Operator Semigroups for Convergence Analysis

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Abstract. The paper serves as a review on the basic results showing how functional analytic tools have been applied in numerical analysis. It deals with abstract Cauchy problems and present how their solutions are approximated by using space and time discretisations. To this end we introduce and apply the basic notions of operator semigroup theory. The convergence is analysed through the famous theorems of Trotter and Kato, Lax, and Chernoff. We also list some of their most important applications.

Keywords: Numerical analysis · Operator semigroups · Convergence analysis · Trotter–kato approximation theorem · Lax equivalence theorem · Chernoff's theorem

1 Introduction

In the present paper we will give an overview on how functional analytic tools have been applied in numerical analysis. In particular, we will consider well-posed partial differential equations and analyse how to ensure the convergence of their numerical solution to the exact one. To this end, we will treat the problem in a functional analytic framework and apply results from operator semigroup theory, for which our main reference is the monograph by Engel-Nagel [5].

We start with an example to motivate what kind of problems are to solved when seeking a numerical solution. In Sect. 3 the corresponding abstract problem and its solution, the operator semigroup, will be introduced. The convergence of the space and time discretisation methods are analysed in Sect. 4.1 and 4.2, respectively, based on the results of Trotter [15], Kato [9], Ito and Kappel [8], Lax and Richtmeyr [11], and Chernoff [4]. In Sect. 5 we show how the previous results can be combined and present the convergence result of Bátkai et al. [1] based on the work of Pazy [14]. Section 6 deals as an outlook on other topics in numerical analysis where operator semigroups play an important role.

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2 Motivation

As a motivating example we consider the one-dimensional heat equation on the interval $[0,\pi]$ with homogeneous Dirichlet boundary condition

$$\begin{cases} \frac{\partial}{\partial t} w(t,x) = \frac{\partial^2}{\partial x^2} w(t,x), & t>0,\ x\in(0,\pi), \\ w(0,x) = w_0(x), & x\in(0,\pi), \\ w(t,0) = w(t,\pi) = 0 & \end{cases} \tag{1}$$

with the given initial function $w_0 \in \mathrm{L}^2(0,\pi)$. Its solution is obtained by separating the variables and has the form

$$w(t,x) = \sum_{j=1}^{\infty} c_j \mathrm{e}^{-j^2 t} \sin(jx) \ \text{ with } \ c_j = \tfrac{2}{\pi} \int_0^{\pi} w_0(x) \sin(jx) \mathrm{d}x,\ j\in\mathbb{N}. \tag{2}$$

2.1 Numerical Solution

We show now two ways how to obtain an approximation to w, that is, the numerical solution to problem (1).

Example 1 (Finite differences). We approximate the partial derivatives in problem (1) by the usual finite difference schemes on equidistant spatial and temporal meshes with grid size $h = \frac{\pi}{m-1} > 0$, for some fixed $m \in \mathbb{N}\setminus\{1\}$, and time step $\tau > 0$:

$$\begin{aligned} \tfrac{\partial}{\partial t} w(t,x) &\approx \tfrac{1}{\tau}\big(w(t+\tau,x) - w(t,x)\big), \\ \tfrac{\partial^2}{\partial x^2} w(t,x) &\approx \tfrac{1}{h^2}\big(w(t,x-h) - 2w(t,x) + w(t,x+h)\big). \end{aligned}$$

This leads to the following discrete problem for $w_j^{(\ell)} \approx w(\ell\tau,(j-1)h)$ for $j = 1,...,m$ and $\ell\in\mathbb{N}$:

$$\tfrac{1}{\tau}(w_j^{(\ell)} - w_j^{(\ell-1)}) = \tfrac{1}{h^2}\big(w_{j+1}^{(\ell-1)} - 2w_j^{(\ell-1)} + w_{j-1}^{(\ell-1)}\big). \tag{3}$$

Due to the initial and boundary conditions $w_j^{(0)} = w_0((j-1)h)$, $j = 1,...,m$, and $w_1^{(\ell)} = w_m^{(\ell)} = 0$, $\ell\in\mathbb{N}$, the solution

$$w_j^{(\ell)} = w_j^{(\ell-1)} + \tfrac{\tau}{h^2}\big(w_{j+1}^{(\ell-1)} - 2w_j^{(\ell-1)} + w_{j-1}^{(\ell-1)}\big) \tag{4}$$

can be computed step by step for all indices $\ell\in\mathbb{N}$ and $j = 2,...,m-1$. Then the approximation of w is obtained by certain interpolation schemes in space and time.

