

Some Results of FEM Schemes Analysis by Finite Difference Method

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Abstract. A method of investigation of numerical schemes deriving from the variational formulation of the problem (variational- difference method and FEM) is discusses. The method is based on the reduction of the numerical schemes to the canonical finite difference form. The resulting numerical scheme standard notation in the form of a grid operator equality is used for analyzing its approximation, stability and other properties. The application of this approach to a wider classes of finite elements (from the simplest ones to the Hermitian elements and serendipities) is discussed. These opportunities are illustrated by the analysis of FEM schemes for Timoshenko shells and elasticity dynamic problems.

1 Conversion of Finite Element Numerical Schemes to Finite Difference Form

The investigation of difference schemes involves the standard scheme notation in the form of a grid operator equality. In the case of uniform difference grids the FEM scheme operator can be written in the final form, which is suitable for the theoretical analysis. The conversion procedure is similar to the constructing the system of Euler differential equations of the variational problem. Lets consider the construction method for the two-dimensional case, which can be naturally is generalized to the n- dimensional case. Let $R^2 = \{x\} = \{(x_1, x_2)\}$ set uniform (possibly oblique), the main grid coordinates of nodes

$$x_{ij} = \begin{bmatrix} x_{ij}^1 \\ x_{ij}^2 \end{bmatrix} = B_h \begin{bmatrix} i \\ j \end{bmatrix} + \begin{bmatrix} x_0^1 \\ x_0^2 \end{bmatrix} \quad i, j \in Z, \quad (1)$$

where B_h – is real nonsingular matrix 2×2 . To refine the concept of certainty uniform finite element mesh. The finite element mesh in R^2 is said to be uniform if the elements and their nodes are periodically with a period given by a grid form (1). Let a functional is given

$$W = \int_{\Omega} F(u, p_1, p_2) d\Omega, \quad (2)$$

where u is an unknown function satisfying the given boundary conditions; $p_1 = \frac{\partial u}{\partial x^1}$, $p_2 = \frac{\partial u}{\partial x^2}$.

It is necessary to find a function u , deliver functional W extreme or stationary value. Solution of this problem satisfies the differential equation of Euler

$$\frac{\partial F}{\partial u} - \frac{\partial}{\partial x^1} \frac{\partial F}{\partial p_1} - \frac{\partial}{\partial x^2} \frac{\partial F}{\partial p_2} = 0 \tag{3}$$

and boundary conditions. Constructing of FEM numerical scheme reduces to the partition of the region into finite elements (construction of the finite element mesh) and the choice of basis functions, then the FEM problem is defined. It further reduces by known methods [3, 4] to an algebraic system.

We assume that the FEM problem is defined, i.e. functional (2) is given, built finite element mesh and selected basis functions. Consider the transformation scheme to FEM finite difference form in the simplest case of Lagrangian elements.

We assume that each cell is divided into r , finite elements, where $r = 1$ to quadrilateral elements and $r = 2$ for the triangular elements (three-dimensional case r can take values from 1 to 6). Then the functional (2) can be written as

$$W_h = \sum_{(i,j) \in \Omega_h} \sum_{k=1}^r \int_{E_{ijk}} F(u, p_1, p_2) d\Omega, \tag{4}$$

where E_{ijk} - k -th element in the (i, j) -th cell of the main foil. Assuming that the k -th element of the type comprising m_k node template has S_k . Unknown function u in the element is given as

$$u = \sum_{l=0}^{m_k-1} C_{lk} \varphi_{lk}, \tag{5}$$

where φ_{lk} ($l = 0, \dots, m_k - 1$)- the basic function of the k -type element. For example, for a 4-node bilinear element

$$f(x^1, x^2) = c_0 + c_1(x^1 - x_c^1) + c_2(x^2 - x_c^2) + c_3(x^1 - x_c^1)(x^2 - x_c^2)$$

(here (x_c^1, x_c^2) - the coordinates of the element center). Coefficients c_m can be expressed through the function node values $c_m = \sum_{k=1}^m \beta_k^m f_k$ (it uses the local node numbering). As a result of the transition to a nodes global numbering for the l -type element the following formula we get:

$$c_m = d_{m,l}^+ f_{ij} = \sum_{(p,q) \in S_l} \beta_{p,q}^{m,l} f_{i+p,j+q}. \tag{6}$$

This formula also defines the basic differential operators $d_{m,l}^+$. Coefficients $\beta_{p,q}^{m,l}$ depend from the element template and matrix B_h . We also introduce the characters taken from different operators adjoint to (6):

$$d_{m,l}^- f_{ij} = (-1)^{K_m} \sum_{(p,q) \in S_l} \beta_{p,q}^{m,l} f_{i-p,j-q}. \tag{7}$$

Here K_{m-} is order of derivative, which is approximated by the operator. C_{lk} coefficients can be expressed through the values of function u at the nodes of the element, and further through the difference operators $d_0^{k+}, \dots, d_{m_k-1}^{k+}$. As a result, the function u in element E_{ijk} may be represented as

$$u = \sum_{l=0}^{m_k-1} \left(d_{l,k}^+ u \right)_{ij} \psi_{lk}, \tag{8}$$

where ψ_{lk} - some linear combinations of functions φ_{lk} .

