in Tori

Let M be a complex torus $T = \mathbb{C}^n/\Gamma$, where Γ is a lattice, i.e., $\Gamma = \sum_{j=1}^{2n} \mathbb{Z}e_j$ for linearly independent vectors e_j over $\mathbb{R}$, equipped with a metric induced from the euclidean metric on $\mathbb{C}^n$. In view of Theorems 12 and 13, one may expect that there is a reasonable classification theory for Levi flat hypersurfaces in T. Indeed there is one. Combining the method of the proof of Theorem 14 with a study of distance functions to Levi flat hypersurfaces in T, the following was proved in [45].

Theorem 17 *Let T be a complex torus and let $A \subset T$ be a closed subset. Assume that there exist a neighborhood $U \supset A$ and a one-codimensional topologically trivial holomorphic foliation $\mathcal{F}$ on U for which A is stable. Then, either there exists a holomorphic 1-form ω on T such that $\mathcal{F} = \mathrm{Ker}(\omega|_U)$, or there exist a complex 2-torus T', a holomorphic map $\pi : T \to T'$, and a closed subset $A' \subset T'$ such that $A = \pi^*(A')$. In the latter case, $\mathcal{F} = \pi^*\mathcal{F}'$ on a neighborhood of A for some holomorphic foliation $\mathcal{F}'$ on a neighborhood of A'.*

The reason why one can apply the method of Theorem 14 in this situation is that the curvature of the leaves of the Levi foliation contributes to the positivity of the Levi form of $-\log\delta$ for the distance δ to A in the following way.

Lemma 18 (cf. [32]) *Let $C \subset \mathbb{C}^2$ be a complex curve defined by $C = \{(t, f(t)); t \in V\}$ for open $V \subset \mathbb{C}$ and holomorphic f. Then, for any $p \in C$ there exists a neighborhood U of p in $\mathbb{C}^2$ such that*

$$\sum_{j,k=1}^{2}(\partial^2(\log\delta)/\partial z_j\partial\bar{z}_k)\xi_j\bar{\xi}_k = A/B$$
$$A = |\partial^2 f/\partial t^2|^2|\xi_1 + (\partial\bar{f}/\partial\bar{t})\xi_2|^2$$
$$B = 2(|\partial f\partial t|^2 + 1)\{(|\partial f/\partial t|^2 + 1)^2 - |\partial^2 f/\partial t^2|^2|z_2 - f(t)|^2\}$$

holds for $(z_1, z_2) \in U \setminus C$ and $\xi \in \mathbb{C}^2$. Here δ denotes the euclidean distance from z to C and the variable t in A and B are specified in such a way that $t = t(z_1, z_2)$ is the solution of $z_1 - t + (\partial\bar{f}/\partial\bar{t})z_2 - f(t) = 0$.

There is unfortunately a fatal gap in the proof of the main theorem in [32]. Nevertheless Lemma 18 is very useful. As for some generalizations of Lemma 18, the reader is referred to subsequent papers of Matsumoto (cf. [30, 31]). As a special case where A is a real analytic Levi flat hypersurface and extends the Levi foliation, Theorem 17 says the following.

Corollary 19 *Let X be a real analytic Levi flat hypersurface in a complex torus T of dimension ≥ 3. Then either X is a parallel translate of a Lie subgroup of T, or there exist a complex 2-torus T', a holomorphic map $\pi : T \to T'$ and $X' \subset T'$ such that $X = \pi^{-1}(X')$.*

Sketch proof of Corollary 19 If X is neither a parallel translate of a Lie subgroup nor the union of preimages of simple closed curves by a holomorphic map onto a one-dimensional torus, the nullity of the Levi form of plurisubharmonic exhaustion functions of $T \setminus X$ defines a trivial subbundle E of $T^{1,0}T$ of corank 2 such that $E|_X$ is tangent to X by Theorem 17, which defines a map π whose fibers are tangent to E. □

Obviously Theorem 17 says nothing about the case of 2-tori. Therefore it may be worthwhile to note here that a definitive thing can be said for some 2-tori.

Theorem 20 *Let X be a real analytic Levi flat hypersurface in a complex 2-torus T. If T is not algebraic, then X is either a translate of a real subtorus of T, or T is a principal 1-torus bundle over a 1-torus T' and X is the closure of the parallel transports of the orbit of a Lie subgroup of the fiber group by a meromorphic connection over T'.*

Proof If X is not a translate of any subtorus, then it is easy to see that $T \setminus X$ is Stein by Lemma 18 and Grauert's theorem in [19]. Hence, by the analytic continuation theorem of Levi, the Gauss map associated to the Levi foliation extends to T as a nonconstant meromorphic map from T to $\mathbb{CP}^1$. (cf. [52]. See also [33].) Since T is not algebraic, f does not have a point of indeterminacy. Hence f induces a smooth fibration say π from T onto a 1-torus T' such that f is constant on each fiber of π.

Then X is of the form as asserted with respect to a meromorphic connection with poles in the set $\{x \in T'; \pi(x) \subset X\}$. □

It has to be mentioned that holomorphic foliations of codimension one without singularities have been classified by Ghys [17] on complex tori of any dimension. Accordingly, all the Levi flat stables sets of such foliations can be described in a similar manner as above (cf. [45], Appendix). Some of them have certain symmetry property. (cf. [42]). Furthermore, the above method exploiting the Gauss map can be generalized to the case of torus bundles over compact Riemann surfaces as long as the algebraic dimension of the total space is ≤ 1. In particular, one observes that what Nemirovski described in [37] was essentially all the real analytic Levi flat hypersurfaces with Stein complement in such surfaces. This is the main idea in extending the classification to that in Hopf surfaces. The results will be explained below.

6 Classification in Hopf Surfaces

By definition, Hopf surfaces are compact complex surfaces whose universal covering is $\mathbb{C}^2 \setminus \{0\}$. It is known that, for any Hopf surface H, there exists a Hopf surface $\mathcal{H}(a, b, \lambda, m)$ defined as the quotient of $\mathbb{C}^2 \setminus \{0\}$ by the action of $\mathbb{Z}$ generated by $(z_1, z_2) \mapsto (az_1 + \lambda z_2^m, bz_2)$, where (z_1, z_2) is the coordinate of $\mathbb{C}^2$, $m \in \mathbb{N}, a, b, \lambda \in \mathbb{C}, 0 < |a| \leq |b| < 1$ and $(b^m - a)\lambda = 0$, and an unramified covering map from $\mathcal{H}(a, b, \lambda, m)$ to H (cf. [26]). $\mathcal{H}(a, b, \lambda, m)$ is called a primary Hopf surface. For the classification of Levi flat hypersurfaces, it suffices to restrict ourselves to the case $\mathcal{H} = \mathcal{H}(a, b, \lambda, m)$. For simplicity we put $\mathcal{H}(a, b) = \mathcal{H}(a, b, 0, m)$. Kim, Levenberg and Yamaguchi [27] first studied Levi flat hypersurfaces in $\mathcal{H}(a, a)$ by applying a variational formula for the Robin function. The result was generalized in [28] as follows.

Theorem 21 *Let X be a real analytic Levi flat hypersurface in $\mathcal{H}(a, b)$ such that $\mathcal{H}(a, b) \setminus X$ is not Stein. Then either $\mathcal{H}(a, b) \setminus X$ is holomorphically convex or the connected components of X are defined by* $|z_1|^{\log |b|} |z_2|^{-\log |a|} = \text{const}$.

In [28], it is noted that $\mathcal{H}(a, b) \setminus X$ is Stein if b is real and $X = \{\text{Im} z_2 = 0\}$. Clearly, $\mathcal{H}(a, b, \lambda, m) \setminus X$ is also Stein if b is real and $X = \{\text{Im} z_2 = 0\}$. This was supplemented in [48, 49] as follows.

Theorem 22 *Let X be a real analytic Levi flat hypersurface in $\mathcal{H}(a, b, \lambda, m)$. If $\mathcal{H}(a, b, \lambda, m) \setminus X$ is Stein, then there exist a surjective holomorphic map π from $\mathbb{C}^2 \setminus \{0\}$ to $\hat{\mathbb{C}}$ with fibers $\mathbb{C} \setminus \{0\}$ such that $\pi|_{\pi^{-1}(\mathbb{C})}$ is a trivial fibration, a closed curve C in a fiber of π, and a meromorphic 1-form η on $\hat{\mathbb{C}}$ such that the image, under $\mathbb{C}^2 \setminus \{0\} \mapsto \mathcal{H}(a, b, \lambda, m)$, of the union of the parallel transports of C induced by η over the set of regular points of η is densely contained in X. If $\mathcal{H}(a, b, \lambda, m) \setminus X$ is not Stein, then $\lambda = 0$.*

The reader is referred to [48] for the proof of the first assertion, as well as how the parallel transport of C is defined with respect to η (see also [41]). For the latter part, see [49].

Remark 23 Recently, Miebach [34] gave an alternative proof of Theorem 21 by analyzing the action of a Lie group on $\mathcal{H}(a, b)$. Actually he determined all the non-Stein pseudoconvex domains in $\mathcal{H}(a, b)$. As for the domains with thin boundary, it seems reasonable to expect, in view of Theorem 22, that such Stein domains in $\mathcal{H}(a, b, \lambda, m)$ can be described explicitly, too. (Added in proof: This was solved by Miebach, recently.)

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Interplay Between CR Geometry and Algebraic Geometry

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1 Introduction

CR manifolds are abstract models of real hypersurfaces in complex spaces. The abstract definition of the boundary as a *CR* structure on a complex manifold is essentially in Cartan [7]. For more detail, see [24, 25]. Strongly pseudoconvex *CR* manifolds have rich geometric and analytic structures. Namely, there is an intrinsic pseudo conformed geometry for which complete local invariants have been obtained, see for example [8, 14, 41], as well as a deep analysis of the $\overline{\partial}_b$ complex, see for example [12, 22, 23, 46]. The harmonic theory for the $\overline{\partial}_b$ complex on compact strongly pseudoconvex *CR* manifolds was developed by Kohn [21]. Using this theory, Boutet de Monvel [4] proved that if X is a compact strongly pseudoconvex *CR* manifold of dimension $2n-1$, $n \geq 3$, then there exist C^∞

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functions $f_1, \ldots, f_N$ on X such that each $\overline{\partial}_b f_j = 0$ and $f = (f_1, \ldots, f_N)$ defines an embedding of X in $\mathbb{C}^N$. Thus, any compact strongly pseudoconvex *CR* manifold of dimension ≥ 5 can be *CR* embedded in some complex Euclidean space. On the other hand, 3-dimensional strongly pseudoconvex compact orientable *CR*-manifolds are not necessarily embeddable. The first example is due to Andreotti according to [37]. This example also appeared in the list of homogeneous structures of Cartan although the embeddability question was not addressed. Nirenberg [35] first proved that 3-dimensional *CR* manifolds might not be locally embeddable. Jacobowitz and Treves [19, 20] showed that in fact non-embeddable *CR* structures are, in some sense, dense in the space of *CR*-structures over a 3-dimensional manifold. The theory of harmonic integrals on strongly pseudoconvex *CR* structures over small balls was due to Kuranishi [23]. Using this theory, Kuranishi [23] proved that any strongly pseudoconvex *CR* manifold of dimension $2n-1$ with $n \geq 5$ can be locally *CR* embedded as a real hypersurface in $\mathbb{C}^n$. For $n = 4$, Akahori [1] proved that Kuranishi's local embedding theorem is also true. However, the 5-dimensional case of local embeddability of *CR* manifolds remains open.

Throughout this paper, our *CR* manifolds are always assumed to be compact orientable and embeddable in some $\mathbb{C}^N$. By a beautiful theorem of Harvey and Lawson [16, 17], these *CR* manifolds are the boundaries of subvarieties in $\mathbb{C}^N$. This allowed the first author [46] to relate *CR* geometry and algebraic geometry of singularities for the first time. The purpose of this paper is to discuss the interplay between *CR* geometry and algebraic geometry. Our paper is organized as follows. In Sect. 2, we shall recall the basic notion of *CR* geometry. In Sect. 3, we show how to use the Bergman function of the first author to give canonical construction of moduli space for complete Reinhardt domains. In Sect. 4, we use algebraic geometry to study the complex Plateau problem. In Sect. 5, we study the minimal embedding dimension of compact *CR* manifolds in complex Euclidean space. Finally in Sect. 6, we study invariants of compact strongly pseudoconvex *CR* manifolds arising from geometry of singularities.

2 Preliminary

Definition 2.1 Let X be a connected orientable manifold of real dimension $2n-1$. A *CR* structure on X is an $(n-1)$-dimensional subbundle S of the complexified tangent bundle $\mathbb{C}TX$ such that

(1) $S \cap \overline{S} = \{0\}$
(2) If L, L' are local sections of S, then so is $[L, L']$.

A manifold with a *CR* structure is called a *CR* manifold. There is a unique subbundle $\mathcal{H}$ of the tangent bundle $T(X)$ such that $\mathbb{C}\mathcal{H} = S \oplus \overline{S}$. Furthermore, there is a unique homomorphism $J : \mathcal{H} \longrightarrow \mathcal{H}$ such that $J^2 = -1$ and $S = \{v - iJv : v \in \mathcal{H}\}$. The pair $(\mathcal{H}, J)$ is called the real expression of the *CR* structure.

Definition 2.2 Let $L_1, \dots, L_{n-1}$ be a local frame of S. Then $\overline{L}_1, \dots, \overline{L}_{n-1}$ is a local frame of $\overline{S}$ and one may choose a local section N of TX such that $L_1, \dots, L_{n-1}$, $\overline{L}_1, \dots, \overline{L}_{n-1}, N$ is a local frame of $\mathbb{C}TX$. The matrix (c_{ij}) defined by

$$[L_i, \overline{L}_j] = \Sigma a_{ij}^k L_k + \Sigma b_{ij}^k \overline{L}_k + \sqrt{-1} c_{ij} N$$

is Hermitian and is called the Levi form of X.

Proposition 2.1 *The number of non-zero eigenvalues and the absolute value of the signature of the Levi form* (c_{ij}) *at each point are independent of the choice of* $L_1, \dots, L_{n-1}, N$.

Definition 2.3 The CR manifold X is called strongly pseudoconvex if the Levi form is definite at each point of X.

Theorem 2.2 (Boutet de Monvel [4]) *If X is a compact strongly pseudoconvex CR manifold of dimension* $(2n-1)$ *and* $n \geq 3$*, then X is CR embeddable in* $\mathbb{C}^N$.

Although there are non-embeddable compact 3-dimensionable *CR* manifolds, in this paper all *CR* manifolds are assumed to be embeddable in complex Euclidean space. The following theorem links *CR* geometry and algebraic geometry together.

Theorem 2.3 (Harvey-Lawson [16, 17]) *For any compact connected embeddable CR manifold X, there is a unique complex variety V in $\mathbb{C}^N$ for some N such that the boundary of V is X and V has only normal isolated singularities.*

3 Bergman Function and Moduli Space of Complete Reinhardt Domains

Recall that a complex manifold M is called strictly pseudoconvex if there is a compact set B in M, and a continuous real valued function ϕ on M, which is strictly plurisubharmonic outside B and such that for each $c \in \mathbb{R}$, the set $M_c = \{x \in M : \phi(x) < c\}$ is relatively compact in M. Note that a strictly pseudoconvex complex manifold is a modification of a Stein space at a finite many points.

Let V be a Stein variety of dimension $n \geqslant 2$ in $\mathbb{C}^N$ with only irreducible isolated singularities. We assume that ∂V is a smooth *CR* manifold. Let $\pi: M \to V$ be a resolution of singularity with E as an exceptional set. We shall define the k-th order Bergman function $B_M^{(k)}(z)$ on M which is a biholomorphic invariant of M.

Definition 3.1 Let F (respectively, F_k) be the set of all L^2 integrable holomorphic n-forms Ψ on M (respectively, vanishing at least the k-th order on the exception set E of M). Let $\{w_j\}$ (respectively, $\{w_j^{(k)}\}$) be a complete orthonormal basis of F (respectively, F_k). The Bergman kernel (respectively Bergman kernel vanishing on the exceptional set of k-th order) is defined to be $K(z) = \sum w_j(z) \wedge \overline{w_j(z)}$ (respectively, $K^{(k)}(z) = \sum w_j^{(k)}(z) \wedge \overline{w_j^{(k)}(z)}$).

Lemma 3.1 *F/F_k is a finite dimensional vector space.*

Lemma 3.2 *Bergman kernel vanishing on the exceptional set of k-th order $K^{(k)}(z)$ is independent of the choice of the complete orthonormal basis of F_k and $K^{(k)}(z)$ is invariant under biholomorphic maps.*

Definition 3.2 Let M be a resolution of a Stein variety V of dimension $n \geqslant 2$ in $\mathbb{C}^N$ with only irreducible isolated singularity at the origin. The k-th order Bergman function $B_M^{(k)}$ on M is defined to be $K_M^{(k)}/K_M$.

Theorem 3.1 *$B_M^{(k)}$ is a global function defined on M which is invariant under biholomorphic maps. Moreover, $B_M^{(k)}$ is nowhere vanishing outside the exceptional set of M. If the canonical bundle is generated by its global sections in a neighborhood of the exceptional set, then the zero set of the k-th order Bergman function $B_M^{(k)}$ is precisely the exceptional set of M.*

Theorem 3.2 *Let M be a strictly pseudoconvex complex manifold of dimension $n \geqslant 2$ with exceptional set E. Let A be a compact submanifold contained in E. Let $\pi: M_1 \to M$ be the blow up of M along A. Then we have $K_{M_1}^{(k)}(z) = \pi^* K_M^{(k)}(z)$ and $K_{M_1}(z) = \pi^* K_M(z)$. Consequently $B_{M_1}^{(k)}(z) = \pi^* B_M^{(k)}(z)$.*

Let $\pi_i: M_i \to V$, $i = 1, 2$, be two resolutions of singularities of V. By Hironaka's theorem [18], there exists a resolution $\tilde{\pi}: \tilde{M} \to V$ of singularities of V such that $\tilde{M}$ can be obtained from M_i, $i = 1, 2$, by successive blowing up along submanifolds in exceptional set. In view of Theorems 3.1 and 3.2, the following definition is well defined if the canonical bundle is generated by its global sections in a neighborhood of the exceptional set.

Definition 3.3 Let V be a Stein variety in $\mathbb{C}^N$ with only irreducible isolated singularities. Let $\pi: M \to V$ be a resolution of singularities of V such that the canonical bundle is generated by its global sections in a neighborhood of the exceptional set. Define the k-th order Bergman function $B_V^{(k)}$ on V to be the push forward of the k-th order Bergman function $B_M^{(k)}$ by the map π.

Theorem 3.3 *Let V be a Stein variety in $\mathbb{C}^N$ with only irreducible isolated singularities. Assume that there exists a resolution M of singularities of V such that the canonical bundle is generated by its global sections in a neighborhood of the exceptional set. Then the k-th order Bergman function $B_V^{(k)}$ on V is invariant under biholomorphic maps and $B_V^{(k)}$ vanishes precisely on the singular set of V.*

For the convenience of the readers, we recall the following two important theorems.

Theorem 3.4 ([13]) *A biholomorphic mapping between two strictly pseudoconvex domains is smooth up to boundary and the induced boundary mapping gives a CR-equivalence between the boundaries.*

Theorem 3.5 ([40]) *Two n-dimensional bounded Reinhardt domains D_1 and D_2 are mutually equivalent if and only if there exists a transformation $\phi : \mathbb{C}^n \to \mathbb{C}^n$ given by $z_i \mapsto r_i z_{\sigma(i)} (r_i > 0, i = 1, \cdots, n$ and σ being a permutation of the indices i) such that $\phi(D_1) = D_2$.*

The following Proposition 3.1 tells us how to use singularity structures to distinguish *CR* structures.

Proposition 3.1 ([50]) *Let X_1, X_2 be two strictly pseudoconvex CR manifolds of dimension $2n-1$ which bound varieties V_1, V_2 respectively in $\mathbb{C}^N$ with only isolated normal singularities. If $\Phi: X_1 \to X_2$ is a CR-isomorphism, then Φ can be extended to a biholomorphic map from V_1 to V_2.*

In view of the above Proposition 3.1, if X_1 and X_2 are two strictly pseudoconvex *CR* manifolds which bound varieties V_1 and V_2 respectively with non-isomorphic singularities, then X_1 and X_2 are not *CR* equivalent. Therefore to study the *CR* equivalence of two strictly pseudoconvex *CR* manifolds X_1 and X_2, it remains to consider the case when X_1 and X_2 are lying on the same variety V. It is known that the global invariant Bergman function of k-th order can be used to study the *CR* equivalence problem of smooth *CR* manifolds lying on the same variety. As an example, we shall show explicitly how *CR* manifolds varies in the A_n-variety $\tilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : f(x, y, z) = xy - z^{n+1} = 0\}$. An explicit resolution $\tilde{\pi}: \tilde{M}_n \to \tilde{V}_n$ can be given in terms of coordinate charts and transition functions as follows:

$$\text{Coordinate charts:} \quad \tilde{W}_k = \mathbb{C}^2 = \{(u_k, v_k)\}, \; k = 0, 1, \cdots, n.$$

$$\text{Transition functions:} \quad \begin{cases} u_{k+1} = \dfrac{1}{v_k} \\ v_{k+1} = u_k {v_k}^2 \end{cases} \quad \text{or} \quad \begin{cases} u_k = {u_{k+1}}^2 v_{k+1} \\ v_k = \dfrac{1}{u_{k+1}} \end{cases}$$

$$\text{Resolution map:} \quad \tilde{\pi}(u_k, v_k) = \left(u_k^{k+1} v_k^k, u_k^{n-k} v_k^{n+1-k}, u_k v_k\right) \quad \text{or}$$

$$(x, y, z) = (u_0, u_0^n v_0^{n+1}, u_0 v_0) = \cdots = ({u_n}^{n+1} v_n^n, v_n, u_n v_n)$$

$$\text{Exceptional set:} \quad E = \tilde{\pi}^{-1}(0) = C_k = \{u_{k-1} = 0\} \cup \{v_k = 0\}, \quad k = 1, \cdots, n.$$

From now on, we suppose V to be a bounded complete Reinhardt domain in $\tilde{V}_n$ (cf. Definition 3.5). Then let $M = \tilde{\pi}^{-1}(V) = \cup_{k=0}^n W_k$, where $W_k = \tilde{\pi}^{-1}(V) \cap \tilde{W}_k$, $k = 0, 1, \cdots, n$. Observe that under $\pi := \tilde{\pi}|_M : M \to V$, $W_0 \backslash C_1$ is mapped biholomorphically onto $V \backslash y$-axis. In particular $M \backslash W_0$ is of measure zero in the obvious sense. Hence, we may compute integrals on M using the (u_0, v_0) coordinate on the chart W_0 alone.

The following proposition is a general consequence of the proof of Proposition 3.2 of [50].

Proposition 3.2 ([10]) *In the above notations, let* $\phi_{\alpha\beta} = u_0^\alpha v_0^\beta \, du_0 \wedge dv_0$, $\alpha, \beta = 0, 1, 2, \ldots$ *Then* $\left\{ \frac{\phi_{\alpha\beta}}{\|\phi_{\alpha\beta}\|_M} : \alpha \geqslant \frac{n}{n+1}\beta \right\}$ *is a complete orthonormal base of* F *and* $\left\{ \frac{\phi_{\alpha\beta}}{\|\phi_{\alpha\beta}\|_M} : \alpha \geqslant \frac{n}{n+1}\beta \text{ and } \alpha \geqslant k \right\}$ *is a complete orthonormal base of* F_k. *Therefore the Bergman kernel vanishing on the exceptional set of k-th order* $K_M^{(k)}$ *and the Bergman kernel* K_M *are given respectively by:*

$$K_M^{(k)}(u_0, v_0) = \Theta_M^{(k)} \, du_0 \wedge dv_0 \wedge d\overline{u_0} \wedge d\overline{v_0}$$

where

$$\Theta_M^{(k)} = \sum_{\substack{\alpha \geqslant \frac{n}{n+1}\beta \\ \alpha \geqslant k}} \frac{|u_0|^{2\alpha} |v_0|^{2\beta}}{\|\phi_{\alpha\beta}\|_M^2},$$

and

$$K_M(u_0, v_0) = \left(\frac{1}{\|\phi_{00}\|_M^2} + \sum_{\substack{\alpha \geqslant \frac{n}{n+1}\beta \\ 1 \leqslant \alpha \leqslant k-1}} \frac{|u_0|^{2\alpha} |v_0|^{2\beta}}{\|\phi_{\alpha\beta}\|_M^2} + \Theta_M^{(k)} \right) du_0 \wedge dv_0 \wedge d\overline{u_0} \wedge d\overline{v_0}.$$

The following results generalize Theorem 3.3 in [50].

Theorem 3.6 ([10]) *In the above notations, the k-th order Bergman function for the strongly pseudoconvex complex manifold M is given by*

$$B_M^{(k)}(u_0, v_0) = \frac{\Theta_M^{(k)}}{\left(\frac{1}{\|\phi_{00}\|_M^2} + \sum_{\substack{\alpha \geqslant \frac{n}{n+1}\beta \\ \alpha \geqslant 1}} \frac{|u_0|^{2\alpha} |v_0|^{2\beta}}{\|\phi_{\alpha\beta}\|_M^2} \right)}$$

The k-th order Bergman function for the variety is given by

$$B_V^{(k)}(x, y) = \frac{\Theta_V^{(k)}}{\left(\frac{1}{\|\phi_{00}\|_M^2} + \Theta_V^{(1)} \right)},$$

where

$$\Theta_V^{(k)} = \sum_{\substack{\alpha \geqslant \frac{n}{n+1}\beta \\ \alpha \geqslant k}} \frac{|x|^{2\alpha - \frac{2n\beta}{n+1}} |y|^{\frac{2\beta}{n+1}}}{\|\phi_{\alpha\beta}\|_M^2}$$

Definition 3.4 An open subset $D \subseteq \mathbb{C}^n$ is a complete Reinhardt domain if, whenever $(z_1, \cdots, z_n) \in D$ then $(\xi_1 z_1, \cdots, \xi_n z_n) \in D$ for all complex numbers ξ_j with $|\xi_j| \leqslant 1$.

It is well known that $\tilde{V_n} = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ is the quotient of $\mathbb{C}^2$ by a cyclic group of order $n+1$, i.e. $\delta.(z_1, z_2) = (\delta z_1, \delta^n z_2)$, where δ is a primitive $(n+1)$-th root of unit. The quotient map $\pi : \mathbb{C}^2 \to \tilde{V}$ is given by $\pi(z_1, z_2) = (z_1^{n+1}, z_2^{n+1}, z_1 z_2)$.

Definition 3.5 An open set V in the A_n-variety $\tilde{V_n} = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ is called a complete Reinhardt domain if $\pi^{-1}(V)$ is a complete Reinhardt domain in $\mathbb{C}^2$.