Example 2 (Spectral method). Let $\widehat{w}_j \in \mathbb{R}$ denote the j^{th} Fourier coefficient of $w(\cdot,x)$ and $\langle\cdot,\cdot\rangle$ the inner product in $\mathrm{L}^2(0,\pi)$. We define further the function

$\varphi_j(x) = \sqrt{\frac{2}{\pi}} \sin(jx)$, $j = 1, ..., m$, satisfying the boundary condition in problem (1). By taking the discrete Fourier transform of both sides of problem (1), one obtains the following initial value problem for the first m Fourier coefficients of w:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} \widehat{w}_j(t) = -j^2 \widehat{w}_j(t), & t \in \mathbb{R},\ j = 1, ..., m, \\ \widehat{w}_j(0) = \langle w_0, \varphi_j \rangle, & j = 1, ..., m \end{cases} \tag{5}$$

with the solution $\widehat{w}_j(t) = \mathrm{e}^{-j^2 t} \widehat{w}_j(0) = \mathrm{e}^{-j^2 t} \langle w_0, \varphi_j \rangle$. Then the approximation of w is obtained with the help of the inverse discrete Fourier transform, that is,

$$w(t, x) \approx \sum_{j=1}^{m} \widehat{w}_j(t) \varphi_j(x) = \sum_{j=1}^{m} \mathrm{e}^{-j^2 t} \langle w_0, \varphi_j \rangle \sqrt{\tfrac{2}{\pi}} \sin(jx). \tag{6}$$

We remark that the formula above really seems to approximate w, since it corresponds to $c_j = \sqrt{\frac{2}{\pi}} \langle w_0, \varphi_j \rangle$ for $j = 1, ..., m$ and $c_j = 0$ for $j > m$ in (2). This means that in this case the inifinite sum is approximated by a finite one.

2.2 Abstract Setting

Problem (1) can also be handled in an abstract way. To this end we define the Banach space $X = \mathrm{L}^2(0, \pi)$, the linear operator $A : X \to X$ as $Af = f''$ for all $f \in \{\eta \in \mathrm{L}^2(0, \pi) :\ \eta(0) = \eta(\pi) = 0\}$, and the function $u : [0, \infty) \to X$ as $(u(t))(x) = w(t, x)$ for all $t \geq 0$ and $x \in [0, \pi]$. Then problem (1) corresponds to the following initial value problem on X:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} u(t) = Au(t), & t > 0, \\ u(0) = u_0 \end{cases} \tag{7}$$

with $u_0 = w_0$. In order to solve problem (7) numerically, for $m \in \mathbb{N}$ one defines Banach spaces X_m, some suitable (for the sake of simplicity linear) operators $P_m : X \to X_m$, $J_m : X_m \to X$, and a linear operator $A_m : X_m \to X_m$. Then the numerical solution $u_m : [0, \infty) \to X_m$ is obtained from the following initial value problem in X_m for all $m \in \mathbb{N}$:

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} u_m(t) = A_m u_m(t), & t > 0, \\ u_m(0) = P_m u_0 \end{cases} \tag{8}$$

with $u_0 = w_0$. Problem (8) corresponds to the spatially discretised version of problem (7). The solution of the original problem (1) is obtained as $w(t, x) = (u(t))(x)$ where it is to analysed whether $u(t) = \lim\limits_{m \to \infty} J_m u_m(t)$ holds uniformly for t in compact intervals. In some cases $u_m(t)$ is further approximated by $u_{m,k}$ by using certain time discretisation methods (see the examples below). Then

$$u(t) = \lim_{m \to \infty} J_m \lim_{k \to \infty} u_{m,k} \tag{9}$$

should hold uniformly for t in compact intervals. In Sect. 5 we will study under which conditions the limit (9) holds. The corresponding choices of the spaces and operators in Examples 1 and 2 are the following.

