Superconvergence of Some Linear and Quadratic Functionals for Higher-Order Finite Elements

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Abstract. This paper deals with the calculation of linear and quadratic functionals of approximate solutions obtained by the finite element method. It is shown that under certain conditions the output functionals of an approximate solution are computed with higher order of accuracy than that of the solution itself. These abstract results are illustrated by two numerical examples for the Poisson equation.

Keywords: Finite element method · Output functionals · Dual problems · Hermite finite elements · Bogner-Fox-Schmit element · Convergence order

1 Introduction

Traditional methods for solving equations of mathematical physics, such as the finite element method, are to find a solution in the entire domain. Meanwhile, in a number of applications, researchers are interested not in the solution as a whole, but only in its goal-oriented output functionals. For example, in air flow around the body, engineers are interested in lift and drag rather than in the solution at every point in the space [1,2]. In such cases, one would be interested in the precision of these output functionals rather than of the entire solution. Moreover, with appropriate triangulation in the finite element method one can achieve a significant increase in the accuracy of the required functionals without increasing the computational time for the problem as a whole [2,3].

It has long been noted that the finite elements of higher degrees provide a higher order of convergence for an approximate solutions (under sufficient smoothness of the exact solution) [4–6]. And the weaker the norm in which the error between the exact and approximate solutions $u - u^h$ is estimated, the higher the rate of convergence. For example, in the norms of the Sobolev spaces $H^m(\Omega)$, the less m, the higher the attainable convergence rate

$$\left\| u - u^h \right\|_{H^m(\Omega)} \le ch^{k+1-m} \left\| u \right\|_{H^{k+1}(\Omega)}, \quad 0 \le m \le k+1,$$

where k is the full degree of the polynomials involved in the approximation of the solution. When solving second-order elliptic equations, the use of linear

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or polylinear polynomials corresponding to $k = 1$ quite simply leads to this estimate for $m = 1$. The Aubin-Nitsche approach (independently discovered and described in [7–9]) leads to this estimate for $m = 0$.

In this paper first we prove some abstract results and then consider a two-dimensional model problem solved by the finite element method with cubic Hermite elements. Unexpectedly the ultrahigh order of convergence was achieved for some output linear and quadratic functionals of an approximate solution which does not directly followed from the accuracy order of an approximate solution. Thus, the paper is devoted to the theoretical justification of this beneficial effect with the help of dual problems.

2 An Abstract Results

Let V and W be the Banach spaces with norms $\|\cdot\|_V$ and $\|\cdot\|_W$ respectively and V_h and W_h be the families of their finite-dimensional subspaces (trial and test subspaces of the finite element method) with a discrete set of h approaching 0.

Let $a(v, w) : \ V \times W \to R$ be a bounded bilinear form

$$|a(v, w)| \le c_1 \|v\|_V \|w\|_W \quad \forall\, v \in V, \ w \in W \tag{1}$$

with a constant c_1 independent of v and u. And let $f(w) : W \to R$ be a linear functional.

Suppose that we solve the problem:
find $u \in V$ such that

$$a(u, \varphi) = f(\varphi) \ \ \forall\, \varphi \in W. \tag{2}$$

But as it mentioned above, let the main purpose consist in the computation of the value $J(u)$ of an (output) linear functional $J(v) : V \to R$.

Instead of problem (2) we solve the following one (for example, by the finite element method):
find $u^h \in V_h$ such that

$$a(u^h, \varphi) = f(\varphi) \ \ \forall\, \varphi \in W^h. \tag{3}$$

Thereafter we compute the approximate value $J(u^h)$.

For a moment, suppose that the functional $J(v)$ is bounded:

$$|J(v)| \le c_2 \|v\|_V \quad \forall\, v \in V \tag{4}$$

with a constant c_2 independent of v. Then we get the estimate

$$\left|J(u) - J(u^h)\right| \le c_2 \left\|u - u^h\right\|_V . \tag{5}$$

We see that this estimate gives a rather modest result.

To improve this situation, consider the auxiliary dual problem:
find $w \in W$ such that

$$a(\psi, w) = J(\psi) \ \ \forall\, \psi \in V. \tag{6}$$

This problem is indeed auxiliary: we need not solve it either analytically or numerically.