Theorem 3.7 ([10]) *Let V_i, $i = 1, 2$, be two bounded complete Reinhardt domains in A_n-variety $\tilde{V_n} = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$. Let*

$$g^{(\alpha,\beta)} = \frac{\|\phi_{10}\|^{\alpha - \frac{n}{n+1}\beta} \|\phi_{n,n+1}\|^{\frac{\beta}{n+1}}}{\|\phi_{\alpha\beta}\| \, \|\phi_{00}\|^{\alpha - \frac{n-1}{n+1}\beta - 1}}.$$

If V_1 is biholomorphic to V_2, then

$$\xi^{(\alpha,\beta)} := g^{(\alpha,\beta)} \cdot g^{(n\alpha-(n-1)\beta,(n+1)\alpha-n\beta)},$$

$$\zeta^{(\alpha,\beta)} := g^{(\alpha,\beta)} + g^{(n\alpha-(n-1)\beta,(n+1)\alpha-n\beta)},$$

$$\eta^{(\alpha,p,q)} := (g^{(\alpha,p)} - g^{(n\alpha-(n-1)p,(n+1)\alpha-np)}) \cdot (g^{(\alpha,q)} - g^{(n\alpha-(n-1)q,(n+1)\alpha-nq)})$$

and

$$\omega^{(\alpha_1,\alpha_2,p_1,p_2)} := (g^{(\alpha_1,p_1)} - g^{(n\alpha_1-(n-1)p_1,(n+1)\alpha_1-np_1)}) \cdot (g^{(\alpha_2,p_2)} - g^{(n\alpha_2-(n-1)p_2,(n+1)\alpha_2-np_2)}),$$

where

$$\alpha \geqslant 1, \alpha \geqslant \frac{n}{n+1}\beta, 0 \leqslant p,\ q \leqslant \left[\frac{n+1}{n}\alpha\right], p \neq q,$$

$$0 \leqslant p_i \leqslant \left[\frac{n+1}{n}\alpha_i\right], \alpha_i \geqslant 1, \alpha_1 \neq \alpha_2, i = 1, 2,$$

are all invariants, i.e.

$$\xi_{V_1}^{(\alpha,\beta)} = \xi_{V_2}^{(\alpha,\beta)}, \zeta_{V_1}^{(\alpha,\beta)} = \zeta_{V_2}^{(\alpha,\beta)}, \eta_{V_1}^{(\alpha,p,q)} = \eta_{V_2}^{(\alpha,p,q)},$$

$$\omega_{V_1}^{(\alpha_1,\alpha_2,p_1,p_2)} = \omega_{V_2}^{(\alpha_1,\alpha_2,p_1,p_2)}.$$

The following Theorem says that these invariants in Theorem 3.7 determine completely the Bergman function up to automorphisms of A_n-variety.

Theorem 3.8 ([10]) *Let V_i, $i = 1, 2$, be two bounded complete Reinhardt strictly pseudoconvex (respectively C^ω-smooth pseudoconvex) domains in $\tilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 \colon xy = z^{n+1}\}$. If*

$$\xi_{V_1}^{(\alpha,\beta)} = \xi_{V_2}^{(\alpha,\beta)}, \zeta_{V_1}^{(\alpha,\beta)} = \zeta_{V_2}^{(\alpha,\beta)}, \eta_{V_1}^{(\alpha,p,q)} = \eta_{V_2}^{(\alpha,p,q)},$$

$$\omega_{V_1}^{(\alpha_1,\alpha_2,p_1,p_2)} = \omega_{V_2}^{(\alpha_1,\alpha_2,p_1,p_2)},$$

where

$$\alpha \geqslant 1, \alpha \geqslant \frac{n}{n+1}\beta, 0 \leqslant p,\ q \leqslant \left[\frac{n+1}{n}\alpha\right], p \neq q,$$

$$0 \leqslant p_i \leqslant \left[\frac{n+1}{n}\alpha_i\right], \alpha_i \geqslant 1, \alpha_1 \neq \alpha_2, i = 1, 2,$$

then there exists an automorphism $\Psi = (\psi_1, \psi_2, \psi_3)$ of A_n-variety $\tilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ given by either

$$(\psi_1, \psi_2, \psi_3) =$$

$$\left(\frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}}x,\ \frac{\|\phi_{n,n+1}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{n,n+1}\|_{M_1}}y,\ \frac{\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}}z\right),$$

or

$$(\psi_1, \psi_2, \psi_3) =$$

$$\left(\frac{\|\phi_{10}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{n,n+1}\|_{M_1}}y,\ \frac{\|\phi_{n,n+1}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{10}\|_{M_1}}x,\ \frac{\|\phi_{11}\|_{M_2}}{\|\phi_{00}\|_{M_2}}\frac{\|\phi_{00}\|_{M_1}}{\|\phi_{11}\|_{M_1}}z\right).$$

such that Ψ sends V_1 to V_2.

As an immediate corollary of Theorem 3.8 above, we have the following theorem.

Theorem 3.9 ([10]) *The moduli space of bounded complete Reinhardt strictly pseudoconvex (respectively C^ω-smooth pseudoconvex) domains in A_n-variety $\tilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ is given by the image of the map $\Phi : \{V : V$ a bounded complete Reinhardt strictly pseudoconvex (respectively C^ω-smooth pseudoconvex) domain in $\tilde{V}_n\} \to \mathbb{R}^\infty$, where the component function of Φ are the invariant functions*

$$\xi^{(\alpha,\beta)}, \zeta^{(\alpha,\beta)}, \eta^{(\alpha,p,q)}, \omega^{(\alpha_1,\alpha_2,p_1,p_2)},$$

$$\alpha \geqslant 1, \alpha \geqslant \frac{n}{n+1}\beta, 0 \leqslant p,\ q \leqslant \left[\frac{n+1}{n}\alpha\right], p \neq q,$$

$$0 \leqslant p_i \leqslant \left[\frac{n+1}{n}\alpha_i\right], \alpha_i \geqslant 1, \alpha_1 \neq \alpha_2, i = 1, 2.$$

defined in Theorem 3.7.

The following theorem says that the biholomorphic equivalence problem for bounded complete Reinhardt domains in A_n-variety $\tilde{V}_n$ is the same as the biholomorphic equivalence problem for the corresponding bounded complete Reinhardt domains in $\mathbb{C}^2$.

Theorem 3.10 ([10]) *Let $\pi : \mathbb{C}^2 \to \tilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$ be the quotient map given by $\pi(z_1, z_2) = (z_1^{n+1}, z_2^{n+1}, z_1 z_2)$. Let V_i, $i = 1, 2$, be bounded complete Reinhardt domains in $\tilde{V}_n$ such that $W_i := \pi^{-1}(V_i), i = 1, 2$, are bounded complete Reinhardt domain in $\mathbb{C}^2$. Then V_1 is biholomorphic to V_2 if and only if W_1 is biholomorphic to W_2. In particular, V_1 is biholomorphic to V_2 if and only if there exists a biholomorphism $\Phi : V_1 \to V_2$ given by $\Phi(x, y, z) = (a^{n+1}x, b^{n+1}y, abz)$ or $\Phi(x, y, z) = (a^{n+1}y, b^{n+1}x, abz)$ where $a, b > 0$.*

As a corollary of Theorems 3.10 and 3.9, we have the following theorem.

Theorem 3.11 ([10])

(1) Let $\mathcal{W} = \{W : W = \pi^{-1}(V)$ where V is a bounded complete Reinhardt domain in A_n-variety$\}$ be the space of bounded complete Reinhardt domains in $\mathbb{C}^2$ which are invariant under the action of the cyclic group of order $n + 1$ on $\mathbb{C}^2$. Then

$$\xi^{(\alpha,\beta)}, \zeta^{(\alpha,\beta)}, \eta^{(\alpha,p,q)}, \omega^{(\alpha_1,\alpha_2,p_1,p_2)},$$

$$\alpha \geqslant 1, \alpha \geqslant \frac{n}{n+1}\beta, 0 \leqslant p,\ q \leqslant \left[\frac{n+1}{n}\alpha\right], p \neq q,$$

$$0 \leqslant p_i \leqslant \left[\frac{n+1}{n}\alpha_i\right], \alpha_i \geqslant 1, \alpha_1 \neq \alpha_2, i = 1, 2,$$

defined in Theorem 3.7 are invariants of $\mathcal{W}$.

(2) *Let $\mathcal{W}_P = \{W : W = \pi^{-1}(V)$ where V is a complete Reinhardt pseudoconvex C^ω-smooth domain in A_n-variety$\}$ and $\mathcal{W}_{SP} = \{W : W = \pi^{-1}(V)$ where V is a complete Reinhardt strictly pseudoconvex domain in A_n-variety$\}$. Then the moduli space of $\mathcal{W}_{\mathcal{P}}$ (respectively $\mathcal{W}_{SP}$) is given by the image of the map $\tilde{\Phi}_P : \mathcal{W}_P \to \mathbb{R}^\infty$ (respectively $\tilde{\Phi}_{SP} : \mathcal{W}_{SP} \to \mathbb{R}^\infty$), where the component functions of $\tilde{\Phi}_P$ (respectively $\tilde{\Phi}_{SP}$) are the invariant functions*

$$\xi^{(\alpha,\beta)}, \zeta^{(\alpha,\beta)}, \eta^{(\alpha,p,q)}, \omega^{(\alpha_1,\alpha_2,p_1,p_2)},$$

$$\alpha \geqslant 1, \alpha \geqslant \frac{n}{n+1}\beta, 0 \leqslant p,\ q \leqslant \left[\frac{n+1}{n}\alpha\right], p \neq q,$$

$$0 \leqslant p_i \leqslant \left[\frac{n+1}{n}\alpha_i\right], \alpha_i \geqslant 1, \alpha_1 \neq \alpha_2, i = 1, 2,$$

defined in Theorem 3.7. In particular, the moduli space of $\mathcal{W}_P$ (respectively $\mathcal{W}_{SP}$) is the same as the moduli space of bounded complete Reinhardt pseudoconvex C^ω-smooth domains (respectively bounded complete Reinhardt strictly pseudoconvex domains) in A_n-variety $\tilde{V}_n = \{(x, y, z) \in \mathbb{C}^3 : xy = z^{n+1}\}$.

It is an interesting question to study the geometry of the moduli space of bounded complete Reinhardt domains in A_n-variety. As an example, we look at two families of domains in A_n-variety and construct the moduli space of these families explicitly. More specifically, consider

$$V^{(d)}_{(a,b,c)} = \{(x, y, z): xy = z^2, a|x|^{2d} + b|y|^{2d} + c|z|^{2d} < \varepsilon_0\}.$$

Here we assume that a, b, c are strictly greater than zero, and d is a fixed integer greater than or equal to one. This is a 3 parameters family of pseudoconvex domains in A_1-variety $\tilde{V}_1 = \{(x, y, z) \in \mathbb{C}^3 : xy = z^2\}$. Using our Bergman function theory, we can write down the explicit moduli space of this family as shown in the following theorem by means of the invariant

$$\nu^{(\alpha,\beta)} = \frac{(\xi^{(\alpha,\beta)})^{\frac{1}{2}} \cdot (\xi^{(n\alpha-(n-1)\beta,(n+1)\alpha-n\beta)})^{\frac{1}{2}}}{(\xi^{(\alpha,\alpha)})^{\frac{1}{2}}}, \quad \textit{for} \quad n = 1.$$

Theorem 3.12 ([10]) *Let*

$$V^{(d)}_{(a,b,c)} = \{(x, y, z) \in \mathbb{C}^3: xy = z^2, a|x|^{2d} + b|y|^{2d} + c|z|^{2d} < \varepsilon_0\}.$$

Let $\sim$ denote the biholomorphic equivalence. Then the map

$$\varphi: \{V^{(d)}_{(a,b,c)}\} \to \mathbb{R}_+,\ V^{(d)}_{(a,b,c)} \mapsto \nu^{(2d-1,d-1)}$$

is injective up to a biholomorphism equivalence. More precisely, the induced map

$$\tilde{\varphi}\colon \{V^{(d)}_{(a,b,c)}\}/\sim\ \to \mathbb{R}_+$$

is one-to-one map from $\{V^{(d)}_{(a,b,c)}\}/\sim$ *onto* $\left(0, \dfrac{2}{\pi}\right)$*. So the moduli space of* $\{V^{(d)}_{(a,b,c)}\}$ *is an open interval* $\left(0, \dfrac{2}{\pi}\right)$.

The biholomorphically equivalent problem of domains in A_1-variety is not only interesting in its own right, but also has application to the classical biholomorphically equivalent problem of domains in $\mathbb{C}^2$. In fact, let

$$W^{(d)}_{(a,b,c)} = \{(x, y)\colon a|x|^{2d} + b|y|^{2d} + c|xy|^d < \varepsilon_0\}$$

Corollary 3.1 ([10]) *The moduli space of* $W^{(d)}_{(a,b,c)}$ *is the same as the moduli space of* $V^{(d)}_{(a,b,c)}$*, which is* $(0, \frac{2}{\pi})$.

As an application to the above theory, it is easy to compute explicitly the invariant $\nu^{(3,1)}$ for two domains $V^{(1)}_{(1,1,1)}$ and $V^{(2)}_{(1,1,1)}$ in A_1-variety. As a consequence, we see that $V^{(1)}_{(1,1,1)}$ is not biholomorphic to $V^{(2)}_{(1,1,1)}$ and the domain $W^{(1)}_{(1,1,1)}$ in $\mathbb{C}^2$ is not biholomorphic to the domain $W^{(2)}_{(1,1,1)}$ in $\mathbb{C}^2$.

One of the basic problems in complex geometry is to find a reasonable object which parametrizes all non-isomorphic complex manifolds. This is the well known moduli problem. Let D_1 and D_2 be two domains in $\mathbb{C}^n$. One of the most fundamental problems in complex geometry is to find necessary and sufficient conditions which will imply that D_1 and D_2 are biholomorphically equivalent. For $n = 1$, the celebrated Riemann mapping theorem states that any simply connected domains in $\mathbb{C}$ are biholomorphically equivalent. For $n \geq 2$, there are many domains which are topologically equivalent to the ball but not biholomorphically equivalent to the ball [36]. Poincaré studied the invariance properties of the *CR* manifolds, which are real hypersurfaces in $\mathbb{C}^n$, with respect to biholomorphic transformations. The systematic study of such properties for real hypersurface was made by Cartan [7] and later by Chern and Moser [8]. A main result of the theory is the existence of a complete system of local differential invariants for *CR*-structures on real hypersurface. In 1974, Fefferman [13] proved that a biholomorphic mapping between two strongly pseudoconvex domains is smooth up to the boundaries and the induced boundary mapping is a *CR*-equivalence on the boundary. Thus, one can use Chern-Moser invariants to study the biholomorphically equivalent problem of two strongly pseudoconvex domains. Using the Chern-Moser theory, Webster [44] gave a complete characterization when two ellipsoids in $\mathbb{C}^n$ are biholomorphically equivalent. In 1978, Burns Shnider and Wells [6] showed that the number of moduli of a moduli space of a strongly pseudoconvex bounded domain has to be infinite. Thus the moduli problem of open manifolds is really a very difficult one.

Lempert [27] made significant progress in the subject. He was able to construct the moduli space of bounded strictly convex domains of $\mathbb{C}^n$ with marking at the origin. Although the theory established by Lempert is beautiful, the computation of his invariants is a hard problem.

In [10], Du and Yau studied the moduli problem of complete Reinhardt domains in $\mathbb{C}^2$. The main tool to solve this moduli problem with geometry information is the new biholomorphic invariant Bergman function defined by Yau [50]. In fact Yau's Bergman function theory can also solve the biholomorphic equivalence problem or moduli problem for complete Reinhardt pseudoconvex domains in $\mathbb{C}^n$ for all $n \geq 2$. In order to describe the complete biholomorphic invariants of bounded complete Reinhardt domains in $\mathbb{C}^n$, we introduce the following notations. Let S_n be the symmetric group of degree n. Recall that group ring $\mathbb{R}[S_n]$ is a ring of the form $\mathbb{R}[\tau_1, \tau_2, \dots, \tau_{n!}]$ with $\tau_i \in S_n$ for $1 \leqslant i \leqslant n!$. Let $\sum_i x_i\tau_i$ and $\sum_j y_j\tau_j$, where x_i, y_j are in $\mathbb{R}$, be two elements in $\mathbb{R}[S_n]$. Then

$$(\sum_i x_i\tau_i)(\sum_j y_j\tau_j) := \sum_{i,j} x_i y_j(\tau_i \cdot \tau_j),$$

where $\tau_i \cdot \tau_j$ is the product in the group S_n. We shall consider $\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n]$ the product of the group ring with itself. Such a product has a natural S_n-module structure in the following manner. Let $\sigma \in S_n$ and $(\sum_i x_i\tau_i, \cdots, \sum_i y_i\tau_i) \in (\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n])$. Then

$$\sigma(\sum_i x_i\tau_i, \cdots, \sum_i y_i\tau_i) = (\sum_i x_i(\tau_i\sigma), \cdots, \sum_i y_i(\tau_i\sigma)).$$

Definition 3.6 Two elements f, g in $\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n]$ are said to be equivalent and denoted by $f \sim g$ if there exists a $\sigma \in S_n$ such that $\sigma(f) = g$.

Let $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$ be an n-tuple of nonnegative integers. Denote $\phi_{\vec{\alpha}} = \left(\prod_{\vec{\alpha}=1}^n z_i^{\alpha_i}\right) dz_1 \wedge dz_2 \wedge \cdots \wedge dz_n$. For a domain D in $\mathbb{C}^n$, we shall use notation $\|\phi_{\vec{\alpha}}\|_D^2 := \int_D \phi_{\vec{\alpha}} \wedge \overline{\phi}_{\vec{\alpha}}$. In [9], the authors showed that all biholomorphic invariants of a bounded complete Reinhardt domains are contained in $(\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n])/\sim$ where there are $n!$ copies of $\mathbb{R}[S_n]$ and $\sim$ is the equivalence relation defined in Definition 3.6.

Theorem 3.13 ([9]) *Let D be a bounded complete Reinhardt domain in $\mathbb{C}^n$. Let $\vec{\alpha} = (\alpha_1, \dots, \alpha_n)$ be a n-tuple of nonnegative integers and $\tau \in S_n$. Denote*

$$g_D^\tau(\vec{\alpha}) = \frac{\|\phi_{\vec{0}}\|_D^{\Sigma\alpha_i - 1} \|\phi_{\tau(\vec{\alpha})}\|_D}{\prod_{i=1}^n \|\phi_{\vec{e}_i}\|_D^{\alpha_{\tau(i)}}}$$

where $\tau(\vec{\alpha}) = (\alpha_{\tau(1)}, \ldots, \alpha_{\tau(n)})$ *and* $\vec{e}_i = (0, \ldots, 0, 1, 0, \ldots, 0)$ *with 1 in the ith component. Then for all n-tuple of nonnegative integers*

$$\vec{\beta}_1, \cdots, \vec{\beta}_{n!},\ \xi_D^{(\vec{\beta}_1, \cdots, \vec{\beta}_{n!})} = (\sum_{\tau \in S_n} g_D^\tau(\vec{\beta}_1)\tau, \cdots, \sum_{\tau \in S_n} g_D^\tau(\vec{\beta}_{n!})\tau)$$

as an element in $(\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n])/\sim$ *is a biholomorphic invariant. In fact, if* D_1 *and* D_2 *are two such domains which are biholomorphically equivalent, then there exists a* $\sigma \in S_n$ *such that*

$$g_D^\tau(\vec{\alpha}) = g_{D_2}^{\tau \cdot \sigma}(\vec{\alpha}) \qquad \forall \tau \in S_n \text{and } \forall\, \vec{\alpha} \text{ n-tuple of nonnegative integers.}$$

The invariants in Theorem 3.13 are complete invariants for bounded complete Reinhardt pseudoconvex domains with C^1 boundaries.

Theorem 3.14 ([9]) *Let* D_i, $i = 1, 2$, *be two bounded complete Reinhardt pseudoconvex domains in* $\mathbb{C}^n$ *with* C^1 *boundaries. If for all* $\vec{\alpha_1}, \cdots, \vec{\alpha_{n!}}$ *n-tuples of non-negative integers,* $\xi_{D_1}^{(\vec{\alpha_1}, \cdots, \vec{\alpha_{n!}})} = \xi_{D_2}^{(\vec{\alpha_1}, \cdots, \vec{\alpha_{n!}})}$ *in* $(\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n])/\sim$, *where*

$$\xi_D^{(\vec{\alpha_1}, \cdots, \vec{\alpha_{n!}})} = (\sum_{\tau \in S_n} g_D^\tau(\vec{\alpha_1})\tau, \cdots, \sum_{\tau \in S_n} g_D^\tau(\vec{\alpha_{n!}})\tau),$$

then there exists $\sigma \in S_n$ *and a biholomorphic map*

$$\Psi_\sigma(z_1, \ldots, z_n) = (a_1 z_{\sigma(1)}, \ldots, a_n z_{\sigma(n)}),$$

where $a_i = \frac{\|\phi_{\vec{0}}\|_{D_1} \|\phi_{\vec{e_i}}\|_{D_2}}{\|\phi_{\vec{e}_{\sigma(i)}}\|_{D_1} \|\phi_{\vec{0}}\|_{D_2}}$, *such that* Ψ_σ *sends* D_1 *onto* D_2.

Theorems 3.13 and 3.14 above give a complete characterization of two bounded complete Reinhardt domains with real analytic boundaries in $\mathbb{C}^n$ to be biholomorphically equivalent in terms of the group ring $(\mathbb{R}[S_n] \times \cdots \times \mathbb{R}[S_n])/\sim$. In case $n = 2$, we can actually write down the complete numerical invariants for two bounded complete Reinhardt domains with real analytic boundaries in $\mathbb{C}^2$ to be biholomorphically equivalent.

Theorem 3.15 ([9]) *Let* D_1, D_2 *be two bounded complete Reinhardt pseudoconvex domains in* $\mathbb{C}^2$ *with* C^1 *boundaries. Then* D_1 *is biholomorphic to* D_2 *if and only if*

(1) $g_{D_1}(\alpha_1, \alpha_2) + g_{D_1}(\alpha_2, \alpha_1) = g_{D_2}(\alpha_1, \alpha_2) + g_{D_2}(\alpha_2, \alpha_1)$

(2) $g_{D_1}(\alpha_1, \alpha_2) g_{D_1}(\alpha_2, \alpha_1) = g_{D_2}(\alpha_1, \alpha_2) g_{D_2}(\alpha_2, \alpha_1)$

(3) $(g_{D_1}(\alpha_1, \alpha_2) - g_{D_1}(\alpha_2, \alpha_1))(g_{D_1}(\beta_1, \beta_2) - g_{D_1}(\beta_2, \beta_1))$
$= (g_{D_2}(\alpha_1, \alpha_2) - g_{D_2}(\alpha_2, \alpha_1))(g_{D_2}(\beta_1, \beta_2) - g_{D_2}(\beta_2, \beta_1))$

for all non-negative integers α_i, β_i, *where*

$$g_{D_i}(\alpha_1, \alpha_2) = \frac{\|\phi_{\vec{0}}\|_{D_i}^{\alpha_1+\alpha_2-1} \|\phi_{(\alpha_1,\alpha_2)}\|_{D_i}}{\prod_{j=1}^{2} \|\phi_{\vec{e}_j}\|_{D_i}^{\alpha_j}}$$

Corollary 3.2 ([9]) *The moduli space of bounded complete Reinhardt domains with* C^1 *boundaries in* $\mathbb{C}^2$ *can be constructed explicitly as the image of the complete family of numerical invariants:* $g_D(\alpha_1, \alpha_2) + g_D(\alpha_2, \alpha_1)$, $g_D(\alpha_1, \alpha_2)g_D(\alpha_2, \alpha_1)$ *and*

$$(g_D(\alpha_1, \alpha_2) - g_D(\alpha_2, \alpha_1))(g_D(\beta_1, \beta_2) - g_D(\beta_2, \beta_1))$$

$\forall\ \alpha_i, \beta_i$ *non-negative integers.*

In order to find the complete numerical biholomorphic invariants of bounded complete Reinhardt domain in $\mathbb{C}^n$ for $n \geqslant 3$, we need to consider the finite symmetric group $S_n = \{\sigma_1, \sigma_2, \ldots, \sigma_{n!}\}$ of degree n acting on the affine space $\mathbb{C}^{n!n!} = \mathbb{C}^{n!} \times \cdots \times \mathbb{C}^{n!}$, which is the product of $n!$ copies of $\mathbb{C}^{n!}$, in the following manner. Let $\tau \in S_n$ and

$$(x_{\sigma_1}, \ldots, x_{\sigma_{n!}}; \cdots ; y_{\sigma_1}, \ldots, y_{\sigma_{n!}}) \in \mathbb{C}^{n!} \times \cdots \times \mathbb{C}^{n!} = \mathbb{C}^{n!n!}.$$

Then

$$\tau \cdot (x_{\sigma_1}, \ldots, x_{\sigma_{n!}}; \cdots ; y_{\sigma_1}, \ldots, y_{\sigma_{n!}}) = (x_{\sigma_1\tau}, \ldots, x_{\sigma_{n!}\tau}; \cdots ; y_{\sigma_1\tau}, \ldots, y_{\sigma_{n!}\tau}).$$

Since S_n is linearly reductive, by Hilbert Theorem, the ring of invariants

$$\mathbb{C}[x_{\sigma_1}, \ldots, x_{\sigma_{n!}}; \cdots ; y_{\sigma_1}, \ldots, y_{\sigma_{n!}}]^{S_n}$$

is finitely generated. Moreover, the generators can be listed explicitly by Göbel's theorem [15]. Before we give the statement of Göbel's theorem, we shall introduce some definitions first.

Definition 3.7 Suppose that a finite group G acts as permutations on a finite set X. We then refer to X together with the G-action as a finite G-set. A subset $B \subset X$ is called an orbit if G permutes the elements of B among themselves and the induced permutation action of G on B is transitive.

Definition 3.8 If $K = (k_1, \cdots, k_n)$ is an n-tuple of non-negative integers, then K is called an exponent sequence. The associated partition of K is the ordered set consisting of the n numbers $k_1, \cdots, k_n$ rearranged in weakly decreasing order. We denote by $\lambda(K)$ the partition associated to K, so

$$\lambda(K) = (\lambda_1(K) \geq \lambda_2(K) \geq \cdots \geq \lambda_n(K))$$

and the n-tuple $(\lambda_1(K), \lambda_2(K), \cdots, \lambda_n(K))$ is a permutation of $k_1, \cdots, k_n$. The monomial x^K is called special if the associated partition $\lambda(K)$ of the exponent sequence K satisfies

(1) $\lambda_i(K) - \lambda_{i+1}(K) \leq 1$ for all $i = 1, \cdots, n-1$ and

(2) $\lambda_n(K) = 0$.

Notice that if two exponent sequences A and B are permutations of each other, then $\lambda(A) = \lambda(B)$.

