Determination of (0, 2)-Regular Sets in Graphs and Applications

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Abstract In this paper, relevant results about the determination of (κ, τ)-regular sets, using the main eigenvalues of a graph, are reviewed and some results about the determination of (0, 2)-regular sets are introduced. An algorithm for that purpose is also described. As an illustration, this algorithm is applied to the determination of maximum matchings in arbitrary graphs.

1 Introduction

All graphs considered throughout this paper are simple (with no loops nor multiple edges), undirected and have order n. $V(G) = \{1, 2, \ldots, n\}$ and $E(G)$ denote, respectively, the vertex and the edge sets of G and ij represents the edge linking nodes i and j of $V(G)$. If $i \in V(G)$, then the vertex set denoted by $N_G(i) = \{j \in V(G) : ij \in E(G)\}$ is called neighbourhood of i. Additionally, $N_G[i]$ denotes the closed neighbourhood of vertex i (that is, $N_G[i] = N_G(i) \cup \{i\}$). Given a graph G and a set of vertices $U \subset V(G)$, the subgraph of G induced by U, $G[U]$, is such that $V(G[U]) = U$ and $E(G[U]) = \{ij : i, j \in U \wedge ij \in E(G)\}$. A (κ, τ)-*regular set* of a graph is a vertex subset inducing a $\kappa-$regular subgraph such that every vertex not in the subset has τ neighbours in it, [2].

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The adjacency matrix $A_G = [a_{ij}]$ of G is the $n \times n$ symmetric matrix such that $a_{ij} = 1$ if $ij \in E(G)$ and $a_{ij} = 0$ otherwise. The n eigenvalues of A_G are usually called the eigenvalues of G and are ordered $\lambda_{max}(G) = \lambda_1(G) \geq \cdots \geq \lambda_n = \lambda_{min}(G)$. These eigenvalues are all real because A_G is symmetric. It is also known that, provided G has at least one edge, we have that $\lambda_{min}(G) \leq -1$ and, furthermore, $\lambda_{min}(G) = -1$ if and only if every connected component of G is complete, [4]. The multiplicity of λ_i as eigenvalue of G (and, consequently, as eigenvalue of A_G) is denoted by $m(\lambda_i)$. Throughout this paper, $\sigma(G)$ will denote the spectrum of G, that is, the set of G's eigenvalues together with their multiplicities. The eigenspace associated to each eigenvalue λ of G is denoted by $\mathscr{E}_G(\lambda)$.

An eigenvalue of a graph G is *main* if its associated eigenpace is not orthogonal to the all-one vector **j**. The vector space spanned by such eigenvectors of G is denoted *Main*(G). The remaining (distinct) eigenvalues of G are referred to as *non-main*. The dimension of $\mathscr{E}_G(\lambda_i)$, the eigenspace associated to each main eigenvalue λ_i of G, is equal to the multiplicity of λ_i. The *index* of G, its largest eigenvalue, is main. The concepts of main and non-main eigenvalue were introduced in [4]. An overview on the subject was published in [5].

Given a graph G, the *line graph* of G, which is denoted by $L(G)$, is constructed by taking the edges of G as vertices of $L(G)$ and joining two vertices in $L(G)$ by an edge whenever the corresponding edges in G have a common vertex. The graph G is called the *root graph* of $L(G)$.

A *stable set* (or independent set) of G is a subset of vertices of $V(G)$ whose elements are pairwise nonadjacent. The stability number (or independence number) of G is defined as the cardinality of a largest stable set and is usually denoted by $\alpha(G)$. A maximum stable set of G is a stable set with $\alpha(G)$ vertices. Given a nonnegative integer k, the problem of determining whether G has a stable set of size k is *NP*-complete and, therefore, the determination of $\alpha(G)$ is, in general, a hard problem.

A *matching* in a graph G is a subset of edges, $M \subseteq E(G)$, no two of which have a common vertex. A matching with maximum cardinality is called a maximum matching. Furthermore, if for each vertex $i \in V(G)$ there is one edge of the matching M incident with i, then M is called a perfect matching. It is obvious that every perfect matching is also a maximum matching. Notice that the determination of a maximum stable set of a line graph, $L(G)$, is equivalent to the determination of a maximum matching of G. There are several polynomial-time algorithms for the determination of a maximum matching of a graph.