Theorem 1. *Let the problems (2), (3), and (6) with the condition (1) have unique solutions* u, u^h, *and* w, *respectively. Besides, the approximation properties of the subspaces* V_h *and* W_h *provide the following estimate:*

$$\left\| u - u^h \right\|_V \leq c_3 h^r \tag{7}$$

and there exists an element $w^h \in W_h$ *such that*

$$\left\| w - w^h \right\|_W \leq c_4 h^s \tag{8}$$

with constants c_3, c_4, $r > 0$, $s > 0$ *independent of* h. *Then*

$$\left| J(u) - J(u^h) \right| \leq c_1 c_3 c_4 h^{r+s}. \tag{9}$$

Proof. Due to linearity

$$\left| J(u) - J(u^h) \right| = \left| J(u - u^h) \right|. \tag{10}$$

From (6) we have

$$J(u - u^h) = a(u - u^h, w). \tag{11}$$

From the problems (2) and (3) it follows that $a(u - u^h, w^h) = 0$. Subtract this from (11):

$$J(u - u^h) = a(u - u^h, w - w^h). \tag{12}$$

Then due to (1), (7), (8) we get

$$\left| J(u - u^h) \right| \leq c_1 \left\| u - u^h \right\|_V \left\| w - w^h \right\|_W \leq c_1 c_3 c_4 h^{r+s}. \qquad \square \tag{13}$$

Thus, the estimate (9) demonstrates a higher order of accuracy which is improved by the order of approximation in the dual problem.

Now consider the case where we need to find a *quadratic* functional of an approximate solution. For this purpose we introduce a *symmetric* bilinear form $b(v, w) : V \times V \to R$ and try to find the value $I(u) = b(u, u)$. Solving the problem (3) we get an the approximate solution $u^h \in V_h$ for which we can compute $I(u^h) = b(u^h, u^h)$. Show that under some simple conditions we again get a higher order of accuracy like for the linear functional.

For this purpose consider the auxiliary dual problem:
find $w \in W$ *such that*

$$a(\psi, w) = b(u + u^h, \psi) \;\; \forall \psi \in V. \tag{14}$$

Again this problem is indeed auxiliary and we need not solve it either analytically or numerically.

Theorem 2. *Let the problems (2), (3), and (14) with the condition (1) have unique solutions* u, u^h, *and* w, *respectively. And let the approximation properties of subspaces* V_h *and* W_h *provide the estimates (7) and (8). Then*

$$\left| I(u) - I(u^h) \right| \leq c_1 c_3 c_4 h^{r+s}. \tag{15}$$

Proof. Due to linearity in each arguments and symmetry between them we get

$$\left|I(u) - I(u^h)\right| = \left|b(u,u) - b(u^h,u^h)\right| = \left|b(u+u^h, u-u^h)\right|. \tag{16}$$

From (14) we have

$$b(u+u^h, u-u^h) = a(u-u^h, w). \tag{17}$$

From the problems (2) and (3) it follows that $a(u-u^h, w^h) = 0$. Subtract this from (17):

$$b(u+u^h, u-u^h) = a(u-u^h, w-w^h).$$

Then due to (1), (7), (8) we get

$$\left|b(u+u^h, u-u^h)\right| \le c_1 \left\|u-u^h\right\|_V \left\|w-w^h\right\|_W \le c_1 c_3 c_4 h^{r+s}.$$ □

Note that we did not use in the direct way the boundedness of the bilinear form

$$|b(v,w)| \le c_5 \|v\|_V \|w\|_V \quad \forall\, v,w \in V.$$

From here on, constants c_i are independent of functions in the right-hand side and of h. The usage of the above inequality gives a much weaker order of accuracy:

$$\begin{aligned}\left|I(u) - I(u^h)\right| = \left|b(u+u^h, u-u^h)\right| \le c_5 \left\|u+u^h\right\|_V \left\|u-u^h\right\|_V \le \\ \le c_3 c_5 h^r \left\|u+u^h\right\|_V.\end{aligned}$$

3 Formulations of Test Problems

Let Ω be the square $[0,1]\times[0,1]$ with the boundary Γ. For our further consideration we use the usual notations for Sobolev spaces [10]. Let $H^0(\Omega) = L^2(\Omega)$ be the Hilbert space of functions Lebesgue measurable on Ω and equipped with the inner product

$$(u,v)_\Omega = \int_\Omega u\, v\, d\Omega, \qquad u,v \in H^0(\Omega),$$

and the finite norm

$$\|u\|_{0,\Omega} = (u,u)_\Omega^{1/2}, \quad u \in H^0(\Omega).$$

For integer *positive* k, $H^k(\Omega)$ is the Hilbert space of functions $u \in H^0(\Omega)$ whose weak derivatives up to order k inclusive belong to $H^0(\Omega)$. The norm in this space is defined by the formula

$$\|u\|_{k,\Omega} = \left(\sum_{0\le s+r\le k} \left| \frac{\partial^{s+r} u}{\partial x^s\, \partial y^r} \right|^2_{0,\Omega} \right)^{1/2}.$$

Introduce also the functional space $H_0^1(\Omega)$ as the closure in the norm $\|\cdot\|_{1,\Omega}$ of all infinitely differentiable functions with support in Ω.