After substituting (8) into (4) integration we obtain

$$W_h = \sum_{(i,j) \in \Omega_h} \sum_{k=1}^r \gamma_k \Phi_k (\xi_{ij0}, \dots, \xi_{ijm_k-1}). \tag{9}$$

where $\xi_{ijl} = (d_l^+ u)_{ij}$.

We write the variation of the functional (9):

$$\delta W_h = \sum_{(i,j) \in \Omega_h} \sum_{k=1}^r \gamma_k (f_{k0} \delta (d_0^{k+} u) + \dots + f_{km_k-1} \delta (d_{m_k-1}^{k+} u))_{ij}. \tag{10}$$

Here we use the notation

$$(f_{kl})_{ij} = \frac{\partial \Phi_k}{\partial \xi_{ijl}}$$

Substituting (10) in the discrete variation equation

$$\delta W_h = 0$$

and applying grid integration by parts, we obtain

$$\sum_{(i,j) \in \Omega_h} \sum_{k=1}^r \gamma_k \sum_{l=0}^{m_k-1} (-1)^{K_l} d_{l,k}^- f_{kl} \delta u_l = 0.$$

The last equality is satisfied if the conditions

$$\sum_{k=1}^r \gamma_k \sum_{l=0}^{m_k-1} (-1)^{K_l} d_{l,k}^- f_{kl} = 0 \tag{11}$$

obey at all nodes of the difference grid. Equation (11) represents a difference scheme standard form. Thus, we obtain the final difference form of the FEM scheme.

This method of FEM schemes conversion to finite differences can be used for theoretical analysis of a wide FEM schemes class for both linear and nonlinear problems. The method of variational-difference schemes transformation to finite differences is similar with not great distinctions. Similar transformations for other types of finite elements (Hermitian, sirendipity etc.) were considered in [1].

2 Analysis of Approximation of Variational-Difference and Finite Element Schemes of Timoshenko Plates Theory

Consider the system of equations describing the transverse vibrations of the Tymoshenko plate [5] 1D case recorded in the dimensionless form. System has the form

$$\begin{aligned} a \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial \psi}{\partial x} \right) - \frac{\partial^2 w}{\partial t^2} &= 0 \\ \frac{\partial^2 w}{\partial x^2} - \frac{12a}{\eta^2} \left(\frac{\partial w}{\partial x} + \psi \right) - \frac{\partial^2 \psi}{\partial t^2} &= 0 \end{aligned} \tag{12}$$

It is equivalent to a single equation of the fourth order

$$\frac{\partial^4 w}{\partial x^4} + \frac{12a}{\eta^2} \frac{\partial^2 w}{\partial t^2} - \left(1 + \frac{1}{a} \right) \frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{1}{a} \frac{\partial^4 w}{\partial t^4} = 0. \tag{13}$$

Consider the difference schemes approximation of the equations system (12). Finite-difference scheme has the form

$$\begin{cases} a (D_{11}w + D_{01}\psi) - D_{tt}w = 0 \\ D_{11}\psi - \frac{12a}{\eta^2} (D_{01}w + \psi) - D_{tt}\psi = 0 \end{cases} \tag{14}$$

variational-difference scheme has the form

$$\begin{cases} a (D_{11}w + D_{01}\psi) - D_{tt}w = 0 \\ D_{11}\psi - \frac{12a}{\eta^2} (D_{01}w + D_{00}\psi) - D_{tt}\psi = 0 \end{cases} \tag{15}$$

linear finite element scheme has the form

$$\begin{cases} a (D_{11}w + D_{01}\psi) - D_{tt}w = 0 \\ D_{11}\psi - \frac{12a}{\eta^2} (D_{01}w + D_0\psi) - D_{tt}\psi = 0 \end{cases} \tag{16}$$

There in (14)–(16) $D_{11}f = \frac{1}{h^2} (f_{i+1} - 2f_i + f_{i-1})$, $D_{00}f = \frac{1}{4} (f_{i+1} + 2f_i + f_{i-1})$, $D_{01}f = \frac{1}{2h} (f_{i+1} - f_{i-1})$, $D_0f = \frac{1}{6} (f_{i+1} + 4f_i + f_{i-1})$, $D_{tt}f = \frac{1}{\tau^2} (f^{j+1} - 2f^j + f^{j-1})$.

