Asymptotic-Numerical Method for Moving Fronts in Two-Dimensional R-D-A Problems

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Abstract. A singularly perturbed initial-boundary value problem for a parabolic equation known in applications as the reaction-diffusion equation is considered. An asymptotic expansion of the solution with moving front is constructed. Using the asymptotic method of differential inequalities we prove the existence and estimate the asymptotic expansion for such solutions. The method is based on well-known comparison theorems and formal asymptotics for the construction of upper and lower solutions in singularly perturbed problems with internal and boundary layers.

Keywords: Singularly perturbed parabolic problems · Reaction-diffusion equation · Internal layers · Fronts · Asymptotic methods · Differential inequalities

1 Statement of the Problem

The purpose of the presented paper is to develop an effective numerical-asymptotic approach to study solutions with internal transition layers – moving fronts – in a mathematical model of reaction-diffusion type in the case of two spatial dimensions. We demonstrate our method for the following problem.

Consider the equation

$$\varepsilon^2 \Delta u - \varepsilon \frac{\partial u}{\partial t} = f\left(u, x, y, \varepsilon\right), \qquad y \in (0, a), \quad x \in (-\infty, +\infty), \quad t > 0 \qquad (1)$$

with the boundary and initial conditions

$$\left.\frac{\partial u}{\partial y}\right|_{y=0;a} = 0, \quad u(x, y, t, \varepsilon) = u(x+L, y, t, \varepsilon), \quad u(x, y, t, \varepsilon)|_{t=0} = u^0(x, y). \quad (2)$$

In the Eq. (1), $\varepsilon > 0$ is small parameter, which is usually a consequence of the parameters of the physical problem. It should be noted that the appearance of the small parameter before the spatial derivatives is determined by the characteristics of the physical system, while the small parameter before the time derivative determines only the scale of the time, convenient for the further consideration. Functions $u^0(x, y)$ and $f(u, x, y, \varepsilon)$ are assumed to be sufficiently smooth and L- periodic in the variable x.

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Stationary solutions of problem (1)–(2) with internal and boundary layers have been thoroughly investigated (see [1] and the references therein). The generation of an internal layer from smooth initial functions has also been studied (see [2,3]). Main purpose of this paper is to study the solution of moving front type and to obtain equations for effective description of its dynamics. We also prove the existence of a solution of such type and construct its asymptotics. The results below extend [4], where the case of one spatial dimension was considered, and the ideas in [5] are used for the proof of the existence of front type solutions.

Suppose the following conditions are satisfied.

($\mathbf{A_1}$). (a) *The function* $f(u,x,y,\varepsilon)$ *is such that the reduced equation* $f(u,x,y,0) = 0$ *has exactly three roots* $u = \varphi^{(\pm)}(x,y)$, $u = \varphi^{(0)}(x,y)$. (b) *Assume that* $\varphi^{(-)}(x,y) < \varphi^{(0)}(x,y) < \varphi^{(+)}(x,y)$ *for all* $(x,y) \in \bar{D} = (-\infty,+\infty) \times [0,a]$ *and* $f_u(\varphi^{(\pm)}(x,y),x,y,0) > 0$, $f_u(\varphi^{(0)}(x,y),x,y,0) < 0$.

It is known from [2,3] that under condition (A_1) and some quite general conditions for the initial function $u^0(x,y)$ at time of order $t_B(\varepsilon) = B\varepsilon\,|\ln\varepsilon|$ the solution of problem (1)–(2) quickly generates a thin internal transition layer between the two levels $\varphi^{(-)}(x,y)$ and $\varphi^{(+)}(x,y)$ located in the neighborhood of some curve $C_0^0 : y = h^0(x)$.

($\mathbf{A_2}$). *Assume that the initial function* $u^0(x,y)$ *has the form of a transition layer:* $u^0(x,y) = \varphi^{(-)}(x,y) + O(\varepsilon)$ *for* $(x,y) \in D_0^{(-)}$, $u^0(x,y) = \varphi^{(+)}(x,y) + O(\varepsilon)$ *for* $(x,y) \in D_0^{(+)}$ *excluding a small neighborhood of the curve* $C_0^0 : y = h^0(x)$.