Consider the following model problem: *find* $u(x,y) \in H^2(\Omega)$ *such that*

$$- \Delta u = f(x,y) \text{ in } \Omega, \tag{18}$$

$$u = 0 \text{ on } \Gamma. \tag{19}$$

Let the solution u be smooth enough: $u \in H^4(\Omega)$. Then $f \in H^2(\Omega)$.

First take the output functional

$$J(u) := \int_\Omega ug \, d\Omega \tag{20}$$

with some function $g \in H^0(\Omega)$ and show that this functional is computed by bicubic finite elements with higher order of accuracy than a solution as a whole.

To get the weak form of this problem, multiply the equation (18) by an arbitrary function $\varphi \in H_0^1(\Omega)$ and integrate by parts with the help of the boundary conditions (19). As a result we get the equality

$$\int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial \varphi}{\partial y} \right) d\Omega = \int_\Omega f\varphi \, d\Omega. \tag{21}$$

In the weak form [5,6,11] this problem is reformulated as follows:
find $u \in H_0^1(\Omega)$ *such that*

$$a(u,\varphi) = (f,\varphi)_\Omega \;\; \forall \varphi \in H_0^1(\Omega) \tag{22}$$

with the bilinear form

$$a(u,\varphi) = \int_\Omega \left(\frac{\partial u}{\partial x}\frac{\partial \varphi}{\partial x} + \frac{\partial u}{\partial y}\frac{\partial \varphi}{\partial y} \right) d\Omega.$$

We construct a uniform triangulation $\Im_h$ by subdividing Ω into N^2 closed "rectangles" by the lines

$$x_i = ih, \;\; i = 0, ..., N; \;\; y_j = jh, \;\; j = 0, ..., N; \;\; \text{where } \; h = 1/N.$$

Here we shall describe a finite element by the triple $(e, \mathrm{P}_e, \Sigma_e)$ [6] where e is a "reference" cell (in this paper we put $e = [0,1]^2$); P_e is a space of polynomials on e; and Σ_e is the set of linear functionals called degrees of freedom (DoF).

Denote by P_k with positive *integer* k the space of all polynomials in two variables of full degree k:

$$\sum_{0 \le i+j \le k} a_{i,j} x^i y^j.$$

And denote by Q_k the space of all polynomials of degree k for each variable:

$$\sum_{0 \le i,j \le k} a_{i,j} x^i y^j.$$

First consider the possible implementation of the *bilinear* finite elements for solving the problem (22) by the Bubnov-Galerkin finite element method. Due to the approximation properties of these elements we can get only

$$\left\|u-u^h\right\|_{1,\Omega} \le c_3 h \quad \text{and} \quad \left\|w-w^h\right\|_{1,\Omega} \le c_4 h$$

with an interpolant w^h. Then from Theorem 1 we obtain

$$\left|J(u)-J(u^h)\right| \le c_1 c_3 c_4 h^2.$$

But this estimate does not provide us an improvement in comparison with the direct analysis by Aubin-Nitsche trick.

The situation is different for finite elements of higher order. First consider Lagrange elements on square $e = [0,1]^2$ (Fig. 1). Introduce the corresponding grid of nodes

$$S^{(2)} = S^{(1)} \times S^{(1)} \text{ where } S^{(1)} = \{a_i : \; a_i = i/3, \; i = 0, ..., 3\}.$$

a) full element b) serendipity element

Fig. 1. Nodes of the full and incomplete "serendipity" Lagrange cubic elements.

Then the bicubic element is described by the triple $(e, \; Q_3, \; \Sigma_3)$ where

$$\Sigma_3 = \{\psi_{i,j} : \; \psi_{i,j}(p) = p(a_i, a_j) \; \forall \, i,j = 0, ..., 3 \; \forall p \in Q_3\}. \tag{23}$$

It has 16 degrees of freedom on one elementary cell.