The present paper introduces an algorithm for the recognition of (0,2)-regular sets in graphs and the application of this algorithm is illustrated with the determination of maximum matchings through an approach involving (0,2)-regular sets.

2 Main Eigenvalues, Walk Matrix and (κ, τ)-Regular Sets

We begin this section recalling a few concepts and surveying some relevant results.

If G has p distinct main eigenvalues $\mu_1, \ldots, \mu_p$, the main characteristic polynomial of G is

$$m_G(\lambda) = \lambda^p - c_o\lambda^{p-1} - c_1\lambda^{p-2} - \cdots - c_{p-2}\lambda - c_{p-1} = \prod_{i=1}^{p}(\lambda - \mu_i).$$

Theorem 1 ([5]) *If G is a graph with p main distinct eigenvalues $\mu_1, \ldots, \mu_p$, then the main characteristic polynomial of G, $m_G(\lambda)$, has integer coefficients.*

Considering A_G, the adjacency matrix of graph G, the entry $a_{ij}^{(k)}$ of A^k is the number of walks of length k from i to j. Therefore, the $n \times 1$ vector $A^k\mathbf{j}$, gives the number of walks of length k starting in each vertex of G. Given a graph G of order n, the $n \times k$ walk matrix of G is the matrix $W_k = (\mathbf{j}, A\mathbf{j}, A^2\mathbf{j}, \ldots, A^{k-1}\mathbf{j})$. If G has p distinct main eigenvalues, the $n \times p$ walk matrix

$$W = W_p = (\mathbf{j}, A\mathbf{j}, A^2\mathbf{j}, \ldots, A^{p-1}\mathbf{j})$$

is referred to as the *walk matrix* of G. The vector space spanned by the columns of W is called *Main(G)* and it coincides with the vector space spanned by $v_1, \ldots, v_p$ with $v_i \in \mathscr{E}_G(\mu_i)$ and $v_i^t\mathbf{j} \neq 0, i = 1, \ldots, p$. The orthogonal complement of *Main(G)* is denoted, as expected, *Main(G)*$^\perp$. Notice that both *Main(G)* and *Main(G)*$^\perp$ are invariant under A_G.

Taking into account that

$$m_G(A_G) = 0 \Leftrightarrow A_G^p\mathbf{j} - c_0A_G^{p-1}\mathbf{j} - c_1A_G^{p-2}\mathbf{j} - \cdots - c_{p-2}A_G\mathbf{j} - c_{p-1}\mathbf{j} = 0, \qquad (1)$$

the following result holds.

Theorem 2 ([3]) *If G has p main distinct eigenvalues, then*

$$W\begin{pmatrix} c_{p-1} \\ \vdots \\ c_1 \\ c_0 \end{pmatrix} = A^p\mathbf{j},$$

where c_j, with $0 \leq j \leq p - 1$, are the coefficients of the main characteristic polynomial of G.

It follows from this theorem that the coefficients of the main characteristic polynomial of a graph can be determined solving the linear system

$$Wx = A^p\mathbf{j}.$$

Proposition 1 ([2]) *A vertex subset S of a graph G with n vertices is (κ, τ)-regular if and only if its characteristic vector is a solution of the linear system*

$$(A_G - (\kappa - \tau)I_n)x = \tau\mathbf{j}. \tag{2}$$

It follows from this result that if system (2) has a $(0, 1)$-solution, then such solution is the characteristic vector of a (κ, τ)-regular set. In fact, let us assume that x is a $(0, 1)$-solution of (2). Then, for all $i \in V(G)$,

$$(A_G)_i x = |N_G(i) \cap S| = \begin{cases} \kappa \text{ if } i \in S \\ \tau \text{ if } i \notin S \end{cases}.$$

Next, we will associate to each graph G and each pair of nonnegative numbers, (κ, τ), the following parametric vector, [3]:

$$\mathbf{g}_G(\kappa, \tau) = \sum_{j=0}^{p-1} \alpha_j A_G^j \mathbf{j}, \tag{3}$$

where p is the number of distinct main eigenvalues of G and $\alpha_0, \dots, \alpha_{p-1}$ is a solution of the linear system:

$$\begin{pmatrix} \kappa - \tau & 0 & \dots & 0 & -c_{p-1} \\ -1 & \kappa - \tau & \dots & 0 & -c_{p-2} \\ 0 & -1 & \dots & 0 & -c_{p-3} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & -1 & \kappa - \tau - c_0 \end{pmatrix} \begin{pmatrix} \alpha_0 \\ \alpha_1 \\ \vdots \\ \alpha_{p-2} \\ \alpha_{p-1} \end{pmatrix} = -\tau \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{pmatrix}. \tag{4}$$

The following theorem is a slight variation of a result proven in [3].