Theorem 3.16 *Let G be a finite group, X a finite G-set, and R a commutative ring. Then the ring of invariants $R[X]^G$ is generated as an algebra by $e_{|X|} = \prod_{x \in X} x$, the top degree elementary symmetric polynomial in the elements of X, and the orbit sums of special monomials.*

Theorem 3.17 ([9]) *Let $f_1, \ldots, f_N \in \mathbb{C}[x_{\sigma_1}, \ldots, x_{\sigma_{n!}}; \ldots; y_{\sigma_1}, \ldots, y_{\sigma_{n!}}]^{S_n}$ be the generators of the ring of invariant polynomials computed by Theorem 3.16. Let D be a bounded complete Reinhardt domain in $\mathbb{C}^n$. Then, for $\vec{\alpha}_1, \vec{\alpha}_2, \ldots, \vec{\alpha}_{n!}$ n-tuples of non-negative integers,*

$$f_1(g_D^\sigma(\vec{\alpha}_1), \ldots, g_D^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n}, \ldots, f_N(g_D^\sigma(\vec{\alpha}_1), \ldots, g_D^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n}$$

are biholomorphic invariants, where

$$g_D^\sigma(\vec{\beta}) = \frac{\|\phi_{\vec{0}}\|_D^{\sum \beta_i - 1} \|\phi_{\sigma(\vec{\beta})}\|_D}{\prod_{i=1}^n \|\phi_{\vec{e}_i}\|_D^{\sigma(i)}}, \quad \vec{\beta} = (\beta_1, \beta_2, \ldots, \beta_n)$$

The following theorem says that the above invariants are actually complete in case the domain D is pseudoconvex.

Theorem 3.18 ([9]) *Let D_i, $i = 1, 2$, be two bounded complete Reinhardt pseudoconvex domains in $\mathbb{C}^n$ with C^1 boundaries. Let $f_1, \ldots, f_N \in \mathbb{C}[x_{\sigma_1}, \ldots, x_{\sigma_{n!}}; \ldots; y_{\sigma_i}, \ldots, y_{\sigma_{n!}}]^{S_n}$ be the generators of the ring of invariant polynomials computed by Theorem 3.16. If for all $\vec{\alpha}_1, \ldots, \vec{\alpha}_{n!}$ n-tuples of non-negative integers*

$$f_i(g_{D_1}^\sigma(\vec{\alpha}_1), \ldots, g_{D_1}^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n} = f_i(g_{D_2}^\sigma(\vec{\alpha}_1), \ldots, g_{D_2}^\sigma(\vec{\alpha}_{n!}))_{\sigma \in S_n}, \quad i = 1, 2, \ldots, N,$$

then there exists $\tau \in S_n$ and a biholomorphic map

$$\Psi_\tau : \mathbb{C}^n \to \mathbb{C}^n, \Psi_\tau(z_1, \ldots, z_n) = (a_1 z_{\tau(1)}, \ldots, a_n z_{\tau(n)}),$$

where

$$a_i = \frac{\|\phi_{\vec{o}}\|_{D_1}\|\phi_{\vec{e_i}}\|_{D_2}}{\|\phi_{e_{\sigma(i)}}\|_{D_1}\|\phi_{\vec{o}}\|_{D_2}},$$

such that Ψ_τ *sends* D_1 *onto* D_2.

Corollary 3.3 ([9]) *The moduli space of bounded complete Reinhardt pseudoconvex domains with* C^1 *boundaries in* $\mathbb{C}^n$ *can be constructed explicitly as the image of the complete family of numerical invariants* $f_i(g_D^\sigma(\vec{\alpha_1}), \ldots, g_D^\sigma(\vec{\alpha_{n!}}))_{\sigma\in S_n}$, $1 \leqslant i \leqslant N$, *where* $\vec{\alpha_1}, \ldots, \vec{\alpha_{n!}}$ *are all possible n-tuples of nonnegative integers.*

Remark 3.1 One can compute explicitly the relation of the generators

$$f_1, \ldots, f_N \in \mathbb{C}[x_{\sigma_1}, \ldots, x_{\sigma_{n!}}; \ldots; y_{\sigma_i}, \ldots, y_{\sigma_{n!}}]^{S_n}.$$

These relations define an algebraic variety in R^∞ where the moduli space lies.

For complete Reinhardt pseudoconvex domains with real analytic boundaries, we can use fewer numerical invariants to characterize these domains. More precisely, we have the following theorems.

Theorem 3.19 ([9]) *Let* D_i, $i = 1, 2$, *be two bounded complete Reinhardt pseudoconvex domains in* $\mathbb{C}^n$ *with real analytic boundaries. Then* D_1 *is biholomorphically equivalent to* D_2 *If and only if for all* $\vec{\alpha}$ *n-tuple of non-negative integers,* $\xi_{D_1}^{\vec{\alpha}} = \xi_{D_2}^{\vec{\alpha}}$ *in* $\mathbb{R}[S_n]/\sim$ *where* $\xi_{D_i}^{\vec{\alpha}} = \sum_{\tau\in S_n} g_{D_i}^\tau(\vec{\alpha})\tau$. *In this case, there exists* $\sigma \in S_n$ *and a biholomorphic map*

$$\Psi_\sigma(z_1, \ldots, z_n) = (a_1 z_{\sigma(1)}, \ldots, a_n z_{\sigma(n)}),$$

where $a_i = \frac{\|\phi_{\vec{0}}\|_{D_1}\|\phi_{\vec{e_i}}\|_{D_2}}{\|\phi_{\vec{e}_{\sigma(i)}}\|_{D_1}\|\phi_{\vec{0}}\|_{D_2}}$, *such that* Ψ_σ *sends* D_1 *onto* D_2.

Theorem 3.20 ([9]) *Let* D_1, D_2 *be two bounded complete Reinhardt pseudoconvex domains in* $\mathbb{C}^2$ *with real analytic boundaries. Then* D_1 *is biholomorphic to* D_2 *if and only if*

$$g_{D_1}(\alpha_1, \alpha_2) + g_{D_1}(\alpha_2, \alpha_1) = g_{D_2}(\alpha_1, \alpha_2) + g_{D_2}(\alpha_2, \alpha_1)$$
$$g_{D_1}(\alpha_1, \alpha_2) g_{D_1}(\alpha_2, \alpha_1) = g_{D_2}(\alpha_1, \alpha_2) g_{D_2}(\alpha_2, \alpha_1)$$

for all non-negative integers α_1, α_2, *where*

$$g_{D_i}(\alpha_1, \alpha_2) = \frac{\|\phi_{\vec{0}}\|_{D_i}^{\alpha_1+\alpha_2-1}\|\phi_{(\alpha_1,\alpha_2)}\|_{D_i}}{\prod_{j=1}^{2}\|\phi_{\vec{e_j}}\|_{D_i}^{\alpha_j}}$$

Theorem 3.21 ([9]) *Let D_i, $i = 1, 2$, be two bounded complete Reinhardt pseudoconvex domains in $\mathbb{C}^n$ with real analytic boundaries. Let*

$$f_1, \ldots, f_N \in \mathbb{C}[x_{\sigma_1}, \ldots, x_{\sigma_{n!}}]^{S_n}$$

be the generators of the ring of invariant polynomials computed by Theorem 3.16. Then D_1 is biholomorphically equivalent to D_2 if and only if for all $\vec{\alpha}$ n-tuples of nonnegative integers

$$f_i(g^{\sigma}_{D_1}(\vec{\alpha}))_{\sigma \in S_n} = f_i(g^{\sigma}_{D_2}(\vec{\alpha}))_{\sigma \in S_n}, \quad i = 1, \ldots, N.$$

In this case, there exists $\tau \in S_n$ and a biholomorphic map $\Psi_\tau: \mathbb{C}^n \to \mathbb{C}^n$, $\Psi_\tau(z_1, \ldots, z_n) = (a_1 z_{\tau(1)}, \ldots, a_n z_{\tau(n)})$, where

$$a_i = \frac{\|\phi_{\vec{0}}\|_{D_1} \|\phi_{\vec{e_i}}\|_{D_2}}{\|\phi_{e_{\sigma(i)}}\|_{D_1} \|\phi_{\vec{0}}\|_{D_2}}$$

such that Ψ_τ sends D_1 onto D_2.

4 Complex Plateau Problem

Let X be a compact connected *CR* manifold of dimension $2n-1$ in $\mathbb{C}^N$. The famous complex Plateau problem asks under what conditions on X, X will be a boundary of a complex submanifold in $\mathbb{C}^N$. By a theorem of Harvey and Lawson [16], X is a boundary of a unique complex variety V in $\mathbb{C}^N$. Therefore we need to understand under what conditions on X, V will be a complex submanifold.

In 1963, J.J. Kohn solved the famous $\overline{\partial}$-Neumann problem. Based on this work, Kohn and Rossi [22] in 1965 introduced the fundamental *CR* invariants, the Kohn-Rossi cohomology groups $H^{p,q}_{KR}(X)$. They proved the finite dimensionality of their cohomology groups for $1 \leq q \leq n-2$ if X is strongly pseudoconvex. Following Tanaka [42], we shall recall the definition of Kohn-Rossi cohomology groups as follows.

Let $\{\mathcal{A}^k(X), d\}$ be the De-Rham complex of X with complex coefficients, and let $H^k(X)$ be the De-Rham cohomology groups. There is a natural filtration of the De-Rham complex as follows. For any integer p and k, put $A^k(X) = \Lambda^k(\mathbb{C}T(X)^*)$ and denoted by $F^p(A^k(X))$ the subbundle of $A^k(X)$ consisting of all $\phi \in A^k(X)$ which satisfy the equality

$$\phi(Y_1, \ldots, Y_{p-1}, \overline{Z}_1, \ldots, \overline{Z}_{k-p+1}) = 0$$

for all $Y_1, \ldots, Y_{p-1} \in \mathbb{C}T(X)_x$ and $Z_1, \ldots, Z_{k-p+1} \in S_x, x$ being the origin of ϕ. Then

$$A^k(X) = F^0(A^k(X)) \supseteq F^1(A^k(X)) \supseteq \cdots \supseteq F^k(A^k(X)) \supseteq F^{k+1}(A^k(X)) = 0$$

setting $F^p(\mathcal{A}^k(X)) = \Gamma(F^p(A^k(X)))$, we have

$$\mathcal{A}^k(X) = F^0(\mathcal{A}^k(X)) \supseteq F^1(\mathcal{A}^k(X)) \supseteq \cdots \supseteq F^k(\mathcal{A}^k(X)) \supseteq F^{k+1}(\mathcal{A}^k(X)) = 0.$$

Since clearly $dF^p(\mathcal{A}^k(X)) \subseteq F^p(\mathcal{A}^{k+1}(X))$, the collection $\{F^p(\mathcal{A}^k(X))\}$ gives a filtration of the De-Rham complex.

Definition 4.1 $H^{p,q}_{KR}(X)$, the Kohn-Rossi cohomology group of type (p, q), is defined to be the group $E_1^{p,q}(X)$ of the spectral sequence $\{E_r^{p,q}(X)\}$ associated with the filtration $\{F^p(\mathcal{A}^k(X))\}$.

More explicitly, let

$$A^{p,q}(X) = F^p(A^{p+q}(X)), \qquad \mathcal{A}^{p,q}(X) = \Gamma(A^{p,q}(X))$$
$$C^{p,q}(X) = A^{p,q}(X)\big/A^{p+1,q-1}(X), \; \mathcal{C}^{p,q}(X) = \Gamma(C^{p,q}(X)).$$

Since $d : \mathcal{A}^{p,q}(X) \to \mathcal{A}^{p,q+1}(x)$ maps $\mathcal{A}^{p+1,q-1}(X)$ into $\mathcal{A}^{p+1,q}(X)$, it induces an operator $\overline{\partial}_b : \mathcal{C}^{p,q}(X) \to \mathcal{C}^{p,q+1}(X)$. $H^{p,q}_{KR}(X)$ are then the cohomology groups of the complex $\{\mathcal{C}^{p,q}(X), \overline{\partial}_b\}$.

Definition 4.2 $H^k_h(X)$, the holomorphic De-Rham cohomology group of degree k, is defined to be the group $E_2^{k,0}(X)$ of the spectral sequence $\{E_r^{p,q}(X)\}$ associated with the filtration $\{F^p(\mathcal{A}^k(X))\}$.

More explicitly, recall $E_0^{p,q}(X) = \mathcal{C}^{p,q}(X)$ and $d_0 : \mathcal{C}^{p,q}(X) \longrightarrow \mathcal{C}^{p,q+1}(X)$ is the map $\overline{\partial}_b$ above. Note that $E_0^{k,0}(X) = \mathcal{C}^{k,0}(X) = \mathcal{A}^{k,0}(X) \subseteq \mathcal{A}^k(X)$. Next,

$$E_1^{p,q}(X) = \frac{\mathrm{Ker}(d_0 : \mathcal{C}^{p,q}(X) \longrightarrow \mathcal{C}^{p,q+1}(X))}{\mathrm{Im}(d_0 : \mathcal{C}^{p,q-1}(X) \longrightarrow \mathcal{C}^{p,q}(X))}$$

and $d_1 : E_1^{p,q}(X) \longrightarrow E_1^{p+1,q}(X)$ is the naturally induced map. In particular,

$$\begin{aligned} E_1^{k,0}(X) &= \ker(d_0 : \mathcal{C}^{k,0}(X) \longrightarrow \mathcal{C}^{k,1}(X)) \\ &= \{\phi \in \mathcal{A}^{k,0}(X) : d\phi \in \mathcal{A}^{k+1,0}(X)\} \end{aligned}$$

and d_1 is just d on $E_1^{k,0}(X) \subseteq \mathcal{A}^k(X)$. $E_1^{k,0}(X)$ is called the space of holomorphic k-forms on X. Denoting $E_1^{k,0}(X)$ by $\mathcal{S}^k(X)$, we have the holomorphic De Rham complex $\{\mathcal{S}^k(X), d\}$. Then

$$E_2^{k,0}(X) = \frac{\text{Ker}(d : \mathcal{S}^k(X) \longrightarrow \mathcal{S}^{k+1}(X))}{\text{Im}(d : \mathcal{S}^{k-1}(X) \longrightarrow \mathcal{S}^k(x))}$$
$$= \frac{\{\text{closed holomorphic } k\text{-forms on } X\}}{\{\text{exact holomorphic } k\text{-forms on } X\}},$$

is the holomorphic De Rham cohomology $H_h^k(X)$.

A strongly pseudoconvex complex manifold M is a modification of a Stein space V with isolated singularities. In 1965, Kohn and Rossi [22] conjectured that in general, either there is no boundary cohomology of the boundary $X = \partial V$ in degree (p, q) for $q \neq 0, n-1$, or it must result from the interior singularities of V. Yau [46] solved the Kohn-Rossi conjecture affirmatively in 1981.

Theorem 4.1 (Yau [46]) *Let X be a compact strongly pseudoconvex CR manifold of dimension* $2n - 1$*,* $n \geq 3$*, which is the boundary of a Stein space V with isolated singularities* $x_1, \ldots, x_m$*. Then for* $1 \leq q \leq n-2$*,*

$$H_{KR}^{p,q}(X) \simeq \bigoplus_{i=1}^{m} H_{\{x_i\}}^{q+1}(V, \Omega_V^p),$$

where Ω_V^p *is the sheaf of germs of holomorphic p-forms on V. If* $x_1, \ldots, x_m$ *are hypersurface singularities, then*

$$\dim H_{KR}^{p,q}(X) = \begin{cases} 0 & p+q \leq n-2, \quad 1 \leq q \leq n-2 \\ \tau_1 + \cdots + \tau_m & p+q = n-1, n, \ 1 \leq q \leq n-2 \\ 0 & p+q = n+1, \quad 1 \leq q \leq n-2 \end{cases}$$

where τ_i *is the number of moduli of V at* x_i*.*

Remark 4.1 Let $f : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0)$ be a holomorphic function. Suppose that $V = \{z \in \mathbb{C}^{n+1} : f(z) = 0\}$ has isolated singularity of the origin. Then the local moduli of V is the dimension of the parameter space of the semi universal deformation space of $(V, 0)$. This number is $\tau = \dim \mathbb{C}\{z_0, \ldots, z_n\} \big/ (f, f_{z_0}, \ldots, f_{z_n})$.

As a result of the above theorem, Yau answers the classical complex Plateau problem for real codimension 3 *CR* in $\mathbb{C}^{n+1}$ satisfactory.

Theorem 4.2 (Yau [46]) *Let X be a compact connected strongly pseudoconvex CR-manifold of real dimension* $2n - 1, n \geq 3$*, in the boundary of a bounded strongly pseudoconvex domain D in* $\mathbb{C}^{n+1}$*. Then X is a boundary of the complex submanifold* $V \subseteq D - X$ *if and only if Kohn-Rossi cohomology groups* $H_{KR}^{p,q}(X)$ *are zero for* $1 \leq q \leq n-2$*.*

For $n = 2$ in Theorem 4.2, X is a 3-dimensional *CR* manifold. The classical complex Plateau problem remains unsolved for over a quarter of a century. The main difficulty is that the Kohn-Rossi cohomology groups are infinite dimensional in this case. Let V be the complex variety with X as its boundary. Then the singularities of V are surface singularities. In order to solve the classical complex Plateau problem for $n = 2$, one would like to ask under what kind of condition on X, V will have only very mild singularities. Our basic observation is the following. Although Kohn-Rossi cohomology groups are infinite dimensional, we can derive from them the holomorphic De Rham cohomology. Let M be a complex manifold. The k-th holomorphic De Rham cohomology $H_h^k(M)$ of M is defined to be the d-closed holomorphic k-forms quotient by the d-exact holomorphic k-forms. It is well known that if M is a Stein manifold, then the holomorphic De Rham cohomology coincides with the ordinary De Rham cohomology.

Definition 4.3 Let (V, x) be an isolated singularity of dimension n. Let $\pi : (M, A) \longrightarrow (V, x)$ be a resolution of singularity with A as exceptional set. Let

$$s = \dim \Gamma(M - A, \Omega^n) \big/ [d\Gamma(M - A, \Omega^{n-1}) + \Gamma(M, \Omega^n)].$$

s is an invariant of the singularity (V, x). It turns out that the s-invariant plays an important role in the relationship between $H_h^n(M - A)$ and $H_h^n(M)$.

Theorem 4.3 (Luk-Yau [31]) *Let X be a compact connected* $(2n - 1)$*-dimensional* $(n \geq 2)$ *strongly pseudoconvex CR manifold. Suppose that X is the boundary of a n-dimensional strongly pseudoconvex complex manifold M which is a modification of a Stein space V with only isolated singularities* $\{x_1, \ldots, x_m\}$*. Let A be the maximal compact analytic set in M which can be blown down to* $\{x_1, \ldots, x_m\}$*. Then*

(1) $H_h^q(X) \cong H_h^q(M - A) \cong H_h^q(M)$ *for* $1 \leq q \leq n - 1$.
(2) $H_h^n(X) \cong H_h^n(M - A)$, $\dim H_h^n(M - A) = \dim H_h^n(M) + s$

where $s = s_1 + \cdots + s_m$ *and* s_i *is the s-invariant of the singularity* (V, x_i)*.*

Remark 4.2 The above theorem in particular asserts that up to degree $n - 1$, the holomorphic De Rham cohomology can extend across the maximal compact analytic set.

Definition 4.4 A normal surface singularity $(V, 0)$ is Gorenstein if these exists a nowhere vanishing holomorphic 2-form on $V - \{0\}$.

Recall that isolated hypersurface or complete intersection singularities are Gorenstein. It is a natural question to ask for a characterization of Gorenstein surface singularities with vanishing s-invariant.

Theorem 4.4 (Luk-Yau [31]) *Let* $(V, 0)$ *be a Gorenstein surface singularity. Let* $\pi : M \longrightarrow V$ *be a good resolution with* $A = \pi^{-1}(0)$ *as exceptional set. Assume that M is contractible to A. If* $s = 0$*, then* $(V, 0)$ *is a quasi-homogeneous singularity,* $H^1(A, \mathbb{C}) = 0$*,* $\dim H^1(M, \Omega^1) = \dim H^2(A, \mathbb{C}) + \dim H^1(M, \mathcal{O})$*, and* $H_h^1(M) = H_h^2(M) = 0$*. Conversely, if* $(V, 0)$ *is a two dimensional quasi-homogeneous Gorenstein singularity and* $H^1(A, \mathbb{C}) = 0$*, then the s-invariant vanishes.*

Let X be a compact CR manifold with CR-structure S. For any C^∞ functions u, there is a section $\overline{\partial}_b u \in \Gamma(\overline{S}^*)$ defined by $(\overline{\partial}_b u)(\overline{L}) = \overline{L}u$ for any $L \in \Gamma(S)$. This can be generalized as follows:

Definition 4.5 A complex vector bundle E over X is said to be holomorphic if there is a differential operator $\overline{\partial}_E : \Gamma(E) \longrightarrow \Gamma(E \otimes \overline{S}^*)$ such that if $\overline{L}u$ denotes $(\overline{\partial}_E u)(\overline{L})$ for $u \in \Gamma(E)$ and $L \in \Gamma(S)$, then for any $L_1, L_2 \in \Gamma(S)$ and any C^∞ function f on X:

(1) $\overline{L}(fu) = (\overline{L}f)u + f(\overline{L}u)$
(2) $[\overline{L}_1, \overline{L}_2]u = \overline{L}_1\overline{L}_2 u - \overline{L}_2\overline{L}_1 u$.

A solution u of the equation $\overline{\partial}_E u = 0$ is called a holomorphic section.

The vector bundle $\hat{T}(X) = \mathbb{C}T(X)\big/\overline{S}$ is holomorphic with respect to the following $\overline{\partial} = \overline{\partial}_{\hat{T}(X)}$. Let ω be the projection from $\mathbb{C}T(X)$ to $\hat{T}(X)$. Take any $u \in \Gamma(\hat{T}(X))$ and express it as $u = \omega(Z)$, $Z \in \Gamma(\mathbb{C}T(X))$. For any $L \in \Gamma(S)$, define $(\overline{\partial}u)(\overline{L}) = \omega([\overline{L}, Z])$. The section $(\overline{\partial}u)(\overline{L})$ of $\hat{T}(X)$ does not depend on the choice of Z and $\overline{\partial}u$ gives a section of $\hat{T}(X) \otimes \overline{S}^*$. Further the operator $\overline{\partial}$ satisfies the conditions in Definition 4.5. The resulting holomorphic vector bundle $\hat{T}(X)$ is called the holomorphic tangent bundle of X.

Lemma 4.1 *If X is a real hypersurface in a complex manifold M, then the holomorphic tangent bundle $\hat{T}(X)$ is naturally isomorphic to the restriction of X of the bundle $T^{1,0}(M)$ of all $(1,0)$ tangent vectors to M.*

Definition 4.6 Let X be a compact CR manifold of real dimension $2n-1$. X is said to be a Calabi-Yau CR manifold if there exists a nowhere vanishing holomorphic section in $\Gamma(\Lambda^n \hat{T}(X)^*)$ where $\hat{T}(X) = \mathbb{C}T(X)\big/\overline{S}$ is the holomorphic tangent bundle of X.

Remark 4.3 (a) Let X be a compact CR manifold of real dimension $2n-1$ in $\mathbb{C}^n$. Then X is a Calabi-Yau CR manifold. (b) let X be a strongly pseudoconvex CR manifold of real dimension $2n-1$ contained in the boundary of a bounded strongly pseudoconvex domain in $\mathbb{C}^{n+1}$. Then X is a Calabi-Yau manifold.

The following theorem is a fundamental theorem toward the complete solution of the classical complex Plateau problem for 3-dimensional strongly pseudoconvex Calabi-Yau CR manifold in $\mathbb{C}^n$. The theorem is interesting in its own right.

Theorem 4.5 (Luk-Yau [31]) *Let X be a strongly pseudoconvex compact Calabi-Yau CR manifold of dimension 3. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in $\mathbb{C}^n$. If the holomorphic De Rham cohomology $H_h^2(X) = 0$, then X is a boundary of a complex variety V in D with boundary regularity and V has only isolated singularities in the interior and the normalizations of these singularities are Gorenstein surface singularities with vanishing s-invariant.*

Corollary 4.1 (Luk-Yau [31]) *Let X be a strongly pseudoconvex compact CR manifold of dimension 3. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in* $\mathbb{C}^3$*. If the holomorphic De Rham cohomology* $H_h^2(X) = 0$*, then X is a boundary of a complex variety V in D with boundary regularity and V has only isolated quasi-homogeneous singularities such that the dual graphs of the exceptional sets in the resolution are star shaped and all the curves are rational.*

Before we proceed further, we need to introduce some invariants of singularities as well as *CR*-invariants.

Let V be a n-dimensional complex analytic subvariety in $\mathbb{C}^N$ with only isolated singularities. In [47], Yau considered four kinds of sheaves of germs of holomorphic p-forms

1. $\bar{\Omega}_V^p := \pi_* \Omega_M^p$, where $\pi : M \longrightarrow V$ is a resolution of singularities of V.
2. $\bar{\bar{\Omega}}_V^p := \theta_* \Omega^p_{V \backslash V_{sing}}$ where $\theta : V \backslash V_{sing} \longrightarrow V$ is the inclusion map and V_{sing} is the singular set of V.
3. $\Omega_V^p := \Omega^p_{\mathbb{C}^N} / \mathscr{K}^p$, where $\mathscr{K}^p = \{f\alpha + dg \wedge \beta : \alpha \in \Omega^p_{\mathbb{C}^N}; \beta \in \Omega^{p-1}_{\mathbb{C}^N}; f, g \in \mathscr{I}\}$ and $\mathscr{I}$ is the ideal sheaf of V in $\mathbb{C}^N$.
4. $\tilde{\Omega}_V^p := \Omega^p_{\mathbb{C}^N} / \mathscr{H}^p$, where $\mathscr{H}^p = \{\omega \in \Omega^p_{\mathbb{C}^N} : \omega|_{V \backslash V_{sing}} = 0\}$.

Clearly Ω_V^p, $\tilde{\Omega}_V^p$ are coherent. $\bar{\Omega}_V^p$ is a coherent sheaf because π is a proper map. $\bar{\bar{\Omega}}_V^p$ is also a coherent sheaf by a theorem of Siu (cf. Theorem A of [38]). If V is a normal variety, the dualizing sheaf ω_V of Grothendieck is actually the sheaf $\bar{\bar{\Omega}}_V^n$.