(a) *Example* 1. We choose $X_m = \mathbb{R}^m$, $(P_m w_0)_j = w_0((j-1)h)$ for $j = 1, ..., m$, and

$$A_m = \frac{1}{h^2} \begin{pmatrix} 1 & 0 \dots 0 & 0 \\ 0 & D_{m-2} & \vdots \\ \vdots & & 0 \\ 0 & 0 \dots 0 & 1 \end{pmatrix} \in \mathbb{R}^{m \times m}$$

with $D_{m-2} = \text{tridiag}(1, -2, 1) \in \mathbb{R}^{(m-2)\times(m-2)}$. Then for all $t \geq 0$ and $j = 1, ..., m$, $(u_m(t))_j \in \mathbb{R}$ correspond to the approximate values at the grid points $(j-1)h$, and $u_m(t) \in \mathbb{R}^m$ is their vector. The solution to problem (8) in this case reads as $u_m(t) = \mathrm{e}^{tA_m} P_m w_0$. Since

$$\mathrm{e}^{tA_m} = \lim_{k\to\infty} \left(I_m + \tfrac{t}{k} A_m\right)^k,$$

where $I_m \in \mathbb{R}^{m\times m}$ denotes the identity matrix, one approximates the exponential matrix and obtaines the numerical solution

$$u_{m,k} = \left(I_m + \tfrac{t}{k} A_m\right)^k P_m w_0 \quad \text{for some } k \in \mathbb{N}.$$

If $k \in \mathbb{N}$ and $\tau = \frac{t}{k} > 0$ are fixed, we have

$$u_{m,k}^{(\ell)} = (I_m + \tau A_m)^\ell P_m w_0 = (I_m + \tau A_m) u_{m,k}^{(\ell-1)} \quad \text{for all } \ell \in \mathbb{N},$$

and this corresponds to formula (4), that is, $\left(u_{m,k}^{(\ell)}\right)_j = w_j^{(\ell)}$. The operator J_m describes an interpolation, such as the Lagrangian polynom, etc.

(b) *Example* 2. We choose $X_m = \mathbb{R}^m$, $(P_m w_0)_j = \langle w_0, \varphi_j \rangle$ for $j = 1, ..., m$, and $A_m \in \mathbb{R}^{m\times m}$ with diagonal elements $(A_m)_{jj} = -j^2$, $j = 1, ..., m$, and zero otherwise. Then for all $t \geq 0$, $(u_m(t))_j = c_j$ is the j^{th} Fourier coefficient of w, and $u_m(t) \in \mathbb{R}^m$ is their vector. The solution to problem (8) reads then as

$$u_m(t) = \mathrm{e}^{tA_m} P_m w_0 = \sum_{j=1}^m \mathrm{e}^{-j^2 t} \langle w_0, \varphi_j \rangle.$$

The operator J_m corresponds now to the inverse discrete Fourier transform, that is,

$$\big(J_m u_m(t)\big)(x) = \sum_{j=1}^m \big(u_m(t)\big)_j \varphi_j(x) \quad \text{for all } t \geq 0,\ x \in [0, \pi],$$

which really gives back formula (6).

In the examples above, problem (8) could be easily solved because the spaces X_m were finite dimensional in both cases. However, problems like (7) and (8) can be treated even if X and X_m are infinite dimensional. Then the corresponding solutions are studied in an abstract way presented in the next section.