Theorem 3 ([3]) *Let G be a graph with p distinct main eigenvalues $\mu_1, \dots, \mu_p$. A vertex subset $S \subset V(G)$ is (κ, τ)-regular if and only if its characteristic vector $x(S)$ is such that*

$$x(S) = \mathbf{g} + \mathbf{q},$$

with

$$\mathbf{g} = \sum_{j=0}^{p-1} \alpha_j A^j \mathbf{j},$$

$(\alpha_0, \dots, \alpha_{p-1})$ *is the unique solution of the linear system (4) and if $(\kappa - \tau) \notin \sigma(G)$ then $\mathbf{q} = 0$ else $\mathbf{q} \in \mathscr{E}_G(\kappa - \tau)$ and $\kappa - \tau$ is non-main.*

3   Main Results

An algorithm for the recognition of $(0, 2)$-regular sets in general graphs is introduced in this section. Such algorithm is not polynomial in general and its complexity depends on the multiplicity of -2 as an eigenvalue of the adjacency matrix of A_G. Particular cases for which the application of the algorithm is polynomial are presented.

Theorem 4 *If a graph G has a $(0, 2)$-regular set S, then $|S| = \mathbf{j}^T \mathbf{g}_G(0, 2)$.*

Proof Supposing that $S \subset V(G)$ is a $(0, 2)$-regular set, according to Theorem 3, its characteristic vector x_S verifies

$$x_S = \mathbf{g}_G(0, 2) + \mathbf{q}.$$

Therefore,

$$|S| = \mathbf{j}^T x_S = \mathbf{j}^T \mathbf{g}_G(0, 2) + \mathbf{j}^T \mathbf{q}.$$

Since $\mathbf{q} = 0$ or $\mathbf{q} \in \mathscr{E}_G(\kappa - \tau)$ with $\kappa - \tau$ non-main, the conclusion follows.   □

The following corollary provides a condition to decide when there are no $(0, 2)$-regular sets in G.

Corollary 1 *If $\mathbf{j}^T \mathbf{g}_G(0, 2)$ is not a natural number, then G has no $(0, 2)$-regular set.*

Now let us consider the particular case of graphs where $m(-2) = 0$.

Theorem 5 *If G is a graph such that $m(-2) = 0$, then G has a $(0, 2)$-regular set if and only if $\mathbf{g}_G(0, 2) \in \{0, 1\}^n$.*

Proof According to Theorem 3, since -2 is not an eigenvalue of G, there is a $(0, 2)$-regular set $S \subset V(G)$ if and only if $x_S = \mathbf{g}$.   □

Considering a $m \times n$ matrix M and a vertex subset $I \subset V(G)$, M^I denotes the submatrix of M whose rows correspond to the indices in I.

Theorem 6 *Let G be a graph of order n such that $m(-2) > 0$ and let U be the $n \times m$ matrix whose columns are the eigenvectors of a basis of $\mathscr{E}_G(-2)$. If there is $v \in V(G)$ such that U^N (where $N = N_G[v] = \{v, v_1, \ldots, v_k\}$) has maximum rank, then it is possible to determine, in polynomial time, if G has a $(0, 2)$-regular set.*

Proof According to the necessary and sufficient condition for the existence of a (κ, τ)-regular set presented in Theorem 3, a vertex subset $S \subset V(G)$ is $(0, 2)$-regular if and only if its characteristic vector x_S is of the form

$$x_S = \mathbf{g}_G(0, 2) + \mathbf{q},$$

where $\mathbf{g}$ is defined by (3) and (4). Setting $\mathbf{q} = \mathbf{U}\beta$, where β is an m-tuple of scalars, such scalars may be determined solving the linear subsystem of $x_S = \mathbf{g} + \mathbf{q}$:

$$x_S^N = \mathbf{g}^N + \mathbf{U}^N\beta,$$

for each of the following possible instances of x_S :

$(x_S)_v = 1$ and then $(x_S)_{v_i} = 0, \forall i = 1, \ldots, k$;
$(x_S)_v = 0$ and then one of the following holds:

$(x_S)_{v_1} = (x_S)_{v_2} = 1$ and $(x_S)_{v_i} = 0, \forall v_i \in N_G[v] \setminus \{v_1, v_2\}$;
$(x_S)_{v_1} = (x_S)_{v_3} = 1$ and $(x_S)_{v_i} = 0, \forall v_i \in N_G[v] \setminus \{v_1, v_3\}$;
$\ldots$

$(x_S)_{v_1} = (x_S)_{v_k} = 1$ and $(x_S)_{v_i} = 0, \forall v_i \in N_G[v] \setminus \{v_1, v_k\}$;
$(x_S)_{v_2} = (x_S)_{v_3} = 1$ and $(x_S)_{v_i} = 0, \forall v_i \in N_G[v] \setminus \{v_2, v_3\}$;
$(x_S)_{v_2} = (x_S)_{v_4} = 1$ and $(x_S)_{v_i} = 0, \forall v_i \in N_G[v] \setminus \{v_2, v_4\}$;
$\ldots$

$(x_S)_{v_2} = (x_S)_{v_k} = 1$ and $(x_S)_{v_i} = 0, \forall v_i \in N_G[v] \setminus \{v_2, v_k\}$;
$\ldots$

$(x_S)_{v_{k-1}} = (x_S)_{v_k} = 1$ and $(x_S)_{v_i} = 0, \forall v_i \in N_G[v] \setminus \{v_{k-1}, v_k\}$;

If for any of the cases described above the solution β is such that the obtained entries of vector x_S are $0 - 1$, then such x_S is the characteristic vector of a $(0, 2)$-regular set. If none of the above instances generates a $0 - 1$ vector x_S, then we may conclude that the graph G has no $(0, 2)$-regular set. Notice that each of the (at most) $1 + \binom{k}{2}$ linear systems under consideration can be solved in polynomial time, therefore it is possible to determine in polynomial time if G has a $(0, 2)$-regular set. $\quad\square$

In order to generalize the procedure for the determination of $(0, 2)$-regular sets to arbitrary graphs, it is worth to introduce some terminology. Let G be a graph with vertex set $V = \{1, \ldots, n\}$ and consider $I \subset V(G) = \{i_1, \ldots, i_m\}$. The m-tuple $x^I = (x_{i_1}, \ldots, x_{i_m}) \in \{0, 1\}^m$ is $(0, 2)$-*feasible* if it can be seen as a subvector of a characteristic vector $x \in \{0, 1\}^n$ of a $(0, 2)$-regular set in G. From this definition the following conditions hold:

(1) $\exists i_r \in I : x_{i_r} = 1 \Rightarrow \begin{cases} \forall i_j \in N_G(i_r) \cap I, \ x_{i_j} = 0 \quad \text{and} \\ \forall j \in N_G(i_r), \ \sum_{k \in N_G(j) \setminus \{i_r\}} x_k = 1; \end{cases}$

(2) $\exists i_s \in I : x_{i_s} = 0 \wedge N_G(i_s) \subseteq I \Rightarrow \sum_{j \in N_G[i_s]} x_j = 2.$

Using the $(0, 2)$-feasible concept and consequently the above conditions, we are able to present an algorithm to determine a $(0, 2)$-regular set in an arbitrary graph or to decide that no such set exists.

In the worst cases, steps 7–11 are executed 2^m times and, therefore, the execution of the algorithm is not polynomial. There is, however, a large number of graphs for which the described procedure is able to decide, in polynomial time, if there is a $(0, 2)$-regular set and to determine it in the cases where it exists.

Algorithm 1 To determine a (0, 2)-regular set or decide that no such set exists

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Input: (Graph G of order n, $m = m(-2)$ and matrix Q whose columns are the eigenvectors of a basis of $\mathscr{E}_G(-2)$).

Output: ((0, 2)-regular set of G or the conclusion that no such set exists).