The usual mapping of the two-dimensional "reference" element into an elementary cell $[x_i, x_{i+1}] \times [y_j, y_{j+1}]$ of the triangulation $\Im_h$ has the form

$$\begin{cases} x' = x_i + hx, \\ y' = y_j + hy. \end{cases} \tag{24}$$

Generally speaking, the numbers of DoF are excessive to obtain the corresponding approximation order. Indeed, to achieve the same order of approximation it is sufficient to take polynomials P_3 on e [6] with the number of DoF equal to 10. Therefore the incomplete Lagrange "serendipity" element is often used. In this case, the DoF are omitted which lie strictly inside the cell e and have no influence on interelement continuity (Fig. 1b) [5,6]. The number of DoF for the serendipity element decreases and becomes equal 12. Since the polynomial spaces satisfy the condition $Q_{3'} \supset P_3$, the serendipity element provides the same order of approximation as the full Lagrange element and is more effective because of less number of DoF.

Now consider a simple Hermite bicubic element [12] (Fig. 2a). The number of DoF and the space of polynomials of this element coincide with those of the cubic serendipity element. Therefore, it may seem that they have identical properties. In fact, this is not the case! The Hermite element appears to be more efficient. To show this, we compare the global number of DoF for the interpolation of a smooth function u by these elements on the triangulation $\Im_h$ of the rectangle Ω.

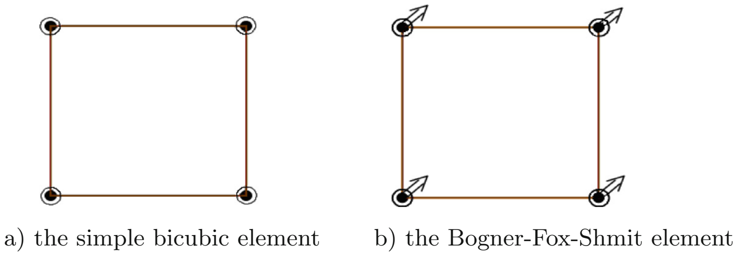

Fig. 2. The cubic Hermite elements. A circle means that DoF involve both first-order derivatives; a double arrow means that DoF involve the second-order mixed derivative.

The global number of DoF of the interpolant u_I^h on the triangulation $\Im_h$ is not proportional to the number of DoF on an element. A part of DoF for different elements coincides along interelement boundaries. Therefore, as the local characteristics of the global number of DoF we take the number $M$of DoF for the element on the half-closed set $[0,1)^2$ (Fig. 3). When mapping the element on the cells of $\Im_h$, these DoF are not repeated and exhaust all nodes inside Ω. Their total number is MN^2.

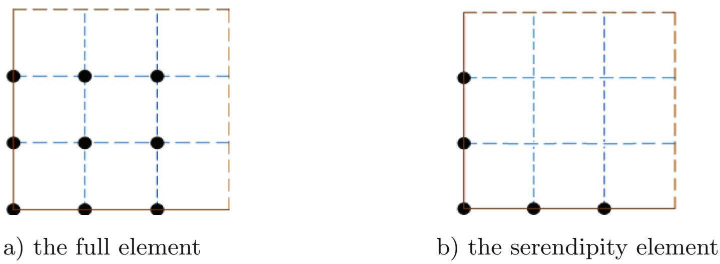

Fig. 3. Nodes of the Lagrange elements on the half-closed set $[0,1)^2$.

An open question remains on the number of nodes on the boundary Γ. In the Dirichlet problem, these DoF are excessive, but they are necessary for the Neumann problem. In both cases their number is of $O(N)$. Thus, MN^2 is the principal term of the asymptotic number of unknowns and equations in the finite element method, e.g., for a second-order elliptic equation.

The index M for the full Lagrange, serendipity, and Hermite bicubic elements is 9, 5, and 3 (Fig. 4a), respectively. Thus, the number of unknowns and equations in the finite element method for the Hermite element is approximately 3 times less than that for the full Lagrange element and 5/3 times less than that for the serendipity element. Such is the case despite the fact that they have the same order of approximation.