3 The Continuous Problem

This section is devoted to introduce the basic notions of operator semigroup theory needed later on. In order to study the convergence of space and time discretisations, the given partial differential equation should be formulated as an abstract Cauchy problem of the form (7) on an appropriate Banach space X with the linear operator $A : D(A) \to X$, where the connection to the unknown function w of a partial differential equation is given by $(u(t))(x) = w(t, x)$ for all $t \geq 0$ and x from the corresponding interval/domain (e.g. for all $x \in [0, \pi]$ for problem (1)). If A were a matrix or any bounded operator on X ($A \in \mathscr{L}(X, X)$ in notation), the solution to problem (7) would be simply the exponential e^{tA} applied to the initial value u_0. Since A is unbounded in general, its exponential cannot be defined as the infinite power series. One suspects, however, that the solutions properties should somehow reflect the properties of the exponential function.

Definition 1 (Definition I.5.1 in [5]). *Let $S : [0, \infty) \to \mathscr{L}(X, X)$ be a mapping with the following properties.*

(i) The identity $S(t+s) = S(t)S(s)$ holds for all $t, s \geq 0$, and one has $S(0) = I$, the identity operator on X (semigroup property).
(ii) The mapping $t \to S(t)f \in X$ is continuous for all $f \in X$ (strong continuity).

Then S is called a strongly continuous one-parameter semigroup of bounded linear operators on the Banach space X.

We note that there always exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that the estimate $\|S(t)\| \leq M\mathrm{e}^{\omega t}$ holds for all $t \geq 0$ (cf. Proposition I.5.5 in [5]). Consider the map $u(t) = S(t)f$ for $f \in X$ and note that if u is differentiable, then one has $\frac{\mathrm{d}}{\mathrm{d}t}u(t) = S(t)(\frac{\mathrm{d}}{\mathrm{d}t}u(t))|_{t=0}$ (cf. Lemma II.1.1 in [5]). Hence, the derivative of the map u at $t = 0$ determines the derivative at each point $t \geq 0$. This suggests us to give this object a name.

Definition 2 (Definition II.1.2 and Lemma II.1.1 in [5]). *The generator $A : D(A) \to X$ of a strongly continuous semigroup S on the Banach space X is the operator*

$$Af := \lim_{\tau \searrow 0} \tfrac{1}{\tau}\big(S(\tau)f - f\big)$$

defined for every f in its domain

$$D(A) := \Big\{ f \in X : \lim_{\tau \searrow 0} \tfrac{1}{\tau}\big(S(\tau)f - f\big) \text{ exists} \Big\}.$$

The next result shows that the semigroup indeed yields the solution to the corresponding abstract Cauchy problem.

Theorem 1 (Theorem II.1.4 and Proposition II.6.2 in [5]). *The generator $A : D(A) \to X$ of a strongly continuous semigroup S has the following properties.*

(a) Operator A is linear, closed, and densely defined, and it determines the semigroup uniquely.
(b) For every $u_0 \in D(A)$, the solution to the abstract Cauchy problem (7) has the form $u(t) = S(t)u_0$.

This means that the solution of a partial differential equation, reformulated as an abstract Cauchy problem (7), is determined through the semigroup S generated by the operator A appearing in (7).

Example 3. Let $X = \mathrm{L}^2(0, \pi)$ and $(Af)(x) = f''(x)$ for all

$$f \in D(A) = \{f \in \mathrm{L}^2(0, \pi) : f(0) = f(\pi) = 0\}$$

as for the heat equation (1). Furthermore, let $\varphi_j(x) = \sqrt{\frac{2}{\pi}} \sin(jx)$ for $j \in \mathbb{N}$. One can show that then A generates the semigroup S of the form

$$S(t)f = \sum_{j=1}^{\infty} \mathrm{e}^{-j^2 t} \langle f, \varphi_j \rangle \varphi_j,$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathrm{L}^2(0, \pi)$. We note that the spectral method introduced in Example 2 follows this idea to approximate the solution to the heat Eq. (1).