1. **If $\mathbf{j}^T g_G(0, 2) \notin \mathbb{N}$ then STOP** (there is no solution) **End If**;
2. **If $m = 0$, then STOP** ($x_S = g_G(0, 2)$) **End If**;
3. **If $\exists v \in V(G) : rank(Q^N) \le d_G(v) + 1$ $(N = N_G[v])$ then STOP** (the output is a consequence of the low multiplicity results) **End If**;
4. Determine $I = \{i_1, \ldots, i_m\} \subset V(G) : rank(Q^I) = m$ and **set $\mathbf{g} := \mathbf{g}_G(0, 2)$**;
5. **Set** *NoSolution := TRUE;*
6. **Set** $X := \{(x_{i_1}, \ldots, x_{i_m})$ which is (0, 2)-feasible for $G\}$;
7. **While** *NoSolution* $\wedge X \ne \emptyset$ **do**
8.   Choose $(x_{i_1}, \ldots, x_{i_m}) \in X$ and **Set** $x^I := (x_{i_1}, \ldots, x_{i_m})^T$;
9.   **Set** $X := X \backslash \{x^I\}$ and determine $\beta : x^I = \mathbf{g}^I + Q^I \beta$;
10.   **If $\mathbf{g} + Q\beta \in \{0, 1\}^n$ then** *NoSolution := FALSE* **End If**;
11. **End While**
12. **If** *NoSolution = FALSE* **then** $x := \mathbf{g} + Q\beta \in \{0, 1\}^n$ **else** return *NoSolution*;
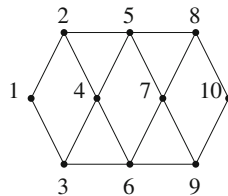13. **End If.**
14. **End.**

Fig. 1 Graph G

Example 1 Consider the graph G depicted in Fig. 1.

We will apply Algorithm 1 to determine a (0, 2)-regular set in G, a graph for which $m = m(-2) = 3$.

Since

$rank(\mathbf{j}) = 1,$
$rank(\mathbf{j} \, A_G \mathbf{j}) = 2,$
$rank(\mathbf{j} \, A_G \mathbf{j} \, (A_G)^2 \mathbf{j}) = 2,$

we have $p = 2$ distinct main eigenvalues of G.

$$W = (\mathbf{j} \, A_G \mathbf{j}) = \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 1 & 3 \\ 1 & 4 \\ 1 & 4 \\ 1 & 4 \\ 1 & 4 \\ 1 & 3 \\ 1 & 3 \\ 1 & 2 \end{pmatrix}.$$

The solution of the linear system $Wx = (A_G)^2\mathbf{j}$ is $C = \begin{pmatrix} -2 \\ 4 \end{pmatrix}$, so the coefficients of the main characteristic polynomial of G are $c_0 = 4, c_1 = -2$.

Next, the coefficients of vector $\mathbf{g} \in Main(G)$ will be determined.

Since

$$(\kappa - \tau)\alpha_0 = -\tau + \alpha_{p-1}c_{p-1}$$

and

$$(\kappa - \tau)\alpha_1 = \alpha_0 + \alpha_{p-1}c_{p-2},$$

we have $\alpha_0 = \frac{6}{7}, \alpha_1 = -\frac{1}{7}$, hence

$$\mathbf{g} = \alpha_0\mathbf{j} + \alpha_1 A_G\mathbf{j} = \begin{pmatrix} 0.2857 \\ 0.4286 \\ 0.2857 \\ 0.2857 \\ 0.4286 \\ 0.5714 \\ 0.4286 \\ 0.2857 \\ 0.4286 \\ 0.5714 \end{pmatrix}.$$

Considering matrix Q whose columns q_1, q_2, q_3 form a basis for the eigenspace associated to eigenvalue -2, we will proceed, searching for a vertex v for which the submatrix of Q corresponding to $N_G[v]$ has full rank.

$$Q = \begin{pmatrix} 0.4962 & 0.0345 & -0.1431 \\ -0.4962 & -0.0345 & 0.1431 \\ -0.4962 & -0.0345 & 0.1431 \\ 0.2335 & -0.1035 & -0.5878 \\ 0.2626 & 0.1380 & 0.4446 \\ 0.2626 & 0.1380 & 0.4446 \\ -0.2626 & 0.3795 & -0.4446 \\ 0 & -0.5175 & 0 \\ 0 & -0.5175 & 0 \\ 0 & 0.5175 & 0 \end{pmatrix}.$$

The obtained results are summarized in the following table.

v	$N_G(v)$	Rank($Q^{N_G[v]}$)
1	2, 3	1
2	1, 4, 5	2
3	1, 4, 6	2
4	2, 3, 5, 6	2
5	2, 4, 7, 8	3