Definition 4.7 The Siu complex is a complex of coherent sheaves $J^\bullet$ supported on the singular points of V which is defined by the following exact sequence

$$0 \longrightarrow \bar{\Omega}^\bullet \longrightarrow \bar{\bar{\Omega}}^\bullet \longrightarrow J^\bullet \longrightarrow 0. \tag{1}$$

Definition 4.8 Let V be a n-dimensional Stein space with 0 as its only singular point. Let $\pi : (M, A) \to (V, 0)$ be a resolution of the singularity with A as exceptional set. The geometric genus p_g, the irregularity q and $g^{(p)}$ invariant of the singularity are defined as follows (cf. [39, 47]):

$$p_g := dim \Gamma(M \backslash A, \Omega^n) / \Gamma(M, \Omega^n), \tag{2}$$

$$q := dim \Gamma(M \backslash A, \Omega^{n-1}) / \Gamma(M, \Omega^{n-1}), \tag{3}$$

$$g^{(p)} := dim \Gamma(M, \Omega_M^p) / \pi^* \Gamma(V, \Omega_V^p). \tag{4}$$

And recall that the s-invariant of the singularity is defined (cf. Definition 4.3) as follows

$$s := dim \Gamma(M \backslash A, \Omega^n) / [\Gamma(M, \Omega^n) + d\Gamma(M \backslash A, \Omega^{n-1})]. \tag{5}$$

Lemma 4.2 ([31]) *Let V be a n-dimensional Stein space with* 0 *as its only singular point. Let* $\pi : (M, A) \to (V, 0)$ *be a resolution of the singularity with A as exceptional set. Let* $J^\bullet$ *be the Siu complex of coherent sheaves supported on* 0. *Then:*

1. $dim J^n = p_g$,
2. $dim J^{n-1} = q$,
3. $dim J^i = 0$, *for* $1 \le i \le n-2$.

Proposition 4.1 ([31]) *Let V be a n-dimensional Stein space with* 0 *as its only singular point. Let* $\pi : (M, A) \to (V, 0)$ *be a resolution of the singularity with A as exceptional set. Let* $J^\bullet$ *be the Siu complex of coherent sheaves supported on* 0. *Then the s-invariant is given by*

$$s = dim H^n(J^\bullet) = p_g - q \tag{6}$$

and

$$dim H^{n-1}(J^\bullet) = 0 \tag{7}$$

Let X be a compact connected strongly pseudoconvex *CR* manifold of real dimension 3, in the boundary of a bounded strongly pseudoconvex domain D in $\mathbb{C}^N$. By Harvey and Lawson [16], there is a unique complex variety V in $\mathbb{C}^N$ such that the boundary of V is X. Let $\pi : (M, A_1, \cdots, A_k) \to (V, 0_1, \cdots, 0_k)$ be a resolution of the singularities with $A_i = \pi^{-1}(0_i), 1 \le i \le k$, as exceptional sets. Then the s-invariant defined in Definition 4.8 is *CR* invariant, which is also called $s(X)$.

In order to solve the classical complex Plateau problem, we need to find some *CR*-invariant which can be calculated directly from the boundary X and the vanishing of this invariant will give the regularity of Harvey-Lawson solution to the complex Plateau problem. For this purpose, we define a new sheaf $\bar{\bar{\Omega}}_V^{1,1}$.

Definition 4.9 Let $(V, 0)$ be a Stein germ of a 2-dimensional analytic space with an isolated singularity at 0. Define a sheaf of germs $\bar{\bar{\Omega}}_V^{1,1}$ by the sheaf associated to the presheaf

$$U \mapsto < \Gamma(U, \bar{\bar{\Omega}}_V^1) \wedge \Gamma(U, \bar{\bar{\Omega}}_V^1) >,$$

where U is an open set of V.

Lemma 4.3 ([11]) *Let V be a* 2*-dimensional Stein space with* 0 *as its only singular point in* $\mathbb{C}^N$. *Let* $\pi : (M, A) \to (V, 0)$ *be a resolution of the singularity with A as exceptional set. Then* $\bar{\bar{\Omega}}_V^{1,1}$ *is coherent and there is a short exact sequence*

$$0 \longrightarrow \bar{\bar{\Omega}}_V^{1,1} \longrightarrow \bar{\bar{\Omega}}_V^2 \longrightarrow \mathscr{G}^{(1,1)} \longrightarrow 0 \tag{8}$$

where $\mathscr{G}^{(1,1)}$ is a sheaf supported on the singular point of V. Let

$$G^{(1,1)}(M\backslash A) := \Gamma(M\backslash A, \Omega_M^2)/ < \Gamma(M\backslash A, \Omega_M^1) \wedge \Gamma(M\backslash A, \Omega_M^1) >, \tag{9}$$

then $dim\mathscr{G}_0^{(1,1)} = dimG^{(1,1)}(M\backslash A)$.

Thus, from Lemma 4.3, we can define a local invariant of a singularity which is independent of resolution.

Definition 4.10 Let V be a 2-dimensional Stein space with 0 as its only singular point. Let $\pi : (M,A) \to (V,0)$ be a resolution of the singularity with A as exceptional set. Let

$$g^{(1,1)}(0) := dim\mathscr{G}_0^{(1,1)} = dimG^{(1,1)}(M\backslash A). \tag{10}$$

We will omit 0 in $g^{(1,1)}(0)$ if there is no confusion from the context.

Let $\pi : (M, A_1, \cdots, A_k) \to (V, 0_1, \cdots, 0_k)$ be a resolution of the singularities with $A_i = \pi^{-1}(0_i), 1 \leq i \leq k$, as exceptional sets, and $A = \cup_i A_i$. In this case, we still let

$$G^{(1,1)}(M\backslash A) := \Gamma(M\backslash A, \Omega_M^2)/ < \Gamma(M\backslash A, \Omega_M^1) \wedge \Gamma(M\backslash A, \Omega_M^1) > .$$

Definition 4.11 If X is a compact connected strongly pseudoconvex CR manifold of real dimension 3 which is the boundary of a bounded strongly pseudoconvex domain D in $\mathbb{D}^N$. Suppose V in $\mathbb{C}^N$ such that the boundary of V is X. Let $\pi : (M, A = \cup_i A_i,) \to (V, 0_1, \cdots, 0_k)$ be a resolution of the singularities with $A_i = \pi^{-1}(0_i)$, $1 \leq i \leq k$, as exceptional sets. Let

$$G^{(1,1)}(M\backslash A) := \Gamma(M\backslash A, \Omega_M^2)/ < \Gamma(M\backslash A, \Omega_M^1) \wedge \Gamma(M\backslash A, \Omega_M^1) > \tag{11}$$

and

$$G^{(1,1)}(X) := \mathscr{S}^2(X)/ < \mathscr{S}^1(X) \wedge \mathscr{S}^1(X) > \tag{12}$$

where $\mathscr{S}^p$ are holomorphic cross sections of $\wedge^p(\hat{T}(X)^*)$. Then we set

$$g^{(1,1)}(M \setminus A) := dimG^{(1,1)}(M \setminus A) \tag{13}$$

$$g^{(1,1)}(X) := dimG^{(1,1)}(X). \tag{14}$$

Lemma 4.4 ([11]) *Let X be a compact connected strongly pseudoconvex CR manifold of real dimension 3 which bounds a bounded strongly pseudoconvex variety V with only isolated singularities $\{0_1, \cdots, 0_k\}$ in $\mathbb{C}^N$. Let $\pi : (M, A_1, \cdots, A_k) \to (V, 0_1, \cdots, 0_k)$ be a resolution of the singularities with $A_i = \pi^{-1}(0_i)$, $1 \leq i \leq k$, as exceptional sets. Then $g^{(1,1)}(X) = g^{(1,1)}(M\backslash A)$, where $A = \cup A_i$, $1 \leq i \leq k$.*

By Lemma 4.4 and the proof of Lemma 4.3, we can get the following lemma easily.

Lemma 4.5 *Let X be a compact connected strongly pseudoconvex CR manifold of real dimension 3, which bounds a bounded strongly pseudoconvex variety V with only isolated singularities* $\{0_1, \cdots, 0_k\}$ *in* $\mathbb{C}^N$. *Then* $g^{(1,1)}(X) = \sum_i g^{(1,1)}(0_i) = \sum_i dim\mathscr{G}^{(1,1)}_{0_i}$.

The following proposition is to show that $g^{(1,1)}$ is bounded above.

Proposition 4.2 ([11]) *Let V be a 2-dimensional Stein space with 0 as its only singular point. Then* $g^{(1,1)} \leq p_g + g^{(2)}$.

The following theorem is the crucial part for the classical complex Plateau problem.

Theorem 4.6 ([11]) *Let V be a 2-dimensional Stein space with 0 as its only normal singular point with* $\mathbb{C}^*$*-action. Let* $\pi : (M, A) \to (V, 0)$ *be a minimal good resolution of the singularity with A as exceptional set, then* $g^{(1,1)} \geq 1$.

In the paper [31], Luk and Yau gave a sufficient condition $H^2_h(X) = 0$ to determine when X can bound some special singularities. However, even if both $H^2_h(X)$ and $H^1_h(X)$ vanish, V still can be singular.

The *CR* invariants in Definition 4.11 (formula 14) can be used to give sufficient and necessary conditions for the variety bounded by X being smooth after normalization.

Theorem 4.7 ([11]) *Let X be a strongly pseudoconvex compact Calabi-Yau CR manifold of dimension 3. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in* $\mathbb{C}^N$. *Then X is a boundary of the complex variety* $V \subseteq D - X$ *with boundary regularity and the variety is smooth after normalization if and only if s-invariant and* $g^{(1,1)}(X)$ *vanish.*

Corollary 4.2 ([11]) *Let X be a strongly pseudoconvex compact CR manifold of dimension 3. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in* $\mathbb{C}^3$. *Then X is a boundary of the complex sub-manifold* $V \subset D - X$ *if and only if s-invariant and* $g^{(1,1)}(X)$ *vanish.*

Corollary 4.3 ([11]) *Let X be a strongly pseudoconvex compact Calabi-Yau CR manifold of dimension 3. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in* $\mathbb{C}^N$ *with* $H^2_h(X) = 0$. *Then X is a boundary of the complex sub-manifold up to normalization* $V \subset D - X$ *with boundary regularity if and only if* $g^{(1,1)}(X) = 0$.

Corollary 4.4 ([11]) *Let X be a strongly pseudoconvex compact CR manifold of dimension 3. Suppose that X is contained in the boundary of a strongly pseudoconvex bounded domain D in* $\mathbb{C}^3$ *with* $H^2_h(X) = 0$. *Then X is a boundary of the complex sub-manifold* $V \subset D - X$ *if and only if* $g^{(1,1)}(X) = 0$.

5 Minimal Embedding Dimension of Compact CR Manifold

Let us first consider a compact strongly pseudoconvex manifold X of dimension $2n-1$ where $n \geq 3$. As mentioned above, X can be CR embedded in some $\mathbb{C}^N$. It is therefore of interest to study the minimal dimensional complex Euclidean space in which X CR embeds. Our starting point is the Theorem 4.1 which provides us with obstruction to CR embedding:

Theorem 5.1 *Let X be a compact strongly pseudoconvex CR manifold of dimension* $2n-1, n \geq 3$. *Then X cannot be CR embedded in* $\mathbb{C}^n$ *unless all* $H_{KR}^{p,q}(X) = 0$, $1 \leq q \leq n-2$. *Further, X cannot be CR embedded in* $\mathbb{C}^{n+1}$ *if one of the following does not hold:*

(1) $H_{KR}^{p,q}(X) = 0$ *for* $p+q \leq n-2$ *and* $1 \leq q \leq n-2$

(2) $\dim H_{KR}^{p,q}(X) = \dim H_{KR}^{p',q'}(X)$ *for* $\left.\begin{matrix} p+q \\ p'+q' \end{matrix}\right\} = n-1, n$ *and* $1 \leq q, q' \leq n-2$

(3) $H_{KR}^{p,q}(X) = 0$ *for* $p+q \geq n+1$ *and* $1 \leq q \leq n-2$.

We next consider an interesting class of CR manifolds.

Definition 5.1 Let X be a CR manifold with structure bundle S. Let α be a smooth S^1-action on X and V be its generating vector field. The S^1-action α is called holomorphic of $\mathcal{L}_V\Gamma(S) \subseteq \Gamma(S)$ where $\mathcal{L}_V$ denotes the Lie derivative. It is called transversal if V is transversal to $S \oplus \overline{S}$ in $\mathbb{C}TX$ at every point of X.

For a CR manifold X which admits a transversal holomorphic S^1-action, the invariant Kohn-Rossi cohomology is defined as follows.

Definition 5.2 With the notation in Definition 5.1, consider first the differential operator on k forms $N : \mathcal{A}^k(X) \to \mathcal{A}^k(X)$ defined by $N\phi = \sqrt{-1}\mathcal{L}_V\phi$, $\phi \in \mathcal{A}^k(X)$. Observe that N leaves invariant the spaces $\mathcal{A}^{p,q}(X)$ and $\mathcal{C}^{p,q}(X)$, and commutes with the operators d and $\overline{\partial}_b$. Hence N acts on the cohomology groups $H_{KR}^{p,q}(X)$. Now define the invariant Kohn-Rossi cohomology by $\tilde{H}_{KR}^{p,q}(X) = \{c \in H_{KR}^{p,q}(X) : Nc = 0\}$.

For a compact strongly pseudoconvex CR manifold X of dimension $2n-1, n \geq 3$, which admits a transversal holomorphic S^1-action, the invariant Kohn-Rossi cohomology $\tilde{H}_{KR}^{p,q}(X)$, for $1 \leq p+q \leq 2n-N-1$, are obstructions to CR embedding in $\mathbb{C}^N$. This is implied by the following theorem.

Theorem 5.2 (Luk-Yau [33]) *Let X be a compact strongly pseudoconvex CR manifold of dimension* $2n-1, n \geq 3$, *which admits a transversal holomorphic* S^1*-action. Suppose that X is CR embeddable in* $\mathbb{C}^N$. *Then* $\tilde{H}_{KR}^{p,q}(X) = 0$ *for all* $1 \leq p+q \leq 2n-N-1$.

The proof of Theorem 5.2 contains two main parts. The first part depends heavily on the work of Lawson-Yau [26], which provides us with topological restrictions on X. In particular it can be shown that the De Rham cohomology groups $H^k(X) = 0$ for $1 \leq k \leq 2n-N-1$. The second part follows Tanaka's differential geometric study on the $\overline{\partial}_b$ cohomology groups [42]. The existence of the vector field V with $[V, \Gamma(S)] \subseteq \Gamma(S)$ entails a formalism analogous to Kähler geometry linking the various cohomology groups via harmonic forms. The details of the proof of Theorem 5.2 are contained in [33].

For 3 dimensional compact strongly pseudoconvex *CR* manifolds, global *CR* embedding in complex Euclidean space may fail and much work has been done on this phenomenon. See for example [3, 5, 28]. We only remark that as a consequence of the global invariants to be discussed in the next section, we find obstructions to *CR* embedding in $\mathbb{C}^3$, assuming that the 3-dimensional strongly pseudoconvex *CR* manifold is *CR* embeddable in some $\mathbb{C}^N$ to begin with. These obstructions provide us with numerous examples of such 3-dimensional *CR* manifolds not *CR* embeddable in $\mathbb{C}^3$.

Remark 5.1 It is interesting to note that there are compact strongly pseudoconvex 3 dimensional *CR* manifolds with arbitrarily large minimal embedding dimensions. For any positive integer N, take any 2-dimensional strongly pseudoconvex complex manifold with maximal compact analytic set A which is a smooth rational curve having self intersection number $-N$. The corresponding weighted dual graph is hence

$$\underset{-N}{\bullet}\ .$$

On blowing down A, one gets a 2-dimensional rational singularity (V, x). The minimal embedding dimension of (V, x) is $-A \cdot A + 1 = N + 1$. Let X be the intersection of V with a small sphere centered at x. Then the minimal embedding dimension of X is $N + 1$.

6 Global Invariants of Compact Strongly Pseudoconvex CR Manifolds

As a first step towards the difficult classification problem of compact strongly pseudoconvex *CR* manifolds [43], it would be useful to understand the following notion of equivalence which is weaker than *CR* equivalence.

Definition 6.1 Assume that X_1, X_2 are compact strongly pseudoconvex embeddable CR manifolds of dimension $2n - 1$, $n \geq 2$. By [16, 17], there are unique complex varieties $V_1 \subseteq \mathbb{C}^{N_1}$ and $V_2 \subseteq \mathbb{C}^{N_2}$ such that $\partial V_1 = X_1$, $\partial V_2 = X_2$, V_1 and V_2 have only isolated normal singularities. X_1, X_2 are called algebraically equivalent if V_1 and V_2 have isomorphic singularities Y_1, Y_2, i.e. $(V_1, Y_1) \cong (V_2, Y_1)$ as germs of varieties.

Remark 6.1 It is not difficult to show that *CR* equivalence implies algebraic equivalence. Hence all algebro-geometric invariants of the singularities of V are *CR* invariants of X.

In case a compact strongly pseudoconvex *CR* manifold X of dimension $2n-1$ embeds in $\mathbb{C}^{n+1}$, $n \geq 2$, it is the boundary of a complex hypersurface V with isolated singularities $x_1, \ldots, x_m$. In this case, an Artinian algebra can be associated to X as follows.

Definition 6.2 With the above notation, let f_i be a defining function of the germ (V, x_i), $1 \leq i \leq m$. Then the $\mathbb{C}$-algebra $A_i = \mathcal{O}_{n+1}\big/(f_i, \frac{\partial f_i}{\partial z_0}, \ldots, \frac{\partial f_i}{\partial z_n})$ is a commutative local Artinian algebra called the moduli algebra of (V, x_i). The moduli algebra is independent of the choice of defining function. We associate to the *CR* manifold X the Artinian algebra $A(X) = \bigoplus_{i=1}^{m} A_i$.

By the work of Mather-Yau [34] on isolated hypersurface singularities, it can be shown that the associated Artinian algebras are complete algebraic *CR* invariants in the following sense.

Theorem 6.1 (Luk-Yau [30]) *Two compact strongly pseudoconvex real codimension 3 CR manifolds X_1, X_2 are algebraically equivalent if and only if the associated Artinian algebras $A(X_1)$, $A(X_2)$ are isomorphic $\mathbb{C}$-algebras.*

Definition 6.3 With the above notation, let $L(X)$ be the algebra of derivations of $A(X)$. Since $A(X)$ is finite dimensional as $\mathbb{C}$-vector space and $L(X)$ is contained in the endomorphism algebra of $A(X)$, consequently $L(X)$ is a finite dimensional Lie algebra with the obvious Lie algebra structure.

Theorem 6.2 (Yau [48, 49]) *With the above notation, $L(X)$ is a finite dimensional solvable Lie algebra.*

We remark that there are Torelli type examples in which the Lie algebras $L(X_t)$ associated to a family of compact strongly pseudoconvex real codimension 3 *CR* manifolds X_t suffice to distinguish *CR* equivalence. For example, in the family $X_t = \{(x, y, z) \in \mathbb{C}^3 : x^6 + y^3 + z^2 + tx^4y = 0 \text{ and } |x|^2 + |y|^2 + |z|^2 = \epsilon^2\}$ where $\epsilon > 0$ is a small fixed number and $t \in \mathbb{C}$ with $4t^2 + 27 \neq 0$, X_{t_1}, X_{t_2} are *CR* equivalent if and only if $L(X_{t_1}), L(X_{t_2})$ are isomorphic Lie algebras.

Question 6.1 *How can one compute $A(X)$ and $L(X)$ directly from X without going through V?*

For the rest of this section we consider embeddable 3 dimensional compact strongly pseudoconvex *CR* manifolds. By taking resolutions of the singularities of the subvariety V bounded by such a *CR* manifold X in complex Euclidean space, numerical invariants under algebraic equivalence may be defined as follows.

Definition 6.4 Let $\pi : M \to V$ be a resolution of the singularities Y of V such that the exceptional set $A = \pi^{-1}(Y)$ has normal crossing, i.e., the irreducible components A_i of A are nonsingular, they intersect transversally and no three meet at a point. According to Artin [2], there exists a unique minimal positive divisor Z, called the fundamental cycle, with support on A, such that $Z \cdot A_i \leq 0$ for all A_i. For any positive divisor $D = \Sigma d_i A_i$, let $\mathcal{O}_M(-D)$ be the sheaf of germs of holomorphic functions on M vanishing to order d_i on A_i, let $\mathcal{O}_D = \mathcal{O}_M/\mathcal{O}_M(-D)$ and let $\chi(\mathcal{O}_D) = \sum_{i=0}^{2}(-1)^i \dim H^i(M, \mathcal{O}_D)$. It can be proved that $p_f(X) := 1-\chi(\mathcal{O}_Z)$, $p_a(X) := \sup(1 - \chi(\mathcal{O}_D))$ where D ranges over all positive divisors with support on A and $p_g(X) := \dim H^1(M, \mathcal{O})$ are defined independent of the resolution π and are invariants of X under algebraic equivalence. The detailed proofs are contained in [32]. We refer to $p_f(X), p_a(X)$ and $p_g(X)$ as the fundamental genus, arithmetic genus and geometric genus of X respectively.

The following facts are known:

- $0 \leq p_f(X) \leq p_a(X) \leq p_g(X)$
- $p_f(X) = 0 \Leftrightarrow p_a(X) = 0 \Leftrightarrow p_g(X) = 0$.

Further numerical invariants under algebraic equivalence are given by $m_Z(X) := Z \cdot Z$, $q(X) := \dim H^0(M-A, \Omega^1)\big/H^0(M, \Omega^1)$, $\chi(X) := K \cdot K + \chi_T(A)$ and $\omega(X) := K \cdot K + \dim H^1(M, \Omega^1)$, where Ω^1 is the sheaf of germs of holomorphic 1-form on M, $\chi_T(A)$ is the topological Euler characteristic of A and K is the canonical divisor on M. These invariants are defined independent of the choice of the resolution π. Since K is a divisor with rational coefficient, $\chi(X)$ and $\omega(X)$ are in general rational numbers.

Using the above invariants, one may attempt a rough algebraic classification of embeddable 3 dimensional compact strongly pseudoconvex *CR* manifolds.

Definition 6.5 An embeddable 3 dimensional compact strongly pseudoconvex *CR* manifold X is called a rational (respectively elliptic) *CR* manifold if $p_a(X) = 0$ (respectively $p_a(X) = 1$).

If X is a rational or an elliptic *CR* manifold embeddable in $\mathbb{C}^3$ and M_0 is the minimal good resolution of the subvariety V bounded by X in $\mathbb{C}^3$, then the weighted dual graph for the exceptional set of M_0 is completely classified. The same also holds for those X embeddable in $\mathbb{C}^3$ and has $p_g(X) = 1$. With the weighted dual graphs classified, the topology of the embedding of the exceptional set in M_0 is well understood.

As an application, one obtains obstructions to embedding in $\mathbb{C}^3$ for the above three classes of *CR* manifolds when their weighted dual graphs fail to have the required forms. For example, a rational *CR* manifolds whose weighted dual graph is not a direct sum of the graphs A_k, D_k, E_6, E_7, E_8 is not embeddable in $\mathbb{C}^3$.

Similarly in view of the following theorem, one obtains numerical obstructions to embedding in $\mathbb{C}^3$ for those *CR* manifolds failing the conditions in the theorem.

Theorem 6.3 (**[29, 32]**) *Let X be a compact strongly pseudoconvex 3-dimensional CR manifold embeddable in* $\mathbb{C}^3$. *Then*

(1) $\chi(X)$ *and* $\omega(X)$ *are integers.*
(2) $10p_g(X) + \omega(X) \geq 0$
(3) *If* $p_a(X) = 1$, *then* $\chi(X) \geq -3$
(4) *If X admits a transversal holomorphic* S^1*-action, then* $6p_g(X) + \chi(X) > 0$.

We remark that (4) depends on the Durfee conjecture which is solved by Xu and Yau [45].

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Fatou Coordinates for Parabolic Dynamics

Tetsuo Ueda

1 Introduction

Parabolic maps have rich structure and their behavior can be described in terms of the Fatou coordinates. In this article, we will first review, in the case of one complex variable, the characterization of intrinsic structure of parabolic fixed points. We will discuss how this description can be generalized for the case of mappings in two variables. Polynomial automorphisms of $\mathbb{C}^2$ provide concrete and interesting examples. We will review some basic definitions and results on polynomial automorphisms and make remarks in the case of the maps with semi-parabolic fixed points. Finally we will make an overview on the recent results in [3] about bifurcations of semi-parabolic mappings.

2 Fatou Coordinates in One Variable

We will consider a holomorphic map in one complex variable of the form

$$x \mapsto x + A_2x^2 + A_3x^3 + \cdots,$$

defined in a neighborhood of $0 \in \mathbb{C}$. Such a map is said to have a *parabolic fixed point* at 0. In this article we will treat only the case where $A_2 \neq 0$. Changing coordinates by $z = -1/(A_2x)$, we can regard this as a mapping defined in a

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neighborhood of $z = \infty$ in the Riemann sphere $\hat{\mathbb{C}}$, of the form

$$z \mapsto f(z) = z + 1 + \frac{a_1}{z} + \cdots . \tag{1}$$

We choose a sufficiently large $R_0 > 0$ and assume that f and its inverse f^{-1} are both injective on $\{R_0 < |z| \leq \infty\}$ and that $|f(z) - z - 1| < \delta < 1$ there.

Theorem 2.1 *There exists a neighborhood U of ∞ and its subdomains B^ι, B^o with the following properties:*

(1) $B^\iota \cup B^o = U \setminus \{\infty\}$.
(2) $f(B^\iota) \subset B^\iota$ *and* $f^n|B^\iota \to \infty$ *when* $n \to \infty$; $f^{-1}(B^o) \subset B^o$ *and* $f^{-n}|B^o \to \infty$ *when* $n \to \infty$;
(3) *The intersection* $B^\iota \cap B^o$ *consists of two connected components* U^+ *and* U^- *which are invariant under* f.

Theorem 2.2 *There exist injective holomorphic functions $\varphi^{\iota/o}$ on $B^{\iota/o}$ satisfying the Abel equation $\varphi^{\iota/o}(f(z)) = \varphi^{\iota/o}(z) + 1$. These functions are determined uniquely up to additive constants by either one of the following conditions:*

(i) $\varphi^{\iota/o}(z) - z - a_1 \log z \to$ const. *when* $\operatorname{Re} z \to \pm\infty$ *(where* log *denotes a single valued branch of the logarithm on* $B^{\iota/o}$;
(ii) $d\varphi^{\iota/o}/dz \to 1$ *when* $\operatorname{Re} z \to \pm\infty$;

The functions $\varphi^{\iota/o}$ are called (incoming and outgoing) *Fatou coordinates* for f and they are given by

$$\varphi^{\iota/o}(z) = \lim_{n\to\infty} \left\{ f^{\pm n}(z) - a_1 \log f^{\pm n}(z) \mp n \right\} + \text{arbitrary const.}$$

For the detail we refer to the original work [8] by Fatou or the standard text [14].

See also [19], where the two variable case is discussed. The argument is easily adapted to one variable case.

