Difference Schemes for Delay Parabolic Equations with Periodic Boundary Conditions

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Abstract. The initial-boundary value problem for the delay parabolic partial differential equation with nonlocal conditions is studied. The convergence estimates for solutions of first and second order of accuracy difference schemes in Hölder norms are obtained. The theoretical statements are supported by a numerical example.

Keywords: Difference schemes · Delay parabolic equation · Hölder spaces · Convergence

1 Introduction

Delay differential equations provide a mathematical model for a physical or biological system in which the rate of change of the system depends on the past history. The theory of these equations is well developed and has numerous applications in natural and engineering sciences. Typical examples that delay differential equations appear are diffusive population models with temporal averages over the past,tumor growth, neural networks, control theory, climate models etc. Numerical solutions of delay ordinary differential equations have been studied mostly for ordinary differential equations (cf., e.g., [1–9] and the references therein). Generally, delay partial differential equations get less attention than delay ordinary differential equations.

In recent years, A. Ashyralyev and P. E. Sobolevskii obtained the stability estimates in Hölder norms for solutions of the initial-value problem of delay differential and difference equations of the parabolic type [10,11]. D.B. Gabriella used extrapolation spaces to solve delay differential equations with unbounded delay operators [12]. Different kinds of problems for delay partial differential equations are solved by using operator approach (see, e.g., [13–20]).

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In this paper, the initial-boundary value problem for the delay differential equation of the parabolic type

$$\begin{cases} u_t(t,x) - a(x)u_{xx}(t,x) + c(x)u(t,x) \\ = d(t)(-a(x)u_{xx}(t-\omega,x) + c(x)u(t-\omega,x)), \\ 0 < t < \infty,\ x \in (0,L), \\ u(t,x) = g(t,x), -\omega \le t \le 0,\ x \in [0,L], \\ u\left(t,0\right) = u\left(t,L\right), u_x\left(t,0\right) = u_x\left(t,L\right),\ t \ge 0 \end{cases} \tag{1}$$

is studied. Here $g(t,x)$ $(t \in (-\infty,0),\ x \in [0,L])$, $a(x), c(x)$ $(x \in (0,L))$ are given smooth bounded functions and $a(x) \ge a > 0,\ c(x) \ge c > 0$. Difference schemes of first and second order of accuracy for the numerical solutions of Problem (1) are presented. The convergence of these difference schemes are studied. The numerical solutions are found by using MATLAB programs.

2 Difference Schemes, Convergence Estimate

The discretization of Problem (1) is carried out in two steps. In the first step, we define the grid space

$$[0,L]_h = \{x = x_n :\ x_n = nh,\ 0 \le n \le M,\ Mh = L\}.$$

To formulate our results, we introduce the Banach space $\overset{\circ}{C}{}_h^{\alpha} = \overset{\circ}{C}{}^{\alpha}[0,L]_h$, $\alpha \in [0,1)$, of all grid functions $\varphi^h = \{\varphi_n\}_{n=0}^M$ defined on $[0,L]_h$ with $\varphi_0 = \varphi_M$, $\varphi_1 - \varphi_0 = \varphi_M - \varphi_{M-1}$ or $3\varphi_0 - 4\varphi_1 + \varphi_2 = -3\varphi_M + 4\varphi_{M-1} - \varphi_{M-2}$ equipped with the norm

$$\left\|\varphi^h\right\|_{\overset{\circ}{C}{}_h^{\alpha}} = \left\|\varphi^h\right\|_{C_h} + \sup_{1 \le n < n+r \le M-1} \left|\varphi_{n+r} - \varphi_n\right| (rh)^{-\alpha},$$

$$\left\|\varphi^h\right\|_{C_h} = \max_{1 \le n \le M-1} \left|\varphi_n\right|.$$