4 Space and Time Discretisations

In Sect. 3 we saw that well-posed partial differential equations can be formulated as abstract Cauchy problems, and their solution is given by a strongly continuous semigroup. In Examples 1 and 2 we introduced two usual ways how the numerical solution to partial differential equations are usually obtained, that is, by using certain spatial and temporal discretisation schemes. We saw then that spatial discretisations mean the approximation of the generator A in problem (7). Discretisation in time is the approximation of the resulting semigroup.

4.1 Generator Approximations as Space Discretisations

Let X_m, $m \in \mathbb{N}$, be Banach spaces, and define some kind of projection and embedding operators as follows, see e.g. in Sect. 4.1 in [8].

Property 1. Let X and $X_m, m \in \mathbb{N}$ be Banach spaces. Consider the bounded linear operators $P_m \in \mathscr{L}(X, X_m)$ and $J_m \in \mathscr{L}(X_m, X)$ for $m \in \mathbb{N}$ with the properties

(i) $P_m J_m = I_m$, the identity on X_m, and
(ii) $\lim_{m\to\infty} \|J_m P_m f - f\| = 0$ for all $f \in X$.

One can show that operators P_m, J_m in Examples 1 and 2 possess Property 1.

The famous result of Trotter [15] and Kato [9] states that, under suitable conditions, if the generator A is approximated by a sequence of another generators A_m, then the corresponding semigroups S_m will approximate the semigroup S generated by A, as well.

Theorem 2 (First Trotter–Kato Approximation Theorem, Theorem 4.2 and Proposition 4.3 in [8], cf. Theorem III.4.8 in [5]). *For all $m \in \mathbb{N}$ let X and X_m be Banach spaces and let the operators P_m, J_m possess Property 1. Suppose that for all $m \in \mathbb{N}$, A and A_m generate the semigroups S and S_m in X and X_m, respectively. Suppose further that there exists constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|S(t)\|, \|S_m(t)\| \leq M\mathrm{e}^{\omega t}$ holds for all $m \in \mathbb{N}$, $t \geq 0$. Then the following assertions are equivalent.*

(i) There is a dense subspace $Y \subset D(A)$ such that there is $\lambda > 0$ with $(\lambda - A)Y$ being dense in X. Furthermore, for all $f \in Y$ there is a sequence with elements $f_m \in D(A_m)$ which satisfies $\lim\limits_{m\to\infty} \|f_m - P_m f\|_{X_m} = 0$ and

$$\lim_{m\to\infty} \|A_m f_m - P_m A f\|_{X_m} = 0\,.$$

(ii) It holds that $\lim\limits_{m\to\infty} \|J_m S_m(t) P_m f - S(t)f\| = 0$ for all $f \in X$ uniformly for t in compact intervals.

Since in both Examples 1 and 2 the sequence A_m converge to A in the sense of Theorem 2(a) and all the other conditions are satisfied as well, Theorem 2(b) implies that u_m converge to u.

4.2 Semigroup Approximations as Time Discretisations

We consider the abstract Cauchy problem (7) where A generates the strongly continuous semigroup S. Since multistep time discretisation schemes can also be treated as one-step methods (see [12]), we only consider one-step time discretisation methods. After some definitions, we will state the convergence results.

Property 2. Let Z be a Banach space, and let the map $V : [0,\infty) \to \mathscr{L}(Z,Z)$ possess the following properties.

(i) The map V is strongly continuous, that is, the function $[0,\infty) \ni \tau \mapsto V(\tau)f \in Z$ is continuous for all $f \in Z$.
(ii) $V(0) = I$, the identity on Z.

Definition 3. *Let S be the semigroup with generator A and consider the abstract Cauchy problem (7) on the Banach space X. Consider further a map $F : [0,\infty) \to \mathscr{L}(X,X)$ with Property 2, which is called then time discretisation.*

(a) The time discretisation F is called consistent with S if

$$\lim_{\tau\to 0} \tfrac{1}{\tau}\big(F(\tau)S(t)f - S(t+\tau)f\big) = 0$$

holds for all $f \in X$ and uniformly for t in compact intervals.