Our further purpose is to study the front type solution of (1)–(2) and describe its dynamics.

Let us consider the following problem, where $f(u,x,y,\varepsilon)$ satisfies the condition (A_1) and x, y are parameters:

$$\frac{\partial^2 p}{\partial \xi^2} + W\frac{\partial p}{\partial \xi} = f(p,x,y,0); \quad p(x,0) = \varphi^{(0)}(x,y),\ p(x,\pm\infty) = \varphi^{(\pm)}(x,y) \quad (3)$$

This problem is well known (see, for example, [6]), and for every x, y there exists a unique pair $(W(x,y), p(\xi;x,y))$ that satisfies problem (3) and the following estimates are valid (C and σ are positive constants)

$$\left|p\,(x,\xi) - \varphi^{(\pm)}\,(x,y)\right| \le Ce^{\sigma|\xi|} \quad \text{for} \quad \xi \to \pm\infty.$$

($\mathbf{A_3}$). *There exists a solution* $h(x,t)$ *of the Cauchy problem*

$$\frac{h_t}{\sqrt{1+h_x^2}} = W(x,h(x,t)), \quad h(x,0) = h^0(x), \ \ h(x,t) = h(x+L,t), \ \ x \in (-\infty;+\infty).$$

Using this solution for fixed t *we define the curve* $C(t) \div \{y = h(x,t)\} \in \bar{D}$ *if* $t \in [0;T]$.

2 Description of the Moving Front

We define the location of the internal layer at fixed t by curve $C_\lambda(t) \div \{y = h^*(x,t,\varepsilon)\}$, which is the intersection of the solution $u(x,y,t,\varepsilon)$ and root $u = \varphi^{(0)}(x,y)$. An asymptotic approximation of $C_\lambda(t)$ will be constructed below. We denote by $D^{(+)}$ and $D^{(-)}$ the domains located at two sides of curve $C_\lambda(t)$.

2a. Formal asymptotic procedure.

To construct the formal asymptotics of the solution (1)–(2) we consider:

$$\varepsilon^2 \Delta u - \varepsilon \frac{\partial u}{\partial t} - f\,(u,x,y,\varepsilon) = 0, \qquad (x,y) \in D^{(\pm)}, \quad t > 0,$$

$$u(x,y,t,\varepsilon) = u(x+L,y,t,\varepsilon), \quad u(x,y,0,\varepsilon) = u^0(x,y,\varepsilon), \quad (x,y) \in D^{(\pm)}$$

$$u(x,h^*(x,t,\varepsilon),t,\varepsilon) = \varphi^{(0)}(x,h^*(x,t,\varepsilon)), \qquad \left.\frac{\partial u}{\partial y}\right|_{y=0} = 0 \tag{4}$$

and

$$u(x,h^*(x,t,\varepsilon),t,\varepsilon) = \varphi^{(0)}(x,h^*(x,t,\varepsilon)), \qquad \left.\frac{\partial u}{\partial y}\right|_{y=a} = 0 \tag{5}$$

To find the location of the internal transition layer $C_\lambda\,(t)$ we introduce local coordinates (r,l) in a neighborhood of some curve $C_0\,(t): \quad \{x = l,\ y = h(l,t)\}$, where r is the distance from $C_0\,(t)$ along the normal to this curve, with the sign "+" in the domain $D^{(+)}$ and with "–" in $D^{(-)}$, l is the coordinate of the point on the curve $C_0(t)$ from which this normal is going. We have

$$x = l + r \cdot n_1\,(l,t)\,, \quad y = h\,(l,t) + r \cdot n_2\,(l,t)\,, \tag{6}$$

where $n_1(l,t) = \frac{-h_l}{\sqrt{1+h_l^2}}$, $n_2(l,t) = \frac{1}{\sqrt{1+h_l^2}}$ are the components of the unit normal vector to $C_0(t)$ at the point $(l,\ h(l,t))$. Note, that in these coordinates the curve $C_0(t)$ is determined by $r = 0$. Further we will show, how to find $C_0(t)$.