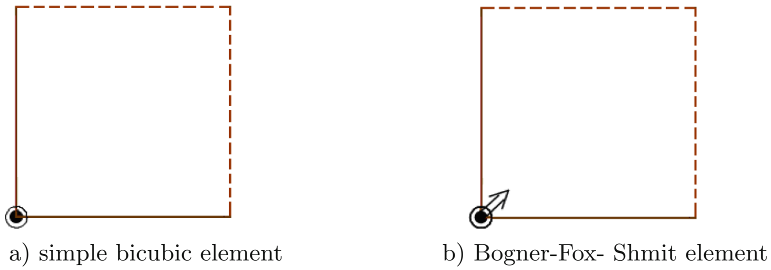

Fig. 4. Nodes of the Hermite elements on the half-closed set $[0, 1)^2$.

The Bogner-Fox-Schmit element [12–14] is a more complicated Hermite bicubic finite element. It is defined by the triple (Fig. 2b)

$$\begin{gathered}(e,\ Q_3,\ \Sigma_{3'}) \text{ where } \Sigma_{3'} = \{\psi_{s,i,j}\ (s = 0, 1, 2, 3):\ \ \psi_{0,i,j}(p) = p(a_i, a_j),\\ \psi_{1,i,j}(p) = \partial p/\partial x(a_i, a_j),\ \ \psi_{2,i,j}(p) = \partial p/\partial y(a_i, a_j),\\ \psi_{3,i,j}(p) = \partial^2 p/\partial x \partial y(a_i, a_j)\ \ \forall\, i, j = 0, 3\big\}\,.\end{gathered} \tag{25}$$

For the triangulation $\Im_h$, it provides continuity of an approximation u^h as well as of its first-order derivatives [12–14]. Thus, this element belongs to $H^2(\Omega)$ [11]. At the same time, it has the index $M = 4$ (Fig. 4b) which is less than the indices M of the Lagrange full and serendipity elements.

Now consider the implementation of any cubic finite elements for solving the problem (22) by the Bubnov-Galerkin finite element method. Due to the approximation properties of these elements (under sufficient smoothness) we can get

$$\left\|u - u^h\right\|_{1,\Omega} \le c_3 h^3 \quad \text{and} \quad \left\|w - w^h\right\|_{1,\Omega} \le c_4 h^3$$

for an interpolant w^h. Then from Theorem 1 we obtain

$$\left|J(u) - J(u^h)\right| \le c_1 c_3 c_4 h^6. \tag{26}$$

4 Numerical Results

Now consider the concrete problem (18)–(19) with the right-hand side

$$f(x, y) = 16(1 - x)(1 - y)(x^2 + y^2)\sin(4xy) + 8(x - x^2 + y - y^2)\cos(4xy). \tag{27}$$

This problem has the exact solution

$$u(x,y) = \sin(4xy)(1-x)(1-y).$$

And assume that we need to compute the output functional

$$J(u) := \int_{\Omega} u \, d\Omega. \tag{28}$$

We solve the problem (22) by the finite element method with the help of the Bogner-Fox-Schmit finite element. And then calculate (28) for an approximate solution u^h :

$$J(u^h) := \int_{\Omega} u^h \, d\Omega. \tag{29}$$

We perform these computations for $h = 1/8, 1/16, 1/32$ and determine the error $\varepsilon_1^h = \left| J(u) - J(u^h) \right|$. We demonstrate this error in Table 1 together with its decreasing exponent $d_1^h = \ln_2 \left| \varepsilon_1^{2h} / \varepsilon_1^h \right|$.

Table 1. The approximation errors and their decreasing exponent.

i	h	$\varepsilon_1^h = \left\lvert J(u) - J(u^h) \right\rvert$	d_1^h	$\varepsilon_2^h = \left\lvert I(u) - I(u^h) \right\rvert$	d_2^h
1	1/8	5.81×10^{-9}	–	3.92×10^{-9}	–
2	1/16	9.64×10^{-11}	5.91	6.94×10^{-11}	5.81
3	1/32	1.53×10^{-12}	5.97	1.15×10^{-12}	5.91

From this Table we can see that $\varepsilon_1^h = \left| J(u) - J(u^h) \right|$ tends to zero asymptotically as $O(h^6)$. But this does not follow from Theorem 1 in the direct way. Indeed, in this case the problem (6) has the form

$$\begin{aligned} -\Delta w &= 1 \text{ in } \Omega, \\ w &= 0 \text{ on } \Gamma. \end{aligned} \tag{30}$$

Despite the smoothness of the right-hand side the solution w does not belong to space $H^3(\Omega)$ because of singularities in four angles of the rectangle [15].