It is obvious that the submatrix of Q corresponding to lines $2, 4, 5, 7$ and 8, the closed neighbourhood of vertex 5, has full rank, so we will consider the subvector of $\mathbf{g}$ and the submatrix of Q corresponding to $I = N_G[5] = \{2, 4, 5, 7, 8\}$.

$$\mathbf{g}^I = \begin{pmatrix} 0.4286 \\ 0.2857 \\ 0.2857 \\ 0.2857 \\ 0.4286 \end{pmatrix}, Q^I = \begin{pmatrix} -0.4962 & -0.0345 & 0.1431 \\ 0.2335 & -0.1035 & -0.5878 \\ 0.2626 & 0.1380 & 0.4446 \\ -0.2626 & 0.3795 & -0.4446 \\ 0 & -0.5175 & 0 \end{pmatrix}.$$

Supposing that G has a $(0, 2)$-regular set S, there are two possibilities to be considered: whether $5 \in S$ or $5 \notin S$ and there are $1 + \binom{4}{2}$ possible instances $(1) - (7)$ for the entries of x_S that correspond to $N_G[5]$ (see next table).

Inst.	$(x_S)_1$	$(x_S)_2$	$(x_S)_3$	$(x_S)_4$	$(x_S)_5$	$(x_S)_6$	$(x_S)_7$	$(x_S)_8$	$(x_S)_9$	$(x_S)_{10}$
(1)	*	0	*	0	1	*	0	0	*	*
(2)	*	1	*	1	0	*	0	0	*	*
(3)	*	1	*	0	0	*	1	0	*	*
(4)	*	1	*	0	0	*	0	1	*	*
(5)	*	0	*	1	0	*	1	0	*	*
(6)	*	0	*	1	0	*	0	1	*	*
(7)	*	0	*	0	0	*	1	1	*	*

Supposing that $5 \in S$, the entries of x_S corresponding to $I = \{2, 4, 5, 7, 8\}$ are

$$x_S^I = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}$$

and the solution of the subsystem

$$x_S^I = \mathbf{g}^I + \beta_1(q^I)_1 + \beta_2(q^I)_2 + \beta_3(q^I)_3$$

is $\beta_1 = 1.0215, \beta_2 = 0.8281, \beta_3 = 0.7461$.

Solving the complete system and calculating $\mathbf{g} + \beta_1 q_1 + \beta_2 q_2 + \beta_3 q_3$ for the evaluated values of β_1, β_2 and β_3, the following result is obtained

$$
x_S = \mathbf{g} + \beta_1 q_1 + \beta_2 q_2 + \beta_3 q_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}
$$

and $S = \{1, 5, 6, 10\}$ is a $(0, 2)$-regular set of G.

4   Application: Determination of Maximum Matchings

In this section, Algorithm 1 is combined with the procedure for maximum matchings described in [1], to provide a strategy for the determination of maximum matchings in arbitrary graphs. Such strategy is based on the determination of $(0, 2)$-regular sets in the correspondent line graphs, in the cases where they occur, or on the addition of extra vertices to the original graphs, in the situations where the line graphs under consideration have no $(0, 2)$-regular sets.

Considering the graph described in Example 1 and the $(0, 2)$-regular set determined by Algorithm 1, it is easily checkable that it corresponds to a maximum matching in graph G whose line graph is $L(G)$. Both graphs are depicted in Fig. 2.

It should be noticed that, according to Theorem 7 in [1], a graph G which is not a star neither a triangle has a perfect matching if and only if its line graph has a $(0, 2)$-regular set.

We will now determine a maximum matching in a graph whose line graph does not have a$(0, 2)$-regular set, which is equivalent to say that the root graph has no perfect matchings, following the algorithmic strategy proposed in [1].
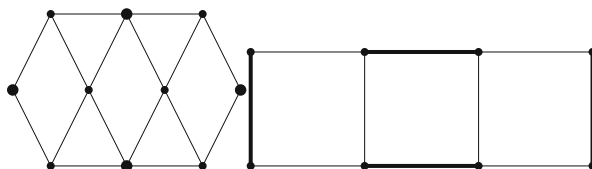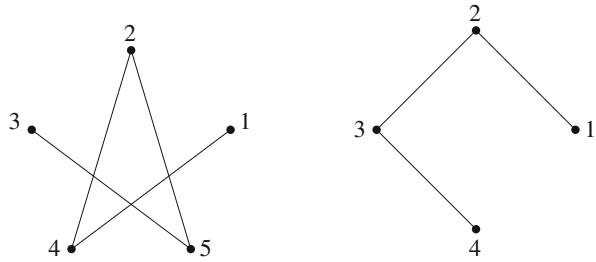

Fig. 2 Graphs $L(G)$ and G

Fig. 3 Graphs G_1 and $L(G_1)$

Example 2 Consider graphs G_1 and $L(G_1)$ both depicted in Fig. 3.