On U^+ and U^- we have Fatou coordinates φ^ι and φ^o. The relation between these two Fatou coordinates are described in the following way:

We can choose $K > 0$ so that the images $\varphi^o(U^+)$ and $\varphi^o(U^-)$ contain $U^o_+ = \{\operatorname{Im} \zeta > K\}$ and $U^o_- = \{\operatorname{Im} \zeta < -K\}$, respectively. We define

$$h^\pm(\zeta) = \varphi^\iota \circ (\varphi^o)^{-1}(\zeta) \qquad \text{on } U^o_\pm.$$

Since we have $h^{\pm}(\zeta+1)=h^{\pm}(\zeta)+1$ the functions $h^{\pm}(\zeta)-\zeta$ are periodic with period 1 and have Fourier expansion:

$$\begin{cases} h^{+}(\zeta)=\zeta+\sum_{k=0}^{\infty} c_k^{+}\exp(2\pi i k\zeta) \quad \text{on } U_{+}^{o}\,, \\ h^{-}(\zeta)=\zeta+\sum_{k=0}^{\infty} c_k^{-}\exp(-2\pi i k\zeta) \quad \text{on } U_{-}^{o}\,. \end{cases}$$

If we choose Fatou coordinates $\tilde{\varphi}^{\iota}(z)=\varphi^{\iota}+\alpha$ and $\tilde{\varphi}^{o}(z)=\varphi^{o}+\beta$, then

$$\tilde{h}^{\pm}(\zeta)=\tilde{\varphi}^{\iota}\circ(\tilde{\varphi}^{o})^{-1}(\zeta)=\varphi^{\iota}\circ(\varphi^{o})^{-1}(\zeta-\beta)+\alpha=h(\zeta-\beta)+\alpha$$

The pair $h=(h^{+},h^{-})$ will be called the *connection* of the mapping f, (for $\varphi^{\iota/o}$). Two connections $h=(h^{+},h^{-})$ and $\tilde{h}=(\tilde{h}^{+},\tilde{h}^{-})$ are said to be equivalent if there exist α and β such that $\tilde{h}^{\pm}(\zeta)=h(\zeta-\beta)+\alpha$.

The following result due to Écalle and Voronin gives an intrinsic characterization of the structure of parabolic fixed points (cf., [7, 22]).

Theorem 2.3 *The map f determines an equivalence class of connections and two mappings f_1 and f_2 are conjugate if and only if their connections are equivalent. Conversely, for a given pair (h^{+},h^{-}), there exists a mapping of the form (1) that has (h^{+},h^{-}) as its connection.*

Voronin's proof is sketched as follows: When a connection (h^{+},h^{-}) is given, we construct a Riemann surface by patching two domains corresponding to B^{ι} and B^{o} using this connection. The crucial point is to show that the resulting Riemann surface is conformally equivalent to a punctured disk. Then, we can fill the puncture and obtain the desired map that has the filled puncture as the fixed point.

Global maps Now suppose that f is a holomorphic self map of the Riemann sphere $\hat{\mathbb{C}}$ (i.e. rational function) with a parabolic fixed point, which we assume to be ∞. By the parabolic basin $\mathcal{B}$ to ∞, we mean the set of points in $\hat{\mathbb{C}}$ whose forward orbits tend to ∞ locally uniformly. We have $\mathcal{B}=\bigcup_{n\geq 1} f^{-n}(B^{\iota})$.

Then the incoming Fatou coordinate φ^{ι} can be continued to a holomorphic function on $\mathcal{B}$, with the use of the Abel's equation $\varphi^{\iota}(f(z))=\varphi^{\iota}(z)+1$. We denote by $H=(\varphi^{o})^{-1}$ the inverse of the outgoing Fatou coordinate φ^{o}. Then, using the equation $H(\zeta+1)=f\circ H(\zeta)$, the map H can be continued to a holomorphic map $H:\mathbb{C}\to\hat{\mathbb{C}}$. Setting $\Omega=H^{-1}(\mathcal{B})\subset\mathbb{C}$ (ζ-plane), the map h can continued to a mapping

$$h=\varphi^{\iota}\circ H\;:\;\Omega\to\mathcal{B}.$$

Simultaneous linearization We can regard a parabolic fixed point as a limit of hyperbolic fixed points, i.e. attracting or repelling fixed points (cf. [21]). Let

$$f_\lambda(z) = \lambda z + a_2 z^2 + \cdots$$

be a family of maps with fixed point 0 depending on the parameter λ. When $0 < |\lambda| < 1$, the origin is an attracting fixed point and there are a neighborhood U_λ of 0 and a holomorphic function $\psi_\lambda(z)$ on U_λ satisfying the Schröder equation

$$\psi_\lambda(f_\lambda(z)) = \lambda \psi_\lambda(z) \qquad (z \in U_\lambda)$$

Suppose that λ tends to 1 from inside of the unit disk non tangentially. We can choose U_λ in such a way that they tend to an open set U with 0 on its boundary. We set

$$\varphi_\lambda(z) = \frac{1}{\psi_\lambda(z)} + \frac{\lambda}{\lambda - 1}$$

Thus the Fatou coordinate is regarded as the limit of modified linearizing coordinates of attracting maps, and $\varphi_\lambda(z)$ tends to the incoming Fatou coordinate for $f_1(z)$.

Problems

1. Give a similar description for 2-dimensional maps with (semi-)parabolic fixed points. Specifically give an intrinsic characterization of conjugacy classes of these maps using (2-dimensional versions of) connections.
   In the next section we see a partial answer for this problem. A complete answer seems to be still not known.
2. When a 2-dimensional map with a semi-attracting fixed point is perturbed in a good direction, this splits into a pair of an attracting fixed point and a saddle point. We will be able to regard the Fatou coordinates as limits of the linearizing coordinates for the attracting point and saddle.

3 Semi-parabolic (Semi-attracting) Fixed Points

Let M be a complex manifold of dimension 2 and $F : M \to M$ be a holomorphic automorphism of M. Suppose that this map has a fixed point $O \in M$, i.e., $F(O) = O$, and that the differential $F'(O)$ of F at O has eigenvalues 1 and b with $0 < |b| < 1$. In this case, the origin is said to be a semi-parabolic fixed point of F. Then we can choose a local coordinate system (x, y) so that F takes the form

$$(x, y) \mapsto \Big(x + \sum_{j+k \geq 2} a_{jk} x^j y^k,\ by + \sum_{j+k \geq 2} b_{jk} x^j y^k\Big).$$

In what follows we pose the condition $a_{20} \neq 0$.

Convergence by forward and backward iterations Let us first look at the dynamical structure of the fixed points of this type. What are the sets of points that converge (pointwise or locally uniformly) to the fixed point O under forward or backward iterations of F? The answer to this question is given in terms of the sets $\mathcal{B}, \mathcal{C}$ and Σ defined as follows. (We refer to [19, 20] for the detail. See also [9].)

We define the *Basin of local uniform convergence* $\mathcal{B}$ by

$$\mathcal{B} = \{p \in M \mid F^n \to O \ (n \to \infty) \ \text{ locally uniformly }\}.$$

This set $\mathcal{B}$ is a connected open set biholomorphic to $\mathbb{C}^2$. The fixed point O is on the boundary $\partial\mathcal{B}$ of $\mathcal{B}$. Further there exists a biholomorphic map $\Phi : \mathcal{B} \to \mathbb{C}^2$ such that $F|\mathcal{B}$ is conjugate to a translation on $\mathbb{C}^2$:

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{F|\mathcal{B}} & \mathcal{B} \\ {\scriptstyle\Phi}\big\downarrow & & \big\downarrow{\scriptstyle\Phi} \\ \mathbb{C}^2 & \xrightarrow[G]{} & \mathbb{C}^2 \end{array}$$

where G stands for the translation $(u, v) \mapsto (u + 1, v)$

The *Strong stable curve* (Stable manifold corresponding to the eigenvalue b) is defined as a holomorphic map $\eta : \mathbb{C} \to M$ with $\eta(0) = O,\ \eta'(0) \neq 0$ satisfying the following commutative diagram.

$$\begin{array}{ccc} M & \xrightarrow{F} & M \\ {\scriptstyle\eta}\big\uparrow & & \big\uparrow{\scriptstyle\eta} \\ \mathbb{C} & \xrightarrow[b]{} & \mathbb{C} \end{array}$$

where b stands for the multiplication $t \to bt$.

We will also call the image $\mathcal{C} := \eta(\mathbb{C})$ the strong stable curve. The set $\mathcal{C}$ lies on the boundary $\partial\mathcal{B}$ of $\mathcal{B}$ and the union $\mathcal{B} \cup \mathcal{C}$ is the sets of all points that are attracted (pointwise) to O by the forward iterations of F.

There exists a holomorphic map $H : \mathbb{C} \to M$ satisfying $F \circ H(\zeta) = F(\zeta + 1)$, i.e.,

$$\begin{array}{ccc} M & \xrightarrow{F} & M \\ {\scriptstyle H}\big\uparrow & & {\scriptstyle H}\big\uparrow \\ \mathbb{C} & \xrightarrow[\tau_1]{} & \mathbb{C} \end{array}$$

where τ_1 stands for the translation by 1.

By the *Asymptotic curve* (or the unstable manifold), we mean either this map or its image $\Sigma = H(\mathbb{C}) \subset M$. Then $\Sigma \cup \{O\}$ is the set of all points that are attracted to O by the backward iterations of F.

Fatou coordinates We define the *incoming Fatou coordinate* on $\mathcal{B}$ by $\varphi^\iota(p) = u \circ \Phi(p)$. Here $\Phi : \mathcal{B} \to \mathbb{C}^2$ is the biholomorphic map and u stands for the first component of this map. The function φ^ι gives the information on the 'future' behavior of points in $\mathcal{B}$. This satisfies the Abel's functional equation

$$\varphi^\iota(F(p)) = \varphi^\iota(p) + 1 \qquad (p \in \mathcal{B}).$$

We define the *outgoing Fatou coordinate* $\varphi^o : \Sigma \to \mathbb{C}$ to be the inverse map of $H : \mathbb{C} \to \Sigma$. This gives the information on the 'past' of points in Σ. This satisfies the equation

$$\varphi^o(F(p)) = \varphi^o(p) + 1 \qquad (p \in \Sigma)$$

Structure of the intersection $\Sigma \cap \mathcal{B}$ If a point is in the intersection of $\mathcal{B}$ and Σ, both of its forward and backward orbits converges to the fixed point O. Investigation of these points are important since, when the map F is perturbed, they can yield periodic points.

We have both incoming and outgoing Fatou coordinates on $\mathcal{B} \cap \Sigma$, and thus its structure can be described using the relation between the two Fatou coordinates. For this purpose we define Ω to be the biholomorphic image of $\mathcal{B} \cap \Sigma$ on the complex plane of the outgoing Fatou coordinate:

$$\Omega = \varphi^o(\mathcal{B} \cap \Sigma) = H^{-1}(\mathcal{B} \cap \Sigma).$$

Proposition 3.1 *(1)* *Ω is invariant under the translation $\zeta \mapsto \zeta + 1$.*
(2) *Ω contains $\{|\operatorname{Im}\zeta| > R\}$ if $R > 0$ is sufficiently large.*
(3) *Suppose that M is a Stein manifold (for example $M = \mathbb{C}^2$). Then Ω has at least two connected components, one of which Ω^+ contains $\{\operatorname{Im}\zeta > R\}$ and another Ω^- contains $\{\operatorname{Im}\zeta < -R\}$. Further all the connected components are simply connected.*

The assertion (1) follows from the fact that $\mathcal{B}\cap\Sigma$ is invariant under F. (2) follows from the local property of a semi-attracting fixed point. (3) follows from the fact that $\mathcal{B}$ is a Runge domain (see [3] for the detail).

Now we consider the function

$$h = \varphi^\iota \circ (\varphi^o)^{-1} : \Omega \to \mathbb{C}$$

This satisfies the functional equation $h(\zeta + 1) = h(\zeta) + 1$. Hence $h(\zeta) - \zeta$ is a periodic function with period 1. We have the expressions on $\Omega^{\pm}$:

$$\begin{cases} h^+(\zeta) = \zeta + c_0^+ + \sum_{n=1}^{\infty} c_n^+ \exp(2n\pi i\zeta) & \zeta \in \Omega^+ \\ h^-(\zeta) = \zeta + c_0^- + \sum_{n=1}^{\infty} c_n^- \exp(-2n\pi i\zeta) & \zeta \in \Omega^- \end{cases}$$

The Fatou coordinates are unique up to additive constants, and the map F determines the equivalence class of the pair (h^+, h^-) as in Sect. 1. We will choose the Fatou coordinates so that $c_0^+ + c_0^- = 0$ holds.

Transition maps For a map F with a semi-attracting fixed point, we define the family of transition maps $T_\alpha : \mathcal{B} \to \Sigma$ with parameter $\alpha \in \mathbb{C}$ by

$$T_\alpha = (\varphi^o)^{-1} \circ \tau_\alpha \circ \varphi^\iota$$

Here τ_α denotes the translation $\zeta \mapsto \zeta + \alpha$.

The reason for making this definition may be unclear: There is no relation between the future behavior of a point in $\mathcal{B}$ and the past behavior of a point on Σ. So there seems to be no reason to make correspondence between these two points—as long as we are looking at only the map F. Though the transition maps are defined only in terms of the map F, they can be regarded as approximations of the dynamics when F is perturbed in adequate direction. Thus the transition maps will serve to give description of the perturbations in terms the unperturbed map F.

We consider the restriction of the map T_α to $\mathcal{B} \cap \Sigma$. The outgoing Fatou coordinate φ^o gives the correspondence of Σ onto $\mathbb{C}$, and hence $\mathcal{B} \cap \Sigma$ onto Ω, So we may think of the map

$$h_\alpha = \varphi^o \circ T_\alpha \circ (\varphi^o)^{-1} : \Omega \to \mathbb{C}$$

as an expression of the map $T_\alpha | \mathcal{B} \cap \Sigma$ in terms of the outgoing Fatou coordinate. We note the following relation:

$$h_\alpha = \varphi^o \circ (\varphi^o)^{-1} \circ \tau_\alpha \circ \varphi^\iota \circ (\varphi^o)^{-1} = \tau_\alpha \circ \varphi^\iota \circ (\varphi^o)^{-1} = \tau_\alpha \circ h$$

4 Polynomial Automorphisms of $\mathbb{C}^2$

Analogs of (Filled-)Julia sets Here we recall some basic results on polynomial automorphisms of $\mathbb{C}^2$. Polynomial automorphisms of $\mathbb{C}^2$ are, in a sense, considered as a natural generalization of polynomial maps in one complex variable. By a result of Friedland and Milnor [11], dynamically interesting polynomial automorphisms of $\mathbb{C}^2$ are those that are conjugate to compositions $F_k \circ \cdots \circ F_1$ of generalized Hénon maps.

$$F_j(x, y) = (y, P_j(y) - a_j x), \qquad j = 1, \ldots, k,$$

where a_j are complex numbers $\neq 0$ and $P_j(y)$ are polynomials of degree ≥ 2. In the following we will consider only polynomial automorphisms of this type.

For a polynomial automorphism F, the analog of the 1-dimensional filled Julia set is the set

$$K^{\pm} = \{(x, y) \in \mathbb{C}^2 \mid \{F^{\pm n}(x, y)\}_{n \geq 0} \text{ are bounded}\}$$

The boundaries these sets correspond to the Julia set:

$$J^{\pm} = \partial K^{\pm}$$

Further we define

$$K = K^+ \cap K^- \text{ and } J = J^+ \cap J^-$$

The sets $K^{\pm}$, $J^{\pm}$ are unbounded and closed. On the other hand their intersections K, J are compact sets. We set

$$J^* = \overline{\{\text{saddle periodic points of } F\}} \qquad \text{(closure)}$$

This set J^* is a subset of J. It is not known whether these two sets coincide.

The Green functions $G^{\pm}(x, y)$ for the map F is defined by

$$G^{\pm}(x, y) = \lim_{n \to \infty} \frac{1}{d^n} \log^+ \|F^{\pm n}(x, y)\|$$

These functions are continuous on $\mathbb{C}^2$. The sets $K^{\pm}$ and $J^{\pm}$ are characterized in terms of $G^{\pm}$:

$$K^{\pm} = \{(x, y) \in \mathbb{C}^2 \mid G^{\pm}(x, y) = 0\} \quad \text{and} \quad J^{\pm} = \operatorname{supp} dd^c G^{\pm}.$$

For the details of these results we refer to [1, 2, 10, 12, 15].

Semi-continuity Let M be a manifold and consider a family (set-valued function) $\varepsilon \mapsto X_\varepsilon$ of subsets $X_\varepsilon \subset M$ depending on the parameter $\varepsilon \in E \subset \mathbb{C}$. The family X_ε is said to be upper semi-continuous at ε_0 if, for any open set U in M with $U \cap X_{\varepsilon_0} = \emptyset$, there is a neighborhood E_0 of ε_0 such that $U \cap X_\varepsilon = \emptyset$ holds for all $\varepsilon \in E_0$. The family X_ε is said to be lower semi-continuous at ε_0 if, for any open set U in M with $U \cap X_{\varepsilon_0} \neq \emptyset$, there is a neighborhood E_0 of ε_0 such that $U \cap X_\varepsilon \neq \emptyset$ holds for all $\varepsilon \in E_0$. The family X_ε is continuous at ε_0 when it is both upper and lower semi-continuous.

Now suppose that F_ε $(\varepsilon \in U)$ is a family of polynomial automorphisms of $\mathbb{C}^2$ depending continuously on the parameter ε. Since the Green function depends continuously on the parameter, we have the following theorem.

Theorem 4.1

(1) The set valued function $\varepsilon \mapsto K^{\pm}(F_\varepsilon)$ is upper semi-continuous.
(2) The set valued functions $\varepsilon \mapsto J^{\pm}(F_\varepsilon)$ and $\varepsilon \mapsto J^*(F_\varepsilon)$ are lower semi-continuous.

Hénon maps with semi-attracting fixed point The simplest of dynamically interesting polynomial automorphisms are the quadratic Hénon maps:

$$F_{a,c} : (x, y) \mapsto (y, y^2 + c - ax),$$

which constitute a family depending on two parameters $(a, c) \in \mathbb{C}^* \times \mathbb{C}$. The map $F_{a,c}$ has two fixed points if $(a+1)^2 - 4c \neq 0$. When $(a+1)^2 - 4c = 0$, the map $F_{a,c}$ has only one fixed point $((a+1)/2, (a+1)/2)$ with eigenvalues of the differential 1 and $|a|$. When $|a| < 1$, this is semi-attracting of the type we discussed in Sect. 3.

Now, for general polynomial automorphisms with a semi-attracting fixed point, we will make some remarks on the relations between the sets $K^{\pm}$, $J^{\pm}$, K, J and $\mathcal{B}$, Σ, $\mathcal{B} \cap \Sigma$. These facts follow easily by combining the results in [19, 20] and [1].

Theorem 4.2

(1) $\mathcal{B}$ is a connected component of $\operatorname{int} K^+$.
(2) $\mathcal{C} \subset \partial\mathcal{B} = J^+ = \partial K^+$ (dense).
(3) $\Sigma \subset J^- = K^-$ (dense).
(4) $\Sigma \cap \mathcal{B} \subset K$.

Since K is bounded, we know from (4) that $\Sigma \not\subset \mathcal{B}$.

5 Bifurcation of Semi-parabolic Maps

This section is devoted to outline the results given in [3], to which we refer the readers for the detail. (See also [4–6, 13, 16–18, 23].)

Let us consider a family $\{F_\varepsilon\}$ of polynomial automorphisms of $\mathbb{C}^2$ depending on the parameter $\varepsilon \in E$, where E is a neighborhood of 0 in $\mathbb{C}$. We will assume the following conditions:

(i) For $\varepsilon = 0$, the map F_0 has a semi-attracting fixed point O and F_0 is of the form $(x, y) \mapsto (x + x^2 + \cdots, by + \cdots)$ with respect to a local coordinate with center O.
(ii) For $\varepsilon \neq 0$, the fixed point splits into two fixed points $(\pm\varepsilon, 0) + O(\varepsilon^2)$.

Proposition 5.1 *By choosing a suitable local coordinates* (x, y) *and the parameter* ε*, we may suppose that the family* F_ε *is of the form*

$$F_\varepsilon \ : \ (x, y) \ \mapsto \ (x + (x^2 + \varepsilon^2)\alpha_\varepsilon(x, y), \ b_\varepsilon(x)y + (x^2 + \varepsilon^2)\beta_\varepsilon(x, y)).$$

where $\alpha_\varepsilon(0, 0) = 1$.

Then $(\pm i\varepsilon, 0)$ are the fixed points and $\{x = \pm i\varepsilon\}$ are the stable curves for these fixed points. The restrictions to these stable curves of the mapping F_ε are scalar multiplications.

α-sequences Let $\alpha \in \mathbb{C}$. A sequence of pairs $\{(\varepsilon_j, n_j)\}$ of $\varepsilon_j \in \mathbb{C}$ and $n_j \in \mathbb{N}$ will be called an α sequence, if $\varepsilon_j \to 0, \ n_j \to \infty$ and $n_j - \pi/\varepsilon_j \to \alpha$ when $j \to \infty$. A sequence $\{\varepsilon_j\}$ will be called an α-sequence if $\{n_j\}$ can be so chosen that $\{(\varepsilon_j, n_j)\}$ is an α-sequence.

From this condition it follow that $\mathrm{Im}\, 1/\varepsilon_j = \mathrm{Im}\, \varepsilon_j/|\varepsilon_j|^2$ is bounded. In other words, an α sequence $\{\varepsilon_j\}$ is quadratically tangent to the positive part of the real axis

The following theorem implies that, when F_0 is a map with a semi-attracting fixed point, the perturbed maps F_ε can be approximated by transition maps T_α for F_0. This is a generalization of the Lavaurs theorem to the 2-dimensional case. Since the tools in one dimension such as the uniformization theorem are not available in our case, we need some detailed considerations.

Theorem 5.2 *If* $\{(\varepsilon_j, n_j)\}$ *is an* α*-sequence, then* $\lim_{j\to\infty} F_{\varepsilon_j}^{n_j} = T_\alpha$ *holds.*

If the image under the transition map T_α of a point p in the attracting basin $\mathcal{B}$ is contained again in $\mathcal{B}$, then we can define its image $T_\alpha^2(p)$. The transition map T_α thus defines a partially defined dynamics on $\mathcal{B}$.

The set $K^+(F_0, T_\alpha)$ We will define analogs of filled-Julia for transition maps T_α. We define $K^+(F_0, T_\alpha)$ to be the set of points p in $K^+(F_0)$ that satisfies one of the following conditions:

(i) $T_\alpha^n(p)$ are well-defined and contained in $\mathcal{B}$ for all $n \geq 0$.
(ii) There exists some $n \geq 0$ such that, $T_\alpha^k(p) \in \mathcal{B}$ for $k < n$, and that $T_\alpha^n(p) \in K^+(F_0) \setminus \mathcal{B}$.

In other words, the set $K^+(F_0, T_\alpha)$ is the complement of the set consisting of the points such that there is an $n \geq 0$ such that $T_\alpha^k(p) \in \mathcal{B}$ for $k < n$ and $T_\alpha^n(p) \notin K^+(F_0)$.

Proposition 5.3 *The set* $K^+(F_0, T_\alpha)$ *has the following properties:*

(1) $K^+(F_0, T_\alpha) \setminus \mathcal{B} = K^+(F_0) \setminus \mathcal{B} = K^+(F_0, T_\alpha) \subset K^+(F_0)$.
(2) $K^+(F_0, T_\alpha) = F(K^+(F_0, T_\alpha)) = K^+(F_0, T_{\alpha+1})$. *Hence* $K^+(F_0, T_\alpha)$ *depends only on* $\alpha \mod 1$.
(3) $K^+(F_0, T_\alpha) \cap \mathcal{B}$ *consists of the fibers of* $\{\varphi^\iota = \mathrm{const.}\}$.

The following theorem implies that the Julia-Lavaurs sets for transition maps can be approximated, in a sense, by the (filled) Julia sets for the perturbed maps.

Theorem 5.4 *If $\{\varepsilon_j\}$ is an α-sequence, then*

$$\mathcal{B} \cap \limsup_{j\to\infty} K^+(F_{\varepsilon_j}) \subset K^+(F_0, T_\alpha)$$

For any point $p \in \mathcal{B}$, we can choose $\alpha \in \mathbb{C}$ so that $p \notin K^+(F_0, T_\alpha)$. For such a point p, we have $p \notin \limsup_{j\to\infty} K^+(F_{\varepsilon_j})$.

The set $\boldsymbol{J^*(F_0, T_\alpha)}$ Now we also define analogs of J^* for transition maps:

$$J^*(F_0, T_\alpha) := \overline{\{\text{repelling periodic points of } T_\alpha\}} \quad \text{(closure)}.$$

Theorem 5.5 *If $\{\varepsilon_j\}$ is an α-sequence, then*

$$\liminf_{j\to\infty} J^*(F_{\varepsilon_j}) \supset J^*(F_0, T_\alpha).$$

We can show that $\mathcal{B} \cap J^*(F_0, T_\alpha)$ is non-empty. If $p \in \mathcal{B} \cap J^*(F_0, T_\alpha)$, then we have $p \in \liminf_{j\to\infty} J^*(F_{\varepsilon_j})$.

Discontinuity By combining the above facts, we know that there are $\alpha, \alpha' \in \mathbb{C}$ such that $\mathcal{B} \cap J^*(F_0, T_\alpha) \not\subset K^+(F_0, T_{\alpha'})$.

Theorem 5.6 *For $X = J^*,\ J,\ J^+,\ K,\ K^+$, the set valued functions $\varepsilon \mapsto X(F_\varepsilon)$ are discontinuous at $\varepsilon = 0$.*

To show this, we choose α and $\alpha' \in \mathbb{C}$ so that $\mathcal{B} \cap J^*(F_0, T_\alpha) \not\subset K^+(F_0, T_{\alpha'})$. Then by the preceding theorem we have

$$J^*(F_0, T_\alpha) \subset \liminf_{j\to\infty} J^*(F_{\varepsilon_j}) \subset \liminf_{j\to\infty} X(F_{\varepsilon_j})$$

$$K^+(F_0, T_\alpha) \supset \mathcal{B} \cap \limsup_{j\to\infty} K^+(F_{\varepsilon_j}) \supset \mathcal{B} \cap \limsup_{j\to\infty} X(F_{\varepsilon_j})$$

If $X(F_\varepsilon)$ is continuous at $\varepsilon = 0$, then $\mathcal{B} \cap J^*(F_0, T_\alpha) \subset K^+(F_0, T_\alpha)$, which is a contradiction.

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Aspects in Complex Hyperbolicity

Sai-Kee Yeung

Dedicated to Professor Yum-Tong Siu on the occasion of his 70th birthday

1 Introduction

Complex hyperbolicity is a notion in complex geometry which could be understood either from the point of view of value distribution of entire holomorphic curves in a complex manifold, or the point of view of existence of non-positive curved metric. The two commonly used notions are Brody hyperbolicity and Kobayashi hyperbolicity. A complex manifold M is said to be Brody hyperbolic if it does not contain the image of any non-trivial holomorphic map from $\mathbf{C}$. M is said to be Kobayashi hyperbolic if the Kobayashi metric on M is non-degenerate, cf. [44]. For simplicity, we regard a pseudo-metric as a metric in this exposition. The Kobayashi metric can be characterized as the largest among all the pseudo-distance functions δ_M on M satisfying $\delta_M(f(a), f(b)) \leq d_P(a, b)$ for all holomorphic maps $f : \Delta \to M$ and $a, b \in \Delta$, where Δ is the unit disc in $\mathbf{C}$ and d_P is the hyperbolic distance function on Δ, cf. [22]. It follows immediately that any Kobayashi hyperbolic manifold is Brody hyperbolic as well, since the image of any entire holomorphic curve on

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M would have degenerate Kobayashi semi-distance. For a compact manifolds, the two notions are equivalent, following a normal family argument as given by Brody reparametrization argument in [9]. For non-compact manifolds, there are examples of Brody hyperbolic manifolds which are not Kobayashi hyperbolic.