There are two ways to avoid these singularities. One of them is in special condensation of mesh in the vicinity of singularities. This is a really productive way in some cases [3]. But in our situation we got the sixth order without any condensation of mesh. This means that the justification must be finer. It may be transformed in different ways. One of them consists in the introduction of weighted norms in spaces $\|\cdot\|_V$ and $\|\cdot\|_W$. The reasoning is very tedious. We simplify it by some transformation of the theorem proof. Take the right-hand side of equality (12) and transform it in following way:

$$a(u - u^h, w - w^h) = \int_{\Omega} -\Delta(w - w^h)(u - u^h) \, d\Omega. \tag{31}$$

Introduce the weight function

$$\rho(x,y) = x(1-x)y(1-y)$$

and use it in the following way:

$$\left| \int_\Omega -\rho\Delta(w-w^h)\rho^{-1}(u-u^h)\, d\Omega \right| \le \left\| \rho\Delta(w-w^h) \right\|_{0,\Omega} \left\| \rho^{-1}(u-u^h) \right\|_{0,\Omega}.$$

The first norm becomes small enough because of weight degenerating in the vicinity of each angle:

$$\left\| \rho\Delta(w-w^h) \right\|_{0,\Omega} \le ch^2.$$

And the second norm becomes small enough because of the Aubin-Nitsche trick and degenerating both functions in the vicinity of the boundary:

$$\left\| \rho^{-1}(u-u^h) \right\|_{0,\Omega} \le ch^4.$$

Therefore

$$\left| J(u-u^h) \right| = \left| a(u-u^h, w-w^h) \right| \le ch^6.$$

This estimate indeed is consistent with numerical results.

For the quadratic functional the situation with Theorem 2 is simpler. Let we need to compute the output functional

$$I(u) := \int_\Omega u^2\, d\Omega = \; b(u,u) \quad \text{where} \quad b(v,w) = \int_\Omega vw\, d\Omega. \tag{32}$$

Let we solved the problem (22) by the finite element method with the help of the Bogner-Fox-Schmit finite element again. And then calculate the required value (32) for an approximate solution u^h :

$$I(u^h) := \int_\Omega (u^h)^2\, d\Omega. \tag{33}$$

We perform there computations for $h = 1/8$, $1/16$, $1/32$ and determine the error $\varepsilon_2^h = \left| I(u) - I(u^h) \right|$. This error is demonstrated in Table 1 together with its decreasing exponent $d_2^h = \ln_2 \left| \varepsilon_2^{2h}/\varepsilon_2^h \right|$. From this Table we can see that $\varepsilon_2^h = \left| I(u) - I(u^h) \right|$ tends to zero asymptotically as $O(h^6)$. Moreover, this follows directly from Theorem 2. Indeed, the function w is a solution of the problem

$$\begin{aligned} &-\Delta w = u + u^h \quad \text{in} \quad \Omega, \\ &w = 0 \quad \text{on} \quad \Gamma. \end{aligned} \tag{34}$$

First, this time the right-hand side of the problem belongs to $H^2(\Omega)$ due to the application of the Bogner-Fox-Schmit element. Second, it equals zero in each angle of the rectangle Ω. These properties ensure that $w \in H^4(\Omega)$ [15] and

$$\left\| w - w^h \right\|_{1,\Omega} \le c_4 h^3.$$

Function u also belongs to $H^4(\Omega)$ and provides the estimate [6]

$$\left\| u - u^h \right\|_{1,\Omega} \le c_4 h^3.$$

Thus, Theorem 2 indeed guarantees the sixth order of accuracy.

5 Resume

By virtue of the dual problems, for some linear and quadratic functionals we prove convergence of higher order than follows from the standard theory of the finite element method. Note that this effect becomes possible for more complicated (for example, cubic) finite elements than linear ones. For linear elements on triangles and for bilinear ones on quadrangles this approach does not give higher order of accuracy than it follows from the usual implementation of the Aubin-Nitsche trick.

Moreover, once again we remind that Hermite finite elements are more effective in comparison with the Lagrange ones of the same degrees of polynomials due to a smaller number of unknowns and of discrete equations in the finite element method. Besides, the Bogner-Fox-Schmit finite element is more effective than the Lagrange cubic elements and belongs to $H^2(\Omega)$ which simplifies the justification of higher order convergence and gives some useful possibilities like direct computation of a residual for an approximate solution u^h. This provides necessary and visual information for condensation of a triangulation. Usually, their use is limited to domains consisting of rectangles. But the complementing of these elements by suitable triangular elements near the boundary [16] extends the possible range of their application.

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