(b) A time discretisation F is called stable, if there are constants $T > 0$ and $M \geq 1$ such that $\|F(\tau)^k\| \leq M$ holds for all $\tau \geq 0$ and $k \in \mathbb{N}$ with $k\tau \leq T$.

(c) A time discretisation F is called convergent, if for all $t \geq 0$, $\tau_n \to 0$, $k_n \to \infty$ with $k_n \tau_n \to t$ we have

$$\lim_{n\to\infty} \|S(t)f - F(\tau_n)^{k_n} f\| = 0 \textit{ for all } f \in X.$$

Theorem 1(b) states that the semigroup S corresponds to the solution operator of the abstract Cauchy problem (7). To get a reliable approximation to S (i.e., a numerical solution), one has to ensure the convergence of the time discretisation scheme F. The next celebrated result is the basic of the numerical convergence analysis.

Theorem 3 (Lax Equivalence Theorem, [11]**).** *A consistent time discretisation is convergent if and only if it is stable.*

One can also say something about the order of the convergence, however, maybe only on a smaller set of initial values.

Definition 4. *Let S be the semigroup with generator A and consider the abstract Cauchy problem (7) on the Banach space X. Consider further a map $F : [0, \infty) \to \mathscr{L}(X, X)$ with Property 2. Suppose that there is a densely and continuously embedded subspace $Y \subset X$, which is invariant under the semigroup, and let $p > 0$.*

(a) The time discretisation F is called consistent with S of order p on Y if there is a constant $C > 0$ such that for all $f \in Y$ we have

$$\|F(\tau)f - S(\tau)f\| \leq C\tau^{p+1}\|f\|_Y.$$

(b) The time discretisation F is called convergent of order p on Y if for all $t \geq 0$ there is a constant $\widetilde{C} > 0$ such that for all $f \in Y$ we have

$$\|F(\tau)^k f - S(k\tau)f\| \leq \widetilde{C} t \tau^p \|f\|_Y \tag{10}$$

for all $k \in \mathbb{N}$, $\tau \geq 0$ with $k\tau \leq t$.

We note that p may depend on the subspace Y. Essentially by the same way as proving Theorem 3, the next result can be shown.

Proposition 1. *Suppose that there is a densely and continuously embedded subspace $Y \subset X$ which is invariant under the semigroup operators $S(t)$ satisfing $\|S(t)\|_Y \leq M e^{\omega t}$ for some $M \geq 1$ and $\omega \in \mathbb{R}$ and for all $t \geq 0$. If there is $p > 0$ such that F is a stable time discretisation scheme which is consistent of order p on Y, then it is convergent of order p on Y.*

Example 4. Let $(A, D(A))$ generate the semigroup S with $\|S(t)\| \leq M e^{\omega t}$ for some $M \geq 1$ and $\omega \in \mathbb{R}$ and for all $t \geq 0$. For all $\tau \in (0, \frac{1}{\omega}]$, we define the implicit Euler time discretisation as $F(\tau) = (I - \tau A)^{-1}$ being consistent. Moreover, if $\omega = 0$ and $Y = D(A^2)$, one has $p = 1$ in (10).

Since the generator property of the operator A is equivalent to the well-posedness of the problem (7) (see Theorem II.6.7 in [5]), Theorem 3 and Proposition 1 concern only well-posed problems. There exist results, however, which prove the generator property of an operator through approximations. They are extremely important in numerical analysis. We present now one of the most famous ones by Chernoff [4].

Theorem 4 (Chernoff Product Formula, Cor. III.5.3 in [5]). *Let X be a Banach space and consider a map $F : [0,\infty) \to \mathscr{L}(X,X)$ with the following properties.*

(a) The map F has Property 2.
(b) There exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|F(t)^k\| \leq M\mathrm{e}^{\omega k t}$ for all $t \geq 0$ and $k \in \mathbb{N}$.
(c) There is a subset $Y \subset X$ such that $(\lambda - A)Y$ is dense for some $\lambda > \omega$ and the limit

$$Af := \lim_{\tau \searrow 0} \tfrac{1}{\tau}\big(F(\tau)f - f\big) \tag{11}$$

exists for all $f \in Y$.