Using these local coordinates we define the unknown curve $C_\lambda(t)$ in the form of a power series in ε:

$$r = \lambda^*\,(l,t,\varepsilon) = \varepsilon \cdot \lambda_1\,(l,t) + \varepsilon^2 \cdot \lambda_2\,(l,t) + ..., \tag{7}$$

The asymptotics of (4), (5) can be constructed in the form including regular and boundary functions

$$U^{(\pm)}(x,y,t,\varepsilon) = \bar{u}^{(\pm)}(x,y,\varepsilon) + P^{(\pm)}(\rho^{(\pm)},x,\varepsilon) + Q^{(\pm)}(\xi,l,\varepsilon), \tag{8}$$

where $\xi = \frac{r-\lambda^*(l,t,\varepsilon)}{\varepsilon}$, $\rho^{(+)} = \frac{a-y}{\varepsilon}$, $\rho^{(-)} = \frac{y}{\varepsilon}$, and the functions $\bar{u}^{(\pm)}(x,y,\varepsilon)$, $P^{(\pm)}(\rho^{(\pm)},x,\varepsilon)$, $Q^{(\pm)}(\xi,l,\varepsilon)$ are power series in ε, which can be find by the standard method of boundary functions [1]. The functions $Q^{(\pm)}\,(\xi,l,\varepsilon)$ describe the internal transition layer (moving front) near the curve $C_\lambda\,(t)$, therefore they depend on the variable t by means of ξ. The functions $P^{(\pm)}(\rho^{(\pm)},x,\varepsilon)$ describe the solution near the boundaries $y = 0$, $y = a$. The regular series $\bar{u}^{(\pm)}(x, y)$ in the domains $D^{(-)}$ and $D^{(+)}$, and also the boundary series $P^{(\pm)}\left(\rho^{(\pm)}, x, \varepsilon\right)$ near the boundaries of D are determined by the standard scheme [1]. Note that the boundary series $P^{(\pm)}\left(\rho^{(\pm)}, x, \varepsilon\right)$ are significant only in a small area near $y = 0$ and $y = a$, and rapidly exponentially decrease and do not influence the behavior of the internal transition layer. By this reason we concentrate only on the describing of the internal layer $Q^{(\pm)}\left(\xi, l, \varepsilon\right)$.

To define the terms of (7) and (8) we must write the asymptotic expansions for the solutions of each problems (4) and (5) according standard scheme [1]. Terms of series (7), (8) will be defined in this process from the conditions of continuous matching for the functions $U^{(-)}\left(x, y, t, \varepsilon\right)$, $U^{(+)}\left(x, y, t, \varepsilon\right)$ - asymptotic expansions in domains $D^{(\pm)}$ - and their normal derivatives on the curve $C_\lambda(t)$ (C^1 - matching conditions):

$$\text{(a)} \quad U^{(-)} = U^{(+)}, \qquad \text{(b)} \quad \varepsilon \frac{\partial U^{(-)}}{\partial n} = \varepsilon \frac{\partial U^{(+)}}{\partial n} \quad \text{on} \quad C_\lambda(t) \tag{9}$$

Conditions (9) must be carried out consistently for zero and all higher degrees of ε.