Schemes (14)–(16) differ by only one equations member approximation - function ψ . They all have second-order approximation. Transform them to the form similar to (13). Finite-difference scheme (14) takes the form:

$$\left(1 + 3a \left(\frac{h}{\eta} \right)^2 \right) D_{11}D_{11}w + \frac{12}{\eta^2} D_{tt}w - \left(1 + \frac{1}{a} \right) D_{11}D_{tt}w + \frac{1}{a} D_{tt}D_{tt}w = 0. \tag{17}$$

variational-difference scheme (15) takes the form:

$$D_{11}D_{11}w + \frac{12}{\eta^2} D_{00}D_{BB}w - \left(1 + \frac{1}{a} \right) D_{11}D_{tt}w + \frac{1}{a} D_{tt}D_{tt}w = 0. \tag{18}$$

linear finite element scheme (16) takes the form:

$$\left(1 + a \left(\frac{h}{\eta}\right)^2\right) D_{11}D_{11}w + \frac{12}{\eta^2}D_0^*D_{tt}w - \left(1 + \frac{1}{a}\right) D_{11}D_{tt}w + \frac{1}{a}D_{tt}D_{tt}w = 0. \tag{19}$$

Comparing (17)-(19) with the original differential equation (13), we conclude that at finite values of the quantity h/η (relationship grid spacing to plate thickness) difference equations (17) and (19) do not approximate equation (13) but, respectively, the equations

$$\left(1 + 3a \left(\frac{h}{\eta}\right)^2\right) \frac{\partial^4 w}{\partial x^4} + \frac{12}{\eta^2} \frac{\partial^2 w}{\partial t^2} - \left(1 + \frac{1}{a}\right) \frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{1}{a} \frac{\partial^4 w}{\partial t^4} = 0. \tag{20}$$

and

$$\left(1 + a \left(\frac{h}{\eta}\right)^2\right) \frac{\partial^4 w}{\partial x^4} + \frac{12}{\eta^2} \frac{\partial^2 w}{\partial t^2} - \left(1 + \frac{1}{a}\right) \frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{1}{a} \frac{\partial^4 w}{\partial t^4} = 0. \tag{21}$$

Thus, we can conclude that all schemes have the convergence in the usual sense, but the finite difference and finite element schemes do not have the uniform convergence by grid problem parameter. Scheme (15) has the uniform convergence of the parameter h/η . Quantitative analysis confirming the findings is given in [1]. A similar analysis was conducted and the approximation for two-dimensional schemes. Analysis results agree well with well known reduced integration technique by O. Zienkiewicz [6], due to which the scheme (16) becomes the scheme (15).

3 Influence of the Mutual Location of Finite Elements on the Accuracy of the Numerical Solution

Consider the effect of finite element mutual location influence to approximation and accuracy of schemes. We take the 3D elastic problem and 4-node linear finite element. Way of the base parallelepiped dividing into tetrahedrons defined by a set of templates elements. Pattern of each element contains four integer vector is a subset of $\{(000), (001), (010), (011), (100), (101), (110), (111)\}$.

Below you can see the types of partitions hexahedron.

(1) 5 tetrahedra (Fig. 1.):

$$\begin{aligned} S_1 &= \{(000), (011), (101), (110)\}, & S_2 &= \{(000), (011), (101), (001)\}, \\ S_3 &= \{(000), (100), (101), (110)\}, & S_4 &= \{(000), (011), (010), (110)\}, \\ & & S_5 &= \{(111), (011), (101), (110)\} \end{aligned}$$

(2) 6 tetrahedra with centrally symmetric partition (Fig. 2.):

$$\begin{aligned} S_1 &= \{(010), (001), (101), (011)\}, & S_2 &= \{(000), (001), (101), (010)\}, \\ S_3 &= \{(000), (100), (101), (010)\}, & S_4 &= \{(101), (110), (010), (100)\}, \\ S_5 &= \{(111), (110), (010), (101)\}, & S_6 &= \{(111), (011), (010), (101)\}. \end{aligned}$$