Then the closure $\overline{A}$ of A generates a strongly continuous semigroup S which is given by

$$S(t)f = \lim_{k\to\infty} F(\tfrac{t}{k})^k f \tag{12}$$

for all $f \in X$ and uniformly for t in compact intervals.

5 The Discrete Problem

In Sect. 3 we saw that the solution to the abstract Cauchy problem (7) is given by the semigroup generated by the operator A appearing in (7). Thus, if one aims to approximate the solution to problem (7), one has to approximate the corresponding semigroup S by the product of appropriate operators F_m depending on $m \in \mathbb{N}$. As already seen in Examples 1 and 2, one chooses a space discretisation scheme which corresponds to the approximation of the generator A by a sequence of generators A_m (cf. Section 4.1), then a time discretisation when the semigroup S_m is approximated by the product of F_m (cf. Section 4.2). The solution of a well-posed problem (7) is given by $u(t) = S(t)u_0$ for all $t \geq 0$. Application of a space discretisation means that $u(t)$ is approximated by $u_m(t) = S_m(t)P_m u_0$, $m \in \mathbb{N}$ (cf. Theorem 2). This is further approximated by using a time discretisation, that is, by $u_{m,k} = F_m(\frac{t}{k})^k P_m u_0$, $m, k \in \mathbb{N}$ (cf. Theorem 3 for the semigroup S_m on X_m).

Definition 5. *Let X be a Banach space and $A : D(A) \to X$ be the generator of the strongly continuous semigroup S on X. Furthermore, let $F_m : [0,\infty) \to \mathscr{L}(X_m, X_m)$ has Property 2 for all $m \in \mathbb{N}$. Then $u_{m,k} = J_m F_m(\frac{t}{k})^k P_m u_0 \in X_m$ is called the numerical solution at time $t \geq 0$ to the corresponding abstract*

Cauchy problem (7) *with initial value* u_0. *The numerical method is called convergent at time level* $t \geq 0$ *if for all* $u_0 \in X$ *one has*

$$\lim_{m,k\to\infty} \|u_{m,k} - u(t)\| = 0, \text{ that is, } \lim_{m,k\to\infty} \left\| J_m F_m(\tfrac{t}{k})^k P_m u_0 - S(t)u_0 \right\| = 0 \quad (13)$$

uniformly for t *in compact intervals.*

we note that the notaion $\lim\limits_{m,k\to\infty}$ stands for the usual limit for the double indexed sequences.

Remark 1. The following conditions are sufficient for the convergence (13).

(i) There exists $\bar{u}_m(t) \in X$ such that $\lim\limits_{k\to\infty} \|u_{m,k} - \bar{u}_m(t)\| = 0$ uniformly for $m \in \mathbb{N}$.
(ii) It holds that $\lim\limits_{m\to\infty} \|\bar{u}_m(t) - u(t)\|$ uniformly for t in compact intervals.

These conditions refer to the convergence of discretisations in time and space, respectively, studied in Sects. 4.2 and 4.1.

When considering well-posed problems (7), the Lax Equivalence Theorem 3 and the First Trotter–Kato Approximation Theorem 2 already imply the convergence.

Proposition 2. *Suppose that* A, A_m *generates the semigroups* S, S_m *on the Banach spaces* X, X_m, *respectively, for all* $m \in \mathbb{N}$, *and that the operators* P_m, J_m, $m \in \mathbb{N}$, *possess Property 1 such that* $P_m X \subset D(A_m)$. *Suppose further that*

$$\lim_{m\to\infty} \|A_m P_m f - P_m A f\|_{X_m} = 0 \quad (14)$$

holds for all $f \in Y$, *where* $Y \subset D(A)$ *and* $(\lambda - A)Y$ *are dense in* X *for some* $\lambda > 0$. *Moreover, let* $F_m : [0,\infty) \to X_m$ *be a stable time discretisation which is consistent with* S_m *for all* $m \in \mathbb{N}$. *Then* F_m *is convergent, more precisely, for all* $f \in X$ *one has*