We briefly describe some details of this asymptotic procedure. Using the local coordinates (6) and introducing the stretched variable $\xi = \frac{r - \lambda^*(l,t,\varepsilon)}{\varepsilon}$, we have for the parabolic operator $\hat{L}u \equiv \varepsilon^2 \Delta u - \varepsilon \frac{\partial u}{\partial t}$ in the form

$$\hat{L}u = \frac{\partial^2 u}{\partial \xi^2} + V_n \frac{\partial u}{\partial \xi} - \varepsilon \left[k \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial t} \right] + O\left(\varepsilon^2\right), \tag{10}$$

where $V_n = V_0 + \lambda_t^*$ is the normal speed of the point on the curve $C_\lambda(t)$ and $V_0 \equiv \frac{h_t}{\sqrt{1 + h_l^2}}$; $k = k(l)$ is the local curvature of $C_\lambda\left(t\right)$, $\lambda^*(l, t, \varepsilon)$ is defined in (7).

We represent $f(u, x, y, \varepsilon)$ in the form $f(u, x, y, \varepsilon) = \bar{f}^{(\pm)}(x, y, \varepsilon) + Q^{(\pm)} f(\xi, l, \varepsilon)$, where the functions $\bar{f}^{(\pm)}(x, y, \varepsilon) = f(\bar{u}^{(\pm)}(x, y), x, y, \varepsilon)$ and

$$Q^{(\pm)} f(\xi, l, \varepsilon) = f(\bar{u}^{(\pm)}(x, y) + Q^{(\pm)}(\xi, l, \varepsilon), x, y, \varepsilon) - \bar{f}^{(\pm)}(x, y, \varepsilon)$$

are power series in ε, and the indices $(\pm)$ correspond to the domains $D^{(\pm)}$. Substituting these functions and the operator $\varepsilon^2 \Delta u - \varepsilon \frac{\partial u}{\partial t}$ in the form (10) into (4), (5) and equating the terms depending on (x, y) and (ξ, l) separately, we obtain the relations to determine the coefficients of the asymptotic expansions:

$$\varepsilon^2 \Delta \bar{u}^{(\pm)} - \bar{f}^{(\pm)}\left(u, x, y, \varepsilon\right) = 0, \tag{11}$$

$$\left(\frac{\partial^2}{\partial \xi^2} + V_n \frac{\partial}{\partial \xi} - \varepsilon \left(\frac{\partial}{\partial t} + k \frac{\partial}{\partial \xi} \right) + O\left(\varepsilon^2\right) \right) Q^{(\pm)} = Q^{(\pm)} f\left(\xi, l, t, \varepsilon\right). \tag{12}$$

2b. Zero order functions (moving front).

At zero order we have for regular part $f(u^{(\pm)}, x, y, 0) = 0$. Thus according to condition (A_1) we can take $u^{(\pm)}\left(x, y\right) = \varphi^{(\pm)}\left(x, y\right)$.

The functions $Q_0^{(\pm)}(\xi, l)$ satisfy the following problem:

$$\left(\frac{\partial^2}{\partial \xi^2} + V_0 \frac{\partial}{\partial \xi}\right) Q_0^{(\pm)}(\xi, l) = f\left(\varphi^{(\pm)}(x, y) + Q_0^{(\pm)}(\xi, l), x, y, 0\right), \tag{13}$$

$$Q_0^{(\pm)}(0, l) = \varphi^{(0)}(x, y) - \varphi^{(\pm)}(x, y) \text{ for } (x, y) \in C_\lambda(t); \quad Q_0^{(\pm)}(\pm\infty, l) = 0.$$

We define the continuous function $\tilde{u}(\xi) = \varphi^{(\pm)}(x, y) + Q_0^{(\pm)}(\xi, l)$ for $(x, y) \in C_\lambda t)$, and rewrite (13) as

$$\frac{\partial^2 \tilde{u}}{\partial \xi^2} + V_0 \frac{\partial \tilde{u}}{\partial \xi} = f(\tilde{u}, x, y, 0); \quad \tilde{u}(\pm\infty) = \varphi^{(\pm)}(x,\, y), \quad \tilde{u}(0) = \varphi^{(0)}(x, y) \tag{14}$$