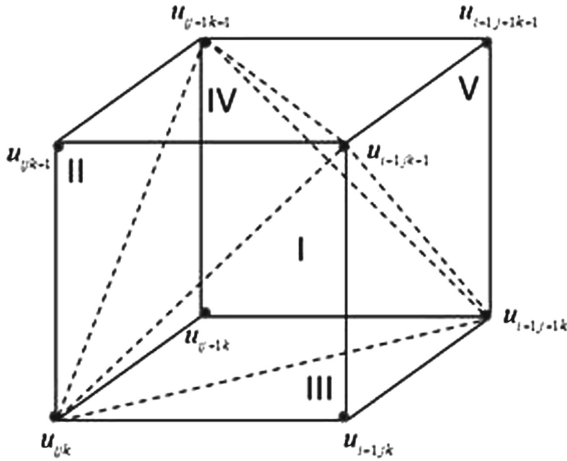


Fig. 1. 5 tetrahedra

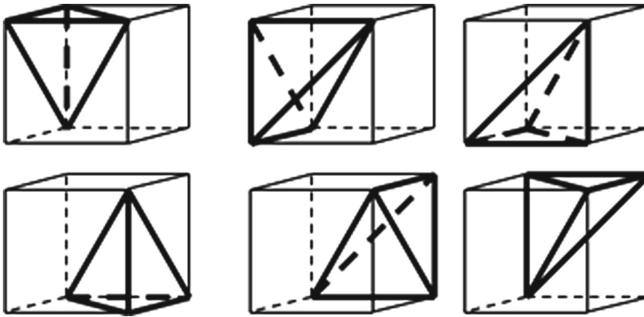


Fig. 2. 6 tetrahedra,

(3) 6 tetrahedra with rotational-symmetric partition:

$$\begin{aligned}
 S_1 &= \{(010), (001), (101), (011)\}, & S_2 &= \{(000), (001), (101), (010)\}, \\
 S_3 &= \{(000), (100), (101), (010)\}, & S_4 &= \{(100), (101), (011), (111)\}, \\
 S_5 &= \{(100), (011), (110), (111)\}, & S_6 &= \{(100), (011), (010), (110)\}.
 \end{aligned}$$

(4) 6 tetrahedra with non-symmetric partition:

$$\begin{aligned}
 S_1 &= \{(010), (001), (101), (011)\}, & S_2 &= \{(000), (001), (101), (010)\}, \\
 S_3 &= \{(000), (100), (101), (010)\}, & S_4 &= \{(100), (010), (110), (111)\}, \\
 S_5 &= \{(100), (010), (101), (111)\}, & S_6 &= \{(010), (101), (011), (111)\}.
 \end{aligned}$$

Unknown functions in linear element represented in the form

$$f(x^1, x^2, x^3) = c_0 + c_1(x^1 - x_c^1) + c_2(x^2 - x_c^2) + c_3(x^3 - x_c^3)$$

(here (x_c^1, x_c^2, x_c^3) - the coordinates of the center of the element).

We write the functional as energy internal of the linearly elastic body:

$$W = \frac{1}{2} \int_{\Omega} \sigma_{\alpha\beta} \varepsilon_{\alpha\beta} d\Omega$$

Further, according to the algorithm described above, we obtain the representation of FEM schemes in the traditional finite-difference form

$$(\lambda + \mu) \begin{vmatrix} D_{11} u_1 + D_{12} u_2 + D_{13} u_3 \\ D_{21} u_1 + D_{22} u_2 + D_{23} u_3 \\ D_{31} u_1 + D_{32} u_2 + D_{33} u_3 \end{vmatrix} + \mu D_{\Delta} \begin{vmatrix} u_1 \\ u_2 \\ u_3 \end{vmatrix} + \rho \begin{vmatrix} F_1 \\ F_2 \\ F_3 \end{vmatrix} = \rho D_{tt} \begin{vmatrix} u_1 \\ u_2 \\ u_3 \end{vmatrix}, \quad (22)$$

a similar system of Lamé equations

$$(\lambda + \mu) \text{grad div } u + \mu \Delta u + \rho F = \rho \frac{\partial^2 u}{\partial t^2} \quad (23)$$

