Recent Advances in Numerical Solution of HJB Equations Arising in Option Pricing

Song Wang(✉) and Wen Li

Department of Mathematics and Statistics, Curtin University,
GPO Box U1987, 6845 Perth, Australia
{Song.Wang,Wen.Li}@curtin.edu.au

Abstract. This paper provides a brief survey on some of the recent numerical techniques and schemes for solving Hamilton-Jacobi-Bellman equations arising in pricing various options. These include optimization methods in both infinite and finite dimensions and discretization schemes for nonlinear parabolic PDEs.

1 Introduction

Financial derivative securities consist of three major parts: *Forwards and Future* (obligation to buy or sell), *Options* (right to buy or sell) and *Swaps* (simultaneous selling and purchasing). The first two form the basis of derivative securities. It is known that an option is a contract which gives to its owner the right, not obligation, to buy (*call*) or sell (*put*) a fixed quantity of assets of a specified stock at a fixed price called *exercise/strike price* on or before a given date (*expiry date*). There are two major types of options – European options which can be exercised only on the expiry date and American options that are exercisable on or before the expiry date.

An option has both intrinsic and time values, and can be traded on a secondary financial market even though it may not be exercisable at the time point. How to accurately price options has long been a hot topic for mathematicians and financial engineers. It was shown by Black and Scholes [7] that the value of a European option on a stock satisfies a second order parabolic partial differential equation with respect to the time t and the underlying asset price S in a complete market with constant volatility and interest rate and without transaction costs on trading the option and its underlying stock. This equation is now known as the Black-Scholes (BS) equation. A more comprehensive discussion of this model can be found in [32]. The BS equation can be solved exactly when the coefficients are constants. However, for problems of practical importance, numerical solutions to them are normally sought. Therefore, efficient and accurate numerical algorithms are essential for solving such a problem accurately.

The value of an American call option is usually the same as that of its European counterpart. However, the value $V(S,t)$ of an American put option on an asset/stock whose price S follows a geometric Brownian motion is governed by the following linear complementarity problem (LCP) (cf., e.g., [45,47])

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$$LV := -\frac{\partial V}{\partial t} - \frac{1}{2}\sigma^2(t)S^2\frac{\partial^2 V}{\partial S^2} - r(t)S\frac{\partial V}{\partial S} + r(t)V \geq 0, \tag{1}$$

$$V - V^*(S) \geq 0, \quad LV \cdot (V - V^*(S)) = 0 \tag{2}$$

for $(S,t) \in \Omega := I \times [0,T)$ almost everywhere ($a.e.$) with the payoff/terminal and boundary conditions

$$V(S,T) = V^*(S), \quad V(0,t) = V^*(0), \quad V(S_{\max},t) = 0, \tag{3}$$

where $I = (0, S_{\max}) \subset \mathbb{R}$ with $S_{\max}$ a positive constant usually much greater than the strike price K of the option, $\sigma(t)$ denotes the volatility of the asset, $r(t)$ the interest rate, and $V^*(S)$ is the final or payoff condition of the option. There are various payoff conditions depending types of options [47]. For example the payoff function for a vanilla American put is

$$V^*(S) = \max\{K - S, 0\}, \quad S \in I.$$

The LCP (1)–(2) can also be written as

$$\min\{LV(S,t), V(S,t) - V^*(S)\} = 0 \quad (S,t) \in \Omega \tag{4}$$

with (3). This equation is called a Hamilton-Jacobi-Bellman (HJB) equation which is usually unsolvable analytically.