If the function $f(u, x, y, \varepsilon)$ satisfies the condition (A_1), thus problem (14) has the unique solution $\tilde{u}(\xi)$, and the estimate $\left|\tilde{u}(\xi) - \varphi^{(\pm)}(x, y)\right| \le M e^{\sigma|\xi|}$ for $\xi \to \pm\infty$ is valid, where M and σ are positive constants [6]. Note, that condition (9a) is fulfilled by the definition of $\tilde{u}(\xi)$. If we suppose $V_0 = W(x, y)$ (see condition (A_3)) and define the curve $C_0(t)$ according to condition (A_3), then the $C^{(1)}$ - matching conditions (9b) in zero order will be fulfilled also. So, the location of the moving front in zero order approximation is the curve $C_0(t)$, which satisfies the following Cauchy problem:

$$\frac{h_t}{\sqrt{1 + h_x^2}} = W(x, h), \quad h(x, 0) = h^0(x), \quad h(x, t) = h(x + L, t) \tag{15}$$

According to condition (A_3), there exists such $T > 0$, that the solution $h(x, t)$ of (15) defines the curve $C_0(t) \div \{y = h(x, t)\} \in \bar{D}$ for $t \in [0, T]$.

2c. First order asymptotics.

Separating terms with ε^1 in (11), we obtain for the regular functions $\bar{u}_1^{(\pm)}$ the equation $f_u(\varphi^{(\pm)}(x, y), x, y, 0) \cdot \bar{u}^{(\pm)} + f_\varepsilon(\varphi^{(\pm)}(x, y), x, y, 0) = 0$, which has a unique solution (see condition (A_1)).

For the transition layer functions $Q_1^{(\pm)}$ we get the linear differential equations

$$\left(\frac{\partial^2}{\partial \xi^2} + V_0 \frac{\partial}{\partial \xi} - f_u(\tilde{u}(\xi), x, y, 0)\right) Q_1^{(\pm)} = q_1(\xi, l, t) \equiv \frac{\partial Q_0^{(\pm)}(\xi, l)}{\partial t} + \tag{16}$$

$$+ (k - (\lambda_1)_t) \frac{\partial Q_0^{(\pm)}(\xi, l)}{\partial \xi} + \tilde{f}_\varepsilon + \left[\tilde{f}_r + \tilde{f}_u \cdot \frac{\partial \varphi^{(\pm)}}{\partial r}\right](\lambda_1 + \xi) + \left(\tilde{f}_u - \bar{f}_u\right) \bar{u}_1^{(\pm)}$$

with the boundary conditions $Q_1^{(\pm)}(\pm\infty, l) = 0$,

$$Q_1^{(\pm)}(0, l) = \left[-u_1^{(\pm)}(x, y) + \lambda_1(l, t)\left(\frac{\partial \varphi^{(0)}}{\partial n} - \frac{\partial \varphi^{(\pm)}}{\partial n}\right)\right]\Bigg|_{(x,y) \in C_0(t)}. \tag{17}$$

In (16), (17) $\lambda_1(l, t)$ is the unknown first term of (7), $k = k(l)$ is the local curvature of $C_0(t)$; $\tilde{g}$ means the function depending on $(\tilde{u}(\xi), (x, y) \in C_0(t), t)$ and $\bar{g}$ - the function depending on $\left(\varphi^{(\pm)}, (x, y) \in C_0(t), t\right)$.

If we mark $\Phi(\xi,t) = \frac{\partial \tilde{u}}{\partial \xi}$, the solution of (16) with boundary conditions from (17) can be written explicitly (function $q_1(\xi,l,t)$ defined in (16)):

$$Q_1^{(\pm)}(\xi,l) = \frac{\Phi(\xi,t)}{\Phi(0,t)} Q_1^{(\pm)}(0,l) - \Phi(\xi,t) \int\limits_0^{\xi} \frac{e^{-V_0 \eta}}{\Phi^2(\eta,t)} \int\limits_{\eta}^{\pm\infty} \Phi(\tau,t) e^{V_0 \tau} q_1(\tau,l,t) d\tau d\eta \tag{18}$$