In recent years, interests in complex hyperbolicity have been kindled by conjectured parallelism between complex hyperbolicity and Mordellic properties in diophantine geometry, due to conjectures of Bombieri, Lang, Osgood and Vojta, cf. [55]. For a smooth projective variety V defined over a number field k, we say that V is Mordellic if the number of rational points in k is at most finite. In case that V is quasi-projective, we say that V is Mordellic if the number of integral points with respect to the infinity divisor is finite. A general conjecture of Lang [27] states that a smooth projective variety V defined over a number field k is complex hyperbolic if and only if it is Mordellic.

It is in general a difficult problem to prove that a complex manifold is complex hyperbolic, and even more so to prove Mordellic properties. The main purpose of this article is to consider some aspects of these topics through some explicit examples.

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1.1 $P^1_{\mathbf{C}} - \{0, 1, \infty\}$ Revisited

The example of $P^1_{\mathbf{C}} - \{0, 1, \infty\}$ is historically among the first interesting examples in complex hyperbolicity. Little Picard Theorem concludes that $P^1_{\mathbf{C}} - \{0, 1, \infty\}$ is hyperbolic.

To give a conceptually simple reason, we recall a handy criterion in the proof of hyperbolicity, the Schwarz Lemma of Ahlfors. Suppose M is a complex manifold equipped with a Hermitian metric h with holomorphic sectional curvature bounded from above by a negative constant. Ahlfors Schwarz Lemma states that a holomorphic map $f : \Delta \to M$ satisfies $f^*h \le c g_P$, where g_P is the Poincaré metric on Δ and c is a positive constant. An immediate consequence is that M as above is complex hyperbolic.

Little Picard Theorem can be explained conceptually from Riemann Uniformization Theorem, which states that the universal covering of $P^1_{\mathbf{C}} - \{0, 1, \infty\}$ is biholomorphic to the unit disk Δ in $\mathbf{C}$. Now the Poincaré metric on Δ has constant negative holomorphic sectional curvature -4, from which hyperbolicity follows after applying Ahlfors Schwarz Lemma.

Another observation is that $P^1_{\mathbf{C}} - \{0, 1, \infty\} = \Delta/\Gamma_2$, where Γ_2 is the second congruence subgroup of $PSL(2, \mathbf{Z})$ of level 2. As such Δ/Γ_2 is naturally a covering of $\Delta/PSL(2, \mathbf{Z})$. On the other hand it is well-known that $\Delta/PSL(2, \mathbf{Z})$ can be considered as the parameter space of the space of all elliptic curves. Hence

$P^1_{\mathbf{C}} - \{0, 1, \infty\} = \Delta/\Gamma_2$ naturally parametrizes a family of elliptic curves. This is the simplest of a moduli space which satisfies hyperbolic properties.

The example and the above two observations lead naturally to two directions which lead to a lot of developments for complex hyperbolicity.

The first one is whether the complement of a divisor of high degree in $P^n_{\mathbf{C}}$ is hyperbolic. Clearly we may also ask for similar or more refined questions for other pairs of manifolds as well. The second one is whether a natural moduli space of some appropriate complex manifolds are hyperbolic or not.

The first direction is well-motivated and has generated a lot of research activities with a vast amount of literature. Since there are already good overviews of this direction in literature such as [45], we would only remark briefly on known results in the Sect. 2, but focus on the second direction as well as some arithmetic considerations in later sections.

In the following we explain some further motivations for the second direction.

1.2 The Moduli Space of Curves $\mathscr{M}_g$ for $g \geqslant 2$

The moduli space of curves $\mathscr{M}_g$ as a topological space is the set of all equivalence classes of Riemann surfaces of genus g with the equivalent relation given by biholomorphism. It is known that $\mathscr{M}_g$ can be given the structure of a complex space with at worst orbifold singularities, or stacks. We may represent $\mathscr{M}_g$ as the quotient of a Teichmüller space $\mathscr{T}_g$ by the mapping class group Γ_g, and the Teichmüller space can be regarded as a bounded domain in $\mathbf{C}^{3g-3}$. The complex structure on $\mathscr{M}_g$ can also be understood in terms of Kodaira-Spencer theory on deformation of complex structures.

On $\mathscr{M}_g$ a natural biholomorphic invariant metric is given by the Weil-Petersson metric. Let $t \in \mathscr{M}_g$ representing a Riemann surface M_t of genus g. A holomorphic tangent vector to $\mathscr{M}_g$ at t can be identified with the Kodaira-Spencer class in $H^1(M_t, \Theta)$, where Θ is the sheaf of holomorphic tangent vector fields on M_t. Denote by $\mathscr{H}^1(M_t, \Theta)$ the set of harmonic representative in $H^1(M_t, \Theta)$. Classically these are known as harmonic Beltrami differentials. Hence a tangent vector at x is represented by $\Phi_x \in \mathscr{H}^1(M_t, \Theta)$. The Weil-Petersson metric g_{WP} is represented by

$$(\Phi_1, \Phi_2)_{WP} := \int_{M_t} \langle \Phi_1, \Phi_2 \rangle \omega_P \tag{1}$$

where $\langle \cdot, \cdot \rangle$ is the pointwise inner product with respect to the Poincaré metric g_P on M_t, and the integral is taken with respect to the volume form ω_P of the Poincaré metric. It follows from the work of Ahlfors that g_{WP} is Kähler. Furthermore, it is known from the work of Ahlfors [2, 3] and Royden [39] that the holomorphic sectional curvature of g_{WP} is negative with negative upper bound. In [58] Wolpert showed that the pointwise curvature of g_{WP} from the point of view of differential geometry can be expressed in closed form as

$$R^{(WP)}_{i\bar{j}k\bar{\ell}}(t) = 2\int_{M_t} ((\Box - 2)^{-1}\langle \Phi_i, \Phi_j\rangle) \cdot \langle \Phi_k, \Phi_\ell\rangle \omega_P \qquad (2)$$
$$+2\int_{M_t} ((\Box - 2)^{-1}\langle \Phi_k, \Phi_j\rangle) \cdot \langle \Phi_i, \Phi_\ell\rangle \omega_P.$$

Here $\Box$ is the Laplace-Beltrami operator. As a result, the holomorphic sectional curvature is bounded above by $-\frac{1}{2\pi(g-1)}$. Note that $2(g-1)$ is the degree of the canonical line bundle on M_t. We conclude that $\mathscr{M}_g$ is Kobayashi hyperbolic from Ahlfors Schwarz Lemma.

1.3 Algebraic Geometric Results in Moduli Spaces of Higher Dimensional Varieties

The results of Sect. 1.1 show that moduli spaces of smooth projective curves with ample or flat canonical line bundle are Kobayashi hyperbolic. The result has the following algebraic geometric interpretation. Suppose that $\pi : M \to P^1_{\mathbf{C}}$ is an algebraic family of generically smooth curves. Then there are at least 3 singular fibers. The reason is that $P^1_{\mathbf{C}}$ minus three or more points is hyperbolic, but $P^1_{\mathbf{C}}$ minus two or fewer points is not hyperbolic, since it contains $\mathbf{C}^*$, or image of $\mathbf{C}$ after the exponential mapping. Similarly if the base curve is an elliptic curve, there is at least one singular fiber.

From an algebraic geometric point of view, it is interesting to know if the above observation is true also for family of higher dimensional varieties of general type. In fact, results in this direction have been proved by Migliorini [32] and Kovacs [25, 26]. A typical result is that for a $P^1_{\mathbf{C}}$ family of canonically polarized projective algebraic varieties, there are at least 3 singular fibers in the family. As $P^1_{\mathbf{C}}$ minus 3 points is hyperbolic, the above result is the consequence of a result of Viehweg and Zuo [53], that the base manifold of any family of non-trivial canonically polarized projective algebraic manifolds is Brody hyperbolic. For a precise statement, we refer the readers to statement in [53].

Two questions arise naturally. The first one is whether the base manifold above is Kobayashi hyperbolic as well. As mentioned before, the notion of Kobayashi hyperbolicity is strictly stronger than Brody hyperbolicity for non-compact manifolds. The second is whether similar results hold for families of polarized Kähler Ricci flat manifolds. We would address the above the problems from Sects. 3 to 4.

2 Some Tools in the Study of Complex Hyperbolicity

The purpose of this section is to explain some examples in the first direction mentioned in the introduction of this paper. We would also explain some techniques known in the study of complex hyperbolicity which directly or indirectly motivate the discussions in the later sections.

A far reaching generalization of Little Picard Theorem is Nevanlinna theory. Nevanlinna theory provides the formalism and techniques for the study of entire holomorphic curve, which is the image of an entire holomorphic map from **C** in a complex manifold. In particular, the classical Little Picard Theorem is a consequence of the formulation that the defect of any entire holomorphic curve on $P^1_{\mathbf{C}}$ is at most $2+\epsilon$ for arbitrarily small $\epsilon > 0$, which itself is consequence of the Second Main Theorem of Nevanlinna.

A direct generalization of Little Picard Theorem to higher dimensions is the question of complex hyperbolicity or its analogs for $P^n_{\mathbf{C}} - D$, where D is the union of a finite number of hyperplanes in general positions. Among many interesting results, we just mention a few below. For statements in value distribution of entire holomorphic curves, there is the result of H. Cartan on Truncated Second Main Theorem [10], a direct generalization of the result of Nevanlinna to higher dimensional projective spaces. There is also the result of Ahlfors [1], who introduced the notion of associated curves and studied their defects. In an inhomogeneous representation of an entire holomorphic curve $f : \mathbf{C} \to \mathbf{P}^{\mathbf{n}}_{\mathbf{C}}$, the associated curve can be considered as a wedge product of the mapping f and its successive derivatives, $f \wedge f' \wedge \cdots f^{(k)}$, regarded as a map from **C** taking values in the Grassmanian resulted. Ahlfors obtained Second Main Theorem and defect relations for the associated maps iteratively. Hyperbolicity of complement of $2n+1$ hyperplanes in generic positions in $P^n_{\mathbf{C}}$ is proved by Greens. Replacing $2n+1$ hyperplanes by $2n+1$ hypersurfaces, the result is also known due to Ru and other people, cf. [40] and the references there. for results in this direction.

The more difficult situation is the question of hyperbolicity of $P^n_{\mathbf{C}} - D$ for a generic divisor D of high degree. For a generic D, defined by a polynomial of degree d, $G(X_1, \cdots X_{n+1}) = 0$ in $P^n_{\mathbf{C}}$, we may consider the branch cover M of $P^n_{\mathbf{C}} - D$ defined by

$$T^d = G(X_1, \cdots X_{n+1}) \tag{3}$$

in $P^{n+1}_{\mathbf{C}}$. In this way, a hyperbolicity problem on $P^n_{\mathbf{C}} - D$ is reduced to the corresponding problem on a hypersurface M on $P^{n+1}_{\mathbf{C}}$. The statement can be made precise. In particular, Kobayashi conjectured that $P^n_{\mathbf{C}} - D$ is hyperbolic if $\deg D \geqslant 2n+1$, similarly for a generic hypersurface $D \subset P^{n+1}$ of degree at least $2(n+1)$. Analogous results for a general manifold have been conjectured by Lang and Vojta, cf. [27, 55]. The precise degree is expected to be dictated by the geometry of the manifold involved, such as the degree of the canonical class.

The Kobayashi Conjecture as mentioned has led to a lot of research activities, though still not solved. In [48], Siu and myself showed that the conjecture is true for $n = 2$ if the degree of D is very large. The degree was lowered greatly by the work of McQuillen [28, 29] and Demailly-Elgoul [14]. In higher dimensions, a breakthrough comes from [46], see also [45], where Siu introduced the method of slanted vector fields on moduli of hypersurfaces and showed that the statements of the conjectures of Kobayashi were true if the degree of D is sufficiently large. Reasonable effective bounds of the method in dimensions 2 and 3 have been given as in Paun [37] and Rousseau [38]. For the analogous problem of algebraic degeneracy of the image of entire holomorphic curves, results with bounds on the degree of D has been obtained by Diverio-Merker-Rousseau [16], see also [30].

As used in [48], there are in general the following steps in proving complex hyperbolicity. The first step is the construction of some non-trivial jet differentials ω vanishing on some ample divisor on M. Once this is available, a suitable Schwarz Lemma for holomorphic jet differentials implies that the image of an entire holomorphic curve $f : \mathbf{C} \to \mathbf{M}$ satisfies $f^*\omega = 0$. This means that the image of f is confined in the sense its jets satisfy a differential equation coming from ω. The second step involves further restriction of the image, by either repeating the construction of sections on the restriction of the jet bundles on the Zariski closure of the image of f in the jet space, or by showing that there are a lot of freedom in the choice of the jet differentials ω so that their common vanishing set could be shown to be small when projected down to the manifold M. For the first step, the usual method is by Riemann-Roch together with estimates of higher cohomology groups as given in [12, 21, 45], or by explicit Siegel Lemma type argument as in [48], Section 2, and [45]. See the results of [13, 31] and the references there for more recent works. For the second step, a direct computation using Riemann-Roch type theorem on the image of jets of entire holomorphic curve works only in special situation or low dimensions. The method of slanted vector fields introduced in [44] is general but the degree involved is still quite large at this stage. The latter approach is parallel to the restrictions of rational curves on a very general hypersurface of large degree in projective spaces as studied by Clemens [11], Ein [17] and Voisin [54].

The Schwarz Lemma for holomorphic jet differentials was proved for special two jet in dimension 2 in [48], here the word special means that the jet differential has invariant form under reparametrization, and is generalized to all situations in [12, 50] and [45].

As mentioned earlier, the formulation of complex hyperbolicity of a complex manifold M in terms of geometry either in the form of M or a pair (M, D) has been generalized by Lang and Vojta, cf. [27, 55]. Apart from $P^n_{\mathbf{C}}$, one may for example ask the same question for ample divisors M in Abelian varieties A and the complement of an ample divisor D in an Abelian variety A. For the case of $M \subset A$, the results of Bloch essentially implies that the Zariski closure of an entire holomorphic curve in M is a translation of some sub-Abelian variety, cf. [7, 45]. The modern use of jet differentials as well as some analogues of Schwarz Lemma can be traced to the work of [7]. The case of $A \backslash D$ has been a conjecture of Lang and was settled in [49, 50]. The corresponding result for semi-Abelian varieties were discussed in

[36]. The situation of Abelian varieties has an analogue in arithmetics, which would be discussed in Sect. 5.

We mention that the use of generalized Weil-Petersson metric on iterated Kodaira-Spencer class in [51] to be explained in Sect. 3 is motivated by this version of Schwarz lemma. The formulation of telescoping estimates of the curvature expressions mentioned in Sect. 3.3 is motivated by the formulation of Ahlfors for associated curves as mentioned above.

3 Moduli of Canonically Polarized Manifolds

In the following two sections, we would explain results in the second direction mentioned in the Sect. 1.1. From a differential geometric point of view, it would be desirable to generalize the computation of curvature for a family of Riemann surfaces of fixed genus as given in Sect. 1.2 to a family of higher dimensional manifolds, from which complex hyperbolicity would follow naturally. We consider a holomorphic family $\pi : \mathscr{X} \to S$ of compact canonically polarized complex manifolds over a complex manifold S. By this we assume that $\pi : \mathscr{X} \to S$ is a surjective holomorphic map of maximal rank between two complex manifolds $\mathscr{X}$ and S, and each fiber $M_t := \pi^{-1}(t)$, $t \in S$, is a compact complex manifold such that K_{M_t} is ample. From results of Aubin [5] and Yau [59], every compact complex manifold with ample canonical line bundle admits a Kähler-Einstein metric of negative Ricci curvature, which is unique up to a positive multiplicative constant. Hence on each M_t, the Ricci curvature tensor of the Kähler-Einstein metric g satisfies $R_{\alpha\bar{\beta}}(t) = kg_{\alpha\bar{\beta}}(t)$ for some constant $k < 0$.

The formulation of the problem and the first breakthrough are given by the paper of Siu in [44].

3.1 Curvature Formula of Siu for the Weil-Petersson Metric

For a holomorphic family $\pi : \mathscr{X} \to S$ of complex manifolds, holomorphic tangent vectors at $t \in S$ are represented by Kodaira-Spencer map $\rho_t : T_tS \to H^1(M_t, TM_t)$. We say that π is effectively parametrized if each ρ_t is injective. One can easily define Weil-Petersson metric with respect to the Kähler metric in the same way as in the case of one dimensional fibers. The curvature formula for higher dimensional fibers is however complicated and the sign is difficult to determine.

Here are some details. Consider an effectively parametrized family $\pi : \mathscr{X} \to S$ of canonically polarized manifolds. For $t \in S$ and a local tangent vector field u (of type $(1, 0)$) on an open subset U of S, there is a unique lifting of u such that $\Phi(u(t))$ is the harmonic representative of Kodaira-Spencer class $\rho_t(u(t))$ for each $t \in S$, which is called the canonical lifting or horizontal lifting of u, cf. [41, 43] When $u = \partial/\partial t^i$ is a coordinate vector field, we will simply denote its canonical lifting by $v_i := v_{\partial/\partial t^i}$ and the associated harmonic Kodaira-Spencer representative by $\Phi_i := \Phi(\partial/\partial t^i)$. The Weil-Petersson metric $h^{(WP)} = \sum_{i,j=1}^n h^{(WP)}_{i\bar{j}} dt^i \otimes d\bar{t}^j$ on S is defined as in Eq. (1) by

$$h^{(WP)}_{i\bar{j}}(t) := \int_{M_t} \langle \Phi_i, \Phi_j \rangle \frac{\omega^n}{n!}, \tag{4}$$

where $\langle \Phi_i, \Phi_j \rangle := (\Phi_i)^{\gamma}_{\bar{\alpha}} \overline{(\Phi_j)^{\delta}_{\bar{\beta}}} g_{\gamma\bar{\delta}} g^{\bar{\alpha}\beta}$ denotes the pointwise Hermitian inner product on tensors.

It follows from Koiso's result [24] that $h^{(WP)}$ is Kähler. Let $R^{(WP)}$ denote the curvature tensor. By [43], p. 296, the components of the curvature tensor $R^{(WP)}$ of $h^{(WP)}$ with respect to normal coordinates (of $h^{(WP)}$) at a point $t \in S$ are given by

$$\begin{aligned} R^{(WP)}_{i\bar{j}k\bar{\ell}}(t) = k &\int_{M_t} ((\Box - k)^{-1} \langle \Phi_i, \Phi_j \rangle) \cdot \langle \Phi_k, \Phi_\ell \rangle \frac{\omega^n}{n!} \\ &+ k \int_{M_t} ((\Box - k)^{-1} \langle \Phi_k, \Phi_j \rangle) \cdot \langle \Phi_i, \Phi_\ell \rangle \frac{\omega^n}{n!} \\ &+ k \int_{M_t} \langle (\Box - k)^{-1} \mathscr{L}_{v_i} \Phi_k, \mathscr{L}_{v_j} \Phi_\ell \rangle \frac{\omega^n}{n!} \\ &+ \int_{M_t} \langle H(\Phi_i \owedge \Phi_k), H(\Phi_j \owedge \Phi_\ell) \rangle \frac{\omega^n}{n!}. \end{aligned} \tag{5}$$

Here by normal coordinates of $h^{(WP)}$ at the point $t \in S$, we mean $h^{(WP)}_{i\bar{j}}(t) = \delta_{ij}$, and $\partial_k h^{(WP)}_{i\bar{j}}(t) = \partial_{\bar{k}} h^{(WP)}_{i\bar{j}}(t) = 0$. The notation $\mathscr{L}_v \Phi$ denotes the Lie derivative of Φ with respect to the vector field v and $H(\Phi_i \owedge \Phi_k)$ denotes the harmonic projection as a bundle-valued form of the wedge product of Φ_i and Φ_k in both the form and tangent vector directions.

The pointwise computation of the curvature formula in (5) is a beautiful formula on which all later curvature computations of Weil-Petersson type metrics built on. The deduction comes from clever grouping of terms and involved loops of integration by parts guided from geometric intuitions.

The holomorphic sectional curvature corresponds to components of form $R^{(WP)}_{i\bar{i}i\bar{i}}$. The first two terms on the right hand side of (5) are negative from our assumption of effective parametrization and the third one is semi-negative. The problem is on control of the fourth term which is semi-positive. Hence beautiful as it is, the formula (5) is not sufficient to deduce hyperbolicity properties of the moduli space except under very restrictive situations corresponding to the vanishing of the fourth term.

For a long time, people have been trying to dominate the fourth term by the first three terms. This seems to be not possible in general (cf. the remark in [52]). In the next subsection, we will introduce the method of [51] to handle the difficulty. At this point, we mention that the approach of (5) has been applied to the case of families of polarized Kähler Ricci flat manifolds in Nannicini [35]. The results of (5) has also been formulated in a sometimes more efficient way in [41]. The work of [51] follows more closely the formulation in [43], but also makes use of some simplifications in [41]. In Sect. 4, we will present the results on hyperbolicity for family of Kähler-Ricci flat manifolds.

3.2 Generalized Weil-Petersson Metric and Curvature Formula

In the next few subsections, we will summarize the results in [51], whose goal is to provide a Finsler metric with holomorphic sectional curvature bounded from above by a negative constant so that the space is Kobayashi hyperbolic. Results in this section is the first step, which is a generalization of the formula of Siu in (5).

We fix a coordinate open subset $U \subset S$ with coordinate functions $t = (t^1, \ldots, t^m)$ such that the origin $t = 0$ lies in U. For each $t \in S$ and each coordinate tangent vector $\frac{\partial}{\partial t^i}$, we recall the horizontal lifting v_i and the harmonic representative Φ_i of $\rho_t(\frac{\partial}{\partial t^i})$ on M_t as given earlier. Fix an integer ℓ satisfying $1 \leqslant \ell \leqslant n$, and let $J = (j_1, \ldots, j_\ell)$ be an ℓ-tuple of integers satisfying $1 \leqslant j_d \leqslant m$ for each $1 \leqslant d \leqslant \ell$. We denote by

$$\Psi_J := H(\Phi_{j_1} \owedge \cdots \owedge \Phi_{j_\ell}) \in \mathscr{A}^{0,\ell}(\wedge^\ell TM_t) \tag{6}$$

the harmonic projection of $\Phi_{j_1} \owedge \cdots \owedge \Phi_{j_\ell}$. As t varies, we still denote the resulting family of tensors by Ψ_J (suppressing its dependence on t), when no confusion arises.

Observe that from definition, the expression $(\Phi_{j_1} \owedge \cdots \owedge \Phi_{j_\ell}) \in \mathscr{A}^{0,\ell}(\wedge^\ell TM_t)$ is symmetric in $j_1, \ldots, j_\ell$. Hence after composing with the Kodaira-Spencer map, we may define a Hermitian metric on $S^\ell(T_S)$ with norm given by

$$\|v_{i_1} \cdots v_{i_\ell}\|_2^2 = \int_{M_t} \langle v_{i_1} \cdots v_{i_\ell}, v_{i_1} \cdots v_{i_\ell} \rangle \frac{\omega^n}{n!}. \tag{7}$$

We call such an expression a generalized Weil-Petersson metric. To compute the curvature, we need to study $\partial_i \|\Psi_J\|_2^2$. The following proposition is a direct generalization of the identity in (5).