Moreover, $C_\tau(E) = C\left([0,\infty)_\tau, E\right)$ is the Banach space of all grid functions $f^\tau = \{f_k\}_{k=1}^\infty$ defined on

$$[0,\infty)_\tau = \{t_k = k\tau,\ k = 0,1,\cdots\}$$

with values in E equipped with the norm

$$\|f^\tau\|_{C_\tau(E)} = \sup_{1 \le k < \infty} \|f_k\|_E.$$

To the differential operator A generated by Problem (1), we assign the difference operators A_h^x, B_h^x by the formulas

$$A_h^x \varphi^h(x) = \left\{ -a(x_n) \frac{\varphi_{n+1} - 2\varphi_n + \varphi_{n-1}}{h^2} + c(x_n)\varphi_n \right\}_1^{M-1},$$

$$B_h^x(t)\varphi^h(x) = d(t)A_h^x\varphi^h(x),$$

acting in the space of grid functions $\varphi^h(x) = \{\varphi_n\}_0^M$ satisfying the conditions $\varphi_0 = \varphi_M, \varphi_1 - \varphi_0 = \varphi_M - \varphi_{M-1}$ for the first order of approximation of difference operator A_h^x and the conditions $\varphi_0 = \varphi_M$, $3\varphi_0 - 4\varphi_1 + \varphi_2 = -3\varphi_M + 4\varphi_{M-1} - \varphi_{M-2}$ for the second order of approximation of difference operator A_h^x. It is well known that A_h^x is a strongly positive operator in C_h. With the help of A_h^x and $d(t)A_h^x$, we arrive at the initial value problem

$$\begin{cases} \frac{du^h(t,x)}{dt} + A_h^x u^h(t,x) = d(t)A_h^x u^h(t-w,x), \ 0 < t < \infty, \\ u^h(t,x) = g^h(t,x), \ -\omega \le t \le 0. \end{cases} \tag{2}$$

In the second step, we consider difference schemes of first and second order of accuracy

$$\begin{cases} \frac{1}{\tau}\left(u_k^h(x) - u_{k-1}^h(x)\right) + A_h^x u_k^h(x) = d(t_k)A_h^x u_{k-N}^h(x), \\ t_k = k\tau, \ 1 \le k, \ N\tau = w, \\ u_k^h(x) = g^h(t_k,x), \ t_k = k\tau, \ -N \le k \le 0, \end{cases} \tag{3}$$

$$\begin{cases} \frac{1}{\tau}(u_k^h(x) - u_{k-1}^h(x)) + (A_h^x + \frac{1}{2}\tau(A_h^x)^2)u_k^h(x) \\ = \frac{1}{2}(I + \frac{\tau}{2}A_h^x)d\left(t_k - \frac{\tau}{2}\right)A_h^x(u_{k-N}^h(x) + u_{k-N-1}^h(x)), \\ t_k = k\tau, 1 \le k, u_k^h = g^h(t_k,x), \ t_k = k\tau, \ -N \le k \le 0. \end{cases} \tag{4}$$

Note that in (4), we assign the difference operator $(A_h^x)^2 = A_h^x \cdot A_h^x$ acting in the space of grid functions $\varphi^h(x) = \{\varphi_n\}_0^M$ satisfying the conditions

$$\varphi_0 = \varphi_M, 3\varphi_0 - 4\varphi_1 + \varphi_2 = -3\varphi_M + 4\varphi_{M-1} - \varphi_{M-2},$$

$$2\varphi_0 - 5\varphi_1 + 4\varphi_2 - \varphi_3 = 2\varphi_M - 5\varphi_{M-1} + 4\varphi_{M-2} - \varphi_{M-3},$$

$$10\varphi_0 - 15\varphi_1 + 6\varphi_2 - \varphi_3 = -10\varphi_M + 15\varphi_{M-1} - 6\varphi_{M-2} + \varphi_{M-3}$$

generated by second order approximations of conditions $\varphi_0 = \varphi_L, \varphi_0' = \varphi_L', \varphi_0'' = \varphi_L''$ and $\varphi_0''' = \varphi_L'''$.