Using the $C^{(1)}$ - matching condition (9) for $Q_1^{(\pm)}(\xi,l)$ we get

$$\left. \frac{\partial Q_1^{(+)}(\xi,l)}{\partial \xi} - \frac{\partial Q_1^{(-)}(\xi,l)}{\partial \xi} \right|_{\xi=0} = \left. \frac{\partial \varphi^{(-)}(x,y)}{\partial n} - \frac{\partial \varphi^{(+)}(x,y)}{\partial n} \right|_{(x,y)\in C_0(t)} \tag{19}$$

Substituting the derivatives $\left. \frac{\partial Q_1^{(\pm)}(\xi,l)}{\partial \xi} \right|_{\xi=0}$, calculated from (18) into (19), we obtain the linear Cauchy problem for $\lambda_1(l,t)$:

$$\frac{d\lambda_1(l,t)}{dt} - k(l) = B(l,t) \cdot \lambda_1(l,t) + R(l,t), \quad \lambda_1(l,0) = 0, \tag{20}$$

where $B(l,t)$ and $R(l,t)$ are known function, which do not depend of $\lambda_1(l,t)$ and of its derivatives, $k(l)$ is the local curvature of the curve $C_0(t)$.

Continuing this procedure for higher orders terms in ε and we get linear problems for all $Q_i^{(\pm)}(\xi,l)$, $i = 2,3,...$ and also linear Cauchy problems of type (20) for $\lambda_i(l,t)$, $i = 2,3,...$

As a result, we obtain a nonlinear equation, which determines the location of the moving front at zero order approximation, and linear equations for higher order terms. Note that now we can estimate the location of the moving front and adequately describe the front dynamics not from the original system (1)–(2), but from problem (15) in zero order approximation in ε and from the problems of type (20) at higher order approximations in ε. We present a comparison of asymptotic and numerical results in Sect. 5.

3 Existence of Solution and the Main Theorem

The proof for the existence of a solution to (1)–(2) is based on the asymptotic method of differential inequalities similarly to the case of one spatial dimension (see [4]) with slight changes. Let define $D_n^{(+)}$ and $D_n^{(-)}$ the domains located at two sides of curve $C_n(t)$, where

$$\Lambda_n(l,t) = \sum_{i=1}^{n+1} \varepsilon^i \lambda_i(l,t), \quad \xi_n = \frac{r - \Lambda_n(l,t)}{\varepsilon}, \quad C_n(t) : r = \Lambda_n(l,t) \tag{21}$$

$$U_n(x,y,t,\varepsilon) = \begin{cases} \sum\limits_{i=0}^{n} \varepsilon^i \left(\bar{u}_n^{(+)}(x,y) + Q_n^{(+)}(\xi_n,l,t) \right), & (x,y) \in D_n^{(+)} \\ \sum\limits_{i=0}^{n} \varepsilon^i \left(\bar{u}_n^{(-)}(x,y) + Q_n^{(-)}(\xi_n,l,t) \right), & (x,y) \in D_n^{(-)} \end{cases} \tag{22}$$

We define upper and lower solution $\alpha(x,y,t,\varepsilon)$, $\beta(x,y,t,\varepsilon)$ as follows:

(1) $\alpha(x,y,t,\varepsilon) \leq \beta(x,y,t,\varepsilon), \qquad \alpha,\, \beta(x,y,t,\varepsilon) = \alpha,\, \beta(x+L,y,t,\varepsilon)$

(2α) $\varepsilon^2 \Delta\alpha - \varepsilon\dfrac{\partial\alpha}{\partial t} - f(\alpha,x,y,\varepsilon) \geq 0, \quad (x,y) \in \bar{D}, \quad t \in (0,T], \quad \varepsilon \in (0,\varepsilon_0]$

(2β) $\varepsilon^2 \Delta\beta - \varepsilon\dfrac{\partial\beta}{\partial t} - f(\beta,x,y,\varepsilon) \leq 0, \quad (x,y) \in \bar{D}, \quad t \in (0,T], \quad \varepsilon \in (0,\varepsilon_0]$