where the operators D_{ij} approximate second derivatives, respectively, for the i -th and j -th coordinates, $D_{tt}f = \frac{1}{\tau^2} (f(t + \tau) - 2f(t) + f(t - \tau))$ approximates the second derivative with respect to time, $D_{\Delta} = D_{11} + D_{22} + D_{33}$ - the grid Laplace operator. D_{ij} operators have different specific form depending on the variant schemes investigated and may be either the first or second order approximation. Schemes for linear finite element we have $D_{ij} = \sum_{l=1}^s \gamma_l d_{i,l}^+ d_{j,l}^-$, where $s = 5, \gamma_1 = \frac{1}{3}, \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = \frac{1}{6}$ the scheme with the partition parallelepiped 5 tetrahedron; $s = 6, \gamma_1 = \gamma_2 = \gamma_3 = \gamma_4 = \gamma_5 = \gamma_6 = \frac{1}{6}$ for schemes with the partition parallelepiped 6 tetrahedron. Analysis grid approximation equation (23) by equation (22) conducted a standard method for the case of orthogonal grid with the coordinates of the grid nodes are equal $x_{ijk}^1 = x_0^1 + h_1 i, x_{ijk}^2 = x_0^2 + h_2 j, x_{ijk}^3 = x_0^3 + h_3 k$) showed that one of this schemes (centrally symmetric partition) has second order approximation, and the other three - the first order approximation. The results of the test problem solutions also showed a different rate of schemes convergence

4 Variational-Difference and Finite Element Schemes on Rare Grids [7]

There in formulas (11) is an overall view of finite difference schemes represent FEM on uniform grids. They contain coefficients $\gamma_k = V_k / \Delta V$, where V_k - the volume (area) element k -type, ΔV - the basic unit of volume of a uniform grid of the form (1). Coefficients γ_k satisfy the obvious equality

$$\sum_{k=1}^p \gamma_k = 1, \quad (24)$$

reflects the continuous filling elements of the computational domain (here p - the number of elements that make up the cell.) Varying set of coefficients γ_k , while maintaining this equality, we obtain new difference schemes, some of which can

be quite successful. In particular, variation in the difference scheme similar to (15) on the coefficients of the triangular cells are equal $\gamma_1 = \gamma_2 = 1/2$. Substituting their values $\gamma_1 = 1, \gamma_2 = 0$, we obtain “rare mesh” variational- difference scheme. The scheme has a much better convergence than the original. Note it is also more economical, because it is actually two times less computational cells. A detailed analysis of this scheme are given in [1,2].

Further developing this method, we arrive at the idea of rare mesh schemes FEM. Under the rare mesh scheme we understand the scheme, in which some of the coefficients γ_k equal to zero. Relevant elements do not contribute to the numerical scheme and may be excluded from the calculations. This approach proved to be very productive in solving the three-dimensional elasticity problems. In particular, the scheme has been proposed on the basis of a linear 4 -node finite element, which for central tetrahedron (Fig. 1.) Ratio γ_k was 1, the remaining tetrahedra - zero. This scheme is significantly more economical than traditional and has better convergence. Also, it has no the drawback of numerical schemes on hexahedral elements - the “hourglass instability”. Detailed description of the scheme, the results of its analysis and testing described are given in [7,8].

5 Conclusion

Method described in the study of numerical schemes based on the variational formulation of problems allows more deeply study their properties and to propose ways to improve, as evidenced by the examples discussed. He partly overcomes the gap between the theory of difference schemes and finite element method. This approach can be applied to the analysis of a wide variety of schemes FEM mathematical physics problems.

References

1. Bazhenov, V.G., Chekmarev, D.T.: Solving the problems of plates and shells dynamics by variational- difference method, Nizhny Novgorod (2000) (in Russian)
2. Bazhenov, V.G., Chekmarev, D.T.: On numerical differentiation index commutation. *Comp. Math. and Math. Phys.* **29**(5), 662–674 (1989). (in Russian)
3. Zienkiewicz, O.C., Morgan, K.: *Finite Elements and Approximation*. Wiley, New York (1983)
4. Strang, G., Fix, G.J.L: *An Analysis of the Finite Element Method*. Series in Aut. Comp. pp. XIV + 306 SM Fig. Prentice-Hall Inc, Englewood Clifs, NJ (1973)
5. Timoshenko, S.: *Vibration problems in engineering*. Van Nostrand, New York (1937)
6. Zienkiewicz, O.C., Too, J., Taylor, R.L.: Reduced integration technique in general analysis of plates and shells. *Int. J. Num. Meth. Engng.* **3**(2), 275–290 (1971)
7. Chekmarev, D.T.: Finite element schemes on rare meshes. *Probl. At. Sci. Techn. Ser. Math. Model. phys. proc.* **2**, 49–54 (2009). (in Russian)
8. Zhidkov, A.V., Zefirov, S.V., Kastalkaya, K.A., Spirin, S.V., Chekmarev, D.T.: Rare mesh scheme for numerical solution of three-dimensional dynamic problems of elasticity and plasticity. *Bull. Nizhny Novgorod Univ.* **4**(4), 1480–1485 (2011). (in Russian)