When selling and buying a put whose underlying stock incurs transaction costs, the price of the put is no longer governed by the LCP or (4). Instead, a Nonlineaar Complementarity Problem (NCP) needs to be solved to determine the value of such an option. More specifically, the NCP is of the same form as (1)–(2) with $\sigma(t)$ replaced with $\sigma(S,t,V_S,V_{SS})$. Various models for the nonlinear volatility have been proposed, for example [4,8,19,23,24]. A notable one is the following nonlinear volatility model proposed in [4]:

$$\sigma^2(t,S,V_{SS}) = \sigma_0^2\left(1 + \Psi\left(e^{r(T-t)}a^2S^2V_{SS}\right)\right) \tag{5}$$

where σ_0 is a constant, $a = \kappa\sqrt{\nu N}$ with κ being the transaction cost parameter, ν a risk aversion factor and N the number of options to be sold. In the rest of this paper, we simply refer to a as the transaction parameter. The function Ψ in (5) is the solution to the following nonlinear initial value problem

$$\Psi'(z) = \frac{\Psi(z) + 1}{2\sqrt{z\Psi(z)} - z} \quad \text{for} \quad z \neq 0 \quad \text{and} \quad \Psi(0) = 0$$

to which an implicit exact solution is derived in [13].

HJB equations also arise in determination of the reservation price of a European or American option under proportional transaction costs [15–17] and valuation of American options under a Levy process [9,10], with uncertain valotility [31,48], or with stochastic valotility [18,53], just to name a few. All of these problems are of the form:

$$\min\{L_1(V) - f_1, L_2V - f_2, ..., L_mV - f_m\} = 0 \tag{6}$$

in a given solution domain with a set of boundary and terminal/payoff conditions, where m is a positive integer and, usually, L_1 is a 2nd-order nonlinear differential operator and L_i and f_i are respectively a linear 1st- or 0th-order differential operator and a given function for each of $i = 2, ..., m$.

Because of an optimization process involved and non-smooth payoff conditions, (6) in general does not have any classic (twice continuously differentiable) solutions. Instead, it has the so-called viscosity solutions [14]. Uniqueness of the solution to (6) can usually be proved. For some simple cases, it is also possible to prove the unique solvability of (6) using a conventional technique. For instance, if we introduce a weighted Sobolev space $H^1_{0,w}(I)$ and a convex set $\mathcal{K} = \{v \in H^1_{0,w}(I) : v \le u^*\}$ with $u^* = e^{\beta t}(V_0 - V^*)$, where $V_0 = (1 - S/S_{\max})K$ and $\beta = \sup_{0<t<T} \sigma^2(t)$, the LCP (1) and (2) can be cast into the following Variational Inequality (VI) (cf. [45] for details).

Problem 1. Find $u(t) \in \mathcal{K}$ such that, for all $v \in \mathcal{K}$,

$$\left(-\frac{\partial u(t)}{\partial t}, v - u(t)\right) + A(u(t), v - u(t); t) \ge (f(t), v - u(t)) \tag{7}$$

a.e. in $(0, T)$, where $A(\cdot, \cdot)$ is a bilinear form defined by

$$A(u, v; t) = \left(aS^2 u' + bSu, v'\right) + (cu, v), \quad u, v \in H^1_{0,w}(I) \tag{8}$$

with $(\cdot, \cdot)$ denoting the usual inner product, $u = -e^{\beta t}(V - V_0)$, $f(t) = e^{\beta t} LV_0$, $a = \sigma^2/2$, $b = r - \sigma^2$, and $c = 2r + \beta - \sigma^2$.

For this VI we can show that the bilinear form $A(\cdot, \cdot)$ is coercive and Lipschitz continuous, and thus Problem 1 has a unique solution by a standard argument.

2 The Penalty Method in Infinite Dimensions

The HJB Eq. (6), particularly (7), may be viewed as a constrained optimization problem and it can be solved using an optimization technique. A popular choice is a penalty approach in which a constrained optimization problem is approximated by an unconstrained one with a penalty term in the objective function. Since the resulting optimization problem is unconstrained, it is easier to solve than the original. The linear penalty method for HJB equations was discussed in detail in [6] and its extension to arbitrary power penalty has been proposed and analyzed in [42,45,52] for various HJB equations.