Theorem 1. Assume that

$$\sup_{0 \le t < \infty} |d(t)| \le \frac{1-\alpha}{M2^{2-\alpha}}. \tag{5}$$

Suppose that Problem (1) has a smooth solution $u(t,x)$ and

$$\int_0^\infty \left[\max_{0 \le x \le L} |u_{ss}(s,x)| + \sup_{0<x<x+y<L} \frac{|u_{ss}(s,x+y) - u_{ss}(s,x)|}{y^{2\alpha}}\right] ds < \infty,$$

$$\int_0^\infty \left[\max_{0 \le x \le L} |u_{xxxx}(s,x)| + \sup_{0<x<x+y<L} \frac{|u_{xxxx}(s,x+y) - u_{xxxx}(s,x)|}{y^{2\alpha}}\right] ds < \infty.$$

Then, for the solution of difference scheme (3), the following convergence estimate holds

$$\sup_k \left\|u_k^h - u^h(t_k,\cdot)\right\|_{\overset{\circ}{C}_h^{2\alpha}} \le M_1\left(\tau + h^2\right)$$

with M_1 is a real number independent of τ, α and h.

Proof. Using notations of A_h^x and $B_h^x(t_k)$, we can obtain the following formula for the solution

$$u_k^h(x) = R^k g^h(0,x) + \sum_{j=1}^{k} R^{k-j+1} B_h^x(t_j) g^h(t_{j-N},x)\tau,\ 1 \le k \le N, \quad (6)$$

and

$$u_k^h(x) = R^{k-nN} u_{nN}^h(x)$$
$$+ \sum_{j=nN+1}^{k} R^{k-j+1} B_h^x(t_j) u_{j-N}^h(x)\tau,\ nN \le k \le (n+1)N, \quad (7)$$

where $R = \left(I + \tau A_x^h\right)^{-1}$. The proof of the Theorem 1 is based on the Formulas (6) and (7) on the convergence theorem for difference schemes in $C_\tau\left(E_\alpha^h\right)$, on the estimate

$$\|\exp\{-t_k A_h^x\}\|_{C_h \to C_h} \le M,\ k \ge 0, \quad (8)$$

and on the fact that in the $E_\alpha^h = E_\alpha(A_h^x, C_h)$- norms are equivalent to the norms $\overset{\circ}{C}{}_h^{2\alpha}$ uniformly in h for $0 < \alpha < \frac{1}{2}$ (see, [13]).

Theorem 2. Assume that assumption (5) of the Theorem 1 and the following conditions hold.

$$\int_0^\infty \left[\max_{0 \le x \le L} |u_{sss}(s,x)| + \sup_{0<x<x+y<L} \frac{|u_{sss}(s,x+y) - u_{sss}(s,x)|}{y^{2\alpha}}\right] ds < \infty,$$

$$\int_0^\infty \left[\max_{0 \le x \le L} |u_{xxss}(s,x)| + \sup_{0<x<x+y<L} \frac{|u_{xxss}(s,x+y) - u_{xxss}(s,x)|}{y^{2\alpha}}\right] ds < \infty,$$

$$\int_0^\infty \left[\max_{0 \le x \le L} |u_{xxxxs}(s,x)| + \sup_{0<x<x+y<L} \frac{|u_{xxxxs}(s,x+y) - u_{xxxxs}(s,x)|}{y^{2\alpha}}\right] ds < \infty.$$

Then for the solution of difference scheme (4), the following convergence estimate is satisfied

$$\sup_k \left\|u_k^h - u^h(t_k,\cdot)\right\|_{\overset{\circ}{C}{}_h^{2\alpha}} \le M_2\left(\tau^2 + h^2\right)$$

with M_2 is a real number independent of τ, α and h.