(3) $\left.\frac{\partial\alpha}{\partial y}\right|_{y=0} \geq 0, \quad \left.\frac{\partial\alpha}{\partial y}\right|_{y=a} \leq 0; \quad \left.\frac{\partial\beta}{\partial y}\right|_{y=0} \leq 0, \quad \left.\frac{\partial\beta}{\partial y}\right|_{y=a} \geq 0$. Suppose also that initial function satisfies $\alpha\,(x,y,0,\varepsilon) \leq u^0\,(x,y,\varepsilon) \leq \beta\,(x,y,0,\varepsilon)$.

Now we can formulate the main result in the following theorem.

Theorem 1. *Under the conditions* $(\mathrm{A}_1) - (\mathrm{A}_3)$ *for sufficiently smooth initial function and sufficiently small* ε *there exists the solution* $u(x,y,t,\varepsilon)$ *of the problem (1)–(2) and satisfies*

1. $\alpha(x,y,t,\varepsilon) \leq u(x,y,t,\varepsilon) \leq \beta(x,y,t,\varepsilon)$,
2. $u(x,y,t,\varepsilon) = U_n(x,y,t,\varepsilon) + O(\varepsilon^{n+1})$ *for* $(x,y) \in \bar{D}$, $t \in (0,T]$, $\varepsilon \in (0,\varepsilon_0]$.

Main ideas, how to prove this theorem you can see in [4]. We construct upper and lower solutions by modification of (22)–(21), verify inequalities (1)–(3) from the definitions of $\alpha\,(x,y,t,\varepsilon)$, $\beta\,(x,y,t,\varepsilon)$ and control proper sign of the jump of first normal derivative at the curve $\bar{C}(t) = C_n(t) - \varepsilon^{n+1}\delta(t)$. Required calculations can be done in the same way as in [4].

4 Examples

In this section we present an example, for which we can calculate some parameters of the front in zero order (e.g., normal speed) explicitly. Consider the problem

$$\varepsilon^2 \Delta u - \varepsilon\frac{\partial u}{\partial t} = \left(u - \varphi^0(x,y)\right)\cdot\left(u^2-1\right), \quad y \in (0,1),\ x \in (-\infty,+\infty),\ t > 0$$

$$u_y\big|_{y=0,y=1} = 0, \quad u(x,y,t,\varepsilon) = u(x+1,y,t,\varepsilon), \quad u(x,y,t,\varepsilon)|_{t=0} = u^0(x,y,\varepsilon),$$

where $-1 < \varphi^0(x,y) < 1$ and $u^0(x,y,\varepsilon)$ satisfies condition (A_2). Note, that in this case $\varphi^{(-)}(x,y) = -1$ and $\varphi^{(+)}(x,y) = 1$, and if we mark $\tilde{u}'_\xi = z$, we can write the problem (15) for the zero order function $\tilde{u}(\xi) = \varphi^{(\pm)}(x,y) + Q_0^{(\pm)}(\xi,l)$ in the form

$$z\frac{dz}{d\tilde{u}} + V_0 z = (\tilde{u} - \varphi^0(x,y))\cdot(\tilde{u}^2-1), \qquad \tilde{u}(\pm\infty) = \pm 1. \tag{23}$$

Solution of (23) exists, if there exists a separatrix going from the saddle point $(0;-1)$ to the saddle point $(0;+1)$. If we find this separatrix in the form $z = A\cdot(\tilde{u}^2-1)$, $A<0$, we obtain $A = \frac{-1}{\sqrt{2}}$ and $A\cdot V_0 = -\varphi^0(x,y)$ so $V_0 = \varphi^0(x,y)\cdot\sqrt{2}$, and Eq. (15) for the moving front at zero order of ε takes the form

$$\frac{h_t}{\sqrt{1+h_x^2}} = \varphi^0(x,h)\cdot\sqrt{2}, \quad h(x,0) = h^0(x), \quad h(x,t) = h(x+1,t).$$