Let us demonstrate the power penalty method using (1) and (2) which can be rewritten in the following standard form

$$\mathcal{L}u(x, t) := \frac{\partial u}{\partial t} + \frac{\partial}{\partial S}\left[a(t)S^2 \frac{\partial u}{\partial S} + b(t)Su\right] - c(t)u \le f(S, t), \tag{9}$$

$$u(S, t) - u^*(S, t) \le 0, \quad (\mathcal{L}u(S, t) - f(S, t)) \cdot (u(S, t) - u^*(S, t)) = 0, \tag{10}$$

in Ω with the boundary and terminal conditions

$$u_\lambda(0, t) = 0 = u_\lambda(X, t) \quad \text{and} \quad u_\lambda(S, T) = u^*(S, T), \tag{11}$$

where the coefficients functions and given data are defined in Problem 1. In fact, (7) is the variational form of this LCP, and (9)–(10) can be viewed as the optimality conditions of a constrained functional optimization problem with the constraint in (10). (We will show this in finite dimensions later.) This motivates us to devise the following penalty equation

$$\mathcal{L}u_\lambda(S,t) + \lambda[u_\lambda(S,t) - u^*(S,t)]_+^{1/k} = f(S,t), \quad (S,t) \in \Omega \tag{12}$$

satisfying (11), where $\lambda > 0$ and $k > 0$ are parameters and $[z]_+ = \max\{z, 0\}$ for any z. In (12) the power penalty term $\lambda[u_\lambda(S,t) - u^*(S,t)]_+^{1/k}$ penalizes the positive part of $u_\lambda - u^*$.

Equation (12) is a nonlinear parabolic PDE even when $k = 1$ and the variational problem corresponding to (12) is

Problem 2. Find $u_\lambda(t) \in H^1_{0,w}(I)$ such that, for all $v \in H^1_{0,w}(I)$,

$$\left(-\frac{\partial u_\lambda(t)}{\partial t}, v\right) + A(u_\lambda(t), v; t) + \lambda\left([u_\lambda(t) - u^*(t)]_+^{1/k}, v\right) = (f(t), v) \tag{13}$$

a.e. in $(0,T)$, where A is the bilinear form defined in (8).

The unique solvability of Problem 2 can be proved by showing that the mapping on the LHS of (13) is strongly monotone and continuous [45].

The solution to Problem 2 is in general not equal to that of Problem 1, but we expect that when $\lambda \to \infty$, $u_\lambda \to u$ at some rate depending on λ and k. A convergence theory for this penalty method is established in [6] for $k = 1$ and in [45] for any $k > 0$, which requires the introduction of some function spaces and norms.

For any Hilbert space $H(I)$, let $L^p(0,T;H(I))$ denote the space defined by

$$L^p(0,T;H(I)) = \{v(\cdot,t) : v(\cdot,t) \in H(I) \text{ a.e. in } (0,T); ||v(\cdot,t)||_H \in L^p((0,T))\},$$

where $1 \le p \le \infty$ and $||\cdot||_H$ denotes the natural norm on $H(I)$. The space $L^p(0,T;H(I))$ is equipped with the norm $||v||_{L^p(0,T;H(I))} = \left(\int_0^T ||v(\cdot,t)||_H^p dt\right)^{1/p}$. Clearly, $L^p(0,T;L^p(I)) = L^p(I \times (0,T)) = L^p(\Omega)$. Using this space, it is possible to establish the following theorem.

Theorem 1. *Let u and u_λ be the solutions to Problems 1 and 2, respectively. If $u_\lambda \in L^{1+1/k}(\Omega)$ and $\frac{\partial u}{\partial t} \in L^{k+1}(\Omega)$, then there exists a constant $C > 0$, independent of u, u_λ and λ, such that*

$$||u - u_\lambda||_{L^\infty(0,T;L^2(I))} + ||u - u_\lambda||_{L^2(0,T;H^1_{0,w}(I))} \le \frac{C}{\lambda^{k/2}}, \tag{14}$$

where k is the parameter used in (13).