Proposition 1 *We have*

$$\begin{aligned} &\partial_i \overline{\partial_i} \log \|\Psi_J\|_2^2 \qquad (8)\\ &= \frac{1}{\|\Psi_J\|_2^2}\big(- k((\Box - k)^{-1}(\overline{\Phi_i} \cdot \Psi_J), \overline{\Phi_i} \cdot \Psi_J) - k((\Box - k)^{-1}\langle \Phi_i, \Phi_i \rangle, \langle \Psi_J, \Psi_J \rangle) \\ &\qquad -k((\Box - k)^{-1}(\mathscr{L}_{v_i}\Psi_J), \mathscr{L}_{v_i}\Psi_J) - \big|(\mathscr{L}_{v_i}\Psi_J, \frac{\Psi_J}{\|\Psi_J\|_2})\big|^2 \\ &\qquad -(H(\Phi_i \owedge \Psi_J), H(\Phi_i \owedge \Psi_J))\big). \end{aligned}$$

In the remaining part of this subsection, we give some ideas for the proof of Proposition 1. The expression that we need to compute is given by

$$\begin{aligned}\partial_i\overline{\partial_i}\log\|\Psi_J\|_2^2 &= \partial_i\Big(\frac{\partial_{\bar{i}}\|\Psi_J\|_2^2}{\|\Psi_J\|_2^2}\Big)\\ &= \frac{\partial_i\partial_{\bar{i}}\|\Psi_J\|_2^2}{\|\Psi_J\|_2^2} - \frac{(\partial_i\|\Psi_J\|_2^2)(\partial_{\bar{i}}\|\Psi_J\|_2^2)}{\|\Psi_J\|_2^4}. \qquad (9)\end{aligned}$$

For this purpose, we observe that

$$\begin{aligned}\partial_i\|\Psi_J\|_2^2 &= \frac{\partial}{\partial t^i}\int_{M_t}\langle\Psi_J,\Psi_J\rangle\frac{\omega^n}{n!}\\ &= \int_{M_t}\langle\mathscr{L}_{v_i}\Psi_J,\Psi_J\rangle\frac{\omega^n}{n!} + \int_{M_t}\langle\Psi_J,\mathscr{L}_{\overline{v_i}}\Psi_J\rangle\frac{\omega^n}{n!}\\ &= \int_{M_t}\langle\mathscr{L}_{v_i}\Psi_J,\Psi_J\rangle\frac{\omega^n}{n!},\end{aligned}$$

where in the last step we have used the fact that Ψ_J is harmonic and that $(\mathscr{L}_{\overline{v_i}}\Psi_J)^{(0,\ell)}_{(\ell,0)}$ is $\overline{\partial}$-exact (cf. [51], Lemma 3), so that

$$\int_{M_t}\langle\Psi_J,\mathscr{L}_{\overline{v_i}}\Psi_J\rangle\frac{\omega^n}{n!} = 0 \qquad (10)$$

Differentiating the complex conjugate of above expression, we get

$$\begin{aligned}0 &= \frac{\partial}{\partial t^i}\int_{M_t}\langle\mathscr{L}_{\overline{v_i}}\Psi_J,\Psi_J\rangle\frac{\omega^n}{n!}\\ &= \int_{M_t}\langle\mathscr{L}_{v_i}\mathscr{L}_{\overline{v_i}}\Psi_J,\Psi_J\rangle\frac{\omega^n}{n!} + \int_{M_t}\langle\mathscr{L}_{\overline{v_i}}\Psi_J,\mathscr{L}_{\overline{v_i}}\Psi_J\rangle\frac{\omega^n}{n!}. \qquad (11)\end{aligned}$$

We obtain

$$\begin{aligned}\partial_i\partial_{\bar{i}}\|\Psi_J\|_2^2 = \partial_{\bar{i}}\partial_i\|\Psi_J\|_2^2 &= \frac{\partial}{\partial\bar{t}^i}\int_{M_t}\langle\mathscr{L}_{v_i}\Psi_J,\Psi_J\rangle\frac{\omega^n}{n!}\\ &= \int_{M_t}\langle\mathscr{L}_{\overline{v_i}}\mathscr{L}_{v_i}\Psi_J,\Psi_J\rangle\frac{\omega^n}{n!} + \int_{M_t}\langle\mathscr{L}_{v_i}\Psi_J,\mathscr{L}_{v_i}\Psi_J\rangle\frac{\omega^n}{n!}.\\ &= I + II + III, \qquad (12)\end{aligned}$$

where

$$I := -\int_{M_t} \langle \mathscr{L}_{\overline{v_i}} \Psi_J, \mathscr{L}_{\overline{v_i}} \Psi_J \rangle \frac{\omega^n}{n!}, \tag{13}$$

$$II := \int_{M_t} \langle \mathscr{L}_{[\overline{v_i}, v_i]} \Psi_J, \Psi_J \rangle \frac{\omega^n}{n!} = (\mathscr{L}_{[\overline{v_i}, v_i]} \Psi_J, \Psi_J),$$

$$III := \int_{M_t} \langle \mathscr{L}_{v_i} \Psi_J, \mathscr{L}_{v_i} \Psi_J \rangle \frac{\omega^n}{n!} = (\mathscr{L}_{v_i} \Psi_J, \mathscr{L}_{v_i} \Psi_J).$$

after applying the identity $\mathscr{L}_{\overline{v_i}} \mathscr{L}_{v_i} = \mathscr{L}_{v_i} \mathscr{L}_{\overline{v_i}} + \mathscr{L}_{[\overline{v_i}, v_i]}$. It remains to compute the expressions I, II and III. After some careful manipulation of the terms similar to [43], we find that

$$\begin{aligned} I = & -k((\Box - k)^{-1}(\overline{\Phi_i} \cdot \Psi_J), \overline{\Phi_i} \cdot \Psi_J) - (\overline{\Phi_i} \cdot \Psi_J, \overline{\Phi_i} \cdot \Psi_J) \\ & + (\overline{\Phi_i} \searrow \Psi_J, \overline{\Phi_i} \searrow \Psi_J) + (\overline{\Phi_i} \nearrow \Psi_J, \overline{\Phi_i} \nearrow \Psi_J), \\ II = & -(\langle \Phi_i, \Phi_i \rangle, \langle \Psi_J, \Psi_J \rangle) - k((\Box - k)^{-1} \langle \Phi_i, \Phi_i \rangle, \langle \Psi_J, \Psi_J \rangle), \\ III = & (\Phi_i \owedge \Psi_J, \Phi_i \owedge \Psi_J) - (H_1(\Phi_i \owedge \Psi_J), H_1(\Phi_i \owedge \Psi_J)) \\ & -k((\Box - k)^{-1}(\mathscr{L}_{v_i} \Psi_J) \mathscr{L}_{v_i} \Psi_J), \end{aligned} \tag{14}$$

where $\overline{\Phi_i} \searrow \Psi_J \in \mathscr{A}^{1,\ell-1}(\wedge^\ell TM_t)$ and $\overline{\Phi_i} \nearrow \Psi_J \in \mathscr{A}^{0,\ell}(\wedge^{\ell-1} TM_t \wedge \overline{TM_t})$ are defined by

$$(\overline{\Phi_i} \searrow \Psi_J)^{\alpha_1 \cdots \alpha_\ell}_{\overline{\delta \beta_1} \cdots \overline{\beta_{\ell-1}}} := \overline{(\Phi_i)^{\sigma}_{\overline{\delta}}} (\Psi_J)^{\alpha_1 \cdots \alpha_\ell}_{\overline{\sigma \beta_1} \cdots \overline{\beta_{\ell-1}}} \quad \text{and} \tag{15}$$

$$(\overline{\Phi_i} \nearrow \Psi_J)^{\alpha_1 \cdots \alpha_{\ell-1} \overline{\gamma}}_{\overline{\beta_1} \cdots \overline{\beta_\ell}} := \overline{(\Phi_i)^{\gamma}_{\overline{\sigma}}} (\Psi_J)^{\alpha_1 \cdots \alpha_{\ell-1} \sigma}_{\overline{\beta_1} \cdots \overline{\beta_\ell}}$$

respectively. Proposition 1 follows by putting the above information together.

We remark that the above calculations follows closely the one in [43], where the case of $\ell = 1$ was treated. Note that the fourth term on the right hand side is controlled by the third term there from spectral decomposition. Hence the first four terms gives rise to a non-negative sign, but the fourth one is of non-positive sign and is the one to be controlled.

After the completion of the paper [51], we noticed that an analogous formula in dual formulation had been obtained independently by [42]. Here by dual formulation, we refer to computation of curvature for the dual bundle in the sense of Kodaira-Serre, cf. [23].

3.3 A Telescopic Formulation

The second step in the construction of the Finsler metric in [51] is to formulate estimates in identity (8) in a way that we may apply a telescopic argument.

Fix $v_i \in TS$ and let Φ_i the corresponding harmonic representative in the Kodaira-Spencer class. For a positive integer ℓ, we define the relative tensor

$$H^{(\ell)} := H(\underbrace{\Phi_i \otimes \cdots \otimes \Phi_i}_{\ell\text{–times}}), \tag{16}$$

so that $H^{(\ell)} = \Psi_J$ with J given by the ℓ-tuple $(i, i, \cdots, i)$, here $H(\cdot)$ refers to the projection to the harmonic component. The second main step of our argument is the following.

Proposition 2 *Suppose* $\|H^{(\ell)}\|_2 > 0$ *that* $\|H^{(\ell-1)}\|_2 > 0$. *Then we have*

$$\partial_i \overline{\partial_i} \log \|H^{(\ell)}\|_2^2 \geq \frac{\|H^{(\ell)}\|_2^2}{\|H^{(\ell-1)}\|_2^2} - \frac{\|H^{(\ell+1)}\|_2^2}{\|H^{(\ell)}\|_2^2}. \tag{17}$$

Here is the outline of proof. From Proposition 1 and the remark there, we conclude that

$$\begin{aligned} \partial_i \overline{\partial_i} \log \|H^{(\ell)}\|_2^2 \geqslant \frac{1}{\|H^{(\ell)}\|_2^2} \big(& - k((\Box - k)^{-1}(\overline{\Phi_i} \cdot H^{(\ell)}), \overline{\Phi_i} \cdot H^{(\ell)}) \\ & - k((\Box - k)^{-1}\langle \Phi_i, \Phi_i \rangle, \langle H^{(\ell)}, H^{(\ell)} \rangle) \\ & - (H(\Phi_i \otimes H^{(\ell)}), H(\Phi_i \otimes H^{(\ell)})) \big). \end{aligned} \tag{18}$$

The key point of our argument in this step is to neglect the second term on the right hand side, and observe that numerator of the first term satisfies

$$\begin{aligned} (-k(\Box - k)^{-1}(\overline{\Phi_i} \cdot H^{(\ell)}), \overline{\Phi_i} \cdot H^{(\ell)}) &\geqslant (H(\overline{\Phi_i} \cdot H^{(\ell)}), \overline{\Phi_i} \cdot H^{(\ell)}) \\ &\geq \Big| \Big(\overline{\Phi_i} \cdot H^{(\ell)}, \frac{H^{(\ell-1)}}{\|H^{(\ell-1)}\|_2} \Big) \Big|^2 \\ &= \frac{\|H^{(\ell)}\|_2^4}{\|H^{(\ell-1)}\|_2^2}. \end{aligned}$$

In the above the first inequality follows from spectral decomposition. The key observation is the second identity which follows from linear algebra. The proposition follows directly from combining the above estimates.

3.4 Construction of the Finsler Metric

Given the above Proposition, we may hope to absorb the bad of term of the right hand side of estimates (17) for ℓ by the good term for $\ell+1$. Observe that the bad term $\frac{\|H^{(\ell+1)}\|_2^2}{\|H^{(\ell)}\|_2^2} = 0$ for $\ell = n$, since $H^{n+1}(M_t, \wedge^{n+1}\Theta) = 0$. However, the term may vanish for some $\ell < n$ and the greatest such ℓ may be different for different base point t. As a result, the search for Finsler metric with negative upper bound in curvature is rather challenging. This is the third step of the proof in [51]. Our result is as follows.

Let $N \geq n$ be a fixed positive integer. Let

$$A := \frac{(2\pi)^n K_{M_t}^n}{k^n n!}. \tag{19}$$

Let $C_1 := \min\left\{1, \frac{1}{A}\right\}$ and $C_\ell = \frac{C_{\ell-1}}{3} = \frac{C_1}{3^{\ell-1}}$ for $2 \leq \ell \leq n$. Let $a_1 = 1$ and $a_\ell = \left(\frac{3a_{\ell-1}}{C_1}\right)^N = \left(\frac{3}{C_1}\right)^{\frac{N(N^{\ell-1}-1)}{N-1}}$ for $2 \leq \ell \leq n$. Define for $u \in T_tS$ and $t \in S$ a function $h : TS \to \mathbb{R}$ given by

$$h(u) = \Big(\sum_{\ell=1}^{n} a_\ell \|u\|_{WP,\ell}^{2N}\Big)^{\frac{1}{2N}} \tag{20}$$

Then

$$\partial_t \partial_{\bar{t}} \log((h(\frac{\partial}{\partial t}))^2) \geq \frac{C_n}{n^{\frac{1}{N}} a_n^{1+\frac{1}{N}}} \cdot (h(\frac{\partial}{\partial t}))^2. \tag{21}$$

This implies that the holomorphic sectional curvature is bounded by a negative constant. Hence we conclude the following theorem after applying Ahlfors Schwarz Lemma.

Theorem 1 *Let $\pi : \mathscr{X} \to S$ be an effectively parametrized holomorphic family of compact canonically polarized complex manifolds over a complex manifold S. Then S admits a C^∞ $Aut(\pi)$-invariant Finsler metric whose holomorphic sectional curvature is bounded above by a negative constant. Hence S is Kobayashi hyperbolic.*

Here we say that a Finsler metric h on S is *Aut*(π)*-invariant* if $f^*h = h$ for any pair of automorphisms $(F, f) \in \mathrm{Aut}(\mathscr{X}) \times \mathrm{Aut}(\mathscr{S})$ satisfying $f \circ \pi = \pi \circ F$.

We remark that the upper bound of the holomorphic sectional curvature in (21) depends only on the degree $K_{M_t}^n$ of the fibers. In complex dimension one, the result is essentially the same as (2), the formula of Wolpert for moduli space of Riemann surfaces.

4 Moduli of Polarized Kähler Ricci-Flat Manifolds

Recall that in complex dimension 1, the space $P^1_{\mathbf{C}} - \{0, 1, \infty\}$ parametrizes a family of elliptic curves, and the base space is Kobayashi hyperbolic. Elliptic curves have trivial canonical line bundle. A higher dimensional analogue of the result is to consider the same problem for a holomorphic family of polarized Kähler Ricci-flat manifolds. Hence a natural question is whether such a family is Kobayashi hyperbolic or not. In particular, one asks if it is possible to study such problems from the point of view of Weil-Petersson metric. The question is answered in [52]

Theorem 2 *Let $\pi : \mathscr{X} \to S$ be an effectively parametrized holomorphic family of compact polarized Ricci-flat Kähler manifolds over a complex manifold S. Then S admits a C^∞ $Aut(\pi)$-invariant Finsler metric whose holomorphic sectional curvature is bounded above by a negative constant coming from generalized Weil-Petersson metrics. As a consequence, S is Kobayashi hyperbolic.*

A holomorphic family of compact complex manifolds $\pi : \mathscr{X} \to S$ over a complex manifold S is said to be a family of polarized Ricci-flat Kähler manifold if it satisfies the following properties. The mapping $\pi : \mathscr{X} \to S$ is a surjective holomorphic map of maximal rank between two complex manifolds $\mathscr{X}$ and S, and each fiber (M_t, ω_t) is a Ricci-flat Kähler manifold polarized by ω_t, where $M_t := \pi^{-1}(t)$, $t \in S$. Moreover, we require that the cohomology class $[\phi_t^* \omega_t] \in H^2(M_0, \mathbf{C})$ is a constant class for all t, where $\phi_t : M_0 \to M_t$ is the restriction of ϕ to $M_0 \times \{t\}$ for a smooth trivialization $\phi : M_0 \times I \to \mathscr{X}$.

Historically, a result analogous to the work of [43] for a family of polarized Kähler Ricci-flat manifolds was obtained by Nannicini [35] as follows.

$$R^{(WP)}_{\bar{i}jk\bar{\ell}}(t) = -\frac{1}{4V}(h_{\bar{i}j}h_{\bar{l}k} + h_{\bar{i}k}h_{\bar{l}j}) - \int_{M_t} \langle \mathscr{L}_{v_i}\Phi_k, \mathscr{L}_{v_j}\Phi_\ell\rangle \frac{\omega^n}{n!}$$
$$+ \int_{M_t} \langle H(\Phi_i \owedge \Phi_k), H(\Phi_j \owedge \Phi_\ell)\rangle \frac{\omega^n}{n!},$$

here V is the volume of M_o.

Modifying the argument of the last section, Wing-Keung To and myself obtained in [52] first the following generalization to the higher dimensional cases.

$$\begin{aligned}
&\partial_i\overline{\partial_i} \log \|\Psi_J\|_2^2 \\
= &\frac{1}{\|\Psi_J\|_2^2}\Big(H(\overline{\Phi_i} \cdot \Psi_J), \overline{\Phi_i} \cdot \Psi_J) + (H(\langle \Phi_i, \Phi_i\rangle), \langle \Psi_J, \Psi_J\rangle) \\
&+((H(\mathscr{L}_{v_i}\Psi_J), \mathscr{L}_{v_i}\Psi_J) - \Big|(\mathscr{L}_{v_i}\Psi_J, \frac{\Psi_J}{\|\Psi_J\|_2})\Big|^2 - (H(\Phi_i \owedge \Psi_J), H(\Phi_i \owedge \Psi_J))\Big).
\end{aligned}$$

The rest of the argument is then a modification of the arguments in Sect. 3. In particular, in place of A chosen in (19), we define

$$A := \frac{(2\pi)^n \omega_t^n}{k^n n!} = \frac{(2\pi)^n \omega_0^n}{k^n n!}. \tag{22}$$

Similar to the case of a family of canonically polarized manifolds, the upper bound on the holomorphic sectional curvature depends only on A.

5 Some Higher Dimensional Examples of Varieties with Finite Number of Rational Points

In this final section, we remark on a few observations related to the arithmetic aspects of complex hyperbolic manifolds. A basic conjecture of Lang states that a smooth projective algebraic manifold defined over a number field k is complex hyperbolic if and only if it has a at most a finite number of rational points over k, and a similar statement for integral points with respect to the divisor given by a compactifying divisor, cf. [27].

The conjecture in complex dimension one for hyperbolic compact Riemann surfaces is verified by the solution of Mordell Conjecture by Faltings [18]. The quasi-projective case in complex dimension one is known earlier in the results of Siegel. An alternative proof of the Mordell Conjecture is given by Vojta [56]. The problem in higher dimensions is wide open. An interesting class of examples known in higher dimension is the result of Faltings [19, 20] on subvarieties X of Abelian varieties A defined over a number field k, which states that the Zariski closure of the set of rational points on X is a the translate of an Abelian subvariety in A. A similar statement for integral points on the complement of an ample divisor is proved in the papers as well. Note that the corresponding results for complex hyperbolicity were proved by Bloch [7] and Siu-Yeung [48] mentioned in Sect. 3. The arithmetic results on semi-abelian varieties was given by Vojta [57].

From Riemann Uniformization Theorem, the universal covering of a compact hyperbolic Riemann surface is just the complex ball of dimension one, $B^1_{\mathbb{C}} \cong \Delta^1$, and similarly for a non-compact Riemann surface of finite volume. From differential geometric point of view, the simplest complex hyperbolic manifolds are provided by complex hyperbolic spaces $B^n_{\mathbb{C}}/\Gamma$ for some discrete group Γ. The hyperbolicity follows from Ahlfors Schwarz Lemma and the existence of a Kähler metric with negative Riemannian sectional curvature. The only other compact complex manifolds known to possess a Kähler metric of negative Riemannian sectional curvature are the examples known as Mostow-Siu surfaces, see [34] and [15]. In the following we describe a few examples studied in [60] for which results in complex geometry and the results of Faltings above allow us to deduce Mordellic properties. We will only consider complex dimension two. First we make the following observation in [60].

Proposition 3 *Let M be a smooth projective algebraic surface defined over a number field F. Assume that there exists an unramified covering $M' \to M$ defined over some number field F' so that the irregularity $q(M') = \dim H^1(M', \mathcal{O}_{M'})$ is at least 3 and that there is no non-constant morphism from a curve of genus 0 or 1 into M, then $M(F)$ has finite cardinality.*

The idea of the proof is to relate rational points $M(F)$ on M to rational points $M'(F')$ for some finite extension F' of F by classical results of Hermite and Chevalley-Weil. Then one considers the Albanese map on M' and apply the results of Faltings in [19] mentioned above.

We have the following immediately corollary, for which conditions in the proposition above can be verified.

Corollary 1 *The number of rational points on a smooth Picard Modular Surface defined over a number field is finite. Similarly the result holds for the number of rational points on a Mostow-Siu surface defined over a number field.*

Proposition 3 shows that Mordellic properties would follow from virtual positivity of the first Betti number for surfaces, at least for complex two ball quotients. In this way, the problem is related to the following problem in cohomology of Lie groups and geometric topology. It has been conjectured by Borel [8], parallel to a corresponding conjecture of Thurston for real hyperbolic spaces, that the first Betti number of of a complex ball quotient is virtually positive. Recall that a property is virtually true on a manifold if it holds after passing to a finite unramified covering if necessary. Hence Mordellic properties of complex two ball quotients can be established if the first Betti number is shown to be virtually at least 5. The conjecture of Borel is open for a general compact complex ball quotient at this point. It is proved in [60] that the conjecture is true for a non-compact complex ball quotient of finite volume.

Theorem 3 *Let $M \cong B^2_{\mathbb{C}}/\Gamma$ be a smooth cofinite complex two ball quotient. Then given any $N > 0$, there exists a finite unramified covering of M' with $b^1(M') \geqslant N$. In particular, M has at most a finite number of integral points with respect to some compactifying divisor.*

The idea of proof is to observe that the Betti number of such a complex two ball quotient increases with the number of cusps, which increases when one goes to some unramified coverings. In the case of an arithmetic quotient of a complex two ball, a compactification can be given by Baily-Borel compactification [6] which adds a point to a cusp and is singular, or by toroidal compactification developed by Ash-Mumford-Rapoport-Tai [4] which adds a torus to an end and is smooth. In the case of non-arithmetic quotients, a differential geometric construction to each of the above two cases has been developed by Siu-Yau [47] and Mok [33]. To find some non-trivial class in H_1, we consider the structure near the toroidal compactification of an end, and in a sense show that some 1-cycle from the compactifying torus lifts to M.

At this point, the situation for other complex ball quotients is still not completely understood.

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A Survey on L^2 Extension Problem

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1 Introduction

It's natural to explain the L^2 extension problem using the Oka-Cartan's global theory on Stein manifolds including Cartan's theorem A and theorem B and Cartan extension theorem, and Hörmander's L^2 method for solving $\bar{\partial}-$equation.

Multiplier ideal sheaf could be regarded as a mixture of sheaf cohomology method and L^2 method. It's used to give a unify treatment of Kodaira embedding theory in complex geometry and some fundamental problems in several complex variables such as Cartan extension theorem, Levi problem, Cousin problems I and II et al.

Given a coherent analytic sheaf on a Stein manifold, Cartan's theorem A asserts that the stalk of the sheaf is generated by the global section of the sheaf; while Cartan's theorem B asserts that degree ≥ 1 cohomology groups of the Stein manifold with value in the sheaf vanish.

Cartan extension theorem says that given a closed complex subvariety S in the Stein manifold M, then any holomorphic function f on the complex subvariety S (or holomorphic section of a holomorphic vector bundle restricting on S) can be holomorphically extended to a holomorphic function F (or holomorphic section) on the Stein manifold M.

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It's known that Cartan's theorem B implies theorem A (an early result of Siu) and Cartan's theorem B is equivalent to Cartan extension theorem. Cartan's theory was established in 1950s by the method of sheaf cohomology.

In 1960s, L^2 method for solving $\bar{\partial}-$equation was established which is also successful to obtain the above Cartan's theorems and to provide another method to solve some fundamental problems in several complex variables such as Levi's problem, Cousin problems I and II, and so on. For basic knowledge on several complex variables including L^2 method, the reader is referred to [15, 16, 18, 20, 32–39, 49–52, 55, 71–74, 82, 83].

In the above setting of Cartan extension theorem, the following question seems to be quite natural:

A natural question *if the holomorphic function or section is of a special property (say, invariant w.r.t. a group action, bounded or L^2), could one find the holomorphic extension which is still of the same special property?*

For a discussion of the above question for the case of group actions, i.e., the case for the extension invariant w.r.t. a group action, the reader is referred to [70, 87, 88]. This is related to Zhou's solution in 1997 of a longstanding problem – the extended future tube conjecture, which was posed by N.N. Bogoliubov, V. S. Vladimirov and A.S. Wightman in 1950s when they studied QFT and Hilbert 6th problem.

In the present paper, we'll focus on discussing the above question for the case of the L^2 extension.

The L^2 extension problem is stated as follows (see, Demailly [21]):

L^2 **extension problem** *for a suitable pair (M, S), where S is a closed complex subvariety of a complex manifold M, given a holomorphic function f (or a holomorphic section of a holomorphic vector bundle) on S satisfying suitable L^2 conditions on S, find an L^2 holomorphic extension F on M together with a good or even optimal L^2 estimate for F on M.*

It's obvious that the problem could be divided into two parts, we call the first one existence part, the latter one optimal-estimate part. In the present paper, we mainly consider the optimal-estimate part of the L^2 extension problem, and also consider an application of the existence part in a solution of the strong openness conjecture.

The famous Ohsawa-Takegoshi L^2 extension theorem (Ohsawa wrote a series of papers on L^2 extension theorem in more general settings) gives an answer to the first part of the problem – existence of L^2 extension. There have been other proofs and a lot of important applications of the theorem in complex geometry and several complex variables, thanks to the works of Y.-T. Siu [75–81], J.P. Demailly [17, 19, 21–26], Ohsawa [59–66], and Berndtsson [2–10] et al. [27, 56, 57, 67].

An unsolved problem remained – the second part of the L^2 extension problem, which we call the L^2 extension problem with optimal estimate or sharp L^2 extension problem. The above-mentioned works of Siu, Demailly, Ohsawa, and Berndtsson et al. not only gave other proofs of the Ohsawa-Takegoshi L^2 extension theorem, but also gave some explicit good estimates which could be also regarded as attempts to the L^2 extension problem with optimal estimate.

One of the motivations to consider the L^2 extension problem with optimal estimate originates from the so-called Suita's conjecture.

Let Ω be an open Riemann surface, which admits a Green function G_Ω. Let κ_Ω be the Bergman kernel for square integrable holomorphic $(1,0)$ forms on Ω.

Let $c_\beta(z)$ be the logarithmic capacity which is locally defined by

$$c_\beta(z) := \exp \lim_{\xi \to z}(G_\Omega(\xi, z) - \log|\xi - z|)$$

on Ω (see [69]).

In [84], Suita conjectured in 1972 the following:

Suita's conjecture *On any open Riemann surface Ω with Green function, then $(c_\beta(z))^2|dz|^2 \leq \pi\kappa_\Omega(z)$, equality holding for $z \in \Omega$ if and only if Ω is conformally equivalent to the unit disc less a (possible) closed set of inner capacity zero.*

It should be noted that Suita's conjecture is originally posed for open Riemann surfaces (not only for the bounded planar domains) and actually consists of two parts: inequality part and equality part, although the inequality part was often referred as Suita's conjecture. The equality part is more difficult.

2 Optimal Estimate for Ohsawa's Paper III

2.1 A Previous Result

We started to consider the L^2 extension problem with optimal estimate in [91], introducing a new method of undetermined function.

Theorem 0 (Guan, Zhou, Zhu) *Let M be a Stein manifold of dimension n. Let φ and ψ be two plurisubharmonic functions on M. Assume that w is a holomorphic function on M such that $\sup_M(\psi + 2\log|w|) \leq 0$ and dw does not vanish identically on any branch of $w^{-1}(0)$. Put $H = w^{-1}(0)$ and $H_0 = \{x \in H : dw(x) \neq 0\}$. Then there exists a uniform constant $\mathbf{C} < 1.954$ independent of M, φ, ψ and w such that, for any holomorphic $(n-1)$-form f on H_0 satisfying*

$$c_{n-1}\int_{H_0} e^{-\varphi-\psi} f \wedge \bar{f} < \infty,$$

where $c_k = (-1)^{\frac{k(k-1)}{2}} i^k$ for $k \in \mathbb{Z}$, there exists a holomorphic n-form F on M satisfying $F = dw \wedge \tilde{f}$ on H_0 with $\theta^*\tilde{f} = f$ and

$$c_n \int_M e^{-\varphi} F \wedge \bar{F} \leq 2\mathbf{C}\pi c_{n-1}\int_{H_0} e^{-\varphi-\psi} f \wedge \bar{f},$$

where $\theta : H_0 \longrightarrow M$ is the inclusion map.

Remark When φ and ψ are psh and $\mathbf{C}$ is large, the above is due to Ohsawa [60], the so-called Ohsawa's L^2 extension theorem with negligible weight which is more general than Ohasawa-Takegoshi L^2 extension theorem.

2.2 Method of Undetermined Functions

A method of introducing undetermined functions with using ODE was initiated to approach the L^2 extension problem with optimal estimate in [91]. This method turns out to be able to reach the solution of the problem.

Method of undetermined functions:

- take undetermined function for twist factor or weight instead of taking explicit function,
- do L^2 estimate according to twisted Bochner-Kodaira Identity or the method of complete Kähler metric, together with a lemma of Berndtsson,
- get suitably divided two sum terms
- via naturally asking one term zero, obtain ODE and decide the undetermined functions by solving the ODE

This method is based on combining of the earlier and advanced works of Ohsawa-Takegoshi, Berndtsson, Demailly, McNeal-Varolin, Ohsawa, Siu (who developed the twisted Bochner-Kodaira Identity and the method of complete Kähler metric).

One key difference between our method and their works is that they choose the explicit functions for the twist factor or the weight, while we choose the undetermined functions.