Since -2 is not an eigenvalue of $L(G_1)$, we will determine the parametric vector $\mathbf{g}_{L(G_1)}(0, 2)$ in order to find out if its coordinates are $0 - 1$. $L(G_1)$ has two main eigenvalues and its walk matrix is

$$W = \begin{pmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 2 \\ 1 & 1 \end{pmatrix}.$$

The coefficients of the main characteristic polynomial of $L(G_1)$, that are the solutions of system $Wx = A_{L(G_1)}\mathbf{j}$, are $c_0 = c_1 = 1$. The corresponding solutions of system (4), that is, the coefficients of $\mathbf{g}_{L(G_1)}(0, 2)$, are $\alpha_0 = \frac{6}{5}$ and $\alpha_1 = -\frac{2}{5}$. Therefore,

$$\mathbf{g}_{L(G_1)}(0, 2) = \alpha_0\mathbf{j} + \alpha_1 A_{L(G_1)}\mathbf{j} = \begin{pmatrix} 0.8 \\ 0.4 \\ 0.4 \\ 0.8 \end{pmatrix}$$

and it can be concluded that the graph $L(G_1)$ has no (0, 2)-regular sets. In order to determine a maximum matching in G_1, we will proceed as it is proposed in [1]. Since G_1 has an odd number of vertices, a single vertex will be added to G_1 and connected to all its vertices. The graph G_2 and its line graph $L(G_2)$, depicted in Fig. 4, are obtained.

Repeating the procedure described in Algorithm 1 (now applied to $L(G_2)$), we have that $m(-2) = 3$ and $p = 4$. It is also easy to verify that $c_0 = 5, c_1 = 1, c_2 = -6$ and $c_3 = 0$ are the coefficients of the main characteristic polynomial of $L(G_2)$. The solution of system (4) is $\alpha_0 = 1, \alpha_1 = -\frac{13}{20}, \alpha_2 = \frac{7}{20}, \alpha_3 = -\frac{1}{20}$ and the corresponding parametric vector $\mathbf{g}_{L(G_2)}(0, 2)$ is
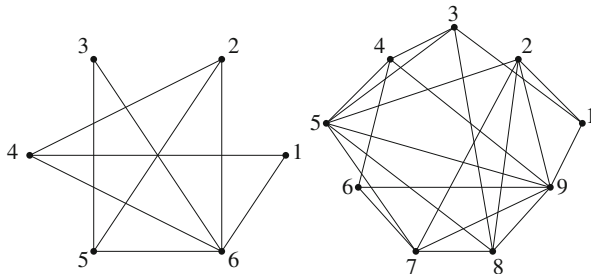
Fig. 4 Graphs G_2 and $L(G_2)$

$$\mathbf{g}_{L(G_2)}(0,2) = \alpha_0\mathbf{j} + \alpha_1 A_{L(G_2)}\mathbf{j} + \alpha_2 (A_{L(G_2)})^2\mathbf{j} + \alpha_3 (A_{L(G_2)})^3\mathbf{j} = \begin{pmatrix} 0.6 \\ 0.4 \\ 0.4 \\ 0.4 \\ 0.2 \\ 0.6 \\ 0.4 \\ 0 \\ 0 \end{pmatrix}.$$

We will now consider matrix Q, whose columns form a basis for the eigenspace associated to the eigenvalue -2 of $L(G_2)$.

$$Q = \begin{pmatrix} 0.2241 & 0.4458 & 0.2260 \\ -0.2241 & -0.4458 & -0.2260 \\ 0.3176 & -0.4525 & -0.3072 \\ -0.4115 & 0.3651 & -0.3121 \\ 0.0939 & 0.0875 & 0.6193 \\ -0.1302 & -0.3583 & 0.3933 \\ 0.1302 & 0.3583 & -0.3933 \\ -0.5416 & 0.0068 & 0.0812 \\ 0.5416 & -0.0068 & -0.0812 \end{pmatrix}.$$