Theorem 1 tells us that $u_\lambda \to u$ at the rate of $\mathcal{O}(\lambda^{-k/2})$ as λ or/and k goes to ∞. Similar results for Nonlinear Complementarity Problems (NCPs) and bounded

NCPs are given in [42,52]. The idea of the above penalty approach can also be used for solving (6). More specifically, (6) can be approximated by the following penalty equation

$$L_1(V_\lambda) - \sum_{i=1}^{m} \lambda_i [L_i V_\lambda - f_i]_-^{1/k} = f_1, \tag{15}$$

where $[z]_- = \max\{0, -z\}$ for any z, $\lambda = (\lambda_1, ..., \lambda_m)^\top$ is a set of penalty parameters and $k > 0$ is a power parameter. (Clearly, we may use different k's for different penalty terms in the above equation.) It would be thought that, as established above for Problem 2, (15) is uniquely solvable and its solution converges exponentially to that of (6). However, for some cases we are only able to prove that (15) has a unique viscosity solution and the solution converges to that of (6), but unable to establish the rates of convergence. For example, the penalty method for the HJB equations arising from determining the reservation prices of European and American options with transaction costs [28–30] in which the constraints contain derivatives of the solution. The main reason for this is that when the penalty terms contain differential operators, we may not be able to prove the strong monotonicity of the operator on the LHS of (15).

We also comment that though the solution to the penalty equation converges to that of the HJB equation, it does not mean that the constraints are strictly satisfied for any fixed (λ, k). Instead, they are satisfied up to an approximation error. Thus, the above method is sometimes called an exterior penalty method. It is possible to construct an interior penalty method such as that proposed in [35] and analyzed in [49] in which an approximation always satisfies the constraints.

3 Discretization Schemes

LCPs and HJB equations in infinite dimensions such as (6, 9, 10 and 13) can hardly be solved exactly unless for some trivial cases. Therefore, a numerical scheme is needed to for the discretization of such a system so that the discretized system can be solved by linear/nonlinear algebraic system solver in finite dimensions. Various discretization schemes can be used for the PDEs depending on the problem in question. Popular spatial discretization schemes for (12) are

- upwind finite difference schemes [11,25,26,29,30,37],
- fitted finite volume method [3,12,22,36,39,46,50], and
- finite element methods [1,2,38].

In designing a descretization scheme for (12), the main requirements are as follows.

1. The scheme should be unconditionally or conditionally stable and the solution to the discretized system should converge to the viscosity solution to (13).
2. The solution to the discretized system should be non-negative irrespectively of choices of mesh or other parameters, as by nature prices are non-negative.
3. The finite dimensional linear/nonlinear system can be solved efficiently by an advanced, usually iterative, solver.

Item 2 is guaranteed if the discretiztion is monotone or the system matrix of the discrtized equation is an M-matrix in the linear case. In this case, a discrete maximum principle is satisfied by the scheme and so the solution to the discrtized system attains its extrema at the boundary of the solution domain.

Note that the BS operator $\mathcal{L}$ in (12) becomes degenerate as $S \to 0^+$. Mathematically, the weak solution to (12) cannot take a trace (boundary condition) at $S = 0$. This is why we needed the introduction of a weighted Sobolev space $H^1_{0,w}(I)$ in the previous section. Also, because of this difficulty, one usually needs to truncate the spatial domain I into $(S_{\min}, S_{\max})$ for a small positive number $S_{\min} < S_{\max}$ if a conventional scheme is used to discretize (12). Equivalently, a common practice is to use $x = \ln S$ to transform I into $-\infty < x < \ln S_{\max}$ and solve the transformed problem on a finite interval. A fitted finite volume is proposed in [39] for solving the BS equation governing European options without this domain transformation or truncation. The scheme has the merit that it is unconditionally stable, has the first-order convergence rate in mesh parameters and yields a system of which the coefficient matrix is an M-matrix. We now demonstrate this scheme using (12).