Proof. Using notations of A_h^x and $B_h^x(t_k)$ again, we can obtain the following formula for the solution

$$u_k^h(x) = R^k g^h(0,x)$$
$$+ \sum_{j=1}^{k} R^{k-j+1}\left(I + \frac{\tau A_h^x}{2}\right)\left(g^h(t_{j-N},x) + g^h(t_{j-N-1},x)\right)\tau,\ 1 \le k \le N, \quad (9)$$

and

$$u_k^h(x) = R^{k-nN} u_{nN}^h(x)$$
$$+ \sum_{j=nN+1}^{k} R^{k-j+1} \left(I + \frac{\tau A_h^x}{2} \right) B_h^x(t_j) \frac{1}{2} \left(u_{j-N}^h(x) + u_{j-N-1}^h(x) \right) \tau,$$
$$nN \le k \le (n+1)N. \tag{10}$$

where $R = \left(I + \tau A_h^x + \frac{\left(\tau A_x^h\right)^2}{2} \right)^{-1}$. The proof of the Theorem 2 is based on the Formulas (9) and (10) on the convergence theorem for difference schemes in $C_\tau\left(E_\alpha^h\right)$, on the estimate (8) and on the equivalence of the norms as in Theorem 1.

Finally, the numerical methods are given in the following section for the solution of delay parabolic differential equation with the nonlocal condition. The method is illustrated by numerical examples.

3 Numerical Applications

We consider the initial-boundary value problem

$$\begin{cases} \frac{\partial u(t,x)}{\partial t} - \frac{\partial^2 u(t,x)}{\partial x^2} = -(0.1) \frac{\partial^2 u(t-1,x)}{\partial x^2}, \ t > 0, \ 0 < x < \pi, \\ u(t,x) = e^{-4t} \sin 2x, -1 \le t \le 0, \ 0 \le x \le \pi, \\ u(t,0) = u(t,\pi), u_x(t,0) = u_x(t,\pi), \ t \ge 0 \end{cases} \tag{11}$$

for the delay parabolic differential equation. The exact solution of this problem for $t \in [n-1, n], \ n = 0, 1, 2, \cdots, \ x \in [0, \pi]$ is

$$u(t,x) = \begin{cases} e^{-4t} \sin 2x, \quad -1 \le t \le 0, \\ e^{-4t} \left\{ 1 + 4(0.1) e^4 t \right\} \sin 2x, \quad 0 \le t \le 1, \\ e^{-4t} \left\{ 1 + 4(0.1) e^4 t + \frac{\left[4(0.1)e^4(t-1)\right]^2}{2!} \right\} \sin 2x, \quad 1 \le t \le 2, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ e^{-4t} \left\{ 1 + 4(0.1) e^4 t + \cdots + \frac{\left[4(0.1)e^4(t-n)\right]^{n+1}}{(n+1)!} \right\} \sin 2x, \quad n \le t \le n+1, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{cases}$$

We get the following first order of accuracy difference scheme for the approximate solution of the initial-boundary value problem for the delay parabolic Eq. (11)

$$\begin{cases} \frac{u_n^k-u_n^{k-1}}{\tau}-\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}=-0.1\frac{u_{n+1}^{k-N}-2u_n^{k-N}+u_{n-1}^{k-N}}{h^2}, \\ mN+1\le k\le (m+1)N,\ m=0,1,\cdots,\ 1\le n\le M-1, \\ u_n^k=e^{-4t_k}\sin 2x_n,\ -N\le k\le 0,\ 0\le n\le M, \\ u_0^k=u_M^k,\ u_1^k-u_0^k=u_M^k-u_{M-1}^k,\ k\ge 0. \end{cases} \tag{12}$$