Searching for a vertex of degree ≥ 2 in $L(G_2)$ whose closed neighbourhood corresponds to a submatrix of Q with maximum rank, the following table is obtained.

v	$N_{L(G_2)}(v)$	rank$(Q^{N_{L(G_2)}[v]})$
1	2, 3, 8	2
2	1, 5, 7, 8, 9	3
3	1, 4, 5, 8	3
4	3, 5, 6, 9	3
5	2, 3, 4, 7, 8, 9	3
6	4, 7, 9	2
7	2, 5, 6, 8, 9	3
8	1, 2, 3, 5, 7, 9	3
9	2, 4, 5, 6, 7, 8	3

It is evident that the closed neighbourhood of vertex 2 verifies the mentioned requirements and we will proceed considering the subvector of **g** and the submatrix of Q whose lines are the elements of $N_{L(G_2)}[2]$.

$$\mathbf{g}^I = \begin{pmatrix} 0.6 \\ 0.4 \\ 0.2 \\ 0.4 \\ 0 \\ 0 \end{pmatrix}, Q^I = \begin{pmatrix} 0.2241 & 0.4458 & 0.2260 \\ -0.2241 & -0.4458 & -0.2260 \\ 0.0939 & 0.0875 & 0.6193 \\ 0.1302 & 0.3583 & -0.3933 \\ -0.5416 & 0.0068 & 0.0812 \\ 0.5416 & -0.0068 & -0.0812 \end{pmatrix}.$$

Supposing that $L(G_2)$ contains a (0, 2)-regular set, there are $1+\binom{5}{2}$ possible instances for the entries of x_S that correspond to $N_{G_2}[2]$. One of them is

$(x_S)_1$	$(x_S)_2$	$(x_S)_3$	$(x_S)_4$	$(x_S)_5$	$(x_S)_6$	$(x_S)_7$	$(x_S)_8$	$(x_S)_9$
0	1	*	*	0	*	0	0	0

Assuming that $2 \in S$, the entries $1, 2, 5, 7, 8, 9$ of x_S must be of the form

$$x_S^I = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and the solution of the subsystem

$$x_S^I = \mathbf{g}^I + \beta_1(q^I)_1 + \beta_2(q^I)_2 + \beta_3(q^I)_3$$

is $\beta_1 = -0.0367, \beta_2 = -1.2566, \beta_3 = -0.1399$.

Solving the complete system $x_S = \mathbf{g} + \beta_1 q_1 + \beta_2 q_2 + \beta_3 q_3$ and computing $\mathbf{g} + \beta_1 q_1 + \beta_2 q_2 + \beta_3 q_3$, we obtain

$$x_S = \mathbf{g} + \beta_1 q_1 + \beta_2 q_2 + \beta_3 q_3 = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

and $S = \{2, 3, 6\}$ is a $(0, 2)$-regular set of $L(G_2)$. The resulting $(0, 2)$-regular set S corresponds to the perfect matching of G_2: $M^* = \{\{1, 4\}, \{2, 5\}, \{3, 6\}\}$. Therefore, intersecting the edges of M^* with the edge set of the root graph G_1, a maximum matching of G_1

$$M = \{\{1, 4\}, \{2, 5\}\},$$

is determined.

5   Final Remarks

The aim of this paper is the introduction of an algorithm for the determination of $(0, 2)$-regular sets in arbitrary graphs. In Sect. 2, an overview of the most relevant results about the determination of (κ, τ)-regular sets using the main eigenspace of a given graph is presented. Such results were introduced [3]. In Sect. 3, several results that lead to the determination of $(0, 2)$-regular sets are introduced and a new algorithm that determines a $(0, 2)$-regular set in an arbitrary graph or concludes that no such set exists is also described. Section 4 is devoted to the application of the introduced algorithm to the determination of maximum matchings.

Despite the interest of the introduced techniques for the determination of $(0, 2)$-regular sets in general graphs, their particular application to the determination of maximum matchings is not efficient in many cases. The use of these techniques in this context is for illustrating the application of the algorithm. It remains as an open problem, to obtain additional results for improving the determination of $(0, 2)$-regular sets in line graphs.

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