Let I be divided into N sub-intervals $I_i := (S_i, S_{i+1})$, $i = 0, 1, .., N-1$ with $0 = S_0 < S_1 < \cdots < S_N = S_{\max}$. For each $i = 0, 1, ..., N-1$, we put $h_i = S_{i+1} - S_i$ and $h = \max_{0 \le i \le N-1} h_i$. Dual to this mesh, we define another mesh with nodes $S_{i-1/2} = (S_{i-1} + S_i)/2$ for $i = 1, 2, ..., N$, $S_{-1/2} = 0$ and $S_{N+1/2} = S_{\max}$. Integrating both sides of (12) over $(S_{i-1/2}, S_{i+1/2})$ and applying the mid-point quadrature rule to the first, third, fourth and last terms, we obtain

$$-\frac{\partial u_i}{\partial t} l_i - \left[S_{i+1/2}\rho(u)|_{S_{i+1/2}} - S_{i-1/2}\rho(u)|_{S_{i-1/2}}\right] + \left[c_i u_i + \lambda[u_i - u_i^*]_+^{1/k}\right] l_i = f_i l_i \tag{16}$$

for $i = 1, 2, ..., N-1$, where $l_i = S_{i+1/2} - S_{i-1/2}$, $c_i = c(S_i, t)$, $f_i = f(S_i, t)$, $u_i^* = u^*(S_i)$, u_i is the nodal approximation to $u(S_i, t)$ to be determined and $\rho(u)$ is a flux associated with u defined by $\rho(u) := aSu_S + bu$.

To derive an approximation to the flux at the two end-points $S_{i+1/2}$ and $S_{i-1/2}$, let us consider the following two-point boundary value problem

$$(\rho_i)' := (aSv' + b_{i+1/2}v)' = 0, \ S \in I_i, \ v(S_i) = u_i, \ v(S_{i+1}) = u_{i+1}, \tag{17}$$

where $b_{i+1/2} = b(S_{i+1/2}, t)$. This is motivated by the technique used for singularly perturbed convection-diffusion equations (cf. [33,34]). When $i \ge 1$, (17) has the exact solution

$$\rho_i = b_{i+1/2}\frac{S_{i+1}^{\alpha_i}u_{i+1} - S_i^{\alpha_i}u_i}{S_{i+1}^{\alpha_i} - S_i^{\alpha_i}}, \quad v = \frac{\rho_i}{b_{i+1/2}} - \frac{u_{i+1} - u_i}{S_{i+1}^{\alpha_i} - S_i^{\alpha_i}}(S_i S_{i+1})^{\alpha_i} S^{-\alpha_i}, \tag{18}$$

where $\alpha_i = b_{i+1/2}/a$. Obviously, ρ_i i provides an approximation to the flux $\rho(u)$ at $S_{i+1/2}$ for $i = 1, ..., N-1$.

When $i = 0$, (17) becomes degenerate at $S = 0$, and we need to look into the asymptotic behaviour of ρ_0 as $S_0 \to 0^+$. This is given in the following two cases.

If $\alpha_0 < 0$, it is easy t see to verify that $\lim_{S_0 \to 0^+} \rho_0 = b_{1/2} u_0$. Similarly, if $\alpha_0 > 0$, we have from (18) $\lim_{x_0 \to 0^+} \rho_0 = b_{1/2} u_1$. Combining these two cases we have

$$\rho_0 = b_{1/2} \frac{1 - \text{sign}(b_{1/2})}{2} u_0 + b_{1/2} \frac{1 + \text{sign}(b_{1/2})}{2} u_1, \tag{19}$$

since $\alpha_0 = b_{1/2}/a$ and $b_{1/2}$ have the same sign pattern.