$$\lim_{m,k\to\infty} \left\| J_m F_m(\tfrac{t}{k})^k P_m f - S(t)f \right\| = 0$$

uniformly for t *in compact intervals.*

Proof. Due to Remark 1, it suffices to study the limits separately. The Lax Equivalence Theorem 3 imply that F_m is convergent in X_m, that is,

$$\lim_{k\to\infty} \left\| F_m(\tfrac{t}{k})^k f_m - S_m(t) f_m \right\|_{X_m} = 0$$

holds for all $f_m \in X_m$. Since operators $J_m : X_m \to X$ are bounded and with the choice $f_m = P_m f$ for $f \in X$, we have that

$$\lim_{k\to\infty} \left\| J_m F_m(\tfrac{t}{k})^k P_m f - J_m S_m(t) P_m f \right\| = 0 \quad (15)$$

uniformly for t in compact intervals. From (14), the First Trotter–Kato Approximation Theorem 2 implies that

$$\lim_{m\to\infty} \|J_m S_m(t) P_m f - S(t) f\| = 0 \tag{16}$$

holds for all $f \in X$ uniformly for t in compact intervals. Hence, limits (15) and (16), and Remark 1 with $\bar{u}_m(t) := J_m S_m(t) P_m f$ yield the convergence. □

The results above all concern well-posed problems. In case when the operator A is not known to be a generator of a semigroup, a modified version of Chernoff Product Formula 4 can be applied. The original theorem, presented in Pazy [14], states the result in the space X_m, however, we formulate it here as a result in the space X.

Theorem 5 (Modified Chernoff Product Formula, [1]**).** *Let X_m, $m \in \mathbb{N}$ be Banach spaces and consider a sequence of maps $F_m : [0,\infty) \to \mathscr{L}(X_m, X_m)$ with the following properties.*

(a) The maps F_m have Property 2 for all $m \in \mathbb{N}$.
(b) There exist constants $M \geq 1$ and $\omega \in \mathbb{R}$ such that $\|F_m(t)^k\| \leq M\mathrm{e}^{\omega k t}$ for all $t \geq 0$ and $m, k \in \mathbb{N}$.
(c) There is a subset $Y \subset X$ such that $(\lambda - A)Y$ is dense for some $\lambda > \omega$ and the limit

$$\lim_{m\to\infty} \tfrac{1}{\tau}\big(J_m F_m(\tau) P_m f - J_m P_m f\big)$$

exists uniformly for τ in compact intervals, and

$$Af := \lim_{\tau\searrow 0} \lim_{m\to\infty} \tfrac{1}{\tau}\big(J_m F_m(\tau) P_m f - J_m P_m f\big)$$

exists for all $f \in Y$.

Then the closure $\overline{A}$ of A generates a strongly continuous semigroup S which is given by

$$S(t)f = \lim_{m,k\to\infty} J_m F_m(\tfrac{t}{k})^k P_m f \tag{17}$$

for all $f \in X$ and uniformly for t in compact intervals.

6 Outlook/Applications

With the help of similar techniques presented in Sect. 5, several numerical treatments can be proved to be convergent. We just mention here a few examples which are of great importance in practice. The convergence of the standard time discretisation methods, such as Runge–Kutta methods, were analysed by using Lax Equivalence Theorem 3. Even more general rational approximations are studied in Brenner and Thomée [3]. The convergence of various operator splitting methods were proved e.g. in Trotter [16], Kato [10], Faragó and Havasi [6], and Bátkai et al. [1] and [2] by using Chernoff Product Formula, Theorem 4 and its modified version, Theorem 5. Exponential integrators were also studied by using operator semigroup approach in Hochbruck and Ostermann [7]. Nonlinear problems are treated in Palencia and Sanz-Serna [13] contaning the Lax Equivalence Theorem 3 as a special case of well-posed linear initial value problems.

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