We can rewrite (12) in matrix form

$$AU^k+BU^{k-1}=R\varphi^k,\ 1\le k\le N,\ U^0=\varphi. \tag{13}$$

From (13) it follows that

$$U^k=-A^{-1}BU^{k-1}+A^{-1}R\varphi^k,\ 1\le k\le N. \tag{14}$$

Second, using the second order of accuracy difference scheme for the approximate solution of Problem (11) we obtain the following system of equations

$$\begin{cases} \frac{u_n^k-u_n^{k-1}}{\tau}-\frac{u_{n+1}^k-2u_n^k+u_{n-1}^k}{h^2}+\frac{\tau}{2}\left(\frac{u_{n+2}^k-4u_{n+1}^k+6u_n^k-4u_{n-1}^k+u_{n-2}^k}{h^4}\right) \\ =-(0.1)\left\{\frac{u_{n+1}^{k-N}-2u_n^{k-N}+u_{n-1}^{k-N}}{2h^2}+\frac{u_{n+1}^{k-1-N}-2u_n^{k-1-N}+u_{n-1}^{k-1-N}}{2h^2}\right. \\ -\frac{\tau}{2}\left[\frac{u_{n+2}^{k-N}-4u_{n+1}^{k-N}+6u_n^{k-N}-4u_{n-1}^{k-N}+u_{n-2}^{k-N}}{2h^4}\right. \\ \left.\left.+\frac{u_{n+2}^{k-1-N}-4u_{n+1}^{k-1-N}+6u_n^{k-1-N}-4u_{n-1}^{k-1-N}+u_{n-2}^{k-1-N}}{2h^4}\right]\right\}=0, \\ mN+1\le k\le (m+1)N,\ m=0,1,\cdots,\ 2\le n\le M-2, \\ u_n^k=e^{-4t_k}\sin 2x_n,\ -N\le k\le 0,\ 0\le n\le M, \\ u_0^k=u_M^k,\ k\ge 0, \\ 3u_0^k-4u_1^k+u_2^k=-3u_M^k+4u_{M-1}^k-u_{M-2}^k,\ k\ge 0, \\ 2u_0^k-5u_1^k+4u_2^k-u_3^k=2u_M^k-5u_{M-1}^k+4u_{M-2}^k-u_{M-3}^k,\ k\ge 0, \\ 10u_0^k-15u_1^k+6u_2^k-u_3^k=-10u_M^k+15u_{M-1}^k-6u_{M-2}^k+u_{M-3}^k,\ k\ge 0. \end{cases} \tag{15}$$

We have again $(M+1)\times(M+1)$ system of linear equations and we rewrite them in the matrix form (13). Now, we give numerical results for different values of N and M and u_n^k represent the numerical solutions of these difference schemes at (t_k,x_n). Tables 1, 2, 3, and 4 are constructed for $N=M=50$, 100, 200 in

Table 1. Comparison of the errors of different difference schemes in $t \in [0, 1]$

Method	N=M=50	N=M=100	N=M=200
Difference scheme (3)	0.1073487831	0.0547768623	0.0276391767
Difference scheme (4)	0.0022364343	0.0005798409	0.0001463692

Table 2. Comparison of the errors of different difference schemes in $t \in [1, 2]$

Method	N=M=50	N=M=100	N=M=200
Difference scheme (3)	0.0044792100	0.0023497584	0.0012013854
Difference scheme (4)	0.0014763957	0.0003821482	0.0000965218

Table 3. Comparison of the errors of different difference schemes in $t \in [2, 3]$

Method	N=M=50	N=M=100	N=M=200
Difference scheme (3)	0.0030208607	0.0015155583	0.0007582183
Difference scheme (4)	0.0003769232	0.0000973214	0.0000245676

Table 4. Comparison of the errors of different difference schemes in $t \in [3, 4]$

Method	N=M=50	N=M=100	N=M=200
Difference scheme (3)	0.0008494067	0.0004217773	0.0002100074
Difference scheme (4)	0.0000762753	0.0000196839	0.0000049727

$t \in [0, 1]$, $t \in [1, 2]$, $t \in [2, 3]$, $t \in [3, 4]$ respectively and the error is computed by the following formula.

$$E_M^N = \max_{\substack{-N \le k \le N \\ 1 \le n \le M}} \left| u\left(t_k, x_n\right) - u_n^k \right|.$$

Thus the numerical results of this section support the theoretical arguments in Theorems 1 and 2.

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