Using (18) and (19), we have from (16)

$$-\frac{\partial u_i}{\partial t} l_i + e_{i,i-1} u_{i-1} + e_{i,i} u_i + e_{i,i+1} u_{i+1} + d_i(u_i) = f_i l_i, \tag{20}$$

where, $d_i(u_i) = \lambda l_i [u_i - u_i^*]_+^{1/k}$,

$$\begin{aligned}
e_{1,0} &= -\frac{x_1}{2} b_{1/2} \frac{1 - \text{sign}(b_{1/2})}{2}, \quad e_{1,2} = -\frac{b_{1+1/2} x_{1+1/2} x_2^{\alpha_1}}{x_2^{\alpha_1} - x_1^{\alpha_1}}, \\
e_{1,1} &= \frac{x_1}{2} b_{1/2} \frac{1 + \text{sign}(b_{1/2})}{2} + \frac{b_{1+1/2} x_{1+1/2} x_1^{\alpha_1}}{x_2^{\alpha_1} - x_1^{\alpha_1}} + c_1 l_1, \\
e_{i,i-1} &= -\frac{b_{i-1} x_{i-1/2} x_{i-1}^{\alpha_{i-1}}}{x_i^{\alpha_{i-1}} - x_{i-1}^{\alpha_{i-1}}}, \quad e_{i,i+1} = -\frac{b_{i+1/2} x_{i+1/2} x_{i+1}^{\alpha_i}}{x_{i+1}^{\alpha_i} - x_i^{\alpha_i}}, \\
e_{i,i} &= \frac{b_{i-1} x_{i-1/2} x_i^{\alpha_{i-1}}}{x_i^{\alpha_{i-1}} - x_{i-1}^{\alpha_{i-1}}} + \frac{b_{i+1/2} x_{i+1/2} x_i^{\alpha_i}}{x_{i+1}^{\alpha_i} - x_i^{\alpha_i}} + c_i l_i
\end{aligned}$$

for $i = 2, 3, ..., N-1$. These form an $N-1$ nonlinear ODE system for $U(t) := (u_1(t), ..., u_N(t))^\top$ with the homogeneous boundary condition $u_0(t) = 0 = u_N(t)$.

Let E_i be a row vector defined by $E_i = (0, .., 0, e_{i,i-1}(t), e_{i,i}(t), e_{i,i+1}(t), 0, ..., 0)$ fir $i = 1, ..., N-1$, where $e_{i,i-1}, e_{i,i}, e_{i,i+1}$ are defined above and those not defined are zeros. Obviously, using E_i, (20) can be rewritten as

$$-\frac{\partial u_i(t)}{\partial t} l_i + E_i(t) U(t) + d_i(u_i(t)) = f_i(t) l_i, \tag{21}$$

for $i = 1, 2, ..., N-1$. This is a first order ODE system.

To discretize (21), we choose t_i $(i = 0, 1, ..., M)$ satisfying $T = t_0 > t_1 > \cdots > t_M = 0$, and apply the two-level implicit time-stepping method with a splitting parameter $\theta \in [1/2, 1]$ to (21) to yield

$$\begin{aligned}
\frac{u_i^{m+1} - u_i^m}{-\Delta t_m} l_i + \theta \left[E_i^{m+1} U^{m+1} + d_i(u_i^{m+1})\right] + (1-\theta) \left[E_i^m U^m + d_i(u_i^m)\right] \\
= (\theta f_i^{m+1} + (1-\theta) f_i^m) l_i
\end{aligned}$$

for $m = 0, 1, ..., M-1$, where $\Delta t_m = t_{m+1} - t_m < 0$, $E_i^m = E_i(t_m)$, $f_i^m = f(x_i, t_m)$ and $U^m = (u_1^m, u_2^m, ..., u_{N-1}^m)^\top$. This nonlinear system can be rewritten as the following matrix form

$$(\theta E^{m+1} + G^m) U^{m+1} + \theta D(U^{m+1}) = F^m + [G^m - (1-\theta) E^m] U^m - (1-\theta) D(U^m)$$

for $m = 0, 1, ..., M - 1$, where the coefficient matrices are self-explanatory. The boundary and terminal conditions are $u_0^m = 0 = u_N^m$ and $U^0 = (u_1^*, u_2^*, ..., u_{N-1}^*)^\top$.