5 Numerical Experiment

The asymptotic approximation will be compared with the results of numerical solution of the problem (1)–(2). For this purpose we use a finite-difference scheme for problem (1)–(2) and for the Eq. (15). Calculations are done in D, representing a rectangle with the sides $L = 1$, $a = 1$, for $\varepsilon = 0.01$ and the function $f(u, x, y, \varepsilon) = \left(u^2 - 1\right) \cdot \left(u - \varphi^0(x, y)\right)$ for some cases of $\varphi^0(x, y)$ and the initial curve $y = h^0(x)$. Results are represented in Figs. 1, 2. Figure 1 shows sequent positions of the front (zero order asymptotics and the numerical solution of full problem (1)–(2)) at different times for $\varphi^0(x, y) = 0.15 \cos 4\pi x$, $h^0(x) = 0.5 - 0.15 \sin 2\pi x$. Figure 2 shows sequent positions of the front at different times for $\varphi^0(x, y) = 0.15 \cos 4\pi x$, $h^0(x) = 0.5 - 0.15 \sin 2\pi x$.

The analysis of the numerical calculations showed a good correspondence between the above asymptotic descriptions of the front behavior by (15) and numerical calculations for problem (1)–(2). Thus, the asymptotic approach allows fully to describe the dynamics and the shape of the moving front, its width and the time process of its formation, which is important for the effective estimate of various parameters of the physical system. In addition, the combination of asymptotic and numerical methods gives the possibility to speed up the process of constructing approximate solutions with a suitable accuracy. As a result, we have more efficient numerical calculations.

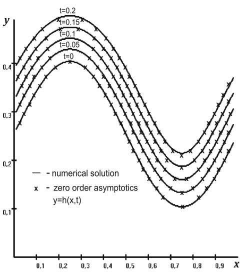

Fig. 1.

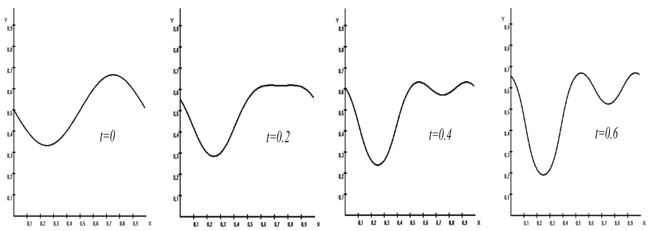

Fig. 2.

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References

1. Vasilieva, A.B., Butuzov, V.F., Nefedov, N.N.: Contrast structures in singularly perturbed problems. J. Fund. Prikl. Math. **4**(3), 799–851 (1998)
2. Volkov, V.T., Grachev, N.E., Nefedov, N.N., Nikolaev, A.N.: On the formation of sharp transition layers in two-dimensional reaction-diffusion models. J. Comp. Math. Math. Phys. **47**(8), 1301–1309 (2007)
3. Butuzov, V.F., Nefedov, N.N., Schneider, K.R.: On generation and propagation of sharp transition layers in parabolic problems. Vestnik MGU **3**(1), 9–13 (2005)

4. Bozhevolnov, Y.V., Nefedov, N.N.: Front motion in parabolic reaction-diffusion problem. J. Comp. Math. Math. Phys. **50**(2), 264–273 (2010)
5. Nefedov, N.N.: The method of diff. inequalities for some classes of nonlinear singul. perturbed problems. J. Diff. Uravn. **31**(7), 1142–1149 (1995)
6. Fife, P.C., Hsiao, L.: The generation and propagation of internal layers. Nonlinear Anal. Theory Meth. Appl. **12**(1), 19–41 (1998)
7. Volkov, V., Nefedov, N.: Asymptotic-numerical investigation of generation and motion of fronts in phase transition models. In: Dimov, I., Faragó, I., Vulkov, L. (eds.) NAA 2012. LNCS, vol. 8236, pp. 524–531. Springer, Heidelberg (2013)