For bounded pseudoconvex domain in $\mathbb{C}^n$ and a special ψ, instead of using the twisted Bochner-Kodaira Identity or the method of complete Kähler metric, Blocki used Chen's proof (arXiv:1105.2430), but this part didn't give a better or optimal estimate. In order to get better estimate, Blocki in [11] used the same method and the same ODE as G-Z-Z's to obtain the same constant as G-Z-Z's, and then developed this equation to obtain first $\mathbf{C} = 1$ in [12] as a continuation of [91].

2.3 Main Result: Optimal Estimate

Continuing our previous work [40, 91], in the framework of Ohsawa series paper III on L^2 extension theorem, Guan and Zhou [41] obtained the optimal estimate version of Ohsawa's L^2 extension theorem with negligible weight.

Theorem 1 (Guan, Zhou) *In the above theorem, we only assume that* $\varphi + \psi$ *and* ψ *be plurisubharmonic on* M*. The same conclusion is true for* $\mathbf{C} = 1$.

We illuminate briefly one idea of our proof on the method of undetermined function which was first introduced in [40].

Lemma (see (Demailly's book)) *Let* (X, ω) *be a Kähler manifold of dimension n with a Kähler metric* ω*,* (E, h) *be an Hermitian holomorphic vector bundle. Suppose that* $\eta, g > 0$ *are smooth functions on X,* $\alpha \in \mathcal{D}(X, \Lambda^{n,q}T_X^* \otimes E)$ *(space of smooth differential forms with values in E with compact support). Denote by* $\bar{\partial}^*$ *the Hilbert adjoint operator of* $\bar{\partial}$*.*

Then we have

$$\begin{aligned}&\|(\eta + g^{-1})^{\frac{1}{2}}\bar{\partial}^*\alpha\|_{\tilde{h}}^2 + \|\eta^{\frac{1}{2}}\bar{\partial}\alpha\|_{\tilde{h}}^2\\ &\geq \langle\langle[\eta\sqrt{-1}\Theta_E - \sqrt{-1}\partial\bar{\partial}\eta - \sqrt{-1}g\partial\eta \wedge \bar{\partial}\eta, \Lambda_\omega]\alpha, \alpha\rangle\rangle_{\tilde{h}}.\end{aligned}$$

We introduce two undetermined functions s, u*, one function s is for the twist factor, another one u is for the weight.*

Take the twist factor η, the weight ϕ, and the new metric $\tilde{h}$ on E in the above lemma as follows.

- v is a smooth increasing convex function on $\mathbb{R}$, such that $v(0) = 0$,
- $\eta := s(-v \circ \Psi)$ and $\phi := u(-v \circ \Psi)$, where $s \in C^\infty((0, +\infty))$, and $u \in C^\infty((0, +\infty))$,
- $\tilde{h} := he^{-\Psi-\phi}$.

Then we have

$$\begin{aligned}&\eta\sqrt{-1}\partial\bar{\partial}\phi - \sqrt{-1}\partial\bar{\partial}\eta - \sqrt{-1}g\partial\eta \wedge \bar{\partial}\eta\\ =&(s' - su')((v' \circ \Psi)\sqrt{-1}\partial\bar{\partial}\Psi + (v'' \circ \Psi)\sqrt{-1}\partial(\Psi) \wedge \bar{\partial}(\Psi))\\ +&((u''s - s'') - gs'^2)\sqrt{-1}\partial(v \circ \Psi) \wedge \bar{\partial}(v \circ \Psi),\end{aligned}$$

Therefore we get two sum terms in the above formula. Asking naturally the second term zero, i.e., we take $g = \frac{u''s-s''}{s'^2} \circ (-v \circ \Psi)$. Because of the first term, we're naturally led $s' - u's = 1$.

In this case, we have $\eta + g^{-1} = (s + \frac{s'^2}{u''s-s''}) \circ (-v \circ \Psi)$.

In summary, using a lemma of Berndtsson [1] (see also [91]), we're led to the following ODE system.

Two undetermined functions $u, s \in C^\infty((0, +\infty))$ satisfy the following two ODE equations:

$$\begin{aligned}&(1)(s + \frac{s'^2}{u''s - s''})e^{u-t} = \mathbf{C},\\ &(2)s' - u's = 1,\end{aligned} \tag{1}$$

where $t \in [0, +\infty)$, and $\mathbf{C} = 1$, $u''s - s'' > 0$.

When $u = 0$, it is just the ODE obtained in G-Z-Z articles [40, 91] where we got $\mathbf{C} < 1.954$.

The method in G-Z's work: take two undetermined functions, one is from the twist factor, another from the weight; get two ODE's with boundary conditions (one developed from G-Z-Z, another one is new which turns out to be important in our sequel works); easy to solve the ODE system.

The idea of using two explicit functions for the twistor and the weight already appeared in Siu's paper. Our idea of using two undetermined functions existed for some time, see Zhou's former student Zhu's Ph.D's thesis (available in some Chinese libraries) finished in 2010 and defended in May 2011.

2.4 Application: Inequality Part in Suita's Conjecture

One has a relation between the L^2 extension theorem with estimate $\mathbf{C}$ and the inequality part of the Suita's conjecture:

the L^2 extension theorem with estimate $\mathbf{C}$ implies the inequality part of the Suita's conjecture with the constant $\mathbf{C}$: $(c_\beta(z))^2|dz|^2 \leq \mathbf{C}\pi\kappa_\Omega(z)$.

For bounded domain in $\mathbb{C}$, Blocki first gave a solution of the inequality part of Suita's conjecture[12] as a continuation of [91].

The inequality part of Suita's conjecture in its original form was obtained independently as a corollary of the above Theorem 1.

Consequently, we solve an equivalent problem relate to the Fuchsian groups given in [68].

2.5 Application: Stability of the Bergman Kernels

The above Theorem 1 (not necessarily asking both φ and ψ are plurisubharmonic, only assuming $\varphi + \psi$ and ψ are plurisubharmonic) was used to study the stability of the Bergman kernels under a deformation of a bounded pseudoconvex domains by Ohsawa in [65].

3 Optimal Estimate for Ohsawa's Paper V

In the framework of Ohsawa series paper V on L^2 extension theorem, Guan and Zhou [42, 43] obtained the L^2 extension theorem with optimal estimate.

3.1 Main Result: Optimal Estimate

We present our main result in this setting as follows.

Theorem 2 (Guan, Zhou) *Let (M, S) satisfy condition (ab), S be of pure codim k, h be a smooth metric on a holomorphic vector bundle E on M with rank r. Then, for any negative polar function Ψ on M such that $\Psi \in C^\infty(M \setminus S)$ and $\Theta_{he^{-\Psi}} \geq 0$ and $\Theta_{he^{-(1+\delta)\Psi}} \geq 0$ on $M \setminus S$ in the sense of Nakano, there exists a uniform constant $\mathbf{C} = 1$ such that, for any holomorphic section f of $K_M \otimes E|_S$ on S satisfying*

$$\frac{\pi^k}{k!} \int_S |f|_h^2 dV_M[\Psi] < \infty,$$

there exists a holomorphic section F of $K_M \otimes E$ on M satisfying $F = f$ on S and

$$\int_M |F|_h^2 dV_M \leq \mathbf{C}(1 + \delta^{-1}) \frac{\pi^k}{k!} \int_S |f|_h^2 dV_M[\Psi].$$

Especially, if Ψ is furthermore psh, there exists a holomorphic section F of $K_M \otimes E$ on M satisfying $F = f$ on S and

$$\int_M |F|_h^2 dV_M \leq \mathbf{C} \frac{\pi^k}{k!} \int_S |f|_h^2 dV_M[\Psi].$$

For the definition of the pair (M, S) satisfying condition (ab), the reader is referred to [43]. It includes:

(1) M is a Stein manifold, and S is any closed complex subvariety of M;
(2) M is a complex projective algebraic manifold, and S is any closed complex subvariety of M;
(3) M is a projective family, and S is any closed complex subvariety of M.

A polar function is an upper-semi-continuous function Ψ from M to the interval $[-\infty, A)$ where $A \in (-\infty, +\infty]$, such that

(1) $\Psi^{-1}(-\infty) \supset S$, and $\Psi^{-1}(-\infty)$ is a closed subset of M;
(2) If S is l–dimensional around a point $x \in S_{reg}$, there exists a local coordinate $(z_1, \cdots, z_n)$ on a neighborhood U of x such that $z_{l+1} = \cdots = z_n = 0$ on $S \cap U$ and

$$\sup_{U \setminus S} |\Psi(z) - (n-l) \log \sum_{l+1}^{n} |z_j|^2| < \infty.$$

For each polar function Ψ, one can associate a positive measure $dV_M[\Psi]$ on S as the minimum element of the partial ordered set of positive measures $d\mu$ satisfying

$$\int_{S_l} f d\mu \geq \limsup_{t\to\infty} \frac{2(n-l)}{\sigma_{2n-2l-1}} \int_M f e^{-\Psi} \mathbb{I}_{\{-1-t<\Psi<-t\}} dV_M$$

for any nonnegative continuous function f with compact support on M.

3.2 Application: A Conjecture of Ohsawa

As a corollary of the above Theorem 2, we obtain a proof of a conjecture of Ohsawa in [63] which is an extension of the Suita's conjecture (inequality part) to high dimensional manifolds and high codimensional subvarieties.

Let $G(\cdot, S)$ be the nontrivial generalized pluricomplex Green function. Let dV_M be a continuous volume form on M and let $\{\sigma_j\}_{j=1}^{\infty}$ (resp. $\{\tau_j\}_{j=1}^{\infty}$) be a complete orthogonal system of $A^2(M, K_M, dV_M^{-1}, dV_M)$ and $A^2(S, K_M|_S, dV_M^{-1}, dV_M[G(\cdot, S)])$

Denote by $\kappa_M = \sum_{j=1}^{\infty} \sigma_j \otimes \bar{\sigma}_j \in C^{\omega}(M, K_M \otimes \bar{K}_M)$ (resp. $\kappa_{M/S} = \sum_{j=1}^{\infty} \tau_j \otimes \bar{\tau}_j \in C^{\omega}(S, K_M \otimes \bar{K}_M)$.)

A conjecture of Ohsawa $\kappa_{M/S}(x) \leq (\pi^k/k!)\kappa_M(x)$ *for any* $x \in S_{n-k}$.

Corollary (Guan, Zhou) *The above conjecture of Ohsawa holds.*

3.3 A Relation to Berndtsson's Log-Plurisubharmonicity

Considering two cases,

(1) M a pseudoconvex domain in $\mathbb{C}^{n+m}$ with coordinate (z, t), Y is a domain in $\mathbb{C}^m$ with coordinate t, $p(z, t) = t$;
(2) M is a projective manifold, and Y is a complex manifold, and p is a fibration.

Let L be an Hermitian holomorphic line bundle over M, e be the local frame of L. Let κ_{M_t} be the Bergman kernel of $K_{M_t} \otimes L$ on the fibre M_t, and $\kappa_{M_t} := B_t(z) dz \otimes e \otimes d\bar{z} \otimes \bar{e}$ locally.

Berndtsson established the following:

Theorem (Berndtsson) $\log B_t(z)$ *is a plurisubharmonic function with respect to* (z, t).

One dimensional case is due to Maitani-Yamaguchi.

We observed the following relation between L^2 extension theorem with optimal estimate and log-plurisubharmonicity of the relative Bergman kernel.

Corollary (Guan, Zhou) *Our L^2 extension theorem with optimal estimate implies the above Berndtsson's theorem.*

Note that L^2 extension theorem without optimal estimate can not do so.

3.4 Weakly Psedoconvex Case

Let $\mathfrak{R}$ be a class of functions defined by

$$\left\{R \in C^\infty(-\infty, 0] : R > 0,\ R' \le 0 \text{ and } \int_{-\infty}^{0} \frac{1}{R(t)} dt < +\infty\right\}.$$

We will denote $\int_{-\infty}^{0} \frac{1}{R(t)} dt$ by C_R.

Theorem (Zhou, Zhu) *Let R be a function in $\mathfrak{R}$. Let (X, g) be a weakly pseudoconvex complex n-dimensional manifold possessing a Kähler metric g, and let L (resp. E) be a Hermitian holomorphic line bundle over X (resp. a Hermitian holomorphic vector bundle of rank m over X), and s a global holomorphic section of E. Let ψ be a smooth plurisubharmonic function on X. Assume that s is generically transverse to the zero section, and let*

$$Y = \{x \in X : s(x) = 0, \wedge^m ds(x) \neq 0\}.$$

Moreover, assume that

(1) $\sqrt{-1}\Theta_L + \sqrt{-1}\partial\bar{\partial}(\psi + m \log |s|^2) \ge 0$

and that there is a continuous function $\alpha \ge 0$ on X such that the following two inequalities hold everywhere on X:

(2) $\alpha\sqrt{-1}\Theta_L + \alpha\sqrt{-1}\partial\bar{\partial}(\psi + m \log |s|^2) + \frac{1}{m}\sqrt{-1}\partial\bar{\partial}\psi \ge \frac{\{\sqrt{-1}\Theta_E s, s\}}{|s|^2}$,

(3) $\frac{1}{m}\psi + \log |s|^2 \le -2\alpha.$

Then for every holomorphic section f on Y with values in the line bundle $K_X \otimes L$ (restricted to Y), such that

$$\int_Y \frac{|f|^2}{|\wedge^m (ds)|^2} e^{-\psi} dV_Y < +\infty,$$

there exists a holomorphic section F on X with values in $K_X \otimes L$, such that $F = f$ on Y and

$$\int_X \frac{|F|^2}{e^{\psi+m\log|s|^2}R(\psi+m\log|s|^2)}dV_X \leq C_R\frac{(2\pi)^m}{m!}\int_Y \frac{|f|^2}{|\wedge^m(ds)|^2}e^{-\psi}dV_Y.$$

Remark The above theorem in [90] could be regarded as an optimal version of the Ohsawa-Takegoshi-Manivel L^2 extension theorem in [19].

4 The Sharp L^2 Extension Problem in a General Setting

Let h be a smooth metric on a holomorphic vector bundle E on M with rank r which is semi-positive in the sense of Nakano. An hermitian metric h on E is said to be semi-positive in the sense of Nakano if Θ_h is semi-positive definite as an hermitian form on $T_X \otimes E$, i.e. if for every $u \in T_X \otimes E$, $u \neq 0$, we have $\Theta_h(u,u) \geq 0$.

Let $\Psi - A$ be a negative plurisubharmonic polar function on M, which is smooth on $M \setminus S$.

4.1 Main Result: A Solution of the Sharp L^2 Extension Problem

We establish an L^2 extension theorem with optimal estimate, solving the L^2 extension problem with optimal estimate in a general setting:

Theorem 3 (Guan, Zhou) *There exists a uniform constant* $\mathbf{C} = 1$ *such that, for any holomorphic section* f *of* $K_M \otimes E|_S$ *on* S *of pure codimension* k *satisfying*

$$\frac{\pi^k}{k!}\int_S |f|_h^2 dV_M[\Psi] < \infty,$$

there exists a holomorphic section F *of* $K_M \otimes E$ *on* M *satisfying* $F = f$ *on* S *and*

$$\int_M c_A(-\Psi)|F|_h^2 dV_M \leq \mathbf{C}(\int_{-A}^{\infty} c_A(t)e^{-t}dt)(\frac{\pi^k}{k!}\int_S |f|_h^2 dV_M[\Psi]).$$

Here, $c_A(t)$ *is a positive smooth function on* $(-A,+\infty)$ $(A \in (-\infty,+\infty])$ *satisfying* $\int_{-A}^{\infty} c_A(t)e^{-t}dt < \infty$*, and*

$$(\int_{-A}^{t} c_A(t_1)e^{-t_1}dt_1)^2 > c_A(t)e^{-t}\int_{-A}^{t}\int_{-A}^{t_2} c_A(t_1)e^{-t_1}dt_1dt_2,$$

for any $t \in (-A,+\infty)$.

It's easy to see that when $c_A(t)e^{-t}$ is decreasing with respect to t and A is finite, the above inequality holds.

4.2 Application 1: Equality Part in Suita's Conjecture

One of the key points in the paper [44, 45] is the introduction of the function $C_A(t)$, which was motivated by Ohsawa's paper [59] and Demailly's paper [19] et al., and plays a key role in the proof of the difficult equality case in Suita's conjecture, by selecting carefully a special $C_A(t)$.

Corollary (Guan, Zhou) *The equality part of Suita's conjecture holds.*
Therefore the full Suita's conjecture holds.

4.3 Application 2: L-conjecture

Let Ω be an open Riemann surface which admits a Green function G_Ω, which is not biholomorphic to unit disc less a (possible) closed set of inner capacity zero.

The adjoint L-kernel is defined by $L_\Omega(z,t) := \frac{2}{\pi}\frac{\partial^2 G_\Omega(z,t)}{\partial z \partial t}$.

***L*-Conjecture** *Assume that $G_\Omega(\cdot, t)$ is exhaustion for any $t \in \Omega$. Then for any $t \in \Omega$, $\exists z \in \Omega$, s.t. $L_\Omega(z,t) = 0$.*

The equality part of Suita's conjecture implies L-conjecture.

Corollary (Guan, Zhou) *L-conjecture is true.*

The L-conjecture was stated for a finite Riemann surface Ω (see [85, 86]). It's known that for such a Riemann surface, the assumption holds, i.e., on a finite Riemann surface Ω, the Green function $G_\Omega(\cdot, t)$ is exhaustion for any $t \in \Omega$.

4.4 Application 3: The Extended Suita's Conjecture

Let Ω be an open Riemann surface, which admits a Green function G_Ω.

Let $p : \Delta \to \Omega$ be the universal covering from unit disc Δ to Ω.

We call the holomorphic function f (resp. holomorphic $(1,0)$ form F) on Δ is a multiplicative function (resp. multiplicative differential (Prym differential)) if there is a character χ, which is the representation of the fundamental group of Ω, such that $g^*f = \chi(g)f$ (resp. $g^*F = \chi(g)F$), where $|\chi| = 1$.

Denote the set of such kinds of functions f (resp. forms F) by $\mathcal{O}^\chi(\Omega)$ (resp. $\Gamma^\chi(\Omega)$).

Note that $F \wedge \bar{F}$ is fibre constant respect to p, then one can define the multiplicative Bergman kernel $\kappa^{\chi}(x, \bar{y})$ for $\Gamma^{\chi}(\Omega)$ on $\Omega \times \Omega$.

For Green function $G_{\Omega}(\cdot, z_0)$, one may find a χ_{z_0} and a multiplicative function $f_{z_0} \in \mathcal{O}^{\chi_{z_0}}(\Omega)$, such that $|f_{z_0}| = p^* e^{G_{\Omega}(\cdot, z_0)}$.

Extended Suita Conjecture *$c_{\beta}^2(z)|dz|^2 \leq \kappa_{\Omega}^{\chi}(z, \bar{z})$, and equality holds if and only if $\chi = \chi_{z_0}$.*

Corollary (Guan, Zhou) *The extended Suita conjecture holds.*

4.5 Application 4: Optimal Versions of Various L^2 Extension Theorems

By taking different $c_A(t)$, obtain optimal estimate versions of various well known L^2 extension theorems due to Ohsawa, Siu, Manivel, Demailly, Berndtsson, McNeal-Varolin, Demailly-Hacon-Paun, et al.

Demailly-Hacon-Paun's L^2 extension theorem plays an important role in [22] on the existence of good minimal models.

As examples, we give two optimal versions of Ohsawa's L^2 extension theorem in [59], by taking $c_A(t)$.

Corollary *For any holomorphic section f of $K_{S_{reg}} \otimes E|_{S_{reg}}$ on S_{reg} satisfying*

$$\frac{\pi^m}{m!} \int_{S_{reg}} \{f, f\}_h < \infty,$$

where $S = \{g_1 = \cdots = g_m = 0\}$, g_i are holomorphic functions on M, which satisfies $\wedge_{j=1}^m dg_j|_{S_{reg}} \neq 0$,

there exists a holomorphic section F of $K_M \otimes E$ on M satisfying $F = f \wedge \bigwedge_{k=1}^m dg_k$ on S_{reg} and

$$\int_M (1 + |g_1|^2 + \cdots + |g_m|^2)^{-m-\varepsilon} \{F, F\}_h$$
$$\leq \mathbf{C}\Big(m \sum_{j=0}^{m-1} C_{m-1}^j (-1)^{m-1-j} \frac{1}{m-1-j+\varepsilon}\Big) \frac{(2\pi)^m}{m!} \int_{S_{reg}} \{f, f\}_h,$$

where the uniform constant $\mathbf{C} = 1$, which is optimal for any m.

To obtain the above result, we take $c_{\infty}(t) := (1 + e^{-\frac{t}{m}})^{-m-\varepsilon}$ in our Theorem 3.

Corollary *For any holomorphic section f of $K_{S_{reg}} \otimes E|_{S_{reg}}$ on S_{reg} satisfying*

$$\frac{\pi^m}{m!}\int_{S_{reg}} \{f, f\}_h < \infty,$$

there exists a holomorphic section F of $K_M \otimes E$ on M satisfying $F = f \wedge \bigwedge_{k=1}^m dg_k$ on S and

$$\int_M (1 + (|g_1|^2 + \cdots + |g_m|^2)^m)^{-1-\varepsilon}\{F, F\}_h$$
$$\leq \mathbf{C}\frac{1}{\varepsilon}\frac{(2\pi)^m}{m!}\int_{S_{reg}} \{f, f\}_h,$$

where uniform constant $\mathbf{C} = 1$, which is optimal for any m.

To obtain the above result, we take $c_\infty(t) := (1 + e^{-t})^{-1-\varepsilon}$ in our Theorem 3.

5 Strong Openness Conjecture

5.1 Statement of the Conjecture

Let X be complex manifold with dimension n and φ be a plurisubharmonic function on X.

Definition (Nadel [58]) The multiplier ideal sheaf $\mathcal{I}(\varphi)$ is the sheaf of germs of holomorphic functions f such that $|f|^2e^{-\varphi}$ is locally integrable (see also [21, 80, 81], etc.).

Nadel proved that the multiplier ideal sheaf is coherent.

Denote by

$$\mathcal{I}_+(\varphi) := \cup_{\varepsilon>0}\mathcal{I}((1+\varepsilon)\varphi).$$

Strong openness conjecture of Demailly *One has $\mathcal{I}_+(\varphi) = \mathcal{I}(\varphi)$.*

Some known cases:

- Under the assumption that $e^{-2\varphi}$ is locally integrable, i.e., when $\mathcal{I}(\varphi) = \mathcal{O}_X$, this was proved by Berndtsson (arXiv:1305.5781).
- When $dimX < 3$, this was proved by Favre, Jonsson, Mustată.

Theorem 4 (Guan, Zhou [46, 47]) *The strong openness conjecture of Demailly holds.*

The effectiveness problem in the strong openness conjecture is also discussed in [48].

5.2 A Sketch of the Proof of the Conjecture

Our proof is by induction method. One main idea of the proof of the strong openness conjecture is a use movably of the L^2 extension theorem.

Since $\mathcal{I}_+(\varphi) \subset \mathcal{I}(\varphi)$, it's sufficient to prove $\mathcal{I}(\varphi) \subset \mathcal{I}_+(\varphi)$.

Let $F \in \mathcal{I}(\varphi)$. Take a series of movable hyperplanes close to the origin. Using Ohsawa-Takegoshi L^2 extension theorem, one can extend the restricted functions of F on the movable hyperplanes, together with upper bounds which are variable for the L^2 norms of the extended holomorphic functions.

By induction method, we get that the extended functions belong to $\mathcal{I}_+(\varphi)$. By contraction, if the conjecture were not true, i.e., $F \notin \mathcal{I}_+(\varphi)$, we would have obtained a contradiction. Therefore, by curve selection lemma (see [21]), the extended functions would have been divided by F at least along some analytic curve.

We observe that the L^2 norms of such extended functions have lower bounds which are also variable. Furthermore, one may observe that the ratios of the lower bounds and upper bounds would have gone to infinity. This is a contradiction.

5.3 A Conjecture of Demailly-Kollár

Demailly and Kollár posed in [24] the following conjecture:

Conjecture D-K *Let φ be a plurisubharmonic function on $\Delta^n \subset \mathbb{C}^n$, and K be compact subset of Δ^n. If $c_K(\varphi) < +\infty$, then*

$$\frac{1}{r^{2c_K(\varphi)}}\mu(\{\varphi < \log r\})$$

has a uniform positive lower bound independent of $r \in (0,1)$, where $c_K(\varphi) = sup\{c \geq 0 : \exp^{-2c\varphi}$ is L^1 on a neighborhood of $K\}$, and μ is the Lebesgue measure on $\mathbb{C}^n$.

The above conjecture implies the openness conjecture.

Theorem 5 (Guan, Zhou) *The above conjecture of Demailly-Kollár is true.*

5.4 A Conjecture of Jonsson-Mustată

Let I be an ideal of $\mathcal{O}_{\Delta^n,o}$, which is generated by $\{f_j\}_{j=1,\cdots,l}$. In [54], Jonsson and Mustată posed the following conjecture for level sets of plurisubharmonic functions (see also [53]):

Conjecture J-M *Let ψ be a plurisubharmonic function on $\Delta^n \subset \mathbb{C}^n$. If $c_o^I(\psi) < +\infty$, then*

$$\frac{1}{r^{2c_o^I(\psi)}}\mu(\{c_o^I(\psi)\psi - \log|I| < \log r\})$$

has a uniform positive lower bound independent of $r \in (0,1)$, where

$$\log|I| := \log \max_{1\leq j\leq l} |f_j|,$$

$c_o^I(\psi) = sup\{c \geq 0 : |I|^2 e^{-2c\psi}$ is L^1 on a neighborhood of $o\}$ is the jumping number in [54], and μ is the Lebesgue measure on $\mathbb{C}^n$.

For $n \leq 2$, the above conjecture was proved by Jonsson and Mustată in [53].

The above conjecture implies the strong openness conjecture.

Theorem 6 (Guan, Zhou) *The above conjecture of Jonsson-Mustată holds.*

5.5 Applications

We present two immediate consequences of the truth of the strongly openness conjecture.

Corollary *One has $\{\mathcal{I}(\varphi)\} = \{\mathcal{I}(\varphi_A)\}$, where φ is a p.s.h. function and φ_A is a p.s.h. function with analytic singularities.*

This follows from the truth of the strong openness conjecture together with Demailly's equi-singular approximation theorem.

Corollary *Vanishing theorem of Kawamata – Vieweg – Nadel type holds on compact Kähler manifolds.*

In a more precise way, we have the following vanishing theorem:

Let (L,φ) be a pseudo-effective line bundle on a compact Kähler manifold X of dimension n, and $nd(L,\varphi)$ be the numerical dimension of (L,φ).

$$H^p(X, K_X \otimes L \otimes \mathcal{I}(\varphi)) = 0,$$

for any $p \geq n - nd(L,\varphi) + 1$.

This follows from the truth of the strong openness conjecture together with Demailly-Peternell's [25] and Cao's results [14].

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