When $\theta = 1/2$, the time-stepping scheme becomes that of the Crank-Nicolson and when $\theta = 1$, it is the backward Euler scheme. Both of the two cases are unconditionally stable. It is easy to show that the linear part of the coefficient matrix of the above system is an M-matrix and the nonlinear part is strongly monotone. Thus, the solution to the system is non-negative. An upper error bound of order $\mathcal{O}(h + \Delta t)$ in a discrete analogue of the norm in (14) for the solution to the above system has been proved in [3] under certain conditions, where h and Δt denote the maximal mesh sizes in space and time. A superconvergent fitted finite volume method for (12) with the linear penalty, based on a judicious choice of the dual mesh in the above scheme, has been recently proposed in [46] which has the merit that the scheme yields a superconvergent derivative (Delta of an option) with almost no additional computational costs.

Upwind finite difference schemes in space have been used for solving HJB equations arising from pricing other types of options such as those in [26, 29, 30]. In these cases, we showed the convergence of the numerical schemes by proving they are consistent, stable and monotone [5]. However, convergence rates for these schemes have not been established. We comment that, the use of upwind finite difference methods for multi-dimensional HJB equations such as those arising from pricing options on multiple assets or with stochastic volatility [22, 51] does not in general yield systems whose coefficient matrices are M-matrix. In this case, the finite volume method has to be used.

4 The Penalty Method in Finite Dimensions

The process for solving HJB equations in the previous sections is to use the penalty equation to approximate an HJB equation and then solve the penalty equation by a discretization scheme. This process is reversible, i.e., one may discretize an HJB equation first to yield a finite-dimensional one and then devise a penalty method for solving the HJB equation in finite-dimensions. Let us demonstrate this procedure using (9) and (10).

The application of the fitted finite volume method and time-stepping scheme in the previous section to the LCP (9) and (10) yields, at each time step, an LCP of the form

$$Ax \leq b, \quad x \leq 0 \quad \text{and} \quad x^\top (Ax - b) = 0, \tag{22}$$

where $x \in \mathbb{R}^n$ for a positive integer n, A is an $n \times n$ positive-definite matrix and $b \in \mathbb{R}^n$ is a known vector. In (22), x represents an approximation of the values of $u - u^*$ at the interior spatial mesh nodes. Let us consider the minimization problem

$$\min_{x \in \mathbb{R}^n} Q(x) \quad \text{subject to} \quad x \leq 0, \tag{23}$$

where $Q(x)$ is a quadratic function of x such that $\nabla Q(x) = Ax - b$. The Karush–Kuhn–Tucker (KKT) conditions for this problem are

$$Ax - b + \mu = 0, \quad \mu^\top x = 0, \quad \mu \geq 0, \quad x \leq 0,$$

where $\mu \in \mathbb{R}^n$ is the multiplier. From the first and third expressions in the above we have $Ax - b = -\mu \leq 0$. Using this inequality and eliminating μ from the above yields (22). Therefore, (22) is an optimality condition for (23) and thus both have the same solutions. To find an approximation to the solution of (23), we consider the following unconstrained problem

$$\min_{x\in\mathbb{R}^n} \left(Q(x) + \frac{\lambda}{1+1/k}[x]_+^{1+1/k} \right),$$

where $\lambda > 1$ is the penalty constant and $k > 0$ is a parameter. The 1st-order necessary optimality condition for this problem is

$$Ax_\lambda - b + \lambda[x_\lambda]_+^{1/k} = 0 \quad \text{or} \quad Ax_\lambda + \lambda[x_\lambda]_+^{1/k} = b.$$

This is a (power) penalty equation approximating (22). Clearly, it is a finite-dimensional analogue of (12). The solution x_λ is an approximation to that of (22).

Discretized HJB equations are often of the form: $f(x) \leq b$, $x \leq 0$ and $x^\top f(x) = 0$ ([27]). Following the above discussion, the penalty equation approximating this NCP is

$$f(x_\lambda) + \lambda[x_\lambda]_+^{1/k} = 0, \tag{24}$$

where $f : \mathbb{R}^n \mapsto \mathbb{R}^n$. If, for some $\alpha, \beta > 0$, $\gamma \in (0, 1]$ and $\xi \in (1, 2]$, f satisfies

1. Holder Continuity: $||f(x_1) - f(x_2)||_2 \leq \beta||x_1 - x_2||_2^\gamma, \quad \forall x_1, x_2 \in \mathbb{R}^n$,
2. ξ-monotonicity: $(x_1 - x_2)^\top(f(x_1) - f(x_2)) \geq \alpha||x_1 - x_2||_2^\xi, \quad \forall x_1, x_2 \in \mathbb{R}^n$,

then one can show that there exist a positive constant C, independent of λ, such that

$$||x_\lambda - x||_2 \leq \frac{C}{\lambda^{k/(\xi-\gamma)}}, \tag{25}$$

where x and x_λ are respectively the solutions to the NCP and (24) and $||\cdot||_2$ denotes the l_2-norm on $\mathbb{R}^n$ (cf. [20,43]).

Note that the convergence rate of the power penalty method in finite-dimensions established above is higher than the one in (14) of its infinite-dimensional counterpart, particularly when $f(x)$ is strongly monotone and Lipschitz continuous (i.e., $\xi = 2$ and $\gamma = 1$). However, the arbitrary constant C in (25) is dependent on the dimensionality n, since in the proof of (25) we used the fact that all norms in $\mathbb{R}^n$ are equivalent which is not true in infinite dimensions.

Discretization of some HJB equations such as those arising from determining reservation price of an option under proportional transaction costs (cf. [29,30]) gives rise to optimization problems with bound constraints on Bx for a non-square matrix B, rather than on x. This kind of HJB equations also arises in optimization problems with bound constraints on derivatives. Power penalty methods have been extended to NCPs and mixed NCPs with either unbounded or bounded linear constraints (cf. [21,40,41,44]) and the upper error bounds in these cases are essentially the same as that in (25).

Note that the nonlinear penalty term in (24) becomes non-Lipschitz when $k > 1$. When solve (24) using a gradient-based method such Newton's method, the penalty term needs to be smoothed out locally in $[0, \varepsilon]$ with ε a small positive number, i.e., we replace $\lambda[x_\lambda]_+^{1/k}$ with $\lambda\phi(x_\lambda)$, where ϕ is given by

$$\phi(z) = \begin{cases} z^{\frac{1}{k}}, & z \geq \varepsilon, \\ \left[\varepsilon^{\frac{1}{k}-2}(3 - \frac{1}{k})z^2 + \varepsilon^{\frac{1}{k}-3}(\frac{1}{k} - 2)z^3\right], & z < \varepsilon, \end{cases}$$

The coefficient matrix of the linearized system of (24) is usually an M-matrix and thus a preconditioned conjugate gradient based iterative method can be used for solving it, particularly when (24) is large-scale.

5 Concluding Remarks

Pricing financial options often involves numerical solution of PDE-constrained nonlinear and non-smooth optimization problems. An efficient numerical technique for option pricing should contain three components - discretization of differential operators, techniques for constrained optimization and numerical solution of nonlinear and non-smooth algebraic systems. In this work we have presented some of our recent advances in the development of efficient and accurate numerical methods for pricing options. Extensive numerical experiments on these methods have been carried out and we refer the reader to the listed references for details.

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